Monash University
Faculty of Information Technology

FIT2014 Theory of Computation

Lecture 11

(A) Closure properties;

(B) Pumping Lemma for Regular Languages

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based in part on previous slides by David Albrecht

Overview

- Closure properties of regular languages
- Circuits in FAs
- Pumping Lemma
- Non-regular Languages

Closure properties of regular languages

Definition

If doing some operation on regular languages always produces another regular language, then we say that the class of regular languages is **closed** under that operation.

We will see that regular languages are closed under:

- complement
- union
- intersection
- concatenation

Theorem.

The complement of a regular language is regular.

We prove this using Kleene's Theorem.

Closure properties of regular languages

Theorem.
The complement of a regular language is regular.

Proof. (outline)
Suppose we have a Regular Language.
There must be a regular expression that defines it.
So, by Kleene's Theorem, there is a Finite Automaton (FA) that defines this language.
We can convert this FA into one that defines the complement of the language. (See Lecture 7.)
So, by Kleene's Theorem, there is a regular expression that defines the complement. □

Closure properties of regular languages

Theorem.
The union of two regular languages is regular.

Proof.
Suppose L_1 and L_2 are regular.

By definition of 'regular language',
there exist regular expressions R_1 and R_2
that describe L_1 and L_2, respectively.

Then $R_1 \cup R_2$ is a regular expression that describes $L_1 \cup L_2$.

- This uses part 3(iii) of the inductive definition of regular expressions in Lecture 6.

So $L_1 \cup L_2$ is regular. $\square$

Theorem.
The intersection of two regular languages is regular.

We can't just mimic the proof that regular languages are closed under union, since there is no $\cap$ operation on regular expressions.

Closure properties of regular languages

Theorem.
The intersection of two regular languages is regular.

Proof.
Suppose L_1 and L_2 are regular.

We know that their complements $\overline{L_1}$ and $\overline{L_2}$ are regular.

So the union of these, $\overline{L_1} \cup \overline{L_2}$, is therefore regular, by the previous Theorem.

Its complement, $\overline{\overline{L_1} \cup \overline{L_2}}$, must also be regular.

But $\overline{\overline{L_1} \cup \overline{L_2}} \; = \; \overline{\overline{L_1}} \cap \overline{\overline{L_2}} \; = \; L_1 \cap L_2$.

So $L_1 \cap L_2$, must also be regular. $\square$

Closure properties of regular languages

Exercises

- Prove that the class of regular languages is closed under concatenation.

$$L_1 L_2 := \{xy : x \in L_1, y \in L_2\}$$

- Prove that the class of regular languages is closed under symmetric difference. (You can use the closure results we've already proved.)

$$L_1 \triangle L_2 := \{\text{strings in } L_1 \text{ but not in } L_2\text{, or in } L_2 \text{ but not in } L_1\}$$

- Is the class of regular languages closed under taking subsets? i.e., is a subset of a regular language necessarily regular?
- Is the class of regular languages closed under taking supersets? i.e., is a superset of a regular language necessarily regular?

Circuits in Finite Automata

Definition

A **circuit** is a directed path which starts and ends at the same state.

The **length** of a circuit is the number of edges in the path.

Observation

Take any Finite Automaton.

Take any string w with has at least as many letters as there are states in that Finite Automaton.

Then the path taken for input w must contain a circuit.

We can divide w up naturally into three parts, $w = xyz$, where:

x := the part *before* the circuit;

y := the part that *goes around* the circuit;

z := the part *after* the circuit;

$w = \underbrace{\texttt{a}}_{\color{cyan}x}\underbrace{\texttt{baaa}}_{\color{green}y}\underbrace{\texttt{abb}}_{\color{magenta}z}$

If w is accepted,
then so are . . .

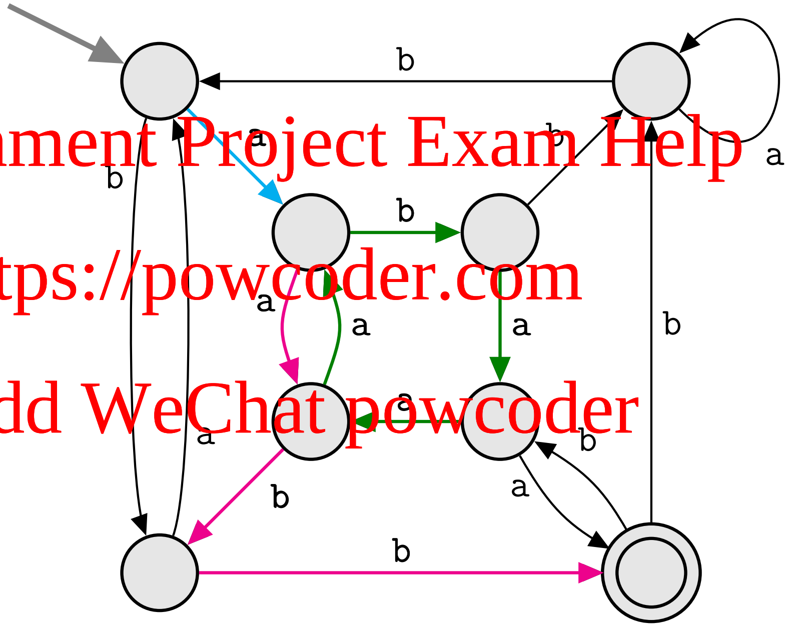
$\underbrace{\texttt{a}}_{x}\underbrace{\texttt{abb}}_{z}$

$\underbrace{\texttt{a}}_{x}\underbrace{\texttt{baaa}}_{y}\underbrace{\texttt{abb}}_{z}$

$\underbrace{\texttt{a}}_{x}\underbrace{\texttt{baaa}}_{y}\underbrace{\texttt{baaa}}_{y}\underbrace{\texttt{abb}}_{z}$

. . .

. . .



$w \quad = \quad \underbrace{\texttt{ba}}_{y}\underbrace{\texttt{bb}}_{z}$

$x \quad = \quad \varepsilon$

If w is accepted,
then so are . . .

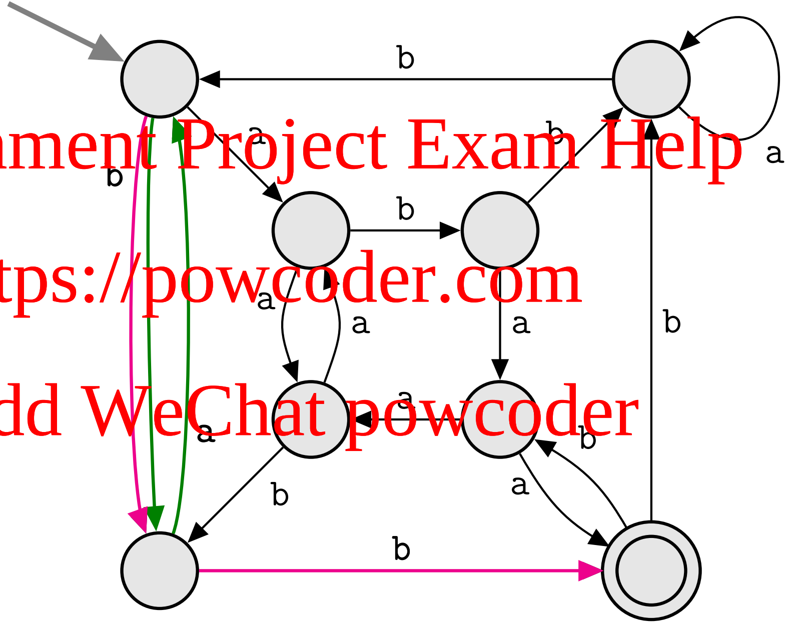
$\underbrace{\texttt{bb}}_{z}$

$\underbrace{\texttt{ba}}_{y}\underbrace{\texttt{bb}}_{z}$

$\underbrace{\texttt{ba}}_{y}\underbrace{\texttt{ba}}_{y}\underbrace{\texttt{bb}}_{z}$

. . .

. . .



Pumping Lemma

Theorem. (Pumping Lemma)
Let L be an infinite regular language, accepted by a FA with N states.
Then for all words $w \in L$ with at least N letters
there exist strings x, y, z, with $y \neq \varepsilon$, such that

- $w = xyz$
- $\text{length}(x) + \text{length}(y) \leq N$
- for all $i \geq 0$, $\quad xy^iz \in L$,
  i.e.,

$$xz, xyz, xyyz, \ldots, xy^nz, \ldots \in L.$$

Symbolically:

$$\forall w \in L \; : \; |w| \geq N \; \Rightarrow \; \big(\exists x, y, z \; : \; (w = xyz) \wedge (y \neq \varepsilon) \wedge (|x| + |y| \leq N) \wedge (\forall i \geq 0 : xy^iz \in L) \big)$$

Pumping Lemma

Proof.

Take any word $w \in L$ with $\geq N$ letters.

By our earlier Observation on circuits in FAs, the path taken by w must include a circuit.

Let

x be the letters of w up to the first circuit.

y be the letters corresponding to the circuit.

z be the remaining letters of w.

We have:

- $w = xyz$ by construction.
- Since the circuit exists, $y \neq \varepsilon$.
- $\text{length}(x) + \text{length}(y) \leq N$, since the FA reads xy without repeating any state.
- Since $w = xyz \in L$, and y starts and finishes at endState(x), and z goes from endState(x) to a Final State, we can repeat y any number of times (or none) and still we end up at the same Final State. $\square$

Pumping Lemma: application

Consequence

Using the Pumping Lemma we can show there are non-regular languages.

Method

Assume L is regular.

Then, by Kleene's Theorem, it is recognised by some FA.

Let N be the number of states in this FA.

Choose a suitable word $w \in L$, of length $\geq N$.

Show that, for any x, $y \neq \varepsilon$, and z such that $w = xyz$ and $|xy| \leq N$ ...

... there exists $i \geq 0$ s.t. $xy^i z \notin L$.

Contradiction.

Compare quantifiers above with those in Pumping Lemma

HALF-AND-HALF:

$L := \{\mathtt{a}^n\mathtt{b}^n : n \geq 0\} = \{\varepsilon, \mathtt{ab}, \mathtt{aabb}, \mathtt{aaabbb}, \ldots\}$.

Theorem.
L is not regular.

Proof. (by contradiction)
Assume that L is regular.
Let $N = \#$ states in an FA for it.

Choose $w := \mathtt{a}^{\lceil N/2 \rceil}\mathtt{b}^{\lceil N/2 \rceil}$. $\overbrace{\mathtt{aaa}\cdots\cdots\mathtt{aa}}^{\lceil N/2 \rceil \text{ letters}}\overbrace{\mathtt{bbb}\cdots\cdots\mathtt{bb}}^{\lceil N/2 \rceil \text{ letters}}$

Observe that $|w| \geq N$.
Consider any x, $y \neq \varepsilon$, and z such that $w = xyz$ and $|xy| \leq N$.
Think: are $xz, xyz, xyyz, \ldots, xy^N z, \ldots$ all in L?

Non-regular languages

$$\overbrace{\mathtt{aaa}\cdots\underbrace{\cdots\cdots}_{y}\cdots\mathtt{aa}}^{\lceil N/2\rceil \text{ letters}}\overbrace{\mathtt{bbb}\cdots\cdots\cdots\mathtt{bb}}^{\lceil N/2\rceil \text{ letters}}$$

Case 1: y is all a's.

Then $xyyz$ has more a's than b's, since $y \neq \varepsilon$.
So $xy^2z \notin L$.

$$\mathtt{aaa}\cdots\cdots\cdots\mathtt{aabbb}\cdots\underbrace{\cdots\cdots}_{y}\cdots\mathtt{bb}$$

Case 2: y is all b's.

Then $xyyz$ has more b's than a's, since $y \neq \varepsilon$.
So $xy^2z \notin L$.

$$\mathtt{aaa}\cdots\cdots\underbrace{\cdots\mathtt{aabbb}\cdots}_{y}\cdots\mathtt{bb}$$

Case 3: y contains an ab.

Then $xyyz$ has two occurrences of ab. This cannot happen for strings in L. So $xy^2z \notin L$.

In every possible case, we have found an i such that $xy^iz \notin L$.
This violates the conclusion of the Pumping Lemma.
Contradiction. $\square$

Non-regular languages

HALF-AND-HALF:

$L \ := \ \{\mathtt{a}^n\mathtt{b}^n : n \geq 0\} \ = \ \{\varepsilon, \mathtt{ab}, \mathtt{aabb}, \mathtt{aaabbb}, \ldots\}$.

Theorem.
L is not regular.

Proof. (by contradiction)
Assume that L is regular. Let $N = \#$ states in an FA for it.
Choose $w = \mathtt{a}^N\mathtt{b}^N$. *[No need for w to be of minimum length.]*
Consider any x, $y \neq \varepsilon$, and z such that $w = xyz$ . . .
. . . and $|xy| \leq N$. *[Previous proof didn't <u>use</u> $|xy| \leq N$. Can it help?]*

Think: are $xz, xyz, xyyz, \ldots, xy^Nz, \ldots$ all in L?

How many cases now?

Non-regular languages

Just one case: y is all a's.

$$\overbrace{\texttt{aaa}\cdots\underbrace{\cdots\cdots}_{y}\cdots\texttt{aa}}^{N\text{ letters}}\overbrace{\texttt{bbb}\cdots\cdots\cdots\texttt{bb}}^{N\text{ letters}}$$

Consider $xyyz$.

$$\overbrace{\texttt{aaa}\cdots\underbrace{\cdots\cdots}_{y}\underbrace{\cdots\cdots}_{y}\cdots\texttt{aa}}^{N+|y|\text{ letters}}\overbrace{\texttt{bbb}\cdots\cdots\cdots\texttt{bb}}^{N\text{ letters}}$$

It has more a's than b's, since $y \neq \varepsilon$.
So $xy^2z \notin L$.
This violates the conclusion of the Pumping Lemma.
Contradiction. $\square$

Non-regular languages

$$\begin{aligned}\mathrm{EQUAL} &:= \{\text{all words which have an equal number of } \mathtt{a}\text{'s and } \mathtt{b}\text{'s}\} \\ &= \{\varepsilon, \mathtt{ab}, \mathtt{ba}, \mathtt{aabb}, \mathtt{abab}, \mathtt{abba}, \mathtt{baba}, \ldots\}\end{aligned}$$

Theorem.
EQUAL is not regular.

Proof. Assume EQUAL is regular.
Observe:

$$\text{HALF-AND-HALF} := \{\mathtt{a}^n\mathtt{b}^n : n \geq 0\} = \mathrm{EQUAL} \cap \mathtt{a}^*\mathtt{b}^*.$$

This implies that HALF-AND-HALF is also regular, since the language defined by $\mathtt{a}^*\mathtt{b}^*$ is regular, and regular languages are closed under intersection.
But we have just seen that HALF-AND-HALF is non-regular.
This is a contradiction.
So our initial assumption, that EQUAL is regular, is wrong.
Therefore EQUAL is non-regular. $\square$

Non-regular languages

$$\begin{aligned}\text{PALINDROME} \quad &:= \quad \{\,\text{all the strings which are the same if they are spelt backwards}\,\} \\ &= \quad \{\varepsilon, \mathtt{a}, \mathtt{b}, \mathtt{aa}, \mathtt{bb}, \mathtt{aaa}, \mathtt{aba}, \mathtt{bab}, \mathtt{bbb}, \dots\}\end{aligned}$$

Theorem.
PALINDROME is non-regular.

Proof. (by contradiction)
Assume PALINDROME is regular.
Then there exists a FA with N states which accepts PALINDROME.

Choose $w = \mathtt{a}^N\mathtt{ba}^N$.

$$\overbrace{\mathtt{aaa}\cdots\cdots\cdots\cdots\mathtt{aa}}^{N\text{ letters}}\,\mathtt{b}\,\overbrace{\mathtt{aaa}\cdots\cdots\cdots\cdots\mathtt{aa}}^{N\text{ letters}}$$

Non-regular languages

Consider all strings x, $y \neq \varepsilon$, and z such that

- $w = xyz$,
- $\text{length}(x) + \text{length}(y) \leq N$.

$$\overbrace{\text{aaa}\cdots\underbrace{\cdots\cdots}_{y}\cdots\text{aa}}^{N\text{ letters}}\,\text{b}\,\overbrace{\text{aaa}\cdots\cdots\text{aa}}^{N\text{ letters}}$$

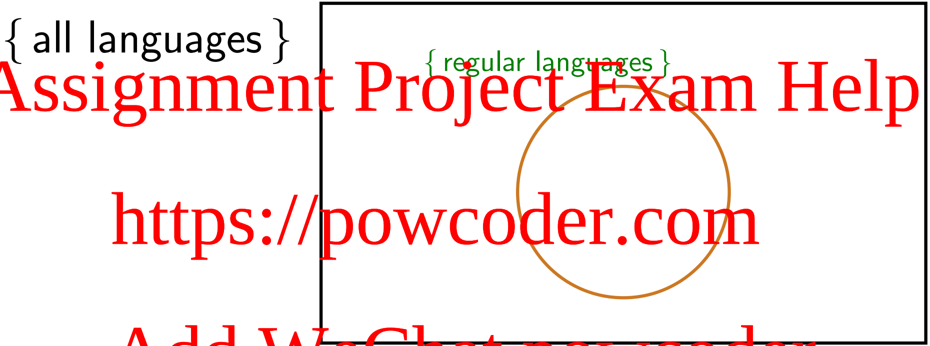
Consider $xyyz$.

$$\overbrace{\text{aaa}\cdots\underbrace{\cdots\cdots}_{y}\underbrace{\cdots\cdots}_{y}\cdots\text{aa}}^{N+|y|\text{ letters}}\,\text{b}\,\overbrace{\text{aaa}\cdots\cdots\text{aa}}^{N\text{ letters}}$$

Since $y \neq \varepsilon$, the solitary b in w is *more than* half-way along xy^2z.

So xy^2z is not a palindrome.
This contradicts the conclusion of the Pumping Lemma applied to PALINDROME.
So our initial assumption, that PALINDROME is regular, is wrong.
Therefore PALINDROME is not regular. □

Revision



Reading: Sipser, Ch. 1.

- closure properties: pp. 58–63.
- Pumping Lemma, non-regular languages: pp. 77–82.