# CSC373



# Weeks 7 & 8:



# Complexity

# Recap



- Linear Programming
  - Standard formulation
  - Slack formulation
  - Simplex
  - Duality
  - Formulating given problems as LPs

# This & Next Week



- Applications of linear programming
  - Shortest path
  - Network flow
- A note about integer programming
- Complexity
  - Turing machines, computability, efficient computation
  - P, NP, and NP-completeness
  - Reductions
  - Idea behind NP-completeness of SAT and 3SAT
  - NP vs co-NP
  - Other complexity classes

# Network Flow via LP



- Problem
  - Input: directed graph $G = (V, E)$, edge capacities $c: E \to \mathbb{R}_{\geq 0}$
  - Output: Value $v(f^*)$ of a maximum flow $f^*$





- Flow $f$ is valid if:
  - Capacity constraints: $\forall (u, v) \in E: 0 \leq f(u, v) \leq c(u, v)$
  - Flow conservation: $\forall u: \sum_{(u,v)\in E} f(u, v) = \sum_{(v,u)\in E} f(v, u)$



- Maximize $v(f) = \sum_{(s,v)\in E} f(s, v)$

Linear constraints

Linear objective!

# Network Flow via LP



$$\text{maximize} \quad \sum_{(s,v)\in E} f_{sv}$$



$$0 \le f_{uv} \le c(u,v) \quad \text{for all } (u,v) \in E$$





$$\sum_{(u,v)\in E} f_{uv} = \sum_{(v,w)\in E} f_{v,w} \quad \text{for all } v \in V\backslash\{s,t\}$$

Exercise: Write the dual of this LP.
What is the dual trying to find?

# Shortest Path via LP



- Problem
  - Input: directed graph $G = (V, E)$, edge weights $w: E \to \mathbb{R}_{\geq 0}$, source vertex $s$, target vertex $t$
  - Output: weight of the shortest-weight path from $s$ to $t$



- Variables: for each vertex $v$, we have variable $d_v$





Why max?

Exercise: prove formally that this works!

$$\begin{aligned} \text{maximize} \quad & d_t \\ \text{subject to} \quad & \\ & d_v \leq d_u + w(u,v) \quad \text{for each edge } (u,v) \in E\,, \\ & d_s = 0\,. \end{aligned}$$

If objective was min., then we could set all variables $d_v$ to 0.

# But...but...



- For these problems, we have different combinatorial algorithms that are much faster and run in strongly polynomial time



- Why would we use LP?





- For some problems, we don't have faster algorithms than solving them via LP

# Multicommodity-Flow



- Problem:
  - Input: directed graph $G = (V, E)$, edge capacities $c: E \to \mathbb{R}_{\geq 0}$, $k$ commodities $(s_i, t_i, d_i)$, where $s_i$ is source of commodity $i$, $t_i$ is sink, and $d_i$ is demand.
  - Output: valid multicommodity flow $(f_1, f_2, \ldots, f_k)$, where $f_i$ has value $d_i$ and all $f_i$ jointly satisfy the constraints



The only known polynomial time algorithm for this problem is based on solving LP!

$$\sum_{i=1}^{k} f_{iuv} \leq c(u, v) \quad \text{for each } u, v \in V,$$

$$\sum_{v \in V} f_{iuv} - \sum_{v \in V} f_{ivu} = 0 \quad \text{for each } i = 1, 2, \ldots, k \text{ and for each } u \in V - \{s_i, t_i\},$$

$$\sum_{v \in V} f_{i,s_i,v} - \sum_{v \in V} f_{i,v,s_i} = d_i \quad \text{for each } i = 1, 2, \ldots, k,$$

$$f_{iuv} \geq 0 \quad \text{for each } u, v \in V \text{ and for each } i = 1, 2, \ldots, k.$$

# Integer Linear Programming



- Variable values are restricted to be integers
- Example:
  - Input: $c \in \mathbb{R}^n, b \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}$
  - Goal:



$$\text{Maximize} \quad c^T x$$



$$\text{Subject to} \quad Ax \leq b$$



$$\boldsymbol{x \in \{0,1\}^n}$$

- Does this make the problem easier or harder?
  - Harder. We'll prove that this is "NP-complete".

# LPs are everywhere...



- Microeconomics
- Manufacturing
- VLSI (very large scale integration) design
- Logistics/transportation
- Portfolio optimization
- Bioengineering (flux balance analysis)
- Operations research more broadly: maximize profits or minimize costs, use linear models for simplicity
- Design of approximation algorithms
- Proving theorems, as a proof technique
- …

# Introduction to Complexity



- You have a problem at hand
- You try every technique known to humankind for finding a polynomial time algorithm, but fail.
- You try every technique known to humankind for proving that there cannot exist a polynomial time algorithm for your problem, but fail.
- What do you do?
  - Prove that it is NP-complete, of course!

# Turing Machines



- "Which problems can a computer (not) solve in a certain amount of time?"
  - How do we mathematically define what a computer is?



- Alan Turing ("Father of Computer Science"), 1936
  - Introduced a mathematical model
  - "Turing machine"
  - All present-day computers can be simulated by a Turing machine
  - Fun fact: So can all the quantum computers
    - But TM might take longer to solve the same problem

# Turing Machines



- We won't formally introduce…but at a high level…
- Turing machine
  - Tape
    - Input is given on tape
    - Intermediate computations can be written there
    - Output must to be written there
  - Head pointer
    - Initially pointing at start of input on tape
  - Maintains an internal "state"
  - A transition function describes how to change state, move head pointer, and read/write symbols on tape

# Computability



- Church-Turing Thesis
  - "Everything that is computable can be computed by a Turing machine"
  - Widely accepted, cannot be "proven"
  - There are problems which a Turing machine cannot solve, regardless of the amount of time available
    - E.g., the halting problem



- What about the problems we *can* solve? How do we define the time required?
  - Need to define an encoding of the input and output

# Encoding



- What can we write on the tape?
  - $S$ = a set of finite symbols
  - $S^* = \bigcup_{n \geq 0} S^n$ = set of all finite strings using symbols from $S$



- Input: $w \in S^*$
  - Length of input = $|w|$ = length of $w$ on tape



- Output: $f(w) \in S^*$



  - Length of output = $|f(w)|$
  - Decision problems: output = "YES" or "NO"
    - E.g. "does there exist a flow of value at least 7 in this network?"

# Encoding



- Example:
  - "Given $a_1, a_2, \ldots, a_n$, compute $\sum_{i=1}^{n} a_i$"
    - Suppose we are told that $a_i \leq C$ for all $i$
  - What $|S|$ should we use?
    - $S = \{0,1\}$ ($|S| = 2$, binary representation)
      - Length of input = $O(\log_2 a_1 + \cdots + \log_2 a_n) = O(n \log_2 C)$
    - What about 3-ary ($|S| = 3$) or 18-ary ($|S| = 18$)?
      - Only changes the length by a constant factor, still $O(n \log C)$
    - What about unary (conceptually, $|S| = 1$)?
      - Length blows up exponentially to $O(nC)$
    - Binary is already good enough, but unary isn't

# Efficient Computability



- Polynomial-time computability
  - A TM solves a problem in polynomial time if there is a polynomial $p$ such that on every instance of $n$-bit input and $m$-bit output, the TM halts in at most $p(n, m)$ steps
  - Polynomial: $n, n^2, 5n^{100} + 1000n^3, n \log^{100} n = o(n^{1.001})$
  - Non-polynomial: $2^n, 2^{\sqrt{n}}, 2^{\log^2 n}$



- Extended Church
  - "Everything that i
    polynomial time"
  - Much less widely accepted, especially today
  - But in this course, efficient = polynomial-time



If you ask the Turing machine to write a $2^n$-bit output, it's only reasonable to let it take $2^n$ time…but usually, we'll look at problems where output is O(length of input), so we can ignore this $m$

# P



- P (polynomial time)
  - The class of all decision problems computable by a TM in polynomial time



- Examples
  - Addition, multiplication, square root



  - Shortest paths
  - Network flow



  - Fast Fourier transform
  - Checking if a given number is a prime [Agrawal-Kayal-Saxena 2002]
  - …

# NP



- NP (nondeterministic polynomial time) intuition
  - Subset sum problem:

    Given an array {–7, –3, –2, 5, 8}, is there a zero-sum subset?



  - Enumerating all subsets is exponential



  - But…given {-3, -2, 5}, we can verify in polynomial time that it is indeed a valid subset and has zero sum



  - A nondeterministic Turing machine could “guess” the solution and then test if it has zero sum in polynomial time

# NP



- NP (nondeterministic polynomial time)
  - The class of all decision problems for which a YES answer can be verified by a TM in polynomial time given polynomial length "advice" or "witness".



  - There is a polynomial-time verifier TM $V$ and another polynomial $p$ such that



    - For all YES inputs $x$, there exists advice $y$ with $|y| = p(|x|)$ on which $V(x, y)$ returns YES



    - For all NO inputs $x$, $V(x, y)$ returns NO for every possible $y$
  - Informally: "Whenever the answer is YES, there's a short proof of it."
    - When the answer is NO, there may not be any short proof for it.

# co-NP



- co-NP
  - Same as NP, except whenever the answer is NO, there is a short proof of it



- Open questions
  - NP = co-NP?
  - P = NP ∩ co-NP?
  - And…drum roll please…





$$P = NP?$$

# P versus NP



- Lance Fortnow in his article on P and NP in Communications of the ACM, Sept 2009



*"The P versus NP problem has gone from an interesting problem related to logic to perhaps the most fundamental and important mathematical question of our time, whose importance only grows as computers become more powerful and widespread."*

# Millenium Problems



- Award of $1M for each problem by the Clay Math institute

1. Birch and Swinnerton-Dyer Conjecture
2. Hodge Conjecture
3. Navier-Stokes Equations
4. P = NP?
5. Poincare Conjecture (Solved)[1]
6. Riemann Hypothesis
7. Yang-Mills Theory



Claim: Worth >> $1M

[1]Solved by Grigori Perelman (2003): Prize unclaimed

# Cook's Conjecture



- Cook's conjecture
  - (And every sane person's belief…)
  - $P$ is likely not equal to $NP$



- Why do we believe this?



  - There is a large class of problems (NP-complete)
  - By now, contains thousands and thousands of problems



  - Each problem is the "hardest problem in NP"
  - If you can efficiently solve *any one of them,* you can efficiently solve *every problem in NP*
    - Despite decades of effort, no polynomial time solution has been found for *any of them*

# Reductions



- Problem $A$ is p-reducible to problem $B$ (denoted $A \leq_p B$) if an "oracle" (subroutine) for $B$ can be used to efficiently solve $A$
  - You can solve $A$ by making polynomially many calls to the oracle and doing additional polynomial-time computation





- Question: If $A$ is p-reducible to $B$, then which of the following is true?



a) If $A$ cannot be solved efficiently, then neither can $B$.
b) If $B$ cannot be solved efficiently, then neither can $A$.
c) Both.
d) None.

# Reductions



- Problem $A$ is p-reducible to problem $B$ (denoted $A \leq_p B$) if an "oracle" (subroutine) for $B$ can be used to efficiently solve $A$
  - You can solve $A$ by making polynomially many calls to the oracle and doing additional polynomial computation





- Question: If I want to prove that my problem $X$ is "hard", I should:



a) Reduce my problem $X$ to a known hard problem.
b) Reduce a known hard problem to my problem $X$.
c) Both.
d) None.

# NP-completeness



- NP-completeness
  - A problem $B$ is NP-complete if it is in NP and every problem $A$ in NP is p-reducible to $B$
  - Hardest problems in NP
  - If one of them can be solved efficiently, every problem in NP can be solved efficiently, implying P=NP



- Observation:



  - If $A$ is in NP, and some NP-complete problem $B$ is p-reducible to $A$, then $A$ is NP-complete too
    - "If I could solve A, then I could solve B, and then I could solve any problem in NP"

# NP-completeness



- But this uses an already known NP-complete problem to prove another problem is NP-complete



- How do we find the *first* NP-complete problem?
  - How do we know there are *any* NP-complete problems at all?
  - Key result by Cook
  - First NP-complete problem: SAT
    - By a reduction from an arbitrary problem in NP to SAT
    - "From first principles"

# CNF Formulas



- Conjunctive normal form (CNF)
  - Boolean variables $x_1, x_2, \ldots, x_n$
  - Their negations $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$
  - Literal $\ell$: a variable or its negation
  - Clause $C = \ell_1 \vee \ell_2 \vee \cdots \vee \ell_r$ is a disjunction of literals
  - CNF formula $\varphi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$ is a conjunction of clauses
    - $k$CNF: Each clause has at most $k$ literals
      - We'll abuse notation a little and assume there are *exactly* $k$
      - Example of 3CNF



$$\varphi = (\bar{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_4) \wedge (\bar{x}_3 \vee \bar{x}_4 \vee x_1)$$

# SAT and 3SAT



- Example of 3CNF

$$\varphi = (\bar{x}_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3) \wedge (\bar{x}_1 \vee x_2 \vee x_4) \wedge (\bar{x}_3 \vee \bar{x}_4 \vee x_1)$$



- "SAT" (Satisfiability) Problem:
  - A CNF formula $\varphi$ is satisfiable if there is an assignment of truth values (TRUE/FALSE) to variables under which the formula evaluates to TRUE
    - That means, in each clause, at least one literal is TRUE
  - SAT: "Given a CNF formula $\varphi$, is it satisfiable?"
  - 3SAT: "Given a 3CNF formula $\varphi$, is it satisfiable?"

# SAT and 3SAT



- Cook-Levin Theorem
  - SAT (and even 3SAT) is NP-complete



- Doesn't use any known NP-complete problem
  - Directly reduces any given NP problem to SAT



  - Reduction is a bit complex, so we'll defer it until later



  - But for now, let's assume SAT and 3SAT are NP-complete and reduce them to other problems (and then those problems to other problems…)

# NP-Complete Examples



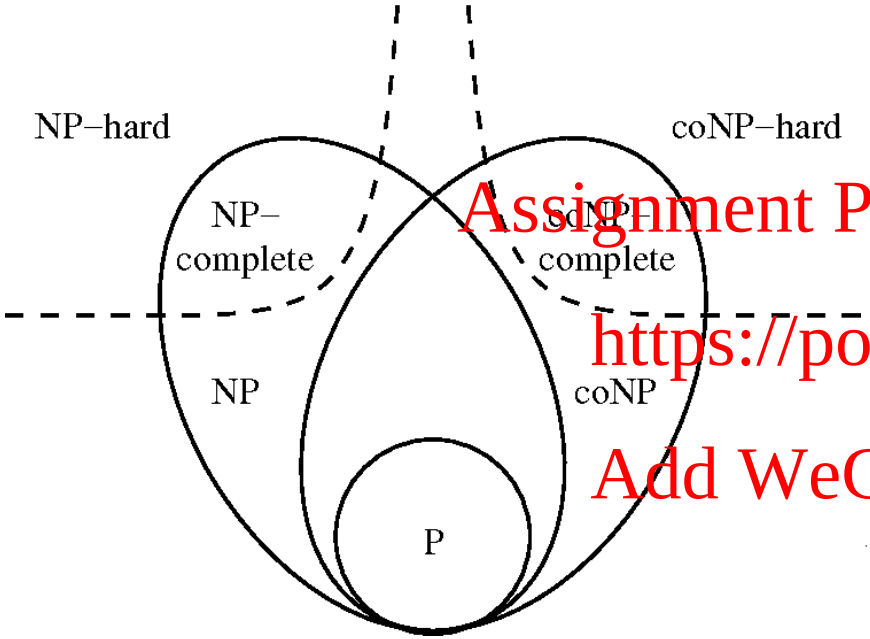
- NP-complete problems
  - SAT = first NP complete problem
  - Decision TSP: Is there a route visiting all $n$ cities with total distance at most $k$?
  - 3-Colorabitility: Can the vertices of a graph be colored with at most 3 colors such that no two adjacent vertices have the same color?
  - Karp's 21 NP-complete problems



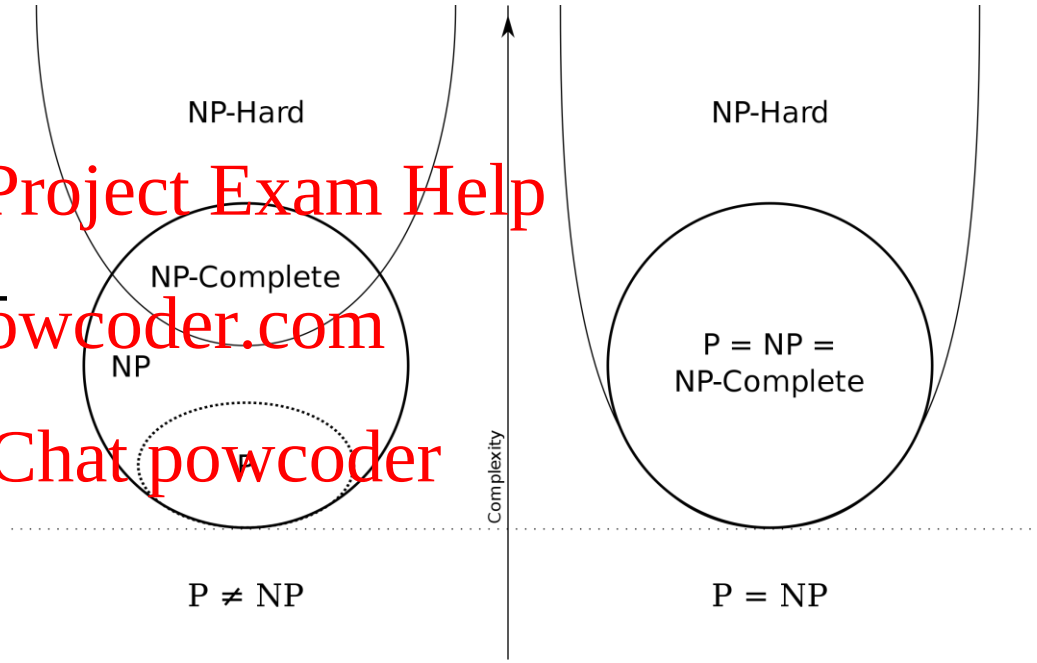
- co-NP-complete
  - Tautology problem ("negation" of SAT):
    - "Given a CNF formula $\varphi$, does it always evaluate to TRUE regardless of variable assignments?"

# Complexity





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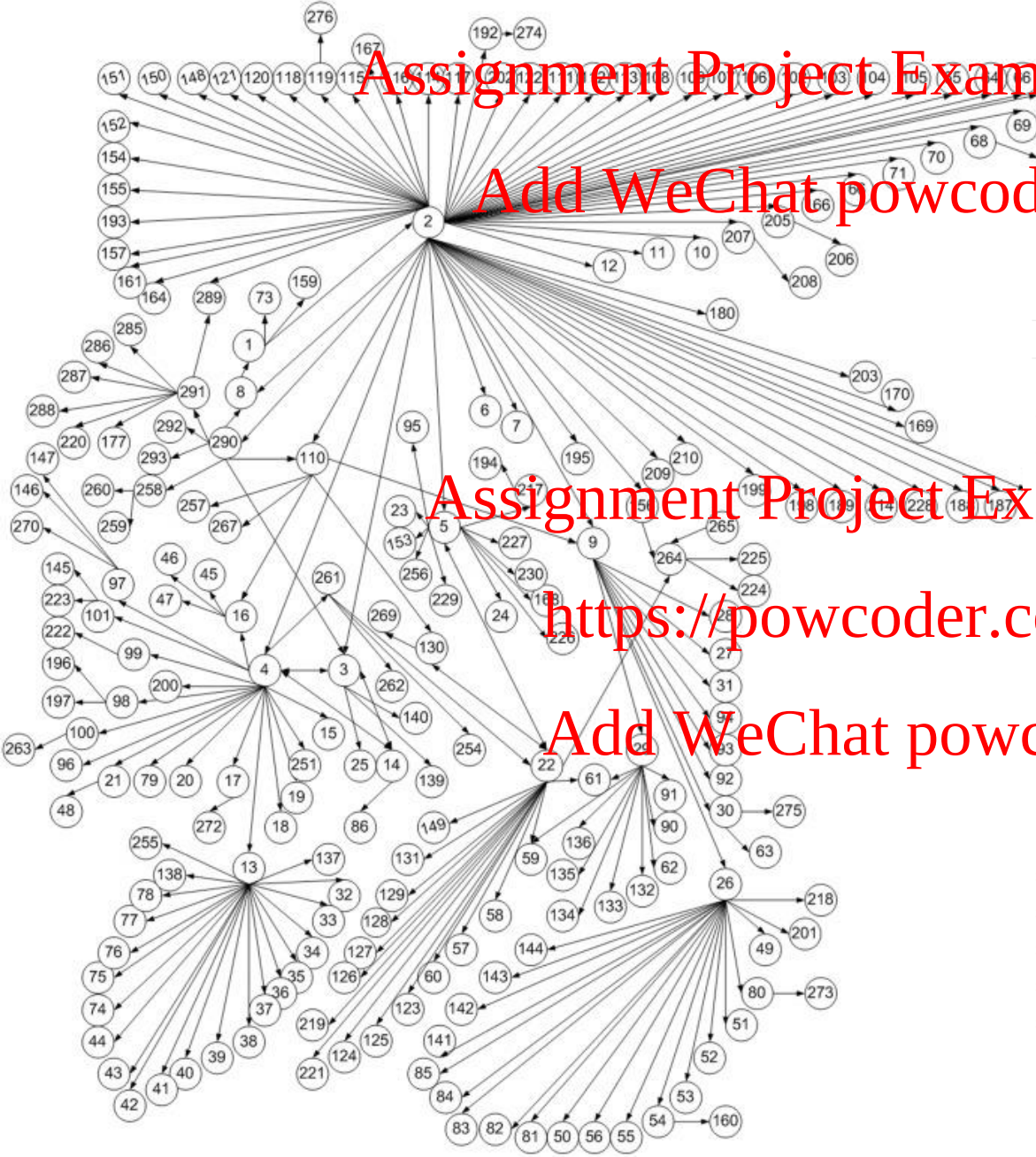

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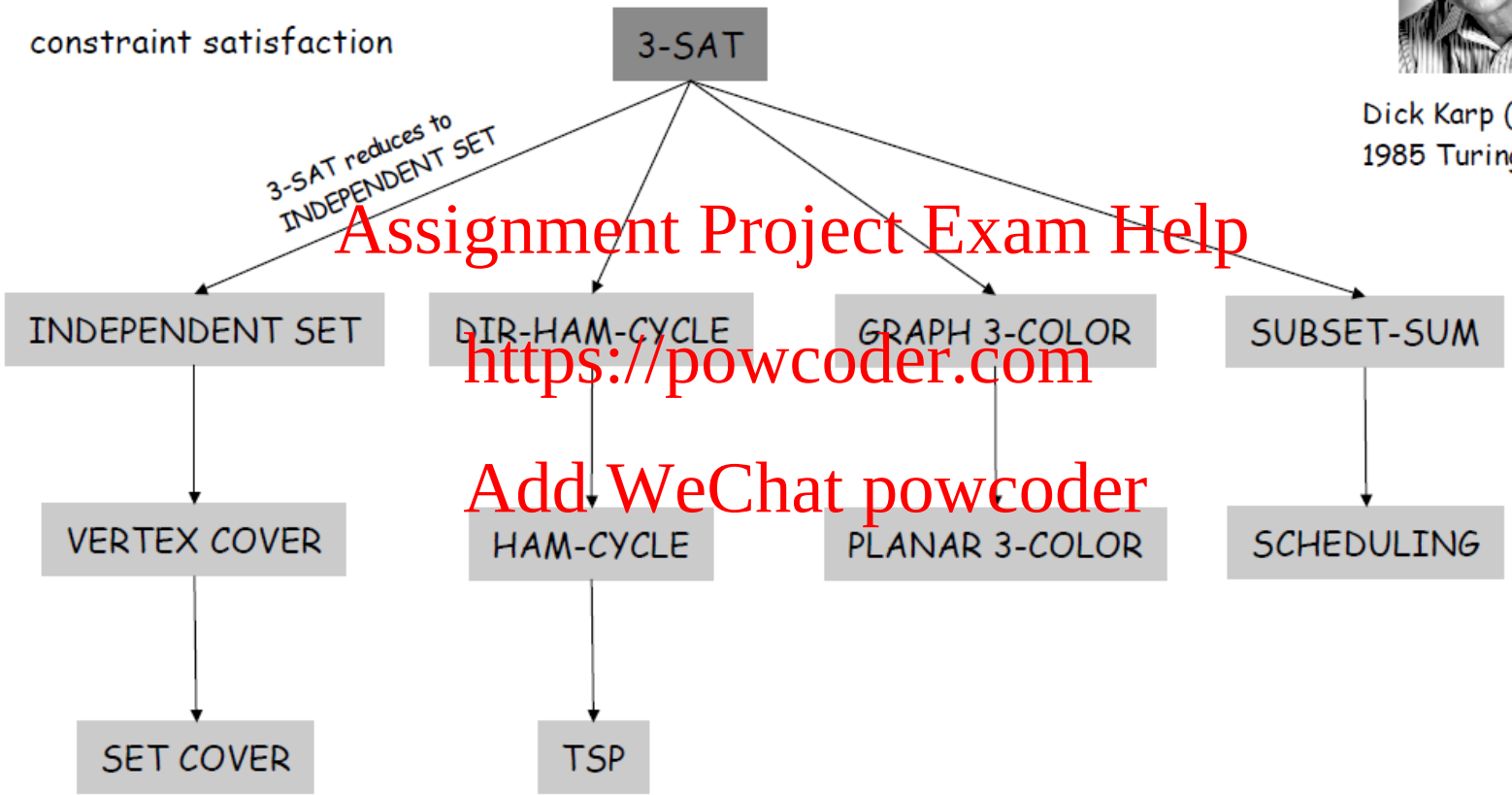
# Survey of polynomial transformations between NP-complete problems

Jorge A. Ruiz-Vanoye [a], Joaquín Pérez-Ortega [b], Rodolfo A. Pazos R. [c], Ocotlán Díaz-Parra [d], Juan Frausto-Solís [e], Hector J. Fraire Huacuja [c], Laura Cruz-Reyes [c], José A. Martínez F. [c]

| Number | Name of problem | Number | Name of problem |
|---|---|---|---|
| 1 | Satisfiability (SAT) | 2 | 3-Satisfiability (3SAT) |
| 3 | Clique (clique cover) | 4 | Vertex cover |
| 5 | Subset sum | 6 | Hitting string |
| 7 | Chinese postman for mixed graphs | 8 | Graph colorability |
| 9 | Three-Dimensional matching (3DM) | 10 | Rectilinear picture compression |
| 11 | Tableau equivalence | 12 | Consistency of databases frequency tables |
| 13 | Hamiltonian Circuit (Directed Hamiltonian path, Undirected Hamiltonian path) | 14 | Independent set |
| 15 | Setbasis | 16 | Hitting set |
| 17 | Comparative containment | 18 | Multiple copy file allocation |
| 19 | Shortest common supersequence | 20 | Longest common subsequence |
| 21 | Minimum cardinality key | 22 | Partition |
| 23 | $K$'th largest subset | 24 | Capacity assignment |
| 25 | Conjunctive Boolean query | 26 | Exact cover by 3-sets (X3C) |
| 27 | Minimum test set | 28 | 3-Matroid intersection |
| 29 | 3-Partition | 30 | Numerical three-dimensional matching |

Dick Karp (1972)
1985 Turing Award



# Just A Tad Bit of History



- [Cook 1971]
  - Proved 3SAT is NP-complete in seminal paper



- [Karp 1972]
  - Showed that 20 other problems are also NP-complete
  - "Karp's 21 NP-complete problems"
  - Renewed interest in this idea





- 1982: Cook won the Turing award

# Independent Set



- Problem
  - Input: Undirected graph $G = (V, E)$, integer $k$
  - Question: Does there exist a subset of vertices $S \subseteq V$ with $|S| = k$ such that for each edge, at most one of its endpoints is in $S$?

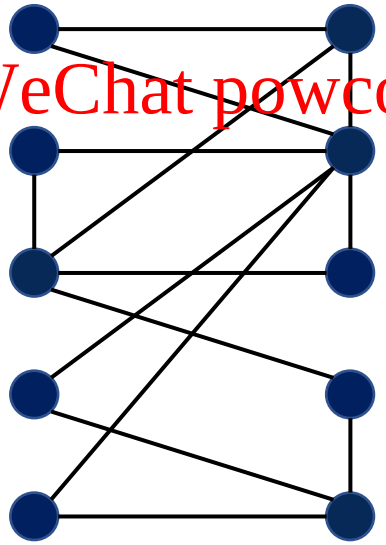




Example:

- Does this graph have an independent set of size 6?
  - Yes!
- Does this graph have an independent set of size 7?
  - No!





= independent set

# Independent Set



- Claim: Independent Set is in NP

- Recall: We need to show that there is a polynomial-time algorithm which
  - Can accept every YES instance with the right polynomial-size advice
  - Will not accept a NO instance with any advice



- Advice: the actual independent set $S$
- Algorithm: check if $S$ is an independent set and if $|S| = k$
- Simple!

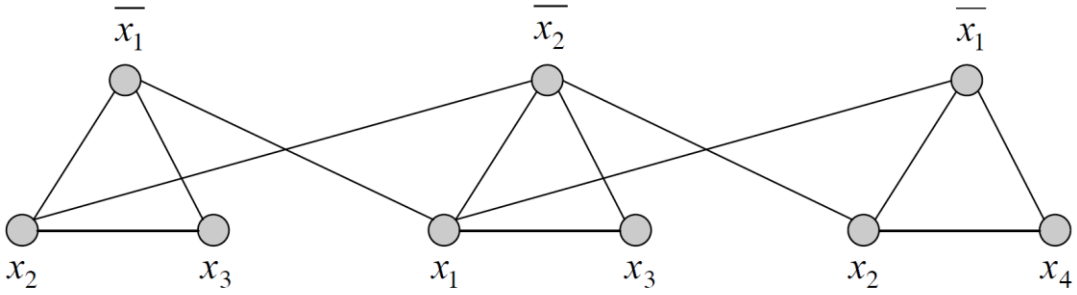
# Independent Set



- Claim: 3SAT $\leq_p$ Independent Set

- Given a formula $\varphi$ of 3SAT with $k$ clauses, construct an instance $(G, k)$ of Independent Set as follows
  - Create 3 vertices for each clause (one for each literal)
  - Connect them in a triangle
  - Connect the vertex of each literal to each of its negations





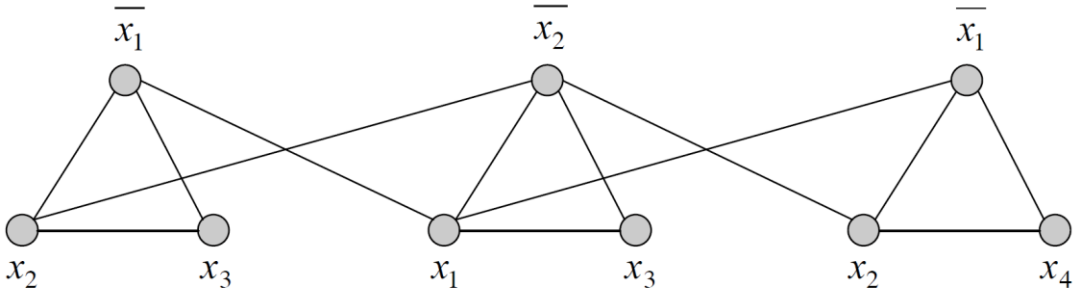

$$\Phi = \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_4\right)$$

# Independent Set



- Why does this work?
  - 3SAT = YES ⇒ Independent Set = YES
    - From each clause, take any literal that is TRUE in the assignment
  - Independent Set = YES ⇒ 3SAT = YES
    - Independent set $S$ must contain one vertex from each triangle
    - No literal and its negation are both in $S$
    - Set literals in $S$ to TRUE, their negations to FALSE, and the rest to arbitrary values






$$\Phi = \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_4\right)$$

# Different Types of Reductions



- $A \leq B$
  - Karp reductions
    - Take an arbitrary instance of $A$, and in polynomial time, construct a single instance of $B$ with the same answer
    - Very restricted type of reduction
    - The reduction we just constructed was a Karp reduction
  - Turing/Cook reductions
    - Take an arbitrary instance of $A$, and solve it by making polynomially many calls to an oracle for solving $B$ and some polynomial-time extra computation
    - Very general reduction
    - In this course, we'll allow Turing/Cook reductions, but whenever possible, see if you can construct a Karp reduction

# Subset Sum



- Problem
  - Input: Set of integers $S = \{w_1, \ldots, w_n\}$, integer $W$
  - Question: Is there $S' \subseteq S$ that adds up to exactly $W$?



- Example
  - $S = \{1, 4, 16, 64, 256, 1040, 1041, 1093, 1284, 1344\}$, $W = 3754$?
  - Yes!
    - $1 + 16 + 64 + 256 + 1040 + 1093 + 1284 = 3754$

# Subset Sum



- Claim: Subset Sum is in NP

- Recall: We need to show that there is a polynomial-time algorithm which
  - Can accept every YES instance with the right polynomial-size advice
  - Will not accept a NO instance with any advice



- Advice: the actual subset $S'$
- Algorithm: check that $S'$ is indeed a subset of $S$ and sums to $W$
- Simple!

# Subset Sum



- Claim: 3SAT $\leq_p$ Subset Sum

- Given a formula $\varphi$ of 3SAT, we want to construct $(S, W)$ of Subset Sum with the same answer
- In the table in the following slide:
  - Columns are for variables and clauses
  - Each row is a number in $S$, represented in decimal
  - Number for literal $\ell$ : has 1 in its variable column and in the column of every clause where that literal appears
    - Number selected = literal set to TRUE
  - "Dummy" rows: can help make the sum in a clause column 4 if and only if at least one literal is set to TRUE

# Subset Sum



- Claim: 3SAT $\leq_p$ Subset Sum

$$C_1 = x \vee y \vee z$$
$$C_2 = \bar{x} \vee \bar{y} \vee \bar{z}$$
$$C_3 = \bar{x} \vee \bar{y} \vee z$$



Decimal representation

| | x | y | z | $C_1$ | $C_2$ | $C_3$ |
|---|---|---|---|---|---|---|
| x | 1 | 0 | 0 | 0 | 1 | 0 |
| ¬ x | 1 | 0 | 0 | 1 | 0 | 1 |
| y | 0 | 1 | 0 | 1 | 0 | 0 |
| ¬ y | 0 | 1 | 0 | 0 | 1 | 1 |
| z | 0 | 0 | 1 | 1 | 1 | 0 |
| ¬ z | 0 | 0 | 1 | 0 | 0 | 1 |
| dummies to get clause columns to sum to 4 | 0 | 0 | 0 | 1 | 0 | 0 |
| | 0 | 0 | 0 | 2 | 0 | 0 |
| | 0 | 0 | 0 | 0 | 1 | 0 |
| | 0 | 0 | 0 | 0 | 2 | 0 |
| | 0 | 0 | 0 | 0 | 0 | 1 |
| | 0 | 0 | 0 | 0 | 0 | 2 |
| W | 1 | 1 | 1 | 4 | 4 | 4 |

# Subset Sum



- Note
  - The Subset Sum instance we constructed has "large" numbers
    - Their values are exponentially large (~$10^{\#variables+\#clauses}$)
    - But the number of bits required to write them is polynomial
  - Can we hope to construct Subset Sum instance with numbers whose values are only $poly(\#variables, \#clasuses)$ large?
    - Unlikely, as that would prove $P = NP$!
    - Like Knapsack, Subset Sum can be solved in pseudo-polynomial time
      - That is, in polynomial time if the numbers are only polynomially large in value

# 3-Coloring

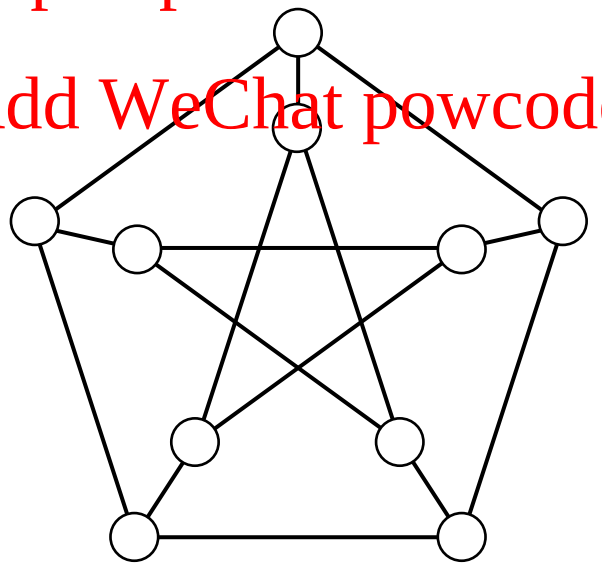


- Problem
  - Input: Undirected graph $G = (V, E)$
  - Question: Can we color each vertex of $G$ using at most three colors such that no two adjacent vertices have the same color?









# 3-Coloring



- Claim: 3-coloring is in NP

- Recall: We need to show that there is a polynomial-time algorithm which
  - Can accept every YES instance with the right polynomial-size advice
  - Will not accept a NO instance with any advice



- Advice: colors of the nodes in a valid 3-coloring
- Algorithm: check that this is a valid 3-coloring
- Simple!

# 3-Coloring



- Claim: 3SAT $\leq_p$ 3-Coloring

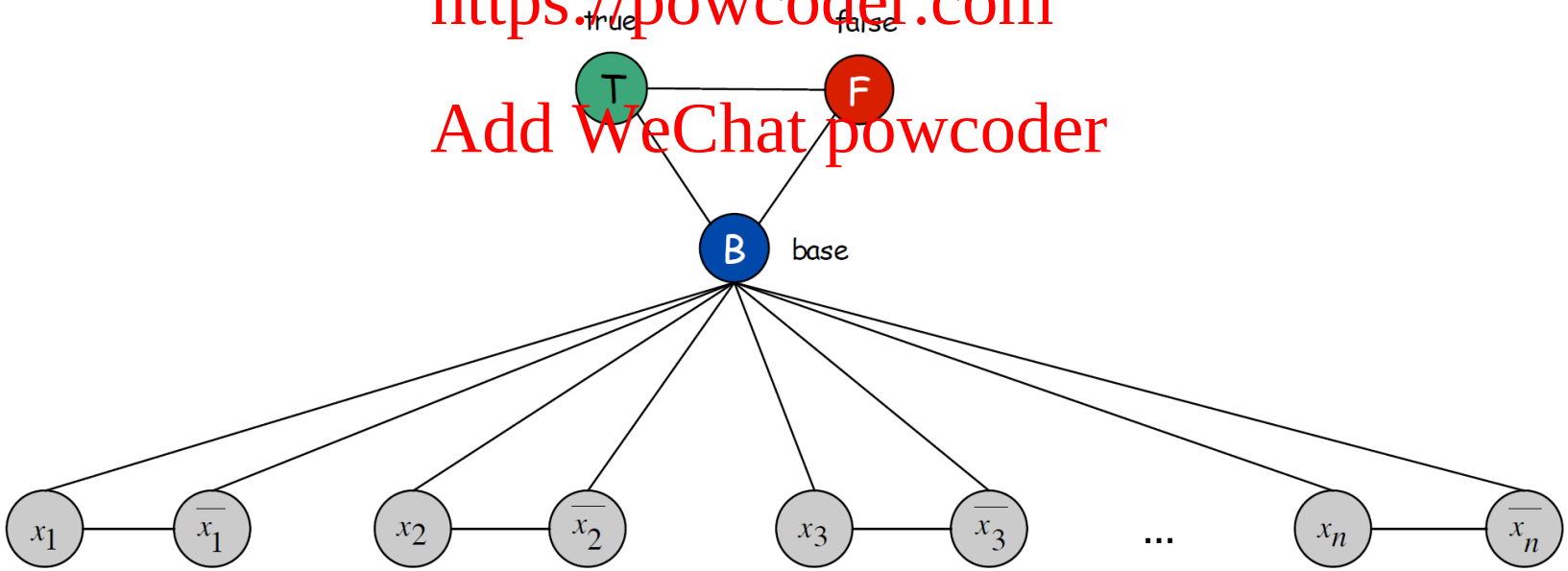
  - Given a 3SAT formula $\varphi$, we want to construct a graph $G$ such that $G$ is 3-colorable if and only if $\varphi$ has a satisfying assignment
  - $G$ will have the following nodes:
    - Type 1: true, false, base, one for each $x_i$, one for each $\overline{x_i}$
    - Type 2: additional nodes for each clause $C_j$
  - 1-1 correspondence between valid 3-colorings of type 1 nodes and valid truth assignments:
    - All literals with the same color as "true" node are set to true
    - All literals with the same color as "false" node are set to false
  - Claim: Fix any colors of type 1 nodes. There exists a valid 3-coloring of $G$ giving these colors to type 1 nodes if and only if the corresponding truth assignment is satisfying for $\varphi$.

# 3-Coloring



- Create 3 new nodes T, F, and B, and connect them in a triangle
- Create a node for each literal, connect it to its negation and to B
- T-F-B must have different colors, and so must B-$x_i$-$\bar{x}_i$
  - Each literal has the color of T or F, its negation has the other color
  - Valid 3-coloring ⇔ valid truth assignment (set all with color T to true)



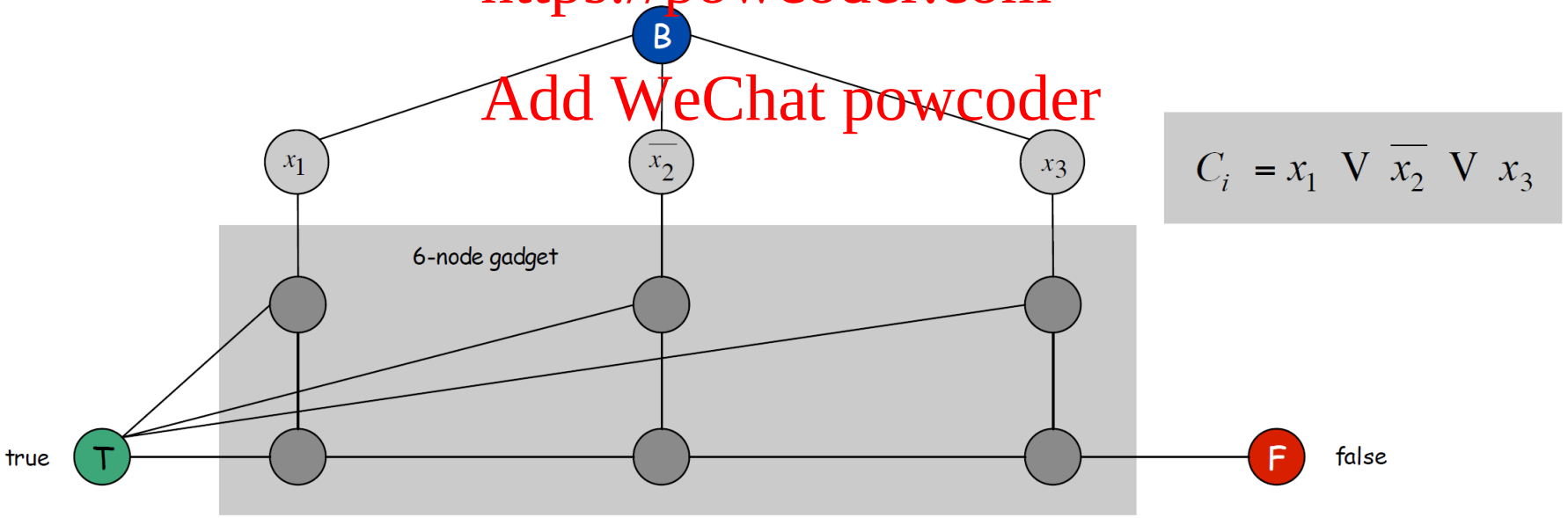


# 3-Coloring



- We also need valid 3-coloring ⇔ *satisfying* truth assignment
  - For each clause, add the following gadget with 6 nodes and 13 edges
  - Claim: Clause gadget is 3-colorable ⇔ at least one of the nodes corresponding to the literals in the clause is assigned color of T





# 3-Coloring

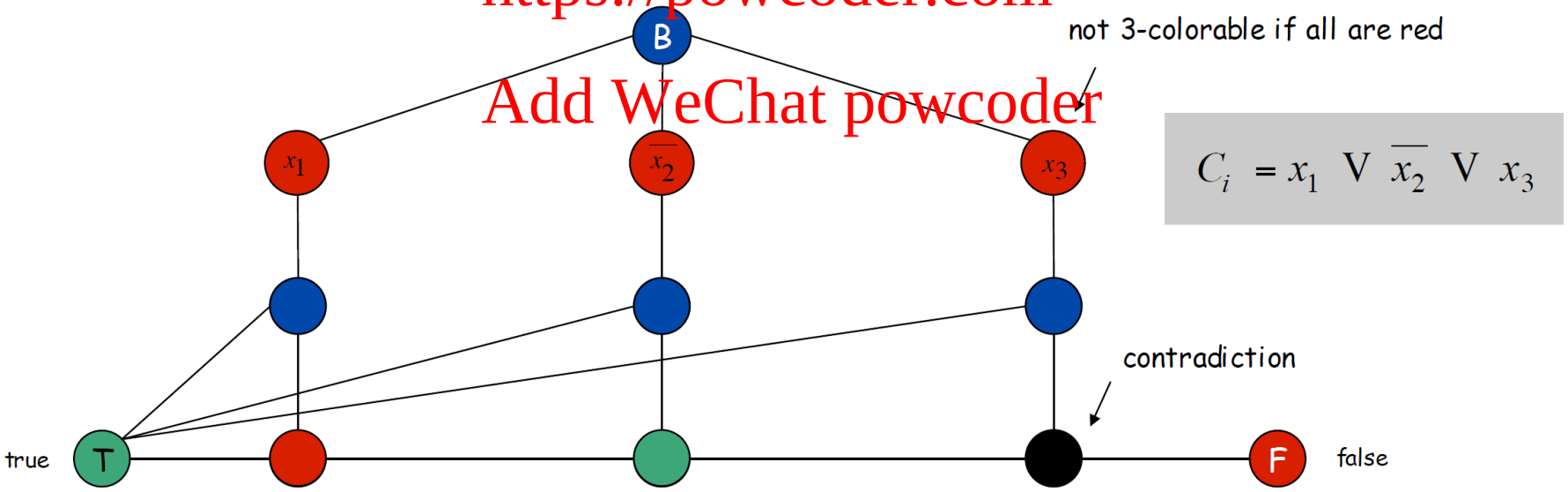


- Claim: Valid 3-coloring ⇒ truth assignment satisfies $\varphi$
  - Suppose a clause $C_i$ is not satisfied, so all its three literals must be F
  - Then the 3 nodes in top layer must be B
  - Then the first two nodes in bottom layer must be F and T
  - No color left for the remaining node ⇒ contradiction!









# 3-Coloring

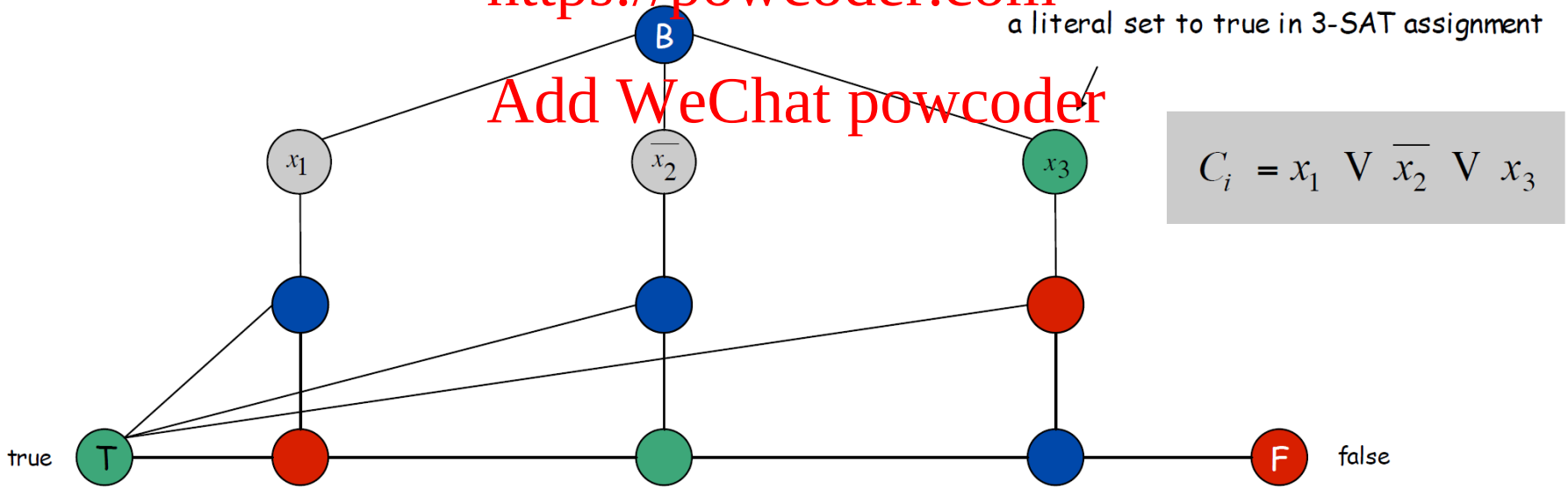


- We just proved: valid 3-coloring ⇒ satisfying assignment
- Claim: satisfying assignment ⇒ valid 3-coloring
  - Each clause has at least one literal with color T
  - Exercise: Regardless of which literal has color T and which color (T/F) the other literals have, the clause widget can always be 3-colored









# Review of Reductions



- If you want to show that problem B is NP-complete
- Step 1: Show that B is in NP
  - Some polynomial-size advice should be sufficient to verify a YES instance in polynomial time
  - No advice should work for a NO instance
  - Usually, the solution of the "search version" of the problem works
    - But sometimes, the advice can be non-trivial
    - For example, to check LP optimality, one possible advice is the values of both primal and dual variables, as we saw in the last lecture

# Review of Reductions



- If you want to show that problem B is NP-complete
- Step 2: Find a known NP-complete problem A and reduce it to B (i.e. show $A \leq_p B$)



  - This means taking an arbitrary instance of A, and solving it in polynomial time using an oracle for B



    - Caution 1: Remember the direction. You are "reducing known NP-complete problem to your current problem".



    - Caution 2: The size of the B-instances you construct should be polynomial in the size of the original A-instance
  - This would show that if B can be solved in polynomial time, then A can be as well
  - Some reductions are trivial, some are notoriously tricky…

# Binary Integer Linear Programming (BILP)

- Problem
  - Input: $c \in \mathbb{R}^n, b \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n}, k \in \mathbb{R}$
  - Question: Does there exist $x \in \{0,1\}^n$ such that $c^T x \geq k$ and $Ax \leq b$?



- Decision variant of "maximize $c^T x$ subject to $Ax \leq b$" but instead of any $x \in \mathbb{R}^n$ with $x \geq 0$, we are restricting $x$ to binary.



- Does restricting search space make the problem easier or harder?
  - This is actually NP-complete!

# BILP Feasibility



- An even simpler problem
  - Special case where $c = k = 0$, so $c^T x \geq k$ is always true



- Problem
  - Input: $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$
  - Question: Does there exist $x \in \{0,1\}^n$ such that $Ax \leq b$?





  - Just need to find a feasible solution
  - This is still NP-complete!

# BILP Feasibility



- Claim: BILP Feasibility is in NP

- Recall: We need to show that there is a polynomial-time algorithm which
  - Can accept every YES instance with the right polynomial-size advice
  - Will not accept a NO instance with any advice



- Advice: simply a vector $x$ satisfying $Ax \leq b$
- Algorithm: Check if $Ax \leq b$
- Simple!

# BILP Feasibility



- Claim: 3SAT $\leq_p$ BILP Feasibility

- Take any formula $\varphi$ of 3SAT
- Create a binary variable $x_i$ for each variable $x_i$ in $\varphi$
  - We'll represent its negation $\bar{x}_i$ with $1 - x_i$
- For each clause $C$, we want at least one of its three literals to be TRUE
  - Just make sure their sum is at least 1
  - E.g. $C = x_1 \vee \bar{x}_2 \vee \bar{x}_3 \Rightarrow x_1 + (1 - x_2) + (1 - x_3) \geq 1$
- Easy to check that
  - this is a polynomial reduction
  - Resulting system has a feasible solution iff $\varphi$ is satisfiable

# So far...



- To establish NP-completeness of problem B, we always reduced 3SAT to B
  - But we can reduce any other problem A that we have already established to be NP-complete
  - Sometimes this might lead to a simpler reduction because A might already be "similar" to B



- Let's see an example!

# Vertex Cover



- Problem
  - Input: Undirected graph $G = (V, E)$, integer $k$
  - Question: Does there exist a vertex cover of size $k$?
    - That is, does there exist $S \subseteq V$ with $|S| = k$ such that every edge is incident to at least one vertex in $S$?

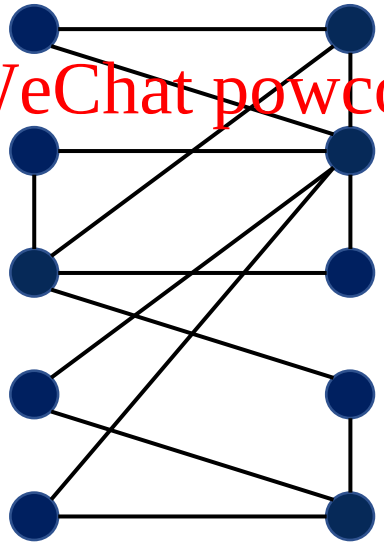




Example:

- Does this graph have a vertex cover of size 4?
  - Yes!
- Does this graph have a vertex cover of size 3?
  - No!





= vertex cover

# Vertex Cover



- Problem
  - Input: Undirected graph $G = (V, E)$, integer $k$
  - Question: Does there exist a vertex cover of size $k$?
    - That is, does there exist $S \subseteq V$ with $|S| = k$ such that every edge is incident to at least one vertex in $S$?

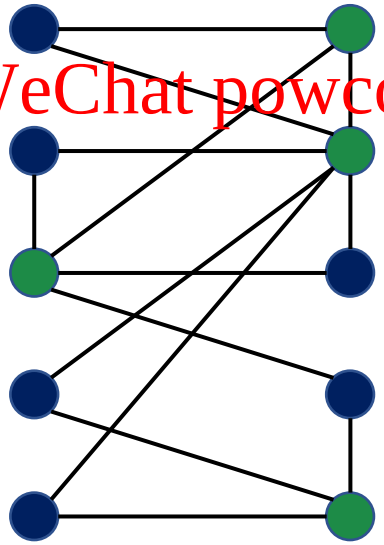




Question:

- Did we see this graph in the last lecture?
  - Yes!
  - For independent set of size 6





= vertex cover

= independent set

# Vertex Cover



- Vertex cover and independent set are intimately connected!

- Claim: $G$ has a vertex cover of size $k$ if and only if $G$ has an independent set of size $n - k$



- Stronger claim: $S$ is a vertex cover if and only if $V \setminus S$ is an independent set

# Vertex Cover



- Claim: $S$ is a vertex cover if and only if $V \backslash S$ is an independent set
- Proof:
  - $S$ is a vertex cover
  - IFF: For every $(u, v) \in E$, at least one of $\{u, v\}$ is in $S$
  - IFF: For every $(u, v) \in E$, at most one of $\{u, v\}$ is in $V \backslash S$
  - IFF: No two vertices of $V \backslash S$ are connected by an edge
  - IFF: $V \backslash S$ is an independent set ■

# Vertex Cover



- Claim: Independent Set $\leq_p$ Vertex Cover
  - Take an arbitrary instance $(G, k)$ of Independent Set
  - We want to check if there is an independent set of size $k$
  - Just convert it to the instance $(G, n - k)$ of Vertex Cover
  - Simple!
    - A reduction from 3SAT would have basically repeated the reduction we already did for 3SAT $\leq_p$ Independent Set
  - Note: I didn't argue that Vertex Cover is in NP
    - This is simple as usual. Just give the actual vertex cover as the advice.

# Set Cover



- Problem
  - Input: A universe of elements $U$, a family of subsets $S$, and an integer $k$
  - Question: Do there exist $k$ sets from $S$ whose union is $U$?



- Example
  - $U = \{1,2,3,4,5,6,7\}$



  - $S = \{\{1,3,7\}, \{2,4,6\}, \{4,5\}, \{1\}, \{1,2,6\}\}$



  - $k = 3$? Yes! $\{\{1,3,7\}, \{4,5\}, \{1,2,6\}\}$
  - $k = 2$? No!

# Set Cover



- Claim: Set Cover is in NP
  - Easy. Let the advice be the actual $k$ sets whose union is $U$.



- Claim: Vertex Cover $\leq_p$ Set Cover

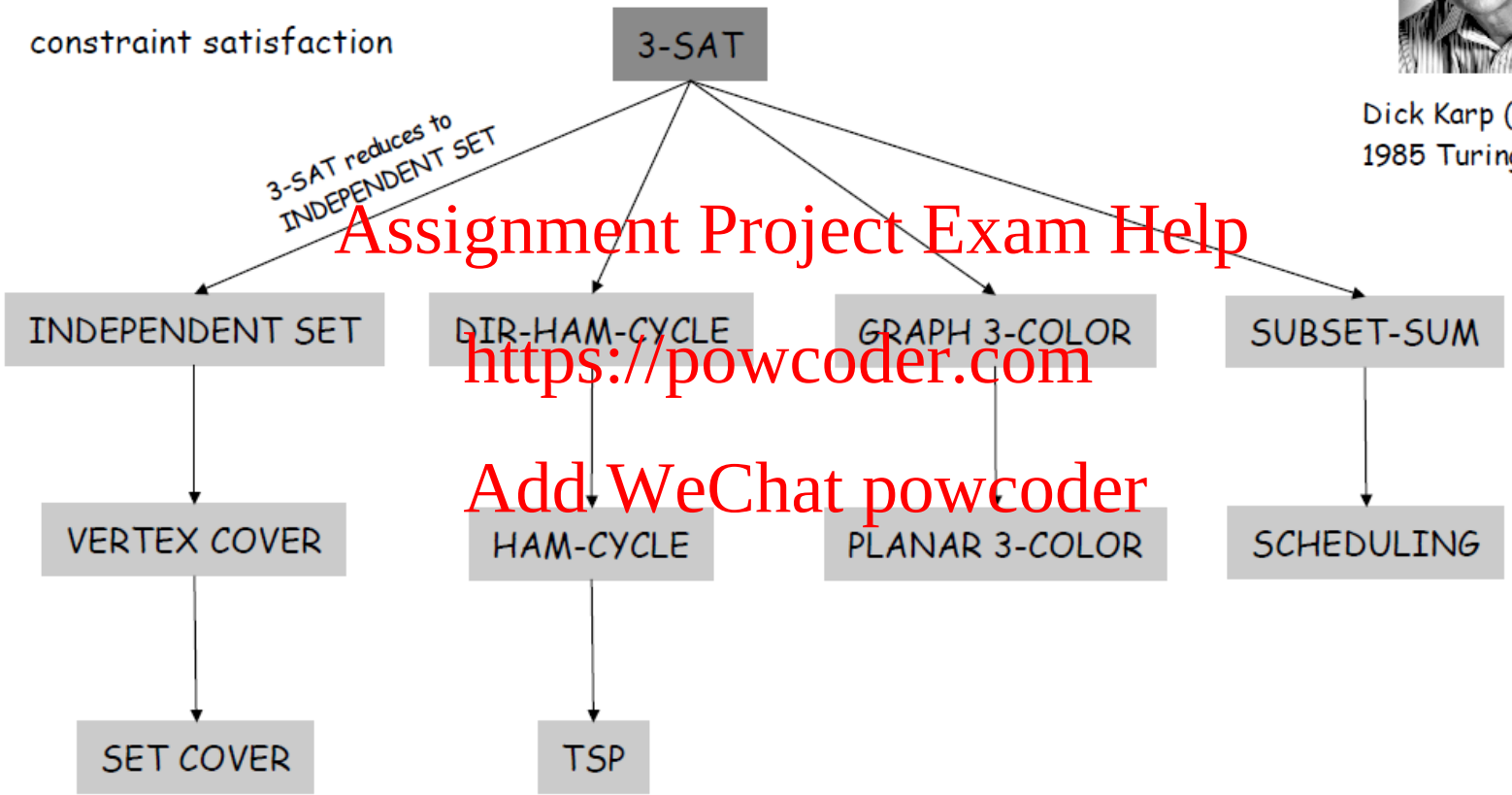


  - Given an instance of vertex cover with graph $G = (V, E)$ and integer $k$, create the following set cover instance



    - Set $U = E$
    - For each $v \in V$, $S$ contains a set $S_v$ of all edges incident on $v$
    - Selecting $k$ set whose union is $U$ = selecting $k$ vertices such that union of their incident edges covers all edges
    - Hence, the two problems obviously have the same answer

Dick Karp (1972)
1985 Turing Award



# Cook-Levin Theorem



- We did not prove "the first NP-completeness" result



- Theorem: 3SAT is NP-complete
  - We need to prove this without using any other "known NP-complete" problem



  - We want to directly show that *every problem in NP* can be reduced to 3SAT



- We will first reduce any NP problem to SAT, and then reduce SAT to 3SAT

# Cook-Levin Theorem



- We're not going to prove it in this class, but the key idea is as follows
  - If a problem is in NP, then $\exists$ Turing machine $T(x, y)$ which
    - takes as input a problem instance $x$ and an advice $y$ of size $p(|x|)$
    - verifies in $q(|x|)$ time whether $x$ is a YES instance
    - both $p$ and $q$ are polynomials
  - $x$ is a YES instance iff $\exists y\ T(x, y) = ACCEPT$

# Cook-Levin Theorem



NOT IN SYLLABUS

- $x$ is a YES instance iff $\exists y\, T(x, y) = ACCEPT$
  - We need to convert $\exists y\, T(x, y) = ACCEPT$ into whether a SAT formula $\varphi$ is satisfiable



- Recall that a Turing machine $T$ consists of a memory tape, a head pointer, a state, and a transition function



- What describes $T$ at any given step of its computation?



  - What is written in each cell of its memory tape?
  - Which cell of the tape is the read/write head currently pointing to?
  - What state is the Turing machine in?

# Cook-Levin Theorem



NOT IN SYLLABUS

- $x$ is a YES instance iff $\exists y\ T(x, y) = ACCEPT$
  - We need to convert $\exists y\ T(x, y) = ACCEPT$ into $\exists z\ \varphi(z) = TRUE$, where $z$ consists of Boolean variables and $\varphi$ is a SAT formula



- Variables:
  - $T_{i,j,k}$ = True if machine's tape cell $i$ contains symbol $j$ at step $k$ of the computation
  - $H_{i,k}$ = True if the machine's read/write head is at tape cell $i$ at step $k$ of the computation
  - $Q_{q,k}$ = True if machine is in state $q$ at step $k$ of the computation





  - Cell index $i$ and computation step $k$ only need to be polynomially large as $T$ works in polynomial time

# Cook-Levin Theorem



NOT IN SYLLABUS

- $x$ is a YES instance iff $\exists y\ T(x, y) = ACCEPT$
  - We need to convert $\exists y\ T(x, y) = ACCEPT$ into $\exists z\ \varphi(z) = TRUE$, where $z$ consists of Boolean variables and $\varphi$ is a SAT formula



- Clauses:
  - Express how the variables must be related using the transition function
  - Express that the Turing machine must reach the state $ACCEPT$ at some step of the computation



- This establishes that SAT is NP-complete.
- Next: SAT $\leq_p$ 3SAT.

# Cook-Levin Theorem



- Claim: SAT $\leq_p$ 3SAT
  - Take an instance $\varphi = C_1 \wedge C_2 \wedge \cdots$ of SAT
  - Replace each clause with multiple clauses with exactly 3 literals each



  - For a clause with one literal, $C = \ell_1$:
    - Add two variables $z_1, z_2$, and replace $C$ with four clauses



$$(\ell_1 \vee z_1 \vee z_2) \wedge (\ell_1 \vee \bar{z}_1 \vee z_2) \wedge (\ell_1 \vee z_1 \vee \bar{z}_2) \wedge (\ell_1 \vee \bar{z}_1 \vee \bar{z}_2)$$



    - Verify that this is logically equivalent to $\ell_1$

  - For a clause with two literals, $C = (\ell_1 \vee \ell_2)$:
    - Add variable $z_1$ and replace it with the following:

$$(\ell_1 \vee \ell_2 \vee z_1) \wedge (\ell_1 \vee \ell_2 \vee \bar{z}_1)$$

    - Verify that this is logically equal to $(\ell_1 \vee \ell_2)$

# Cook-Levin Theorem



- Claim: SAT $\leq_p$ 3SAT
  - For a clause with three literals, $C = \ell_1 \vee \ell_2 \vee \ell_3$:
    - Perfect. No need to do anything!
  - For a clause with 4 or more literals, $C = (\ell_1 \vee \ell_2 \vee \cdots \vee \ell_k)$:
    - Add variables $z_1, z_2, \ldots, z_{k-3}$ and replace it with:

      $(\ell_1 \vee \ell_2 \vee z_1) \wedge (\ell_3 \vee \bar{z}_1 \vee z_2) \wedge (\ell_4 \vee \bar{z}_2 \vee z_3) \wedge \cdots$
      $\wedge (\ell_{k-2} \vee \bar{z}_{k-4} \vee z_{k-3}) \wedge (\ell_{k-1} \vee \ell_k \vee \bar{z}_{k-3})$

    - Check:
      - If any $\ell_i$ is TRUE, then there exists an assignment of $z$ variables to make this TRUE
      - If all $\ell_i$ are FALSE, then no assignment of $z$ variables will make this TRUE

# NP vs co-NP



- Complements of each other
  - NP = short proof for YES, co-NP = short proof for NO
  - If a problem "Does there exist…" is in NP, then its complement "Does there not exist…" is in co-NP, and vice-versa
  - The same goes for NP-complete and co-NP-complete





- Example
  - SAT is NP-complete ("Does there exist $x$ satisfying $\varphi$?")
    - So "Does there exist no $x$ satisfying $\varphi$?", i.e., "Is $\varphi$ always FALSE?" is coNP-complete
  - Then, Tautology ("Is $\varphi$ always TRUE?") is also coNP-complete

# NP ∩ co-NP



- Clearly, P ⊆ NP ∩ co-NP
  - No advice needed; can just solve the problem in polytime
  - Major open question: Is P = NP ∩ co-NP?



- NP ∩ co-NP: Short proof of both YES and NO
  - Hunt for problems not known in P but still in NP ∩ co-NP

# NP ∩ co-NP



- Linear programming
  - [Gale–Kuhn–Tucker 1948]: LP is in NP ∩ co-NP
  - Question: max objective value ≥ threshold?
  - Proof of YES: Provide a feasible solution with objective ≥ threshold
  - Proof of NO: Provide optimal primal and dual solutions





CHAPTER XIX

LINEAR PROGRAMMING AND THE THEORY OF GAMES [1]



BY DAVID GALE, HAROLD W. KUHN, AND ALBERT W. TUCKER [2]

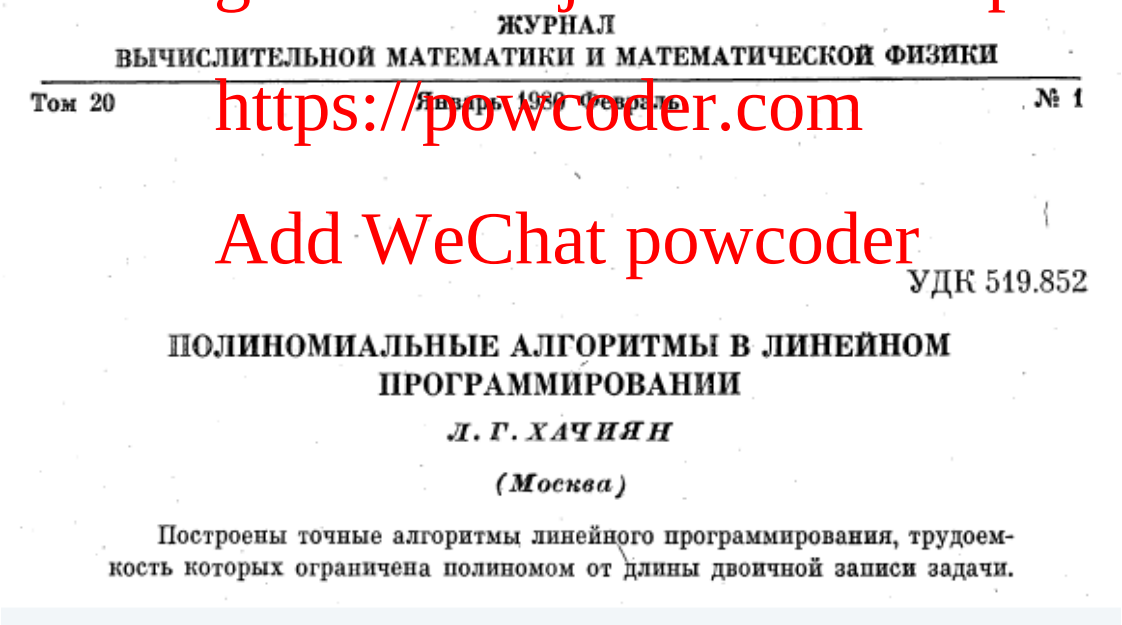
The basic "scalar" problem of *linear programming* is to maximize (or minimize) a linear function of several variables constrained by a system of linear inequalities [Dantzig, II]. A more general "vector" problem calls for maximizing (in a sense of partial order) a system of linear functions of several variables subject to a system of linear inequalities and, perhaps, linear equations [Koopmans, III]. The purpose of this chapter is to establish theorems of duality and existence for general "matrix" problems of linear programming which contain the "scalar" and "vector" problems as special cases, and to relate these general problems to the theory of zero-sum two-person games.

# NP ∩ co-NP



- Linear programming
  - But later, Khachiyan [1979] proved that LP is in P





ЖУРНАЛ
ВЫЧИСЛИТЕЛЬНОЙ МАТЕМАТИКИ И МАТЕМАТИЧЕСКОЙ ФИЗИКИ

Том 20 Январь 1980 Февраль № 1





УДК 519.852

## ПОЛИНОМИАЛЬНЫЕ АЛГОРИТМЫ В ЛИНЕЙНОМ ПРОГРАММИРОВАНИИ

*Л. Г. ХАЧИЯН*

*(Москва)*

Построены точные алгоритмы линейного программирования, трудоемкость которых ограничена полиномом от длины двоичной записи задачи.

# NP ∩ co-NP



- Primality testing ("Is $n$ a prime?")
  - [Pratt 1975]: PRIMES is in NP ∩ co-NP
  - Proof of NO: Easy, provide a non-trivial factor
  - Proof of YES: relies on interesting math





SIAM J. COMPUT.
Vol. 4, No. 3, September 1975

## EVERY PRIME HAS A SUCCINCT CERTIFICATE*



VAUGHAN R. PRATT†

**Abstract.** To prove that a number $n$ is composite, it suffices to exhibit the working for the multiplication of a pair of factors. This working, represented as a string, is of length bounded by a polynomial in $\log_2 n$. We show that the same property holds for the primes. It is noteworthy that almost no other set is known to have the property that short proofs for membership or nonmembership exist for all candidates without being known to have the property that such proofs are easy to come by. It remains an open problem whether a prime $n$ can be recognized in only $\log_2^{\alpha} n$ operations of a Turing machine for any fixed $\alpha$.

The proof system used for certifying primes is as follows.

AXIOM. $(x, y, 1)$.

INFERENCE RULES.

$R_1$: $(p, x, a), q \vdash (p, x, qa)$ provided $x^{(p-1)/q} \not\equiv 1 \pmod p$ and $q|(p-1)$.

$R_2$: $(p, x, p-1) \vdash p$ provided $x^{p-1} \equiv 1 \pmod p$.

THEOREM 1. *$p$ is a theorem $\equiv$ $p$ is a prime.*

THEOREM 2. *$p$ is a theorem $\supset$ $p$ has a proof of $\lceil 4 \log_2 p \rceil$ lines.*

# NP ∩ co-NP



- Primality testing ("Is $n$ a prime?")
  - Later, Agrawal, Kayal, and Saxena [2004] proved that PRIMES is in P
    - Milestone result!



Annals of Mathematics, 160 (2004), 781–793

**PRIMES is in P**

By Manindra Agrawal, Neeraj Kayal, and Nitin Saxena*

**Abstract**

We present an unconditional deterministic polynomial-time algorithm that determines whether an input number is prime or composite.

# NP ∩ co-NP



- Factoring ("Does $n$ have a factor $\leq k$?")
  - FACTOR is in NP ∩ co-NP
    - Proof of YES: Just present such a factor
    - Proof of NO:
      - Present the entire prime factorization of $n$ along with a short proof that each presented factor is a prime
      - Verifier TM can check that each factor is indeed a prime, their product is indeed $n$, and none of the factors is $\leq k$
        - Actually, proofs of primality are not required anymore since we know the TM can just run the AKS algorithm to check if the factors are prime

# NP ∩ co-NP



- Factoring ("Does $n$ have a factor $\leq k$?")
  - Major open question: Is FACTOR in P?
    - Basis of several cryptographic procedures



  - Challenge: Factor the following number.





7403756347956171282804679609742957314259318888923128908493623263897276503402826627689199641962511784399589433050212758537011896809828673317327310893090055250511687706329907239638078671008609696253793465056379635 9

RSA-704
($30,000 prize if you can factor it)

# NP ∩ co-NP



- Factoring ("Does $n$ have a factor $\leq k$?")
  - [Shor 1994]: We can factor an $n$-bit integer in $O(n^3)$ steps on a quantum computer.
  - *Scalable* quantum computers can help
    - 2001: Factored 15 = 3 x 5
    - 2012: Factored 21 = 3 x 7

# Other Complexity Classes



- Based on the exact time complexity
  - DTIME($n$), NTIME($n^2$), …
    - Deterministic / nondeterministic time complexity
- Based on space complexity
  - DSPACE($n$), NSPACE($\log n$)
- Using randomization
  - ZPP (expected polynomial time, no errors)
    - Is P = ZPP?
- Allowing probabilistic errors
  - RP (polynomial time, one-sided error)
  - BPP (polynomial time, two-sided erros)