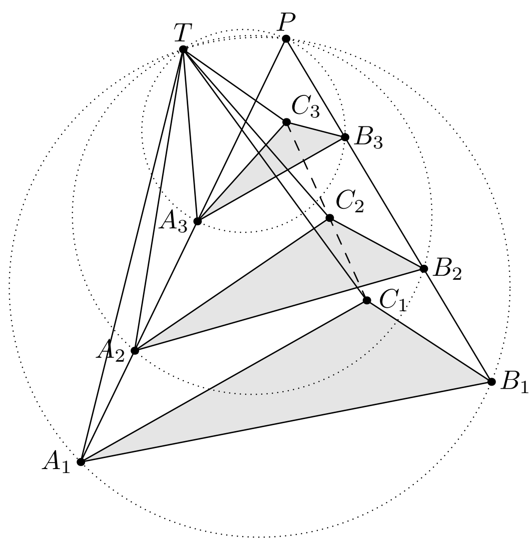
This article will be about a lemma which I discovered fairly recently. I initially wanted to call it "moving points" but unfortunately that name is already taken by another technique so I'm just gonna call it "the lemma" in this handout. You can call it whatever you want. This is a pretty exotic lemma but it is nevertheless good to know and can be useful in some circumstances. Anyway here it is in all its glory.

Lemma. Let A and B be two points moving with constant velocity on two fixed lines (not necessarily distinct) l_1 and l_2. Let C be also a point such that the shape of $\triangle ABC$ is fixed while both A and B are moving. Then C is also moving along a fixed line with constant velocity.

Proof. This is a pretty weird lemma isn't it? Let t_1, t_2 and t_3 be three instances of time, and let X_i denote the position of point X at time t_i. Then it suffices to show that C_1, C_2 and C_3 are collinear. We will consider three cases as follows.

Case 1. l_1, l_2 are distinct and not parallel.



First notice that

$$\frac{A_1A_2}{A_2A_3} = \frac{t_2 - t_1}{t_3 - t_2} = \frac{B_1B_2}{B_2B_3}$$

where the ratios are directed. Let T be the center of spiral similarity that sends segment A_1A_3 to segment B_1B_3. Then this also sends A_2 to B_2. Now T is also the center of spiral similarity which sends A_1B_1 to A_2B_2, and this sends C_1 to C_2 as $\triangle A_1B_1C_1 \sim \triangle A_2B_2C_2$. Therefore,

$$\measuredangle TC_1C_2 = \measuredangle TA_1A_2.$$

Similarly, we have $\measuredangle TC_1C_3 = \measuredangle TA_1A_3 = \measuredangle TA_1A_2$, so C_1, C_2 and C_3 are collinear.

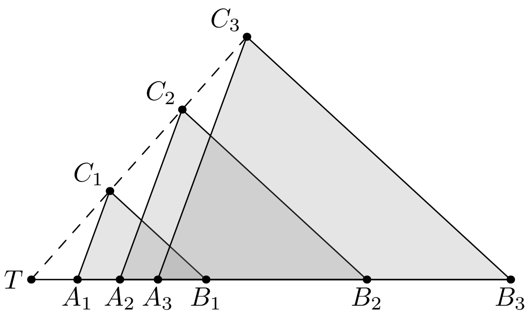
Case 2a. l_1, l_2 are distinct, parallel, and A and B have different velocities.

This case is the same as Case 1.

Case 2b. l_1, l_2 are distinct, parallel, and A and B have the same velocity.

This case can be seen as translating $\triangle ABC$ in the direction of l_1 and l_2 with constant velocity. Thus C obviously moves along a fixed line too.

Case 3a. $l_1 \equiv l_2$, and A and B have different velocities.



Let T be the center of homothety which maps segment A_1B_1 to segment A_2B_2. This obviously sends C_1 to C_2, so T, C_1, C_2 are collinear. As in Case 1,

$$\frac{A_1A_2}{A_2A_3} = \frac{B_1B_2}{B_2B_3}$$

so T is the center of another homothety which sends A_1B_1 to A_3B_3. Therefore, T, C_1, C_2 are collinear and the result follows.

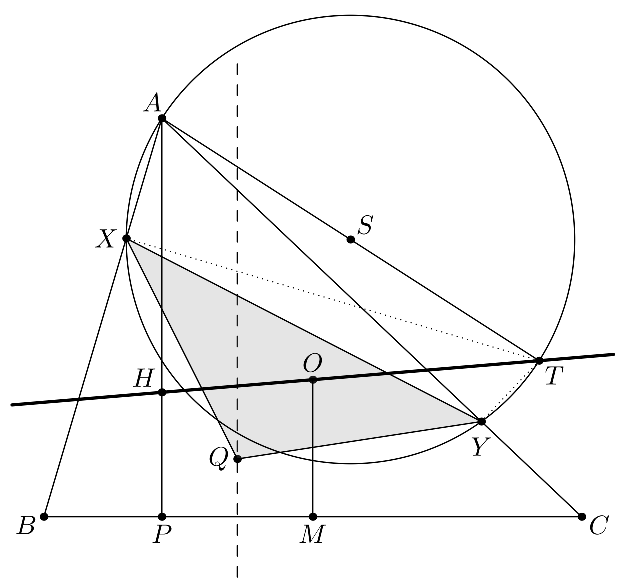
Case 3b. $l_1 \equiv l_2$, and A and B have the same velocity.

This case can be seen as moving segment AB along the line with constant velocity. Thus, obviously, C moves along a line too.

In all cases, it is easy to check that $\frac{C_1C_2}{C_2C_3} = \frac{t_2-t_1}{t_3-t_2}$, so C indeed moves with constant velocity. Also, in Case 1, Case 2a and Case 3a, T can coincide with C. In that case C would be fixed, but this can be also seen as moving along a fixed line with velocity $\vec{0}$. Thus we are finally done. $\square$

Now let's destroy some problems with our lemma. The first problem is a G5 from IMO Shortlist 2016.

Problem 1. (IMO SL 2016 G5) Let D be the foot of perpendicular from A to the Euler line (the line passing through the circumcentre and the orthocentre) of an acute scalene triangle ABC. A circle ω with centre S passes through A and D, and it intersects sides AB and AC at X and Y respectively. Let P be the foot of altitude from A to BC, and let M be the midpoint of BC. Prove that the circumcentre of triangle XSY is equidistant from P and M.



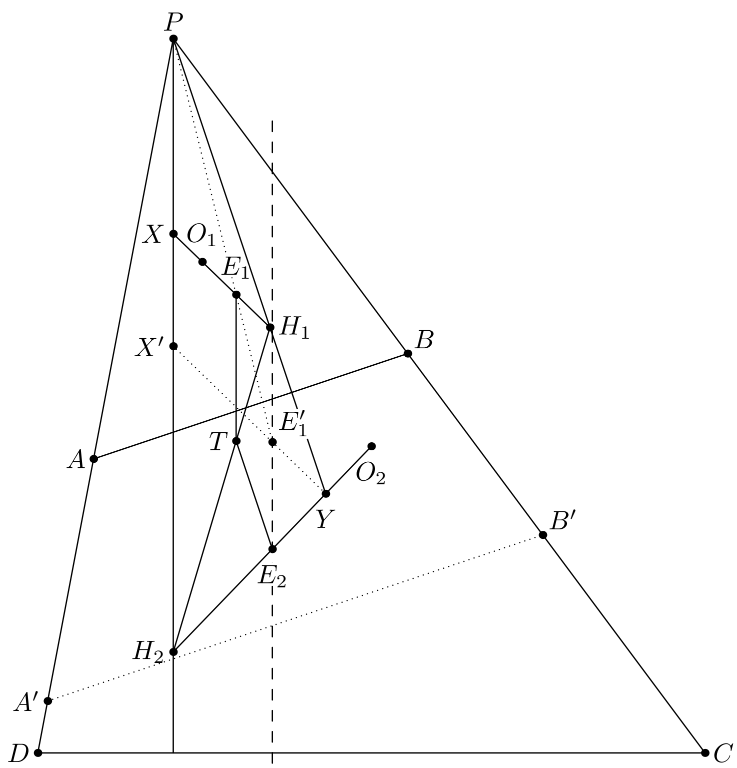
Solution. Let ω intersect OH again at T, and let's move T uniformly along the Euler line. Then both X and Y are the foots of perpendiculars from T, which means that they also move uniformly respectively. Moreover, if we let Q be the circumcenter of $\triangle XSY$, the shape of $\triangle XQY$ is fixed. This is because $\measuredangle QXY = 90^\circ - \measuredangle YSX = 90^\circ - 2\measuredangle CAB$ and similarly, $\measuredangle QYX = 90^\circ - 2\measuredangle CBA$. Therefore, by our lemma, Q moves uniformly along a fixed line too. We will show that this line is the perpendicular bisector of PM. It suffices to check for two cases that this is true. Notice that O and H are different as $\triangle ABC$ is scalene.

We will first check the case when $T = H$. Then ω is the nine-point circle of $\triangle ABC$, so Q lies on the perpendicular bisector of PM.

Now let's check the case when $T = O$. Then X and Y are midpoints of AB and AC respectively. Since $XYMP$ is an isosceles trapezoid, the perpendicular bisectors of XY and PM coincide, and hence Q lies on the perpendicular bisector of PM in this case as well. This completes the proof. $\square$

The following problem is a bit harder but still not that bad for a G6.

Problem 2. (IMO SL 2009 G6) Let the sides AD and BC of the quadrilateral $ABCD$ (such that AB is not parallel to CD) intersect at point P. Points O_1 and O_2 are circumcenters and points H_1 and H_2 are orthocenters of $\triangle ABP$ and $\triangle CDP$, respectively. Denote the midpoints of segments O_1H_1 and O_2H_2 by E_1 and E_2 respectively. Prove that the perpendicular from E_1 on CD, the perpendicular from E_2 on AB and the lines H_1H_2 are concurrent.

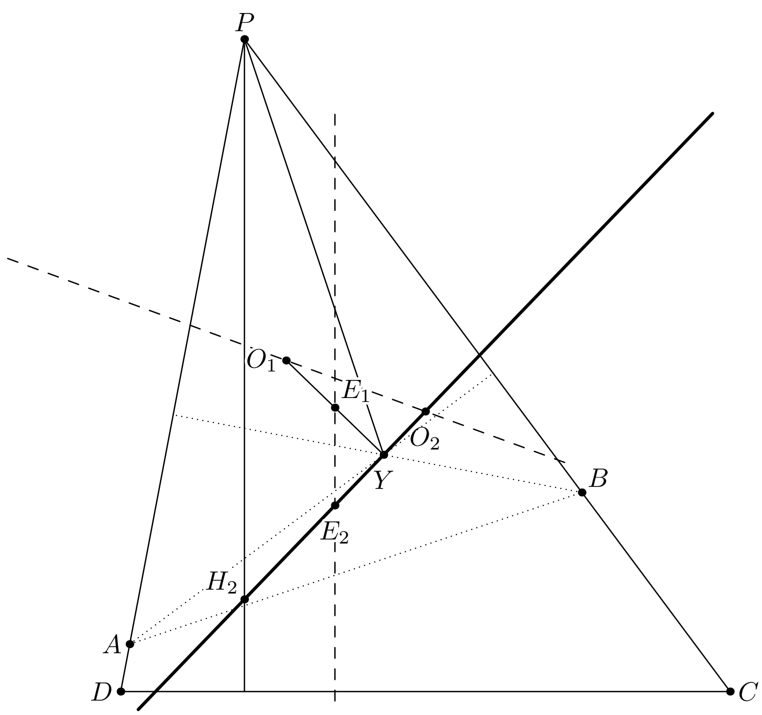


Solution. Let l_1 be the line through E_1 perpendicular to CD, and define l_2 similarly for E_2. We will first consider the case when $O_1 \neq H_1$ and $O_2 \neq H_2$. Let O_1H_1 cut the line through P perpendicular to CD at X, and define Y analogously for $\triangle PDC$. Since PH_2 and PH_1 are perpendicular to CD and AB, they are parallel to l_1 and l_2 respectively, and thus to show the concurrency, it suffices to show that

$$\frac{H_1E_1}{E_1X_1} = \frac{YE_2}{E_2H_2}.$$

Now consider a homothety centered at P which sends H_1 to Y, and denote the images of other points with $\bullet'$. Since homotheties preserve ratios, we have $\frac{H_1E_1}{E_1X} = \frac{YE_1'}{E_1'X'}$, or in other words, we must show that $E_1'E_2$ is parallel to PH_2. We can now rephrase the problem like so.

Rephrased Version. Let $\triangle PDC$ be a triangle with circumcenter O_2, orthocenter H_2 and nine point center E_2. Let Y be a point on $\overline{O_2H_2}$, and let A and B be points on PD and PC such that Y is the orthocenter of $\triangle PAB$. If O_1 and E_1 are the circumcenter and nine point center of that triangle, show that E_1E_2 is parallel to PH_2.



Let's move Y uniformly on $\overline{O_2H_2}$. Both A and B move uniformly since YA remains perpendicular to PC and similarly for YB. Moreover, $\triangle O_1AB$ has fixed shape since $\measuredangle APB$ is fixed, so applying the lemma shows that O_1 moves uniformly along a fixed line. This means that E_1 which is the midpoint of O_1Y also moves uniformly along a fixed line. Thus it suffices to check that this line is the line through E_2 parallel to PH_2. When $Y = H_2$, $O_1 = O_2$, which implies that $E_1 = E_2$. When $Y = O_2$, O_1 lies on PH_2 and therefore, E_1E_2 is parallel to PH_2 and we are done for this case.

Now WLOG, assume that $O_2 = H_2$. ($O_1 = H_1$ and $O_2 = H_2$ can't happen at the same time because that would imply that $AB \parallel CD$.) In that case, $\triangle ABP$ is equilateral, and hence we have $PO_1 = PH_1$ (why?), and so O_1H_1 is actually parallel to CD. Then the line through E_1 perpendicular to CD passes through $O_2 = H_2$ and hence the three lines are concurrent at this point. We are finally done. $\square$