CS 205: Final Exam Question 4 - Modulus and Diophantine 16:198:205

In class and in posted notes, we consider problems like trying to find integer solutions x, y to the equation

$$3x + 5y = D \tag{1}$$

for various values of D. Using congruences, we were able to decouple the x and y variables, determine the 'form' that x and y must have, and then return to the original equation to discover how those two forms were related. Thinking of the above equation as a line, and noting that integer solutions must fall on that line, we were able to construct a 1-dimensional parameterization in terms of an integer parameter k, such that for any integer value of k,

$$\begin{aligned} x &= x_0 + ak \\ y &= y_0 + bk, \end{aligned} \tag{2}$$

represented an integer solution to the original equation.

1) For a given value of D, give an explicit formula for an (x_0, y_0) and a, b to parameterize the integer solutions to the above. The formula should be in terms of D, and an integer parameter k.

2) Are you confident that your parameterization captures *all* integer solutions to $3x + 5y = D$? For any D? Why?

Consider now the equation:

$$3x + 5y + 7z = 1. \tag{3}$$

3) Prove that for any integer value of z, there are integer solutions for x and y.

4) Parameterize the set of integer solutions (x, y, z) in terms of an integer z and an integer parameter k. Note, because the above equation represents a plane in 3-D space, the solutions are two dimensional, and thus require two parameters (in this case, z and k).

5) Are you confident that your parameterization captures *all* integer solutions (x, y, z)? Why?

Now consider the system of equations:

$$\begin{aligned} 3x + 5y + 7z &= 1 \\ 7x + 3y + 5z &= 1. \end{aligned} \tag{4}$$

6) Are there any integer solutions (x, y, z) that satisfy both these equations simultaneously? The intersection of two planes is a line, so give a 1-D integer parameterization of the integer solutions to this system.

Bonus

Adapt your work here to solve for the integer solutions to:

$$21x + 15y + 35z = 1. \tag{5}$$

What complicates the solution here, and how can you approach solving it?