Question 10 Solution

COMP3121/9101 21T3 Final Exam

This document gives a model solution to question 10 of the final exam. Note that alternative solutions may exist.

1. There are n monsters planning to take over the city, but only one hero guards the city. The hero has combat effectiveness a_0 and initially has b_0 health points, while the ith monster has combat effectiveness a_i and initially has b_i health points. Both combat effectiveness and health points are positive integers. Both the monsters and the hero die when they reach zero health points or less.

   In order to protect the people in the city, the hero will fight monsters until either all monsters are killed or the hero dies. In each fight, the hero can fight any living monster. If the ith monster is selected, then that monster loses a_0 health points and the hero loses a_i health points. After any fight, it is possible that neither the hero nor the monster dies, and it is also possible that both are killed. However, each time the hero kills the selected monster without dying, the hero gains h health points as their success inspires them to keep fighting.

   Design an algorithm which determines whether the hero can successfully kill all the monsters (surviving all fights) and runs in $O(n \log n)$ time.
   You must provide reasoning to justify the correctness and time complexity of your algorithm.

   The input consists of the positive integers n and h, as well as $2n+2$ positive integers $a_0, b_0, a_1, b_1, \ldots, a_n, b_n$.

   The output is either YES or NO.

   For example, suppose $n = 2$ and $h = 10$. Suppose the hero has combat effectiveness $a_0 = 5$ and initial health $b_0 = 9$, and that the monsters have:

   1. combat effectiveness $a_1 = 3$ and initial health $b_1 = 14$.
   2. combat effectiveness $a_2 = 4$ and initial health $b_2 = 8$.

   The correct answer for this example is YES.

   For each monster i, first work out the amount of health c_i that must be expended in order to kill that monster. This is given by the number of fights required $\lceil b_i/a_0 \rceil$ multiplied by the combat effectiveness a_i of the monster.

   We sort the monsters by increasing order of c_i. This takes $O(n \log n)$ time. Now, for each monster in this order, the hero should select that monster and fight it until the

monster is killed. The hero spends c_i health points to kill monster i, and if the hero survives, they gain h health points. The health points of the hero can be updated in $O(1)$ for each monster. If the hero's health points remain positive at the conclusion of every fight, the answer is YES. Otherwise, the answer is NO.

The total time complexity is $O(n \log n + n) = O(n \log n)$.

Proof of correctness: First, observe that the hero should fight one monster until it is dead, rather than switching between living monsters. The latter approach rearranges the order in which the hero takes damage, and delays the bonus h points for killing a monster. This is clearly no better than fighting the monsters one by one.

It remains to prove that this sequence of monsters is optimal, using a 'greedy stays ahead' proof. Since we expend the fewest possible health points to kill the first monster, our algorithm maximises the health points of the hero immediately after killing one monster. We then maximise the health points of the hero after killing two monsters and so on until all monsters are killed, and if these maxima are positive the answer is YES.