Monash University
Faculty of Information Technology
Semester 2, 2022

FIT2014
Tutorial 1
Languages and Logic

ASSESSED PREPARATION: Question 6.
You must provide a serious attempt at this entire question at the start of your tutorial (for on-campus classes), or submit it online by the start of your tutorial using the class-specific link in the appropriate week in Moodle (for online classes).

1. Let ODD-ODD be the language of strings, over the alphabet {a,b}, that contain an odd number of a's and an odd number of b's. Let $\overline{\text{ODD-ODD}}$ be its complement.

Prove that PALINDROMES $\subseteq \overline{\text{ODD-ODD}}$.

2. Distributive Law for propositional logic:

(a) Prove that

$$P \vee (Q \wedge R) \quad \text{is logically equivalent to} \quad (P \vee Q) \wedge (P \vee R).$$

(b) Prove that

$$P \wedge (Q \vee R) \quad \text{is logically equivalent to} \quad (P \wedge Q) \vee (P \wedge R),$$

using part (a) and a rule named after a friend of Charles Babbage.

3. Prove that

$$(P_1 \wedge \cdots \wedge P_n) \Rightarrow C \quad \text{is logically equivalent to} \quad \neg P_1 \vee \cdots \vee \neg P_n \vee C$$

4. A meeting about moon mission software is held at NASA in 1969. Participants may include Judith Cohen (electrical engineer), Margaret Hamilton (computer scientist), and Katherine Johnson (mathematician). Let Judith, Margaret and Katherine be propositions with the following meanings.

Judith	Judith Cohen is in the meeting.
Margaret	Margaret Hamilton is in the meeting.
Katherine	Katherine Johnson is in the meeting.

For each of the following statements, write a proposition in Conjunctive Normal Form with the same meaning.

(a) Judith and Margaret are not both in the meeting.
(b) Either Judith or Margaret, but not both of them, is in the meeting. (This is the "exclusive-OR".)
(c) At least one of Judith, Margaret and Katherine is in the meeting.
(d) At most one of Judith, Margaret and Katherine is in the meeting.
(e) Exactly one of Judith, Margaret and Katherine is in the meeting.
(f) At least two of Judith, Margaret and Katherine are in the meeting.
(g) At most two of Judith, Margaret and Katherine are in the meeting.

(h) Exactly two of Judith, Margaret and Katherine are in the meeting.
(i) Exactly three of Judith, Margaret and Katherine are in the meeting.
(j) None of Judith, Margaret and Katherine is in the meeting.

5. Suppose we have propositions Tree, Leaf, Bipartite, Internal about a connected graph with at least two vertices, with the following meanings.[1]

Tree	The graph is a tree.
Bipartite	The graph is bipartite.
Leaf	The graph has a leaf.
Internal	The graph has a vertex of degree ≥ 2. (Such a vertex is sometimes called an *internal vertex*.)

Using these propositions, write a proposition in CNF with the following meaning:

> If the graph is a tree, then it's bipartite and has a leaf, but if it's not a tree, then it has a vertex of degree ≥ 2.

6. For each past or present Monash unit, we'll use its unit code to denote the proposition that you have passed the unit. So, if ABC1234 is a unit code, then we'll also use ABC1234 for a proposition with the following meaning:

$$\text{ABC1234} = \begin{cases} \text{True}, & \text{if you have passed unit ABC1234;} \\ \text{False}, & \text{if you have not passed unit ABC1234} \\ & \text{(either through never having enrolled in it, or only failing it,} \\ & \text{or doing it currently so that you haven't finished it yet).} \end{cases}$$

Here is an edited extract from the Monash Handbook 2022, specifying the conditions under which you may enrol in FIT2014:

Prerequisite:

- One of FIT1045, FIT1048, FIT1051, FIT1053, ENG1003, ENG1013 or (FIT1040 and FIT1029)

AND

- One of MAT1830, MTH1030, MTH1035, ENG1005

Prohibition:

- CSE2303

(a) Using these rules and the propositions corresponding to all these unit codes, construct an expression in Conjunctive Normal Form that specifies the conditions under which you may enrol in FIT2014.

Now consider how you would construct an equivalent expression in <u>Disj</u>unctive Normal Form:

$$\underbrace{(\ldots \wedge \ldots \wedge \ldots\ldots)}_{\text{disjunct}} \vee \underbrace{(\ldots \wedge \ldots \wedge \ldots\ldots)}_{\text{disjunct}} \vee \cdots\cdots\cdots \vee \underbrace{(\ldots \wedge \ldots \wedge \ldots\ldots)}_{\text{disjunct}}$$

(b) Give *three* of the disjuncts in such an expression.

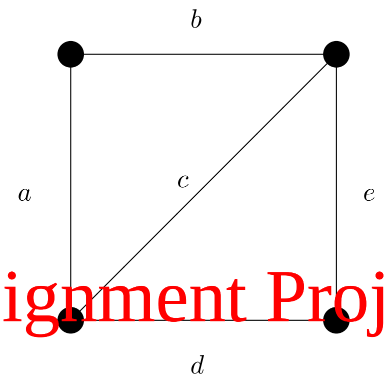
[1] Thanks to FIT2014 tutor Ben Jones for detecting and reporting a small error in an earlier version of this question.

(c) How many disjuncts would such an expression have?

7. *(mostly from FIT2014 Final Exam, 2015)*

A **perfect matching** in a graph G is a subset X of the edge set of G that meets each vertex exactly once. In other words, no two edges in X share a vertex, and each vertex of G is incident with exactly one edge in X.

For example, in the following graph, the edge set $\{a, e\}$ is a perfect matching. But $\{a, b, e\}$ is not a perfect matching (since, for example, a and b share a vertex), and $\{a\}$ is not a perfect matching (since some vertices are not incident with the edge in this set).



Let W be the above graph.
Construct a Boolean expression E_W in Conjunctive Normal Form such that the satisfying truth assignments for E_W correspond to perfect matchings in the above graph W.

When doing this, use Boolean variables a, b, c, d, e which are each True if and only if the edge with the same name belongs to the perfect matching.

8. Using the function symbol **father**, the predicate **taller**, and the constant symbols **claire** and **max**, convert the following sentences to Predicate Logic. Assume that **taller(X,Y)** means **X** is taller than **Y** and the universe of discourse is "all people".

ii. Max's father is taller than Max but not taller than Claire's father.

ii. Someone is taller than Claire's father.

iii. Everyone is taller than someone else.

iv. Everyone who is taller than Claire is taller than Max.

9. Suppose you have access to the following functions and relations:

- $|x|$ = the length of the string x
- equality and inequality relations: $=, <, \leq, >, \geq$
- set membership, denoted by $\in$ as usual.

Let L be a language.

(a) Using quantifiers, write down a logical statement about L that is True if and only if L is finite.

(b) Now write a statement about L that is True if and only if L is infinite.

Supplementary exercises

10. *Three boys, Adam, Brian and Claude, are caught, suspected of breaking a glass window. When the boys were questioned by police:*

Adam said: 'Brian did it; Claude is innocent'.

Brian said: 'If Adam is guilty then so is Claude'.

Claude said: 'I didn't do it; one of the others did'.

The police believed that all the boys were telling the truth, and therefore concluded that Brian broke the window and the others didn't.

Using the following propositions express the statements of the boys and the police conclusion in propositional logic.

A: **Adam broke the window.**

B: **Brian broke the window.**

C: **Claude broke the window.**

Assuming that all the boys were telling the truth, was the police conclusion logically valid?

11. Recall the logical expression given near the end of Lecture 2 for your dinner party guest list:

$$
\begin{aligned}
&(\text{Harry} \vee \text{Ron} \vee \text{Hermione} \vee \text{Ginny}) \\
&\quad \wedge \quad (\neg\text{Hagrid} \vee \text{Norberta}) \\
&\quad \wedge \quad (\neg\text{Fred} \vee \text{George}) \ \wedge \ (\text{Fred} \vee \neg\text{George}) \\
&\quad \wedge \quad (\neg\text{Voldemort} \vee \neg\text{Bellatrix}) \ \wedge \ (\neg\text{Voldemort} \vee \neg\text{Dolores}) \ \wedge \ (\neg\text{Bellatrix} \vee \neg\text{Dolores})
\end{aligned}
$$

How long would an equivalent DNF expression be? Specifically, how many disjuncts — smaller expressions combined using $\vee$ to make the whole expression — would it have?

12. Leonard Books, Cedric Smith, Arthur Stone and Bill Tutte[2] used the pseudonym Blanche Descartes for some of their writings. Each work by Blanche Descartes was written by some nonempty subset of the four. Let Leonard, Cedric, Arthur and Bill be propositions about one of Blanche Descartes's works with the following meanings.

Leonard	Leonard Books was one of the authors.
Cedric	Cedric Smith was one of the authors.
Arthur	Arthur Stone was one of the authors.
Bill	Bill Tutte was one of the authors.

Write a proposition in Conjunctive Normal Form meaning that exactly two of the four were authors of the work.

13. One-dimensional Go.

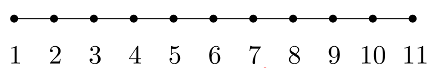
This question uses some concepts from the ancient east Asian board game known as *Go* in the West, *Wéiqí* in China, *Go* or *Igo* in Japan, and *Baduk* in Korea. This game is over 2,000 years old,

[2]While they were still undergraduate students at Cambridge, these four wrote a paper which solved a famous open problem of recreational mathematics, helped create modern graph theory and influenced the course of history: R. L. Brooks, C. A. B. Smith, A. H. Stone, and W. T. Tutte, The dissection of rectangles into squares, *Duke Mathematical Journal* **7** (1940) 312–340.

and is generally regarded as harder than Chess. Indeed, until very recently, computer programs for Go could not defeat human professionals (in contrast to Chess, where the best computer players have been stronger than human world champions since the late 1990s).

The situation changed dramatically early in 2016, with stunning performances by the program AlphaGo, created by Google DeepMind. (This company began as a start-up in London in 2010 and was acquired by Google in 2014.) In October 2015, it defeated the European champion Fan Hui in every game of a five-game match: `http://www.nature.com/news/google-ai-algorithm-masters-ancient-game-of-go-1.19234`. Then in March 2016 it defeated Lee Sedol of Korea, generally regarded as the best player in the world in recent times: `https://www.theatlantic.com/technology/archive/2016/03/the-invisible-opponent/475611/`. The final score in that five-game match was 4-1 in favour of AlphaGo. In May 2017 it defeated the top-ranked player in the world, Ke Jie of China, 3-0. See `https://deepmind.com/research/alphago/` or `http://361points.com/articles/thoughtsonalphago/`.

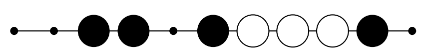
Go is played on a graph, usually a square lattice (grid) of 19×19 vertices. But we will use much simpler graphs in this question, namely path graphs with n vertices and $n-1$ edges, where $n \geq 1$. For example, with $n = 11$ we get the following path graph with 11 vertices and 10 edges.



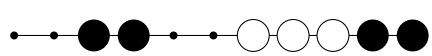
A *position* consists of a placement of black and white stones on some of the vertices of the graph. Each vertex may have a black stone, or a white stone (but not both), or be uncoloured (i.e., vacant). A position is *legal* if every vertex with a stone can be linked to an uncoloured vertex by a path consisting entirely of vertices with stones of that same colour (except for the uncoloured vertex at the end of that path).

For example, the following position is legal, since each of its three "chains" of consecutive vertices of the same colour either starts or ends with an uncoloured vertex.

But the following position is illegal, since it has a chain of white vertices with black vertices at each end. (The position has four chains altogether, and three are ok. But it only takes one without an uncoloured neighbour to make the position illegal.)



We number the vertices on the path graph from 1 to n, from left to right. We say that a position on this path graph is *almost legal* if vertex n is coloured (i.e., has a stone) and its chain is not next to an uncoloured vertex, but every other chain is next to an uncoloured vertex. In other words, it is illegal, but the only chain making it illegal is the chain containing vertex n; all other chains are ok. The two positions given above are *not* almost legal: the first is legal (so it isn't *almost* legal), while the second is illegal but the illegality is not due to the last vertex (which in this case is uncoloured). The following position is almost legal. All its chains are ok except the last one on the right.



Let $V_{B,n}$, $V_{W,n}$, $V_{U,n}$, $L_{B,n}$, $L_{W,n}$, $L_{U,n}$, $A_{B,n}$, $A_{W,n}$ be the following propositions about a position on the n-vertex path graph.

$V_{B,n}$	Vertex n is Black.
$V_{W,n}$	Vertex n is White.
$V_{U,n}$	Vertex n is Uncoloured.
$L_{B,n}$	The position is legal, and vertex n is Black.
$L_{W,n}$	The position is legal, and vertex n is White.
$L_{U,n}$	The position is legal, and vertex n is Uncoloured.
$A_{B,n}$	The position is almost legal, and vertex n is Black.
$A_{W,n}$	The position is almost legal, and vertex n is White.

(a) Use the propositions $L_{B,n}$, $L_{W,n}$, $L_{U,n}$ (together with appropriate connectives) to write a logical expression for the proposition that the position is legal.

Now consider how legality and almost-legality on the n-vertex path graph are affected by extending the path to vertex $n + 1$.

(b) If $L_{B,n}$ is true, what possible states (Black/White/Uncoloured) can vertex $n + 1$ be in, if we want the position to be legal on the $n + 1$-vertex path as well?
Do the same for $L_{W,n}$ and $L_{U,n}$.

(c) If $A_{B,n}$ is true, what possible states can vertex $n + 1$ be in, if we want the position to be legal on the $n + 1$-vertex path?
Do the same for $A_{W,n}$.
Why is there no line for $A_{U,n}$ in the table?

(d) Construct a logical expression for $L_{B,n+1}$ using some of the propositions $V_{\_,n+1}, L_{\_,n}, A_{\_,n}$ in the above table. (In other words, you can only use the L-propositions and A-propositions for the n-vertex path graph, and the V-propositions for vertex $n + 1$.)
Do the same for $L_{W,n+1}$, $L_{U,n+1}$, $A_{B,n+1}$, $A_{W,n+1}$.

14. A *vertex cover* in a graph G is a set VC of vertices such that every edge of G is incident with some vertex in VC.

A *clique* in a graph is a set of vertices that are pairwise adjacent. (I.e., every pair of vertices in the clique is linked by an edge.)

The *complement* of a graph G, written $\overline{G}$, is defined as follows. It has the same vertex set of G, and its edge set consists of every pair of vertices that are *not* adjacent in G.

Let n denote the number of vertices of the graph under discussion.

Suppose we have a graph, and that **chosen** is a unary predicate that takes a vertex of our graph as its argument. This predicate therefore defines a subset of the vertex set of the graph (the "chosen vertices"). Suppose also that we have the following predicates, with the indicated meanings.

vertex(X)	**X** is a vertex in our graph
edge(X,Y)	there is an edge between vertices **X** and **Y**

(a) Write a statement in predicate logic, using the predicates **vertex**, **edge** and **chosen**, to say that the chosen vertices form a vertex cover.

(b) Write a statement in predicate logic, using the same predicates, to say that the chosen vertices form a clique.

(c) Prove that, for any k, the number of vertex covers of size k in G equals the number of cliques of size $n - k$ in $\overline{G}$.

(d) Give the relationship between the size of the smallest vertex cover in G and the size of the largest clique in $\overline{G}$.