Computational Complexity and Computability

Lecture 3 - Algorithms & Computable Functions

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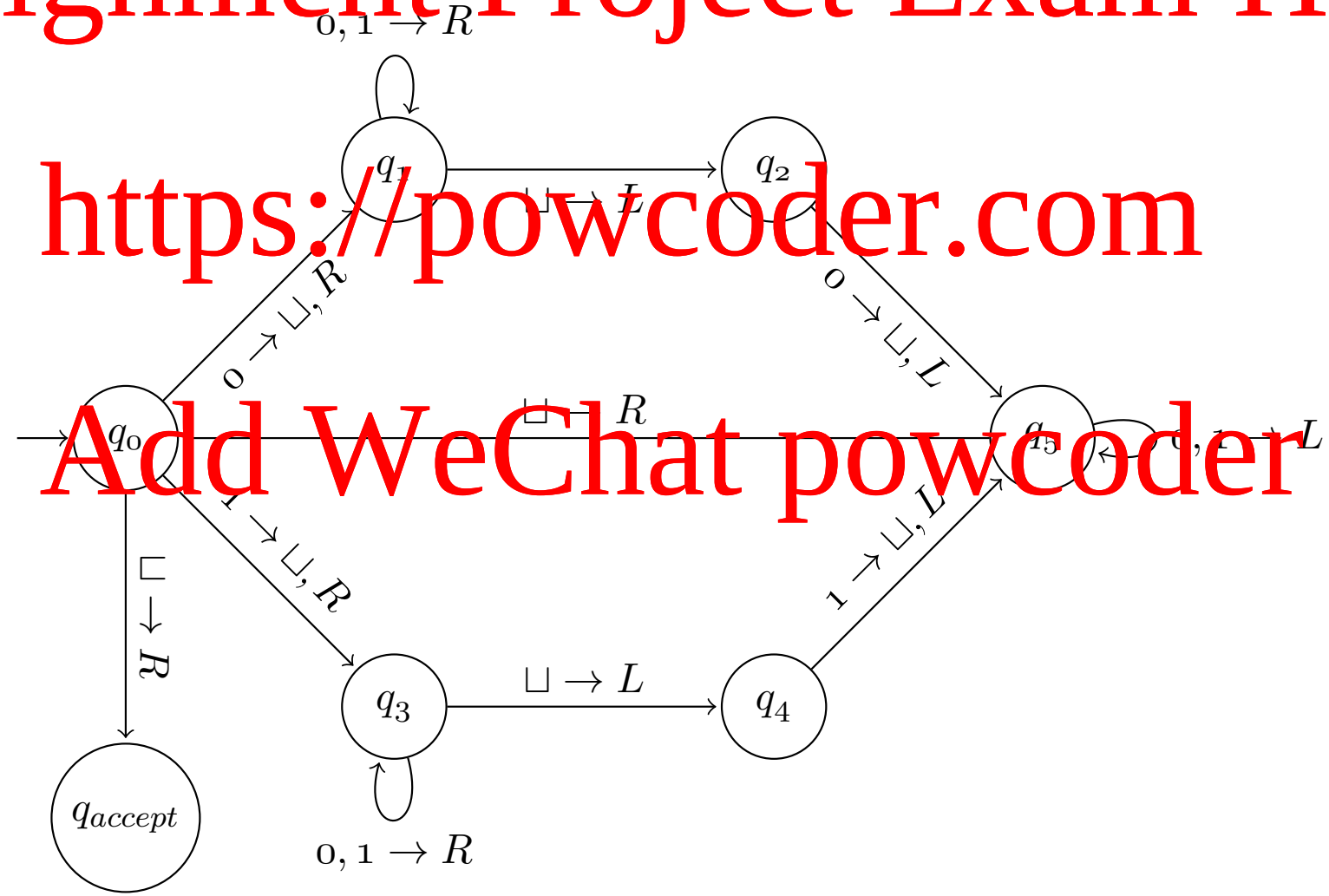
Example 1 - PAL

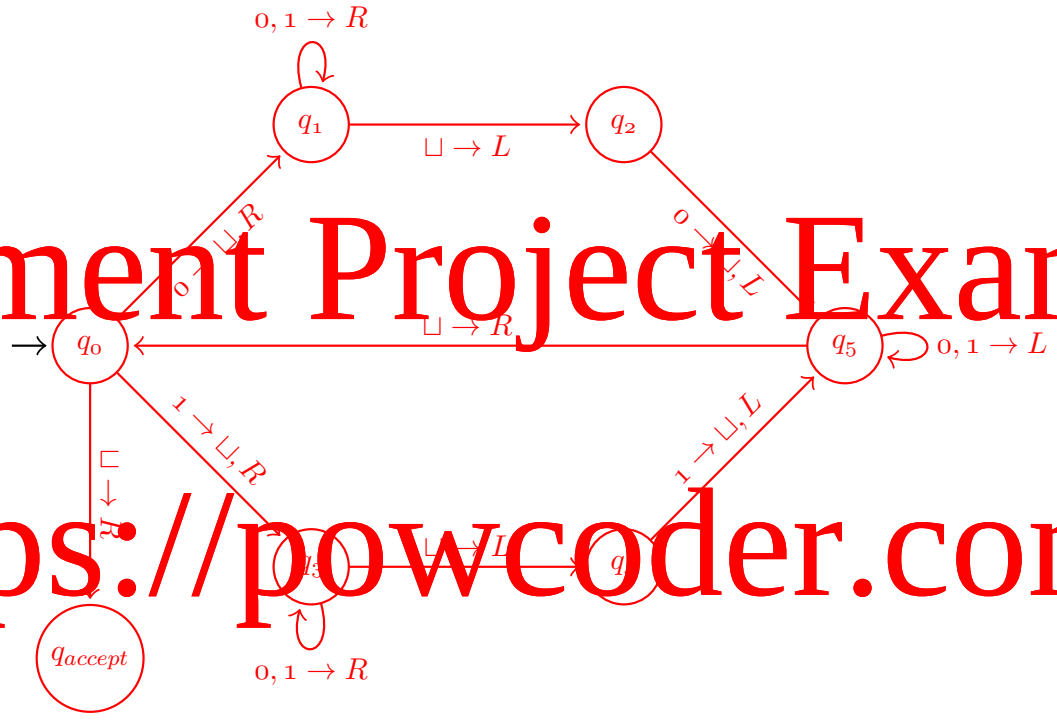
Goal: Describe a TM on the alphabet $\{0, 1\}$ for the languauge

$PAL = \{\text{set of even length palindromes}\} = \{yy^{reverse} \mid y \in \{0, 1\}^*\}$.

Solution:



Execution on input 011110



1. $q_0 011110$
2. $q_1 11110$
3. $1 q_1 1110$
4. $11 q_1 110$
5. $111 q_1 10$
6. $1111 q_1 0$
7. $1111 q_1$
8. $1111 q_2 0$
9. $111 q_5 1$
10. $11 q_5 11$
11. $1 q_5 111$
12. $q_5 1111$
13. $q_5 \sqcup 1111$
14. $q_0 1111$
15. $q_3 111$
16. $1 q_3 11$
17. $11 q_3 1$
18. $111 q_3$
19. $11 q_4 1$
20. $1 q_5 1$
21. $q_5 11$
22. $q_5 \sqcup 11$
23. $q_0 11$
24. $q_3 1$
25. $1 q_3$
26. $q_4 1$
27. q_5
28. q_0
29. q_{accept}

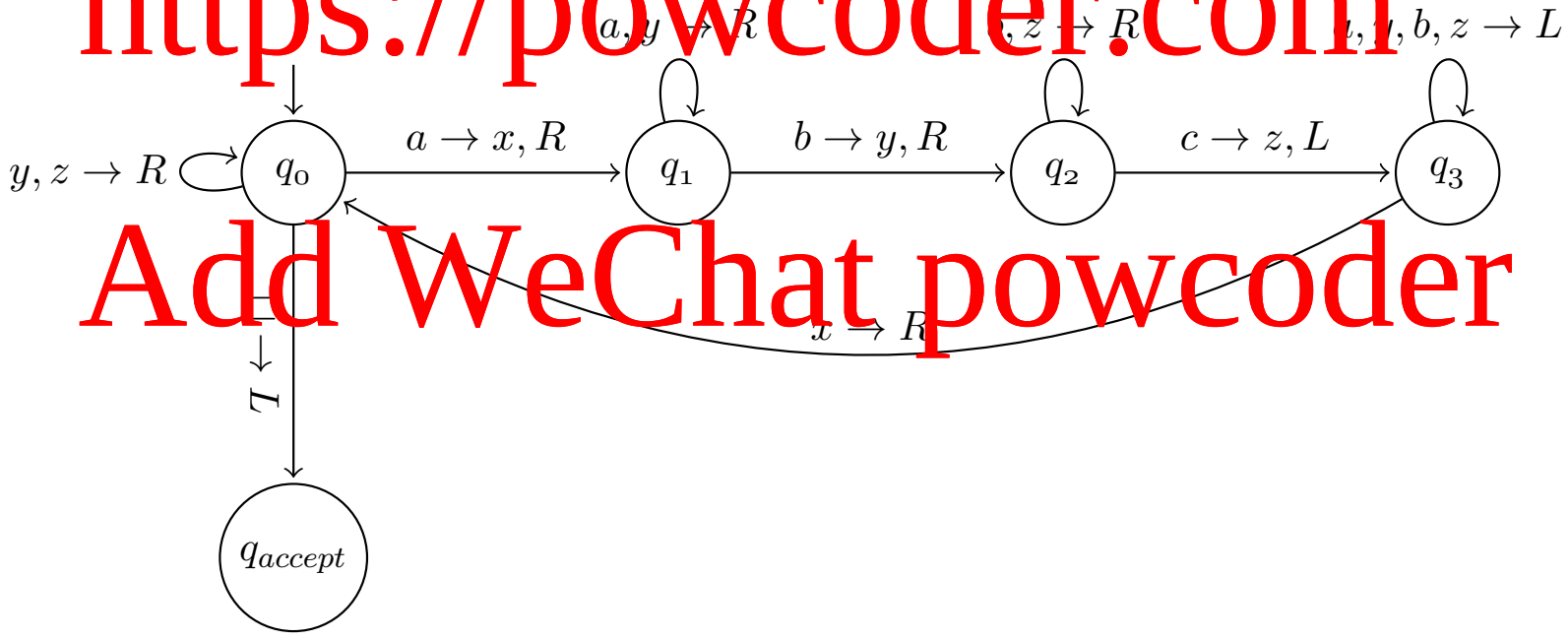
Example 2

Goal : Describe a TM on the alphabet $\{a, b, c\}$ for the languauge

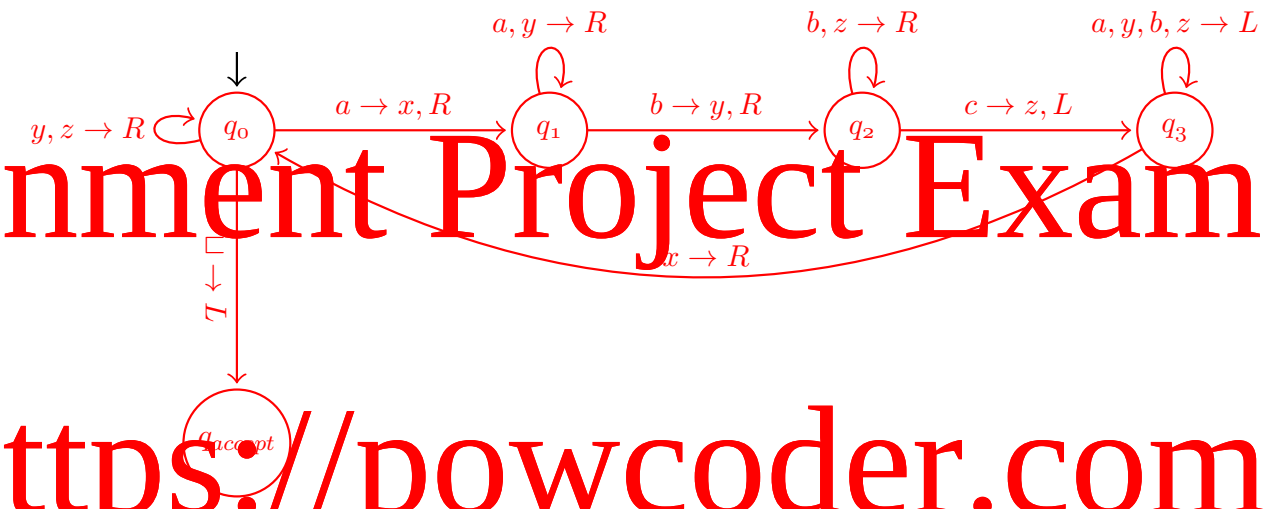
$$\mathcal{L} = \{a^n b^n c^n \mid n \in \mathbb{N}\}$$

Solution:

$$\Sigma = \{a, b, c\}, \qquad \Gamma = \{a, b, c, x, y, z, \sqcup\}$$



Execution on input $aabbcc$



1. $q_0aabbcc$
2. xq_1abbcc
3. xaq_1bbcc
4. $xayq_2bcc$
5. $xaybq_2cc$
6. $xayq_3bzc$
7. xaq_3ybzc
8. xq_3aybzc
9. $q_3xaybzc$
10. xq_0aybzc
11. xxq_1ybzc
12. $xxyq_1bzc$
13. $xxyyq_2zc$
14. $xxyyzq_2c$
15. $xxyyq_3zz$
16. $xxyq_3yzz$
17. xxq_3yyzz
18. xq_3xyyzz
19. xxq_0yyzz
20. $xxyq_0yzz$
21. $xxyyq_0zz$
22. $xxyyzq_0z$
23. $xxyyzzq_0$
24. $xxyyzq_{accept}z$

Computable Functions

Definition

A function $f : A \to B$ is computable if there is a TM M such that for all input $w \in A$, the machine halts with only $f(w) \in B$ on the tape and the head pointing to the beginning of the output.

In other words,

Example

Goal: Show that the function $f(x, y) = x + y$ is computable.

Integer Domain

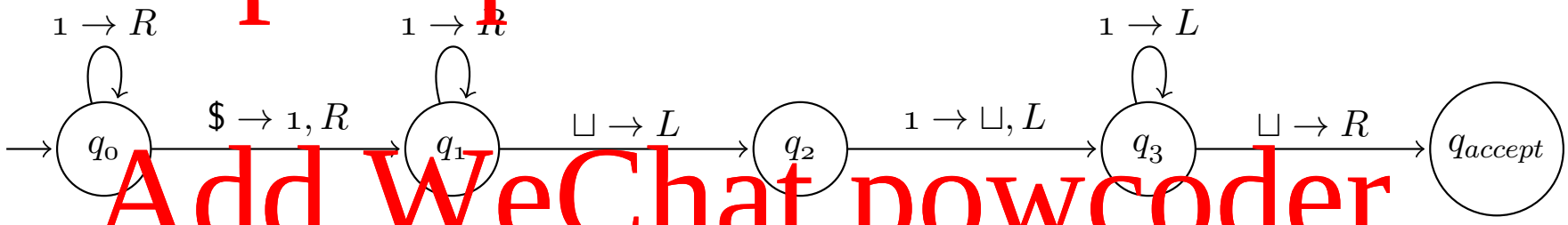
- Decimal, e.g., 5
- Binary, e.g., 101
- Unary, e.g., 11111

For this problem, we will take the input in unary. The input alphabet is $\Sigma = \{1, \$\}$, and the input is represented as $x\$y$.

Example

Goal: Show that the function $f(x, y) = x + y$ is computable.

Solution.



Execution on input 11111$1111

1. $q_0 11111\$1111$
2. $1q_0 1111\$1111$
3. $11q_0 111\$1111$
4. $111q_0 11\$1111$
5. $1111q_0 1\$1111$
6. $11111q_0\$1111$
7. $111111q_1 1111$
8. $1111111q_1 111$
9. $11111111q_1 11$
10. $111111111q_1 1$
11. $1111111111q_1$
12. $111111111q_2 1$
13. $11111111q_3 1$
14. $1111111q_3 11$
15. $111111q_3 111$
16. $11111q_3 1111$
17. $1111q_3 11111$
18. $111q_3 111111$
19. $11q_3 1111111$
20. $1q_3 11111111$
21. $q_3 111111111$
22. $q_3 \sqcup 111111111$
23. $q_{accept} 111111111$

Church-Turing Thesis

Turing's Thesis

Any computation carried out by mechanical means can be performed by a Turing Machine.

Definition

An algorithm for a function $f(w)$ is a Turing Machine that computes $f(w)$.

Note: When we say "There is an algorithm", what we mean is "There exists a Turing Machine that executes the algorithm".