FIT2014 Theory of Computation

Lecture 3
Predicate Logic

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Lecture overview

- Statements with variables
- Predicates
- Definitions and terminology
- Existential quantifier
- Universal quantifier
- Doing logic with quantifiers

Statements with variables

Consider these statements:

- W is negative.
- X passed this subject.
- $Y = Z$.

These do not yet have truth values.

The variables are **free**, in that no value is (yet) given to them.

You can, if you wish, assign values to them.

Each set of values you give to the variables creates a different specific proposition.

Statements with variables

For example, in the statement W is negative, the variable W is free.
If we assign values to it, we can create specific propositions:

$\vdots$	$\vdots$
-2 is negative	True
-1 is negative	True
0 is negative	False
1 is negative	False
2 is negative	False
$\vdots$	$\vdots$

Predicates

Definitions

A **predicate** is a statement with variables such that, for any values of the variables, it is either True or False.

- i.e., it becomes a proposition

We treat each variable as ranging over some **domain**.

- For the predicate N is negative, we've just been using the domain $\mathbb{Z}$.

The variables of a predicate are also called its **arguments**.

A predicate is called k-**ary** if it has k arguments. Special cases: unary, binary, ternary, ...

Some alternative terminology:

# arguments	terminology
1	*property*
≥ 2	*relation*

Predicates: examples

# args.	example	domain
1	isNegative(X)	$\mathbb{N}$
1	isNegative(X)	$\mathbb{Z}$
2	$=$ *[always available]*	objects
2	$X < Y$	numbers
2	isMotherOf(X, Y), meaning "X is the mother of Y"	people
3	gives(X, Y, Z), meaning "X gives Y to Z"	X, Z are people, Y is a gift
$\vdots$	$\vdots$	$\vdots$

Predicates may be thought of as *truth-valued functions*, i.e., functions whose value is always in $\{$True, False$\}$.

Functions

We'll also use functions whose values aren't necessarily just True or False.

# args.	example	domain	codomain
1	$\sqrt{X}$	nonnegative numbers	numbers
1	motherOf(X)	people	people
2	$X + Y$	numbers	numbers
$\vdots$	$\vdots$	$\vdots$	

Functions with no arguments are called **constants**.

- Examples: 5, Annie, ...

A function's arguments can be: constants; variables; functions.

Existential quantifier

There's a fly in my soup. $\exists X : (X \text{ is a fly}) \wedge (X \text{ is in my soup})$.

There exists W: W is negative. $\exists W : W < 0$

If domain of W is $\mathbb{N}$: it's False. $\exists W \in \mathbb{N} : W < 0$

If domain of W is $\mathbb{Z}$: it's True. $\exists W \in \mathbb{Z} : W < 0$

Existential quantifier

Someone did it. $\exists X : X$ did it.

It's *sort-of* like a disjunction . . .

$\cdots\cdots \vee (\text{Annie did it}) \vee (\text{Edward did it}) \vee (\text{Henrietta did it}) \vee (\text{Radhanath did it}) \vee \cdots\cdots$

. . . *but:*

- often the domain of a variable is infinite;
- keep the variables, they're useful.

The variables are now **bound**.
You can no longer give specific values to the variables to create specific propositions.
The quantifier has turned the statement into a single proposition about the entire domains of the variables.

Quantifiers can only be used with *variables*.
Using them with constant objects *makes no sense:* $\exists 5$, $\exists$Annie.

Existential quantifier

Some computer is human. *i.e.,* There exists a human computer.

If the domain of X is $\{\text{computers}\}$

Predicate:

- human(X): X is human.

$$\exists X : \text{human}(X)$$

But what if the domain of X is $\{\text{everything on Earth}\}$?

Predicates:

- human(X): X is human.
- computer(X): X is a computer.

Existential quantifier

Some computer is human. *i.e.,* There exists a human computer.

If the domain of X is $\{$everything on Earth$\}$

Predicates:

- human(X): X is human.
- computer(X): X is a computer.

Correct:

$\exists X : \text{computer}(X) \wedge \text{human}(X)$

- "There exists something that is both computer and human."
- "There exists a human computer."
- "Some computer is human."

Incorrect:

$\exists X : \text{computer}(X) \Rightarrow \text{human}(X)$

- "There exists something which is not a computer or is human."
- "There exists something which is not both a computer and non-human."
- "Not everything is a nonhuman computer."

Universal quantifier

Everyone can pass this subject. For every X : X can pass this subject.
$\forall X$: canPass(X).

All numbers are interesting. $\forall X$: X is interesting. $\forall X$: isInteresting(X).

- True — and we'll prove it!

For all W : W is negative $\forall W : W < 0$. False.

Again, the variables are now **bound**.

Universal quantifier

Every computer is human.

If the domain of X is $\{\text{computers}\}$
Predicate:

- human(X): X is human.

$$\forall X : \text{human}(X)$$

But what if the domain of X is $\{\text{everything on Earth}\}$?

Predicates:

- human(X): X is human.
- computer(X): X is a computer.

Universal quantifier

Every computer is human.

If the domain of X is $\{$everything on Earth$\}$

Predicates:

- human(X): X is human.
- computer(X): X is a computer.

Incorrect: $\forall X$: computer(X) $\wedge$ human(X)	*Correct:* $\forall X$: computer(X) $\Rightarrow$ human(X)
▶ "Everything is both computer and human." ▶ "Everything is a human computer."	▶ "For everything, if it's a computer, then it's human." ▶ "Everything that's a computer is also human." ▶ "Every computer is human."

Multiple quantifiers

Thinking of graphs . . .

Suppose we have a predicate adj(X, Y) meaning that vertices X and Y are adjacent.

Some two vertices are not adjacent. $\exists X \, \exists Y : \; \neg(X = Y) \wedge \neg \mathrm{adj}(X, Y).$

$\exists (X, Y) : \; \neg(X = Y) \wedge \neg \mathrm{adj}(X, Y).$

Every pair of vertices is adjacent. $\forall X \, \forall Y : \; \neg(X = Y) \Rightarrow \mathrm{adj}(X, Y).$

$\forall (X, Y) : \; \neg(X = Y) \Rightarrow \mathrm{adj}(X, Y).$

Some vertex is adjacent to all other vertices. $\exists X \, \forall Y : \; \neg(X = Y) \Rightarrow \mathrm{adj}(X, Y).$

Every vertex has a neighbour. $\forall X \, \exists Y : \; \mathrm{adj}(X, Y).$

Multiple quantifiers

Six degrees of separation

Suppose we have a predicate $\text{knows}(X, Y)$ meaning that person X knows person Y.

It has been claimed that, in the human social network,
the distance between any two people is at most 6.

Exercise: write this claim in predicate logic, using just the predicate knows.

Doing logic with quantifiers

If we know that

$$\forall X \; \text{blah}(X)$$

and obj is any specific object (in the domain of X),
then we can deduce that

$$\text{blah}(\text{obj})$$

We have:

$$(\forall X \; \text{blah}(X)) \;\Rightarrow\; \text{blah}(\text{obj})$$

Also:

$$\text{blah}(\text{obj}) \;\Rightarrow\; (\exists X \; \text{blah}(X))$$

Doing logic with quantifiers

$\forall X\,(p(X) \wedge q(X))$ is logically equivalent to $(\forall X\ p(X)) \wedge (\forall X\ q(X))$

$\exists X\,(p(X) \vee q(X))$ is logically equivalent to $(\exists X\ p(X)) \vee (\exists X\ q(X))$

What about the logical relationship between . . .

$\forall X\,(p(X) \vee q(X))$ and $(\forall X\ p(X)) \vee (\forall X\ q(X))$ . . . ?

. . . etc

Relationship between quantifiers

$\neg\,\forall Y$ means the same as $\exists Y\,\neg$

"Not all dogs are happy." is the same as ... "There exists an unhappy dog."

$$
\begin{aligned}
& \neg\forall X\,(\text{dog}(X) \Rightarrow \text{happy}(X)) && \text{Not all dogs are happy} \\
= \; & \exists X\,\neg(\text{dog}(X) \Rightarrow \text{happy}(X)) && \\
= \; & \exists X\,\neg(\neg\text{dog}(X) \vee \text{happy}(X)) && \text{(see last lecture)} \\
= \; & \exists X\,(\neg\neg\text{dog}(X) \wedge \neg\text{happy}(X)) && \text{(by De Morgan)} \\
= \; & \exists X\,(\text{dog}(X) \wedge \neg\text{happy}(X)) && \text{There exists an unhappy dog}
\end{aligned}
$$

Relationship between quantifiers

Similarly,

$\neg \exists Y$ means the same as $\forall Y \neg$

$\neg \forall Y \neg$ means the same as ______

$\neg \exists Y \neg$ means the same as ______