COLLAPSED PRIMAL-DUAL APPROXIMATION OF LARGE BINARY INTEGER LINEAR PROGRAMS

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We have been using the collapsed primal-dual approach to approximately solve large marketing optimization (generalized assignment) problems since 2004, including marketing campaigns to millions of customers at Discover Financial Services in the late 2000s. Our motivation was the high cost and limitations of commercial solvers. More recently, we extended this methodology to constrained k-means clustering. Here we make a natural extension to large binary integer linear programs (BILPs), including significant applications like the 0-1 multidimensional knapsack problem. **Williamson** provides an overview of the general primal-dual approach:

- Formulate the primal binary integer linear program
- Relax the primary BILP to a primary linear program with 0-1 constraints replaced by interval $[0,1]$
- Formulate and solve the dual of the relaxed primal LP
- Use the dual LP solution together with complementary slackness to approximate the BILP solution
- Dual LP objective function provides a bound to the primal BILP objective function, thereby providing an approximation ratio measuring BILP solution precision

We follow Williamson's general recipe except that we approximately solve the dual in collapsed form, thereby avoiding a large-scale LP solver. **Li, Sun, and Ye** have recently advocated the collapsed dual approach, but they seem to conclude that a special algorithm is required. Actually, the collapsed dual can be solved via a simple, plain-vanilla nonlinear minimizer and the BILP can then be approximated with good large-problem precision via a simple sort using dual shadow prices associated with binary constraints. Indeed, the simplicity and precision of this approach for large, unwieldy problems is its primary attraction.

The standard BILP is

Maximize: $$\sum_{j=1}^{n} c_j x_j$$

Subject to: $$\sum_{j=1}^{n} a_{ij} x_j \le b_i \; , \; i = 1,\ldots,m$$

$$x_j \in \{0,1\} \; , \; j = 1,\ldots,n$$

where normally $n >> m$ and we assume that $c_j > 0$ and $b_i \geq 0$ so that the initial solution with all $x_j = 0$ is feasible. This BILP can be relaxed to a linear program by replacing the binary constraints with $0 \leq x_j \leq 1$, $j = 1,\ldots,n$. The dual of this relaxed LP is

Minimize: $\sum_{i=1}^{m} b_i\, y_i + \sum_{j=1}^{n} z_j$

Subject to: $\sum_{i=1}^{m} a_{i\,j}\, y_i + z_j \geq c_j$, $j = 1,\ldots,n$

y_i , $z_j \geq 0$, $i = 1,\ldots,m$, $j = 1,\ldots,n$

but clearly at a minimum $z_j = max\left(0, c_j - \sum_{i=1}^{m} a_{i\,j}\, y_i \right)$ leading to the collapsed dual

Minimize: $\sum_{i=1}^{m} b_i\, y_i + \sum_{j=1}^{n} max\left(0, c_j - \sum_{i=1}^{m} a_{i\,j}\, y_i \right)$

Subject to: $y_i \geq 0$, $i = 1,\ldots,m$

This objective function is not everywhere smooth, but it is continuous and convex, and it is solvable via a simple, plain-vanilla nonlinear minimizer like *nlminb* in R. If y^* denotes the quasi-optimal dual solution, we then rank-order $c_j - \sum_{i=1}^{m} a_{i\,j}\, y_i^*$ descending and make binary $x_j = 1$ allocations until they are no longer feasible. For large BILPs, approximations are quasi-optimal, as guaranteed by the dual upper bound $\sum_{i=1}^{m} b_i\, y_i^* + \sum_{j=1}^{n} max\left(0, c_j - \sum_{i=1}^{m} a_{i\,j}\, y_i^* \right)$. We were able to approximate a large BILP with 100 constraints and 50,000 variables in about four minutes on a beefy Dell PC, starting with all $x_j = 0$ (assuming all $b_i \geq 0$). The resulting BILP value was within 99.9% of the dual upper bound. We provide a simple R-function *bilp.max* for this task along with corresponding R-function *bilp.min* for a minimization task, starting with all $x_j = 1$ (assuming this is feasible).

References

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