Section 5: Advanced Dynamic Programming

CS 124

March 3, 2021

Outline

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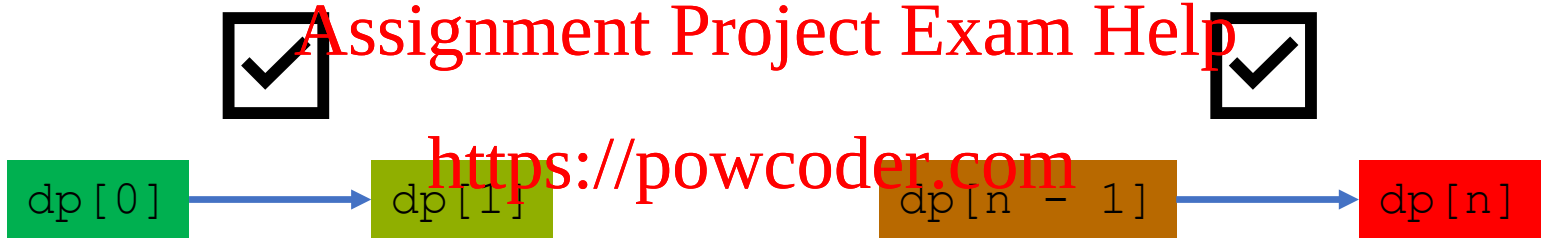
Dynamic Programming

The central idea of dynamic programming is to break a large problem into many sub-problems, and then solve those subproblems in order to build up to a solution.

`dp[0]`

`dp[n]`



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Team Formation

- A corporate restructuring is underway, and we're reorganizing the company to optimize our profits. $\mathcal{X}$ is the set of employees.

- We have a real-valued function $p : 2^{\mathcal{X}} \setminus \{\varnothing\} \to \mathbb{R}$, which tells us the profits that can be generated by a team whose members are a subset of the employees $s \subseteq \mathcal{X}$.

- The profits from the partition $\mathcal{X} = S_1 \uplus S_2 \ldots \uplus S_k$ are

$$p(S_1) + p(S_2) + \ldots + p(S_k).$$

How do we maximize our profits?

Brute-Force

- In order for our algorithm's input size to be n, we let $|\mathcal{X}| = \log_2 n$.
- One approach is to use brute force, and iterate over all partitions of $\mathcal{X}$.
- **Challenge:** Prove there are $n^{\Theta(\log \log n)}$ distinct partitions of $\mathcal{X}$.

Subset Dynamic Programming

- For a faster algorithm, we will use dynamic programming.
- Let $\mathrm{dp}[S]$ be the most profit we can get from our teammates in the subset $S \subseteq \mathcal{X}$.

- Then, we have

$$\mathrm{dp}[\varnothing] = 0, \mathrm{dp}[S] = \max_{\substack{S' \subseteq S \\ S' \neq \varnothing}} p(S') + \mathrm{dp}[S \setminus S']$$

- What's the run-time of this algorithm?

Analysis

- There are n states, and each transition takes at most n time, so the run time is $O(n^2)$.

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- There are n states, and each transition takes at most n time, so the run time is $O(n^2)$.

- However, the analysis actually gets us a little more than that. Namely, note that there are $\binom{\log_2 n}{i}$ sets of size i, and for each of those states they take 2^i iterations, so

$$\sum_{i=0}^{\log_2 n} \binom{\log_2 n}{i} 2^i = (1+2)^{\log_2 n} = n^{\log_2(3)} \approx n^{1.585},$$

which is a smaller polynomial.

Implementation Notes

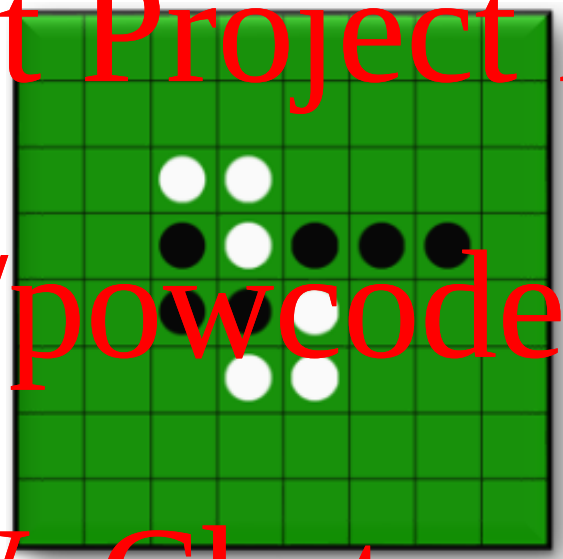
- In practical applications, we have to decide how to represent each subset. Simply using the set to, say, key a map is quite inefficient.

- For example, in this problem we'd like iterate through each $S \subseteq \mathcal{X}, S' \subseteq S$ efficiently.

- In order to obtain memory savings, we represent subsets with binary representations. This is a somewhat common trick which appears in many settings. One name for this trick is called "bitmasks".

A story

In high school, I was part of an AI Othello tournament.

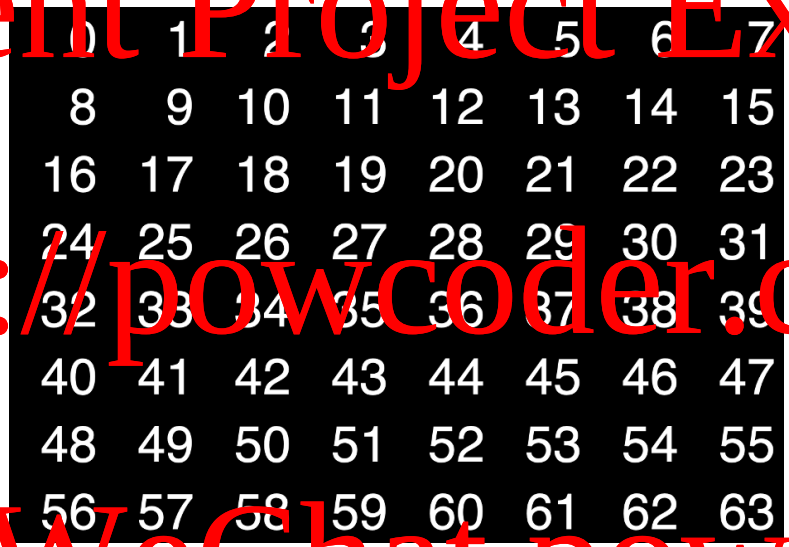


How do we represent a grid efficiently? The most standard approach is to represent the positions of the light counters with an array, say

$$[(3,3),(3,4),(4,4),(5,5),(6,4),(6,5)]$$

Using Binary

One approach is to use a *bitboard*, so the light counters define a number (with binary) and the dark counters also define a number.



The light counters correspond to the squares

$$[18, 19, 27, 36, 43, 44].$$

The binary representation is thus

$$2^{18} + 2^{19} + 2^{27} + 2^{36} + 2^{43} + 2^{44}.$$

Set operations as binary operations

Another benefit of this integer representation is that certain set operations can be expressed as bitwise operations. For example, $S \subseteq S'$ becomes

$$f(s, s') := (b(s) \,\&\sim b(s')) = 0,$$

where b is the binary form of the subset s. One way to see this is to look at this bit-by-bit.

For example, something being zero means that each bit is set to zero.

For $f(s, s')_i = 1$, we would need $b(s)_i = 1$ and $b(s')_i = 0$ for some i – which is exactly precluded by $S \subseteq S'$, as then $i \in S$ and $i \notin S'$.

An Implementation

In what follows let $x = 2$.

```
//set all dp[i] to negative infinity
dp[0] = 0
for (i = 0; i < 2^x; i++){
 for (j = 1; j < 2^x; j++){
  if (j & ~i == 0)
  dp[i] = max(dp[i], dp[i - j] + p[j])
 }
}
return dp[2^x - 1]
```

How do we optimize this?

Note that we are blindly checking each j to see if it a subset of i, and this algorithm is still $O(n^2)$. Surely there's a better way to do this?

```
//set all dp[i] to negative infinity
dp[0] = 0
for (i = 0; i < 2^x; i++) {
 for (j = i; j > 0; j = (j - 1) & i)
  dp[i] = max(dp[i], dp[i - j] + p[j])
 }
}
return dp[2^x - 1]
```

Challenge: Show that this iterates through all desired j. Can you think of a way of replacing $i - j$ with something else, to make it faster?

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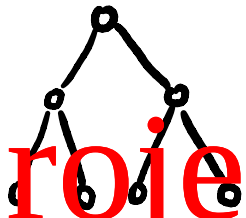
Tree-based dynamic programming



Many problems are particularly natural on trees. Dynamic Programming is often a very useful technique on these problems, where the subproblems are given over the subtrees.

- Each of the subproblems is solving the problem over a given subtree.
- At each step, we solve the problem over each "direct" subtree, and then aggregate those results into a solution for this subtree.

Pseudocode, first attempt


```
dfs(v){
  for (vertex j: N(v)){
    dfs(j);
    //calculate dp[v] with dp[j]
  }
}
dfs(root)
```


What's the problem with this algorithm?

Pseudocode, fixed


```
dfs(v, p){
  for (vertex j: N(v)){
    if (j != p) dfs(j, v);
    //calculate dp[v] with dp[j]
  }
}
dfs(root, nil)
```

Maximum Weighted Matching

We have a tree. What's the highest-weight matching in this tree? A *matching* is a set of edges with no vertex overlap.

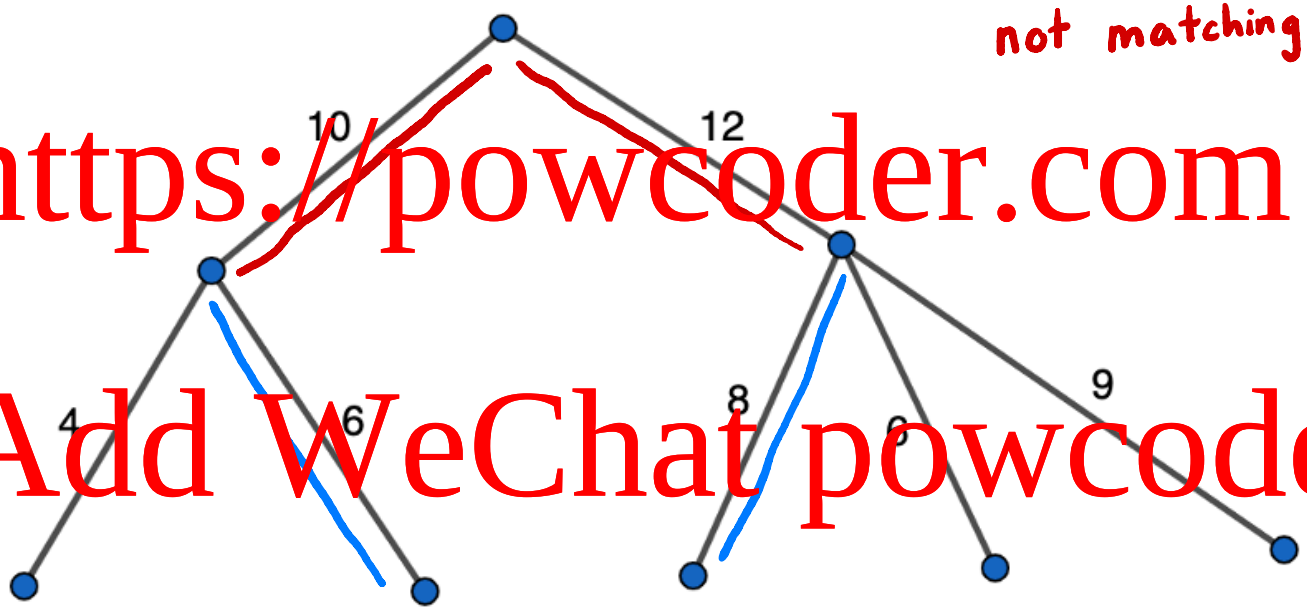
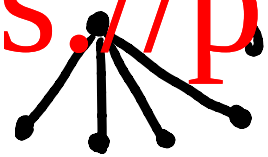


Figure: A tree.

Algorithm

To solve this problem, we write dp[v][0] to be the largest matching without using the root, and dp[v][1] to be the largest matching using the root.

$$dp[v][0] = \sum_{j \in N(v)} \max(dp[j][0], dp[j][1])$$



$$dp[v][1] = \max_{j \in N(v)} \left(w_{v,j} + dp[j][0] + \sum_{k \in N(v), k \neq p, j} \max(dp[k][0], dp[k][1]) \right)$$

Algorithm

To solve this problem, we write $\text{dp}[v][0]$ to be the largest matching without using the root, and $\text{dp}[v][1]$ to be the largest matching using the root.

Then, we have that

$$\text{dp}[v][1] = \max_{\substack{u \in N(v) \\ u \neq p}} \left(w_{u,v} + \text{dp}[u][0] + \sum_{\substack{j \in N(v) \\ j \neq u,p}} \max(\text{dp}[j][0], \text{dp}[j][1]) \right)$$

and

$$\text{dp}[v][0] = \sum_{\substack{j \in N(v) \\ j \neq p}} \max(\text{dp}[j][0], \text{dp}[j][1]).$$

and the answer is $\max(\text{dp}[r][0], \text{dp}[r][1])$.

Runtime Analysis

What's the runtime of this algorithm?

$O(n^2)$

n times

$O(n)$ operations

```
dfs(v, p){
  for (vertex j: N(v)){
    if (j != p) dfs(j, v);
  }
  // calculate dp[v] with dp[j]
}
dfs(root, nil);
```

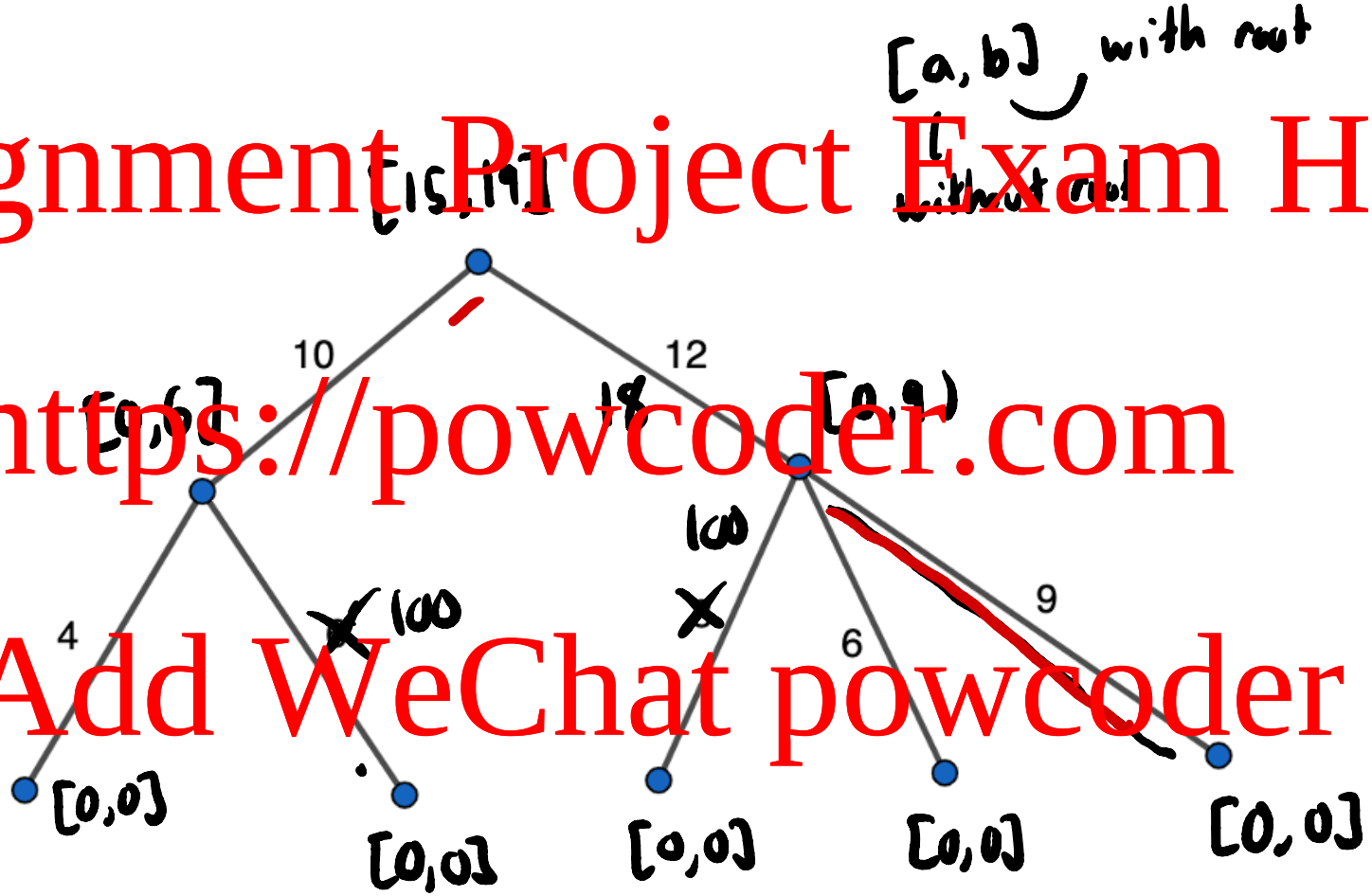
Runtime Analysis

What's the runtime of this algorithm?

```
dfs(v, p){
    for (vertex j: N(v)){
      if (j != p) dfs(j, v);
      //calculate dp[v] with dp[j]
    }
}
dfs(root, nil)
```


Since each vertex j which is not the root has dfs called on it exactly once, the total amount of calls across all internal loops is is $O(n)$. However, we need to make sure that the internal processing is also $O(d)$ where d is the number of neighbors, or we might be in trouble!

Worked Example



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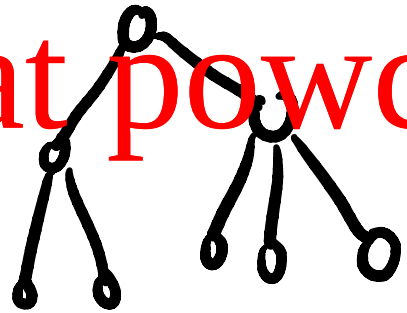
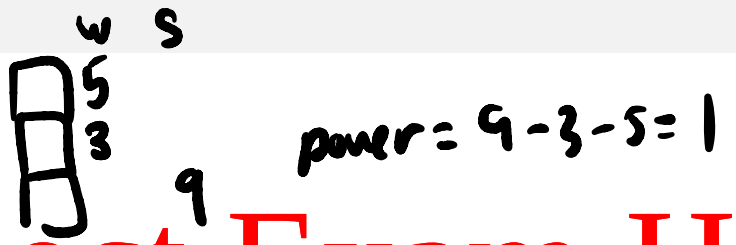
Problem 1: Optimal Taxation

You are the king of a kingdom that's set up like a tree, and you want to collect some taxes. You know that each city has a "tax potential" t_i that you can collect. However, you don't want to tax adjacent cities, because they might talk to each other and begin a revolt. Find the maximum tax you can collect in linear time.



Problem 2: Stack the Blocks



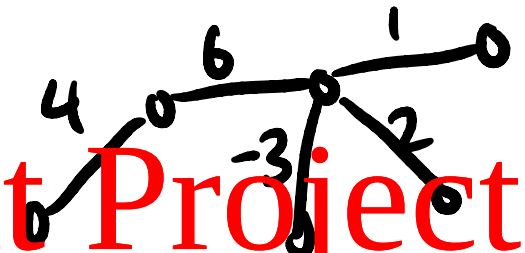
We have n blocks, each with weight w_i and strength b_i. We stack the blocks on top of each other in some order.

The *power* of a block is its strength minus the sum of the weights which lie on it.

The *strength factor* of a stack is the minimum power of any particular block.

Give a $\mathcal{O}(n \cdot 2^n)$ algorithm to find the maximum possible strength factor across all permutations containing all of the blocks (note: answer might be negative).

Problem 3: Longest Path in a Tree



Find the length of the longest path in a tree, where the edges are weighted.