CSC373

Weeks 9 & 10: Approximation Algorithms & Local Search

NP-Completeness

- NP-complete problems
  - Unlikely to have polynomial time algorithms to solve them
  - What do we do?

- One idea: approximation

  - Instead of solving them exactly, solve them approximately
  - Sometimes, we might want to use an approximation algorithm even when we can compute an exact solution in polynomial time (WHY?)

Approximation Algorithms

- Decision versus optimization problems

  - Decision variant: "Does there exist a solution with objective $\geq k$?"
    - E.g. "Is there an assignment which satisfies at least k clauses of a given CNF formula φ?"

  - Optimization variant: "Find a solution maximizing objective"
    - E.g. "Find an assignment which satisfies the maximum possible number of clauses of a given CNF formula φ."

  - If a decision problem is hard, then its optimization version is hard too

  - We'll focus on optimization variants

Approximation Algorithms

- Objectives
  - Maximize (e.g. "profit") or minimize (e.g. "cost")

- Given problem instance I
  - $ALG(I)$ = solution returned by our algorithm
  - $OPT(I)$ = some optimal solution
  - Approximation ratio of ALG on instance I is

$$\frac{profit\big(OPT(I)\big)}{profit\big(ALG(I)\big)} \quad \text{or} \quad \frac{cost\big(ALG(I)\big)}{cost\big(OPT(I)\big)}$$

  - Convention: approximation ratio ≥ 1
    - "2-approximation" = half the optimal profit / twice the optimal cost

Approximation Algorithms

- Worst-case approximation ratio
  - Worst approximation ratio across all possible problem instances I

  - ALG has worst-case c-approximation if for each problem instance I…

$$profit\big(ALG(I)\big) \geq \frac{1}{c} \cdot profit\big(OPT(I)\big)\ or$$

$$cost\big(ALG(I)\big) \leq c \cdot cost\big(OPT(I)\big)$$

  - By default, we will always refer to approximation ratios in the worst case
  - Note: In some textbooks, you might see the approximation ratio flipped (e.g. 0.5-approximation instead of 2-approximation)

PTAS and FPTAS

- Arbitrarily close to 1 approximations

- PTAS: Polynomial time approximation scheme
  - For every $\epsilon > 0$, there is a $(1+\epsilon)$-approximation algorithm that runs in time $poly(n)$ on instances of size n
    - Note: Could have exponential dependence on $1/\epsilon$

- FPTAS: Fully polynomial time approximation scheme
  - For every $\epsilon > 0$, there is a $(1+\epsilon)$-approximation algorithm that runs in time $poly(n, 1/\epsilon)$ on instances of size n

Approximation Landscape

Impossibility of better approximations assuming widely held beliefs like P $\neq$ NP

n = parameter of problem at hand

- An FPTAS
  - E.g. the knapsack problem
- A PTAS but no FPTAS
  - E.g. the makespan problem (we'll see)
- c-approximation for a constant $c > 1$ but no PTAS
  - E.g. vertex cover and JISP (we'll see)
- $\Theta(\log n)$-approximation but no constant approximation
  - E.g. set cover
- No $n^{1-\epsilon}$-approximation for any $\epsilon > 0$
  - E.g. graph coloring and maximum independent set

Approximation Techniques

- Greedy algorithms
  - Make decision on one element at a time in a greedy fashion without considering future decisions

- LP relaxation
  - Formulate the problem as an integer linear program (ILP)
  - "Relax" it to an LP by allowing variables to take real values
  - Find an optimal solution of the LP, "round" it to a feasible solution of the original ILP, and prove its approximate optimality

- Local search
  - Start with an arbitrary solution
  - Keep making "local" adjustments to improve the objective

Greedy Approximation

Makespan Minimization

Makespan

- Problem
  - Input: m identical machines, n jobs, job j requires processing time t_j
  - Output: Assign jobs to machines to minimize makespan

- Let $S[i]$ = set of jobs assigned to machine i in a solution

- Constraints:
  - Each job must run contiguously on one machine
  - Each machine can process at most one job at a time

- Load on machine i: $L_i = \sum_{j \in S[i]} t_j$
- Goal: minimize the *maximum* load, i.e., makespan $L = \max_i L_i$

Makespan

- Even the special case of $m = 2$ machines is already NP-hard by reduction from PARTITION

- PARTITION
  - Input: Set S containing n integers
  - Question: Does there exist a partition of S into two sets with equal sum? (A partition of S into S_1, S_2 means $S_1 \cap S_2 = \emptyset$ and $S_1 \cup S_2 = S$)

- Exercise!
  - Show that PARTITION is NP-complete by reduction from SUBSET-SUM
  - Show that Makespan with $m = 2$ is NP-complete by reduction from PARTITION

Makespan

- Greedy list-scheduling algorithm
  - Consider the n jobs in some "nice" sorted order
  - Assign each job j to a machine with the smallest load so far

- Note: Implementable in $O(n \log m)$ using priority queue

- Back to greedy…?
  - But this time, we can't hope that greedy will be optimal
  - We can still hope that it is approximately optimal

- Which order?

Makespan

- Theorem [Graham 1966]
  - Regardless of the order, greedy gives a 2-approximation.
  - This was one of the first worst-case approximation analyses

- Let optimal makespan = L^*

- To show that makespan under the greedy solution is not much worse than L^*, we need to show that L^* cannot be too low

Makespan

- Theorem [Graham 1966]
  - Regardless of the order, greedy gives a 2-approximation.

- Fact 1: $L^* \geq \max_j t_j$
  - Some machine must process job with highest processing time

- Fact 2: $L^* \geq \frac{1}{m} \sum_j t_j$
  - Total processing time is $\sum_j t_j$
  - At least one machine must do at least $1/m$ of this work (the pigeonhole principle)

Makespan

- Theorem [Graham 1966]
  - Regardless of the order, greedy gives a 2-approximation.

- Proof:

  - Suppose machine i is the bottleneck under greedy (so $L = L_i$)
  - Let j^* be the last job scheduled on machine i by greedy

  - Right before j^* was assigned to i, i had the smallest load
    - Load of the other machines could have only increased from then
    - $L_i - t_{j^*} \le L_k, \forall k$

  - Average over all k : $L_i - t_{j^*} \le \frac{1}{m} \sum_j t_j$
  - $L_i \le t_{j^*} + \frac{1}{m} \sum_j t_j \le L^* + L^* = 2L^*$

Fact 1

Fact 2

Makespan

- Theorem [Graham 1966]
  - Regardless of the order, greedy gives a 2-approximation.

- Is our analysis tight?
  - Essentially.
  - By averaging over $k \neq i$ in the previous slide, one can show a slightly better $2 - 1/m$ approximation
  - There is an example where greedy has approximation as bad as $2 - 1/m$
  - So $2 - 1/m$ is exactly tight.

Makespan

- Tight example:
  - $m(m-1)$ jobs of length 1, followed by one job of length m
  - Greedy evenly distributes unit length jobs on all m machines, and assigning the last heavy job makes makespan $m-1+m=2m-1$
  - Optimal makespan is m by evenly distributing unit length jobs among $m-1$ machines and putting the single heavy job on the remaining

- Idea:
  - It seems keeping heavy jobs at the end is bad.
  - So let's just start with them first!

Makespan Revisited

- Greedy LPT (Longest Processing Time First)
  - Run the greedy algorithm but consider jobs in a non-increasing order of their processing time
  - Suppose $t_1 \geq t_2 \geq \cdots \geq t_n$

- Fact 3: If the bottleneck machine i has only one job j, then the solution is optimal.

  - Current solution has $L = L_i = t_j$

  - We know $L^* \geq t_j$ from Fact 1
- Fact 4: If there are more than m jobs, then $L^* \geq 2 \cdot t_{m+1}$
  - The first $m + 1$ jobs each have processing time at least t_{m+1}
  - By the pigeonhole principle, the optimal solution must put at least two of them on the same machine

Makespan Revisited

- Theorem
  - Greedy LPT achieves $3/2$-approximation

- Proof:

  - Similar to the proof for arbitrary ordering

  - Consider a bottleneck machine i and the job j^* that was last scheduled on this machine by the greedy algorithm

  - Case 1: Machine i has only one job j^*
    - By Fact 3, greedy is optimal in this case (i.e. 1-approximation)

Makespan Revisited

- Theorem
  - Greedy LPT achieves $3/2$-approximation

- Proof:

  - Similar to the proof for arbitrary ordering

  - Consider a bottleneck machine i and the job j^* that was last scheduled on this machine by the greedy algorithm

  - Case 2: Machine i has at least two jobs
    - Job j^* must have $t_{j^*} \leq t_{m+1}$
    - As before, $L = L_i = \underbrace{\left(L_i - t_{j^*}\right)}_{\leq L^*} + \underbrace{t_{j^*}}_{\leq L^*/2} \leq 1.5\, L^*$

Same as before

$t_{j^*} \leq t_{m+1}$ and Fact 4

Makespan Revisited

- Theorem
  - Greedy LPT achieves 3/2-approximation
  - Is our analysis tight? No!

- Theorem [Graham 1966]
  - Greedy LPT achieves $\left(\frac{4}{3} - \frac{1}{3\mathrm{m}}\right)$-approximation
  - Is Graham's approximation tight?
    - Yes.
    - In the upcoming example, greedy LPT is as bad as $\frac{4}{3} - \frac{1}{3m}$

Makespan Revisited

- Tight example for Greedy LPT:
  - 2 jobs each of lengths $m, m+1, \ldots, 2m-1$
  - One more job of length m
  - Greedy-LPT has makespan $4m-1$ (verify!)
  - OPT has makespan $3m$ (verify!)
  - Thus, approximation ratio is at least as bad as $\frac{4m-1}{3m} = \frac{4}{3} - \frac{1}{3m}$

Weighted Set Packing

Weighted Set Packing

- Problem
  - Input: Universe of m elements, sets $S_1, \dots, S_n$ with values $v_1, \dots, v_n \geq 0$
  - Output: Pick disjoint sets with maximum total value
    - That is, pick $W \subseteq \{1, \dots, n\}$ to maximize $\sum_{i \in W} v_i$ subject to $S_i \cap S_j = \emptyset$ for all $i, j \in W$

- What's known about this problem?
  - It's NP-hard
  - For any constant $\epsilon > 0$, you cannot get $O\left(m^{1/2-\epsilon}\right)$ approximation in polynomial time unless NP=ZPP (widely believed to be not true)

Greedy Template

- Sort the sets in some order, consider them one-by-one, and take any set that you can along the way.

- Greedy Algorithm:
  - Sort the sets in a specific order.
  - Relabel them as $1,2,\dots,n$ in this order.
  - $W \leftarrow \emptyset$
  - For $i = 1,\dots,n$:
    - If $S_i \cap S_j = \emptyset$ for every $j \in W$, then $W \leftarrow W \cup \{i\}$
  - Return W.

Greedy Algorithm

- What order should we sort the sets by?

- We want to take sets with high values.
  - $v_1 \geq v_2 \geq \cdots \geq v_n$? Only m-approximation ☹

- We don’t want to exhaust many items too soon.
  - $\frac{v_1}{|S_1|} \geq \frac{v_2}{|S_2|} \geq \cdots \frac{v_n}{|S_n|}$? Also m-approximation ☹

- $\sqrt{m}$-approximation : $\frac{v_1}{\sqrt{|S_1|}} \geq \frac{v_2}{\sqrt{|S_2|}} \geq \cdots \frac{v_n}{\sqrt{|S_n|}}$?

[Lehmann et al. 2011]

Proof of Approximation

- Definitions
  - OPT = Some optimal solution
  - W = Solution returned by our greedy algorithm
  - For $i \in W$, $OPT_i = \{j \in OPT : j \geq i, S_i \cap S_j \neq \emptyset\}$

- Claim 1: $OPT \subseteq \cup_{i \in W} OPT_i$

- Claim 2: It is enough to show that $\forall i \in W$

$$\sqrt{m} \cdot v_i \geq \Sigma_{j \in OPT_i} v_j$$

- Observation: For $j \in OPT_i$, $v_j \leq v_i \cdot \frac{\sqrt{|S_j|}}{\sqrt{|S_i|}}$

Proof of Approximation

- Summing over all $j \in OPT_i$:

$$\Sigma_{j \in OPT_i} v_j \le \frac{v_i}{\sqrt{|S_i|}} \cdot \Sigma_{j \in OPT_i} \sqrt{|S_j|}$$

- Using Cauchy-Schwarz ($\Sigma_i x_i y_i \le \sqrt{\Sigma_i x_i^2} \cdot \sqrt{\Sigma_i y_i^2}$)

$$\Sigma_{j \in OPT_i} \sqrt{1. |S_j|} \le \sqrt{|OPT_i|} \cdot \sqrt{\Sigma_{j \in OPT_i} |S_j|}$$
$$\le \sqrt{|S_i|} \cdot \sqrt{m}$$

Unweighted Vertex Cover

Unweighted Vertex Cover

- Problem
  - Input: Undirected graph $G = (V, E)$
  - Output: Vertex cover S of minimum cardinality

  - Recall: S is vertex cover if every edge has at least one of its two endpoints in S

  - We already saw that this problem is NP-hard

- Q: What would be a good greedy algorithm for this problem?

Unweighted Vertex Cover

- Greedy edge-selection algorithm:
  - Start with $S = \emptyset$
  - While there exists an edge whose both endpoints are not in S, add *both* its endpoints to S

- Hmm…
  - Why are we selecting edges rather than vertices?
  - Why are we adding both endpoints?
  - We'll see..

Unweighted Vertex Cover

GREEDY-VERTEX-COVER(G)

$S \leftarrow \varnothing$.

$E' \leftarrow E$.

WHILE ($E' \neq \varnothing$)

every vertex cover must take at least one of these; we take both

Let $(u, v) \in E'$ be an arbitrary edge.

$M \leftarrow M \cup \{(u, v)\}$. ← M is a matching

$S \leftarrow S \cup \{u\} \cup \{v\}$.

Delete from E' all edges incident to either u or v.

RETURN S.

Unweighted Vertex Cover

- Theorem:
  - Greedy edge-selection algorithm for unweighted vertex cover achieves 2-approximation.

- Observation 1:
  - For any vertex cover S^* and any matching M, $|S^*| \geq |M|$, where $|M|$ = number of edges in M

  - Proof: S^* must contain at least one endpoint of each edge in M

- Observation 2:
  - Greedy algorithm finds a vertex cover of size $|S| = 2 \cdot |M|$
- Hence, $|S| \leq 2 \cdot |S^*|$, where S^* = min vertex cover

Unweighted Vertex Cover

- Corollary:
  - If M^* is a *maximum* matching, and M is a *maximal* matching, then $|M| \geq \frac{1}{2}|M^*|$
- Proof:
  - By design, $|M| = \frac{1}{2}|S|$
  - $|S| \geq |M^*|$ (Observation 1)
  - Hence, $|M| \geq \frac{1}{2}|M^*|$ ■

- This greedy algorithm is also a 2-approximation to the problem of finding a maximum cardinality matching
  - However, the max cardinality matching problem can be solved exactly in polynomial time using a more complex algorithm

Unweighted Vertex Cover

- What about a greedy vertex selection algorithm?
  - Start with $S = \emptyset$
  - While S is not a vertex cover:
    - Choose a vertex v which maximizes the number of uncovered edges incident on it
    - Add v to S
  - Gives $O(\log d_{\max})$ approximation, where $d_{\max}$ is the maximum degree of any vertex
    - But unlike the edge-selection version, this generalizes to set cover
    - For set cover, $O(\log d_{\max})$ approximation ratio is the best possible in polynomial time unless P=NP

Unweighted Vertex Cover

NOT IN SYLLABUS

- Theorem [Dinur-Safra 2004]:
  - Unless P = NP, there is no polynomial-time ρ-approximation algorithm for unweighted vertex cover for any constant $\rho < 1.3606$.

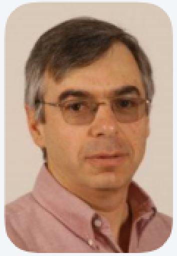
On the Hardness of Approximating Minimum Vertex Cover

Irit Dinur* Samuel Safra†

May 26, 2004

Abstract

We prove the Minimum Vertex Cover problem to be NP-hard to approximate to within a factor of 1.3606, extending on previous PCP and hardness of approximation technique. To that end, one needs to develop a new proof framework, and borrow and extend ideas from several fields.



Unweighted Vertex Cover

NOT IN SYLLABUS

- Theorem [Khot-Regev 2008]:
  - Unless the "unique games conjecture" is violated, there is no polynomial-time ρ-approximation algorithm for unweighted vertex cover for any constant $\rho < 2$.

Vertex Cover Might be Hard to Approximate to within $2 - \varepsilon$

Subhash Khot * Oded Regev †

Abstract

Based on a conjecture regarding the power of unique 2-prover-1-round games presented in [Khot02], we show that vertex cover is hard to approximate within any constant factor better than 2. We actually show a stronger result, namely, based on the same conjecture, vertex cover on k-uniform hypergraphs is hard to approximate within any constant factor better than k.

Unweighted Vertex Cover

NOT IN SYLLABUS

- How does one prove a lower bound on the approximation ratio of any polynomial-time algorithm?
  - We prove that if there is a polynomial-time ρ-approximation algorithm for the problem with $\rho <$ some bound, then some widely believed conjecture is violated
  - For example, we can prove that given a polynomial time ρ-approximation algorithm to vertex cover for any constant $\rho < 1.3606$, we can use this algorithm as a subroutine to solve the 3SAT decision problem in polynomial time, implying P=NP
  - Similar technique can be used to reduce from other widely believed conjectures, which may give different (sometimes better) bounds
  - Beyond the scope of this course

Weighted Vertex Cover

Weighted Vertex Cover

- Problem
  - Input: Undirected graph $G = (V, E)$, weights $w : V \to R_{\geq 0}$
  - Output: Vertex cover S of minimum total weight

- The same greedy algorithm doesn't work
  - Gives arbitrarily bad approximation
  - Obvious modifications which try to take weights into account also don't work
  - Need another strategy…

LP Relaxation

ILP Formulation

- For each vertex v, create a binary variable $x_v \in \{0,1\}$ indicating whether vertex v is chosen in the vertex cover
- Then, computing min weight vertex cover is equivalent to solving the following integer linear program

$$\min \Sigma_v \, w_v \cdot x_v$$

subject to

$$x_u + x_v \geq 1, \qquad \forall (u,v) \in E$$

$$x_v \in \{0,1\}, \qquad \forall v \in V$$

LP Relaxation

- What if we solve the "LP relaxation" of the original ILP?
  - Just convert all integer variables to real variables

ILP with binary variables		**LP with real variables**	
$\min \Sigma_v w_v \cdot x_v$		$\min \Sigma_v w_v \cdot x_v$	
subject to		subject to	
$x_u + x_v \geq 1,$	$\forall (u,v) \in E$	$x_u + x_v \geq 1,$	$\forall (u,v) \in E$
$x_v \in \{0,1\},$	$\forall v \in V$	$x_v \geq 0,$	$\forall v \in V$

Rounding LP Solution

- What if we solve the "LP relaxation" of the original ILP?
  - Let's say we are minimizing objective $c^T x$
  - Since the LP minimizes this over a larger feasible space than the ILP, optimal LP objective value $\le$ optimal ILP objective value
  - Let x_{LP}^* be an optimal LP solution (which we can compute efficiently) and x_{ILP}^* be an optimal ILP solution (which we can't compute efficiently)
    - $c^T x_{LP}^* \le c^T x_{ILP}^*$
    - But x_{LP}^* may have non-integer values
    - Efficiently round x_{LP}^* to an ILP feasible solution $\hat{x}$ without increasing the objective too much
    - If we prove $c^T \hat{x} \le \rho \cdot c^T x_{LP}^*$, then we will also have $c^T \hat{x} \le \rho \cdot c^T x_{ILP}^*$
    - Thus, our algorithm will achieve ρ-approximation

Rounding LP Solution

- What if we solve the "LP relaxation" of the original ILP?
  - If we are maximizing $c^T x$ instead of minimizing, then it's reversed:
    - Optimal LP objective value ≥ optimal ILP objective value, i.e., $c^T x_{LP}^* \geq c^T x_{ILP}^*$
    - Efficiently round x_{LP}^* to an ILP feasible solution $\hat{x}$ without decreasing the objective too much
    - If we prove $c^T \hat{x} \geq (1/\rho) \cdot c^T x_{LP}^*$, then $c^T \hat{x} \geq (1/\rho) \cdot c^T x_{ILP}^*$
    - Thus, our algorithm will achieve ρ-approximation

Weighted Vertex Cover

- Consider LP optimal solution x^*
  - Let $\hat{x}_v = 1$ whenever $x_v^* \geq 0.5$ and $\hat{x}_v = 0$ otherwise
  - Claim 1: $\hat{x}$ is a feasible solution of ILP (i.e. a vertex cover)
    - For every edge $(u, v) \in E$, at least one of $\{x_u^*, x_v^*\}$ is at least 0.5
    - So at least one of $\{\hat{x}_u, \hat{x}_v\}$ is 1 ∎

ILP with binary variables

$\min \Sigma_v w_v \cdot x_v$
subject to
$x_u + x_v \geq 1, \quad \forall (u, v) \in E$
$x_v \in \{0,1\}, \quad \forall v \in V$

LP with real variables

$\min \Sigma_v w_v \cdot x_v$
subject to
$x_u + x_v \geq 1, \quad \forall (u, v) \in E$
$x_v \geq 0, \quad \forall v \in V$

Rounding LP Solution

- Consider LP optimal solution x^*
  - Let $\hat{x}_v = 1$ whenever $x_v^* \geq 0.5$ and $\hat{x}_v = 0$ otherwise
  - Claim 2: $\sum_v w_v \cdot \hat{x}_v \leq 2 * \sum_v w_v \cdot x_v^*$
    - Weight only increases when some $x_v^* \in [0.5,1]$ is rounded *up* to 1
    - At most doubling the variable, so at least doubling the weight ■

ILP with binary variables

$$\min \textstyle\sum_v w_v \cdot x_v$$
subject to
$$x_u + x_v \geq 1, \qquad \forall (u,v) \in E$$
$$x_v \in \{0,1\}, \qquad \forall v \in V$$

LP with real variables

$$\min \textstyle\sum_v w_v \cdot x_v$$
subject to
$$x_u + x_v \geq 1, \qquad \forall (u,v) \in E$$
$$x_v \geq 0, \qquad \forall v \in V$$

Rounding LP Solution

- Consider LP optimal solution x^*
  - Let $\hat{x}_v = 1$ whenever $x_v^* \geq 0.5$ and $\hat{x}_v = 0$ otherwise
  - Hence, $\hat{x}$ is a vertex cover with weight at most 2 * LP optimal value $\leq 2 *$ ILP optimal value

ILP with binary variables

$\min \Sigma_v w_v \cdot x_v$
subject to
$x_u + x_v \geq 1, \qquad \forall (u,v) \in E$
$x_v \in \{0,1\}, \qquad \forall v \in V$

LP with real variables

$\min \Sigma_v w_v \cdot x_v$
subject to
$x_u + x_v \geq 1, \qquad \forall (u,v) \in E$
$x_v \geq 0, \qquad \forall v \in V$

General LP Relaxation Strategy

- Your NP-complete problem amounts to solving
  - Max $c^T x$ subject to $Ax \leq b, x \in \mathbb{N}$ (need not be binary)

- Instead, solve:
  - Max $c^T x$ subject to $Ax \leq b, x \in \mathbb{R}_{\geq 0}$ (LP relaxation)
    - LP optimal value $\geq$ ILP optimal value (for maximization)
  - x^* = LP optimal solution
  - Round x^* to $\hat{x}$ such that $c^T \hat{x} \geq \frac{c^T x^*}{\rho} \geq \frac{\text{ILP optimal value}}{\rho}$
  - Gives ρ-approximation
    - Info: Best ρ you can hope to get via this approach for a particular LP-ILP combination is called the *integrality gap*

Local Search Paradigm

Local Search

- Heuristic paradigm
  - Sometimes it might provably return an optimal solution
  - But even if not, it might give a good approximation

- Template
  - Start with some initial feasible solution S
  - While there is a "better" solution S' in the **local neighborhood** of S
  - Switch to S'

- Need to define:
  - Which initial feasible solution should we start from?
  - What is "better"?
  - What is "local neighborhood"?

Local Search

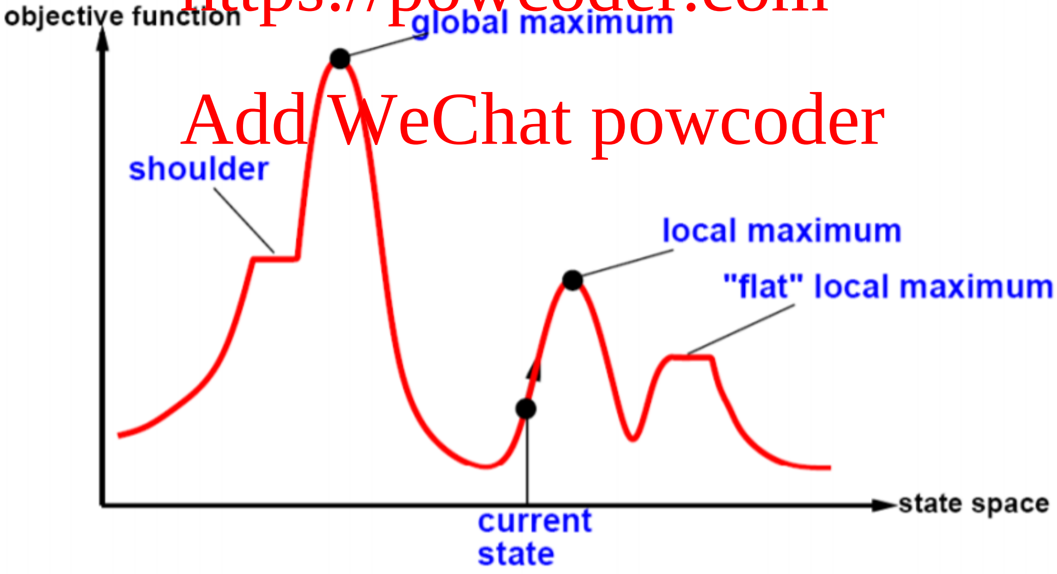
- For some problems, local search provably returns an optimal solution

- Example: network flow
  - Initial solution: zero flow
  - Local neighborhood: all flows that can be obtained by augmenting the current flow along a path in the residual graph
  - Better: Higher flow value

- Example: LP via simplex
  - Initial solution: a vertex of the polytope
  - Local neighborhood: neighboring vertices
  - Better: better objective value

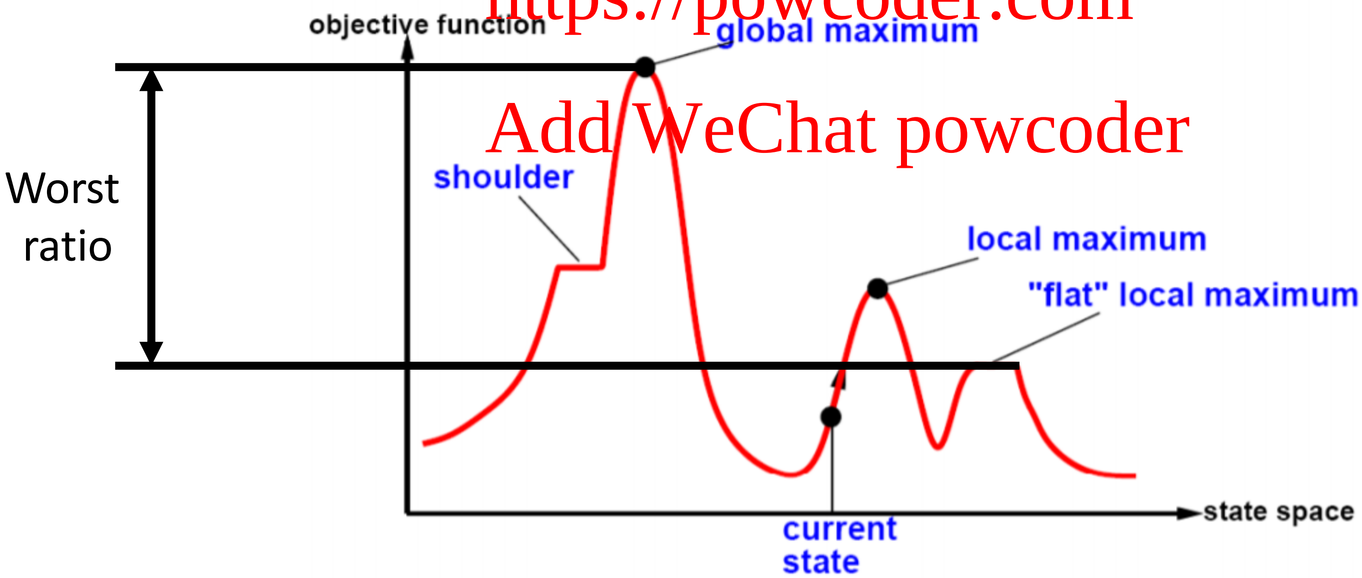
Local Search

- But sometimes it doesn't return an optimal solution, and "gets stuck" in a local maxima



Local Search

- In that case, we want to bound the worst-case ratio between the global optimum and the worst local optimum (the worst solution that local search might return)



Max-Cut

Max-Cut

- Problem
  - Input: An undirected graph $G = (V, E)$
  - Output: A partition (A, B) of V that maximizes the number of edges going across the cut, i.e., maximizes $|E'|$ where $E' = \{(u, v) \in E \mid u \in A, v \in B\}$

- This is also known to be an NP-hard problem

- What is a natural local search algorithm for this problem?
  - Given a current partition, what small change can you do to improve the objective value?

Max-Cut

- Local Search
  - Initialize (A, B) arbitrarily.
  - While there is a vertex u such that moving u to the other side improves the objective value:
    - Move u to the other side.

- When does moving u, say from A to B, improve the objective value?
  - When u has more incident edges going within the cut than across the cut, i.e., when $|\{(u, v) \in E \mid v \in A\}| > |\{(u, v) \in E \mid v \in B\}|$

Max-Cut

- Local Search
  - Initialize (A, B) arbitrarily.
  - While there is a vertex u such that moving u to the other side improves the objective value:
    - Move u to the other side.

- Why does the algorithm stop?
  - Every iteration increases the number of edges across the cut by at least 1, so the algorithm must stop in at most $|E|$ iterations

Max-Cut

- Local Search
  - Initialize (A, B) arbitrarily.
  - While there is a vertex u such that moving u to the other side improves the objective value:
    - Move u to the other side.

- Approximation ratio?
  - At the end, every vertex has at least as many edges going across the cut as within the cut
  - Hence, at least half of all edges must be going across the cut
    - Exercise: Prove this formally by writing equations.

Weighted Max-Cut

- Variant
  - Now we're given integral edge weights $w: E \to \mathbb{N}$
  - The goal is to maximize the total *weight* of edges going across the cut

- Algorithm

  - The same algorithm works…

  - But we move u to the other side if the total *weight* of its incident edges going within the cut is greater than the total *weight* of its incident edges going across the cut

Weighted Max-Cut

- Number of iterations?
  - Unweighted case: #edges going across the cut must increase by at least 1, so it takes at most $|E|$ iterations

  - Weighted case: total *weight* of edges going across the cut must increase by at least 1, but this could take up to $\sum_{e \in E} w_e$ iterations, which can be *exponential* in the input length

    - There are examples where the local search actually takes exponentially many steps

    - Fun exercise: Design an example where the number of iterations is exponential in the input length.

Weighted Max-Cut

- Number of iterations?
  - But we can find a $2 + \epsilon$ approximation in time polynomial in the input length and $\frac{1}{\epsilon}$

  - The idea is to only move vertices when it "sufficiently improves" the objective value

Weighted Max-Cut

- Better approximations?
  - Theorem [Goemans-Williamson 1995]:

    There exists a polynomial time algorithm for max-cut with approximation ratio $\frac{2}{\pi} \cdot \min_{0 \leq \theta \leq \pi} \frac{\theta}{1 - \cos\theta} \approx 0.878$

    - Uses "semidefinite programming" and "randomized rounding"
    - Note: The literature from here on uses approximation ratios ≤ 1, so we will follow that convention in the remaining slides.

  - Assuming the unique games conjecture, this approximation ratio is tight

Exact Max-k-SAT

Exact Max-k-SAT

- Problem
  - Input: An exact k-SAT formula $\varphi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$, where each clause C_i has exactly k literals, and a weight $w_i \geq 0$ of each clause C_i

  - Output: A truth assignment τ maximizing the total weight of clauses satisfied under τ

- Let us denote by $W(\tau)$ the total weight of clauses satisfied under τ

- What is a good definition of "local neighborhood"?

Exact Max-k-SAT

- Local neighborhood:
  - $N_d(\tau)$ = set of all truth assignments τ' which differ from τ in the values of at most d variables

- Theorem: The local search with $d = 1$ gives a $^2/_3$ approximation to Exact Max-2-SAT.

Exact Max-k-SAT

- Theorem: The local search with $d = 1$ gives a ${}^{2}/{}_{3}$ approximation to Exact Max-2-SAT.
- Proof:

  - Let τ be a local optimum
    - S_0 = set of clauses not satisfied under τ
    - S_1 = set of clauses from which exactly one literal is true under τ
    - S_2 = set of clauses from which both literals are true under τ
    - $W(S_0), W(S_1), W(S_2)$ be the corresponding total weights
    - Goal: $W(S_1) + W(S_2) \geq {}^{2}/{}_{3} \cdot \big(W(S_0) + W(S_1) + W(S_2)\big)$
      - Equivalently, $W(S_0) \leq {}^{1}/{}_{3} \cdot \big(W(S_0) + W(S_1) + W(S_2)\big)$

Exact Max-k-SAT

- Theorem: The local search with $d = 1$ gives a ${}^{2}/_{3}$ approximation to Exact Max-2-SAT.
- Proof:
  - We say that clause C "involves" variable j if it contains x_j or $\overline{x_j}$
  - A_j = set of clauses in S_0 involving variable j
    - Let $W(A_j)$ be the total weight of such clauses
  - B_j = set of clauses in S_1 involving variable j such that it is the literal of variable j that is true under τ
    - Let $W(B_j)$ be the total weight of such clauses

Exact Max-k-SAT

- **Theorem:** The local search with $d = 1$ gives a $^2/_3$ approximation to Exact Max-2-SAT.
- **Proof:**
  - $2\,W(S_0) = \sum_j W(A_j)$
    - Every clause in S_0 is counted twice on the RHS
  - $W(S_1) = \sum_j W(B_j)$
    - Every clause in S_1 is only counted once on the RHS for the variable whose literal was true under τ
  - For each j : $W(A_j) \le W(B_j)$
    - From local optimality of τ, since otherwise flipping the truth value of variable j would have increased the total weight

Exact Max-k-SAT

- Theorem: The local search with $d = 1$ gives a ${}^{2}/_{3}$ approximation to Exact Max-2-SAT.
- Proof:
  - $2\,W(S_0) \le W(S_1)$
    - Summing the third equation on the last slide over all j, and then using the first two equations on the last slide
  - Hence:
    - $3\,W(S_0) \le W(S_0) + W(S_1) \le W(S_0) + W(S_1) + W(S_2)$
    - Precisely the condition we wanted to prove…
    - QED!

Exact Max-k-SAT

- Higher d?
  - Searches over a larger neighborhood
  - May get a better approximation ratio, but increases the running time as we now need to check if any neighbor in a large neighborhood provides a better objective

  - The bound is still $2/3$ for $d = o(n)$
  - For $d = \Omega(n)$, the neighborhood size is exponential
  - But the approximation ratio is…
    - At most $4/5$ with $d < {}^{n}/_{2}$
    - 1 (i.e. optimal solution is always reached) with $d = {}^{n}/_{2}$

Exact Max-k-SAT

- Better approximation ratio?
  - We can learn something from our proof
  - Note that we did not use anything about $W(S_2)$, and simply added it at the end

  - If we could also guarantee that $W(S_0) \leq W(S_2)$…
    - Then we would get $4\,W(S_0) \leq W(S_0) + W(S_1) + W(S_2)$, which would give a $3/4$ approximation

  - Result (without proof):
    - This can be done by including just one more assignment in the neighborhood: $N(\tau) = N_1(\tau) \cup \{\tau^c\}$, where τ^c = complement of τ

Exact Max-k-SAT

- What if we do not want to modify the neighborhood?
  - A slightly different tweak also works
  - We want to weigh clauses in $W(S_2)$ more because when we get a clause through S_2, we get more robustness (it can withstand changes in single variables)

- Modified local search:

  - Start at arbitrary τ
  - While there is an assignment in $N_1(\tau)$ that improves the potential $1.5\,W(S_1) + 2\,W(S_2)$
    - Switch to that assignment

Exact Max-k-SAT

- Modified local search:
  - Start at arbitrary τ
  - While there is an assignment in $N_1(\tau)$ that improves the potential $1.5\, W(S_1) + 2\, W(S_2)$
    - Switch to that assignment

- Note:
  - This is the first time that we're using a definition of "better" in local search paradigm that does not quite align with the ultimate objective we want to maximize
  - This is called "non-oblivious local search"

Exact Max-k-SAT

- Modified local search:
  - Start at arbitrary τ
  - While there is an assignment in $N_1(\tau)$ that improves the potential $1.5\, W(S_1) + 2\, W(S_2)$
    - Switch to that assignment

- Result (without proof):
  - Modified local search gives $^3/_4$-approximation to Exact Max-2-SAT

Exact Max-k-SAT

- More generally:
  - The same technique works for higher values of k
  - Gives $\frac{2^k-1}{2^k}$ approximation for Exact Max-k-SAT
    - In the next lecture, we will achieve the same approximation ratio much more easily through a different technique

- Note: This ratio is $^7/_8$ for Exact Max-3-SAT
  - Theorem [Håstad]: Achieving $^7/_8 + \epsilon$ approximation where $\epsilon > 0$ is NP-hard.
    - Uses PCP (probabilistically checkable proofs) technique