Computational Complexity and Computability

Lecture 7 - Closure Properties & Reducibility

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Closure Properties

Theorem

The class D is closed under union, intersection, concatenation, complementation and Kleene star.

Proof.

Let $A, B \in D$. By definition, there are deciders M_A and M_B that recognize A and B, respectively.

Define a TM M as follows:

On input w:

- *Run M_A on w; if M_A rejects, reject.*
- *Run M_B on w; if M_B rejects, reject.*
- *Otherwise accept.*

Clearly, M is a decider and $\mathcal{L}(M) = A \cap B$. Hence, $A \cap B \in D$.

Exercise: Prove the rest. $\square$

Closure Properties

Theorem

The class SD is closed under union, intersection, concatenation and Kleene star, but not under complementation.

Proof.

Let $A, B \in SD$. By definition, there are TMs M_A and M_B that recognize A and B, respectively.

Define a TM M as follows:

On input w:

- *Run M_A and M_B alternately on w step by step; if either M_A or M_B accepts, accept.*

Clearly, $\mathcal{L}(M) = A \cup B$. Hence, $A \cup B \in SD$.

Exercise: Prove the rest. $\square$

Closure Properties

Definition

$coSD := \{A \mid A^c \in SD\}$.

Theorem

$D = SD \cap coSD$

Proof.

$(\longrightarrow)$

If $A \in D$, then $A = \mathcal{L}(M)$ for some decider M. Naturally, $A \in SD$. Define a TM M' as follows:

On input w:

- *Run M on w.*
- *If M accepts, reject; if M rejects, accept.*

Clearly, $\mathcal{L}(M') = A^c$, which implies $A \in coSD$.

Closure Properties

($\longleftarrow$)

Suppose $A \in SD \cap coSD$. Let E and E' be enumerators for A and A^c respectively. Then A is decided by a TM which, on input w, runs E and E' in parallel and accepts or rejects based on which enumerator outputs w. $\square$

$DIAG$ and A_{TM}

Definition

- $DIAG := \{\langle M \rangle \mid M$ is a TM and $\langle M \rangle \notin \mathcal{L}(M)\}$.
- $A_{TM} := \{\langle M, w \rangle \mid M$ is a TM and $w \in \mathcal{L}(M)\}$.

Theorem

- $DIAG \notin SD$
- $A_{TM} \in SD \setminus D$.

$DIAG^c$

Definition

$$DIAG^c := \{\langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \in \mathcal{L}(M)\} \bigcup \{w \mid w \text{ does not encode a TM}\}.$$

Theorem

$DIAG^c \in SD \setminus D$.

Proof.

$$\langle M \rangle \in DIAG^c \iff \langle M \rangle \in \mathcal{L}(M) \iff \langle M, \langle M \rangle \rangle \in A_{TM}.$$

Since $A_{TM} \in SD$, it follows that $DIAG^c \in SD$ as well.

On the other hand $DIAG^c \notin D$, since otherwise $DIAG \in D$ as D is closed under complementation. $\square$

A_{TM}^c

Definition

$$A_{TM}^c := \{\langle M, w\rangle \mid M \text{ is a TM and } w \notin \mathcal{L}(M)\}$$
$$\bigcup \{w \mid w \text{ does not encode a pair of TM and input}\}.$$

Theorem

$A_{TM}^c \notin SD$.

Proof.

We have already shown that $A_{TM} \in SD$ and $D = SD \cap coSD$.

If $A_{TM}^c \in SD$, then $A_{TM} \in coSD$, and consequently, $A_{TM} \in D$.

$\lightning$

Reducibility

Definition

Let A, B be two languages. We say A is mapping reducible to B, written $A \leq_m B$, if there is a computable function $f : \Sigma^* \to \Sigma^*$ such that for every $w \in \Sigma^*$,

$$w \in A \iff f(w) \in B.$$

The function f is called the reduction from A to B.

Properties of reduction

Theorem (Properties of $\leq_m$)

1. *Reflexive:* $A \leq_m A$.
2. *Transitive: If* $A \leq_m B$ *and* $B \leq_m C$*, then* $A \leq_m C$.
3. *If* $A \leq_m B$*, then* $A^c \leq_m B^c$.
4. *If* $A \leq_m B$ *and* B *is (semi-) decidable, then* A *is (semi-) decidable.*

Proof.

1, 2 and 3 : Exercise!

Properties of reduction

4. Let M_B be a (semi-) decider for B, let f be a reduction from A to B, and let M_f be a TM that computes f.

Consider the following TM M_A:

On input w:

- *Run M_f on w to obtain $f(w)$.*
- *Run M_B on $f(w)$ and output whatever M_B outputs.*

As f is a reduction from A to B, $w \in A \iff f(w) \in B$. Equivalently, M_B accepts $f(w)$ iff $w \in A$.

Consequently, $\mathcal{L}(M_A) = A$, and hence, A is (semi-) decidable. $\square$

Corollary

If $A \leq_m B$ and A is not (semi-) decidable, then neither is B.

Application

Problem

Show that $A_{TM}^c \notin SD$ by reducing from $DIAG$.

Proof.

Define $f : \Sigma^* \to \Sigma^*$ as follows:

- $f(\langle M \rangle) = \langle M, \langle M \rangle \rangle$;
- $f(w) = \langle M_{accept}, \varepsilon \rangle$ for strings w that do not encode a TM. (M_{accept} is a machine that always accepts.)

It is easy to see that f is computable.

Also, notice that
$\langle M \rangle \in DIAG \iff \langle M \rangle \notin \mathcal{L}(M) \iff \langle M, \langle M \rangle \rangle \in A_{TM}^c$.
(The reverse direction holds because $\langle M_{accept}, \varepsilon \rangle \notin A_{TM}^c$.)

Hence, $DIAG \leq_m A_{TM}^c$.

Since $DIAG \notin SD$, it follows that $A_{TM}^c \notin SD$. $\square$