Agent-Based Systems

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Coalitional Games

The value of a group

Plan for Today

The coming lectures are about **cooperative game theory**, where we study so-called **coalitional games** and the **formation of coalitions**.

The first step is to talk about **transferable-utility games**.

Today we focus on **stability** for such games:

- definition of transferable-utility games
- examples for transferable-utility games
- the core: set of surplus divisions that are stable

Much of this is also covered in Chapter 8 of the *Essentials*.

K. Leyton-Brown and Y. Shoham.
Essentials of Game Theory: A Concise, Multidisciplinary Introduction
Claypool Publishers, 2008. Chapter 8.

Coalitional Games

A **transferable-utility coalitional game** in characteristic-function form (or simply: a **TU game**) is a tuple $\langle N, v \rangle$, where

- **N** $= \{1, \ldots, n\}$ is a finite set of **players** and
- **v** : $2^N \to \mathbb{R}_{\geqslant 0}$, with $v(\{\}) = 0$, is a **characteristic function** mapping every possible **coalition** $C \subseteq N$ to its **surplus** $v(C)$.

Note:

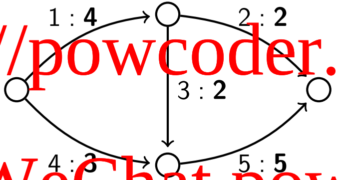
The surplus $v(C)$ is also known as the **value** or the **worth** of C.

The players are assumed to form coalitions (thereby partitioning N). Each coalition C receives its surplus $v(C)$ and—*somehow*—divides it amongst its members (possible due to utility being transferable).

Remark: We'll see a type of **nontransferable-utility games** later on.

Example: A Network Flow Game

Each pipeline is owned by a different player (1, 2, ...). Each pipeline is annotated as (*owner*) : (**capacity**). The surplus $v(C)$ for coalition C is the amount of oil it can pump through the part of the network it owns.



Thus: We obtain $v(12) = \mathbf{2}$, $v(45) = \mathbf{3}$, $v(15) = \mathbf{0}$, $v(134) = \mathbf{0}$, $v(135) = \mathbf{2}$, $v(1345) = \mathbf{5}$, $v(12345) = \mathbf{7}$, and so forth.

Exercise: *What coalition(s) will form? How to divide the surplus?*

Example: A Bankruptcy Game

Alice goes bankrupt. She owes £30k, £60k, £90k to three creditors. But the combined worth of her remaining estate is just £100k.

Model this as a TU game $\langle N, v\rangle$, with $N = \{1, 2, 3\}$, and use v to represent the amount a coalition C of creditors is **guaranteed** to get:

$$v(C) \quad = \quad \max\left(0,\ E - \sum_{i \in N \setminus C} d_i\right)$$

Here $E = £100k$ is the value of the **estate** and d_i is the **debt** owed to creditor $i \in N$ (i.e., $d_1 = £30k$, and so forth).

Exercise: *Decide on an amount x_i to award to each creditor i that seems* **fair** *and is* **stable** *(no coalition should want to break off).*

Simple Games

A **simple game** is a TU game $\langle N, v\rangle$ for which it is the case that $v(C) \in \{0, 1\}$ for every possible coalition $C \subseteq N$.

Thus: every coalition is either **winning** or **losing**.

Voting Games

A (weighted) **voting game** is a tuple $\langle N, \boldsymbol{w}, q \rangle$, where

- $\mathbf{N} = \{1, \ldots, n\}$ is a finite set of **players**;
- $\boldsymbol{w} = (w_1, \ldots, w_n) \in \mathbb{R}^n_{\geqslant 0}$ is a vector of **weights**; and
- $\mathbf{q} \in \mathbb{R}_{\geqslant 0}$ is a **quota**.

Coalition $C \subseteq N$ is **winning**, if the sum of the weights of its members meets or exceeds the quota. Otherwise it is **losing**.

Thus, a voting game $\langle N, \boldsymbol{w}, q \rangle$ is in fact a simple game $\langle N, v \rangle$ with:

$$v(C) = \begin{cases} 1 & \text{if } \sum_{i \in C} w_i \geqslant q \\ 0 & \text{otherwise} \end{cases}$$

Power and the EU

In the Treaty of Rome (1957) the founding countries of the EU fixed the voting rule to be used in the Council of the European Commission:

- To pass, a proposal had to get at least 12 votes in favour.
- France, Germany, and Italy each had 4 votes. Belgium and the Netherlands each had 2 votes. Luxembourg had 1 vote.

This is a weighted voting game $\langle N, \boldsymbol{w}, q \rangle$ with

- $N = \{\mathrm{BE}, \mathrm{DE}, \mathrm{FR}, \mathrm{IT}, \mathrm{NL}, \mathrm{LU}\}$
- $w_{\mathrm{DE}} = w_{\mathrm{FR}} = w_{\mathrm{IT}} = \mathbf{4}$, $w_{\mathrm{BE}} = w_{\mathrm{NL}} = \mathbf{2}$, and $w_{\mathrm{LU}} = \mathbf{1}$
- $q = \mathbf{12}$

Question: *Is this fair? What about Luxembourg in particular?*

Properties of Coalitional Games

Some TU games $\langle N, v\rangle$ have certain properties (for all $C, C' \subseteq N$):

- **additive**: $C \cap C' = \{\}$ implies $v(C \cup C') = v(C) + v(C')$
- **superadditive**: $C \cap C' = \{\}$ implies $v(C \cup C') \geqslant v(C) + v(C')$
- **convex**: $v(C \cup C') \geqslant v(C) + v(C') - v(C \cap C')$
- **cohesive**: $N = C_1 \uplus \cdots \uplus C_k$ implies $v(N) \geqslant v(C_1) + \cdots + v(C_k)$
- **monotonic**: $C \subseteq C'$ implies $v(C) \leqslant v(C')$

Remark: Additive games are not interesting. No synergies between players: every coalition structure is equally good for everyone.

Exercise: *Show that convexity can equivalently be expressed as* $v(S' \cup \{i\}) - v(S') \geqslant v(S \cup \{i\}) - v(S)$ *for* $S \subseteq S' \subseteq N \setminus \{i\}$.

Exercise: *Show that additive $\Rightarrow$ convex $\Rightarrow$ superadditive $\Rightarrow$ cohesive, and also that superadditive $\Rightarrow$ monotonic.*

Examples

What are the properties of the special types of games we have seen?

- **Network flow games** are easily seen to be **monotonic** as well as **superadditive**, but they usually are not **convex**.
- **Voting games**, with weights $\boldsymbol{w} = (w_1, \ldots, w_n)$ and quota q, are **monotonic**, but **not** necessarily **convex** or even **cohesive**.

  Yet, they are **convex** in the natural case of $q > \frac{1}{2} \cdot (w_1 + \cdots + w_n)$.
- **Bankruptcy games**, where $v(C)$ represents the part of the estate the coalition C can guarantee for its members, are **convex**, and thus also **superadditive**, **cohesive**, and **monotonic**.

Issues

The central questions in coalitional game theory are:

- *Which coalitions will* **form***?*
- *How should the members of coalition C* **divide their surplus** $v(C)$*?*
  - *What would be a division that ensures* **stability***?* (this lecture)
  - *What division would be* **fair***?* (next lecture)

Often, the forming of the **grand coalition** N is considered the goal.
This is particularly reasonable for games that are superadditive.

Consider the following 3-player TU game $\langle N, v\rangle$, with $N = \{1,2,3\}$, in which no single player can generate any surplus on her own:

$$
\begin{array}{lll}
v(\{1\}) = 0 & v(\{1,2\}) = 7 & v(N) = 10 \\
v(\{2\}) = 0 & v(\{1,3\}) = 6 & \\
v(\{3\}) = 0 & v(\{2,3\}) = 5 &
\end{array}
$$

Exercise: *What coalition(s) will form? How to divide the surplus?*

Payoff Vectors and Imputations

Suppose the grand coalition has formed. *How to divide its surplus?*
Recall: $v(N)$ is the surplus of the grand coalition N, and $\mathbf{n} = |N|$.

A **payoff vector** is a vector $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_{\geqslant 0}$. Properties:

- $\boldsymbol{x}$ is **feasible** if $\sum_{i \in N} x_i \leqslant v(N)$: do no allocate more than there is.
- $\boldsymbol{x}$ is **efficient** if $\sum_{i \in N} x_i = v(N)$: allocate all there is.
- $\boldsymbol{x}$ is **individually rational** if $x_i \geqslant v(\{i\})$ for all players $i \in N$: nobody should be able to do better on her own.

An **imputation** is a payoff vector that is both individually rational and efficient (and thus also feasible). Reasonable to focus on imputations.

The Core

Which imputations incentivise players to form the grand coalition?

Probably the most important solution concept for coalitional games, formalising this kind of **stability** notion, is the so-called "core" ...

An imputation $\boldsymbol{x} = (x_1, \ldots, x_n)$ is **in the core** of the game $\langle N, v\rangle$ if no coalition $C \subseteq N$ can benefit by breaking away from the grand coalition:

$$\sum_{i \in C} x_i \geq v(C)$$

Remark: Individual rationality is a special case of this (with $C = \{i^\star\}$).

D.B. Gillies.
Some Theorems on n-Person Games
PhD thesis, Department of Mathematics, Princeton University, 1959..

Example: Game with an Empty Core

Consider the following 3-player TU game $\langle N, v \rangle$, with $N = \{1, 2, 3\}$, in which no single player can generate any surplus on her own:

$$
\begin{array}{lll}
v(\{1\}) = 0 & v(\{1,2\}) = 7 & v(N) = \mathbf{8} \\
v(\{2\}) = 0 & v(\{1,3\}) = 6 & \\
v(\{3\}) = 0 & v(\{2,3\}) = 5 &
\end{array}
$$

For an imputation $\boldsymbol{x} = (x_1, x_2, x_3)$ to be in the core, we must have:

- for stability: $x_1 + x_2 \geqslant 7$ and $x_1 + x_3 \geqslant 6$ and $x_2 + x_3 \geqslant 5$
- for efficiency: $x_1 + x_2 + x_3 = 8$

But this clearly is impossible. So the core is empty.

Question: *What games have a* **nonempty core**? *Characterisation?*
Remark: The above game happens to be superadditive but not convex.

Characterisation for Simple Games

<u>Recall:</u> $\langle N, v\rangle$ is a **simple game** if $v(C) \in \{0, 1\}$ for all $C \subseteq N$.
Player $i \in N$ is said to be a **veto player** in the simple game $\langle N, v\rangle$,
if for all $C \subseteq N$ it is the case that $i \notin C$ implies $v(C) = 0$.

Proposition

*A **simple game** (and thus also a **voting game**) has a **nonempty core** <u>iff</u> it has at least one **veto player**.*

($\Leftarrow$) Suppose there are $k \geqslant 1$ veto players. Choose imputation $\boldsymbol{x}$ s.t. $x_i = \frac{1}{k}$ if i is a veto player, and $x_i = 0$ otherwise. Then $\boldsymbol{x}$ is in the core:

- for every **winning** coalition C: $\sum_{i\in C} x_i = k \cdot \frac{1}{k} = 1 = v(C)$
- for every **losing** coalition C: $\sum_{i\in C} x_i \geqslant 0 = v(C)$ holds vacuously

($\Rightarrow$) Suppose the core is nonempty and $\boldsymbol{x}$ is in it. Let $\mathbf{i}^\star \in \operatorname{argmax}_{i\in N} x_i$.

Will show that $i^\star$ can **veto**. By efficiency, $x_{i^\star} > 0$ and $\sum_{i\in N} x_i = 1$.
Take any $C \subseteq N$ s.t. $i^\star \notin C$. Then $\sum_{i\in C} x_i < 1$. Hence, $v(C) = 0$. ✓

Convexity is a Sufficient Condition

Recall: $\langle N, v\rangle$ is **convex** if $v(C \cup C') \geqslant v(C) + v(C') - v(C \cap C')$.

Theorem (Shapley, 1971)

Every TU game that is convex has got a nonempty core.

Proof: Let $[i] := \{1, \ldots, i\}$ for any $i \in N$. Define an imputation $\boldsymbol{x}$:

$$\mathbf{x_i} = v([i]) - v([i-1])$$

By monotonicity, $x_i \geqslant 0$. And: $\sum x_i = v(N)$, i.e., $\boldsymbol{x}$ is efficient.
Now take any $C \subset N$ and $i^\star \in N$ such that $[i^\star - 1] \subseteq C$ and $i^\star \notin C$.
By convexity: $v(C \cup \{i^\star\}) \geqslant v(C) + \mathbf{v([i^\star])} - \mathbf{v([i^\star - 1])}$. Thus:

$$-v(C) \geqslant \mathbf{x_{i^\star}} - v(C \cup \{i^\star\})$$

$$\textstyle\sum_{\mathbf{i \in C}} x_i - v(C) \geqslant \sum_{\mathbf{i \in C} \cup \{i^\star\}} x_i - v(C \cup \{i^\star\})$$

Repeat: $\sum_{\mathbf{i \in C}} \mathbf{x_i} - \mathbf{v(C)} \geqslant \cdots \geqslant \sum_{i \in N} x_i - v(N) = \mathbf{0}$. ✓

L.S. Shapley.
Cores of Convex Games.
International Journal of Game Theory, 1(1):11–26, 1971.

Cohesiveness is a Necessary Condition

Recall: $\langle N, v \rangle$ is **cohesive** if $v(N) \geqslant v(C_1) + \cdots + v(C_K)$ for every possible partition $C_1 \cup \cdots \cup C_K = N$.

Proposition

Every TU game with a **nonempty core** *is* **cohesive**.

Proof: Consider any game $\langle N, v \rangle$ that is *not* cohesive. So there exists a partition $C_1 \uplus \cdots \uplus C_K = N$ with $v(C_1) + \cdots + v(C_K) > v(N)$.

For the sake of contradiction, suppose the core *is* nonempty. So there's an imputation $\boldsymbol{x} = (x_1, \ldots, x_n)$ with $\sum_{i \in C_k} x_i \geqslant v(C_k)$ for all $k \leqslant K$.
Putting everything together, we get:

$$\sum_{\mathbf{i} \in \mathbf{N}} \mathbf{x_i} = \sum_{i \in C_1} x_i + \cdots + \sum_{i \in C_K} x_i \geqslant v(C_1) + \cdots + v(C_K) > \mathbf{v(N)}$$

But this contradicts efficiency of $\boldsymbol{x}$, i.e., it cannot be an imputation. ✓

Exercise: *Find a game with a nonempty core that is not superadditive.*

The Bondareva-Shapley Theorem

Thus: convex $\Rightarrow$ has nonempty core $\Rightarrow$ superadditive. *Getting close.*
We can do better and give a **complete characterisation** (w/o proof):

Theorem (Bondareva, 1962; Shapley 1967)

A TU game has a **nonempty core** *iff that game is* **balanced**.

A collection of weights $\lambda_C \in [0, 1]$, one for each coalition $C \subseteq N$, is called **balanced** if, for all players $i \in N$, we have $\sum_{C \ni i} \lambda_C = 1$.

A TU game $\langle N, v \rangle$ is called **balanced** if, for all balanced collections of weights λ_C, we have $\sum_{C \subseteq N} \lambda_C \cdot v(C) \leqslant v(N)$.
Interpretation: "Grand coalition beats dividing time over coalitions."

O.N. Bondareva.
The Theory of the Core of an n-Person Game (in Russian)
Vestnik Leningrad University, 17(13):141–142, 1962.

L.S. Shapley.
On Balanced Sets and Cores.
Naval Research Logistics Quarterly, 14(4):453–460, 1967.

Summary

This has been an introduction to coalitional games, which include:

- **network flow games**: surplus = capacity of coalition's network
- **bankruptcy games**: surplus = coalition's guaranteed payment
- **voting games**: surplus = coalition's ability to win vote

We've modelled them as **transferable-utility games**, i.e., the members of a coalition can freely distribute the surplus amongst themselves.
We've then asked: when will the grand coalition form and be stable?
Arguably, if we can find a **payoff vector** that is **in the core**.
So **nonemptiness of the core** is important. Results:

- for **simple (and voting) games**: possible iff there is a veto player
- for **general TU games**: possible iff the game is balanced

What next? Matching