G6021 Comparative Programming

Part 3 - foundations

The Lambda Calculus

- The λ-calculus is a computational model based on the mathematical notion of a function.
- Defined by the mathematician Alonzo Church in the 1930's as a precise notation for anonymous functions. He noticed that an expression $x + y$ was sometimes interpreted as:
  - the number $x + y$
  - the function $f : x \mapsto x + y$
  - the function $g : y \mapsto x + y$
  - the function $h : x, y \mapsto x + y$

  With the lambda-calculus notation, these can be easily distinguished:
  - the number $x + y$ is written just $x + y$
  - the function $f : x \mapsto x + y$ is written $\lambda x.x + y$
  - the function $g : y \mapsto x + y$ is written $\lambda y.x + y$
  - the function $h : x, y \mapsto x + y$ is written $\lambda x, y.x + y$

The λ-calculus is used:

- to study computability (as an alternative to Turing Machines),
- to define models (denotational semantics) of programming languages,
- to study strategies and implementation techniques for functional languages (abstract machines),
- to encode proofs in a variety of logics,
- to design automatic theorem provers and proof assistants.

λ-calculus: Syntax

Definition:

Assume an infinite set $\mathcal{X}$ of variables denoted by $x, y, z, \ldots,$ then the set of λ-terms is the least set satisfying:

$$M ::= \mathcal{X} \mid (\lambda \mathcal{X}.M) \mid (MM)$$

which are called variable, abstraction and application

Some examples

- x, $(\lambda y.y)$, $(\lambda x.(\lambda y.x))$, $((\lambda z.z)(\lambda y.y))$

An intuition. The following are all the same:

- $f\ x\ y = x + y$
- $f\ x = \lambda y.x + y$
- $f = \lambda x.\lambda y.x + y$

λ-calculus: Conventions

- write as few parentheses as possible:

$$(\lambda y.(xy)) = \lambda y.xy$$

- application associates to the *left*:

$$xyz = ((xy)z)$$

- abstractions bind as far as possible to the *right*

$$\lambda x.(\lambda y.y)x = (\lambda x.((\lambda y.y)x))$$

- abstractions can be abbreviated:

$$\lambda x.\lambda y.M = \lambda xy.M$$

Examples of λ-terms

- x, $\lambda x.x$, xy, $\lambda x.z$, $xz(yz)$, $\lambda x.\lambda y.yx$
- $\lambda xy.x$, $\lambda xyz.z(xy)$, $\lambda xyz.xz(yz)$
- $\lambda x.\lambda y.x$, $\lambda x.\lambda y.y$
- $(\lambda x.x)y$, $(\lambda x.\lambda y.xy)(\lambda x.x)$
- $\lambda f.\lambda x.x$, $\lambda f.\lambda x.fx$, $\lambda f.\lambda x.f(fx)$, $\lambda f.\lambda x.f(f(fx))$
- $\lambda x.xx$

Note: Haskell syntax:

`\x -> M` is the same as $\lambda x.M$

Exercise: Write the above examples in Haskell syntax. Are they all valid in Haskell?

Variables

A variable is *free* in a λ-term if it is not bound by a λ.
More precisely, the set of free variables of a term is defined as:

$$
\begin{array}{lcl}
FV(x) & = & \{x\} \\
FV(\lambda x.M) & = & FV(M) - \{x\} \\
FV(MN) & = & FV(M) \cup FV(N)
\end{array}
$$

Terms without free variables are called *closed terms*.

We can define:

$$
\begin{array}{lcl}
BV(x) & = & \emptyset \\
BV(\lambda x.M) & = & \{x\} \cup BV(M) \\
BV(MN) & = & BV(M) \cup BV(N)
\end{array}
$$

Question: What is BV defining?

Exercise: Check the FV and BV of the examples.

α-conversion

λ-terms that differ only in the names of their bound variables will be equated. More precisely: If y is not free in M:

$$\lambda x.M =_\alpha \lambda y.M\{x \mapsto y\}$$

where $M\{x \mapsto y\}$ is the term M where each occurrence of x is replaced by y (i.e. we rename every free occurrence of x to y).

IMPORTANT:

- λ-terms are defined modulo α-conversion, so $\lambda x.x$ and $\lambda y.y$ are the SAME term.
- α-equivalent terms represent the same computation (see below).

Computation

- Abstractions represent functions, which can be applied to arguments.
- The main computation rule is β-*reduction*, which indicates how to find the result of the function for a given argument.
- A *redex* is a term of the form: $(\lambda x.M)N$
- It reduces to the term $M\{x \mapsto N\}$ where $M\{x \mapsto N\}$ is the term obtained when we substitute x by N taking into account bound variables.

β-reduction:

$$(\lambda x.M)N \to_\beta M\{x \mapsto N\}$$

- Note that we use the word "reduce", but this does not mean that the term on the right is any simpler. Why?
- Notation: if $M \to_\beta M_1 \to_\beta M_2 \cdots M_n$ then we write $M \to^*_\beta M_n$

Substitution

Substitution is a special kind of replacement: $M\{x \mapsto N\}$ means replace all *free* occurrences of x in M by the term N.

Question: Why only the free occurrences? What happens if we replace all occurrences?

A very useful property of substitution is the following, known as the Substitution Lemma:

If $x \notin FV(P)$:

$$(M\{x \mapsto N\})\{y \mapsto P\} = (M\{y \mapsto P\})\{x \mapsto N\{y \mapsto P\}\}$$

Examples (conversion, reduction, substitution)

α-conversion:

- $\lambda x.x =_\alpha \lambda y.y$
- $\lambda x.\lambda y.xy =_\alpha \lambda z_1.\lambda z_2.z_1 z_2$

and β-reduction:

- $(\lambda x.\lambda y.xy)(\lambda x.x) \rightarrow_\beta \lambda y.(\lambda x.x)y \rightarrow_\beta \lambda y.y$

Normal forms

When do we stop reducing?

- Normal form (NF): Stop reducing when there are no redexes left to reduce.

- A *normal form* is a term that does not contain any redex.
- A term that can be reduced to a term in normal form is said to be *normalisable*.

  Example:
  $$(\lambda x.a(\lambda y.xy))\ b\ c \rightarrow_\beta a(\lambda y.by)c$$
  which is a normal form (recall that application associates to the left).

- Weak Head Normal Form (WHNF). Stop reducing when there are no redexes left, but without reducing under an abstraction.

Exercises

1. What is the difference between a term having a normal form, and being a normal form? Write down some example terms.
2. If a closed term is a weak head normal form, it has to be an abstraction $\lambda x.M$. Why?
3. Does the term $(\lambda x.xx)(\lambda x.xx)$ have a normal form?

The β-reduction graph of a term M, written $G_\beta(M)$, is the set:

$$\{N \mid M \to_\beta^* N\}$$

directed by $\to_\beta$. If several redexes give rise to $M_0 \to_\beta M_1$, then that many directed arcs connect M_0 to M_1.

Example
$G_\beta(WWW)$ with $W \equiv \lambda xy.xyy$ is:

$$\begin{tikzcd}
WWW \arrow[d, shift left] & (\lambda y.yyy)W \arrow[l] \\
(\lambda y.Wyy)W \arrow[u, shift left] \arrow[r, shift left] & (\lambda y.(\lambda z.yzz)y)W \arrow[l, shift left] \arrow[u]
\end{tikzcd}$$

Reduction graph examples

Exercise: Draw the reduction graph for $(II)(II)$, where $I = \lambda x.x$.

$$
\begin{tikzcd}
 & (II)(II) \arrow[dl] \arrow[dr] & \\
I(II) \arrow[dr, "*"'] & & III \arrow[dl] \\
 & II \arrow[d] & \\
 & I &
\end{tikzcd}
$$

Why is one arrow marked “$*$”?

Reduction graph examples

$G_\beta((\lambda x.xx)(\lambda x.xx))$ is:

$$\begin{tikzcd}
(\lambda x.xx)(\lambda x.xx) \arrow[loop below]
\end{tikzcd}$$

Exercise: Draw $G_\beta((\lambda x.x(II))(II))$, where $I = \lambda x.x$

- Confluence: If $M \to_\beta^* M_1$ and $M \to_\beta^* M_2$ then there exists a term M_3 such that $M_1 \to_\beta^* M_3$ and $M_2 \to_\beta^* M_3$
- Normalisation: there exists a sequence of reductions which terminates
- Strong Normalisation (or Termination): All reduction sequences terminate

- The λ-calculus is confluent but not normalising (or strongly normalising).
- Confluence implies unicity of normal forms: Each λ-term has at most one normal form.

Exercise:
Find a term that is not strongly normalising (i.e. a term that does not terminate).

Strategies for reduction

- There can be many different ways in which a term can be reduced to a normal form (resp. WHNF)
- The choice that we make can make a huge difference in how many reduction steps are needed.
- The leftmost-outermost strategy finds the normal form, if there is one. But it may be inefficient.

Exercise:

Indicate whether the following λ-terms have a normal form:

- $(\lambda x.(\lambda y.yx)z)v$
- $(\lambda x.xxy)(\lambda x.xxy)$

Remark

Most functional programming languages adopt the following ideas:

1. reduce to WHNF (do not reduce under an abstraction). Exercise: Why?
2. evaluate arguments in a specific way

The difference between many functional languages lies in the choice taken for the second point.

Evaluating Arguments

1. Call-by-Value (Applicative order of reduction): evaluate arguments first so that we substitute the reduced terms (avoid duplication of work).
2. Call-by-name (Normal order of reduction): evaluate arguments each time they are needed.
3. Lazy Evaluation: evaluate arguments at most once.

Question:
Which is the best/worst strategy? Give examples to support your claims.

Arithmetic in the λ-calculus: Church Numerals

We can define the natural numbers as follows:

- $\overline{0} = \lambda x.\lambda y.y$
- $\overline{1} = \lambda x.\lambda y.x\ y$
- $\overline{2} = \lambda x.\lambda y.x(x\ y)$
- $\overline{3} = \lambda x.\lambda y.x(x(x\ y))$
- $\ldots$

Using this representation we can define arithmetic functions. Example, $\overline{n} \mapsto \overline{n+1}$, is defined by the λ-term S:

$$\lambda x.\lambda y.\lambda z.y((x\ y)z)$$

To check it:

- $S\overline{n} = (\lambda x.\lambda y.\lambda z.y((x\ y)z))(\lambda x.\lambda y.x \ldots (x(x\ y))$
- $\rightarrow_\beta \lambda y.\lambda z.y((\lambda x.\lambda y.x \ldots (x(x\ y))\ y)z)$
- $\rightarrow^*_\beta \lambda y.\lambda z.y(y \ldots (y(y\ z)) = \overline{n+1}$

In general, to define an arithmetic function

$$f : Nat^k \mapsto Nat$$

we will use a λ-term $\lambda x_1 \ldots x_k.M$, which will be applied to k numbers: $(\lambda x_1 \ldots x_k.M)\overline{n_1} \ldots \overline{n_k}$

For example, the following term defines addition:

$$ADD = \lambda x.\lambda y.\lambda a.\lambda b.(x\ a)(y\ a\ b)$$

Exercise:

Check that this term applied to two numbers computes their sum. Hint: reduce the term $\lambda x.\lambda y.\lambda a.\lambda b.(x\ a)(y\ a\ b)\overline{n}\ \overline{m}$

Exercises:

1. Show that the λ-term MULT = $\lambda x.\lambda y.\lambda z.x(yz)$ applied to two Church numerals m and n computes their product $m \times n$.
2. What does the term $\lambda n.\lambda m.m\ (MULT\ \ n)\ \overline{1}$ compute?

Booleans

We can represent Boolean values:

- $False = \lambda x.\lambda y.y$
- $True = \lambda x.\lambda y.x$

and Boolean functions. The function NOT is defined by

$$NOT = \lambda x.(x\ False)\ True$$

We can check that this definition is correct:

- $NOT\ \ False = (\lambda x.(x\ False)\ True) False$
- $\rightarrow_\beta (False\ False) True$
- $\rightarrow^*_\beta True$

and

- $NOT\ \ True = (\lambda x.(x\ False)\ True) True$
- $\rightarrow_\beta (True\ False) True$
- $\rightarrow^*_\beta False$

Conditionals

The following term implements an if-then-else:

$$IF = \lambda x.\lambda y.\lambda z.(x\ y)z$$

Note that

$$IF\ B\ E_1\ E_2 \to_\beta^* E_1 \text{ if } B = True$$

and

$$IF\ B\ E_1\ E_2 \to_\beta^* E_2 \text{ if } B = False$$

Example:

The function *is-zero?* can be defined as:

$$\lambda n.n(True\ False)True$$

Then

$is\text{-}zero?\ \overline{0} \to_\beta^* True$

and

$is\text{-}zero?\ \overline{n} \to_\beta^* False \quad \text{if } n > 0.$

The cost of computing

- We have seen that different reduction strategies can change the efficiency of the computation (also termination)
- We can transform algorithms into more efficient versions. We look at one way in this course:

  Continuation Passing Style

  Note: tail recursive, or accumulating parameter style.
- Program transformation is a very rich topic. Many open research topics here...

Continuations

- Continuations were originally introduced in the study of semantics of programming languages: to allow the formal definition of *control structures*

  - Jumps in programs are considered a “bad” thing (difficult to reason about)
  - Many constructs allow controlled jumps (conditional, loops, case, etc)

  - Do not allow jumping into the middle of a block or function body

  Continuations allow some of these features to be captured in a “clean” way:

  - Exceptions to leave a block or function early
  - `callcc` allows a point in the program to be “marked”. `throw` returns to that point to continue the evaluation.
- They are an advanced control construct available in some functional languages (notably Standard ML and Scheme).

Continuation Passing Style (CPS)

- The idea of CPS is that every function takes an extra argument: a continuation.
- A continuation is a function which consumes the result of a function, and produces the final answer.
- Thus, a continuation represents the remainder of the current computation.

The simplest way to understand CPS is to think about evaluating a simple functional application.

Example CPS: Factorial

```
fact n = if n==0 then 1 else n*fact(n-1)
fact 4
```


Consider now the CPS form:

```
factcps n k = if n==0 then k 1
                  else factcps (n-1) (\r -> k (n*r))
factcps 4 (\x -> x)
```


The second argument `k` is the continuation.

Exercise:

1. What is the relationship between:

   `k (fact n)` and `factcps n k`

2. What is one *main* difference between `fact` and `factcps`?

Factorial: evaluation

```
fact 4 = if 4==0 then 1 else 4*fact(4-1)
       = 4*fact(3)
       = 4*(if 3==0 then 1 else 3*fact(3-1))
       = 4*(3*fact(2))
  ...  = 4*(3*(2*(1*1)))

factcps 4 (\x -> x)
  = factcps 3 (\r -> (\x -> x) (4*r))
  = factcps 3 (\r -> (4*r))
  = factcps 2 (\r -> (\r -> (4*r)) (3*r))
  = factcps 2 (\r -> (4*(3*r)))
  = factcps 1 (\r -> (4*(3*(2*r))))
  = factcps 0 (\r -> (4*(3*(2*(1*r)))))
  = (\r -> (4*(3*(2*(1*r)))))1
  = (4*(3*2*(1*1)))
```

Tail Recursion

- It is generally well-understood in compiler technology that *tail* recursive programs can be implemented more efficiently (because they can be transformed into a simple loop).
- A well known example: Compare the following two functions:

```
rev [] = []
rev (h:t) = rev t ++ [h]

revacc [] acc = acc
revacc (h:t) acc = revacc t (h:acc)
```

- Nothing remains to be done *after* the recursive call (recall the definition of ++).
- Formally, we can show that `rev l = revacc l []`

```
rev [] = []
rev (h:t) = rev t ++ [h]
```



```
revcps [] k = k []
revcps (h:t) k = revcps t (\r -> k (r++[h]))
```

Exercise: Verify that `rev l = revcps l (\x -> x)`
Note that all the continuations here can be represented by lists: `\x -> x++l` for some list `l`. Thus `revcps` can be simplified to `revacc`.

- The previous examples are part of a rich theory of *program transformation*.
- Many advanced compilers perform this transformation automatically (when possible).

- In addition to eliminating recursion, these transformations add additional control in the form of strategies.
- On a negative note, programs become higher-order, and we might loose termination properties.

Worked Example: factorial again

```
fact n = if n==0 then 1 else n*fact(n-1)
fact 4
```


Consider now the CPS form:

```
factcps n k = if n==0 then k 1
              else factcps (n-1) (\r -> k (n*r))
factcps 4 (\x -> x)
```


We can simplify the continuation:


```
factacc n acc = if n==0 then acc
                else factacc (n-1) (n*acc)
factacc 4 1
```

Other uses of CPS

Many programming languages have features like:

- goto (in pascal like languages)
- longjmp/setjmp in C
- exceptions in Java, Haskell, SML, etc.

which allow for the change of control of a program (to exit the current block).

- Continuations are a way of expressing these issues
- Achieved by passing a stack as a value to functions: this stack allows the state of the computation to be reinstated at any point—we can move to any past state in a safe way.
- Such stacks are known as reified control stacks.

However, this is beyond the scope of this course...

Summary of CPS

- All functions can be written in CPS style.
- Some continuations have nice representations as accumulating parameters.
- Tail recursive functions can be compiled into a loop: more efficient than a recursion.
- Many other program transformation techniques for functional programming

Summary

- The λ-calculus is the foundation of functional programming (and also the foundation of many programming concepts).
- It is possible to program using only the λ-calculus, but easier if we allow data types (pattern matching, richer syntax, etc.)

- Test out examples in the notes, and do exercises.
- Try writing some of the λ-terms in Haskell
- Can you write a data type in Haskell for representing λ-terms?