Monash University
Faculty of Information Technology

FIT2014 Theory of Computation

Lecture 17

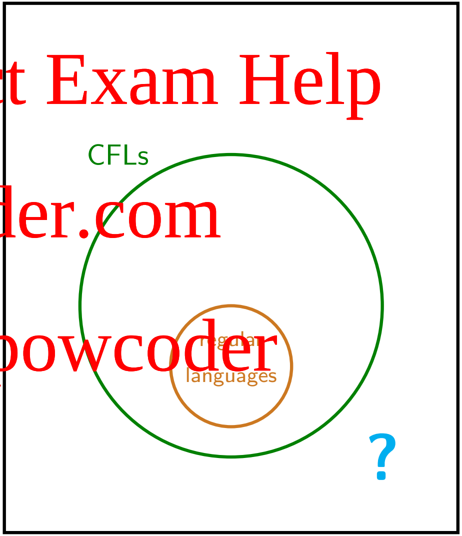
The Pumping Lemma for Context-Free Languages

slides by Graham Farr

Overview

- Pumping Lemma for CFLs
- Proof
- application:
  showing that some languages
  are *not* context-free

all languages



Pumping Lemma for Regular Languages (paraphrased)

Recall:

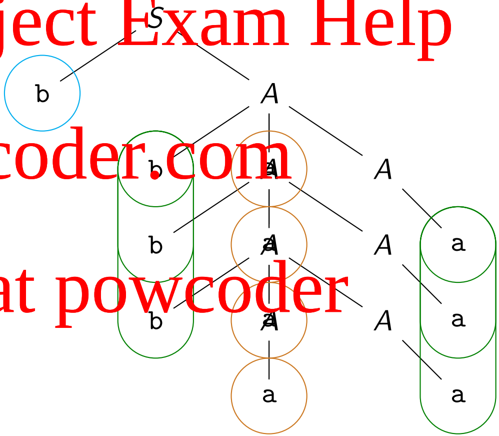
If a Finite Automaton
with N states
accepts
a sufficiently long string,
then the path taken by the string
in the FA
contains a repeated state.

This enables us to "pump" the string —
by repeating one substring —
to generate an infinite family
of members of the language.

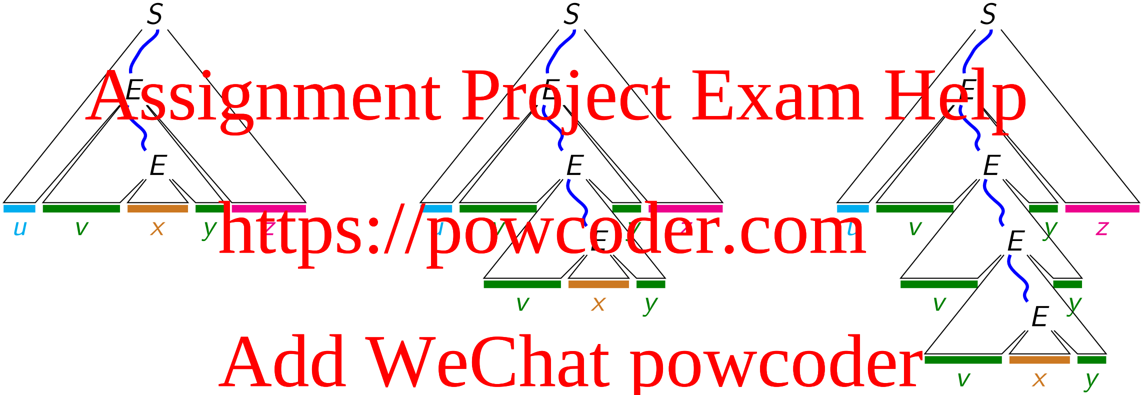
EQUAL

Parse tree for bbbbaaaa

1. $S \to \varepsilon$
2. $S \to \mathtt{b}A$
3. $S \to \mathtt{a}B$
4. $A \to \mathtt{a}$
5. $A \to \mathtt{a}S$
6. $A \to \mathtt{b}AA$
7. $B \to \mathtt{b}$
8. $B \to \mathtt{b}S$
9. $B \to \mathtt{a}BB$

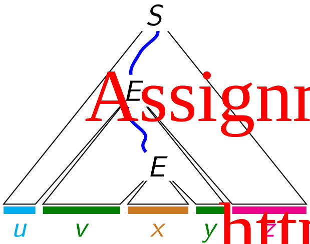


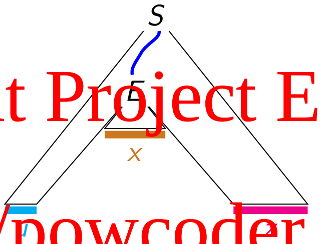
Nonterminal repetition in parse tree paths



$uvxyz$ $\qquad$ uv^2xy^2z $\qquad$ uv^3xy^3z

Nonterminal repetition in parse tree paths





$uvxyz$ uxz

Nonterminal repetition in parse tree paths

Nonterminals: S, E, T, F

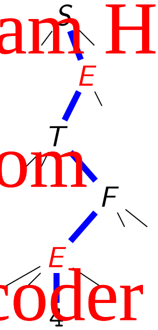
In a parse tree:

If

length of a root-to-leaf path $>$ # nonterminals

then

some nonterminal appears twice on that path.



Nonterminal repetition in parse tree paths

How can we *ensure* that this happens?

How to guarantee that the parse tree for a *sufficiently long* string has a path with a repeated state?

Consider:

$$\text{length of a path from root to leaf} \quad = \quad \# \text{ non-leaf nodes in that path.}$$

Each non-leaf node has a nonterminal symbol.

If

$$\text{max root-to-leaf path length} \quad > \quad \# \text{ nonterminal symbols in the grammar}$$

then some nonterminal symbol occurs twice on that path.

How to guarantee that the parse tree for a *sufficiently long* string has a *sufficiently long* root-to-leaf path?

Nonterminal repetition in binary parse tree paths

Let's use *binary* parse trees

Binary parse tree:

$$2^{\text{max path length}} \geq \# \text{ leaves} = \text{word length}$$

If

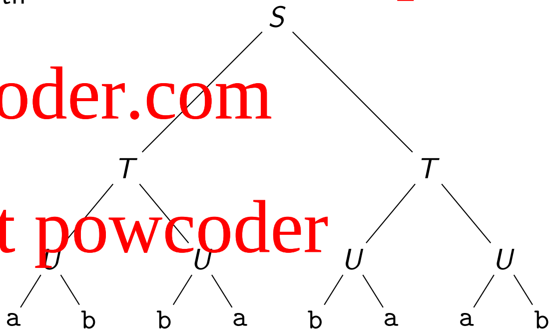
$$\text{word length} > 2^{\# \text{ nonterminals}}$$

then

$$2^{\text{max path length}} > 2^{\# \text{ nonterminals}},$$

$$\therefore \text{ max path length} > \# \text{ nonterminals}$$

and so there will be a repeated symbol
in a root-to-leaf path.

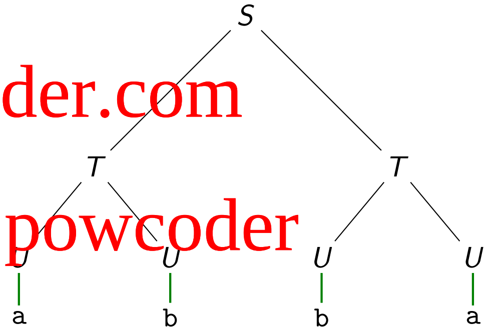


Nonterminal repetition in binary parse tree paths

Let's use *binary* parse trees, **from Chomsky Normal Form grammars!**

Binary **CNF** parse tree:

$$2^{\text{max path length}-1} \;\geq\; \#\text{ leaves} \;=\; \text{word length}$$



If

$$\text{word length} \;>\; 2^{\#\text{ nonterminals}-1}$$

then

$$2^{\text{max path length}-1} \;>\; 2^{\#\text{ nonterminals}-1},$$

$$\therefore\; \text{max path length} \;>\; \#\text{ nonterminals}$$

and so there will be a repeated symbol
in a root-to-leaf path.

Pumping Lemma for CFLs

Let L be any context-free language that has a CFG in CNF with k non-terminal symbols.
Then for every word $w \in L$ with $> 2^{k-1}$ letters,
there exist strings u, v, x, y, z with $\underbrace{vy \neq \varepsilon}_{\text{(i.e., } v,\, y \text{ not both } \varepsilon\text{)}}$

such that

- $w = uvxyz$
- $|vxy| \leq 2^k$, and
- for all $i \geq 0$, $uv^ixy^iz \in L$,
  i.e.,

$$uxz,\ uvxyz,\ uvvxyyz,\ \ldots,\ uv^nxy^nz,\ \ldots \in L.$$

Symbolically:

$$\forall w \in L \ :\ |w| > 2^{k-1} \ \Rightarrow\ \big(\exists u, v, x, y, z \ :\ (w = uvxyz) \wedge (vy \neq \varepsilon) \wedge (|vxy| \leq 2^k)$$
$$\wedge\ (\forall i \geq 0 : uv^ixy^iz \in L)\big)$$

Pumping Lemma for CFLs

Proof. (outline)
Take any word $w \in L$ with $> 2^{k-1}$ letters.

Let T be a parse tree for w, using the CNF CFG for L.

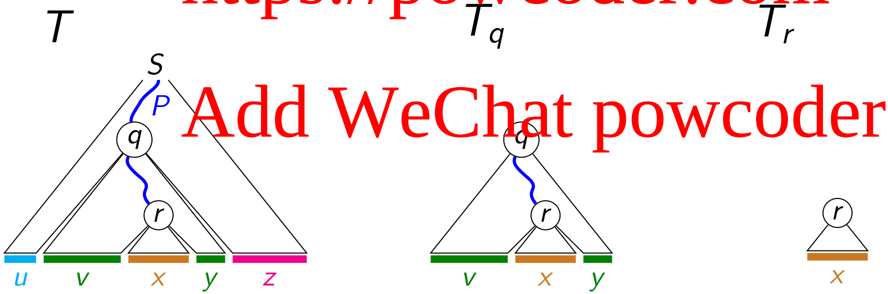
By our earlier Observations on root-to-leaf paths in CNF parse trees, some root-to-leaf path P in T contains a repeated nonterminal symbol.

Among all pairs of nodes in P containing the same nonterminals, choose the pair q, r, with q above r, such that q is as far as possible down the path P.
This ensures all nonterminals below q on P are distinct.
(Reason to be revealed later.)

Pumping Lemma for CFLs

Reading the letters of w from left to right, from the leaves of the tree, define:

- u be the letters of w to the left of the subtree T_q rooted at node q.
- v be the letters at the leaves of T_q that are to the left of the subtree T_r rooted at r.
- x be the letters at the leaves of T_r.
- y be the letters of T_q that are to the right of the subtree T_r.
- z be the remaining letters of w, i.e., those to the right of T_q.



We have:

- $w = uvxyz$ by construction.
- Since q, r are distinct nodes of the path P with q above r, the tree T_r is a proper subtree of T_q.
- Furthermore, since the grammar is in CNF, q has two children, and only one of them is above r, so T_q has some leaves that do not belong to T_r.
- Therefore $vy \neq \varepsilon$.

Pumping Lemma for CFLs

- By our choice of q and r, all nonterminals appearing below q on P are distinct.
- Since we have k nonterminals altogether,
  the subpath of P from q downwards has $\leq k + 2$ nodes
  (being q, then at most k nonterminals, then the leaf).
- Therefore it has length $\leq k + 1$.
  Therefore T_q has $\leq 2^k$ leaves.
  These leaves are the strings v, x, y, in order.
- Therefore $|vxy| \leq 2^k$.

- Replacement of T_q by T_r gives a parse tree for uxz.
- Replacement of T_r by T_q in T gives a parse tree for $uvvxyyz$.
- The new copy of T_q contains a copy of T_r.
  Replacing that copy of T_r by T_q gives a parse tree for $uvvvxyyyz$.
- Any parse tree with a copy of T_r can be enlarged, to be a parse tree of a longer string, by replacing T_r by T_q.
- These observations can be turned into a full formal proof by induction. □

A Tale of Two Pumping Lemmas

If a Finite Automaton
with N states
accepts
a sufficiently long string,
then the path taken by the string
in the FA
contains a repeated state.

This enables us to "pump" the string —
by repeating one substring —
to generate an infinite family
of members of the language.

If a Context-Free Grammar in CNF
with k nonterminals
generates
a sufficiently long string,
then some root-to-leaf path
in the parse tree
contains a repeated nonterminal.

This enables us to "pump" the string —
by repeating two substrings —
to generate an infinite family
of members of the language.

Pumping Lemma for CFLs: application

Consequence

Using the Pumping Lemma for CFLs we can show there are non-context-free languages.

Method

Assume L is context-free.

Then it has a Context-Free Grammar in CNF.

Let k be the number of nonterminal symbols in this CFG.

Choose a suitable word $w \in L$, of length $> 2^{k-1}$.

Show that, for any u, v, x, y, z such that $w = uvxyz$ and $vy \neq \varepsilon$ and $|vxy| \leq 2^k$ …

… there exists $i \geq 0$ s.t. $uv^ixy^iz \notin L$.

Contradiction.

Compare quantifiers above with those in the Pumping Lemma for CFLs.

Non-context-free languages

$$L \;:=\; \{\mathtt{a}^n\mathtt{b}^n\mathtt{a}^n : n \geq 0\} \;=\; \{\varepsilon, \mathtt{aba}, \mathtt{aabbaa}, \mathtt{aaabbbaaa}, \ldots\}.$$

Theorem.
L is not context-free.

Proof. (by contradiction)
Assume that L is context-free. Then it has a CFG.
Then there is a CFG in Chomsky Normal Form that generates $L \setminus \{\varepsilon\}$.
Let $k = \#$ nonterminals in this CNF CFG.
Take $N > 2^{k-1}/3$.
Choose $w = \mathtt{a}^N\mathtt{b}^N\mathtt{a}^N$.
Consider any u, v, x, y, z such that

- $vy \neq \varepsilon$,
- $|vxy| \leq 2^k$, and
- $w = uvxyz$.

Think: are $uxz, uvxyz, uvvxyyz, \ldots, uv^ixy^iz, \ldots$ all in L?

Non-context-free languages

Case 1: v and y are each all a's, or all b's, or empty.

For example: $\overbrace{\mathtt{aaa}\cdots\underbrace{\cdots\cdots}_{v}\cdots\mathtt{aa}}^{N\text{ letters}}\overbrace{\mathtt{bbb}\cdots\underbrace{\cdots}_{y}\cdots\mathtt{bb}}^{N\text{ letters}}\overbrace{\mathtt{aaa}\cdots\cdots\cdots\mathtt{aa}}^{N\text{ letters}}$

Then $uvvxyyz$ can no longer have three equal-length stretches of a's and b's, since:

- The two strings v, y must each lie entirely within one stretch, and there are three stretches, so one of these stretches is unaltered by pumping.
- But at least one of the other stretches *is lengthened*, because $vy \neq \varepsilon$.

So $uv^2xy^2z \notin L$.

Non-context-free languages

Case 2: Either v or y contains ab.

For example: $\texttt{aaa}\cdots\cdots\cdots\underbrace{\cdot\,\texttt{aabbb}\,\cdots}_{v}\cdots\cdots\texttt{bbaaa}\cdots\cdots\cdots\texttt{aa}$

Then $uvvxyyz$ has two occurrences of ab.
This cannot happen for strings in L. So $uv^2xy^2z \notin L$.

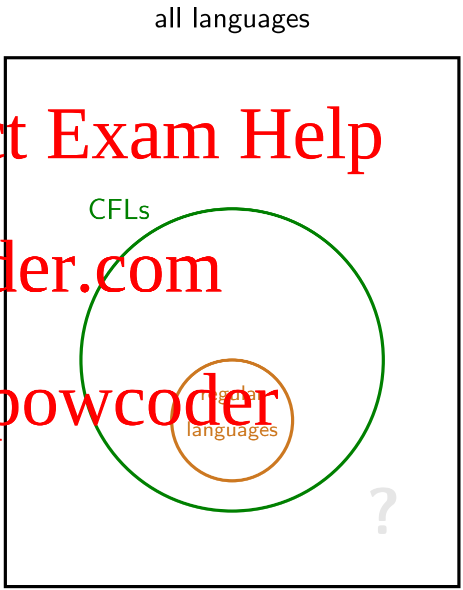
Case 3: Either v or y contains ba.

For example: $\texttt{aaa}\cdots\cdots\cdots\texttt{aabbb}\cdots\cdots\underbrace{\cdots\texttt{bbaaa}\cdots}_{v}\cdots\cdots\texttt{aa}$

Similar argument to Case 2.

In every possible case, we have found an i such that $uv^ixy^iz \notin L$.
This violates the conclusion of the Pumping Lemma for CFLs.
Contradiction. $\square$

Revision



Reading: Sipser, pp. 125–129.