CS 205: Final Exam Question 1 - Functions

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Let $S_N = \{0, 1, 2, \ldots, N-1\}$, i.e., the integers greater than or equal to 0, and strictly less than N.

1) Define a function $f : S_{13} \mapsto S_{13}$ by $f(x) = 6x \mod 13$.

   a) What is the domain of f? The range of f? The image of f?

   b) Prove that f is injective, using the formal definition of injectivity.

   c) Is f invertible? If so, give the values of its inverse function.

2) Define a function $g : S_{15} \mapsto S_{15}$ by $g(x) = 6x \mod 15$.

   a) What is the domain of g? The range of g? The image of g?

   b) Show that g is *not* injective. If x_1 and x_2 in S_{15} map to the same value under g, what must be true about x_1 and x_2?

   c) Show that g is *not* surjective. If $y \in S_{15}$ is not mapped to by g, what must be true of y?

   d) Give a set $A \subset S_{15}$ and $B \subset S_{15}$ such that g is an invertible map from A to B. Find the largest A, B you can.

Given a function $f : S \mapsto S$ (unrelated to the above), a value $x \in S$ is a *fixed point* or has *order* 1 if $x = f(x)$. Similarly, x has *order* 2 if $x = f(f(x))$, and *order* 3 if $x = f(f(f(x)))$. A point x has *order* k if applying f k-times to x yields x. If x *never* returns to x under repeated applications of f, then x has infinite order, or is *transitory*.

3) For any $N > 1$, give an example of an $f : S_N \mapsto S_N$ that has no fixed points.

4) For any $N > 1$, give an example of an $f : S_N \mapsto S_N$ that has only a single point of finite order.

5) Argue that for any $N > 1$, given $f : S_N \mapsto S_N$, if x has finite order, then x has an inverse under f, i.e., if *every* point $x \in S_N$ has a finite order, then f is invertible. *Consider small examples first.*

Suppose you had a program that, given an image file as input, returned a tag from a finite set of tags $\text{Tags} = \{\text{cat}, \text{dog}, \text{orange}, \ldots, \text{UFO}, \text{CannotBeDetermined}\}$, describing the contents of that image (a standard computer vision problem).

6) Describe this program as a function, mapping one set to another set. What is the domain, what is the range? Be as precise as you can be. Is this function invertible?

7) Suppose that the program were restructured so that it returned a collection of tags describing the contents of the image. For instance, a picture of a dog and a cat sitting together might return $\{\text{dog}, \text{cat}\}$. Describe this program as a function, mapping one set to another set. What is the domain, what is the range? Be as precise as you can be. Is this function invertible?

Bonus

A computer can generally be thought of as a device for executing sequences of commands known as programs. A program can be thought of as a description of a function mapping inputs to outputs. By running a computer over a potentially infinite amount of time, producing an infinite sequence of characters as output, we may interpret their computations as producing real numbers.

Thinking in terms of a program as a finite sequence of characters or commands and similarly assuming that the program input must also be finite - argue that there are real numbers that cannot be computed by any program.

Why is this argument independent of such facts as computer speed or architecture/design?