Agent-Based Systems

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Today

We are going to look at the very basics of game theory, in particular:

- Pure and mixed strategies
- Nash equilibria

We are also going to play a game.

K. Leyton-Brown and Y. Shoham
Essentials of Game Theory: A Concise, Multidisciplinary Introduction
Morgan & Claypool Publishers, 2008. Chapters 1 & 2

Why?

Game theory entered **AI** when it became clear that we can use it to study interaction between the software agents in a **multiagent system**.

Nowadays, the study of "**economic paradigms**" is all over AI.

The influential *One Hundred Year Study on AI* (2016) singles out the following eleven "**hot topics**" in AI:

> *large-scale machine learning* | *deep learning* | *reinforcement learning* | *robotics* | *computer vision* | *natural language processing* | *collaborative systems* | *crowdsourcing and human computation* | *algorithmic* **game theory** *and computational social choice* | *internet of things* | *neuromorphic computing*

P.Stone
Artificial Intelligence and Life in 2030.
One Hundred Year Study on Artificial Intelligence, 2016.

The Prisoner's Dilemma

Two hardened criminals, Rowena and Colin, got caught by police and are being interrogated in separate cells. The police only has evidence for some of their minor crimes. Each is facing this dilemma:

- If we cooperate (C) and don't talk, then we each get 10 years for the minor crimes.
- If I cooperate but my partner defects (D) and talks, then I get 25 years.
- If my partner cooperates but I defect, then I go free (as crown witness).
- If we both defect, then we share the blame and get 20 years each.

	C	D
C	-10, -10	-25, 0
D	0, -25	-20, -20

What would you do? Why?

A **normal-form game** is a tuple $\langle N, \boldsymbol{A}, \boldsymbol{u} \rangle$, where

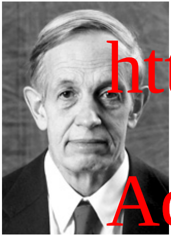
- $\mathbf{N} = \{1, \ldots, n\}$ is a finite set of **players** (or **agents**);
- $\boldsymbol{A} = A_1 \times \cdots \times A_n$ is a finite set of **action profiles** $\boldsymbol{a} = (a_1, \ldots, a_n)$, with A_i being the set of actions available to player i; and
- $\boldsymbol{u} = (u_1, \ldots, u_n)$ is a profile of **utility functions** $u_i : \boldsymbol{A} \rightarrow \mathbb{R}$.

Every player i chooses an action, say, a_i, giving rise to the profile $\boldsymbol{a}$. Actions are played *simultaneously*. Player i then receives payoff $u_i(\boldsymbol{a})$.

Remark: We use boldface italics to denote vectors (i.e., profiles) and Cartesian products (i.e., sets of profiles).

Later we will allow players to randomise over actions. But today we restrict attention to **pure strategies**: strategy ≡ action

Notation: $(a_i', \boldsymbol{a}_{-i})$ is short for $(a_1, \ldots, a_{i-1}, a_i', a_{i+1}, \ldots, a_n)$.



John F. Nash Jr.
(1928–2015)

We say that $a_i^\star \in A_i$ is a **best response** for player i to the (partial) action profile $\boldsymbol{a}_{-i}$, if $u_i(a_i^\star, \boldsymbol{a}_{-i}) \geqslant u_i(a_i', \boldsymbol{a}_{-i})$ for all $a_i' \in A_i$.

We say that action profile $\boldsymbol{a} = (a_1, \ldots, a_n)$ is a **pure Nash equilibrium**, if a_i is a best response to $\boldsymbol{a}_{-i}$ for every agent $i \in N$.

Thus, pure Nash equilibria are **stable** outcomes: no player has an incentive to unilaterally deviate from her assigned strategy.

Exercise: How Many Pure Nash Equilibria?

	L	R
T	2, 2	2, 1
B	1, 3	3, 2

	L	R
T	2, 2	2, 2
B	2, 2	2, 2

	L	R
T	1, 2	2, 1
B	2, 1	1, 2

Zero-Sum Games

A **zero-sum game** is a two-player normal-form game $\langle N, \boldsymbol{A}, \boldsymbol{u} \rangle$ with $u_1(\boldsymbol{a}) + u_2(\boldsymbol{a}) = 0$ for all action profiles $\boldsymbol{a} \in \boldsymbol{A}$. Example:

	0 / 0	-1 / 1	1 / -1
	1 / -1	0 / 0	-1 / 1
	-1 / 1	1 / -1	0 / 0

What are the pure NE of this game? Intuitively, how should you play?

So far, the space of strategies available to player i has simply been her set of actions A_i (pure strategy $=$ action).

We now generalise and allow player i to play any action in A_i with a certain probability.

For any finite set X, let

$$\Pi(X) = \{p : X \to [0,1] \mid \sum_{x \in X} p(x) = 1\}$$

be the set of all **probability distributions** over X.

A **mixed strategy** s_i for player i is a probability distribution in $\Pi(A_i)$.
The set of all her mixed strategies is $S_i = \Pi(A_i)$.

A **mixed-strategy profile** $\boldsymbol{s} = (s_1, \ldots, s_n)$ is an element of $S_1 \times \cdots \times S_n$.

The **expected utility** of player i for the mixed-strategy profile $\boldsymbol{s}$ is:

$$u_i(\boldsymbol{s}) = \sum_{\boldsymbol{a} \in \boldsymbol{A}} \left[u_i(\boldsymbol{a}) \cdot \prod_{j \in N} s_j(a_j) \right]$$

The **support** of strategy s_i is the set of actions $\{a_i \in A_i \mid s_i(a_i) > 0\}$.

- A mixed strategy s_i is **pure** iff its support is a singleton.
- A mixed strategy s_i is **truly mixed** if it is not pure.
- A mixed strategy s_i is **fully mixed** if its support is the full set A_i.

Example: Battle of the Sexes

Traditionally minded Rowena and Colin are planning a social activity. Worst of all would be not to agree on a joint activity; but if they do manage, Colin prefers *auto racing* and Rowena *really* prefers *ballet*.

	A	B
A	2, 4	0, 0
B	0, 0	8, 3

Suppose Rowena chooses to go to the ballet with 75% probability, while Colin chooses to go to the races with certainty (pure strategy):

$$s_1 = (\tfrac{1}{4}, \tfrac{3}{4}) \qquad s_2 = (1, 0)$$

Thus: $u_1(\boldsymbol{s}) = 2 \cdot (\frac{1}{4} \cdot 1) + 0 \cdot (\frac{1}{4} \cdot 0) + 0 \cdot (\frac{3}{4} \cdot 1) + 8 \cdot (\frac{3}{4} \cdot 0) = \frac{1}{2}$

Mixed Nash Equilibria

Consider a game $\langle N, A, \boldsymbol{u} \rangle$ with associated (mixed) strategies $s_i \in S_i$.

- We say that strategy $s_i^\star \in S_i$ is a **best response** for player i to the (partial) strategy profile $\boldsymbol{s}_{-i}$ if $u_i(s_i^\star, \boldsymbol{s}_{-i}) \geqslant u_i(s_i', \boldsymbol{s}_{-i})$ for all $s_i' \in S_i$.
- We say that profile $\boldsymbol{s} = (s_1, \ldots, s_n)$ is a **mixed Nash equilibrium** if s_i is a best response to $\boldsymbol{s}_{-i}$ for every player $i \in N$.

Thus: no player has an incentive to unilaterally change her strategy.

Remark: Note how this definition mirrors that of pure Nash equilibria.

Theorem (Nash, 1951)

Every (finite) normal-form game has at least one (truly mixed or pure) Nash equilibrium.

Let's Play: Numbers Game

Let's play the following game:

> *Every player submits a (rational) number between* **0** *and* **100**. *We then compute the average of all the numbers submitted and multiply that average with 2/3. Whoever got closest to this latter number wins the game.*

Summary

- Game Theory: Decision Theory with many agents
- Equilibria, in particular Nash equilibria, in pure and mixed strategies.

Coming next: Extensive Games.