Integer Linear Programming

Sanjay Dominik Jena

Master of Business Administration

MBA 8419 - Decision Making Technology

Overview of the presentation

- Integrality constraints
  - Definitions and importance
- Applications of integer linear programming

  - Revenue management
    - Airline Company
  - Schedule planning
    - Call center

  - 0-1 Formulations
    - Location problem
    - Product design problem

Integrality constraints

Definitions and importance

Integrality constraint categories

- General integer variables

  Description : Decision that are represented using general discrete variables.

  Examples : Production decisions expressed in number of lots (skus); Assignment decisions (employees $\rightarrow$ schedules); etc.

- Binary variables

  Description : Decision that are represented using discrete variables that can take one of two values, either 0 or 1.

  Examples : Decisions that represent choices; Design decisions; etc.

Integrality constraints

Definitions and importance

Why are these decisions important ?

Example 1 : consider a plan that would call for 1 054.75 chairs to be produced.

2 rounding options :

- 1 054 chairs
- 1 055 chairs

Impact : the production of 1 extra chair has a relatively small marginal impact for the company.

Example 2 : consider a plan that would call for 14.33 houses to be built.

2 rounding options :

- 14 houses
- 15 houses

Impact : the construction of 1 extra house will have a much higher marginal impact for the developer.

Integrality constraints

Definitions and importance

Consider the following optimization problem

$\max z = 100x_1 + 150x_2$

Subject to

$$80x_1 + 40x_2 \leq 400 \quad (1)$$
$$15x_1 + 30x_2 \leq 200 \quad (2)$$
$$x_1, x_2 \geq 0 \quad (3)$$
$$x_1, x_2 \text{ integers} \quad (4)$$

If the problem is solved by excluding the integrality requirements :

$x_1 = 2,222$, $x_2 = 5,555$ et $z = 1055,556$

Simple solution method

1. Find all the rounded solutions
2. Identify the best integer solutions

Integrality constraints

Definitions and importance

Rounded solutions :

- $(x_1 = 2, x_2 = 5) \Rightarrow$ Feasible and $z = 950$
- $(x_1 = 2, x_2 = 6) \Rightarrow$ Infeasible $15(2) + 30(6) = 210 \not\leq 200$
- $(x_1 = 3, x_2 = 5) \Rightarrow$ Infeasible $80(3) + 40(5) = 440 \not\leq 400$
- $(x_1 = 3, x_2 = 6) \Rightarrow$ Infeasible, constraints (3) and (4)

Only one solution is feasible :

$x_1 = 2$, $x_2 = 5$ et $z = 950$

However,

The optimal solution :

$x_1 = 1$, $x_2 = 6$ et $z = 1000$

Simple methods do not necessarily produce optimal solutions. Also, the number of rounded solutions can grow rapidly.

Applications of integer linear programming

Revenue management

Description : A discipline that aims to understand customers' perception of product value and accurately aligning product prices, placement and availability with each customer segment with the objective of maximizing revenues.

Examples :

- Airline industry
- Railway industry
- Hotels
- etc

Question : How should the rates of products be set such as to maximize the revenues generated ?

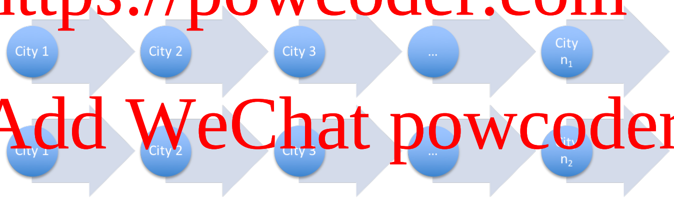
Specificities : pricing strategies vs. overbooking policies vs. managing supplies

Applications of integer linear programming

Revenue management

Aircrafts:

Flight planning:



Etc.

FIGURE – General context - flight planning

Applications of integer linear programming

Revenue management

Leisure Air : Fare type optimization

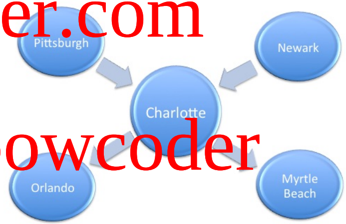
Illustration of the legs

Context :

Ressources :

- Boeing 737-400 (132 seats E)
  Currently in Pittsburgh

- Boeing 737-400 (132 seats E)
  Current in Newark



Operations :

- Leg no.1 : P $\rightarrow$ C,
- Leg no.2 : N $\rightarrow$ C,
- Leg no.3 : C $\rightarrow$ M,
- Leg no.4 : C $\rightarrow$ O.

Applications of integer linear programming

Revenue management

Leisure Air

Context (cont'd) :

The companies proposes 2 types of fares for its economy class :

- discount-fare Q
- full-fare Y

Reservations using the discount-fare Q class must be made 14 days in advance and must include a Saturday night stay in the destination city.

Reservations using the full-fare Y class may be made anytime, with no penalty for changing the reservation at a later date.

The company is interested in planning the itineraries and tarifs that it should propose to its clientele. To determine the itineraries and fares, the company would like to know :

- How many seats should be assigned to each O-D itinerary and fare type ? ODIF $\Rightarrow$ Origin-Destination-Itinerary Fare

Applications of integer linear programming

Revenue management

Tickets, Prices and Predicted Demand

No.	ODIF	Price	Demand
1	PCQ	178$	33
2	PMQ	268$	44
3	POQ	228$	45
4	PCY	380$	16
5	PMY	456$	6
6	POY	560$	11
7	NCQ	199$	26
8	NMQ	249$	56
9	NOQ	349$	39
10	NCY	385$	15
11	NMY	444$	7
12	NOY	580$	9
13	CMQ	179$	64
14	CMY	380$	8
15	COQ	224$	46
16	COY	582$	10

Applications of integer linear programming

Revenue management

Optimization Model

Definitions

- P=Pittsburgh,
- N=Newark,
- C=Charlotte,
- O=Orlando,
- M=Myrtle Beach

Decision Variables

$$
\begin{aligned}
PCQ &= \text{nb. of seats assigned to flight P-C for fare Q}\\
PMQ &= \text{nb. of seats assigned to flight P-M for fare Q}\\
POQ &= \text{nb. of seats assigned to flight P-O for fare Q}\\
PCY &= \text{nb. of seats assigned to flight P-C for fare Y}\\
&\vdots\\
NCQ &= \text{nb. of seats assigned to flight N-C for fare Q}\\
&\vdots\\
COY &= \text{nb. of seats assigned to flight C-O for fare Y}
\end{aligned}
$$

Applications of integer linear programming

Revenue management

Optimization Model (cont'd)

Objective Function

$$\max\ 178PCQ + 268PMQ + 228POQ + 380PCY + \ldots + 224COQ + 582COY$$

Subject to

Aircraft capacity

4 Legs

$$
\begin{aligned}
\text{P-C}: &\quad PCQ + PMQ + POQ + PCY + PMY + POY \le 132\\
\text{N-C}: &\quad NCQ + NMQ + NOQ + NCY + NMY + NOY \le 132\\
\text{C-M}: &\quad PMQ + PMY + NMQ + NMY + CMQ + CMY \le 132\\
\text{C-O}: &\quad POQ + POY + NOQ + NOY + COQ + COY \le 132
\end{aligned}
$$

Demands

$PCQ \le 33$	$PMQ \le 44$	$POQ \le 45$	$PCY \le 16$	$PMY \le 6$	$POY \le 11$
$NCQ \le 26$	$NMQ \le 56$	$NOQ \le 39$	$NCY \le 15$	$NMY \le 7$	$NOY \le 9$
$CMQ \le 64$	$CMY \le 8$	$COQ \le 46$	$COY \le 10$		

Non-negativity and integrality for all decision variables

Applications of integer linear programming

Schedule planning

Call Center

- Operators daily work shifts that last 9 hours
- Shifts can start at the beginning of every 3 hour period
- Minimum number of operators for each period :

Period	0-3	3-6	6-9	9-12	12-15	15-18	18-21	21-24
Need	6	4	12	20	20	24	14	14

- Salaries :
  - Base : 75$ per shift of 9h

- Premiums
  - 11 $ for shifts starting at 0h, 3h ou 6h
  - 5 $ for shifts starting at 18h ou 21h

Q : How many operators to hire to start at each of the periods ?

Applications of integer linear programming

Schedule planning

Decision variables

- x_j = nb. of operators that will begin their shifts at hour j, where $j = 0, 3, 6, 9, 12, 15, 18, 21$.

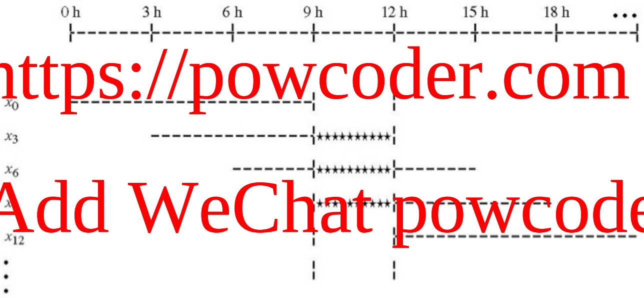


FIGURE – Graphical Representation

Applications of integer linear programming

Schedule planning

Optimization Model

$$\min z = 86x_0 + 86x_3 + 86x_6 + 75x_9 + 75x_{12} + 75x_{15} + 80x_{18} + 80x_{21}$$

subject to

$$x_0 + x_{18} + x_{21} \geq 6$$
$$x_0 + x_3 + x_{21} \geq 4$$
$$x_0 + x_3 + x_6 \geq 12$$
$$x_3 + x_6 + x_9 \geq 20$$
$$x_6 + x_9 + x_{12} \geq 20$$
$$x_9 + x_{12} + x_{15} \geq 24$$
$$x_{12} + x_{15} + x_{18} \geq 14$$
$$x_{15} + x_{18} + x_{21} \geq 14$$

$x_j \geq 0$ and integer for $j = 0, 3, 6, 9, 12, 15, 18, 21$.

Applications of integer linear programming

Schedule planning

Call Center (cont'd)

Description : A company that provides an after-sales service for its clients needs to plan the number of operators to adequately cover the demand over the period of a normal day of operations.

Minimum requirements for switchboard operators

Period	0-1	1-2	2-3	3-4	4-5	5-6	6-7	7-8
Need	6	5	2	2	3	3	4	12
Period	8-9	9-10	10-11	11-12	12-13	13-14	14-15	15-16
Need	20	23	24	24	20	22	24	25
Period	16-17	17-18	18-19	19-20	20-21	21-22	22-23	23-24
Need	22	20	18	16	15	14	9	7

Daily work shift ⇒ 8h. = 7h. of work and 1h. break (meal)
Beginning hour for shifts : 7 h., 8 h., 9 h., 15 h., 16 h., 23 h. and midnight Salaries

- Base ⇒ 80$
- Premiums ⇒ 5$ (shifts beginning at 23 h.) et 10$ (shifts beginning at midnight)

Meals : the meal break can be taken either 3 or 4 hours after the beginning of the shift. However, breaks can only be taken when the company's cafeteria is open.

Cafeteria's opening hours : from 11 h. to 14 h., from 17 h. to 20 h. and from 2 h. to 4 h.

Applications of integer linear programming

Schedule planning

Optimization Model

Decision Variables

Beginning	Break + 3h.	Break + 4h.
7h.	Break scheduled at 10h. ⇒ Caf. closed Impossible	Break scheduled at 11h. ⇒ Caf. opened y_7
8h.	Break scheduled at 11h. ⇒ Caf. opened x_8	Break scheduled at 12h. ⇒ Caf. opened y_8
9h.	Break scheduled at 12h. ⇒ Caf. opened x_9	Break scheduled at 13h. ⇒ Caf. opened y_9
⋮	⋮	⋮

Therefore,

Examples

x_8 = nb. of operators that will begin their shifts at 8 h. and that will take a meal break between 11 h. and 12 h.

y_8 = nb. of operators that will begin their shifts at 8 h. and that will take a meal break between 12 h. and 13 h.

Applications of integer linear programming

Schedule planning

General Representation

Salaire	90	80	80	80	80	80	80	80	80	85	85		
Variable	x_0	y_7	x_8	y_8	x_9	y_9	x_{15}	y_{15}	x_{16}	x_{23}	y_{23}	Min	Dom
0 h – 1 h	×									×	×	6	
1 h – 2 h	×									×	×	5	0 h – 1 h
2 h – 3 h	×										×	2	
3 h – 4 h										×		2	
4 h – 5 h	×									×	×	3	0 h – 1 h
5 h – 6 h	×									×	×	3	0 h – 1 h
6 h – 7 h	×									×	×	4	0 h – 1 h
7 h – 8 h	×	×										12	
8 h – 9 h		×	×	×								20	
9 h – 10 h		×	×	×	×	×						23	11 h – 12 h
10 h – 11 h		×	×	×	×	×						24	11 h – 12 h
11 h – 12 h		×	×	×	×	×						24	
12 h – 13 h		×	×	×	×	×						20	
13 h – 14 h		×	×	×	×	×						22	
14 h – 15 h		×	×	×	×	×						24	11 h – 12 h
15 h – 16 h			×	×	×	×	×	×				25	
16 h – 17 h					×	×	×	×	×			22	
17 h – 18 h							×	×	×			20	
18 h – 19 h								×	×			18	
19 h – 20 h							×					16	
20 h – 21 h							×	×	×			15	19 h – 20 h
21 h – 22 h							×	×	×			14	19 h – 20 h
22 h – 23 h							×	×	×			9	19 h – 20 h
23 h – 24 h									×	×	×	7	

Applications of integer linear programming

Schedule planning

Optimization Model (cont'd)

Objective function

$\min z = 90x_0 + 80(y_7 + x_8 + y_8 + x_9 + y_9 + x_{15} + y_{15} + x_{16}) + 85(x_{23} + y_{23})$

Subject to

$x_0 + x_{23} + y_{23} \geq 6$

$x_0 + y_{23} \geq 2$

$x_{23} \geq 2$

$x_0 + y_7 \geq 12$

$y_7 + x_8 + y_8 \geq 20$

$\vdots$

$x_{16} + x_{23} + y_{23} \geq 7$

Non-negativity and integrality for all decision variables

Applications of integer linear programming

Location coverage planning

Location Problems

Description : The long-range planning department for the Ohio Trust Company bank is considering expanding its operation into a 20-county region in the northeastern Ohio.

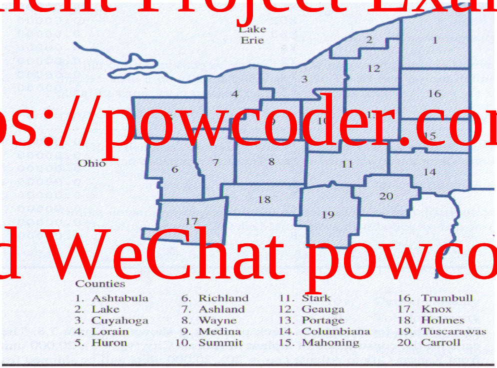


FIGURE – Region considered for expansion

Applications of integer linear programming

Location coverage planning

Description (cont'd) :

Ohio Trust does not have a principal place of business (PPB) in any of the 20 counties.

According to the banking laws in Ohio, if a bank establishes a PPB in any county, branch banks can be established in that county and in any adjacent county.

However, to establish a new PPB, Ohio Trust must either obtained approval for a new bank from the state's superintendent of banks or purchase an existing bank.

Question : Ohio Trust would like to determine the minimum number of PPBs necessary to do business throughout the 20-county region.

Applications of integer linear programming

Location coverage planning

Description (cont'd)

Counties Under Consideration	Adjacent Counties (by Number)
1. Ashtabula	2, 12, 16
2. Lake	1, 3, 12
3. Cuyahoga	2, 4, 9, 10, 12, 13
4. Lorain	3, 5, 7, 9
5. Huron	4, 6, 7
6. Richland	5, [illegible], 1[illegible]
7. Ashland	4, [illegible], 6, 8, 9, 17, 18
8. Wayne	7, 9, 10, 11, 18
9. Medina	3, 4, 7, 8, 10
10. Summit	3, 8, 9, 11, 12, 13
11. Stark	8, 10, 13, 14, 15, 18, 19, 20
12. Geauga	1, 2, 3, 10, 13, 16
13. Portage	3, 10, 11, 12, 15, 16
14. Columbiana	11, 15, 20
15. Mahoning	11, 13, 14, 16
16. Trumbull	1, 12, 13, 15
17. Knox	6, 7, 18
18. Holmes	7, 8, 11, 17, 19
19. Tuscarawas	11, 18, 20
20. Carroll	11, 14, 19

FIGURE – Counties and adjacent ones

Applications of integer linear programming

Location coverage planning

Optimization Model

Decision Variables

x_i = 1 if a PPB is established in county i ; 0 otherwise.

For $i = 1, \ldots, 20$

Fonction objectif :

Minimize the number of PPBs that are necessary to achieve the necessary to cover the considered region

$$\min x_1 + x_2 + \ldots + x_{20}$$

Applications of integer linear programming

Location coverage planning

Optimization Model (cont'd)

Subject to :

Ohio Trust must cover each county to be able to do business :

$$
\begin{array}{rr}
\text{Ashtabula} & x_1 + x_2 + x_{12} + x_{16} \geq 1 \\
\text{Lake} & x_1 + x_2 + x_3 + x_{12} \geq 1 \\
\text{Cuyahoga} & x_2 + x_3 + x_4 + x_9 + x_{10} + x_{12} + x_{13} \geq 1 \\
\vdots & \vdots \\
\text{Carroll} & x_{11} + x_{14} + x_{19} + x_{20} \geq 1
\end{array}
$$

Integrality constraints : $x_i = 0$ or 1, $i = 1, \ldots, 20$

Applications of integer linear programming

Location coverage planning

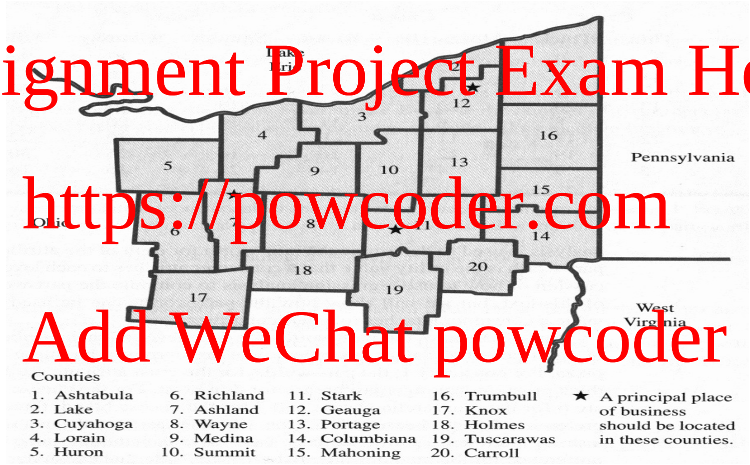


FIGURE – Optimal solution - 3 PPBs

Applications of integer linear programming

0-1 Formulations

Product Design and Market Share Optimization

General Context Technique that can be used to learn how prospective buyers of a product valued the product's attributes.

Salem Foods

Company that is planning to enter the frozen pizza market. There are currently two existing brands, Antonio's and King's, that have the major share of the market.

Four important attributes to define the porduct :

- crust (thin and thick)
- cheese (mozzarella and blend)
- sauce (smooth and chunky)
- sausage (mild, medium and hot)

Applications of integer linear programming

0-1 Formulations

Salem Foods (cont'd)

The two competitors, which are currently in the market, propose the following products :

Description of the proposed pizzas :

Types of pizza	crust	cheese	sauce	sausage
Antonio's	thick	mozzarella	chunky	medium
King's	thin	blend	smooth	mild

Applications of integer linear programming

0-1 Formulations

Salem Foods (cont'd)

Part-worths for the Salem Foods Problem

	crust		cheese		sauce		sausage flavor		
Consumer	thin	thick	mozzarella	blend	smooth	chunky	mild	medium	hot
1	11	2	6	7	3	17	26	27	8
2	11	7	15	17	16	26	14	1	10
3	7	5	8	14	16	7	29	16	19
4	13	20	20	17	17	14	25	29	10
5	2	8	6	11	30	20	15	5	12
6	12	17	11	9	2	30	22	12	20
7	9	19	12	16	16	25	30	23	19
8	5	9	4	14	23	16	16	30	3

8 potential consumers expressed their preference (utility) for specially prepared pizzas with chosen levels for the attributes. A regression analysis → part-worth for each of the attribute levels.

Interpretation

ideal pizza

consummer 1	⇒	thin crust	+ cheese blend	+ sauce chunky	+ sausage medium	total utilitity
		11	+ 7	+ 17	+ 27	= 62

pizza Antonio's

consummer 1	⇒	crust thick	+ cheese mozzarella	+ sauce chunky	+ sausage medium	total utility
		2	+ 6	+ 17	+ 27	= 52

pizza King's

consumer 1	⇒	crust thin	+ cheese blend	+ sauce smooth	+ sausage mild	total utility
		11	+ 7	+ 3	+ 26	= 47

Applications of integer linear programming

0-1 Formulations

Salem Foods (cont'd)

General Objective :

1. Salem is interested in designing a pizza which will please potential consumers such that the company will obtain a majority of the market.

2. In order to be profitable for Salem, the proposed pizza will have to generate a maximum utility for the largest number of potential consumers.

Hypothesis : the considered sample of potential consumers is representative of the market that is pursued.

Applications of integer linear programming

0-1 Formulations

Optimization Model

Decision Variables

Product design :

x_{ij} = 1 if Salem chooses level i for attribute j; 0 otherwise

Market share :

y_k = 1 if consumer k chooses the Salem brand, 0 otherwise

Objective Function

The objective for the company is to carve out the highest possible market share.

$$\max y_1 + y_2 + \ldots + y_8$$

Applications of integer linear programming

0-1 Formulations

Optimization Model (cont'd)

Subject to

Product design

Attributes	choice restrictions
crust	$x_{11} + x_{21} = 1$
cheese	$x_{12} + x_{22} = 1$
sauce	$x_{13} + x_{23} = 1$
sausage flavor	$x_{14} + x_{24} + x_{34} = 1$

Applications of integer linear programming

0-1 Formulations

Optimization Model (cont'd)

Subject to (cont'd)

Defining the market share

Example for consumer 1 :

Total utility function for Salem's pizza :

$11x_{11} + 2x_{21} + 6x_{12} + 7x_{22} + 3x_{13} + 17x_{23} + 26x_{14} + 27x_{24} + 8x_{34}$

Joint analysis :

Types of pizza	Total utility
Ideal	62
Antonio's	52*
King's	47

To modify the present choice of consumer 1 :

$11x_{11} + 2x_{21} + 6x_{12} + 7x_{22} + 3x_{13} + 17x_{23} + 26x_{14} + 27x_{24} + 8x_{34} > 52$

Therefore,

$11x_{11} + 2x_{21} + 6x_{12} + 7x_{22} + 3x_{13} + 17x_{23} + 26x_{14} + 27x_{24} + 8x_{34} \geq 1 + 52y_1$

Applications of integer linear programming

0-1 Formulations

Optimization Model (cont'd)

Subject to (cont'd)

Consumer	pizza	Constraint
1	Antonio's	$11x_{11} + 2x_{21} + 6x_{12} + 7x_{22} + 3x_{13} + 17x_{23} + 26x_{14} + 27x_{24} + 8x_{34} \geq 1 + 52y_1$
2	King's	$11x_{11} + 7x_{21} + 15x_{12} + 17x_{22} + 16x_{13} + 26x_{23} + 14x_{14} + 1x_{24} + 10x_{34} \geq 1 + 58y_2$
3	King's	$7x_{11} + 5x_{21} + 8x_{12} + 14x_{22} + 16x_{13} + 7x_{23} + 29x_{14} + 16x_{24} + 19x_{34} \geq 1 + 66y_3$
4	Antonio's	$13x_{11} + 20x_{21} + 20x_{12} + 17x_{22} + 17x_{13} + 14x_{23} + 25x_{14} + 29x_{24} + 10x_{34} \geq 1 + 83y_4$
5	King's	$2x_{11} + 8x_{21} + 6x_{12} + 11x_{22} + 30x_{13} + 20x_{23} + 15x_{14} + 5x_{24} + 12x_{34} \geq 1 + 58y_5$
6	Antonio's	$12x_{11} + 17x_{21} + 11x_{12} + 9x_{22} + 2x_{13} + 30x_{23} + 22x_{14} + 12x_{24} + 20x_{34} \geq 1 + 70y_6$
7	Antonio's	$9x_{11} + 19x_{21} + 2x_{12} + 16x_{22} + 6x_{13} + 25x_{23} + 30x_{14} + 23x_{24} + 19x_{34} \geq 1 + 79y_7$
8	Antonio's	$5x_{11} + 9x_{21} + 4x_{12} + 14x_{22} + 23x_{13} + 16x_{23} + 16x_{14} + 30x_{24} + 3x_{34} \geq 1 + 59y_8$

Integrality constraints

x_{ij} = 0 or 1, for all i and j

y_k = 0 or 1, for $k = 1, \ldots, 8$