Monash University
Faculty of Information Technology

FIT2014 Theory of Computation

Lecture 2

Propositional Logic

slides by Graham Farr

Lecture overview

- Propositions
- Logical operations
- Tautologies, logical equivalence
- Disjunctive Normal Form
- Conjunctive Normal Form
- Representing logical statements

Propositions

Definition: A **proposition** is a statement which is either *true* or *false*.

Examples

$1 + 1 = 2$ — a proposition which is true.

The earth is flat. — a proposition which is false.

It will rain tomorrow. — a proposition.

'Twas brillig, and the slithy toves
did gyre and gimble in the wabe. — *not* a proposition.

From: Lewis Carroll, *Through the Looking Glass, and What Alice Found There*, Macmillan, London, 1871.

Come and work for us! — *not* a proposition.

This statement is false. — *not* a proposition.

For brevity, a proposition may be given a name, which has a **truth value**, True or False. For example, let X be the proposition $1 + 1 = 2$. Then the truth value of X is True.

Logical operations

Not	$\neg$	$(\sim, \bar{\ }, -)$
And	$\wedge$	$(\&)$
Or	$\vee$	
Implies	$\Rightarrow$	$(\rightarrow)$
Equivalence	$\Leftrightarrow$	$(\leftrightarrow)$

A **connective** is a binary logical operation. E.g.: $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$.

Negation

P: You have prepared for next week's tutorial.
$\neg P$: You have not prepared for next week's tutorial.

Other notation: $\sim P$, $\overline{P}$, $-P$

Truth table:

P	$\neg P$
F	**T**
T	**F**

Conjunction

Radhanath Sikdar (1813–1870)
http://news.bbc.co.uk/2/hi/south_asia/3193576.stm

P Radhanath was a computer.
Q Radhanath was a person.

$P \wedge Q$ Radhanath was a computer and a person.

Truth table:

P	Q	$P \wedge Q$
F	F	**F**
F	T	**F**
T	F	**F**
T	T	**T**

Disjunction

P	I will study FIT3155 Advanced Data Structures & Algorithms.
Q	I will study MTH3170 Network Mathematics.

$P \vee Q$	I'll study FIT3155 **or** I'll study MTH3170.
	I'll study *at least one of* FIT3155 and MTH3170.

Disjunction is sometimes called *inclusive-OR*, and sometimes written as $+$.

Truth table:

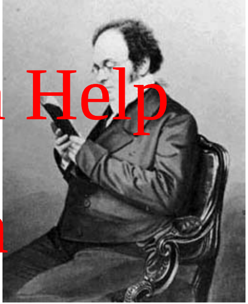
P	Q	$P \vee Q$
F	F	**F**
F	T	**T**
T	F	**T**
T	T	**T**

De Morgan's Laws

$$\neg(P \vee Q) = \neg P \wedge \neg Q$$
$$\neg(P \wedge Q) = \neg P \vee \neg Q$$

Can be proved using truth tables:

P	Q	$P \vee Q$	$\neg(P \vee Q)$	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$
F	F	F	T	T	T	T
F	T	T	F	T	F	F
T	F	T	F	F	T	F
T	T	T	F	F	F	F



Augustus De Morgan (1806–1871)
https://mathshistory.st-andrews.ac.uk/Biographies/De_Morgan/

Conditional

P Stars are visible.

Q The sun has set.

$P \Rightarrow Q$ **If** stars are visible **then** the sun has set.

Stars being visible **implies** the sun has set.

Stars are visible **only if** the sun has set.

Stars are visible is **sufficient** for the sun to have set.

$Q \Leftarrow P$ same as $P \Rightarrow Q$

Also called *implication*.

Conditional

Truth table:

P	Q	$P \Rightarrow Q$
F	F	**T**
F	T	**T**
T	F	**F**
T	T	**T**

Grace Hopper (1906–1992)

`https://www.cs.vassar.edu/history/hopper`

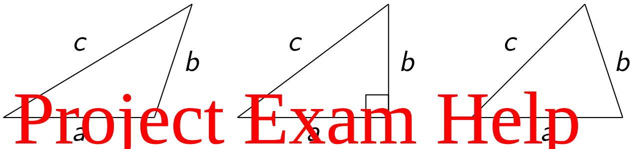
P	Grace is a COBOL expert.
Q	Grace can program.

$P \Rightarrow Q$ **If** Grace is a COBOL expert **then** she can program.

Biconditional

P The triangle is right-angled.

Q The side lengths satisfy $a^2 + b^2 = c^2$



$a^2 + b^2 < c^2$ $\quad$ $a^2 + b^2 = c^2$ $\quad$ $a^2 + b^2 > c^2$

$P \Leftrightarrow Q$ The triangle is right-angled **if and only if** $a^2 + b^2 = c^2$.

The triangle being right-angled is a **necessary and sufficient condition** for $a^2 + b^2 = c^2$.

$Q \Leftrightarrow P$ $a^2 + b^2 = c^2$ is a **necessary and sufficient condition** for the triangle being right-angled.

Truth table:

P	Q	$P \Leftrightarrow Q$
F	F	**T**
F	T	**F**
T	F	**F**
T	T	**T**

Tautologies, logical equivalence

Definitions

A **tautology** is a statement that is always true.

In other words, the right-hand column of its truth table has every entry True.

Two statements P and Q are **logically equivalent** if their truth tables are identical.

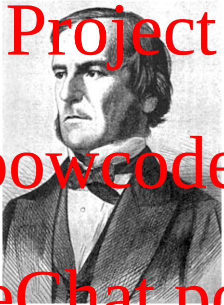
In other words, $P \Leftrightarrow Q$ is a tautology.

Examples

$\neg\neg P$	is logically equivalent to	P
$\neg(P \vee Q)$	is logically equivalent to	$\neg P \wedge \neg Q$
$\neg(P \wedge Q)$	is logically equivalent to	$\neg P \vee \neg Q$
$P \Rightarrow Q$	is logically equivalent to	$\neg P \vee Q$
$P \Leftrightarrow Q$	is logically equivalent to	$(P \Rightarrow Q) \wedge (P \Leftarrow Q)$
	and to	$(\neg P \vee Q) \wedge (P \vee \neg Q)$

These can all be proved using truth tables.

We usually denote logical equivalence by "$=$". So we write $\neg\neg P = P$, etc.



George Boole (1815–1864)
https://mathshistory.st-andrews.ac.uk/Biographies/Boole/

Distributive Laws

$$P \wedge (Q \vee R) \;=\; (P \wedge Q) \vee (P \wedge R)$$

$$P \vee (Q \wedge R) \;=\; (P \vee Q) \wedge (P \vee R)$$

Compare with ordinary algebra:

$$p \times (q + r) \;=\; (p \times q) + (p \times r)$$

but

$$p + (q \times r) \;\neq\; (p + q) \times (p + r)$$

Laws of Boolean algebra

$$\neg\neg P = P$$

$$\neg\text{True} = \text{False} \qquad \neg\text{False} = \text{True}$$

$$P \wedge Q = Q \wedge P \qquad P \vee Q = Q \vee P$$

$$(P \wedge Q) \wedge R = P \wedge (Q \wedge R) \qquad (P \vee Q) \vee R = P \vee (Q \vee R)$$

$$P \wedge P = P \qquad P \vee P = P$$

$$P \wedge \neg P = \text{False} \qquad P \vee \neg P = \text{True}$$

$$P \wedge \text{True} = \text{P} \qquad P \vee \text{False} = \text{P}$$

$$P \wedge \text{False} = \text{False} \qquad P \vee \text{True} = \text{True}$$

Distributive Laws

$$P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R) \qquad P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$$

De Morgan's Laws

$$\neg(P \vee Q) = \neg P \wedge \neg Q \qquad \neg(P \wedge Q) = \neg P \vee \neg Q$$

Disjunctive Normal Form (DNF)

X	Y	P	
F	F	**T**	$\neg X \wedge \neg Y$
F	T	**T**	$\neg X \wedge Y$
T	F	**F**	
T	T	**T**	$X \wedge Y$

$$P = \underbrace{\overbrace{(\neg X \wedge \neg Y)}^{\text{conjunction}} \vee \overbrace{(\neg X \wedge Y)}^{\text{conjunction}} \vee \overbrace{(X \wedge Y)}^{\text{conjunction}}}_{\text{disjunction}}$$

Exercise: simplify this as much as possible, using Boolean algebra.

Disjunctive Normal Form (DNF)

X	Y	Z	P	
F	F	F	**T**	$\neg X \wedge \neg Y \wedge \neg Z$
F	F	T	**F**	
F	T	F	**T**	$\neg X \wedge Y \wedge \neg Z$
F	T	T	**F**	
T	F	F	**F**	
T	F	T	**T**	$X \wedge \neg Y \wedge Z$
T	T	F	**T**	$X \wedge Y \wedge \neg Z$
T	T	T	**F**	

$$P = (\neg X \wedge \neg Y \wedge \neg Z) \vee (\neg X \wedge Y \wedge \neg Z) \vee (X \wedge \neg Y \wedge Z) \vee (X \wedge Y \wedge \neg Z)$$

Disjunctive Normal Form (DNF)

$$P = (\neg X \wedge \neg Y \wedge Z) \vee (\neg X \wedge Y \wedge \neg Z) \vee (X \wedge \neg Y \wedge Z) \vee (X \wedge Y \wedge \neg Z)$$

- A **literal** is an appearance of a variable in which it is either unnegated or negated just once.
  - Example: there are 12 literals in the above expression.

- A logical expression is in **DNF** if it is a **dis**junction of conjunctions of literals.
- Every logical expression is equivalent to one in DNF.
  - To see this: "just" use the truth table.
  - BUT this can be exponentially large (in # of variables).

- In effect, DNF enumerates all situations in which P is True.
- There is another Normal Form that is much more useful for us . . .

Conjunctive Normal Form (CNF)

- A logical expression is in **CNF** if it is a **con**junction of disjunctions of literals.
  - E.g.:

$$\underbrace{\overbrace{(\neg P \lor Q)}^{\text{disjunction}} \land \overbrace{(P \lor \neg Q)}^{\text{disjunction}}}_{\text{conjunction}}$$

  - Each disjunction of literals is called a **clause**.
- Every logical expression is equivalent to one in CNF.
- One way to see this:

  Given P,
  find the DNF of *its negation*, $\neg P$,
  then negate it
  and use De Morgan's Laws.

- BUT it is usually *much faster*, and *much less error-prone*, to work directly from the stated conditions that P must satisfy.
- In this unit, CNF will be *much* more important than DNF.

Representing logical statements

Example:

You are planning a dinner party. Your guest list must have:

- at least one of: Harry, Ron, Hermione, Ginny

$$\text{Harry} \vee \text{Ron} \vee \text{Hermione} \vee \text{Ginny}$$

- Hagrid *only if* it also has Norberta

$$\text{Hagrid} \Rightarrow \text{Norberta} \qquad \textit{...can rewrite as:} \qquad \neg\text{Hagrid} \vee \text{Norberta}$$

- none, or both, of Fred and George

$$\text{Fred} \Leftrightarrow \text{George} \qquad \textit{...can rewrite as:} \qquad (\neg\text{Fred} \vee \text{George}) \wedge (\text{Fred} \vee \neg\text{George})$$

- no more than one of: Voldemort, Bellatrix, Dolores.

$$(\textit{not both } \text{Voldemort \& Bellatrix}) \wedge (\textit{not both } \text{Voldemort \& Dolores}) \wedge (\textit{not both } \text{Bellatrix \& Dolores})$$

$$(\neg\text{Voldemort} \vee \neg\text{Bellatrix}) \wedge (\neg\text{Voldemort} \vee \neg\text{Dolores}) \wedge (\neg\text{Bellatrix} \vee \neg\text{Dolores})$$

Representing logical statements

$$
\begin{aligned}
&(\text{Harry} \vee \text{Ron} \vee \text{Hermione} \vee \text{Ginny}) \\
&\quad \wedge \quad (\neg\text{Hagrid} \vee \text{Norberta}) \\
&\quad \wedge \quad (\neg\text{Fred} \vee \text{George}) \;\; \wedge \;\; (\text{Fred} \vee \neg\text{George}) \\
&\quad \wedge \quad (\neg\text{Voldemort} \vee \neg\text{Bellatrix}) \;\; \wedge \;\; (\neg\text{Voldemort} \vee \neg\text{Dolores}) \;\; \wedge \;\; (\neg\text{Bellatrix} \vee \neg\text{Dolores})
\end{aligned}
$$

This is now in CNF.

Challenge: how long would an equivalent DNF expression be?

Reading

See Sipser, pp. 14–15, and top paragraph of p. 302.