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STOCHASTIC LOSS RESERVING USING BAYESIAN MCMC MODELS,

2ND EDITION

Glenn Meyers, FCAS, MAAA, CERA, Ph.D.

CASUALTY ACTUARIAL SOCIETY

The emergence of Bayesian Markov Chain Monte-Carlo (MCMC) models has provided actuaries with an unprecedented flexibility in stochastic model development. Another recent development has been the posting of a database on the CAS website that consists of hundreds of loss development triangles with outcomes. This monograph begins by first testing the performance of the Mack model on incurred data, and the Bootstrap Overdispersed Poisson model on paid data. It then proposes Bayesian MCMC models that improve the performance over the above models. The features examined include (1) recognizing correlation between accident years in incurred data, (2) allowing for a change in the claim settlement rate in paid data, and (3) a unified model combining paid and incurred data. This monograph continues with an investigation of dependencies between lines of insurance and proposes a way to calculate a cost of capital risk margin.

Keywords. Stochastic loss reserving, Bayesian MCMC models

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About the Author

Glenn Meyers, FCAS, MAAA, CERA, and Ph.D, retired from ISO at the end of 2011 after a 37-year career as an actuary. He holds a B.S. in Mathematics and Physics from Alma College, an M.A. in Mathematics from Oakland University and a Ph.D. in Mathematics from SUNY at Albany. A frequent speaker at Casualty Actuarial Society (CAS) meetings, he has served the CAS by participating on several education and research committees. He has also served on the CAS Board of Directors. His research contributions have been recognized by the CAS through his being a three-time winner of the Woodward-Fondiller Prize, a two-time winner of the Dorweiler Prize, the Dynamic Financial Analysis Prize, the Ronald Bornhuetter Loss Reserves Prize, the American Risk and Insurance Association Prize, the Charles A. Hachemeister Prize, the Matthew Rodermund Service Award and the Michelbacher Significant Achievement Award. In retirement he continues to spend some of his time on his passion for actuarial research.

Preface to the 2nd Edition

Since the original publication of this monograph in January 2015, a number of events transpired.

- The Bayesian MCMC software, Stan, currently appears to be the software of choice by many CAS members.
- In Meyers (2017) and Meyers (2018) I have done research that applies the techniques described in the original monograph in an attempt to advance the state of the art in (1) quantifying dependencies between lines of insurance and (2) calculating a cost of capital risk margin.
- I have presented this material in over a dozen conferences and workshops. Each time that I did this, I spotted and made improvements in both the presentation and the methodology.
- I have had a number of in-depth exchanges with interested individuals. Out of these exchanges, some very good ideas have emerged.

As time progressed, it became clear that the additions and improvements generated by this ongoing research created inconsistencies between the various publications that could inhibit the adoption of this research. So I asked the CAS if a second edition of the monograph could be done and received the go-ahead. What this edition does is present the highlights of my research in this area in an integrated fashion.

Here is a summary of the major changes.

- The Bayesian MCMC modeling has been done using the Stan software instead of JAGS.
- My research on dependencies and risk margins is included.
- The set of loss triangles analyzed has changed. In the original edition, there were a number of loss triangles that, had I actually looked at them, would have been discarded. The triangle selection process for this edition was more rigorous.
- There is a different set of models. The new models include a Bayesian MCMC version of the Cape Cod model and an integrated Paid/Incurred loss model. Gone are the attempts at an incremental paid loss model. (I spent a good amount of time trying to get a new incremental paid model, but my attempts did not yield a model that validated.)
- There is a more thoughtful selection of prior distributions for the Bayesian models.

- The only tests on the first edition are the *p-p* plots. This second edition adds prospective tests and an additional retrospective test for the models. These tests will include a criterion for adding/deleting a particular parameter to the model.
- While I hope that the conclusions in this monograph, and the reasoning behind these conclusions, will be understandable by most actuaries, I removed some of the text designed to introduce actuaries to the tools needed to actually perform the analyses. There is a lot of good introductory material out there that interested actuaries can find quickly. The http://mc-stan.org Stan website is a good place to start.

1. Introduction

The attempts to apply enterprise risk management principles to insurance has placed a high degree of importance on quantifying the uncertainty in the various necessary estimates with stochastic models. For general insurers, the most important liability is the reserve for unpaid losses. Over the years a number of stochastic models have been developed to address this problem. Two of the more prominent non-proprietary models are those of Mack (1993, 1994) and England and Verrall (2002).

While these and other models provide predictive distributions of the outcomes, very little work has been done to retrospectively test, or validate, the performance of these models in an organized fashion on a large number of insurers. In 2011, with the permission of the National Association of Insurance Commissioners (NAIC), Peng Shi and I, in Meyers and Shi (2011), were able to assemble a database consisting of a large number of Schedule P triangles for six lines of insurance. These triangles came from insurer NAIC Annual Statements reported in 1997. Using subsequent annual statements we "completed the triangle" so that we could examine the outcomes and validate the predictive distribution for any proposed model.

Sections 3 and 4 attempt to validate the models of Mack (1993, 1994) and England and Verrall (2002). As it turns out, these models do not accurately predict the distribution of outcomes for the data included in the subject database. Explanation for these results include the following.

- The insurance loss environment is too dynamic to be captured in a single stochastic loss reserve model, i.e., there could be different "black swan" events that invalidate any attempt to model loss reserves.[1]
- There could be other models that better fit the existing data.
- The data used to calibrate the model is missing crucial information needed to make a reliable prediction. Examples of such changes could include changes in the way the underlying business is conducted, such as changes in claim processes or changes in the direct/ceded/assumed reinsurance composition of the claim values in triangles.

One way to rule out the first item above is to (1) find a better model; and/or (2) find better data. This monograph examines a number of different models and data

[1] The term "black swan," as popularized by Taleb (2007), has come to be an oft-used term representing a rare high-impact event.

Here is the situation we now face. First, we are able to construct a wide variety of proposed models and predict their distribution of outcomes with the Bayesian MCMC methodology. Second, we are able to validate a proposed stochastic loss reserve model using a large number of insurers on the CAS Loss Reserve Database. If the insurance loss environment is not dominated by a series of unique "black swan" events, it should be possible to systematically search for models and data that successfully validate. This monograph describes the results I have obtained to date in my pursuit of this goal.

Before introducing Bayesian MCMC models, this monograph will start with examining the Mack (1993, 1994) and the England and Verrall (2002) models. It will identify shortcomings based on the validation results on the holdout data. It will then apply the Bayesian MCMC methodology to proposed models that validate on the holdout data.

While I believe I have made significant progress in identifying models that do successfully validate on the data I selected from the CAS Loss Reserve Database, it should be stressed that more work needs to be done to confirm/reject these results for different data taken from different time periods.

The intended audience for this monograph consists of general insurance actuaries who are familiar with the Mack (1993, 1994) and the England and Verrall (2002) models. While I hope that most sections will be readable by a "generalist" actuary, those desiring a deeper understanding should work with the companion R/Stan scripts to this monograph.[3]

The computer script used to implement these models is written in the R programming language. To implement the MCMC calculations the R script contains another script that is written in Stan. Like R, Stan is an open source programming language one can download for free. For readers who are not familiar with R and Stan, here are some links to help the reader get started.

- http://r-project.org The home page of the R-Project.
- http://mc-stan.org A link to the Stan home page. This website provides instructions for installing and for getting started with Stan.
- http://www.rstudio.com/ A currently popular editor for R scripts.

[3] These scripts are available at https://www.casact.org/pubs/monographs/meyers/Appendix.zip.

2. The CAS Loss Reserve Database

In order to validate a model, one needs not only the data used to build the model, but also the data with outcomes that the model was built to predict. Schedule P of the NAIC Annual Statement contains insurer-level run-off triangles of aggregated losses by line of insurance. Triangles for both paid and incurred losses (net of reinsurance) are reported in Schedule P.[4] To get the outcomes, one must look at subsequent Annual Statements.

To illustrate the calculations described in this monograph, I selected incurred and paid loss triangles from insurers in the database. The data from one of these insurers, insurer group #353 for Commercial Auto, are in Tables 2.1, 2.2, and 2.3. The data in the loss triangles above the diagonal lines are available in the 1997 Annual Statement. These data are used to build the models discussed below. The outcome data below the diagonal lines were extracted, by row, from the Annual Statements listed in the "Source" column. These data are used to validate the models.

The database, along with a complete description of how it was constructed and how the insurers were selected, is available on the CAS website at http://www.casact.org/research/index.cfm?fa=loss_reserves_data.

This monograph will fit various loss reserve models, and test the predictive distributions, to a set of 200 insurer loss triangles taken from four Schedule P (50 from each of Commercial Auto, Personal Auto, Workers' Compensation and Other Liability) lines of insurance. An underlying assumption of these models is that there have not been any substantial changes in the insurer's operation. In our real world, insurers are always tinkering with their operations. Schedule P provides two hints of possible insurer operational changes.

- Changes in the net premium from year-to-year
- Changes in the ratio of net to direct premium from year to year

The criteria for selecting the 200 insurer loss triangles rests mainly on controlling for changes in the above two items. The Appendix gives the group codes for the selected insurers by line of insurance and gives a detailed description of the selection algorithm.

Key summary statistics from all the models considered in the monograph will be available from the CAS website at https://www.casact.org/pubs/monographs/meyers/Appendix.zip.

[4] Paid losses are reported in Part 3 of Schedule P. Incurred losses are the losses reported in Part 2 minus those reported in Part 4 of Schedule P.

Table 2.1. Illustrative Loss Triangle Net Written Premium

AY	1	2	3	4	5	6	7	8	9	10
Premium	5812	4908	5454	5165	5214	5230	4992	5466	5226	4962

Table 2.2. Paid Illustrative Loss Triangle Net of Reinsurance

AY \ Lag	1	2	3	4	5	6	7	8	9	10	Source
1988	952	1529	2813	3647	3724	3832	3899	3907	3911	3912	1997
1989	849	1564	2202	2432	2468	2487	2513	2526	2531	2527	1998
1990	983	2211	2830	3832	4039	4065	4102	4155	4268	4274	1999
1991	1657	2685	3169	3600	3900	4320	4332	4338	4341	4341	2000
1992	932	1940	2626	3332	3368	3491	3531	3540	3540	3583	2001
1993	1162	2402	2799	2996	3034	3042	3230	3238	3241	3268	2002
1994	1478	2980	3945	4714	5462	5680	5682	5683	5684	5684	2003
1995	1240	2080	2607	3080	3678	2004	4117	4125	4128	4128	1997
1996	1326	2412	3367	3843	3965	4127	4133	4141	4142	4144	2005
1997	1413	2683	3173	3674	3805	4005	4020	4095	4132	4139	2006

Table 2.3. Incurred Illustrative Loss Triangle Net of Reinsurance

AY \ Lag	1	2	3	4	5	6	7	8	9	10	Source
1988	1722	3830	3603	3835	3873	3895	3918	3918	3917	3917	1997
1989	1581	2192	2528	2533	2528	2530	2534	2541	2538	2532	1998
1990	1834	3009	3488	4000	4105	4087	4112	4170	4271	4279	1999
1991	2305	3473	3713	4018	4295	4334	4343	4340	4342	4341	2000
1992	1832	2625	3086	3493	3521	3563	3542	3541	3541	3587	2001
1993	2289	3160	3154	3204	3190	3206	3351	3289	3267	3268	2002
1994	2881	4254	4841	5176	5551	5689	5683	5688	5684	5684	2003
1995	2489	2956	3382	3755	4148	4123	4126	4127	4128	4128	2004
1996	2541	3307	3789	3973	4031	4157	4143	4142	4144	4144	2005
1997	2203	2934	3608	3977	4040	4121	4147	4155	4183	4181	2006

3. Validating the Mack Model on Incurred Losses

Probably the two most popular nonproprietary stochastic loss reserve models are those of the Mack (1993, 1994) chain ladder model and the England and Verrall (2002) bootstrap ODP model. This section describes an attempt to validate the Mack model on the incurred loss data from several insurers that are included in the CAS database. Validating the bootstrap ODP model will be addressed in the following section.

Let's begin with the classic chain ladder model. Let $C_{w,d}$ denote the accumulated loss amount, either incurred or paid, for accident year, w, and development lag, d, for $1 \le w \le K$ and $1 \le d \le K$. $C_{w,d}$ is known for the "triangle" of data specified by $w + d \le K + 1$. The goal of this model is to estimate the loss amounts in the last column of data, $C_{w,K}$ for $w = 2, \ldots, K$. To use the chain ladder model, one first calculates the age-to-age factors given by

$$f_d = \frac{\sum_{w=1}^{K-d} C_{w,d+1}}{\sum_{w=1}^{K-d} C_{w,d}} \qquad \text{for } d = 1, \cdots, K-1. \tag{3.1}$$

The chain ladder estimate of $C_{w,K}$ is the product of the latest reported loss, $C_{w,K+1-w}$, and the subsequent age-to-age factors $f_{K+1-w} \cdot \ldots \cdot f_{K-1}$. Putting this together we have

$$C_{w,K} = C_{w,K+1-w} \cdot f_{K+1-w} \cdot \ldots \cdot f_{K-1}. \tag{3.2}$$

Taylor (1986, p.40) discusses the origin of the chain ladder model and concludes that "It appears that it probably originated in the accounting literature, and was subsequently absorbed in to, or rediscovered in, the actuarial." He goes on to say that "Of course, one must bear in mind that both the chain ladder model and estimation method are fairly obvious and might have been derived several times in past literature." Taylor believes that the rather whimsical name of the model was first used by Professor R.E. Beard as he championed the method in the early 1970s while working as a consultant to the U.K. Department of Trade.

Mack (1993, 1994) turns the deterministic chain ladder model into a stochastic model by first treating $\tilde{C}_{w,d}$ as a random variable that represents the accumulated loss amount in the (w, d) cell. He then makes three assumptions:[5]

1. $E[\tilde{C}_{w,d+1}|C_{w,1}, \ldots, C_{w,d}] = C_{w,d} \cdot f_d$
2. For any given d, the random variables $\tilde{C}_{v,d}$ and $\tilde{C}_{w,d}$ are independent for $v \neq w$.
3. $Var[\tilde{C}_{w,d+1}|C_{w,1}, \ldots, C_{w,d}] = C_{w,d} \cdot \alpha_d$

The Mack estimate for $E[\tilde{C}_{w,K}]$ for $w = 2, \ldots, K$ is given by

$$\hat{C}_{w,K} = C_{w,K+1-w} \cdot \hat{f}_{K+1-w} \cdot \ldots \cdot \hat{f}_{K-1} \tag{3.3}$$

where

$$\hat{f}_d = \frac{\sum_{w=1}^{K-d} C_{w,d+1}}{\sum_{w=1}^{K-d} C_{w,d}} \tag{3.4}$$

Given his assumptions above, Mack then derives expressions for the standard deviations $SD[\tilde{C}_{w,K}]$ and $SD[\sum_{w=2}^{K} \tilde{C}_{w,K}]$. Table 3.1 applies Mack's expressions to the illustrative insured data in Table 2.3 using the R "ChainLadder" package.

In addition to the loss statistics calculated by the Mack expressions, Table 3.1 contains the outcomes $\{C_{w,10}\}$ from Table 2.3. Following Mack's suggestion, I calculated the percentile of $\sum_{w=1}^{10} C_{w,10}$, assuming a lognormal distribution with matching the mean and the standard deviation.

Taken by itself, an outcome falling in the 86th percentile gives us little information, as that percentile is not unusually high. If the percentile was, say, above the 99.5th percentile, suspicion might be warranted. My intent here is to test the general applicability of the Mack model on incurred loss triangles. To do this I selected 200 incurred loss triangles, 50 each from four different lines of insurance, and calculated the percentile of the $\sum_{w=1}^{10} C_{w,10}$ outcome for each triangle. My criteria for "general applicability of the model" is that these percentiles should be uniformly distributed. And for a sufficiently large sample, uniformity is testable! Klugman et. al. (2012, Section 16.3) describe a variety of tests that can be applied in this case.

Probably the most visual test for uniformity is a plot of a histogram. If the percentiles are uniformly distributed, we should expect the height of the bars to be equal. Unless the sample size is very large, this will rarely be the case because of random fluctuations. A visual test of uniformity that allows one to test for statistical significance is the *p-p* plot combined with the Kolmogorov-Smirnov (KS) test. Here is how it works.

[5] Depending on the context, various quantities, such as $C_{w,d}$, will represent observations, estimates or random variables. In situations where it might not be clear, let's adopt the convention that for a quantity X, $\tilde{X}$ indicates that X is being treated as a random, or simulated, variable, $\hat{X}$ will denote an estimate of X, and a bare X will be treated as a fixed observation or parameter.

Table 3.1. Mack Model Output for the Incurred Illustrative Loss Triangle

w	Estimate	SD	CV	Outcome	Percentile
1	3917	0	0.000	3917	
2	2538	0	0.000	2532	
3	4167	3	0.001	4279	
4	4367	37	0.009	4341	
5	3597	34	0.010	3587	
6	3236	40	0.012	3268	
7	5358	146	0.027	5684	
8	3765	225	0.060	4128	
9	4013	412	0.103	4144	
10	3955	878	0.222	4181	
Total	38914	1057	0.027	40061	86.03

Suppose one has a sample of n predicted percentiles ranging from 0 to 100 and sort them into increasing order. The expected value of these percentiles is given by $\{e_i\} = 100 \cdot \{1/(n+1), 2/(n+1), \ldots, n/(n+1)\}$. One then plots the expected percentiles on the horizontal axis against the sorted predicted percentiles on the vertical axis. If the predicted percentiles are uniformly distributed, we expect this plot to lie along a 45° line. According to the KS test as described by Klugman et. al. (2012, p. 331) one can reject the hypothesis that a set of percentiles $\{p_i\}$ is uniform at the 5% level if $D \equiv \max|p_i - f_i|$ is greater than its critical value, $136/\sqrt{n}$ where $\{f_i\} = 100 \cdot \{1/n, 2/n, \ldots, n/n\}$. This is represented visually on a *p-p* plot by drawing lines at a distance $136/\sqrt{n}$ above and below the 45° line.[6] We reject the hypothesis of uniformity if the *p-p* plot lies outside the band defined by those lines. For the purposes of this monograph, a model will be deemed "validated" if it passes the KS test at the 5% level.

Klugman (2012, p. 332) also discusses a second test of uniformity that is applicable in this situation. The Anderson-Darling (AD) test is similar to the Kolmogorov-Smirnov test, but it is more sensitive to the fit in the extreme values (near the 0^{th} and the 100^{th} percentile) of the distribution. I applied the AD test along with the KS test on the models described in this monograph with the result that almost all AD tests failed. If in the future someone develops a more refined model, we can raise the bar to the more stringent AD test. Until that happens, I think the KS test is the best tool to differentiate between models.

Figure 3.1 shows both histograms and *p-p* plots for simulated data with $n = 100$. The plots labeled "Uniform" illustrate the expected result. The KS D statistic accompanies each *p-p* plot. The " *" indicates that the D statistic is above its critical value.

[6] This is an approximation as $f_i \approx e_i$.

Figure 3.1. *p-p* plots Test for Uniformity

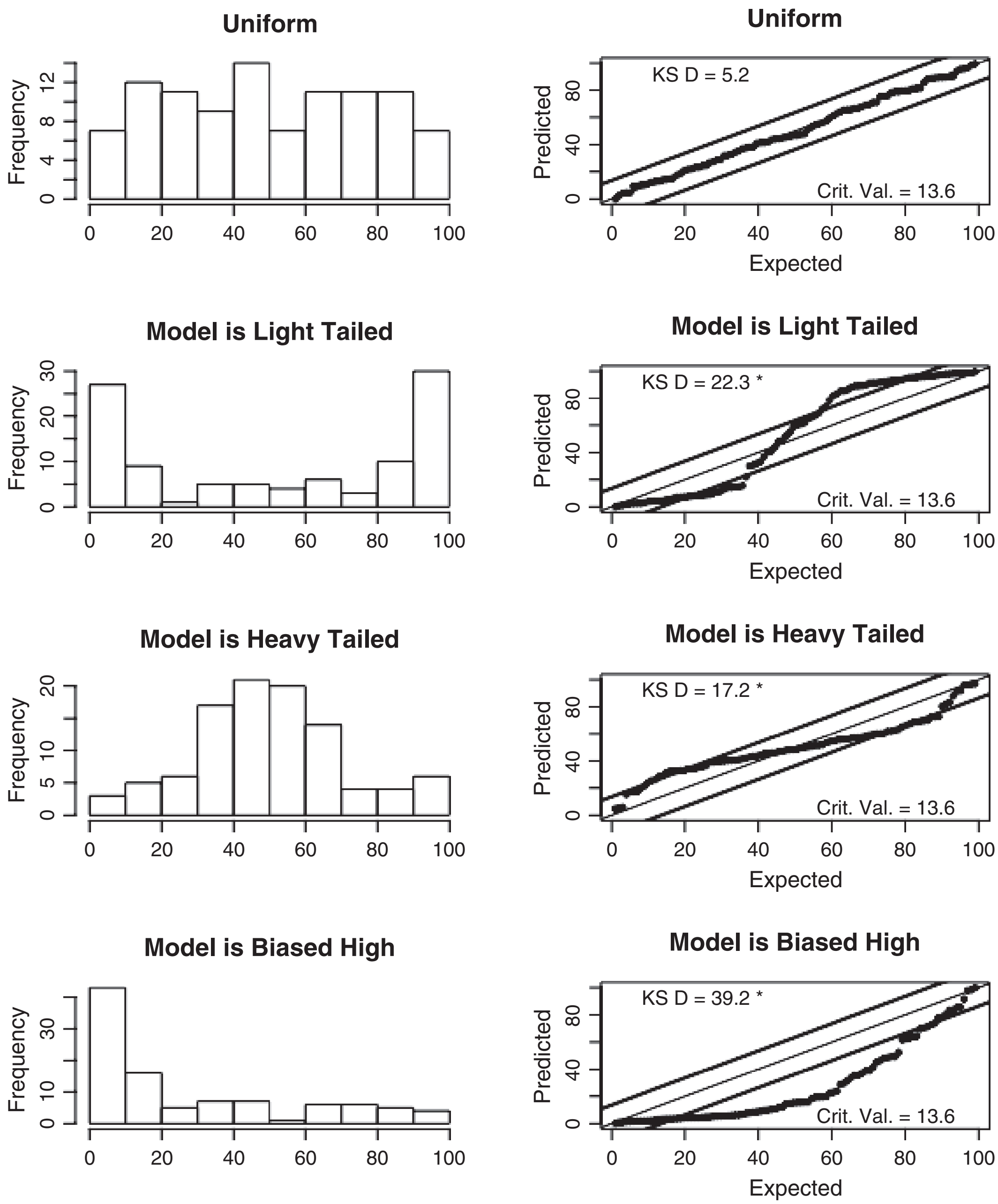
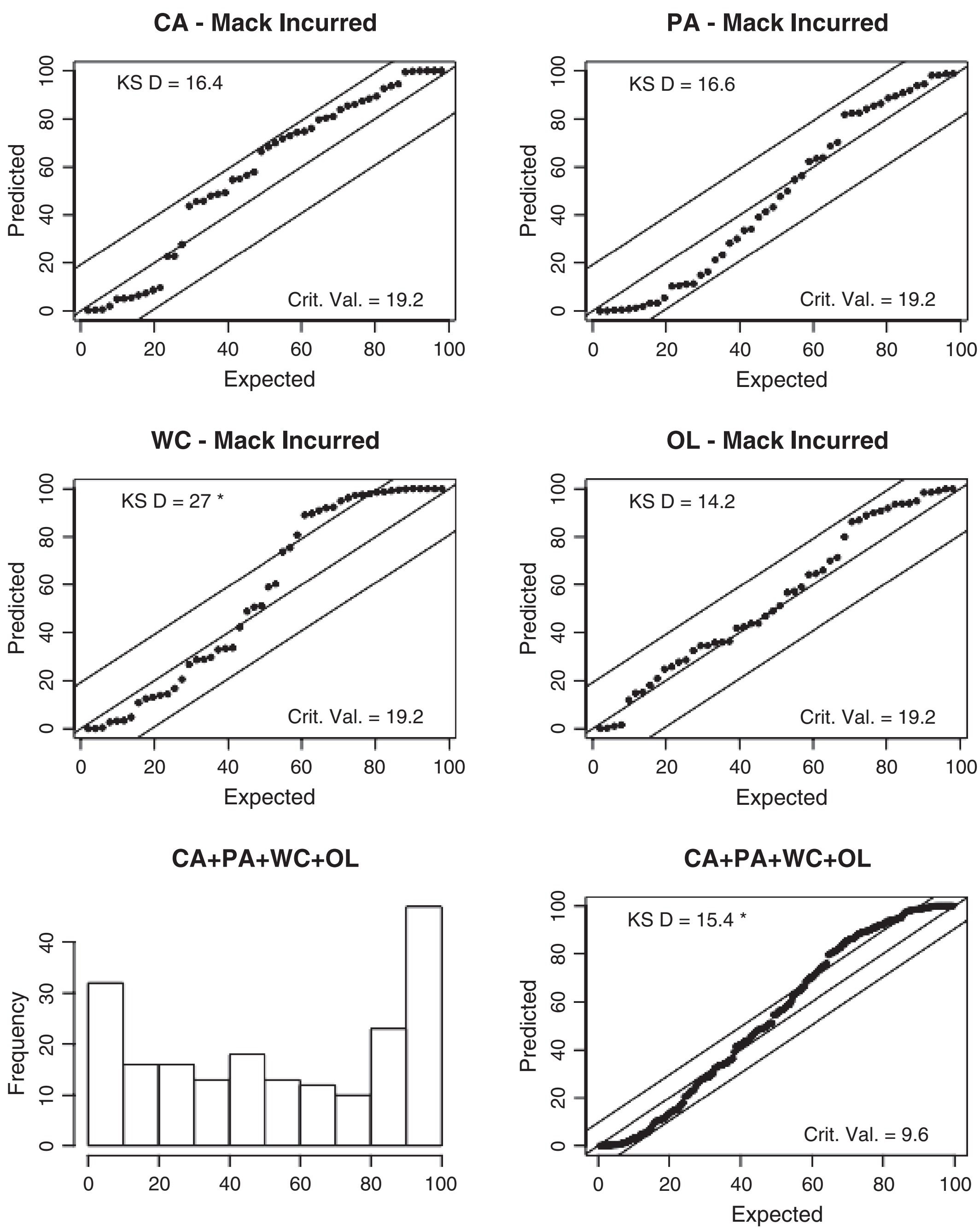


Figure 3.1 also shows *p-p* plots for various departures from uniformity. For example, if the predicted distribution is too light in the tails, there are more than expected high and low percentiles in the predicted outcomes and we see a *p-p* plot that looks like a slanted "S" curve. If the predicted distribution is too heavy in the tails, there are more than expected middle percentiles in the predicted outcomes and we see a *p-p* plot that looks like a slanted backward "S" curve. If the model predicts results that are in general too high, predicted outcomes in the low percentiles will be more frequent.

To validate the Mack model I repeated the calculations for the 200 selected incurred loss reserve triangles.

Figure 3.2 shows the *p-p* plots for the Mack model. The plots were first done separately for the outcome percentiles in each line of insurance. Although the plots fall inside the KS band for three of the four lines, the plots for all four of the lines resemble the slanted "S" curve that is characteristic of a light-tailed predicted distribution. When we combine the outcome percentiles of all four lines, the *p-p* plot lies outside the KS band and we conclude that the distribution predicted by the Mack model is too light in the tails for these data. In all the validation

Figure 3.2. *p-p* plots for the Mack Model on Incurred Loss Triangles



plots below the KS critical values are 19.2 and 9.6 for the individual lines and all lines combined, respectively.

Actuaries might justifiably complain that the performance of a model on different loss triangles from a different time period *may* not be relevant to their current problem. This complaint is duly noted and acknowledged. However, it has been my experience that actuaries choose a model based as much on its reputation as much as its goodness of fit to current data.

Given this history, let's define the term "reputation" to mean the result of a retrospective analysis based on the set of 200 loss triangles taken from the CAS Loss Reserve Database, *within this monograph.* We can then summarize the above conclusion by saying that the Mack model has a reputation for predicting light tails in the distribution of all possible outcomes.

Below, we will introduce a diagnostic for Bayesian MCMC models can be used with the upper triangle data that are available the the time the analyses are being performed.

4. Validating the Bootstrap ODP and Mack Models on Paid Losses

This section does an analysis similar to that done in the last section for the bootstrap ODP model as described by England and Verrall (2002) and implemented by the R "ChainLadder" package. This model was designed to work with incremental losses, $I_{w,d}$, rather than the cumulative losses $C_{w,d}$, where $I_{w,1} = C_{w,1}$ and $I_{w,d} = C_{w,d} - C_{w,d-1}$ for $d > 1$.

A key assumption made by this model is that the incremental losses are described by the overdispersed Poisson distribution with:

$$E\left[\tilde{I}_{w,d}\right] = \alpha_w \cdot \beta_d \quad \textit{and} \quad Var\left[\tilde{I}_{w,d}\right] = \phi \cdot \alpha_w \cdot \beta_d$$

The parameters of the model can be estimated by a standard generalized linear model (GLM) package.[7] They then use a bootstrap resampling procedure to quantify the volatility of the estimate.

England and Verrall point out that the using the ODP model on incremental losses almost all but requires one to use paid, rather than incurred, losses since the over-dispersed Poisson model is defined only for nonnegative losses. Incurred losses include estimates by claims adjusters that can (and frequently do) get adjusted downward. Negative incremental paid losses occasionally occur because of salvage and subrogation, but a feature of the GLM estimation procedure allows for negative incremental losses as long as all column sums of the loss triangle remain positive.

Table 4.1 gives the estimates of the mean, the standard deviation for both the ODP (with 10,000 bootstrap simulations) and Mack models on the data in Table 2.2. The predicted percentiles of the outcomes are also given for each model.

The validation *p-p* plots, similar to those done in the previous section, for both the ODP and the Mack models on paid data are in Figures 4.1 and 4.2. The results for both models are quite similar. Neither model validates well on the paid triangles. A comparison of the *p-p* plots in Figures 4.1 and 4.2 with the illustrative plots in Figure 3.1 suggests that the these models deserve a reputation for overestimating the ultimate losses.

[7] England and Verrall (2002) use a log link function in their GLM. They also note that the GLM for the ODP maximizes the quasi-likelihood, allowing the model to work with continuous (non-integer) losses.

Table 4.1. ODP and Mack Model Output for the Illustrative Loss Triangle Paid Losses

	ODP			Mack			
w	Estimate	SE	CV	Estimate	SE	CV	Outcome
1	3912	0	0	3912	0	0.0000	3912
2	2532	21	0.0083	2532	0	0.0000	2527
3	4163	51	0.0123	4162	3	0.0007	4274
4	4369	85	0.0195	4370	28	0.0064	4341
5	3554	96	0.0270	3555	35	0.0098	3583
6	3211	148	0.0461	3213	157	0.0489	3268
7	5161	240	0.0465	5167	251	0.0486	5684
8	3437	332	0.0966	3442	385	0.1119	4128
9	4220	572	0.1355	4210	750	0.1781	4144
10	4635	1048	0.2261	4616	957	0.2073	4139
Total	39193	1389	0.0354	39177	1442	0.0368	40000
Percentile		73.91			72.02		

Let's now consider the results of this and the prior section. These sections show that two popular models do not validate on outcomes of the 200 Schedule P triangles drawn from the CAS Loss Reserve Database. These models do not validate in different ways when we examine paid and incurred triangles. For incurred triangles, the distribution predicted by the Mack model has a light tail. For paid triangles, the distributions predicted by both the Mack and the bootstrap ODP models tend to produce expected loss estimates that are too high. There are two plausible explanations for these observations.

1. The insurance loss environment has experienced changes that are not observable at the current time.
2. There are other models that can be validated.

To disprove the first explanation, one can develop models that do validate. Failing to develop a model that validates may give credence to, but does not necessarily confirm, that the first explanation is true. This monograph now turns to describing some efforts to find models that do validate.

Figure 4.1. *p-p* Plots for the Bootstrap ODP Model on Paid Loss Triangles

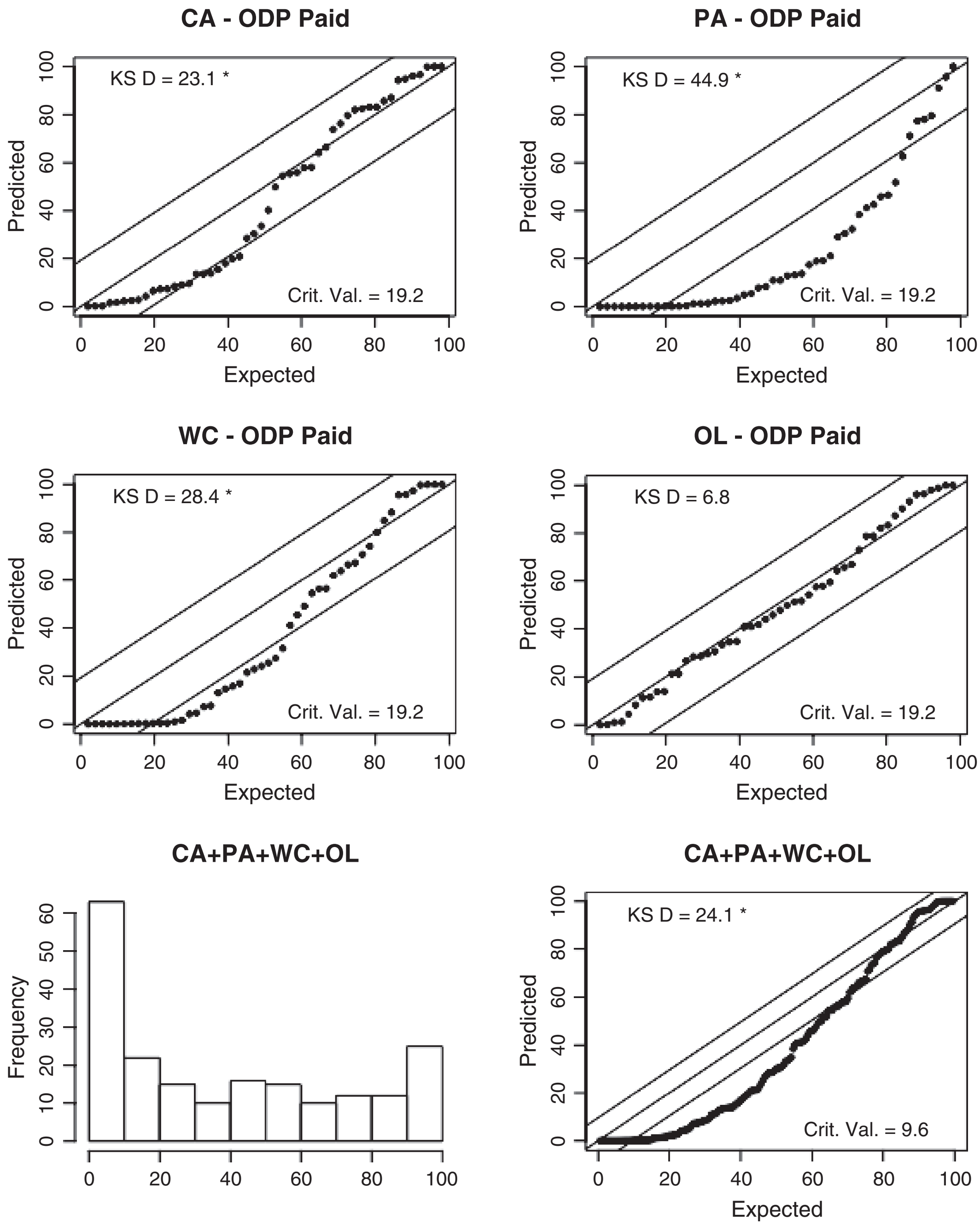
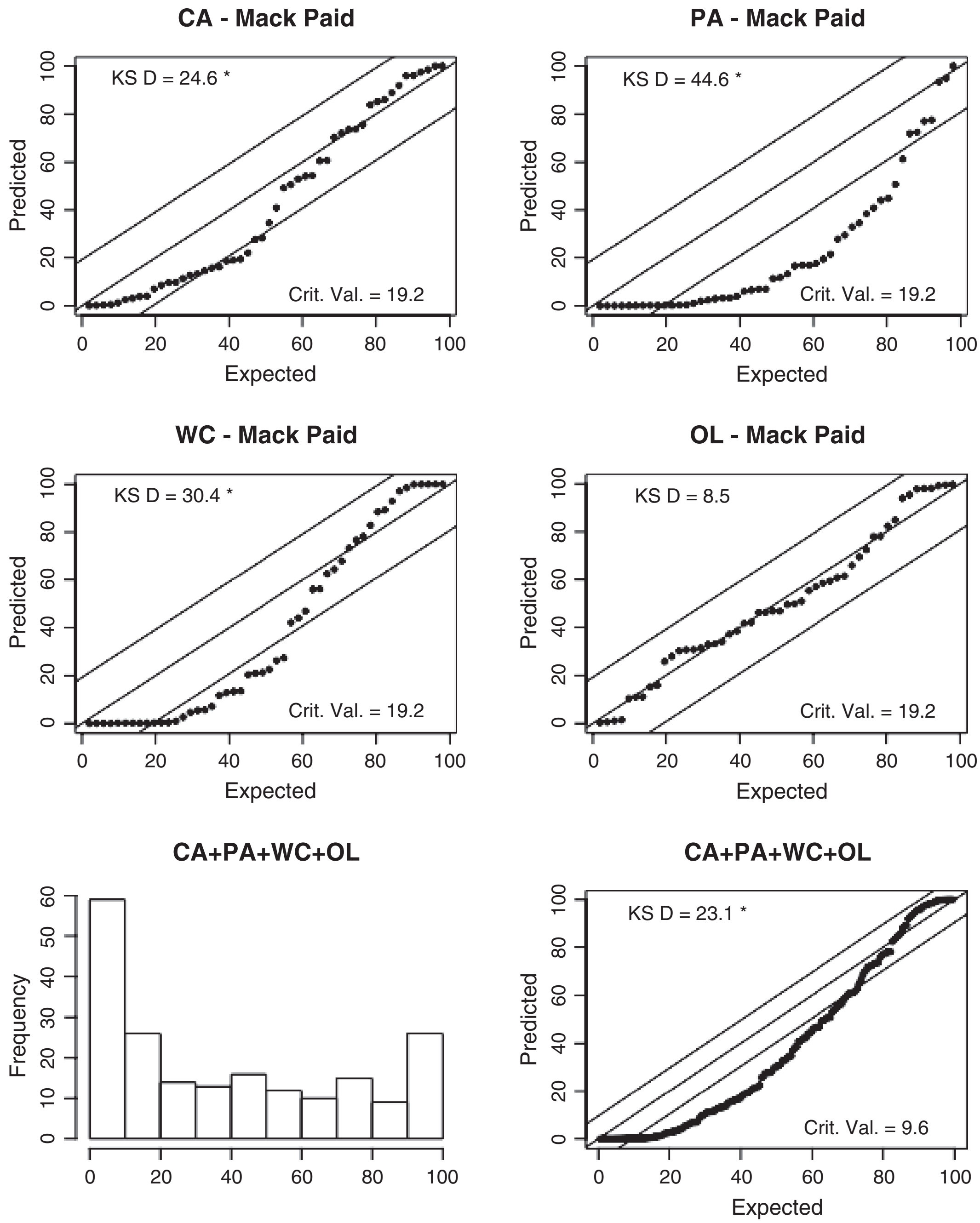


Figure 4.2. *p-p* Plots for the Mack Model on Paid Loss Triangles



5. The Cross Classified Model

The purpose of this section is to introduce a basic Bayesian MCMC that is appropriate for stochastic loss reserving. It assigns an independent parameter for each accident year and development year. The section will then describe some diagnostics to test the model assumptions.

The CRoss Classified (CRC) Model

1. $logelr \sim \text{normal}(-0.4, \sqrt{10}$.
2. $\alpha_w \sim \text{normal}(0, \sqrt{10})$ for $w = 2, \ldots, 10$. Set $\alpha_1 = 0$.
3. $\beta_d \sim \text{normal}(0, \sqrt{10})$ for $d = 1, \ldots, 9$. Set $\beta_{10} = 0$.
4. $a_i \sim \text{uniform}(0, 1)$ for $i = 1, \ldots, 10$.
5. Set $\sigma_d^2 = \sum_{i=d}^{10} a_i$ for $d = 1, \ldots, 10$. Note that this forces $\sigma_1^2 > \cdots > \sigma_{10}^2$.
6. Set $\mu_{w,d} = \log(\text{Premium}_w) + logelr + \alpha_w + \beta_d$.
7. Then $C_{w,d} \sim \text{lognormal}(\mu_{w,d}, \sigma_d)$.

The constraint in line 5 deserves an explanation. The losses $C_{w,d}$ consist of claims with a mixture of settlement dates. The proportion of settled claims increases as the development period, d, increases. Hence the decrease in the variance σ_d^2 as d increases.

Note that there are 29 initial parameters ($logelr$, α_w for $w = 2, \ldots, 10$, β_d, for $d = 1, \ldots, 9$ and a_i for i = 1, . . . , 10) in this model. We will refer to the parameters $\mu_{w,d}$ and σ_d for $w = 1, \ldots, 10$ and $d = 1, \ldots, 10$ as transformed parameters.

As this is the first Bayesian MCMC model in this monograph, let me describe my underlying philosophy that has evolved over the years as I have been building such models. This evolution came from successes, failures and advice from many sources.

- I regard the prior distributions I choose for the model are not a statement of my prior belief, but should be regarded as a feature of the model. The selection of the form of the model is every bit as subjective as the selection of the prior distributions.
- Whenever possible, I try to formulate models and model parameters in terms of quantities that are familiar to the intended users of the model. For example, I expect users will have some familiarity with the expected loss ratio. Given that $\beta_{10} = 0$, the user will have no trouble in interpreting the final loss ratio in the CRC model at $w = 1$ and $d = 10$ as approximately e^{logelr}. Recall that $\alpha_1 = 0$ and, as we shall see, σ_{10} is usually small. The magnitude of the α_ws gives an indication of how much the loss ratio varies from year to year.

- The prior distributions I choose are wider that what I personally believe. One should leave room for surprises.
- But on the other hand, I do not like improper priors, or priors that are heavy-tailed in regions that I consider impossible. Major (2017) shows examples of weird behavior that can occur with heavy-tailed priors.
- Technical note—Often for numerical purposes, I avoid the use of hard boundaries in my choice of prior distributions. Sometimes this can be tricky. As an example, Line 4 in the CRC model above may seem like a contradiction to this practice, but here is what I did in the Stan script. The initial parameter, which I call a_i^{ig} was sampled from an inverse gamma distribution. The transformed parameter a_i is then set equal to the cumulative probability of $1/a_i^{ig}$ from the corresponding gamma distribution, with the result that the prior distribution of a_i is uniformly distributed. The reason for this is, the a_i can often be very close to zero, causing the Stan software to issue warnings. This transformation gives the sampler more room to work with.

Given the CRC model and the data for the illustrative insurer, I used the Stan software to produce a sample of size 10,000 from the posterior distribution of the model. Table 5.1 gives the mean and standard deviation for the posterior distribution of the key parameters in that sample.

Table 5.1. CRC Model Parameter Summary for the Paid Illustrative Loss Triangle

	Mean	Std. Dev.		Mean	Std. Dev.
logelr	−0.3965	0.0233	β_6	−0.0170	0.0413
α_1	0.0000	0.0000	β_7	−0.0060	0.0388
α_2	−0.2541	0.0283	β_8	−0.0038	0.0377
α_3	0.1217	0.0326	β_9	−0.0056	0.0351
α_4	0.2152	0.0395	β_{10}	0.0000	0.0000
α_5	0.0149	0.0466	σ_1	0.2965	0.1034
α_6	−0.0343	0.0637	σ_2	0.2073	0.0543
α_7	0.4354	0.0775	σ_3	0.1334	0.0374
α_8	−0.0199	0.1161	σ_4	0.0946	0.0293
α_9	0.2060	0.1813	σ_5	0.0730	0.0249
α_{10}	0.3435	0.3316	σ_6	0.0576	0.0219
β_1	−1.1999	0.1156	σ_7	0.0472	0.0197
β_2	−0.5751	0.0839	σ_8	0.0384	0.0175
β_3	−0.2825	0.0607	σ_9	0.0300	0.0152
β_4	−0.0954	0.0509	σ_{10}	0.0202	0.0128
β_5	−0.0628	0.0461			

Table 5.2. CRC Model Output for the Paid Illustrative Loss Triangle

w	Premium	Estimate	SE	CV	Outcome	Percentile
1	5812	3912	0	0.0000	3912	
2	4908	2564	114	0.0445	2527	
3	5454	4149	193	0.0465	4274	
4	5165	4315	223	0.0517	4341	
5	5214	3566	203	0.0569	3583	
6	5230	3410	249	0.0730	3268	
7	4992	5208	445	0.0854	5684	
8	5466	3630	442	0.1218	4128	
9	5226	4392	817	0.1860	4144	
10	4962	4976	1762	0.3541	4139	
Total	52429	40121	2487	0.0620	40000	51.88

Our initial objective is to obtain the predictive distribution of the loss outcome at development year 10, by accident year and in total for all the accident years. To to this we simulate for each of the 10,000 parameter vectors:

1. Set $\mu_{w,10} = \log(\text{Premium}_w) + logelr + \alpha_w$ for $w = 2, \ldots, 10$.
2. Simulate $\tilde{C}_{w,10} \sim \text{lognormal}(\mu_{w,10}, \sigma_{10})$ for $w = 2, \ldots, 10$.
3. Calculate $\tilde{C}_{Tot,10} = C_{1,10} + \sum_{w=2}^{10} \tilde{C}_{w,10}$.

Table 5.1 gives a summary of the parameters for the illustrative insurer paid losses. Tables 5.2 and 5.3 give the model output for the illustrative insurer paid and incurred

Table 5.3. CRC Model Output for the Incurred Illustrative Loss Triangle

w	Premium	Estimate	SE	CV	Outcome	Percentile
1	5812	3917	0	0.0000	3917	
2	4908	2549	68	0.0267	2532	
3	5454	4110	120	0.0292	4279	
4	5165	4308	139	0.0323	4341	
5	5214	3546	127	0.0358	3587	
6	5230	3329	147	0.0442	3268	
7	4992	5315	284	0.0534	5684	
8	5466	3776	304	0.0805	4128	
9	5226	4203	579	0.1378	4144	
10	4962	4095	1216	0.2969	4181	
Total	52429	39147	1642	0.0419	40061	74.75

losses respectively. Here, the term "Estimate" is defined as the mean of the corresponding simulated outcomes at development year 10.

Our second objective is to calculate the predicted percentile of the outcome. This is done by counting the number of $\tilde{C}_{Tot,10}$s that are less than or equal the actual outcome. These percentiles are shown in the far right columns of Tables 5.2 and 5.3.

Figures 5.1 and 5.2 give the *p-p* plots for the set of 200 selected paid and incurred triangles. The paid CRC model *p-p* plots look a bit worse than the the corresponding plots for the Mack and ODP models, but they indicate a similar pattern as Mack and

Figure 5.1. *p-p* Plots for the CRC Model on Paid Loss Triangles

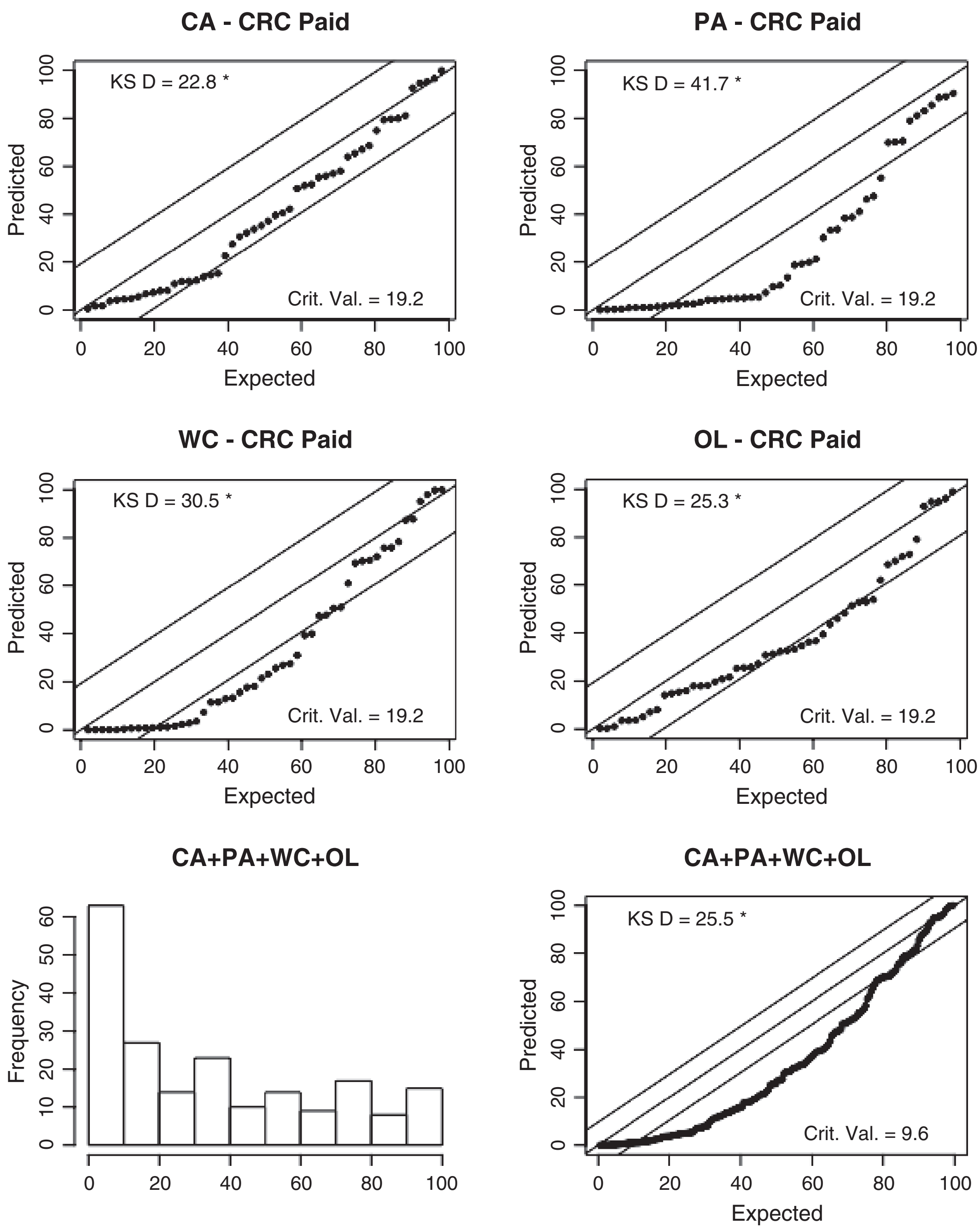
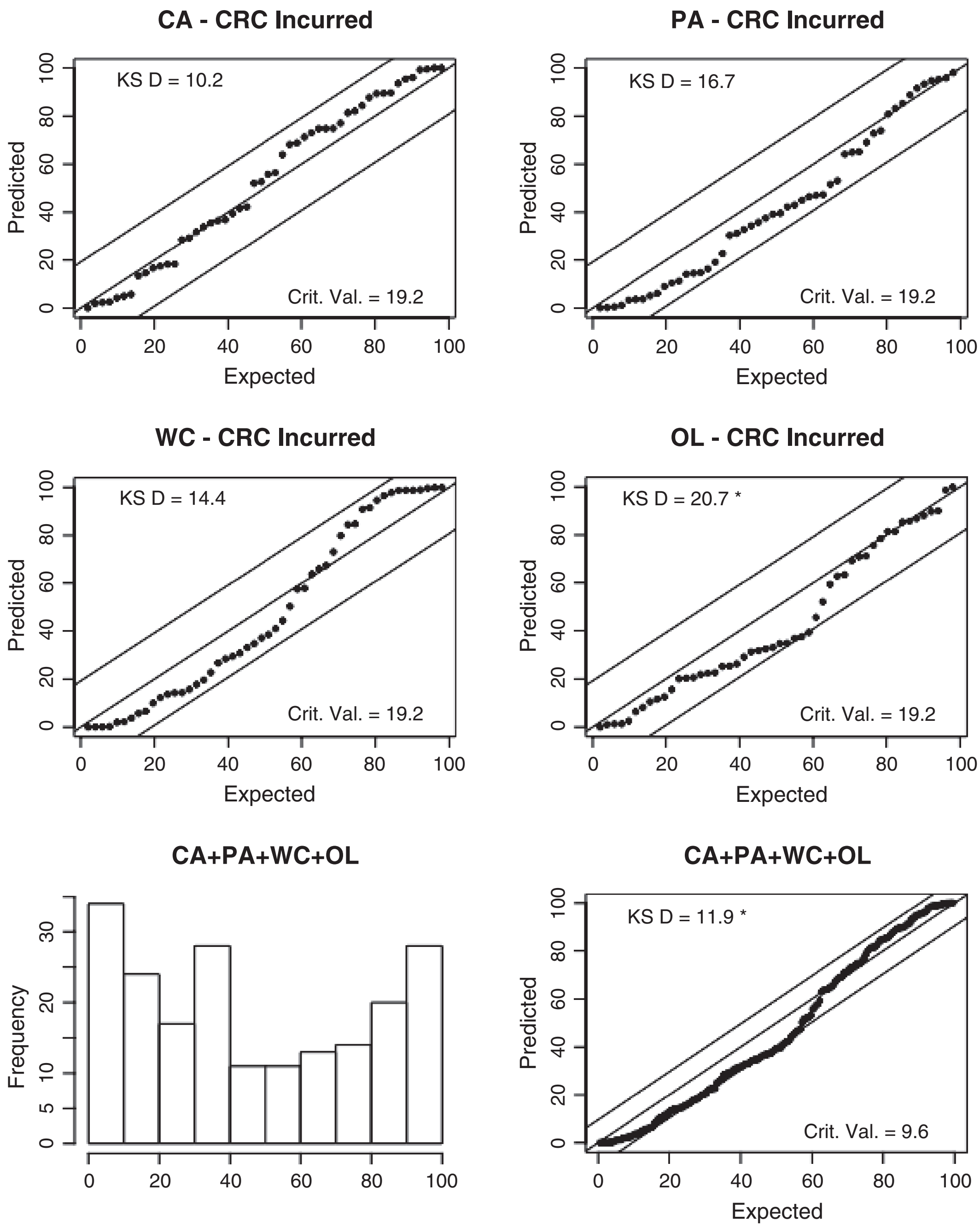


Figure 5.2. *p-p* Plots for the CRC Model on Incurred Loss Triangles



ODP—the paid CRC models share the reputation of predicting losses that are too high. The incurred CRC incurred *p-p* plot is better than the corresponding plot for the Mack model, but it still indicates that it still understates the variability of the predicted losses.

The *p-p* plots in Figures 3.2 – 5.2 are indicators of the reputation of a model. While the reputation of a model is good to know, the actuary who is setting reserves now will want other diagnostics such as residual plots and goodness of fit measures that are applicable to data we have now—the upper triangle. Let's now look at residual plots. We will defer our goodness of fit measure until the next section where we compare the fits of two different models.

Bayesian MCMC models differ from the standard frequentist models in that we have a large sample of parameter vectors, rather than a single parameter vector that arises from, say, a maximum likelihood estimate. To deal with this, one can take a random sample of 100 *j*s and use the corresponding parameter vectors, $\mu_{w,d}^{j}$ and σ_{d}^{j} from the posterior distribution and calculate the standardized residuals, $r_{w,d}^{j}$, for the log of all losses in the training (upper) loss triangle.

$$r_{w,d}^{j} = \frac{\log(C_{w,d}) - \mu_{w,d}^{j}}{\sigma_{d}^{j}} \qquad \text{for } w = 1,\ldots,10 \quad d = 1,\cdots,11-w \text{ and for each } j. \quad (5.1)$$

There are 5,500 values of $r_{w,d}^{j}$. We then put these values into a series of Box plots, organized first by accident year, *w*, and then by development year, *d*.
In this monograph, the Box plot (a.k.a Box and whisker plot) plots the interquartile range as solid bars. The "whiskers" of the plot span the region starting with the top (bottom) plus (minus) 1.5 times the length of the interquartile range. Any points outside the range of the whiskers are plotted individually.

The expected interquartile range for standardized normal residuals are indicated by thin black lines on the plot. In general we should expect the interquartile range of the data to fall pretty close to those black lines.

Figures 5.3 and 5.4 show the standardized residual Box plots for the paid and incurred losses of the CRC model run with the illustrative insurer. They may not seem all that bad to one who has looked at similar plots for other loss reserve models. Generally, one should feel some comfort if the interquartile range contains 0. But I suggest that the reader withhold judgment until we have seen the corresponding plots for other models.

Figure 5.3. CRC Standardized Residual Box Plots for the Paid Illustrative Loss Triangle

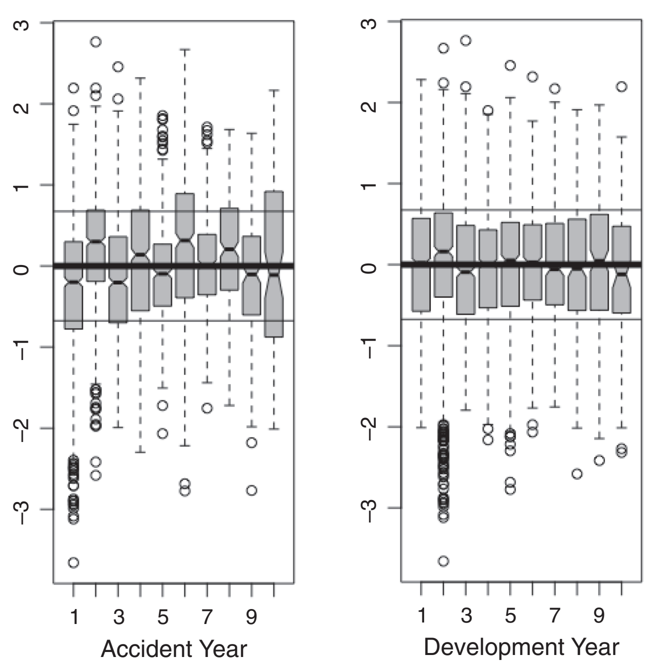
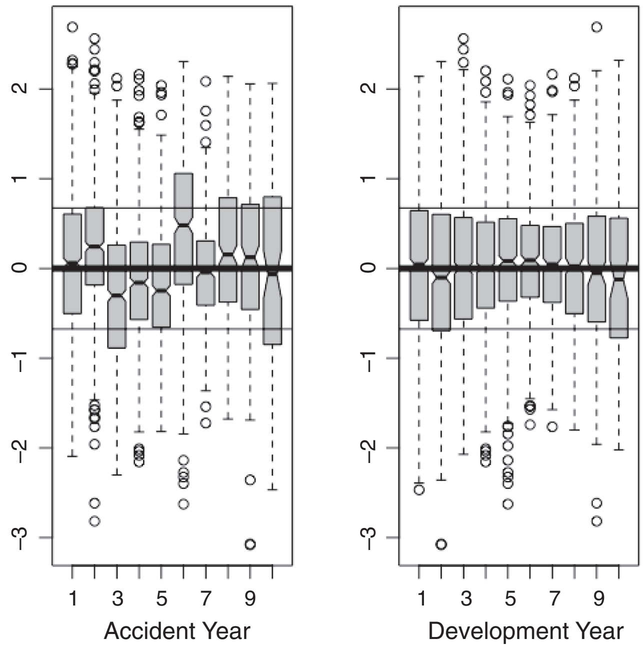


Figure 5.4. CRC Standardized Residual Box Plots for the Incurred Illustrative Loss Triangle



6. The Stochastic Cape Cod Model

One of the most popular loss reserving methodologies is given by Bornhuetter-Ferguson (BF) (1972). A key input to the loss reserve formula given in that paper is the expected loss ratio, which must be judgmentally selected by the actuary. Presentations by Clark (2013) and Leong (2013) suggest that the BF method that assumes a constant loss ratio provides a good fit to industry loss reserve data.

The general idea behind BF is to first use a standard method, e.g., the chain ladder model, to estimate the proportion of losses that are not reported at the end of the most recent development period $11 - w$. Call this quantity, V_{11-w}. Using the notation in this monograph, the projected ultimate loss for accident year w is then given by

$$\hat{C}_{w,10} = C_{w,11-w} + \text{Premium}_w \cdot ELR \cdot V_{11-w} \text{ for } w = 2, \cdots, 10 \tag{6.1}$$

where ELR is the judgmentally selected expected loss ratio.

Over the years many actuaries have been uncomfortable with the sensitivity of the ultimate loss estimate to the judgmentally selected expected loss ratio. To address this concern, James Stanard (1985) and Hans Bühlmann independently, according to Patrik (2001), proposed a model in which the expected loss ratio is estimated from the data. As Bühlmann first proposed this model at a meeting in Cape Cod, it has come to be know as the "Cape Cod" model.

Let's now examine one way to describe a stochastic Cape Cod model.

The Stochastic Cape Cod (SCC) Model

1. $logelr \sim \text{normal}(-0.4, \sqrt{10})$.
2. $\beta_d \sim \text{normal}(1, \sqrt{10})$ for $d = 1, \ldots, 9$. Set $\beta_{10} = 0$.
3. $a_i \sim \text{uniform}(0, 1)$ for $i = 1, \ldots, 10$.
4. Set $\sigma_d^2 = \sum_{i=d}^{10} a_i$ for $d = 1, \ldots, 10$. Note that this forces $\sigma_1^2 > \cdots > \sigma_{10}^2$.
5. Set $\mu_{w,d} = \log(\text{Premium}_w) + logelr + \beta_d$.
6. Then $C_{w,d} \sim \text{lognormal}(\mu_{w,d}, \sigma_d)$.

Given the SCC model and the paid data for the illustrative insurer, I used the Stan software to produce a sample of size 10,000 from the posterior distribution of the model. Table 6.1 gives the mean and standard deviation for the posterior distribution of the relevant parameters in that sample.

Table 6.1. SCC Parameter Summary Paid Illustrative Loss Triangle

	Mean	Std. Dev.		Mean	Std. Dev.
logelr	–0.4033	0.1123	σ_1	0.4608	0.1201
β_1	–1.0897	0.1870	σ_2	0.3691	0.0735
β_2	–0.4926	0.1691	σ_3	0.3183	0.0585
β_3	–0.2155	0.1604	σ_4	0.2853	0.0501
β_4	–0.0170	0.1577	σ_5	0.2579	0.0448
β_5	–0.0439	0.1547	σ_6	0.2351	0.0412
β_6	0.0109	0.1559	σ_7	0.2132	0.0385
β_7	0.0214	0.1565	σ_8	0.1887	0.0368
β_8	–0.0418	0.1586	σ_9	0.1572	0.0372
β_9	–0.1251	0.1603	σ_{10}	0.1051	0.0451
β_{10}	0.0000	0.0000			

Our initial objective is to obtain the predictive distribution of the loss outcome at development year 10, by accident year and in total for all the accident years. To to this we simulate for each of the 10,000 parameter vectors:

1. Set $\mu_{w,10} = \log(\text{Premium}_w) + logelr$ for $w = 2, \ldots, 10$.
2. Simulate $\tilde{C}_{w,10} \sim \text{lognormal}(\mu_{w,10}, \sigma_{10}) - e^{\mu_{w,11-w}+\sigma^2_{11-w}/2} + C_{w,11-w}$ for $w = 2, \ldots, 10$.
3. Calculate $\tilde{C}_{Tot,10} = C_{1,10} + \sum_{w=2}^{10} \tilde{C}_{w,10}$.

Step 2 in the above differs from the corresponding step in the CRC model. It first simulates a loss at development year 10. It then subtracts the expected value of the current reported loss from the SCC model[8] and then adds the current reported loss from the SCC model. Tables 6.2 and 6.3 give the SCC model output for the illustrative insurer.

Figures 6.3 and 6.4 indicate a decidedly worse reputation (as defined Section 3) for the SCC model than for the other models described above. The standardized residual plots in Figures 6.1 and 6.2 suggest that the assumption of a fixed expected loss ratio across accident years may explain the poor performance of the SCC model. More will be said about this poor performance at the end of this section.

Given that we now have two models, we now discuss how we compare models using only the upper triangle data. Let's start the discussion with a review of the Akaike Information Criteria (AIC).

[8] Recall the mean of a lognormal distribution is $e^{\mu+\sigma^2/2}$

Table 6.2. SCC Model Output for the Paid Illustrative Loss Triangle

w	Premium	Estimate	SE	CV	Outcome	Percentile
1	5812	3912	0	0.0000	3912	
2	4908	2905	640	0.2203	2527	
3	5454	4265	724	0.1698	4274	
4	5165	4199	703	0.1674	4341	
5	5214	3376	694	0.2056	3583	
6	5230	3097	690	0.2228	3268	
7	4992	4645	665	0.1432	5684	
8	5466	3180	700	0.2201	4128	
9	5226	3639	657	0.1805	4144	
10	4962	3506	591	0.1686	4139	
Total	52429	36725	3950	0.1075	40000	83.38

Table 6.3. SCC Model Output for the Incurred Illustrative Loss Triangle

w	Premium	Estimate	SE	CV	Outcome	Percentile
1	5812	3917	0	0.0000	3917	
2	4908	2914	627	0.2152	2532	
3	5454	4273	704	0.1648	4279	
4	5165	4213	686	0.1628	4341	
5	5214	3420	675	0.1974	3587	
6	5230	3109	686	0.2206	3268	
7	4992	4907	671	0.1367	5684	
8	5466	3350	723	0.2158	4128	
9	5226	3514	709	0.2018	4144	
10	4962	3319	659	0.1986	4181	
Total	52429	36936	3941	0.1067	40061	82.69

Figure 6.1. SCC Standardized Residual Box Plots for the Paid Illustrative Loss Triangle

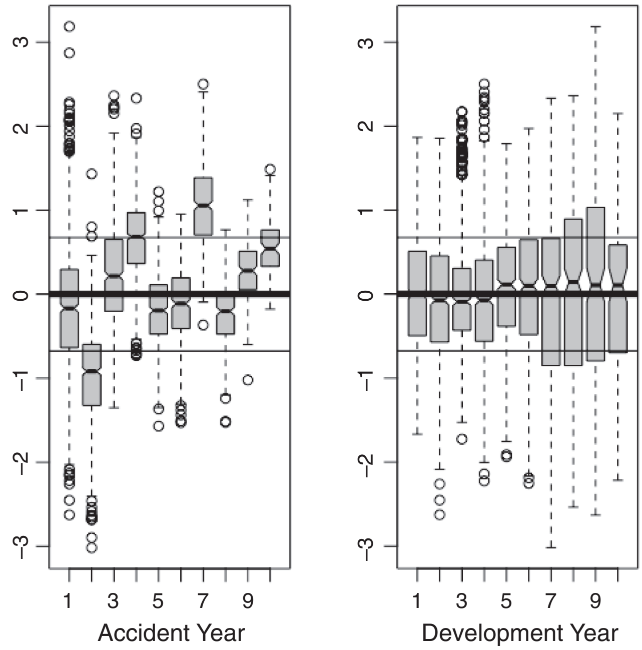


Figure 6.2. SCC Standardized Residual Box Plots for the Incurred Illustrative Loss Triangle

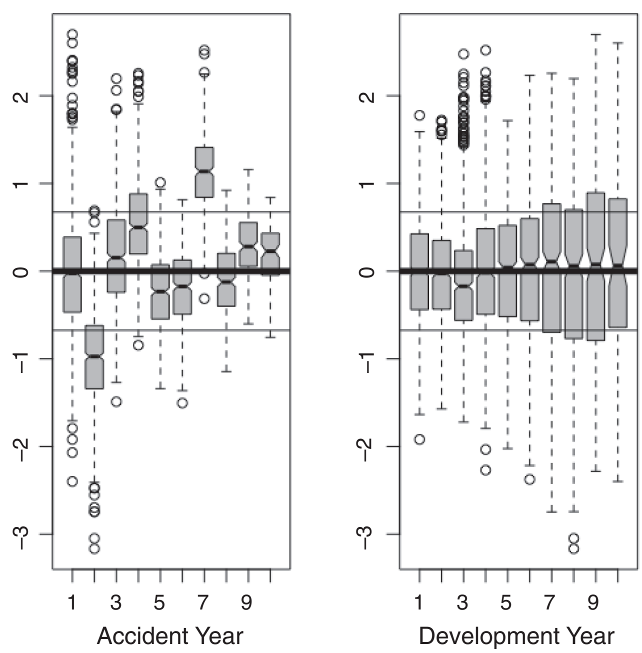


Figure 6.3. *p-p* Plots for the SCC Model on Paid Loss Triangles

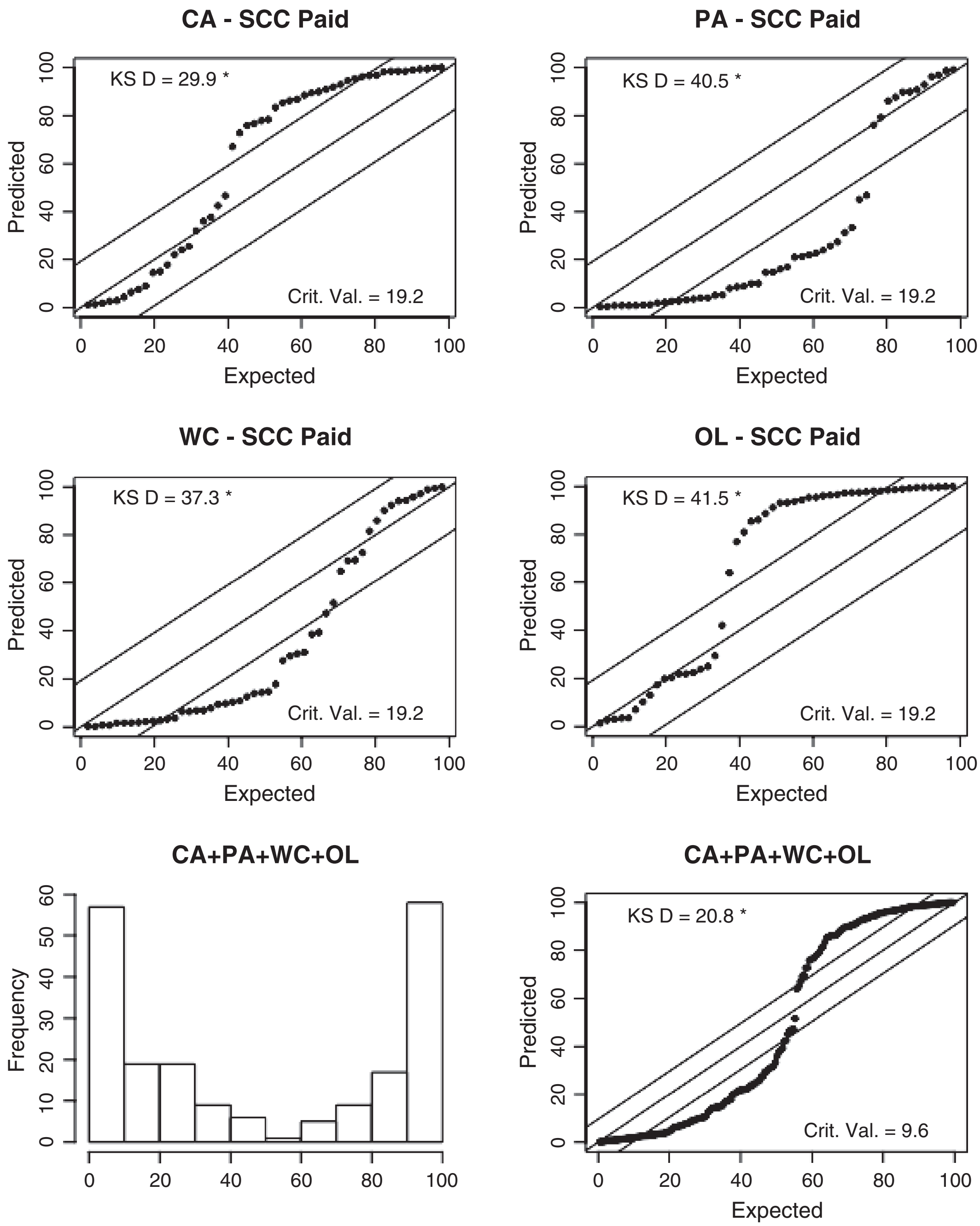
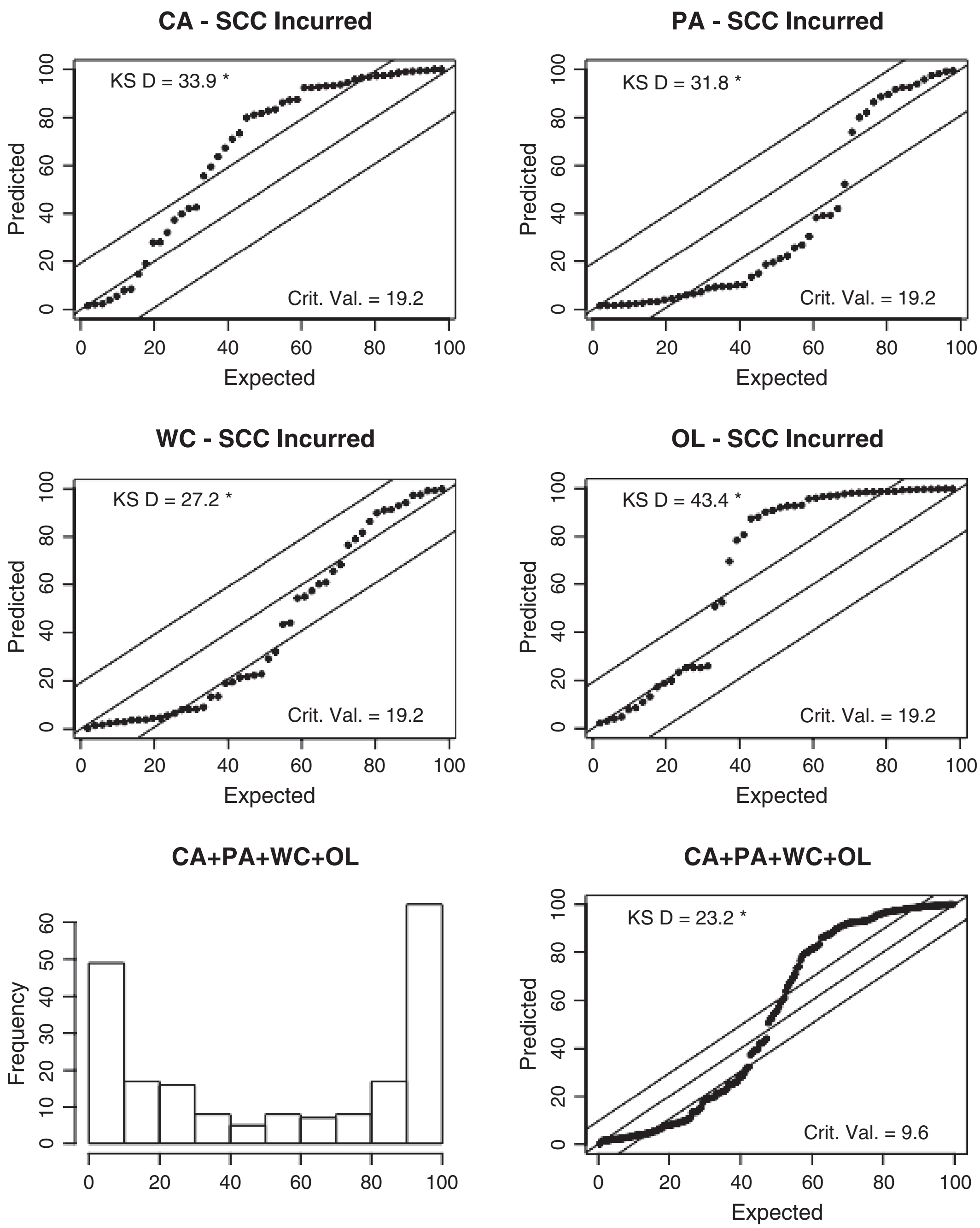


Figure 6.4. *p-p* Plots for the SCC Model on Incurred Loss Triangles



Suppose that we have a model with a data vector, $x = \{x_i\}_{i=1}^N$, and a parameter vector θ, with p parameters. Let $\hat{\theta}$ be the parameter value that maximizes the log-likelihood, L, of the data, **x**. Then the AIC is defined as

$$\text{AIC} = 2 \cdot p - 2 \cdot L(\mathbf{x}|\hat{\theta}) \tag{6.2}$$

Given a choice of models, the model with the lowest AIC is to be preferred. This statistic rewards a model for having a high log-likelihood, but it penalizes the model for having more parameters.

There are problems with the AIC in a Bayesian environment. Instead of a single maximum likelihood estimate of the parameter vector, there is an entire sample of parameter vectors taken from the model's posterior distribution. There is also the sense that the penalty for the number of parameters should not be as great in the presence of strong prior information. To address these concerns, Gelman et. al. (2014, Chapter 7) describe statistics that generalize the AIC in a way that is appropriate for Bayesian MCMC models. Here is a brief overview of one of these statistics.[9]

First, given a stochastic model, $p(x|\theta)$, define the expected log predictive density as

$$elpd = \sum_{i=1}^{I} \log\left(\int p(x_i|\theta) \cdot f(\theta)\, d\theta\right) \tag{6.3}$$

where f is the unknown density of θ.

If $\{\theta_j\}_{j=1}^J$ is a random sample from the posterior distribution of θ, define the computed log predicted density as

$$\widehat{lpd} = \sum_{i=1}^{I} \log\left(\frac{1}{J}\sum_{j=1}^{J} p(x_i|\theta_j)\right) \tag{6.4}$$

Note that if we replace $\{\theta_j\}_{j=1}^J$ with the maximum likelihood estimate, $\hat{\theta}$, $\widehat{lpd}$ is equal to $L(x|\hat{\theta})$ in Equation 6.2.

If the data vector, **x**, come from a holdout sample, i.e., **x** was not used to generate the parameters, $\{\theta_j\}_{j=1}^J$, then the $\widehat{lpd}$ is an unbiased estimate of *elpd*. But if the data vector, **x**, comes from the training sample, i.e., **x** was used to generate the parameters, $\{\theta_j\}_{j=1}^J$, then we expect $\widehat{lpd}$ to be higher than *elpd*. The amount of that bias is called the "effective number of parameters."

Now let's consider what is called "leave one out cross validation" or "loo" for short. For the data point, x_i, one might obtain a sample of parameters $\{\theta^{(-i)}\}$ by an MCMC simulation using all values of **x** except x_i. After doing this calculation for all observed data points in **x**, one can then use Equation 6.4 to calculate an unbiased estimate of the *elpd*.

$$\widehat{elpd}_{loo} = \sum_{i=1}^{I} \log\left(\frac{1}{J}\sum_{j=1}^{J} p\left(x_i\middle|\theta_j^{(-i)}\right)\right) \tag{6.5}$$

[9] Other popular statistics include the Deviance Information Criterion (DIC) and the Wantanabe-Akaike Information Criterion (WAIC). Gelman et. al. (2014, Chapter 7) make the case that the LOOIC is a superior measure.

Table 6.4. Model Comparison Statistics for the Illustrative Loss Triangle

Model	$\widehat{elpd}_{loo}$	p_{loo}	LOOIC
CRC-Paid	47.80	14.97	–95.60
SCC-Paid	–5.14	8.75	10.28
CRC-Incurred	70.97	15.07	–141.93
SCC-Incurred	–2.85	9.13	5.69

While the speed of recent MCMC software packages is impressive, rerunning an MCMC model for each data point would tax the patience of most practitioners. Methods to efficiently calculate $\widehat{elpd}_{loo}$ have been developed. Rather than redo the MCMC simulation, these methods estimate the $\widehat{elpd}_{loo}$ using the $I \times J$ matrix of log-likelihoods of each observation, x_i given each parameter vector, θ_j. Vehtari et. al. (2017) provide the most up-to-date approaches that are incorporated in the R "loo" package. That is what this monograph uses.

When comparing two models, the model with the highest $\widehat{elpd}_{loo}$ should be preferred. For historical reasons, many prefer to state the results on the deviance scale, which similar to that of the AIC in Equation 6.2. This is done by writing

$$\text{LOOIC} = -2 \cdot \widehat{elpd}_{loo} = 2 \cdot p_{loo} - 2 \cdot \widehat{lpd}_{loo} \tag{6.6}$$

Table 6.4 provide these model comparison statistics for the illustrative insurer. These statistics strongly favor the CRC model. Moreover, when you compare the statistics for the models applied to the entire set of 200 loss triangles, the CRC model is favored for all 200 paid and incurred loss triangles.

To be fair, one should note that the Bornhuetter-Ferguson/Cape Cod literature stresses the importance of adjusting the premium a level consistent with the expected losses. No such adjustment was made in the SCC model as presented here. One should view the results above as an indication of the sensitivity of the BF method to the premium adjustment.

That being said, one should also note that the CRC model above, along with the newer models described below allow an actuary to, by the choice of prior distributions, judgmentally influence the expected loss ratio by accident year.

7. The Changing Settlement Rate Model

The *p-p* plots in Figure 5.1 for paid losses and Figure 5.2 for incurred losses indicate that the problem for the CRC model differs for paid and incurred losses. For paid losses, the CRC model tends to overestimate the ultimate losses. For incurred losses, the CRC model tends to understate the variability of the ultimate loss estimate. This section proposes a model that attempts to correct the overestimate of paid ultimate losses by adjusting for a change in the claim settlement rate. The next section attempts to correct for the underestimate of the variability of the ultimate loss estimate for incurred losses.

The Changing Settlement Rate (CSR) Model

1. $logelr \sim \text{normal}(-0.4, \sqrt{10})$.
2. $\alpha_w \sim \text{normal}(0, \sqrt{10})$ for $w = 2, \ldots, 10$. Set $\alpha_1 = 0$.
3. $\beta_d \sim \text{normal}(0, \sqrt{10})$ for $d = 1, \ldots, 9$. Set $\beta_{10} = 0$.
4. $\gamma \sim \text{normal}(0, 0.05)$.
5. $a_i \sim \text{uniform}(0, 1)$ for $i = 1, \ldots, 10$.
6. Set $\sigma_d^2 = \sum_{i=d}^{10} a_i$ for $d = 1, \ldots, 10$. Note that this forces $\sigma_1^2 > \cdots > \sigma_{10.}^2$
7. Set $\mu_{w,d} = \log(\text{Premium}_w) + logelr + \alpha_w + \beta_d \cdot (1 - \gamma)^{w-1}$.
8. Then $C_{w,d} \sim \text{lognormal}(\mu_{w,d}, \sigma_d)$.

The difference between the CSR and the CRC models is seen in line 7 of the above model description. If γ is positive the "log-development factors", i.e., $\beta_d \cdot (1 - \gamma)^{w-1}$, move toward zero for each d as w increases. This indicates a speedup in the claim settlement rate over time. Conversely, a negative γ indicates a slowdown in the claim settlement rate over time.

Note that the CSR model reduces to the CRC model when $\gamma \equiv 0$.

Table 7.1 shows a parameter summary of the CSR model applied to the illustrative insurer. The mean of the γ parameter is 0.0446 indicating a speedup in claim settlement. Figure 7.1 shows the posterior distribution of the γ parameter. Figure 7.2 shows that a speedup in claim settlement rates is fairly common in our set of 200 loss triangles. As we should expect, the estimated ultimate loss for the CSR model in Table 7.2 is less than the ultimate estimated ultimate loss for the CRC model in Table 5.3. Note that since $\beta_{10} = 0$, the calculations of the estimates and outcome percentile are identical to that for the CRC model.

Table 7.1. CSR Model Parameter Summary for the Paid Illustrative Loss Triangle

	Mean	Std. Dev.		Mean	Std. Dev.
logelr	–0.3956	0.0246	β_6	–0.0151	0.0440
α_1	0.0000	0.0000	β_7	–0.0057	0.0409
α_2	–0.2541	0.0272	β_8	–0.0041	0.0396
α_3	0.1188	0.0308	β_9	–0.0062	0.0381
α_4	0.2089	0.0373	β_{10}	0.0000	0.0000
α_5	–0.0002	0.0445	γ	0.0446	0.0282
α_6	–0.0581	0.0617	σ_1	0.2817	0.0980
α_7	0.3881	0.0787	σ_2	0.1893	0.0497
α_8	–0.1097	0.1166	σ_3	0.1277	0.0342
α_9	0.0462	0.1914	σ_4	0.0920	0.0271
α_{10}	0.0645	0.3467	σ_5	0.0714	0.0236
β_1	–1.3794	0.1667	σ_6	0.0568	0.0214
β_2	–0.6479	0.0989	σ_7	0.0467	0.0194
β_3	–0.3032	0.0670	σ_8	0.0382	0.0175
β_4	–0.0928	0.0549	σ_9	0.0299	0.0154
β_5	–0.0608	0.0483	σ_{10}	0.0201	0.0128

Figure 7.1. CSR Posterior Distribution of γ for the Paid Illustrative Loss Triangle

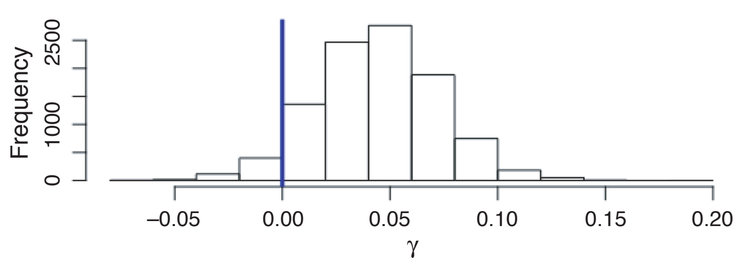


Figure 7.2. CSR Posterior Mean γ for the Set of 200 Loss Triangles

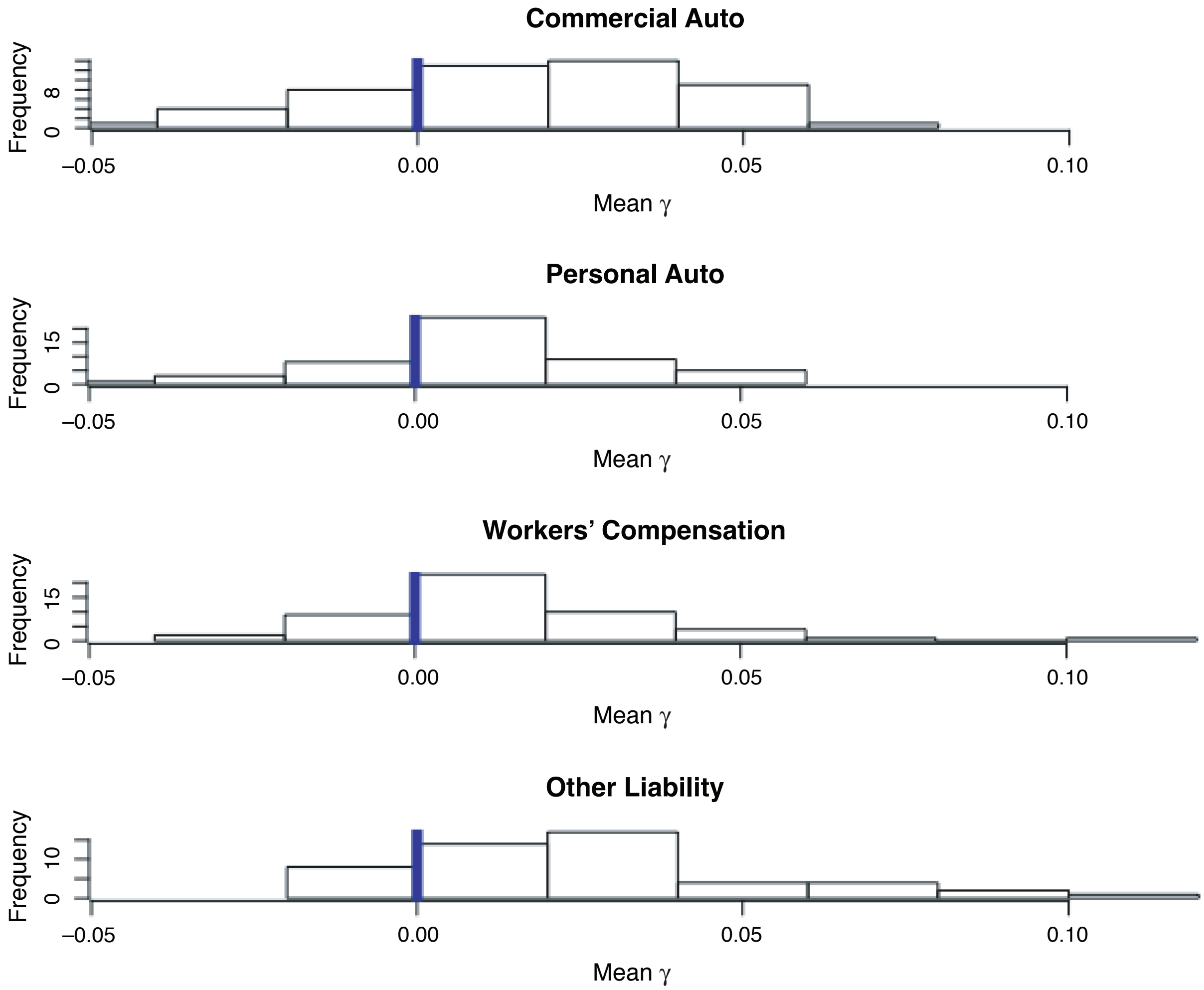


Table 7.2. CSR Model Output for the Paid Illustrative Loss Triangle

w	Premium	Estimate	SE	CV	Outcome	Percentile
1	5812	3912	0	0.0000	3912	
2	4908	2566	113	0.0440	2527	
3	5454	4139	189	0.0457	4274	
4	5165	4292	215	0.0501	4341	
5	5214	3516	192	0.0546	3583	
6	5230	3332	235	0.0705	3268	
7	4992	4971	426	0.0857	5684	
8	5466	3323	407	0.1225	4128	
9	5226	3756	742	0.1976	4144	
10	4962	3790	1416	0.3736	4139	
Total	52429	37597	2401	0.0639	40000	86.26

Table 7.3. Model Comparison Statistics for the Paid Illustrative Loss Triangle

Model	$\widehat{elpd}_{loo}$	p_{loo}	LOOIC
CSR-Paid	49.76	15.09	–99.53
CRC-Paid	47.80	14.97	–95.60

Table 7.3 show that the $\widehat{elpd}_{loo}$ statistic favors the CSR model over the CRC model for the illustrative insurer. The CSR model is favored for 105 of our set of 200 loss triangles. One would expect the CRC model to be favored for those insurers that are not changing their claim settlement rate.

If one looks closely, one will see that the standardized residual Box plot for the CSR modelin Figure 7.3 is slightly better than that of the CRC model in Figure 5.3.

A comparison of the *p-p* plot of the CSR model, Figure 7.4, with that of the CRC model, Figure 5.1, indicates that the CSR model gives a better fit to the holdout data in the lower triangles for our 200 loss triangles. That is to say, it has a better reputation.

Let's now introduce the use of the *elpd* statistic on test, i.e., lower triangle, data. If **x** is a vector containing outcome data that was not used in fitting the model, then the test statistic

$$\widehat{elpd}_{test} = \sum_{i=1}^{N} \log\left(\frac{1}{S}\sum_{j=1}^{J} p(x_i|\theta_j)\right) \tag{7.1}$$

can be used to compare models.

Figure 7.3. CSR Standardized Residual Box Plots for the Paid Illustrative Loss Triangle

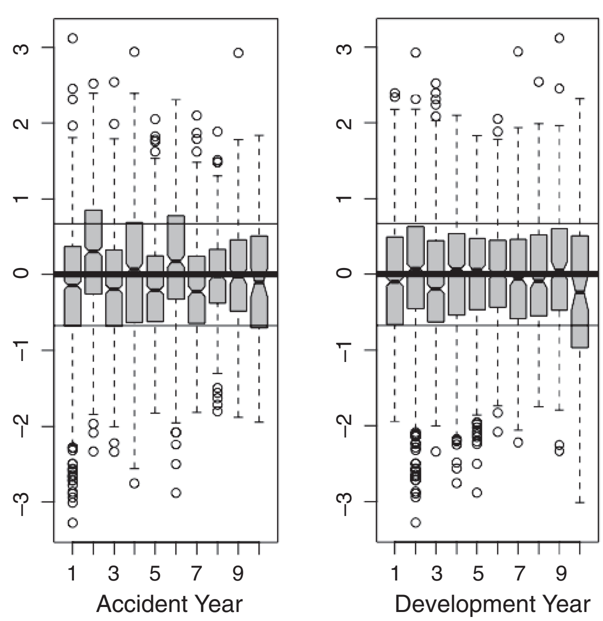
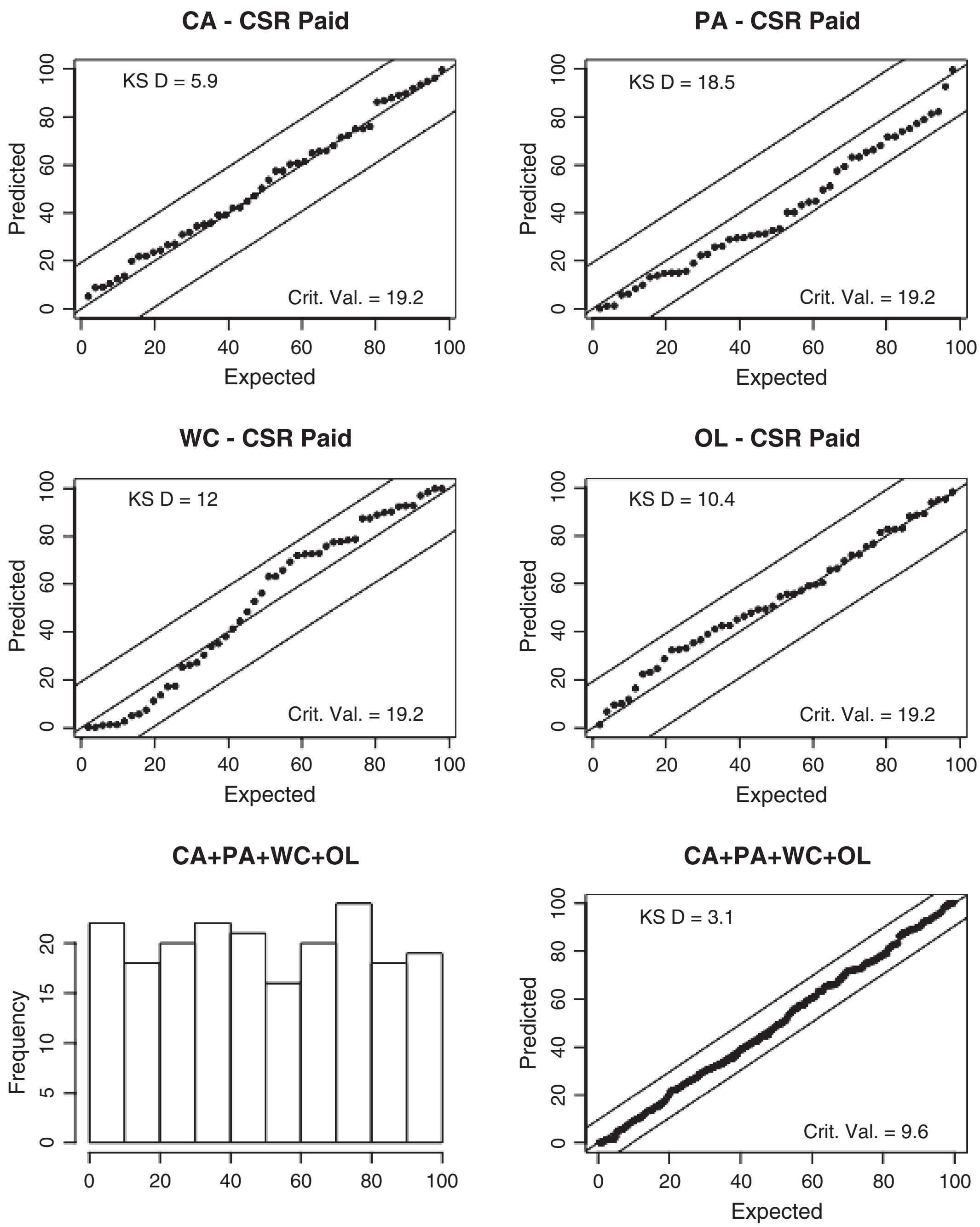


Figure 7.4. ***p-p* Plots for the CSR Model on Paid Loss Triangles**



For actuaries seeking to post a loss reserve liability, such a statistic cannot be used on current data since the lower triangle test data is what they are trying to predict. However we can use that statistic to our set of 200 triangles to compare the performance for each model to provide another indicator of a model's reputation. In pairwise comparisons, the more often the *elpd* statistic favors a model, the better its reputation.

Note that if it turns out there is not a significant change in the claim settlement rate, one should not expect the CSR model to have a better $\widehat{elpd}_{loo}$ statistic.

Table 7.4. $\widehat{elpd}$ Pairwise Comparisons

Line	CSR > CRC(loo)	CSR > CRC(test)
CA	27	27
PA	26	30
WC	27	30
OL	26	32
Total	106	119

Table 7.4 counts the number of times that the $\widehat{elpd}$ statistic favors the CSR model for both the "loo" (upper triangle) and the test (lower triangle) statistics. This table indicates that a significant change in the claim settlement rate occurs fairly often in all four lines of insurance.

The $\widehat{elpd}_{test}$ statistic and the *p-p* plots offer two different perspectives of a model's reputation. This statistic looks at a model's fit to the entirety of the lower triangle's data and provide a metric to compare one model with another. The *p-p* plots examine only the sum of the last column of the lower triangles data. But, as in the case of the CRC model on paid data, they provides hints about the model's shortcomings. Namely that the paid CRC model was biased upward.

8. The Correlated Accident Year Model

We have seen in Figure 3.2 for the Mack model, and in Figure 5.2 for the CRC model, that these models predict a tail that is too light when applied to incurred losses. One way to thicken the tail is to allow for some correlation between the accident years. If it turns out that the correlation is positive, the tails should thicken.

The Correlated Accident Year (CAY) Model

1. $logelr \sim \text{normal}(-0.4, \sqrt{10})$.
2. $\alpha_w \sim \text{normal}(0, \sqrt{10})$ for $w = 2, \ldots, 10$. Set $\alpha_1 = 0$.
3. $\beta_d \sim \text{normal}(0, \sqrt{10})$ for $d = 1, \ldots, 9$. Set $\beta_{10} = 0$.
4. Set $\rho = 2 \cdot \rho_{pos} - 1$, where $\rho_{pos} \sim \text{beta}(2,2)$. Note that this allows ρ to take on any value in the interval (-1,1).
5. $a_i \sim \text{uniform}(0,1)$ for $i = 1, \ldots, 10$.
6. Set $\sigma_d^2 = \sum_{i=d}^{10} a_i$ for $d = 1, \ldots, 10$. Note that this forces $\sigma_1^2 > \cdots > \sigma_{10}^2$.
7. Set $\mu_{1,d} = \log(\text{Premium}_1) + logelr + \beta_d$.
8. Set $\mu_{w,d} = \log(\text{Premium}_w) + logelr + \alpha_w + \beta_d + \rho \cdot (\log(C_{w-1,d}) - \mu_{w-1,d})$ for $w > 1$.
9. Then $C_{w,d} \sim \text{lognormal}(\mu_{w,d}, \sigma_d)$.

Note that the CAY model reduces to the CRC model when $\rho \equiv 0$. Including the ρ parameter in the model creates a correlation between successive accident years.

Proposition: $\text{Corr}[\log(C_{w,d}), \log(C_{w-k,d})] = \rho / (1 + \rho^2)$ if $k = 1$ and is equal to 0 if $k > 1$.[10]

Proof. Without loss of generality, we can ignore the β_ds and refer to σ_d as σ. Also, refer to $\log(C_{w,d})$ as c_w, $\log(\text{Premium}_w) + logelr + \alpha_w$ as α_w, and let Z_w be a unit independent normally distributed random variable.

$$\begin{aligned}
\mu_1 &= \alpha_1 \\
c_1 &= \mu_1 + \sigma \cdot Z_1 \\
&= \alpha_1 + \sigma \cdot Z_1 \\
\mu_2 &= \alpha_2 + \rho \cdot (c_1 - \mu_1) \\
&= \alpha_2 + \rho \cdot \sigma \cdot Z_1
\end{aligned}$$

[10] This proposition and its proof were communicated privately to me by John Major.

$$\begin{aligned}
c_2 &= \mu_2 + \sigma \cdot Z_2 \\
&= \alpha_2 + \sigma \cdot (\rho \cdot Z_1 + Z_2) \\
&\quad \ldots \\
c_w &= \alpha_w + \sigma \cdot (\rho \cdot Z_{w-1} + Z_w)
\end{aligned} \tag{8.1}$$

From Equation 8.1 it follows that

$$\begin{aligned}
\text{Cov}[c_w, c_{w-k}] = \sigma^2 \cdot \rho^2 \cdot \text{Cov}[Z_{w-1}, Z_{w-k-1}] \\
+ \sigma^2 \cdot \text{Cov}[\rho \cdot Z_{w-1}, Z_{w-k}] \\
+ \sigma^2 \cdot \text{Cov}[Z_w, \rho \cdot Z_{w-k-1}] \\
+ \sigma^2 \cdot \text{Cov}[Z_w, Z_{w-k}]
\end{aligned} \tag{8.2}$$

When $k = 1$, the second term of Equation 8.2 is equal to $\rho \cdot \sigma^2$. Since Z_w and Z_{w-k} are independent, the remaining terms are equal to zero. When $k > 1$, all terms are equal to zero.

From Equation 8.1 we have that $Var[c_w] = \sigma^2 \cdot (1 + \rho^2)$. Thus

$$\begin{aligned}
\text{Corr}[c_w, c_{w-k}] &= \frac{\rho}{1+\rho^2} \text{ when } k = 1 \\
&= 0 \text{ when } k > 1
\end{aligned} \tag{8.3}$$

We can obtain the predictive distribution of the loss outcomes at development year 10, by accident year and in total for all the accident years by the following simulation for each of the 10,000 parameter vectors:

1. Set $\mu_{1,10} = \log(\text{Premium}_1) + logelr$ (Note that $\alpha_1 = \beta_{10} = 0$).
2. Set $\mu_{2,10} = \log(\text{Premium}_2) + logelr + \alpha_2 + \rho \cdot (\log(C_{1,10}) - \mu_1)$.
3. Simulate $\tilde{C}_{2,10} \sim \text{lognormal}(\mu_{2,10}, \sigma_{10})$.
4. Set $\mu_{w,10} = \log(\text{Premium}_w) + logelr + \alpha_w + \rho \cdot (\log(\tilde{C}_{w-1,10}) - \mu_{w-1})$ for $w = 3, \ldots, 10$.
5. Simulate $\tilde{C}_{w,10} \sim \text{lognormal}(\mu_{w,10}, \sigma_{10})$ for $w = 3, \ldots, 10$.
6. Calculate $\tilde{C}_{Tot,10} = C_{1,10} + \sum_{w=2}^{10} \tilde{C}_{w,10}$.

Table 8.1 shows a parameter summary of the CAY model applied to the illustrative insurer. The mean of the ρ parameter is 0.1709 indicating a positive correlation between accident years. Thus it should come as no surprise that the standard error of the ultimate loss estimate of the illustrative insurer in Table 8.2 is larger than that produced by the CRC model in Table 5.3. However, one should note that Figure 8.2 indicates a fairly wide posterior distribution of the ρ parameter. When we look at the LOOIC statistics in Table 9.4 we see that the CRC model is favored over the CAY model for the Illustrative Loss Triangle.

This Box plot shows a slightly better fit than that of the CRC model in Figure 5.4.7.

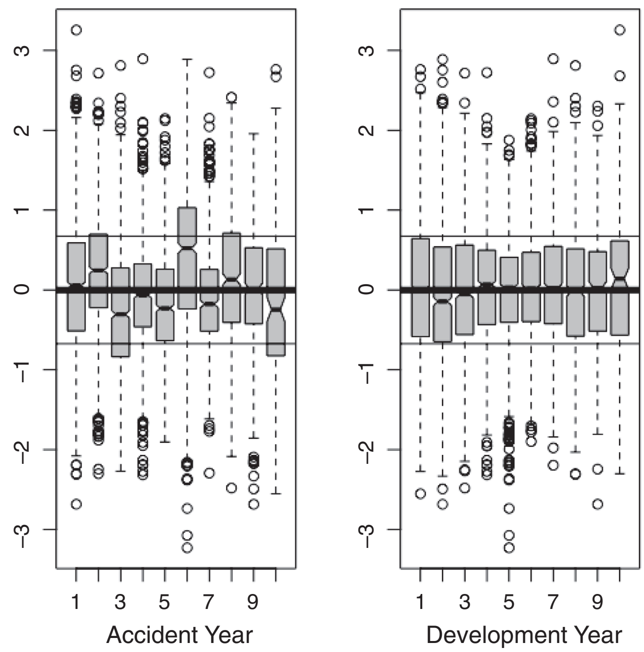
Table 8.1. CAY Model Parameter Summary for the Incurred Illustrative Loss Triangle

	Mean	Std. Dev.		Mean	Std. Dev.
logelr	–0.3945	0.0150	β_6	–0.0008	0.0281
α_1	0.0000	0.0000	β_7	0.0017	0.0269
α_2	–0.2619	0.0156	β_8	0.0050	0.0254
α_3	0.1105	0.0213	β_9	–0.0013	0.0237
α_4	0.2124	0.0253	β_{10}	0.0000	0.0000
α_5	0.0083	0.0308	ρ	0.1709	0.2071
α_6	–0.0586	0.0401	σ_1	0.2632	0.1002
α_7	0.4499	0.0521	σ_2	0.1511	0.0439
α_8	0.0248	0.0819	σ_3	0.0905	0.0280
α_9	0.1601	0.1453	σ_4	0.0586	0.0211
α_{10}	0.1779	0.2984	σ_5	0.0442	0.0178
β_1	–0.5976	0.1212	σ_6	0.0357	0.0154
β_2	–0.1896	0.0699	σ_7	0.0295	0.0135
β_3	–0.0993	0.0476	σ_8	0.0243	0.0117
β_4	–0.0268	0.0353	σ_9	0.0190	0.0100
β_5	–0.0121	0.0301	σ_{10}	0.0128	0.0083

Table 8.2. CAY Model Output for the Incurred Illustrative Loss Triangle

w	Premium	Estimate	SE	CV	Outcome	Percentile
1	5812	3917	0	0.0000	3917	
2	4908	2547	65	0.0255	2532	
3	5454	4107	127	0.0309	4279	
4	5165	4308	144	0.0334	4341	
5	5214	3547	133	0.0375	3587	
6	5230	3329	152	0.0457	3268	
7	4992	5285	296	0.0560	5684	
8	5466	3790	323	0.0852	4128	
9	5226	4180	621	0.1486	4144	
10	4962	4183	1373	0.3282	4181	
Total	52429	39193	1859	0.0474	40061	73.24

Figure 8.1. CAY Standardized Residual Box Plots for the Incurred Illustrative Loss Triangle



Now let's examine some statistics calculated over the set of 200 loss triangles. Figure 8.3 shows that the positive mean ρs are in the overwhelming majority of the triangles. Figure 8.4 shows that the CAY model tends to increase the standard error of the estimates. The *p-p* plots for the CAY model in Figure 8.5 show a noticeable improvement over the *p-p* plots for the CRC model in Figure 5.2 for the CA, PA and WC lines of business. The *p-p* plots for OL line of are almost identical for each figure.

The $\widehat{elpd}$ comparisons are given in Table 8.4. When comparing individual triangles with the model selection statistics, It turns out that, with the $\widehat{elpd}_{loo}$ statistic, the CAY model is favored over the CRC model in only 26 of the 200 loss triangles we examined. But with the $\widehat{elpd}_{test}$ statistic, the CAY models is favored for 121 out of the 200 loss triangles. It would appear that the relatively wide posterior distribution of ρ like that pictured in Figure 8.2 makes it difficult to distinguish between the CAY and the CRC models. However, in the test data, there are many instances that favor the CAY model. This backs up what we see in the *p-p* plots in Figure 8.5.

Thus, it appears that a choice to use the CAY model rests mainly with its reputation.

Figure 8.2. CAY Model Posterior Distribution of ρ for Incurred Illustrative Loss Triangle

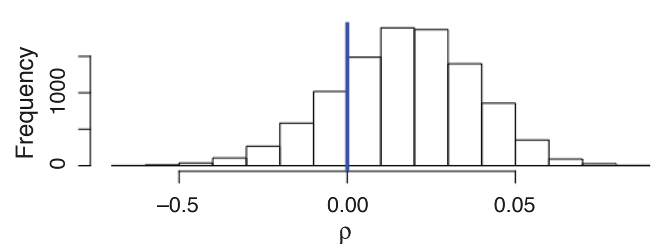


Table 8.3. Model Comparison Statistics for the Incurred Illustrative Loss Triangle

Model	$\widehat{elpd}_{loo}$	p_{loo}	LOOIC
CAY-Incurred	68.65	15.64	−137.30
CRC-Incurred	70.97	15.07	−141.93

Figure 8.3. CAY Posterior Mean of ρ for the Set of 200 Paid Loss Triangles

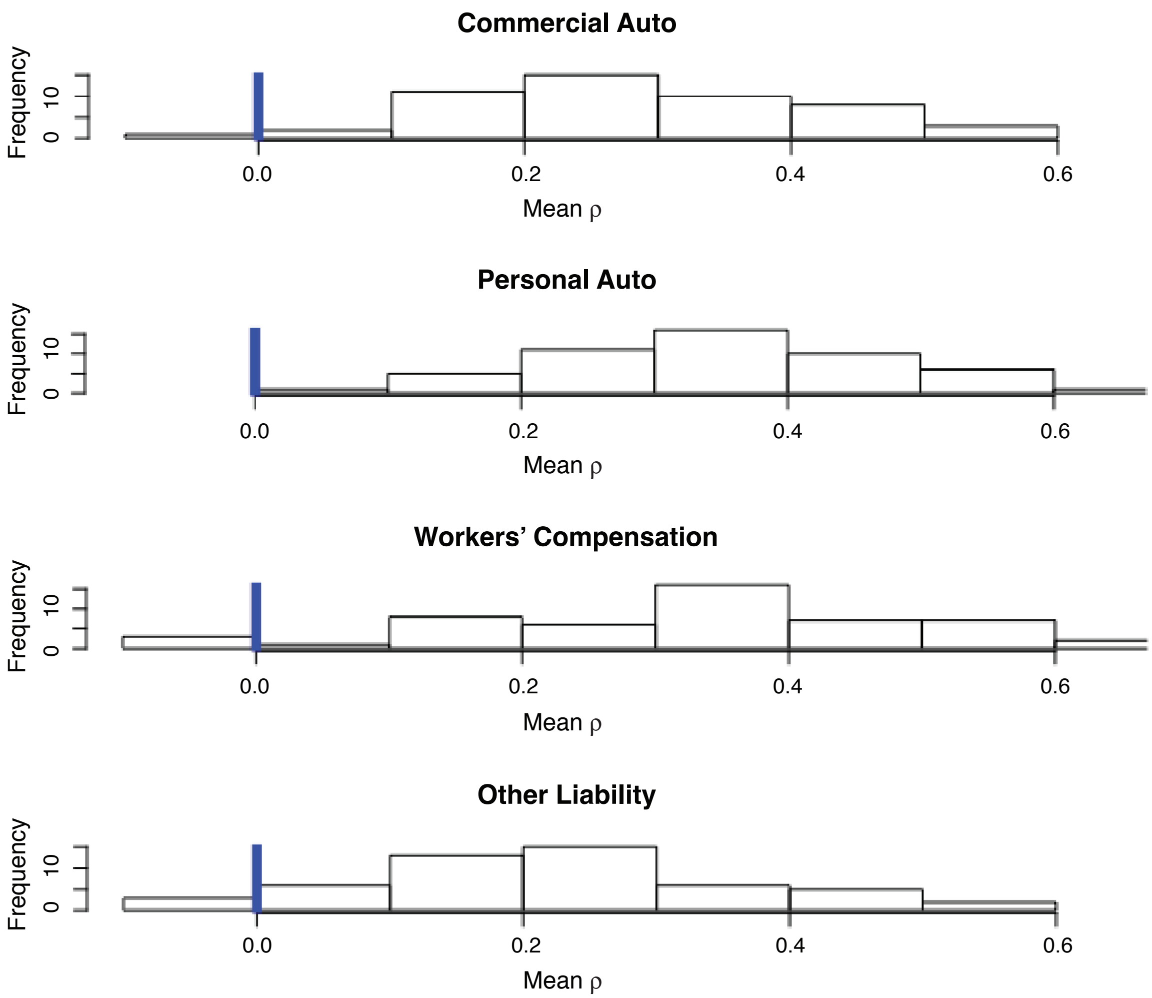


Figure 8.4. CAY to CRC Standard Error Ratios for the Set of 200 Loss Triangles

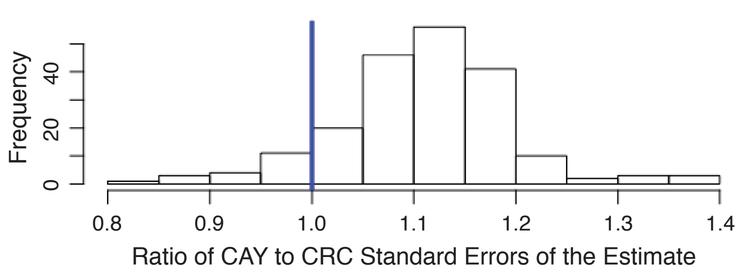


Figure 8.5. *p-p* Plots for the CAY Model on Incurred Loss Triangles

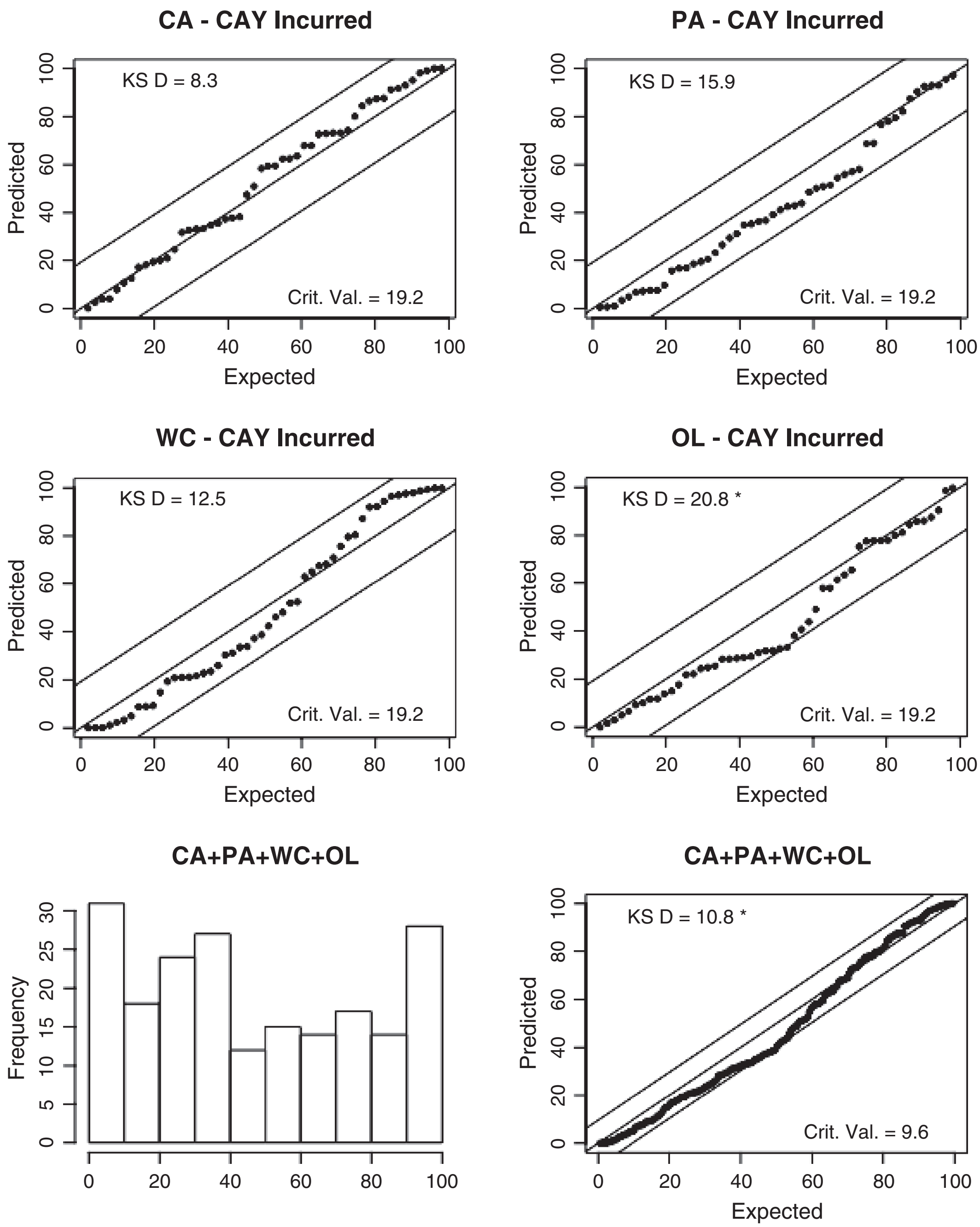


Table 8.4. $\widehat{elpd}$ Pairwise Comparisons

Line	CAY > CRC(loo)	CAY > CRC(test)
CA	5	31
PA	7	28
WC	11	38
OL	3	24
Total	26	121

9. Combining the CAY and CSR Models[11]

Of the models we have considered, the previous sections made the case that the CSR model performs best on the paid data, and the CAY performs best on the incurred data. As one considers the structure of these models, they will notice that the *logelr* and the α_w parameters have the same interpretation, and possibly the same values, in both models. One should expect the remaining parameters to be different as each model is fit to a different, but related, set of losses. This section investigates the relationship between the two models with the idea that there may be some synergy that we can exploit to make more accurate estimates.

A model that does this is specified below. Notice that

- Steps 1 and 2 are the same as in the CSR and CAY model.
- Steps 4 to 8 are the same as in the CSR model, with the prefix "P" denoting the parameters specific to the paid data.
- Steps 9 to 15 are the same as steps 3 to 9 in the CAY model, with the prefix "I" denoting the parameters specific to the incurred data.

Step 3 in this model differs from the corresponding step in the CSR model in that the requirement that ${}_p\beta_{10} = 0$ is dropped. This reduces any distortion caused by the *logelr* parameter for the paid loss triangle being significantly different from the *logelr* parameter for the incurred loss triangle. This frequently occurs in the Workers' Compensation line of business with our data.

The Integrated Paid and Incurred (IPI) Model

1. $logelr \sim \text{normal}(-0.4, \sqrt{10})$.
2. $\alpha_w \sim \text{normal}(0, \sqrt{10})$ for $w = 2, \ldots, 10$. Set $\alpha_1 = 0$.
3. ${}_p\beta_d \sim \text{normal}(0, \sqrt{10})$ for $d = 1, \ldots, 10$.
4. $\gamma \sim \text{normal}(0, 0.05)$.
5. ${}_pa_i \sim \text{uniform}(0, 1)$ for $i = 1, \ldots, 10$.
6. Set ${}_p\sigma_d^2 = \sum_{i=d}^{10} {}_pa_i$ for $d = 1, \ldots, 10$. Note that this forces ${}_p\sigma_1^2 > \cdots > {}_p\sigma_{10}^2$.
7. Set ${}_p\mu_{w,d} = \log(\text{Premium}_w) + logelr + \alpha_w + {}_p\beta_d \cdot (1-\gamma)^{w-1}$.
8. Then ${}_pC_{w,d} \sim \text{lognormal}({}_p\mu_{w,d}, {}_p\sigma_d)$.

[11] The motivation for this section arose out of a series of conversations I had with Ned Tyrrell, FCAS. Mr. Tyrrell's insight was that the estimates obtained by using both paid and incurred data would have lower variability.

9. ${}_I\beta_d \sim \text{normal}(0, \sqrt{10})$ for $d = 1, \ldots, 9$. Set ${}_I\beta_{10} = 0$.
10. Set $\rho = 2 \cdot \rho_{pos} - 1$, where $\rho_{pos} \sim \text{beta}(2,2)$. Note that this allows ρ to take on any value in the interval $(-1,1)$.
11. ${}_Ia_i \sim \text{uniform }(0,1)$ for $i = 1, \ldots, 10$.
12. Set ${}_I\sigma_d^2 = \sum_{i=d}^{10} {}_Ia_i$ for $d = 1, \ldots, 10$. Note that this forces ${}_I\sigma_1^2 > \cdots > {}_I\sigma_{10}^2$.
13. Set ${}_I\mu_{1,d} = \log(\text{Premium}_1) + logelr + {}_I\beta_d$.
14. Set ${}_I\mu_{w,d} = \log(\text{Premium}_w) + logelr + \alpha_w + {}_I\beta_d + \rho \cdot (\log({}_IC_{w-1,d}) - {}_I\mu_{w-1,d})$ for $w > 1$.
15. Then ${}_IC_{w,d} \sim \text{lognormal}({}_I\mu_{w,d}, {}_I\sigma_d)$.

Table 9.1 gives the *logelr* and the α_w parameters for the IPI model with the data from the Illustrative Loss Triangle, and for reference, the corresponding parameters for the CSR and CAY models. The parameters are very similar for the earlier accident years, with some divergence in the last few accident years. What is interesting to note is that the standard deviations of the *logelr* and the α_w parameters are noticeably smaller for the IPI model. These lower standard deviations translate into lower standard errors of the estimates, as noted in Table 9.2 when compared with Table 7.2, and Table 9.3 when compared with Table 8.2.

This reduction in the standard errors extends to almost all of the set of 200 loss triangles as can be seen in Figure 9.3.

The standardized residual Box plots for Illustrative loss triangle with the IPI paid and incurred losses in Figures 9.1 and 9.2 are slightly worse than the corresponding plots for the CSR and CAY model, but they are still reasonable in that zero is within the interquartile range of the standardized residuals for all accident years.

Table 9.1. Summary of the *logelr* and α_w Parameters for the Illustrative Loss Triangle

	CSR Model		CAY Model		IPI Model	
	Mean	Std. Dev.	Mean	Std. Dev.	Mean	Std. Dev.
logelr	–0.3956	0.0246	–0.3945	0.0150	–0.3951	0.0109
α_1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
α_2	–0.2541	0.0272	–0.2619	0.0156	–0.2618	0.0090
α_3	0.1188	0.0308	0.1105	0.0213	0.1157	0.0119
α_4	0.2089	0.0373	0.2124	0.0253	0.2140	0.0153
α_5	–0.0002	0.0445	0.0083	0.0308	0.0091	0.0186
α_6	–0.0581	0.0617	–0.0586	0.0401	–0.0657	0.0263
α_7	0.3881	0.0787	0.4499	0.0521	0.4319	0.0383
α_8	–0.1097	0.1166	0.0248	0.0819	–0.0207	0.0619
α_9	0.0462	0.1914	0.1601	0.1453	0.1248	0.1056
α_{10}	0.0645	0.3467	0.1779	0.2984	0.1571	0.1947

Table 9.2. IPI Model Output for the Paid Illustrative Loss Triangle

w	Premium	Estimate	SE	CV	Outcome	Percentile
1	5812	3912	0	0.0000	3912	
2	4908	2543	58	0.0228	2527	
3	5454	4124	98	0.0238	4274	
4	5165	4309	111	0.0258	4341	
5	5214	3544	97	0.0274	3583	
6	5230	3299	109	0.0330	3268	
7	4992	5180	223	0.0431	5684	
8	5466	3612	234	0.0648	4128	
9	5226	4009	429	0.1070	4144	
10	4962	3986	796	0.1997	4139	
Total	52429	38518	1253	0.0325	40000	88.50

Table 9.3. IPI Model Output for the Incurred Illustrative Loss Triangle

w	Premium	Estimate	SE	CV	Outcome	Percentile
1	5812	3912	0	0.0000	3917	
2	4908	2545	44	0.0173	2532	
3	5454	4126	81	0.0196	4279	
4	5165	4311	95	0.0220	4341	
5	5214	3546	87	0.0245	3587	
6	5230	3301	103	0.0312	3268	
7	4992	5183	216	0.0417	5684	
8	5466	3613	231	0.0639	4128	
9	5226	4012	429	0.1069	4144	
10	4962	3987	795	0.1994	4181	
Total	52429	38541	1225	0.0318	40061	89.66

Figure 9.1. IPI Standardized Residual Box Plots for the Paid Illustrative Loss Triangle

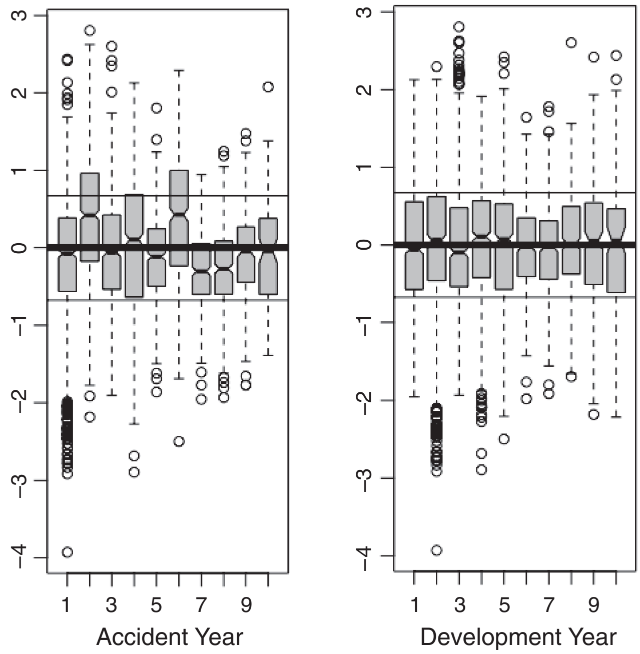


Figure 9.2 IPI Standardized Residual Box Plots for the Incurred Illustrative Loss Triangle

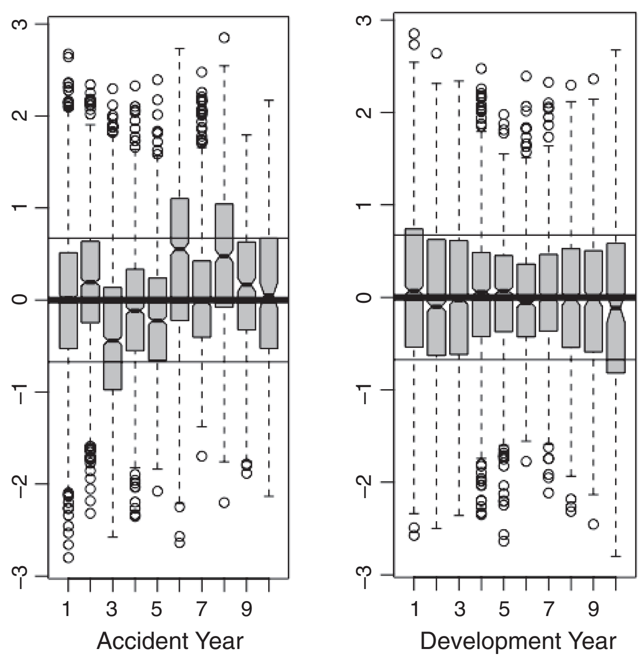
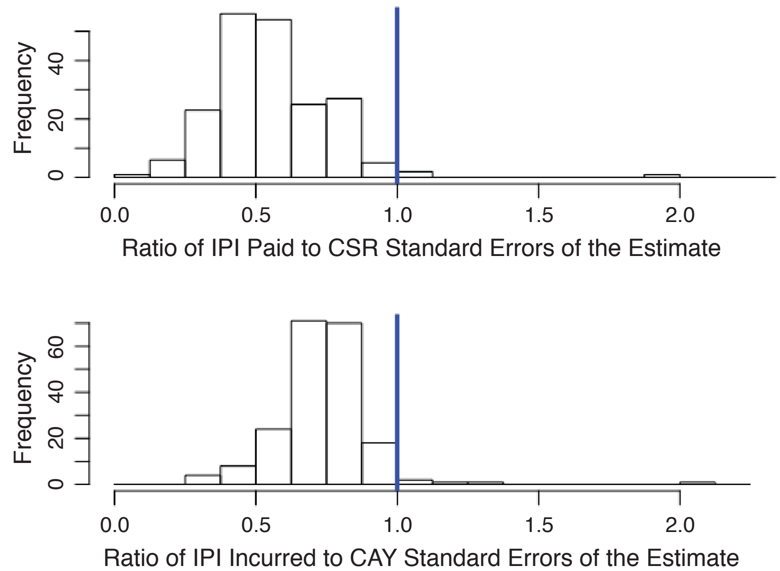


Figure 9.3. Standard Error Reductions by the IPI Model



To calculate the model comparison statistics for the IPI model, we first calculate the $55 \times 10{,}000$ log-likelihood matrices given the parameters specific to the paid, and then the incurred, data. We then use the "loo" package to get the model selection statistics. As we will not introduce any more models in this monograph, the following table gives the model selection statistics for each of the MCMC models considered above.

Now let's examine the $\widehat{elpd}$ comparisons over the set of 200 triangles. The following tables provide comparisons for all the models given in this monograph.

Tables 9.5–9.8 identify several instances where the IPI model has a better fit than the corresponding CSR, CAY or CRC models. The IPI model's advantage is stronger for the paid models than the incurred models.

Table 9.4. Model Comparison Statistics Illustrative Loss Triangles

Model	$\widehat{elpd}_{loo}$	p_{loo}	LOOIC
IPI-Paid	63.54		–127.08
CSR-Paid	49.76	15.09	–99.53
CRC-Paid	47.80	14.97	–95.60
SCC-Paid	–5.14	8.75	10.28
IPI-Incurred	78.36		–156.72
CAY-Incurred	68.65	15.64	–137.30
CRC-Incurred	70.97	15.07	–141.93
SCC-Incurred	–2.85	9.13	5.69

Table 9.5. $\widehat{elpd}_{loo}$ Paid Model Pairwise Comparisons

Line	IPI > CSR	IPI > CRC	CSR > CRC	CRC > SCC
CA	46	45	26	50
PA	41	42	27	50
WC	18	22	25	50
OL	41	40	23	50
Total	146	149	100	200

Table 9.6. $\widehat{elpd}_{test}$ Paid Model Pairwise Comparisons

Line	IPI > CSR	IPI > CRC	CSR > CRC	CRC > SCC
CA	43	44	27	49
PA	42	44	30	47
WC	32	40	30	48
OL	39	42	32	47
Total	156	170	119	191

Table 9.7. $\widehat{elpd}_{loo}$ Incurred Model Pairwise Comparisons

Line	IPI > CAY	IPI > CRC	CAY > CRC	CRC > SCC
CA	17	15	6	50
PA	31	29	7	50
WC	37	37	10	50
OL	22	23	3	50
Total	107	104	26	200

Table 9.8. $\widehat{elpd}_{test}$ Incurred Model Pairwise Comparisons

Line	IPI > CAY	IPI > CRC	CAY > CRC	CRC > SCC
CA	25	31	31	48
PA	35	34	28	50
WC	22	25	38	48
OL	29	31	24	46
Total	111	121	121	192

Figures 9.4 and 9.5 give the *p-p* plots for the paid and incurred IPI models. While seven of the eight *p-p* plots fall within the Kolmogorov-Smirnov critical values, the Workers' Compensation plot indicates that the models tend to be light-tailed. It might be worth noting that the Workers' Compensation line of business frequently provides benefits in the form of long-term annuities and it is more likely that significant differences between the incurrred and the and paid losses remain after 10 years. See Figure 9.6.

This completes the set of models that are considered in this monograph. We now turn to making use of these models to post a loss reserve liability. The current

Figure 9.4. *p-p* Plots for the IPI Model on Paid Loss Triangles

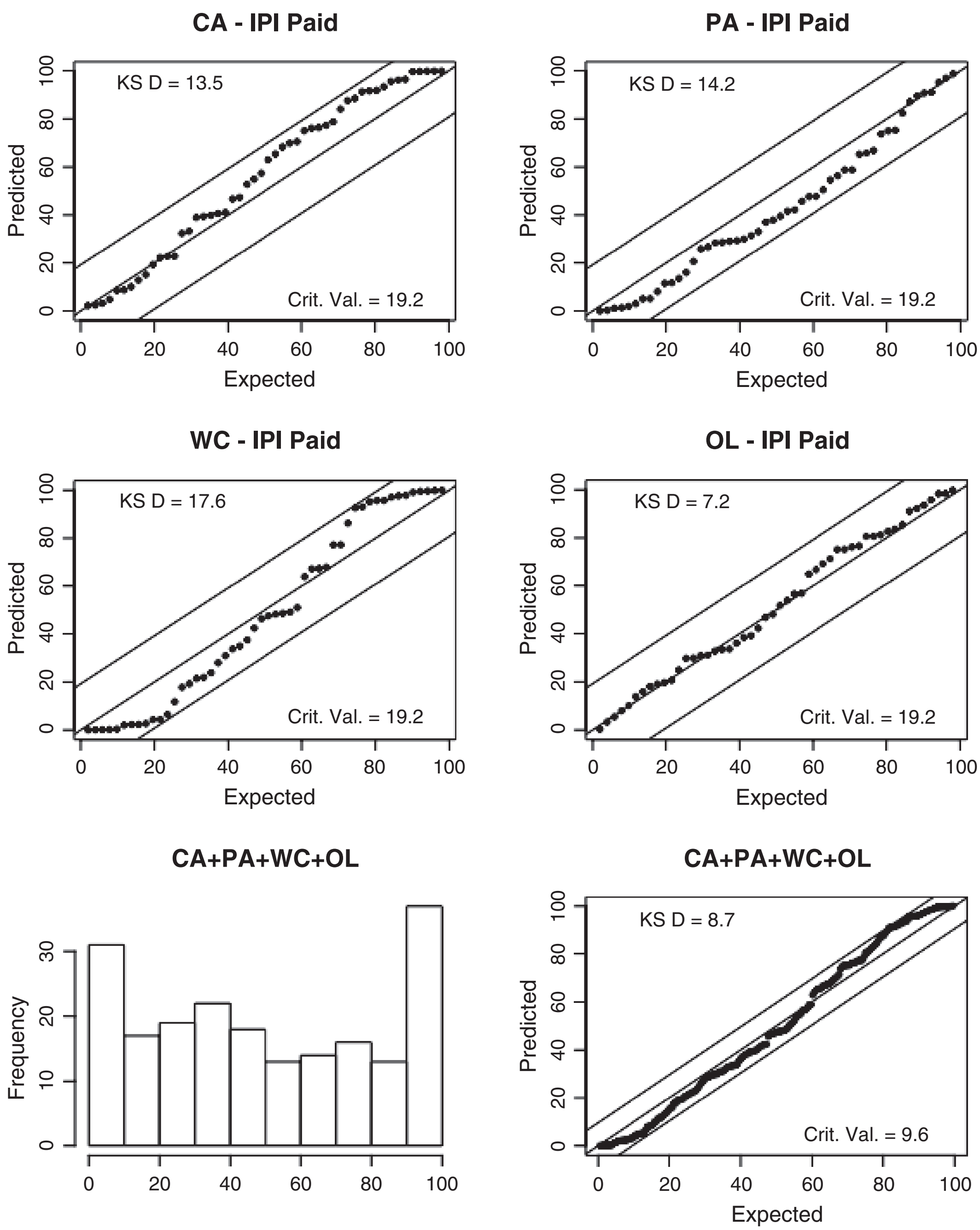
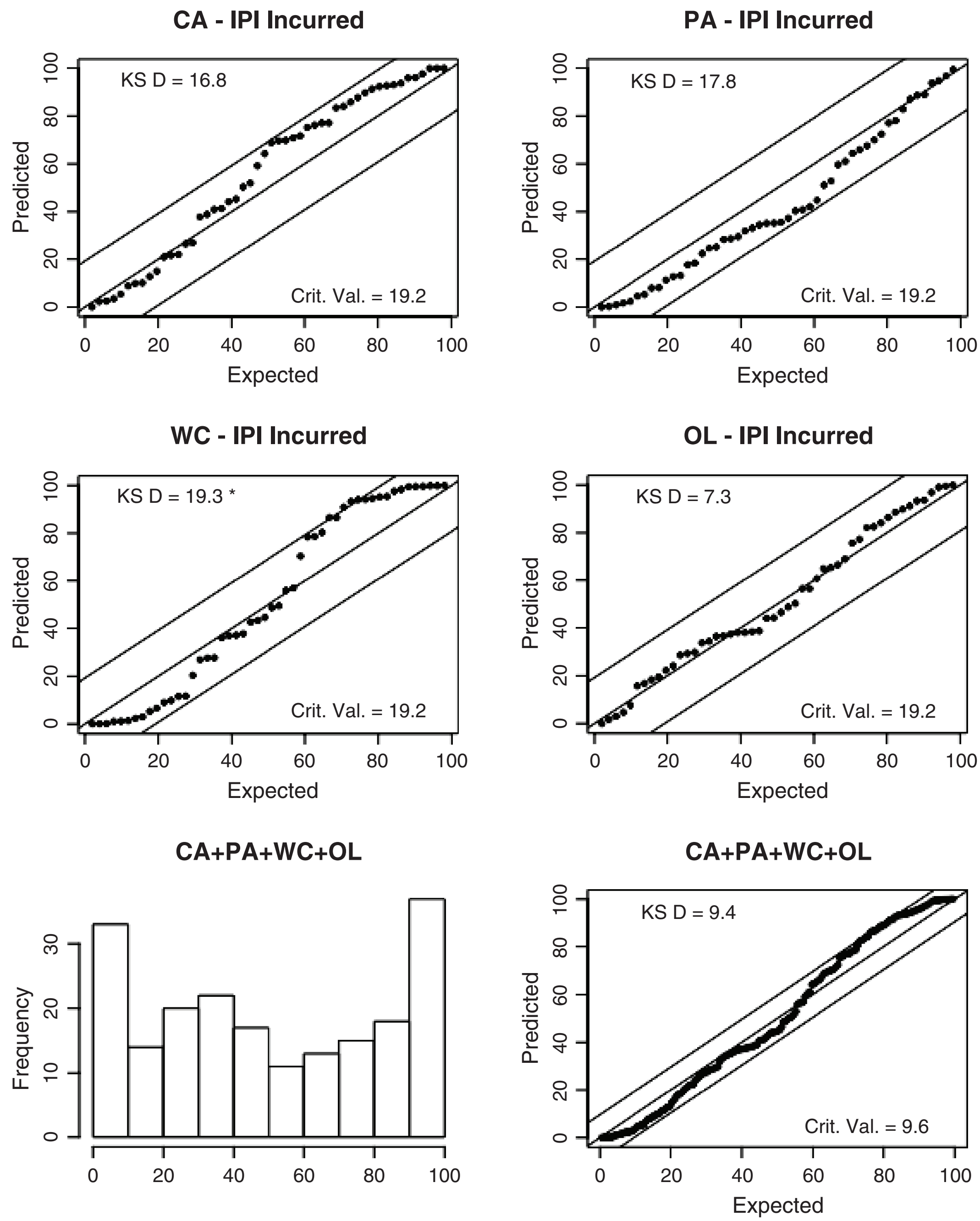


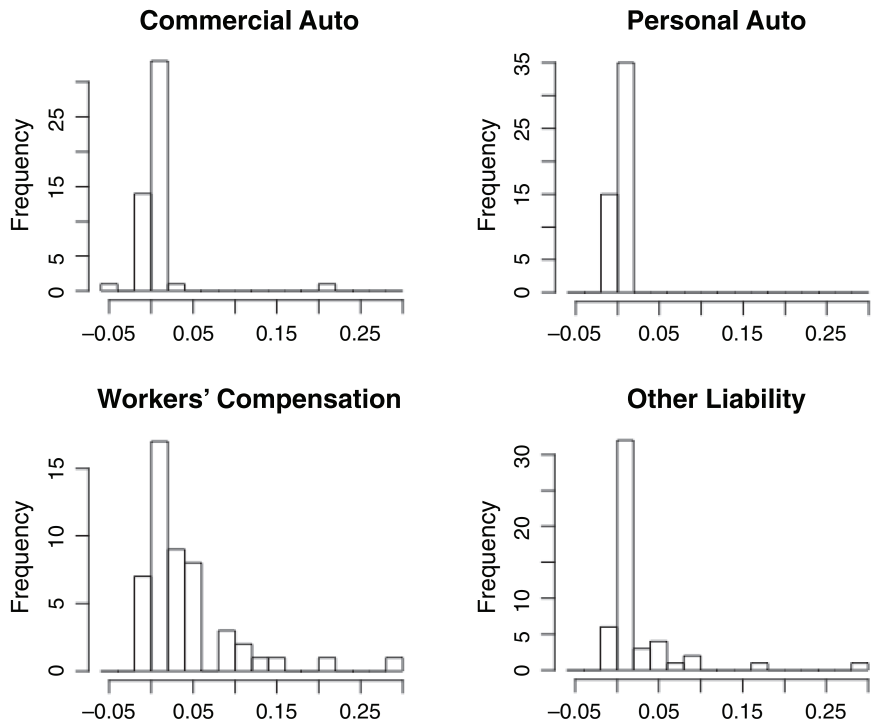
Figure 9.5. *p-p* plots for the IPI Model on Incurred Loss Triangles



consensus under IFRS 17 is that the liability should represented a discounted "best estimate" and a risk margin. To fulfill the discounting requirement, we need a model that estimates the payout pattern. To fulfill the risk margin requirement, we need a stochastic model. This monograph will turn to demonstrating how to use the CSR and the IPI models to address these needs.

An important issue with the risk margin is that of diversification. So, before discussing risk margins, we need to address the issue of dependencies between lines of insurance.

Figure 9.6. Differences in the *logelr* Parameters for the CAY and CSR Models



10. Dependencies Between Lines of Insurance[12]

As actuaries fit their stochastic loss reserve models to the various lines of insurance underwritten by their insurance company, they will ultimately be given the task of posting a single loss reserve liability for that company. An important consideration in evaluating this liability is that of diversification. To quantify the effect of diversification, we first need to address the issue of correlation, or to be more general, dependencies between lines of insurance.

Of interest are the dependencies that remain after a model has been fit to our data. Let's begin with the preliminary observation that dependencies are model dependent. A simple example illustrating this is in Figure 10.1. If one fits the correct parabolic model to each of the dependent y_1 and y_2 variables, we have uncorrelated residuals. But if one fits the incorrect constant model, the residuals are correlated. A similar point is made by Avanzi, Taylor and Wong (2016). To quote their abstract, "We show with some real examples that, sometimes, most (if not all) of the correlation can be 'explained' by an appropriate methodology."

Prior to considering dependencies, our models have taken the form:

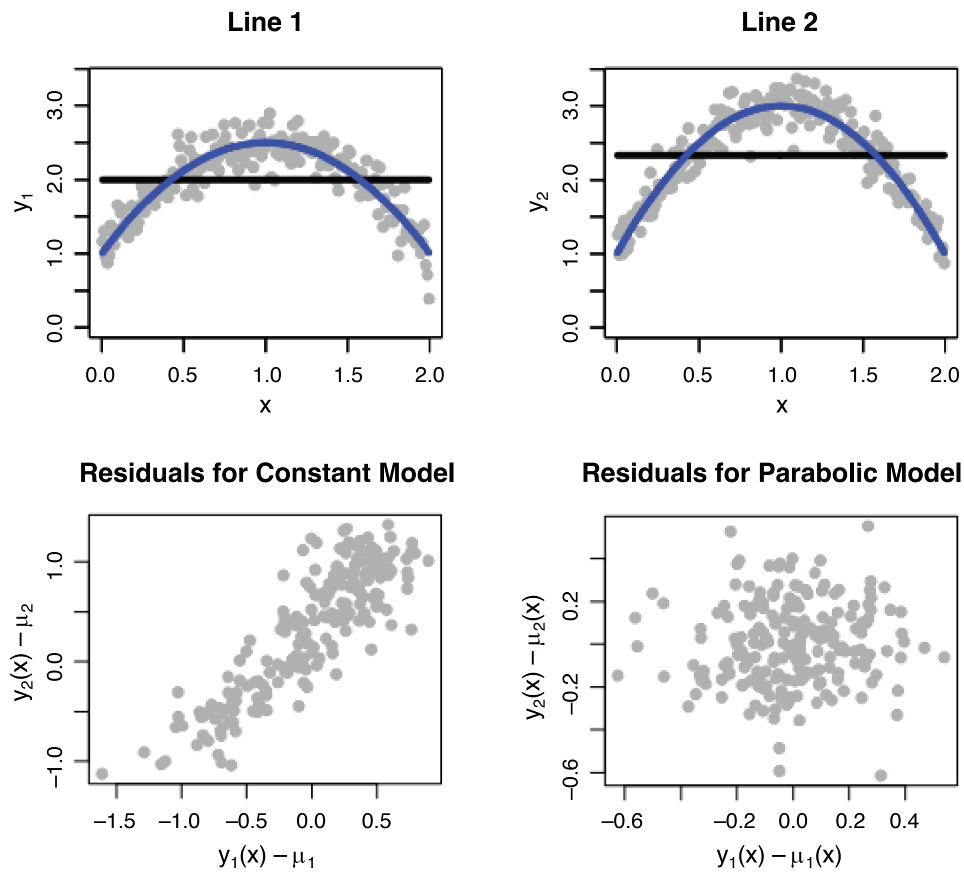
$$\log\left(C_{w,d}\right) \sim \text{normal}\left(\mu_{w,d}^{j}, \sigma_{d}^{j}\right) \tag{10.1}$$

for each w and d, with j being one of J = 10,000 simulations. The transformed parameters $\mu_{w,d}$ and σ_d can come from any of the paid SCC, CSR or IPI models. For a given simulation j, we can then propose the bivariate model for lines of insurance X and Y

$$\begin{pmatrix} \log\left({}_{X}C_{w,d}\right) \\ \log\left({}_{Y}C_{w,d}\right) \end{pmatrix} \sim \text{normal}\left(\begin{pmatrix} {}_{X}\mu_{w,d}^{j} \\ {}_{Y}\mu_{w,d}^{j} \end{pmatrix}, \begin{pmatrix} \left({}_{X}\sigma_{d}^{j}\right)^{2} & \rho^{j} \cdot {}_{X}\sigma_{d}^{j} \cdot {}_{Y}\sigma_{d}^{j} \\ \rho^{j} \cdot {}_{X}\sigma_{d}^{j} \cdot {}_{Y}\sigma_{d}^{j} & \left({}_{Y}\sigma_{d}^{j}\right)^{2} \end{pmatrix}\right) \tag{10.2}$$

[12] An earlier version of this section is in Meyers (2017).

Figure 10.1. Model Dependency Illustration



where ρ^j is a single parameter representing the coefficient of correlation between the two lines for the simulation j. So, to get a "posterior" distribution[13] one proceeds as follows.

1. Use Bayesian MCMC to obtain a sample,

$$\left\{{}_X\mu_{w,d}^j,\ {}_X\sigma_d^j,\ {}_Y\mu_{w,d}^j,\ {}_Y\sigma_d^j\right\}_{j=1}^{10{,}000}$$

   from the posterior distributions for lines of insurance X and Y.
2. For each j use Bayesian MCMC to take a sample of size one from the posterior distribution of

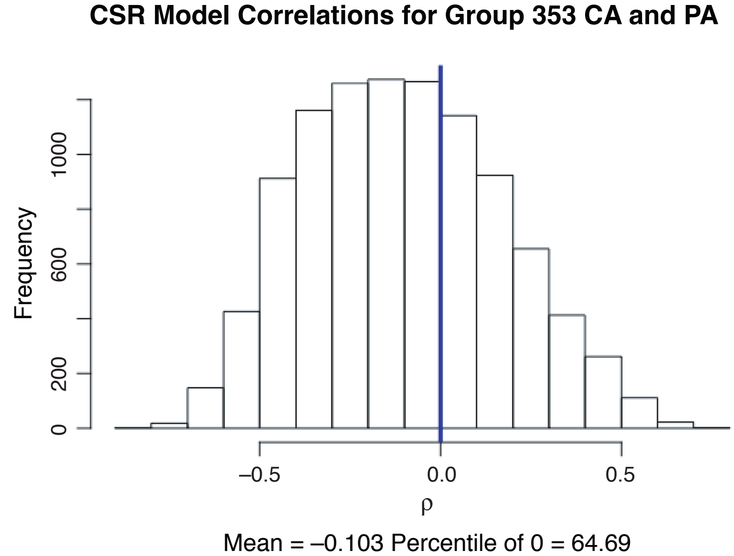
$$\rho^j\big|_X\,\mu_{w,d}^j,\ {}_X\sigma_d^j,\ {}_Y\mu_{w,d}^j,\ {}_Y\sigma_d^j$$

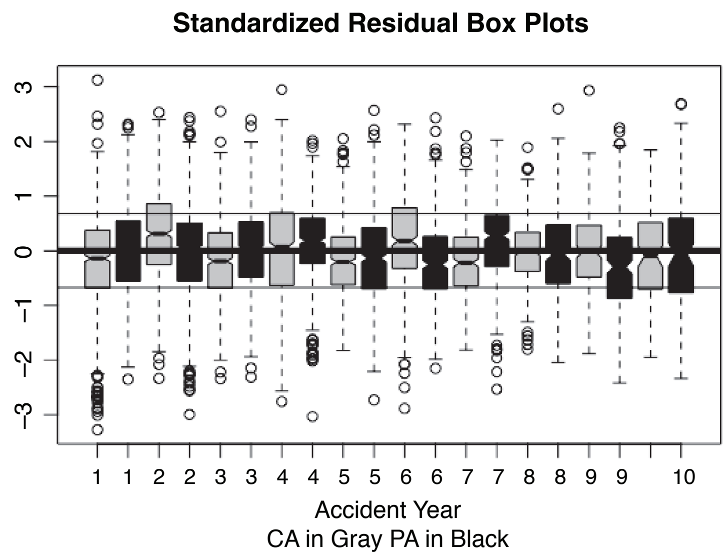
This process is fairly time consuming.[14] But it produces 10,000 equally likely simulations for Equation 10.2.

[13] The term "posterior" is put in quotes to distinguish it from a one-step MCMC model with all parameters ρ^j, ${}_X\mu_{w,d}^j$, ${}_X\sigma_d^j$, ${}_Y\mu_{w,d}^j$, and ${}_Y\sigma_d^j$. See Meyers (2017) for a discussion of this one-step approach.

[14] While the first step runs in less than a minute on my quad core laptop, the second step takes about 12 minutes, making use of R parallel package and compiling the Stan script in advance.

Figure 10.2. Summary Correlation Plots

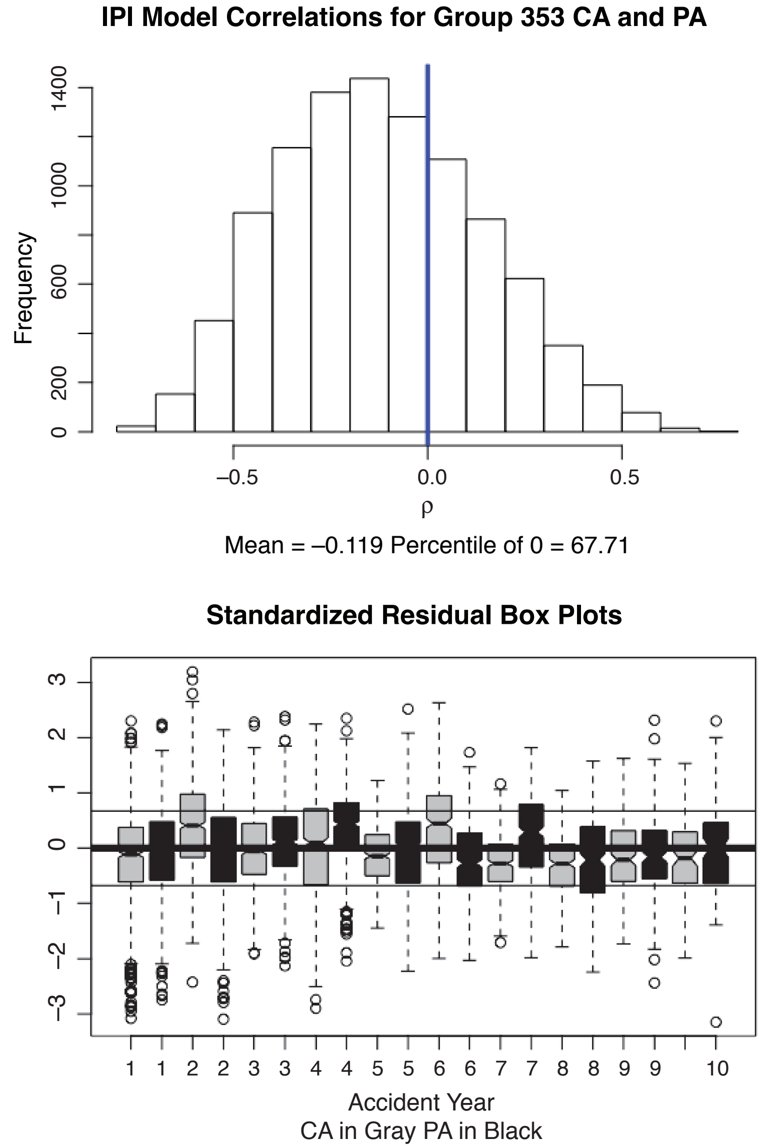




Figures 10.2–10.4 summarize the posterior distributions for the illustrative insurer using the CSR, IPI and SCC models. These figures consist of a histogram of the posterior distribution of ρ and side-by-side accident year standard residual Box plots. Here are some of the highlights of these figures.

- The posterior distribution of ρ is fairly wide, and the point $\rho = 0$ is not close to the tails of the distributions.
- One can see the effect of model dependence as we progress from the CSR model to the SCC model. The CSR model does the best at capturing the accident year effect, and the IPI model captures it nearly as well. The SCC model does not adequately capture the accident year effects. In the side-by-side accident year Box plots, the six of the corresponding medians for each line of insurance are on opposite sides

Figure 10.3. Summary Correlation Plots



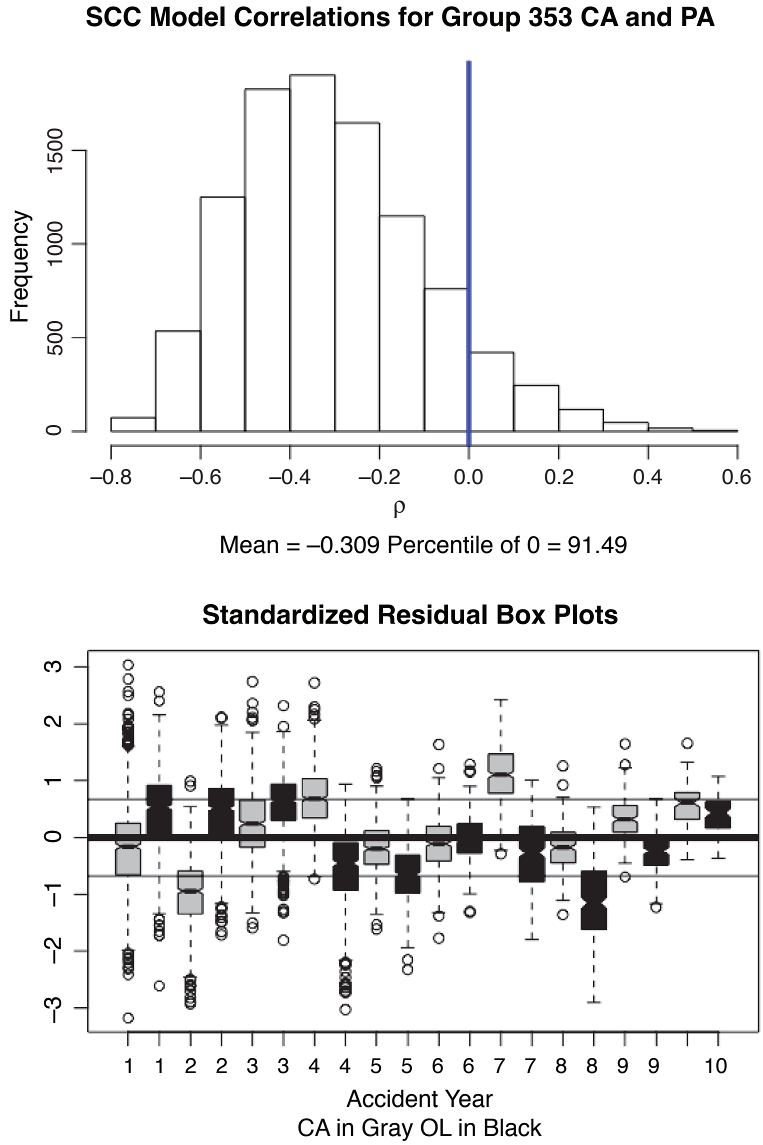
of zero. For the other four, they are fairly close to zero. This results in the posterior mean of ρ being negative.

Now let's turn to comparing the fits of the dependent assumption, specified by Equation 10.2 with the independent assumption specified by setting $\rho = 0$ in Equation 10.2 with the $\widehat{elpd}_{loo}$ statistic. To do this, we need to calculate the log-likelihoods of each observation in the upper triangle using the parameters

$$\left\{\rho^j, {}_X\mu_{w,d}^j, {}_X\sigma_d^j, {}_Y\mu_{w,d}^j, {}_Y\sigma_d^j\right\}_{j=1}^{10,000}$$

and Equation 10.2.

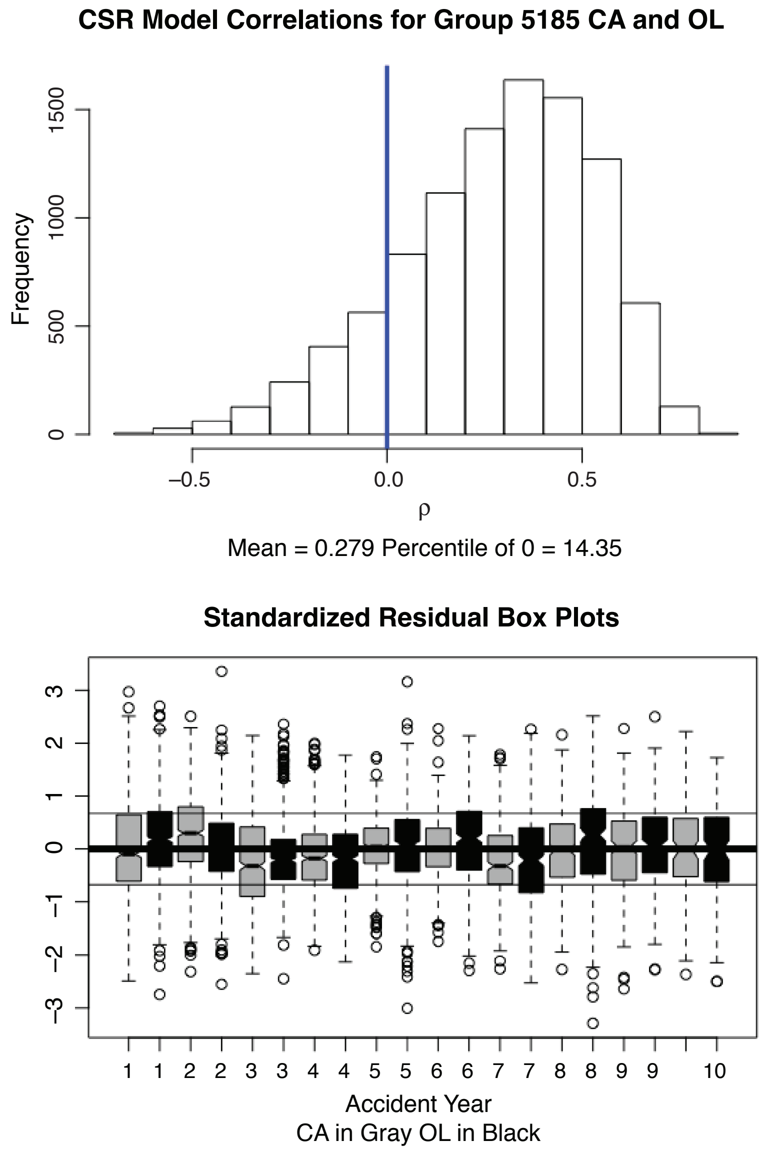
Figure 10.4. Summary Correlation Plots



In our set of 200 loss triangles, there are 119 lines of insurance pairs within the same insurer. I calculated the $\widehat{elpd}_{loo}$ statistics under the dependent and independent assumptions, with the result that for the CSR and IPI models, the independent assumption was favored in *all* 119 cases. for the SCC model, the independent assumption was favored in *all but one* of the 119 cases — Insurer 715 for CA and PA. A close call, where the independent assumption was barely favored, Insurer 5185 for CA and OL has an interesting and informative summary correlation plot. See Figures 10.5–10.7.

- As we saw with the illustrative insurer, the posterior distribution of the ρ parameter is quite wide. The strength of the positive correlation for this insurer decreases with the ability of the model to capture the accident year effect.
- Ignoring the accident year effect in the SCC model completely flipped the correlation from positive to negative. In looking at the standardized residual Box

Figure 10.5. Summary Correlation Plots



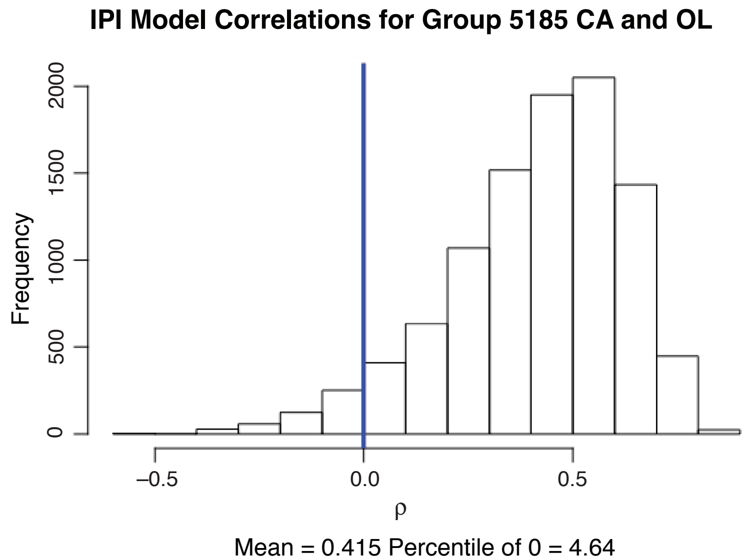
plots, we see that the two lines of insurance appear to be counter-cyclical if we fail to recognize the accident year effect.

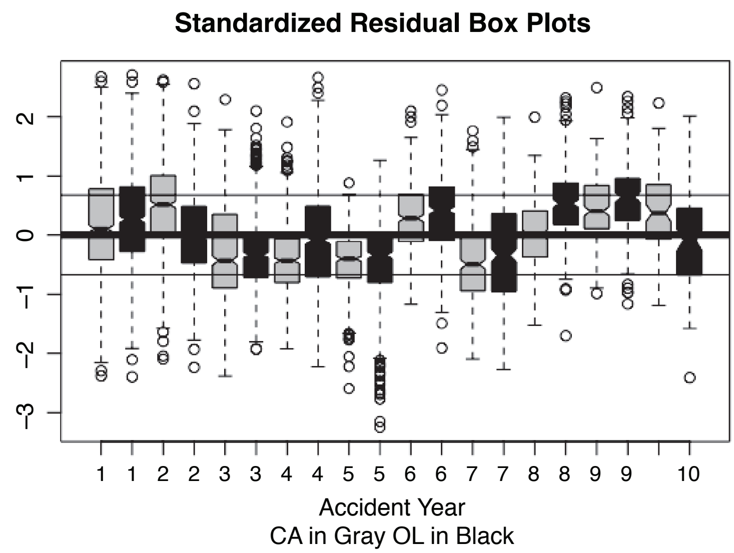
At this point, we drop any additional examination of the SCC model.

There was a reversal when comparing the $\widehat{elpd}_{test}$ statistics on the lower triangle data. The dependent assumption was favored over the independent assumption in 75 and 73 out of the 119 pairs of lines for the CSR and IPI models respectively.[15] I spent a fair amount of time looking for an explanation, but found none. The only conclusion I could draw was that there were some unknown variables influencing the

[15] In many cases the differences were smaller than the differences for the $\widehat{elpd}_{test}$ statistics. I did a bootstrap analysis of the MCMC sampling error and found that many of the differences were too large to be explained by that error.

Figure 10.6. Summary Correlation Plots





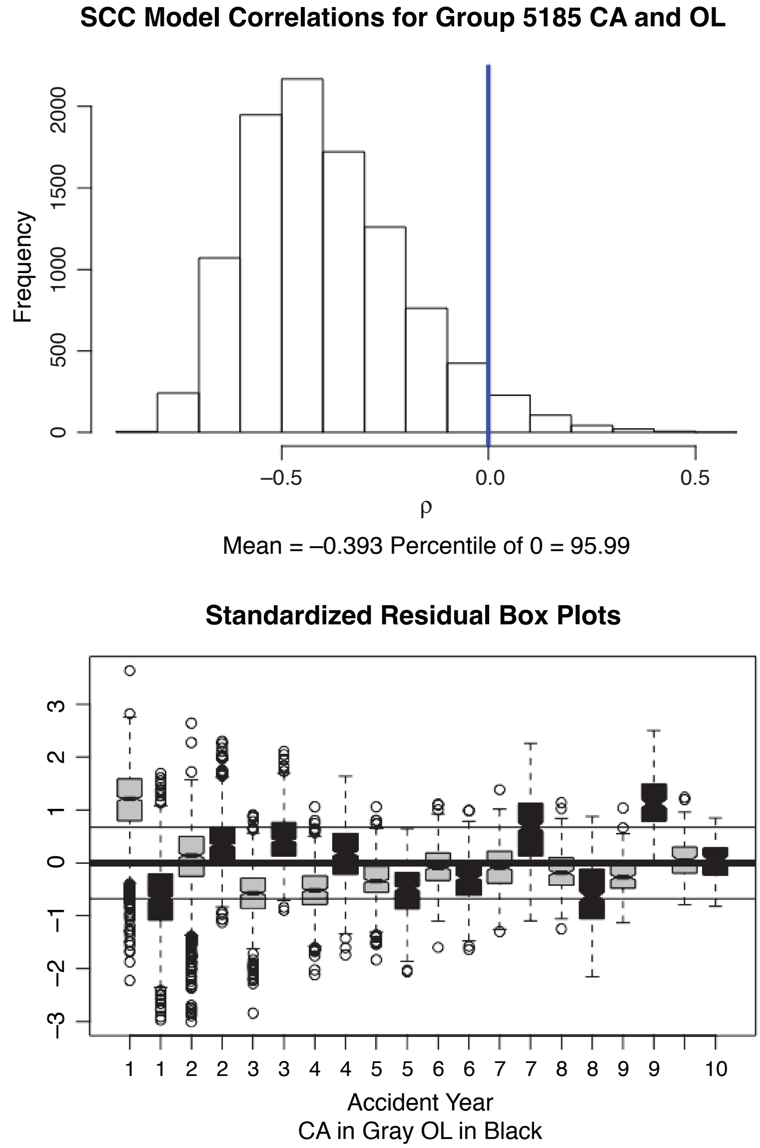
lower triangle data that were not captured by the CSR and IPI models when fit to the upper triangle data.

While it is tempting to justify the independent assumption with $\widehat{elpd}_{loo}$ statistics, in light of the $\widehat{elpd}_{test}$ statistics, I do not think that is prudent at this time. However, we can do some sensitivity tests to see how bad things can get.

Figure 10.8 summarizes the posterior mean of ρ and the effect of the correlation of the standard error of the estimated ultimate loss.[16] It turns out that the risk of a gross understatement of the standard error is fairly small for both models.

[16] Because of the skewness of the lognormal distribution, one should not always expect a positive ρ to increase the standard error.

Figure 10.7. Summary Correlation Plots



In summary, it turns out that dependencies between lines of business are very hard to detect given only the data in the upper triangle. The main source of dependency identified in this section is the failure to recognize the accident year effect. There may be other external effects. A plot similar to the side-by-side standardized residual plot along a different variable may help find other causes of dependency. If such a variable can be found, it should be included in the stochastic loss reserve model.

Figure 10.8. Summary Statistics Over 119 Lines of Business Pairs

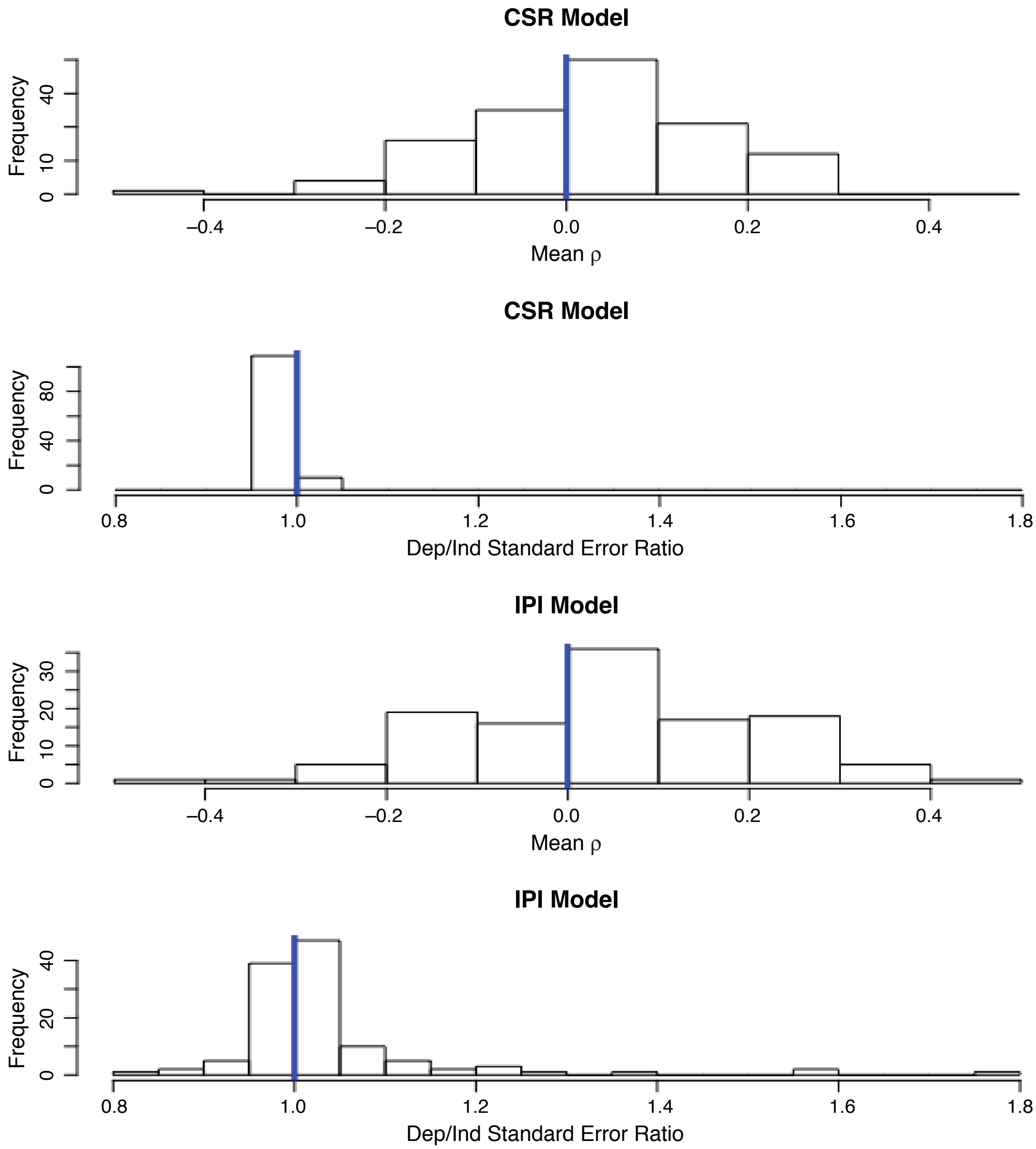


Table 10.1. ρ ⇔ Coefficient of Correlation

ρ	Coefficient of Correlation
0.0	0.000
0.1	0.060
0.2	0.125
0.3	0.205

11. Risk Margin[17]

Now that we have covered a variety of models that attempt to describe a predictive distribution of possible outcomes, we now turn to a principal reason for fitting such a model—posting a liability on an insurer's balance sheet for unpaid claims.

There is a growing consensus, supported by IFRS 17,[18] emerging in the insurance industry that the liability should consist of the expected present value of the unpaid claims, plus a risk margin. One expression of this consensus can be found in the "technical provisions" of the European Solvency II directive.[19] This directive was first published in 2009, and, after a number of amendments, was finally put into effect on January 1, 2016.

These technical provisions refer to the insurer's liability for unpaid losses. Specifically:

1. "The value of the technical provisions shall be equal to the sum of a best estimate and a risk margin."
2. "The best estimate shall correspond to the probability-weighted average of future cash flows, taking account of the time value of money using the relevant risk-free interest rate term structure."
3. "The risk margin shall be calculated by determining the cost of providing an amount of eligible own funds equal to the Solvency Capital Requirement necessary to support the insurance obligations over the lifetime thereof."
4. "Insurance undertakings shall segment their insurance obligations into homogeneous risk groups, and as a minimum by lines of business, when calculating the technical provisions."

This section illustrates a way to implement the principles expressed in the above provisions of the act. While the act goes on to provide some specific recommendations on how to implement those provisions, the scope of this section is more to show how to implement the principles underlying Solvency II using the Bayesian MCMC technology.

A Bayesian MCMC stochastic loss reserve model provides an arbitrarily large number of equally likely simulations that enable one to simulate future cash flows of the liability. From these simulations, it is possible to describe any future state in the model's time

[17] An earlier version of this section is in Meyers 2018.

[18] See the International Actuarial Association (2018) publication on risk adjustments.

[19] The provisions quoted here are stated in Section 2, Article 77 and Article 80, of Chapter VI of the act, p. 222. http://register.consilium.europa.eu/pdf/en/09/st03/st03643-re01.en09.pdf.

horizon, including those states necessary to calculate the technical provisions. That is what this section will do.

As the focus of this monograph is on the stochastic features of estimating loss liabilities, it makes simple assumptions about other relevant parameters, such as the solvency criteria, interest rates and the timing of loss payments.

Here is a high-level description of that cash flow.

1. At the end of the current calendar year (call this time $t = 0$), the insurer posts its best estimate of the liability. The insurer also posts the amount of capital, K_0, needed to contain the uncertainty in this estimate. It invests K_0 in a fund that earns income at the risk-free interest rate i.
2. At the end of the next calendar year, at time $t = 1$, the insurer uses its next year of loss experience to reevaluate its liability.[20] It then posts its updated estimate of the liability and the capital, K_1, needed to contain the uncertainty in this estimate. The difference between $K_0 \cdot (1 + i)$ and K_1 is returned to the investor. If that difference is negative, as it occasionally will be, the investor is expected to contribute an amount to make up that difference.
3. The process continues for future calendar years, t, with the amount,

$$K_{t-1} \cdot (1+i) - K_t,$$

being returned to (or being contributed by) the investor.
4. At some time $t = u$, the loss is deemed to be at ultimate, i.e., no significant changes in the loss is anticipated and so we set $K_t = 0$ for $t > u$. For the examples in this section, $u = 9$.

The present value, discounted at the risky rate r, of the amount returned is equal to

$$\sum_{t=1}^{\mu+1} \frac{K_{t-1} \cdot (1+i) - K_t}{(1+r)^t} \tag{11.1}$$

Since $r > i$, this present value will be less than the initial capital investment of K_0. To adequately compensate the investor for taking on the risk of insuring policyholder losses, the difference can be made up at time $t = 0$ by what we now define as the cost of capital risk margin, R_{COC}.

$$R_{COC} \equiv K_0 - \sum_{t=1}^{\mu+1} \frac{K_{t-1} \cdot (1+i) - K_t}{(1+r)^t} = (r-i) \cdot \sum_{t=0}^{u} \frac{K_t}{(1+r)^{t+1}} \tag{11.2}$$

with the second equality coming after some algebraic manipulations.[21]

[20] As the risk margin is for the current liability, the risk margin does not consider new business in future calendar years.

[21] Note that R_{COC} is similar to, but not identical to, the Solvency II risk margin: $R_{SII} \equiv (r-i) \cdot \sum_{t=0}^{u} \frac{K_t}{(1+i)^t}$.

This monograph's proposed risk margin repeats this calculation for each of the 10,000 MCMC simulations produced by the CSR and IPI models. The posted risk margin will be the arithmetic average of the risk margins calculated for each cash flow.

The examples that follow assume that the risk-free rate, $i = 4\%$ and the risky rate, $r = 10\%$.

The problem that now needs to be addressed is the calculation of the K_ts for each simulated cash flow. A straightforward way to project a future cash flow for this process would be to take a fitted Bayesian MCMC model and simulate an additional calendar year of losses for $t = 1$. Then fit another Bayesian MCMC model to the original data and the simulated data to get the loss estimate and capital requirements for $t = 1$. Then continue this process for $t = 2, \ldots, u$.

While the execution speed of Bayesian MCMC software has significantly increased in recent years, repeating this for 10,000 simulated future cash flows would undoubtedly strain the patience of most practicing actuaries. This section will propose a faster way to simulate the future cash flows to calculate the capital requirements for this process.

Now that we have defined the cost of capital risk margin, here is an outline of the remainder of this section.

- First we show how to use the Bayesian MCMC simulations to calculate the cash flows and corresponding loss estimates implied by the model.
- Then we show how to calculate the best estimate and the risk margins from the cash flows.
- We then investigate the effect of insurer size and line of business on risk margins.
- Then we address the effect of diversification by line of business.
- The calculations above assume that the required capital was calculated for what one might call an "ultimate" time horizon. In the final part of this section, we show how to incorporate a one-year time horizon into the calculations.

The examples will use the CSR model and the parameters of the IPI model that describe paid losses. What is relevant is that, given the loss triangle, T_0, the model uses Bayesian MCMC to obtain a sample of 10,000 equally likely lognormal, $\{\mu_{w,d}^j, \sigma_d^j\}_{j=1}^{10,000}$, simulations from the posterior distribution, $\{\mu_{w,d}, \sigma_d | T_0\}$. This section uses these simulations to describe a sample from the set of possible future cash flows.

With these simulations we can calculate the best estimate of the liability, as specified by Solvency II, as the probability weighted average of the present value of expected future cash flows. Recalling that the mean of a lognormal distribution is equal to $e^{\mu+\sigma^2/2}$, this will be equal to the expected value of the differences in the cumulative payments. To be specific we define the expected payment during development year d for simulation j as:

$$P_{w,d}^j = e^{\mu_{w,d}^j + \left(\sigma_d^j\right)^2/2} - e^{\mu_{w,d-1}^j + \left(\sigma_{d-1}^j\right)^2/2} \qquad \text{for } d = 2, \ldots, 10$$

When using the IPI model,[22] we anticipate that the expected incurred loss at $d = 10$ will be different from the expected paid loss. So we define $P^j_{w,11} = e^{{}_I\mu^j_{w,10} + ({}_I\sigma^2_{10})/2} - e^{\mu^j_{w,10} + (\sigma^2_{10})/2}$.

For simulation j the present value of the liability for the CSR (IPI) model is equal to

$$E^j_{best} = \sum_{w=2(1)}^{10} \sum_{d=12-w}^{10(11)} \frac{P^j_{w,d}}{(1+i)^{w+d-11.5}}$$

Then, since all simulations, j, are equally likely, the "probability-weighted average of future cash flows, taking account of the time value of money" is

$$E_{Best} = \frac{1}{10{,}000} \sum_{j=1}^{10{,}000} E^j_{Best} \tag{11.3}$$

This calculation assumes that the losses are paid one half-year before the end of future calendar year $t = w + d - 11$.

For the scope of this paper, let's also select the ultimate loss, U_j, associated with the *jth* simulation set to be the sum of the expected values of the losses for $d = 10$ over all the accident years.[23] I.e.,

$$U_j = \sum_{w=1}^{10} e^{\left(\mu^j_{w,10} + \left(\sigma^j_{10}\right)^2/2\right)} \tag{11.4}$$

For the lower triangle of $\{\tilde{C}^j_{w,d}\}_{j=1}^{10{,}000}$, define the simulated loss trapezoid for future calendar year t that includes the upper loss triangle, T_0, and the first t diagonals of from the lower loss triangle, i.e.,

$$T^j_t \equiv \begin{cases} C_{w,d} & \text{for } w = 1, \ldots, 10 \text{ and } d = 1, \ldots, 11 - w \\ \tilde{C}^j_{w,d} & \text{for } w = t+1, \ldots, 10 \text{ and } d = 12 - w, \ldots, \min(11 - w + t, 10) \end{cases} \tag{11.5}$$

where $\tilde{C}^j_{w,d}$ is simulated from a lognormal distribution with parameters $\mu^j_{w,d}$ and σ^j_d.

In practice, all we have is an observed loss trapezoid, T_t. Then using Bayes' Theorem and the fact that, initially, all j are equally likely, the probability that the simulation index is equal to j given T_t, for $t > 0$, is given by

$$\Pr[J = j | T_t] = \frac{\prod_{C_{w,d} \in T_t} \phi\left(\log(C_{w,d}) | \mu^j_{w,d}, \sigma^j_d\right)}{\sum_{k=1}^{10{,}000} \prod_{C_{w,d} \in T_t} \phi\left(\log(C_{w,d}) | \mu^k_{w,d}, \sigma^k_d\right)} \tag{11.6}$$

where ϕ is the probability density function for the normal distribution.

---

[22] When the parameters in this section come from an incurred model, we use the left subscript "I" before the μ and σ parameters. There will be no left subscript if the parameters come from a paid model.

[23] For the CSR model, paid $\mu_{w,10}$ and $\sigma_{w,10}$. For the IPI model we use incurred ${}_I\mu_{w,10}$ and ${}_I\sigma_{w,10}$.

At this point, there are a number of options one can choose to calculate the various statistics that are of interest to insurer risk managers. For example, given T_t, one could calculate the ultimate loss estimate, E_t as

$$E_t \equiv E\left[\sum_{w=1}^{10} C_{w,10} \middle| T_t\right] = \sum_{j=1}^{10{,}000} \Pr[J = j | T_t] \cdot U_j. \tag{11.7}$$

If one accepts that the Bayesian MCMC output is representative of all future scenarios, Equation 11.7 is exactly the right calculation for the loss estimate given T_t. But let's consider what one should do to calculate, say, the 99.5th percentile. First one should sort the MCMC simulations in order of increasing U_j. It is not uncommon to find a case where there is a simulation, j, with $\Pr\,[J \le j | T_9] = 0.9900$ and $\Pr[J \le j + 1 | T_9] = 0.9960$.

To deal with this, I decided to calculate the statistics of interest by first taking a random sample of size 10,000 (with replacement), $\{S_t\}$, of the U_js with sampling probabilities $\Pr[J = j | T_t]$. It was easy to implement and surprisingly fast in R. This is subject to an additional simulation error, but it should be small.

The "statistics of interest" for risk margin are, for $t = 0, \ldots, 9$:

1. The mean, E_t, which is equal to the arithmetic average of $\{S_t\}$.
2. The Tail Value-at-Risk at the α level (TVaR@α), which is equal to the arithmetic average of the $(1 - \alpha) \cdot 10{,}000$ highest values of $\{S_t\}$.[24]

Let's denote the total required capital by $K_t \equiv \text{TVaR@}\alpha - E_t$.

We summarize the above in Algorithm 1.

Algorithm 1. Calculate Capital Simulations

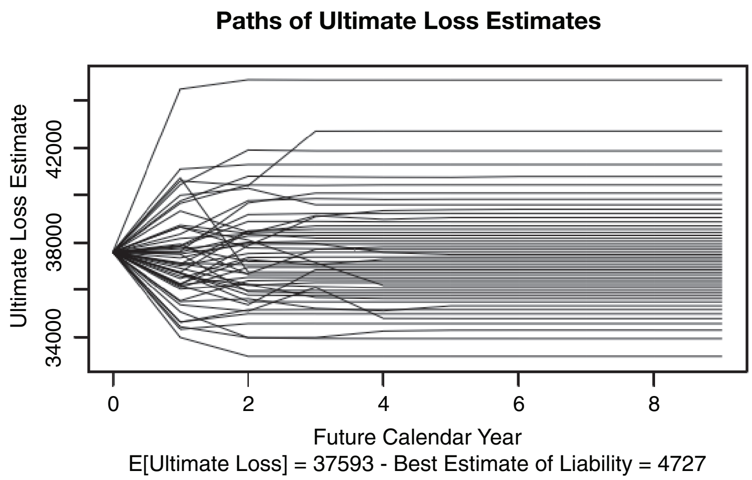
for $k = 1, \ldots, 10{,}000$ **do**
  for $t = 0, \ldots, 9$ **do**
    Simulate cash flows $\{T_t^k\}$ using the $\{(\mu_{w,d}^k, \sigma_d^k)\}$
    Use Equation 11.6 to calculate $\Pr[J = j | T_t^k]$ for each $j = 1, \ldots, 10{,}000$
    Take a random sample of size 10,000 with replacement, $\{S_t^k\}$, of $\{U_j\}_{j=1}^{10{,}000}$ with sampling probabilities $\Pr[J = j | T_t^k]$.
    Set E_t^k equal to the arithmetic average of $\{S_t^k\}$.
    Set K_t^k equal to the arithmetic average of the highest $(1 - \alpha) \cdot 10{,}000$ highest values of $\{S_t^k\}$, minus E_t^k.
  end for
end for

The examples in this paper use $\alpha = 97\%$.

Calculating E_t^j for $t = 0, \ldots, 9$ yields the *jth* path that the loss estimate takes as it moves toward its ultimate value. Of interest for what follows is the set of possible paths that the loss estimate can take. Figures 11.1 and 11.2 show the paths for the paths that contain the 100*th*, the 300*th*, . . . , and the 9,900*th* highest E_9^js of the illustrative

[24] While this section does not use the Value-at-Risk (VaR) in its examples, one could calculate the VaR@α as the $(1 - \alpha) \cdot 10{,}000^{th}$ highest value of $\{S_t\}$.

Figure 11.1. CSR Model



insurer for the CSR and IPI models. These figures illustrate that the E_t^js tend to become more certain over time. These figures also show the best estimate.

Also of interest are the paths of the required capital, K_t^j, for $t = 0, \ldots, 9$. Figures 11.3 and 11.4 show the paths of K_t^j that correspond to the paths taken by E_t^j in Figures 11.1 and 11.2. These figures illustrate that as the estimates of the E_t^js become more more certain, the required capital, K_t^j, tends to decrease over time.

What stands out in these figures is the significant reduction in the necessary capital that result from using the more accurate IPI model. Do not be distracted by the increase in the best estimate for this particular example. Note that the estimated ultimate loss produced by the IPI model is closer to the actual outcome of 40,000.

We now apply the cost of capital risk margin formula, given by Equation 11.2, to each of the required capital paths, $\{K_0^j, \ldots, K_9^j\}_{j=1}^{10,000}$. Recall that the formula defined the cost of capital risk margin as the present value of the capital released as the loss reserve liability becomes more certain. Figures 11.5 and 11.6 show the paths of released capital that correspond to the paths taken by the K_t^js in Figures 11.3 and 11.4. In general,

Figure 11.2. IPI Model

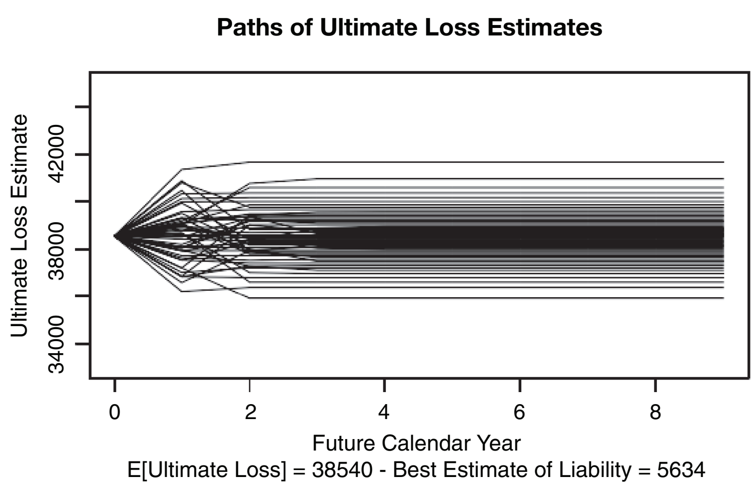


Figure 11.3. CSR Model

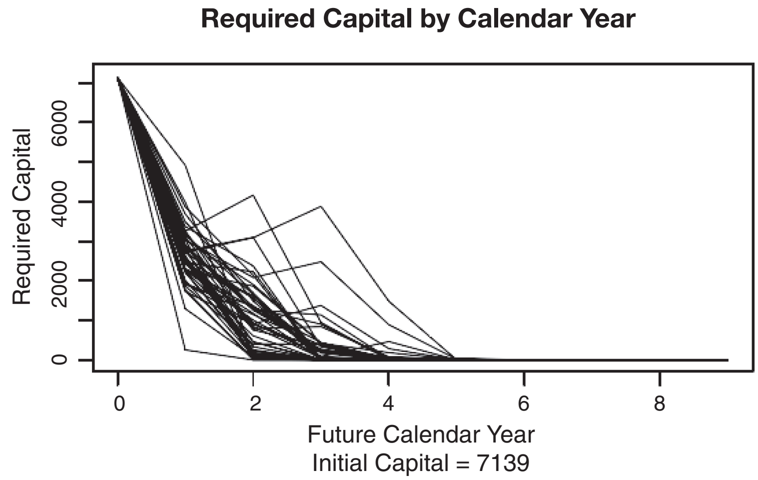


Figure 11.4. IPI Model

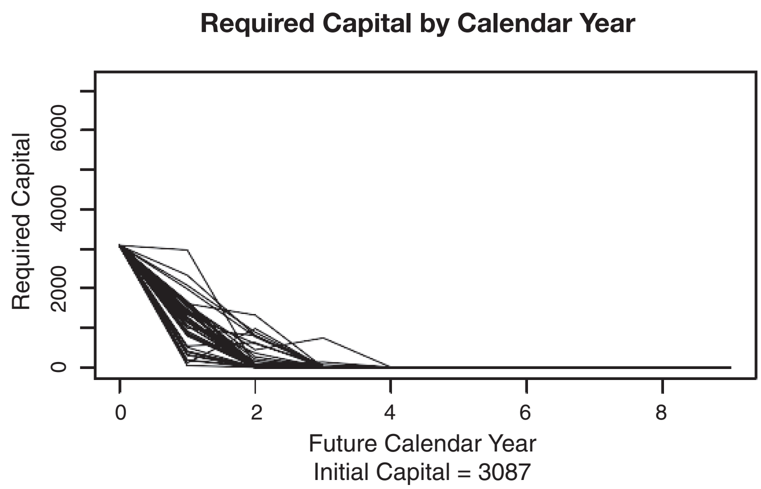


Figure 11.5. CSR Model

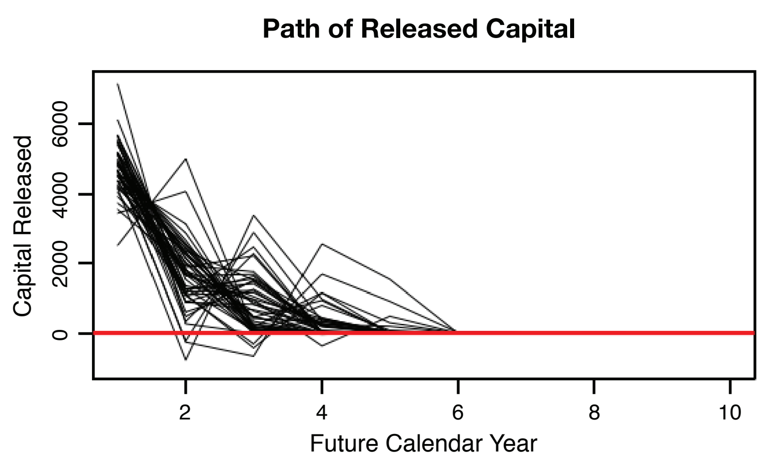
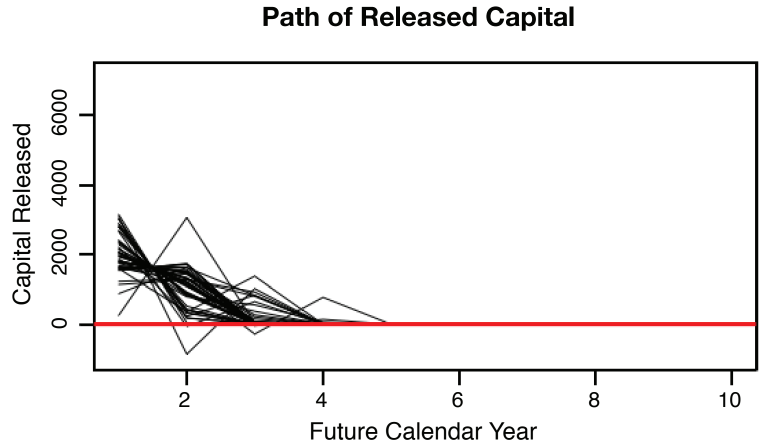


Figure 11.6. IPI Model



these figures show that most of the capital gets released early on, and that occasionally it is necessary to add capital. Figures 11.7 and 11.8 show the recommended posted risk margin calculation for each model.

Of interest is the ratio of the risk margin and the size of the best estimate. To investigate, I calculated the risk margins for all 200 loss triangles in our data. After some exploratory analysis, I concluded that: (1) there are significant differences by line of business; and (2) there is an approximate linear relationship between the log of the risk margin and the log of the best estimate. Figure 11.9 shows the plots of the $log(R_{COC})$ against $\log(E_{Best})$, along with the coefficients and their standard errors of an ordinary linear regression of the form

$$\log(R_{COC}) = a + b \cdot \log(E_{Best}) \tag{11.8}$$

We can rewrite Equation 11.8 in the form

$$\frac{R_{COC}}{E_{Best}} = e^{a} \cdot (E_{Best})^{b-1} \tag{11.9}$$

Figure 11.7. CSR Model

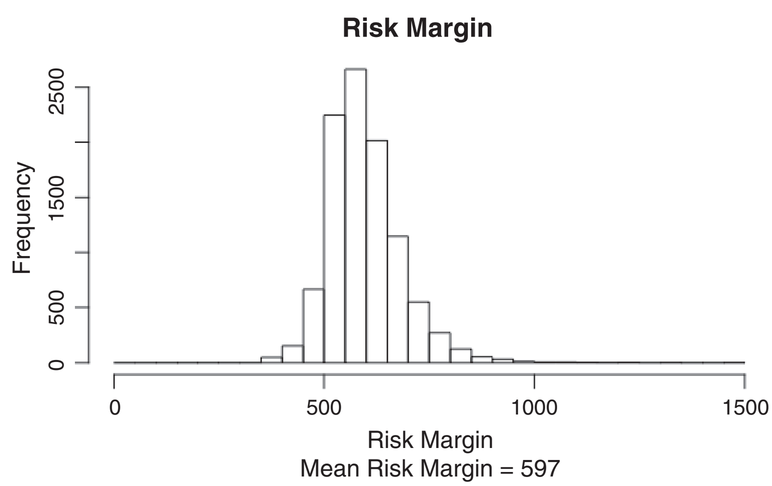


Figure 11.8. IPI Model

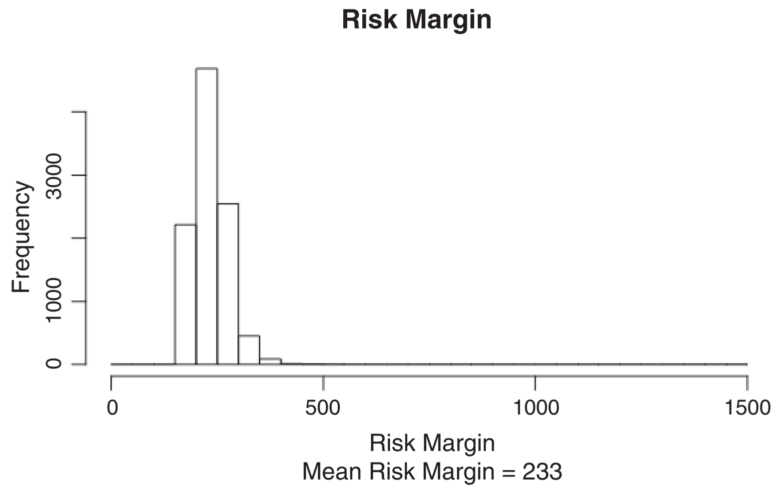
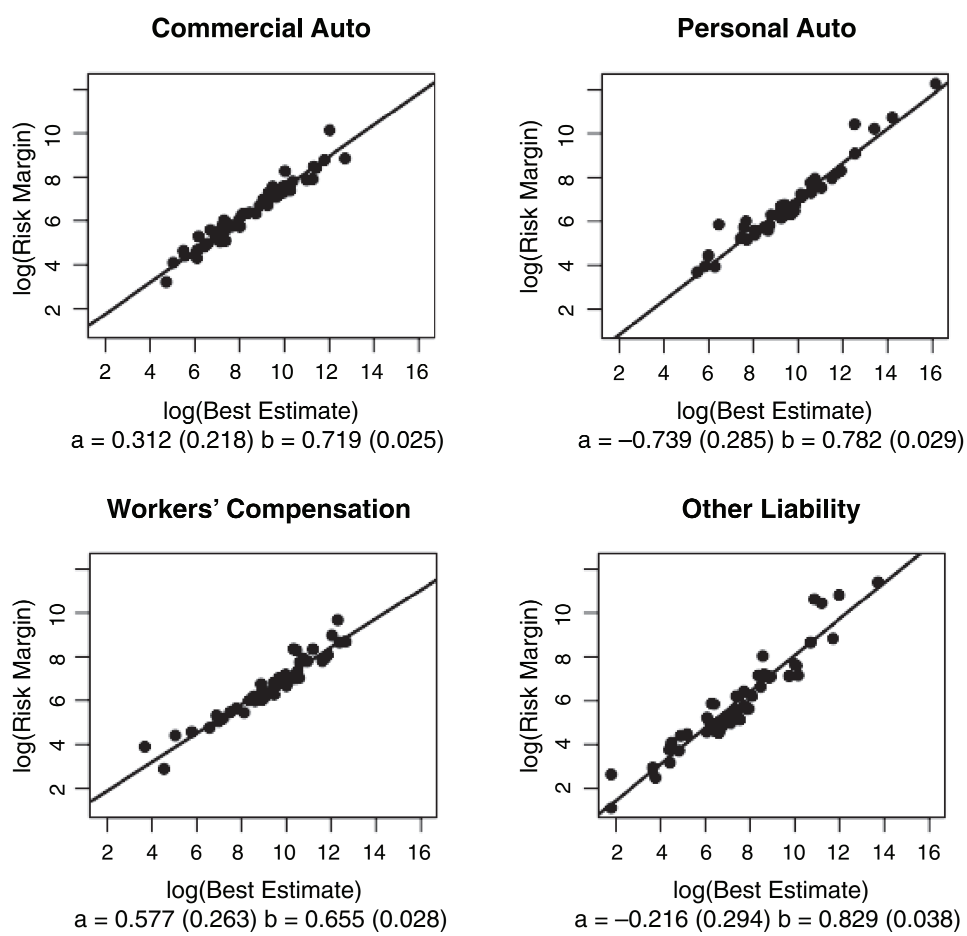


Figure 11.9. CSR Model



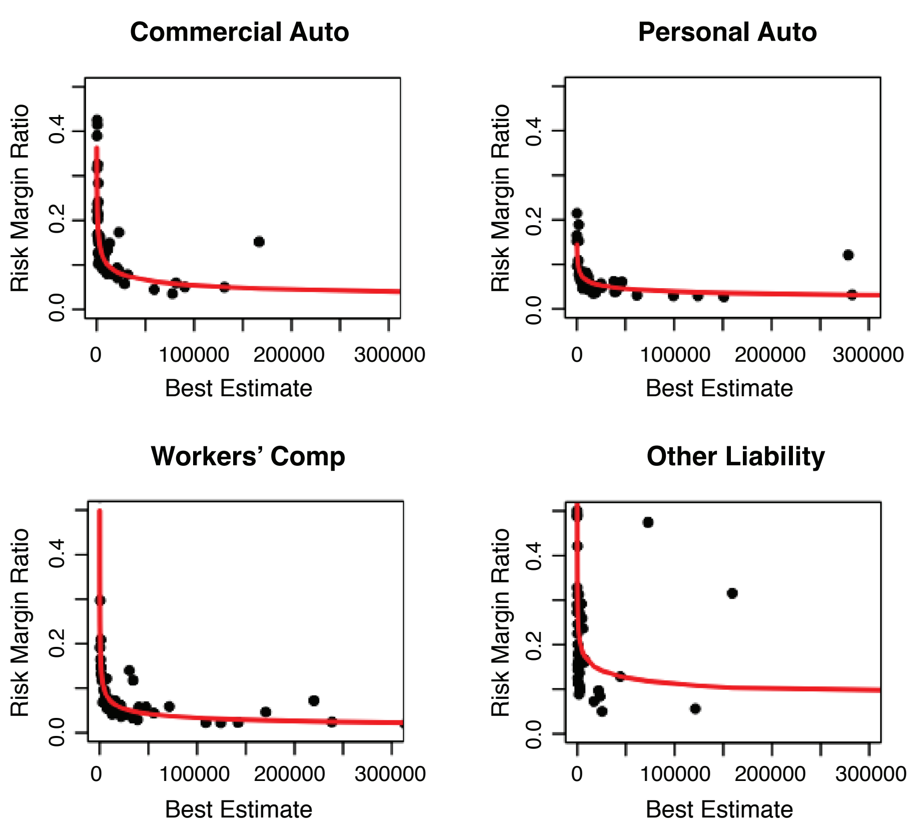
Note from Figure 11.8 that $b < 1$ for all four lines of insurance. This implies that the risk margin to best estimate ratio decreases as the best estimate increases. As Figure 11.10 shows, the ratio can be quite high for insurers with small best estimates. Some insurers might object, especially if the line with the high ratio is a small part of the insurer's book of business.

As noted earlier, the EU Solvency II provision states explicitly that "Insurance undertakings shall segment their insurance obligations into homogeneous risk groups, and as a minimum by lines of business, when calculating the technical provisions." This means that the total risk margin for a multiline insurer is the sum of the risk margins over its individual lines of business.

Longtime observers of the insurance business have recognized that multiline insurers benefit from the diversification of their risk of loss. This being the case, they might well want to reflect the benefits of diversification in their risk margins. The problem with a formal recognition of diversification is that the benefits have been difficult to quantify. What many are afraid of is the possibility that significant losses from the different lines of business could happen at the same time. This possibility is often referred to as the "dependency problem."

As such, the Solvency II non-recognition of diversification may appear to some to be prudent. But to others, especially in light of the results in the last section, it may seem like overkill. If there is some dependency, it is likely to be noticeably less severe than what the Solvency II directive specifies. So let's look at a possible compromise approach.

Figure 11.10. CSR Model



Algorithm 2. Calculate Samples for Dependent Lines

```
for k = 1, . . . , 10,000 do
    for t = 1, . . . , 9 do
        Simulate an L-tuple vector {p_l^k}_{l=1}^L of uniform(0,1) numbers from the normal copula 𝒞
        with coefficient of correlation, ρ.
        For each line of business, l, select _lQ_t^k to be the {p_l^k} • 10,000 highest value of {_lS_t^k}.
    end for
    Set the total ultimate loss sample _TS_t^k = _1Q_t^k + · · · + _LQ_t^k.
end for
```

Mathematical tools that can be used to describe dependency have been available for quite some time. See, for example, Frees and Valdez (1998) and Wang (1998). The main tool described in these papers is called a copula, which is a multivariate distribution on an L-dimensional unit hypercube in which the marginal distributions have a uniform(0,1) distribution. Given a copula $\mathcal{C}$ and samples $\{_lS_t\}$, (see Algorithm 1) for each line l of L lines of business, one begins to calculate R_{COC} by first executing Algorithm 2.

Use the output of this algorithm to calculate $\{_TS_t^k\}$ for $t = 1, \ldots, 9$. Then place $\{_TS_t^k\}$ into Step 5 of Algorithm 1.

There are seven insurer groups with loss triangles in each line of insurance in our set of loss triangles. Table 11.1 shows the effect of diversification for each of the seven groups.[25] The combined risk margin for selected correlation coefficients, ρ, was calculated using the Algorithm 2 sample, $_TS_t^k$, inserted into Algorithm 1.

One can see from Table 11.1 that the diversification effect can vary significantly by insurer. The effect depends on the line mix of the insurer. This can be best seen in the case of Insurance Group 1767, a very large personal lines insurer. This insurer's book of business is dominated by the Personal Auto. To see the effect of an individual line of insurance, I calculated marginal risk margin by subtracting the combined risk margin for all other lines, from the all-line combined risk margin. I then allocated the all-line risk margin to the individual line in proportion to the marginal risk margin. The results of this calculation are in Table 11.2.

Here we see that the dominant Personal Auto line gets allocated a very small diversification credit, whereas the minor Worker's Compensation line gets allocated a very large diversification credit.

To close out this section, we address the issue of time horizon. Required capital, as dictated by Solvency II, is determined by the value-at-risk at the 99.5*th* percentile of the ultimate loss estimate after one year. To change from the TVaR criteria used in this monograph to a value-at-risk criteria is a minor input adjustment to the input of the R-scripts. It is the change to the one-year time horizon that requires some work.

A high-level description of the methodology is to use a Bayesian MCMC model to obtain 10,000 equally likely scenarios that represent the future evolution of the line of business that produced the loss triangle. Then, as new losses come in, it uses Bayes'

[25] The standalone risk margins will differ slightly from those calculated using Algorithm 1. They were calculated using the uniformly distributed p_l^k's that came from the copula used by the other calculations in Tables 11.1 and 11.2.

Table 11.1. Diversification Effect—CSR Model

Group	Risk Margin	Amount	Diversification Credit%
715	Standalone Total	5519	—
—	Combined $\rho = 0.0$	2825	48.8
—	Combined $\rho = 0.1$	3171	42.6
—	Combined $\rho = 0.2$	3491	36.9
—	Combined $\rho = 0.3$	3798	31.6
1538	Standalone Total	3964	—
—	Combined $\rho = 0.0$	2150	45.8
—	Combined $\rho = 0.1$	2372	40.3
—	Combined $\rho = 0.2$	2582	35.2
—	Combined $\rho = 0.3$	2782	30.3
1767	Standalone Total	315085	—
—	Combined $\rho = 0.0$	236532	24.9
—	Combined $\rho = 0.1$	245264	22.4
—	Combined $\rho = 0.2$	253632	19.9
—	Combined $\rho = 0.3$	261936	17.4
3240	Standalone Total	6256	—
—	Combined $\rho = 0.0$	4405	29.6
—	Combined $\rho = 0.1$	4615	26.4
—	Combined $\rho = 0.2$	4821	23.4
—	Combined $\rho = 0.3$	5022	20.3
5185	Standalone Total	6363	—
—	Combined $\rho = 0.0$	3357	47.2
—	Combined $\rho = 0.1$	3736	41.4
—	Combined $\rho = 0.2$	4091	36.0
—	Combined $\rho = 0.3$	4426	30.9
13439	Standalone Total	913	—
—	Combined $\rho = 0.0$	510	44.1
—	Combined $\rho = 0.1$	552	39.6
—	Combined $\rho = 0.2$	595	35.2
—	Combined $\rho = 0.3$	639	30.6
14176	Standalone Total	3986	—
—	Combined $\rho = 0.0$	2196	44.9
—	Combined $\rho = 0.1$	2404	39.8
—	Combined $\rho = 0.2$	2608	34.8
—	Combined $\rho = 0.3$	2809	30.1

Table 11.2. Diversification Effect By Line for Group 1767— CSR Model, $\rho = 0$

		Risk Margin			
Line	Best Estimate	Marginal	Allocate	Standalone	Div. Credit
CA	336,301	660	927	6,985	86.7%
PA	10,280,570	147,566	207,448	215,633	3.8%
WC	238,395	114	161	5,676	97.2%
OL	909,123	19,915	27,996	86,791	67.7%
Total	11,764,389	168,255	236,532	315,085	24.9%

Theorem to update the probability of each scenario. From these updated probabilities, one then calculates the statistics that are needed to calculate the risk margin.

Under a one-year time horizon capital requirement, the capital is determined by the estimate of the ultimate losses after one more calendar year of loss experience. A key step in this methodology is to determine the ultimate loss estimate associated with each MCMC simulation. For the ultimate time horizon it is simply U_j, given by Equation 11.4. However, as Figures 11.1 and 11.2 illustrate, with only one year of losses from a given MCMC simulation, there may be several MCMC simulations with a significant positive probability.

To get a good estimate, $O_{t,j}$, of the expected ultimate loss for the *jth* MCMC simulation, one can simulate future loss experience, given the *jth* MCMC simulation, and calculate the ultimate loss estimate M times. Then set $O_{t,j}$ equal to the average of those estimates. The details are in Algorithm 3.

Both the accuracy of the estimate of $O_{t,j}$ and the computer run time increase with M. I experimented with different values of M and found that $M = 12$ obtained results that were sufficiently accurate given the intrinsic variation of the underlying MCMC simulations.

Algorithm 3. Calculate $O_{t,j}$ Estimates by Calendar Year

for $m = 1, \ldots, M$ **do**
  for $j = 1, \ldots, 10{,}000$ **do**
    for $t = 1, \ldots, 9$ **do**
      Simulate T_t using the parameters $(\mu^j_{w,d}, \sigma^j_d)$.
      Use Equation 11.6 to calculate $\{\Pr[J = j|T_t]\}_{j=1}^{10,000}$.
      Use Equation 11.7 to calculate the ultimate loss estimate, $O^m_{t,k}$.
    end for
    Set $O^m_{10,j} = O^m_{9,k}$
  end for
end for
for $j = 1, \ldots, 10{,}000$ **do**
  for $t = 1, \ldots, 10$ **do**
    Set $O_{t,j} = \text{mean}\ (O^m_{t,k})$.
  end for
end for

Algorithm 4. Calculate Capital Scenarios for a One-Year Time Horizon

```
for k = 1, . . . , 10,000 do
    for t = 0, . . . , 9 do
        Simulate cash flows {T_t^k} using {(μ_{w,d}^k, σ_d^k)}
        Use Equation 11.6 to calculate Pr[J = j|T_t^k] for each j = 1, . . . , 10,000.
        Take a random sample of size 10,000 with replacement, {S_t^k}, of the {O_{t+1,j}}_{j=1}^{10,000} with
        sampling probabilities Pr[J = j|T_t^k].
        Set E_t^k equal to the arithmetic average of {S_t^k}.
        Set C_t^k equal to the arithmetic average of the highest (1 – α) • 10,000 highest values
        of {S_t^k}, minus E_t^k.
    end for
end for
```

Use Algorithm 4 to calculate the risk margin for the one-year time horizon. In this algorithm, one simply substitutes $O_{t+1,\,j}$ for U_j in the 5*th* step of Algorithm 1. Given the output of Algorithm 4, one then calculates risk margins for each MCMC simulation by using Equation 11.2. The posted risk margin will then be the unweighted average of the risk margins for each simulation. Figures 11.11 to 11.16 are analogous to Figures 11.3 to 11.8 for the ultimate time horizon.

Figure 11.11. CSR Model

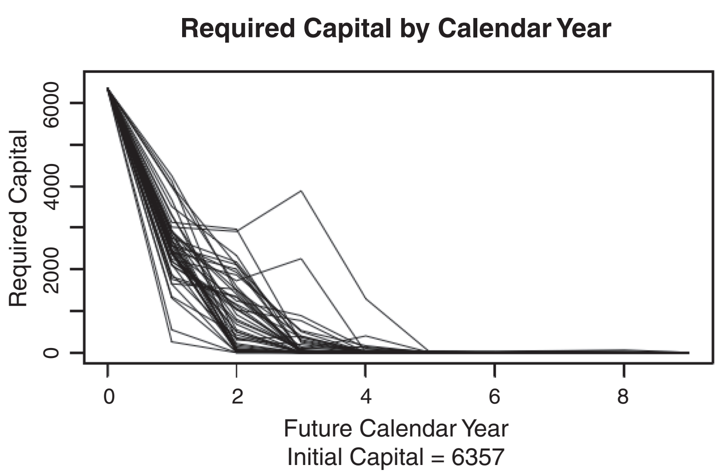


Figure 11.12. IPI Model

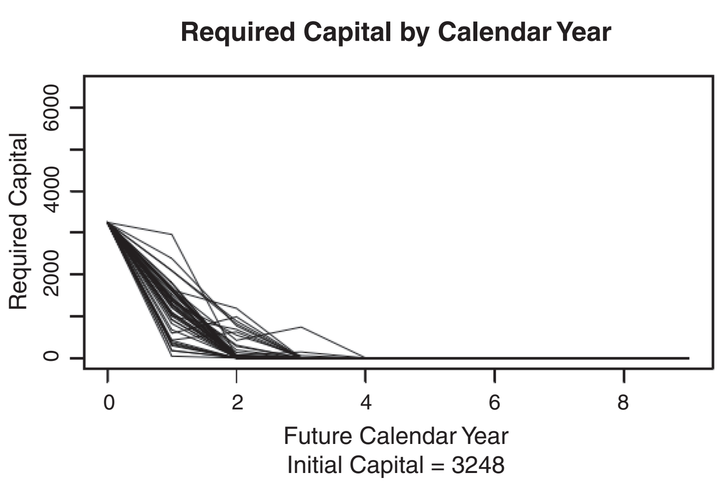


Figure 11.13. CSR Model

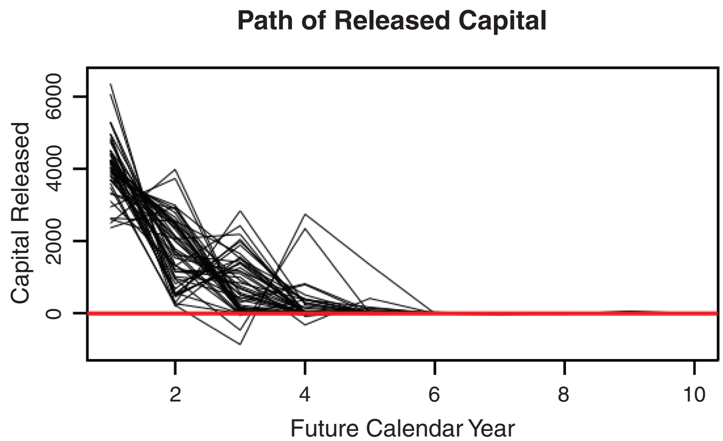


Figure 11.14. IPI Model

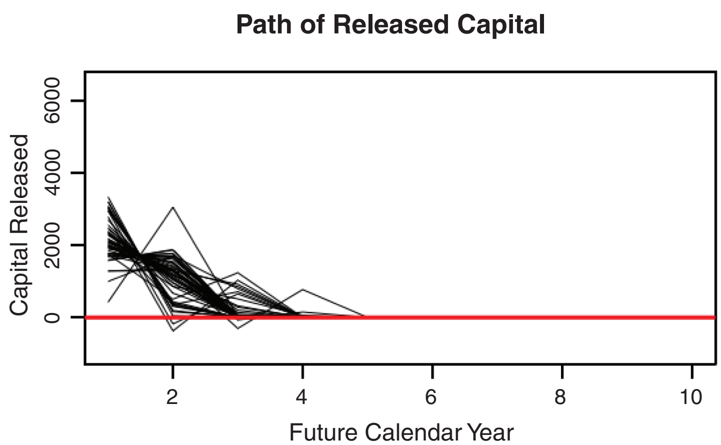


Figure 11.15. CSR Model

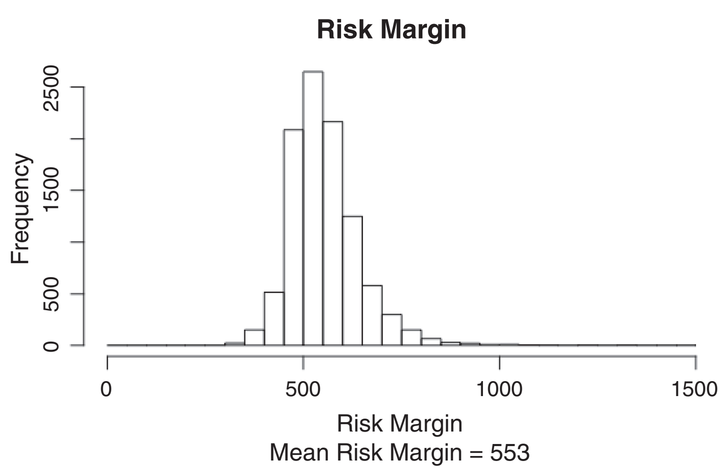
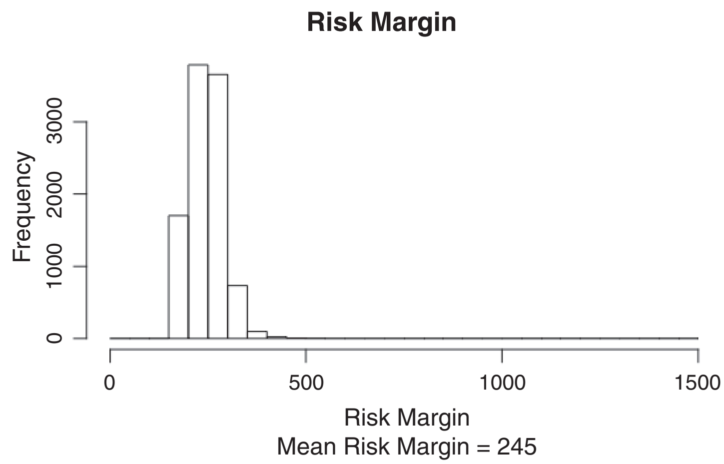


Figure 11.16. IPI Model



Note that initial IPI capital and the IPI risk margin are higher for the one-year time horizon than for the ultimate time horizon for the illustrative insurer, showing that we cannot automatically assume that the one-year time horizon will produce lower risk margins.

12. Summary and Conclusions

The main ideas behind this monograph are as follows.

- It implements the idea of large-scale retrospective testing of stochastic loss reserve models on real data. The goal is not to comment on the reserves of individual insurers. Instead, the goal is to test the predictive accuracy of specific models.
- As shortcomings in existing models are identified, it proposes new Bayesian MCMC models that attempt to overcome these shortcomings.
- It proposes prospective tests of Bayesian MCMC models that can be used to choose between competing models when estimating current liabilities.
- Finally, it shows how to use the output of a Bayesian MCMC model to calculate a cost of capital risk margin.

The data used in this study comes from the CAS Loss Reserve Database. It consists of hundreds of paid and incurred loss triangles that Peng Shi and I obtained from a proprietary database maintained by the NAIC. We are grateful that the NAIC allowed us to make these data available to the public. The data we used to build the models came from the 1997 NAIC Annual Statements. The outcomes came from subsequent statements. I selected 50 loss triangles from each of four lines of insurance. Details on how I selected the loss triangles for this study are in the Appendix.

There were two ways that we retrospectively tested a model. First, we used a model to predict a distribution of outcomes that we will observe in the future. When we do observe outcomes for a large number of predictions, we expect the percentiles of the outcomes to be uniformly distributed. Testing a set of percentiles for uniformity is a standard statistical procedure. We used *p-p* plots as a graphical test, and the Kolmogorov-Smirnov test statistic in this monograph.

The second retrospective test for comparing models was to calculate the expected log posterior density, $\widehat{elpd}_{test}$, on the lower triangle data. While this test is not meaningful in isolation, it does provide a way to compare the performance of different models.

There were two prospective tests in this monograph. The first test is graphical. It consists of standardized residual Box plots using parameters from a sample of MCMC simulations from the posterior distribution. These plots were done separately for each accident year and the development year. All Bayesian MCMC models discussed in this monograph performed fairly well for the plots by development year. What distinguished the models were the Box plots by accident year.

The second prospective test consisted of the "leave one out" cross-validation test statistic, $\widehat{elpd}_{loo}$, described by Vehtari et. al. (2017) as the expected log predictive density when each data point is left out. Like the $\widehat{elpd}_{test}$ statistic, it is used to compare the performance of different models.

Here is a high-level summary of the results obtained with these data using two currently popular models.

- The Mack model on incurred loss triangles, when tested by *p-p* plots, tended to under-predict the variability of the outcomes.
- Both the Mack and overdispersed Poisson (ODP) models on paid loss triangles, when tested by *p-p* plots, tended to over-predict the outcomes.

Moving on to some Bayesian MCMC models, here is a summary of results.

- The CRoss-Classified (CRC) by accident and development year model performed very similarly to the Mack and ODP models on incurred and paid loss triangles. It tended to under-predict the variability of the outcomes for incurred losses and tended to over-predict the outcomes for the paid losses.
- The Stochastic Cape Cod (SCC) model, a Bayesian MCMC version of the Bühlmann-Stanard Cape Cod model, performed very poorly on all prospective and retrospective tests. The standardized residual Box plot was the most revealing, indicating problems with the assumption of a constant expected loss ratio.
- The Changing Settlement Rate (CSR) model on paid loss triangles adjusts the development year parameters to account for a changing claim settlement rate. The result was that the CSR model performed better than the CRC model in over half the cases according to the $\widehat{elpd}_{loo}$ and $\widehat{elpd}_{test}$ statistics. One should expect the CRC model to perform better if the insurer did not actually change its settlement rate. The *p-p* plots indicated an excellent fit to the lower triangle data.
- The Correlated Accident Year (CAY) model on incurred loss triangles adjusts, on the log scale, the expected loss for the current year as a multiple of the prior year's residual. This adjustment usually generates a positive correlation between the losses of adjacent accident years.[26] Prospectively, the $\widehat{elpd}_{loo}$ statistic favors the CAY over the CRC model for a relatively small number of loss triangles. As was the case above, the correlation is not necessarily present in all loss triangles. But then, the CAY is favored over the CRC model for over half the the loss triangles using the $\widehat{elpd}_{test}$ statistic. The *p-p* plots look noticeably better for the CAY model than they do for the CRC model.
- The Integrated Paid and Incurred (IPI) model assumes that the accident year parameters are the same for both the paid and incurred loss triangles. If true, this allows for more accurate estimates, as there is more data contributing to the estimation of the accident year parameters. For the development year parameters, it uses the CSR model assumptions for the paid data, and the CAY model assumptions for the incurred data. The idea works. The standard errors of the loss estimates almost always reduced by a significant amount. The $\widehat{elpd}_{loo}$ and $\widehat{elpd}_{test}$

[26] The model does not automatically favor the sign of the correlation parameter.

statistics favor the IPI model over the corresponding CSR and CAY models over half the time. While the *p-p* plots for each line of insurance are all within the 95% Kolmogorov-Smirnov critical values, the IPI model works best for the auto liability lines of insurance where the paid and incurred losses are almost equal by the tenth development year. The model works less well for Workers' Compensation where a large portion of the losses remain unpaid after 10 years.

While the focus of the preceding analyses was to evaluate the predictive accuracy of various models, the prospective tests described above can be used to evaluate models proposed for setting an insurer's current loss reserve liability. Here is how I envision a loss reserve analysis might proceed with current data.[27]

1. Start by fitting the Mack model using the R "ChainLadder" package for both the paid and the incurred loss data. Use the "plot" function in that package to visually check that the triangle data is reasonable.
2. Fit each of the following models.[28] Initially, I suggest using wide prior distributions as described in Section 5 in order to see if any surprises are lurking.
   – The CRC and CSR models for paid data.
   – The CRC and CAY models for incurred data.
3. Create the standardized residual Box plots for each model.
4. Calculate the $\widehat{elpd}_{loo}$ statistics for each model.
5. Based on the standardized residual Box plots and the $\widehat{elpd}_{loo}$ statistics, select both a paid, and an incurred loss model.
6. At this point, one might want to consider changing the prior distributions to match their true prior expectations, or make other model modifications. Some examples of a modification would be to:
   – Tighten the prior distributions for the *logelr* and the α_w parameters.
   – Force the last few β_d parameters to increase toward zero.
   – Allow the ρ parameter to exponentially move toward zero with increasing d in the CAY model.
   – Allow the γ parameter to change linearly with increasing w in the CSR model.
7. Redo the standardized residual Box plots and recalculate the $\widehat{elpd}_{loo}$ statistics.
8. Examine the fit of the modified model in light of the diagnostics in Step 7. If one sees a bias in the standardized residual plots and/or the $\widehat{elpd}_{loo}$ statistic decreases, when compared to the model in Step 5, one might want to reconsider the changes made in Step 6.
9. With the paid and incurred models deemed satisfactory, and in close agreement as to the ultimate loss, use these models as part of an IPI model to obtain a more accurate estimate of the ultimate liability.

[27] Before trying this, I suggest that you run the companion scripts that accompany this monograph for the various models. They should run as is, with the necessary R packages installed and with the CAS Loss Reserve Database downloaded.

[28] These models are my current favorites. Note that favorites can change over time. If one has other models in mind, they could, of course, substitute their own favorites.

As part of an ongoing loss reserving practice, one should retain the MCMC model fits and retrospectively test new data from future years with the standardized residual Box plots and the $\widehat{elpd}_{test}$ statistics as the data comes in.

Having completed an evaluation of some Bayesian MCMC stochastic loss reserve models, the focus of the monograph turns to calculating risk margins that were motivated by the principles underlying Solvency II and IFRS 17. The general idea behind the risk margin is that an insurer posts an amount of capital that is sufficient to support the business. As time goes on, claims are settled and the insurer's investors receive a cash flow consisting of capital reductions due to the decreased risk. The cost of capital risk margin is defined as the initial capital, minus the present value of that cash flow.

As we are talking about present values, the models we use for risk margins are the CSR model and the parameters of the IPI model that apply to paid losses.

When considering risk margins, the issue of diversification arises. So before addressing risk margins, the monograph addresses dependency between lines of insurance.

Given the predictive distributions of the parameters of each of two lines of insurance, the monograph shows how use Bayesian MCMC to obtain the predictive distribution of correlation coefficients between the losses, on the log scale, of the two lines of insurance. The results are surprising. Prospectively, the $\widehat{elpd}_{loo}$ statistics overwhelmingly favor a model that assumes independence. But retrospectively, the $\widehat{elpd}_{test}$ statistic favors a model that allows for dependency in a solid majority of the cases. This suggests that some unknown variables may be driving the dependency in the later holdout time period.

This being the case, it seems prudent to allow for some degree of correlation between lines of insurance when calculating the diversification credits for risk margins.

The risk margin calculation for individual lines of insurance was done for all 200 loss triangles in our data. It was done for both the CSR and IPI models under a range of dependency assumptions. The IPI risk margins were noticeably less than the CSR risk margins.

There were seven insurers that have loss triangles in all four lines. Diversification credits were calculated for these insurers under a set of reasonable, in light of the results obtained for dependencies, correlation coefficients.

In preparing this monograph I have made every effort to adhere to the "open source" philosophy. The data is publicly available. The software is publicly available for free. The R and Stan scripts used in creating these models are to be made publicly available. I have purposely restricted my methods to widely used software (R, Stan and RStudio) in order to make it easy for others to duplicate and improve on these results.

The models proposed in this monograph are offered as demonstrated improvements over current models. I expect to see further improvements over time. The Bayesian MCMC methodology offers a flexible framework with which one can make these improvements.

Finally, it was the presence of the CAS Loss Reserve Database that made this monograph possible. As conditions change over time, I strongly recommend that the Casualty Actuarial Society sponsors a study such as this from time to time.

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Appendix—The Data Selection Process

When selecting the loss triangles to use in this monograph, my overriding consideration was that the process should be mechanical and well defined. There are two potential mistakes one can make in selecting the insurers to analyze.

- If one were to take all the insurers in the database, or randomly select the insurers, there could be some insurers who made significant changes in their business operations that could violate the assumptions underlying the models.
- If one is too selective, one runs the risk of selecting only those data that best fits a chosen model. For example, let's suppose that I wanted the CAY model to fit the incurred data even better than it does. As an extreme case, noting that the CAY model still appears to be a bit light in the tails, I could have replaced some of the insurers that have outcomes in the tail, with other insurers that have outcomes in the middle.

While I did not have inside information on any changes in the business operations, Schedule P provides some hints in their reporting of both net and direct earned premium by accident year. Both of these data elements are in the CAS Loss Reserve Database.

- If an insurer makes significant changes in its volume of business over the ten-year period covered by Schedule P, a change in business operation could be inferred.
- If an insurer makes significant changes in its net to direct premium ratio over the ten-year period, a change in its reinsurance strategy could be inferred.

To carry out an analysis of this sort, I needed a large number of insurers. After looking at the quality and consistency of the data available in the CAS Loss Reserve Database, I decided to use 50 insurers in each of four major lines of insurance — Commercial Auto, Personal Auto, Worker's Compensation, and Other Liability. Early on I concluded that there was an insufficient number of insurers in the Products Liability and the Medical Malpractices lines to obtain an adequately sized selection.

To implement these considerations, after preliminary exploration I decided to use the loss triangles with lines of business that were "sufficiently large." This included all loss triangles with minimum annual premium of greater than \$20,000 and minimum annual incurred loss of greater than \$4,000.[29] I then calculated the

[29] The Schedule P entries are in \$1,000s.

Table A.1. CV Limits for Insurer Triangles

	Commercial Auto	Personal Auto	Workers' Comp	Other Liability
CV1(Premium)	<0.795	<1.003	<0.772	<0.628
CV2(Net/Direct)	<0.125	<0.125	<0.300	<0.15

coefficients of variation (CV1) for the net earned premiums and CV2 for the net to direct premium ratios over the ten available years. By trial and error, I then set up limits for CV2, and took the top 50 loss triangles sorted in increasing order of CV1. This procedure should have eliminated some of the insurers that changed their business operations.

After some provisional testing, I eliminated insurer group 38997. The final CV limits are given in Table A.1. The final list of the selected insurer groups are in Table A.2.

Table A.2. Group Codes for Selected Loss Triangles

Commercial Auto	Personal Auto		Workers' Comp		Other Liability	
8672	43	13587	86	11347	620	13501
9466	353	13595	337	11703	671	13668
10022	388	13641	353	13439	683	13919
10308	620	13889	388	13501	715	13994
11037	692	14044	671	13528	833	14044
11118	715	14176	715	14176	1252	14176
13420	1066	14257	965	14320	1279	14257
13439	1090	14311	1066	14508	1538	14370
13528	1538	14443	1252	14974	1767	14451
13641	1767	15024	1538	15148	2003	14885
13889	2003	15199	1767	15199	2135	15113
14044	2143	15393	2135	15334	2143	15148
14176	3240	15660	2712	18309	2208	15210
14257	4839	15997	3034	18767	3000	15571
14311	5185	16373	3240	18791	3240	16373
14320	6807	16799	5185	21172	5185	16799
14508	6947	18163	6408	23108	5320	18163
14974	7080	18791	6807	23140	6459	18686

Table A.2. Group Codes for Selected Loss Triangles (*Continued*)

Commercial Auto	Personal Auto		Workers' Comp		Other Liability	
15024	8427	23574	7080	26433	6947	26797
15199	8559	23876	8559	27529	7625	27065
18163	10022	25275	8672	30589	10657	28550
18767	13420	26808	9466	34576	11126	30139
18791	13439	27022	10385	37370	11150	30651
19020	13501	27065	10699	38687	11231	32875
19780	13528	27499	11126	38733	13439	34606