Agent-Based Systems

Paolo Turrini

www.dcs.warwick.ac.uk/~pturrini p.turrini@warwick.ac.uk

More Solution Concepts

thinking about thinking about

Plan for Today

Pure and mixed Nash equilibria are examples for **solution concepts**: formal models to predict what might be the outcome of a game.

Today we are going to see some more such solution concepts:

- **equilibrium in dominant strategies**: do what's definitely good
- **elimination of dominated strategies**: don't do what's definitely bad
- **correlated equilibrium**: follow some external recommendation

For each of them, we are going to see some **intuitive motivation**, then a **formal definition**, and then an example for a relevant **technical result**.

Most of this (and more) is also covered in Chapter 3 of the *Essentials*.

K. Leyton-Brown and Y. Shoham.
Essentials of Game Theory: A Concise, Multidisciplinary Introduction
Claypool Publishers, 2008. Chapter 3.

Example: Prisoner's Dilemma Again

Let's have look at it once more:

	C	D
C	-10 -10	0 -25
D	-25 0	-20 -20

<u>Notice</u> *DD is a Nash Equilibrium, but there is more...*

<u>Discussion:</u> Is this true for all pure strategy Nash Equilibria in all games?

Dominant Strategies

Have we maybe missed the most obvious solution concept? ...

You should play the action $a_i^\star$ that gives you a better payoff than any other action a_i', whatever the others do (such as playing $\boldsymbol{s}_{-i}$):

$$u_i(\mathbf{a}_i^\star, \boldsymbol{s}_{-i}) > u_i(a_i', \boldsymbol{s}_{-i}) \quad \text{for all } a_i' \in A_i \text{ and all } \boldsymbol{s}_{-i} \in \boldsymbol{S}_{-i}$$

Action $a_i^\star$ is called a **strictly dominant strategy** for player i.

Profile $\boldsymbol{a}^\star \in \boldsymbol{A}$ is called an **equilibrium in strictly dominant strategies** if, for every player $i \in N$, action $a_i^\star$ is a strictly dominant strategy.

Downside: This does not always exist (in fact, it usually does not!).

Remark: Equilibria don't change if we define this for mixed strategies. If some best strategy exists, then some pure strategy is (also) best.

	C	D
C	-10, -10	-25, 0
D	0, -25	-20, -20

	L	R
T	1, 0	0, 0
B	0, 1	1, 1

	L	R
T	0, 0	0, 1
B	0, 1	1, 100

Exercise: *Is there an equilibrium in strictly dominant strategies?*

Exercise:
Show that an equilibrium in strictly dominant strategies is also a pure Nash equilibrium.

Action a_i is **strictly dominated** by a strategy $s_i^\star$ if, for all $\boldsymbol{s}_{-i} \in \boldsymbol{S}_{-i}$:

$$u_i(s_i^\star, \boldsymbol{s}_{-i}) > u_i(a_i, \boldsymbol{s}_{-i})$$

Then, if we assume i is **rational**, action a_i can be **eliminated**.
This induces a solution concept:

> *all mixed-strategy profiles of the reduced game that survive iterated elimination of strictly dominated strategies (IESDS)*

Simple example (where the dominant*ing* strategies happen to be pure):

	L	R
T	4, 4	1, 6
B	6, 1	2, 2

	R
T	1, 6
B	2, 2

	R
B	2, 2

Order Independence of IESDS

Suppose $A_i \cap A_j = \{\}$. Then we can think of the **reduced game** G^t after t eliminations simply as the subset of $A_1 \cup \cdots \cup A_n$ that survived.

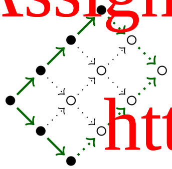
IESDS says: players will actually play G^∞. *Is this well defined?* Yes!

Theorem (Gilboa et al., 1990)

Any order *of eliminating strictly dominated strategies leads to the* ***same reduced game****.*

Proof

Proof: Write $G \twoheadrightarrow G'$ if game G can be reduced to G' by eliminating *one* action. We are done if we can show that $\twoheadrightarrow^*$ (trans. closure) is Church-Rosser. So need to show that $\twoheadrightarrow$ is C-R.



Enough to show: if $G \overset{a_i}{\twoheadrightarrow} G'$ and $G \overset{b_j}{\twoheadrightarrow} G''$, then $G' \overset{b_j}{\twoheadrightarrow} G'''$ for some G'''.

This is immediate: $G \overset{b_j}{\twoheadrightarrow} G''$ means $u_j(s_j^\star, \mathbf{s}_{-j}) > u_j(b_j, \mathbf{s}_{-j})$ for all $\mathbf{s}_{-j}$. Take $\mathbf{s}_{-j} = (a_i', \mathbf{s}_{-ij})$ with $a_i' \neq a_i$: $u_j(s_j^\star, a_i', \mathbf{s}_{-ij}) > u_j(b_j, a_i', \mathbf{s}_{-ij})$. ✓

Remark: This only works due to finiteness of A (induction!).

I. Gilboa, E. Kalai, and E. Zemel.
On the Order of Eliminating Dominated Strategies.
Operations Research Letters, 9(2):85–89, 1990.

Let's Play: Numbers Game (Again!)

Let's play again:

> *Every player submits a (rational) number between* **0** *and* **100***. We then compute the average (arithmetic mean) of all the numbers submitted and multiply that number with 2/3. Whoever got closest to this latter number wins the game.*

Analysis

IESDS results in a reduced game where everyone's only action is 0.
So, we happen to find the only pure Nash equilibrium this way.

IESDS works on the assumption of **common knowledge of rationality**.

In the **Numbers Game**, we have seen:

- Playing 0 usually is not a good strategy in practice, so assuming common knowledge of rationality must be unjustified.
- When we played the second time, the winning number got closer to 0. So by discussing the game, both your own rationality and your confidence in the rationality of others seem to have increased.

Even More Solution Concepts

There are several other solution concepts in the literature. Examples:

- **Iterated elimination of weakly dominated strategies**: eliminate a_i in case there is a strategy $s_i^\star$ such that $u_i(s_i^\star, \boldsymbol{s}_{-i}) \geqslant u_i(a_i, \boldsymbol{s}_{-i})$ for all $\boldsymbol{s}_{-i} \in \boldsymbol{S}_{-i}$ and this inequality is strict in at least one case.
- **Iterated elimination of very weakly dominated strategies**: eliminate a_i in case there is a strategy $s_i^\star$ such that $u_i(s_i^\star, \boldsymbol{s}_{-i}) \geqslant u_i(a_i, \boldsymbol{s}_{-i})$ for all $\boldsymbol{s}_{-i} \in \boldsymbol{S}_{-i}$.

- **ϵ-Nash equilibrium**: no player can gain more than ϵ in utility by unilaterally deviating from her assigned strategy.

  Exercise: *How does the standard definition of NE relate to this?*

K. Leyton-Brown and Y. Shoham.
Essentials of Game Theory: A Concise, Multidisciplinary Introduction.
. Morgan & Claypool Publishers, 2008. Chapter 3.

Summary

We have reviewed several **solution concepts** for normal-form games:

- **equilibrium in dominant strategies**: great *if* it exists
- **IESDS**: iterated elimination of strictly dominated strategies

What next? Extensive games