Solutions for Myanmar Mathematical Olympiads (Tenth Standard)

Organized by the Mathematical Society of Myanmar

Kyaw Shin Thant

v1.0.0

The style of this document is heavily inspired by *An Infinitely Large Napkin* by Evan Chen.

The home page of this document is `https://radiuszero.github.io/solutions/`. Feel free to contact me at `kyawshinthant234@gmail.com` for mistakes, alternate solutions or other suggestions!

Preface

This is a compilation of solutions of all the problems that have appeared previously in the Grade-11 (Tenth Standard) level mathematical olympiads organized by the Mathematical Society of Myanmar (MSM). Most of the solutions here are my own work, while the others were either suggested to me by my friends or found online. I've tried my best to make these solutions as accessible and self-contained as possible. However, you should ideally learn the theory from other more comprehensive textbooks; what I've provided here is just a brief review.

Document Properties

Format

The first part of the document contains all the theorems that I've used in the solutions that may not be part of the standard curriculum. The latter two parts contains the question papers and solutions from 2015 to 2019. As a side note, the IMO team members for year N were chosen from students who had passed the Regional Round and the National Round of year $N-1$; the team selection results were usually announced around the start of May of year N. As of 2023, this system has changed completely.

Colors

You may notice that some of the problems are colored red or yellow. The problems colored in red contain major typos that cannot be fixed reasonably, whereas the ones colored in yellow contain only minor ones that have been fixed. In such cases, I will usually include the original statement as well.

Navigation

For any problem in the question papers, you can go straight to its solution by clicking the boxed number beside it. Similarly, you can go back to the questions by clicking the problem number of the solution. Here's a diagram in case you are not sure about what to click:

$$\boxed{1} \quad \longleftrightarrow \quad \textbf{Problem 1.}$$

Further Reading

If you are a complete beginner at math olympiads and you want to learn more theory before solving problems, I highly recommend taking a look at Evan's recommendations at `https://web.evanchen.cc/wherestart.html`. Here is a collection of all the past question papers by ko Phyoe Min Khant. I've also collected a bunch of handouts and books specifically on geometry that you can find here. Of course, you need to be somewhat familiar with English to make use of these, but this is inevitable since most math books that you can buy at bookstores in Myanmar are written in English anyway. However, I don't recommend buying most of them since they are usually badly written and contain a lot of typos, and you can easily find much higher quality books and resources online without paying a dime.

Personal Notes

I started writing these solutions in the summer of 2021. At the time, there was no concise compilation of solutions for past mathematical olympiads, and the only way to find the question papers and solutions was by scrolling through countless Facebook posts. Since one of my friends had already started writing the solutions for the Grade-10 level olympiads, I decided to take on the Grade-11 ones instead.

In the next few months, I spend countless hours finding the shortest solutions, deciding on the best ways to present them, and tinkering with LaTeX/Asymptote. The progress was slow but steady, and I was feeling that I could release the notes by the end of August. However, in the middle of July, I got an offer from the only university to which I had applied. Suddenly, I had a long list in my hand of tasks to do, and this project ended up at the very bottom of that list.

Fast forward a few years to this summer, I suddenly remembered about the project while browsing through GitHub. Even though two years have passed, the situation in the olympiad community here remains almost unchanged: resources are scarce, hard to find, and often locked behind paywalls. This gave me an incentive to finish what I had started writing long ago. Competing in math olympiads changed my life, and even though I am no longer an active participant, I still wanted to give something back to the community. These notes are my best attempt at this.

Acknowledgements

I would like to give special thanks to Hein Thant for finding mistakes in several problems and suggesting alternate solutions, ko Hein Thant Aung for teaching me some graph theory and in particular the method of deletion-contraction, and ko Phyoe Min Khant for compiling all the question papers which made my life much easier.

Contents

Part I.

Theorems and Techniques

Algebra

Theorem 1.1 (Remainder Theorem). Let $f(x)$ be a polynomial. Then the remainder of f when divided by $x-r$ is $f(r)$.

A special case of this theorem when the remainder is 0 is called the factor theorem. It says that $x-r$ is a factor of f if and only if $f(r)=0$.

Theorem 1.2 (RMS-AM-GM-HM Inequality). For any positive real numbers $a_1, a_2, \ldots, a_n$,

$$\sqrt{\frac{x_1^2+x_2^2+\cdots+x_n^2}{n}} \geq \frac{x_1+x_2+\cdots+x_n}{n} \geq \sqrt[n]{x_1x_2\cdots x_n} \geq \frac{n}{1/x_1+1/x_2+\cdots+1/x_n}$$

with equality if and only if $a_1=a_2=\cdots=a_n$.

Theorem 1.3 (Cauchy-Schwarz Inequality). For any two sequences of real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$, the following inequality holds:

$$(a_1b_1+a_2b_2+\cdots+a_nb_n)^2 \leq (a_1^2+a_2^2+\cdots+a_n^2)(b_1^2+b_2^2+\cdots+b_n^2)$$

with equality if and only if $a_i=kb_i$ for some real constant k.

Theorem 1.4 (Vieta's Formulas). Let $a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0$ be a polynomial with roots $r_1, r_2, \ldots, r_n$. Then the roots and coefficients are related as follows:

$$\begin{aligned} r_1+r_2+\cdots+r_n &= -\frac{a_{n-1}}{a_n} \\ (r_1r_2+r_1r_3+\cdots+r_1r_n)+(r_2r_3+\cdots+r_2r_n)+\cdots+r_{n-1}r_n &= \frac{a_{n-2}}{a_n} \\ &\vdots \\ r_1r_2\cdot\ldots\cdot r_n &= (-1)^n\frac{a_0}{a_n}. \end{aligned}$$

Theorem 1.5 (Descartes' Rule of Signs). Let $f(x)$ be a polynomial with real coefficients, and let k be the number of sign changes in the sequence of coefficients. Then the number of positive real roots of f is either equal to k, or less than it by an even number.

For example, the polynomial x^2+1 has no sign changes since its sequence of coefficients is $1,1$, so it has no positive real roots. On the other hand, the polynomial x^3+4x^2-5x-6 has exactly one sign change (its sequence of coefficients is $1,4,-5,-6$ so the sign changes from $+$ to $-$ between 4 and -5), and so it has exactly one positive real root.

As a corollary, if we let ℓ be the number of sign changes in the coefficients of $f(-x)$, then the number of negative real roots of f is either equal to ℓ, or less than it by an even number.

Theorem 1.6 (Fundamental Theorem of Algebra). Let $f(x)$ be a polynomial of degree n with complex coefficients. Then f splits into n linear factors with complex coefficients. i.e., there exist complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ (not necessarily distinct) such that

$$f(x)=(x-\alpha_1)(x-\alpha_2)\cdot\ldots\cdot(x-\alpha_n).$$

In other words, f has exactly n roots in complex numbers when counted with multiplicity.

For example, the polynomial $x^2 + 1$ has no real roots, but it has 2 roots, namely i and $-i$, in complex numbers[1]. The polynomial $x^3 - 1$ has only one real root, 1, but it has two complex roots, which are $(-1 + \sqrt{3}i)/2$ and $(-1 - \sqrt{3}i)/2$ respectively.

Technique 1.1 (Some Useful Identities). For any real numbers a, b and c, the following identities hold.

1. $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$.
2. $(a + 1)(b + 1)(c + 1) = abc + ab + bc + ca + a + b + c + 1$.
3. $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$.

Technique 1.2 (Mathematical Induction). Sometimes, while proving something, we may notice that a problem can be reduced to a smaller version of the same problem. In those cases, we can use mathematical induction. A proof using induction generally consists of two parts. Suppose that we are trying to prove a statement P.

1. We first show that $P(i)$ is true for some natural number i, usually 0 or 1. This is called the base case.
2. We then show that if $P(n)$ is true, than $P(n + 1)$ must also be true. This is called the inductive step.

If we can do both of these steps, then we can conclude that the statement P is true for all natural numbers $n \geq i$. It is easy to see why this holds: from step 1, $P(i)$ is true, and from step 2, $P(i + 1)$ must be true. Since $P(i + 1)$ is true, by step 2 again, $P(i + 2)$ is also true, and so on. This is often illustrated by the classic example of falling dominoes. Usually, the second step is a lot more important and difficult to prove than the first one.

There are also many other variants of induction. One important and generally more useful version is 'strong induction', where we assume in step 2 that $P(k)$ is true for all $i \leq k \leq n$, rather than just $P(n)$. Another version which is used less often is 'forward - backward induction' which is described below as technique 1.3.

Technique 1.3 (Forward-Backward Induction). This is a rarely used variant of induction, but it can handle some problems where normal induction fails. The procedure is generally like this:

1. Similarly to usual induction, we first show that $P(i)$ is true for some natural number i, usually 0 or 1.
2. We then show that if $P(n)$ is true, $P(2n)$ must also be true.
3. Finally we show that if $P(n)$ is true, $P(n - 1)$ must also be true.

You can see why this technique is called forward - backward induction. We use step 2 to skip over a chunk of numbers, and then use step 3 to show that P is also true for the numbers that we skipped. You can read more about this technique here.

[1] In fact, this is one way to define complex numbers, by adjoining the roots of $x^2 + 1$ to $\mathbb{R}$.

Combinatorics

Theorem 2.1 (Inclusion-Exclusion Principle). The inclusion-exclusion principle is a useful technique to count the number of elements in a union of sets, which may not be disjoint. In its simplest form, it states that given two sets A and B, the number of elements in the set $A \cup B$ is

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

This can be proven easily by using a Venn diagram. More generally, suppose that A_1, A_2, ..., A_n are n sets. Then the number of elements in the union $A_1 \cup A_2 \cup \cdots \cup A_n$ is the alternating sum

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \cdots + (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|.$$

For example, the above formula for $n = 3$ would be

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_2 \cap A_3| - |A_1 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

Theorem 2.2 (Pigeonhole Principle). Suppose that you have n objects and k buckets, where $k < n$. The pigeonhole principle states that no matter how you distribute the objects into the buckets, there will at least be one bucket with more than one object in it. This is quite a trivial statement but it can be used to prove pretty counterintuitive stuff.

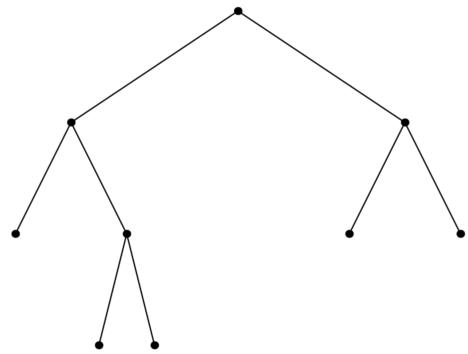
Technique 2.1 (Stars and Bars). Suppose that we have n identical stars and k buckets, and we want to count the number of ways to distribute the stars into buckets such that *each bucket contains at least one star*. One way to decide how to distribute the stars is by laying all the stars in a line, and dividing them into k parts by inserting $k-1$ bars in the gaps between them. Note that there must be at least one star between consecutive bars; this means that only one bar can be inserted into one gap. There are $n-1$ gaps, and the number of ways to insert $k-1$ bars in $n-1$ gaps is

$$\binom{n-1}{k-1}.$$

Now suppose that we allow the buckets to be empty. i.e., we no longer need the each bucket to contain at least one star. In this case, there can be multiple bars in each gap, and it is no longer easy to count the number of ways to insert the bars. Instead, we can view each configuration as a permutation of stars and bars *combined*. Since there are n identical stars and $k-1$ identical bars, the number of ways to permute them is

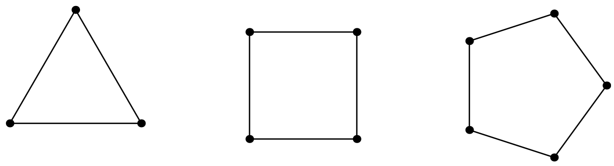
$$\binom{n+k-1}{k-1}.$$

Definition 2.1 (Graph). A graph is a structure consisting of a set of *vertices*, and a set of *edges* connecting those vertices. Usually, the vertices represent objects that we are interested in a problem (e.g., people), and the edges represent the relations between the objects (e.g., two people are adjacent). Below is an example of a graph. The dots are vertices and the line segments are edges.



For example, there is the path graph with n vertices, denoted as P_n, which is just a chain. Here is a (boring) diagram of P_1 (which is just a point), P_2, P_3 and P_4.

There's also the cycle graph with n vertices, denoted as C_n, which as the name suggests is a cycle with n vertices. Here are the diagrams of C_3, C_4 and C_5 respectively.



Definition 2.2 (Chromatic Polynomial)**.** Let G be a graph and k be any positive integer. A k-colouring of G is a colouring of the vertices of G, such that any two vertices which are connected by an edge have different colours. The *chromatic polynomial* of G, often denoted as $P(G, k)$, is the number of k-colourings of G. (The fact that this function is a polynomial in k is not obvious; however, it can be proven using the next technique.)

Technique 2.2 (Deletion-Contraction)**.** Suppose that we have a set of objects in a specific configuration (e.g., in a circle) and we want to count the number of ways to colour them such that no two adjacent objects have the same colour. Such a problem can be quite annoying to tackle as you can miss a case or two and get the wrong answer, but it becomes a lot easier once you translate it into a graph colouring problem. First, start by drawing a vertex for each object, and then draw an edge between any two objects that are adjacent to each other. Then the task of finding the number of colourings merely becomes the task of finding the chromatic polynomial of the resulting graph.

Deletion-contraction gives a way to recursively find the chromatic polynomial of any graph. Let G be a graph and k be a positive integer. Pick any edge e with endpoints u and v. Let $G - e$ be the graph obtained by removing the edge e. Then the set of k-colourings of G' can be divided as follows:

$$\begin{aligned}\{k\text{-colourings of } G - e\} = &\{k\text{-colourings of } G - e \text{ where } u \text{ and } v \text{ have the same colour}\} \\ &\cup \{k\text{-colourings of } G - e \text{ where } u \text{ and } v \text{ have different colours}\}\end{aligned}$$

Note that the number of elements in the first set is just the number of k-colourings of G/e, which is the graph obtained by deleting the edge e of G and merging u and v together. Also, the number of elements in the second set is the number of k-colourings of the original graph G. In other words, we have the following equality:

$$P(G - e, k) = P(G/e, k) + P(G, k),$$

which can be rearranged to form

$$P(G, k) = P(G - e, k) - P(G/e, k).$$

In this way, we can recursively find $P(G, k)$ for any k.

For an example, see the second solution of 2016 Regional Round problem 11.

Geometry

There's not a lot of theorems here since most of them are already covered in the standard curriculum!

Theorem 3.1 (Ptolemy's Theorem)**.** If $ABCD$ is a convex cyclic quadrilateral,

$$AB \cdot CD + AD \cdot BC = AC \cdot BD.$$

Number Theory

Technique 4.1 (Modular Arithmetic Basics)**.** Fix a positive integer n. For any two integers a and b, we say that "a and b are equivalent *modulo* n" if and only if $a - b$ is divisible by n. In other words, a and b are equivalent mod n if and only if they leave the same remainder when divided by n. We write

$$a \equiv b \pmod{n}$$

to symbolize this equivalence. For example,

$$a \equiv 0 \pmod{n}$$

means that a is divisible by n.

This equivalence symbol $\equiv$ really behaves like the equality symbol $=$ that we use all the time. In particular, it respects all of the arithmetic operations except for division. The latter requires special care; we can only divide both sides of the equivalence by an integer k if and only if k is relatively prime to n. (Think about why this is true!) If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then we have

$$a + b \equiv c + d \pmod{n}, \quad ab \equiv cd \pmod{n}, \quad \text{and} \quad a^k \equiv b^k \pmod{n} \text{ for all } k \in \mathbb{N}.$$

This is a very useful notation for dealing with divisibility problems, and it will be used freely in the solutions.

Theorem 4.1 (Fermat's Little Theorem)**.** For any prime p and positive integer a,

$$a^p \equiv a \pmod{p}.$$

If a and p are relatively prime in addition,

$$a^{p-1} \equiv 1 \pmod{p}.$$

This is because we can divide both sides of the equivalence by a.

Lemma 4.2. Let a, b and n be integers. Then a and b both divide n if and only if their least common multiple also divides n. A useful case of this is when a and b are relatively prime. In this case, their least common multiple is ab, so a and b both divide n if and only if ab divides n.

More generally, if we have k integers $a_1, a_2, \ldots, a_n$, then they all divide n if and only if their least common multiple also divides n.

Lemma 4.3. Let a and b be integers, and n be a positive integer. Then

$$\frac{a^n - b^n}{a - b} = a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + b^{n-1}.$$

In particular, $a^n - b^n$ is divisible by $a - b$. Think about how this is related to the last identity in the equation in Technique 4.1.

Lemma 4.4 (Euclid)**.** Let a, b and c be positive integers and suppose that a divides bc. If a and b are relatively prime, then a divides c. This innocent looking lemma is surprisingly hard to prove without being circular!

Part II.

Past Papers

Sample Problems

1 Given the sequence $u_n = n^2 - 6n + 13$, what is the smallest term in the sequence?

2 A function $f(x) = \frac{p}{x-q}$, $q > 0$, $q \neq x$ is such that $f(p) = p$ and $f(2q) = 2q$. Find the values of p and q. If $(f \circ f)(x) = x$, show that $x^2 - x - 2 = 0$. [1]

3 Let $f(x) = 1 + x + x^2 + \cdots + x^n$. The remainder when $f(x)$ is divided by $2x - 1$ is $\frac{341}{256}$ more than the remainder when $f(x)$ is divided by $2x + 1$. Find the value of n.

4 Show that $\mathbf{A} = \begin{bmatrix} \cos^2\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ satisfies $\mathbf{A}^2 + \mathbf{I} = 2\mathbf{A}\cos\theta$, where $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. If $\theta = 30°$, show that $\mathbf{A}' - \mathbf{A}^{-1} = \mathbf{0}$.

5 In the binomial expansion of $(1 + x)^n$, the coefficients of 5th, 6th and 7th terms are consecutive terms of an AP. Find the first three terms of the binomial expansion.

6 Find the number of integers between 100 and 999 such that the sum of the three digits is 12.

7 Determine the solution set for which

$$\frac{(x+1)(x-2)}{1-2x} > 0.$$

8 If a_1, a_2, a_3 are in AP, a_2, a_3, a_4 are in GP and a_3, a_4, a_5 are in HP, then prove a_1, a_3, a_5 are in GP.

9 Prove by mathematical induction that

$$\sum_{r=1}^{n} r^4 = \frac{n(n+1)(6n^3 + 9n^2 + n - 1)}{30}.$$

10 If the equation $x^4 - 4cx^3 + 6x^2 + x + 1 = 0$ has a repeated root p, show that $3c = \frac{p^2+3}{p}$. Hence or otherwise, prove that there is only one positive value c giving a repeated root, and that this value of c is $(\frac{4}{3})^{\frac{3}{4}}$.

[1] The problem originally stated $f(q) = q$, but this does not give $x^2 - x - 2 = 0$ when $(f \circ f)(x) = x$.

$\boxed{11}$ If the base BC of $\triangle ABC$ is trisected at P and Q, show that

$$AB^2 + AC^2 = AP^2 + AQ^2 + 4PQ^2.$$

$\boxed{12}$ From a point O, two straight lines are at any angle. On one of these lines, points A and B are taken such that $OA = \frac{5}{2}$ inch and $AB = \frac{3}{2}$ inch. Find the point on the other line at which AB subtends the greatest angle.

$\boxed{13}$ Given any seven distinct real numbers x_1, x_2, …, x_7, prove that we can always find the numbers $x_i, 1 \leq i \leq 7$ and $x_j, 1 \leq j \leq 7$ such that

$$0 < \frac{x_i - x_j}{1 + x_i x_j} < \frac{1}{\sqrt{3}}.$$

$\boxed{14}$ Prove that $6 \mid n^3 - n$ for all integers n.

$\boxed{15}$ $ABCD$ is a semicircle, A and B being the extremities of the diameter and C and D being points in the arc in which the ratio of arcs $AB : BC : CD = 2 : 3 : 4$. Find $\angle ADC$.

$\boxed{16}$ The 5-digit number $\overline{A986B}$ is divisible by 72. What is the value of $A + B$?

$\boxed{17}$ AB is a chord of a circle and P is any point on the arc of one of the segments cut off. Prove that the bisection of the $\angle APB$ passes through a fixed point on the circumference.

$\boxed{18}$ Two circles touch each other internally at A. Through B, a point on the circumference of the inner circle, a tangent is drawn which meets the circumference of the outer at P and Q. Show that $AP : AQ = BP : BQ$.

2015 Regional Round

$\boxed{1}$ The operation $\triangle$ is defined as $a \triangle b = ab + 2a + b$. Find x, if $\frac{12}{6-x} \triangle 2 = 3 \triangle 5$.

$\boxed{2}$ If $x + \frac{1}{x} = 7$, find the value of $x^3 + \frac{1}{x^3}$.

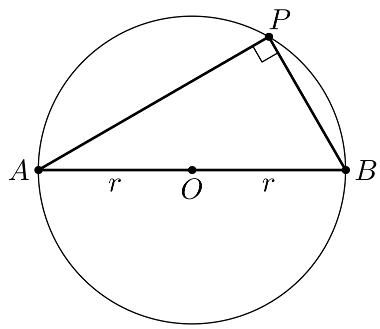
$\boxed{3}$ Find the value of

$$\frac{\frac{1}{2}-\frac{1}{3}}{\frac{1}{3}-\frac{1}{4}} \cdot \frac{\frac{1}{4}-\frac{1}{5}}{\frac{1}{5}-\frac{1}{6}} \cdot \frac{\frac{1}{6}-\frac{1}{7}}{\frac{1}{7}-\frac{1}{8}} \cdot \ldots \cdot \frac{\frac{1}{2014}-\frac{1}{2015}}{\frac{1}{2015}-\frac{1}{2016}}.$$

$\boxed{4}$ Let p and q be the remainders when the polynomials $f(x) = x^3 + 2x^2 - 5ax - 7$ and $g(x) = x^3 + ax^2 - 12x + 6$ are divided by $x + 1$ and $x - 2$ respecively. If $2p + q = 6$, find the value of a.

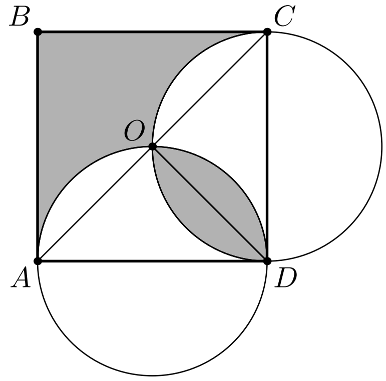
$\boxed{5}$ Prove that $(a + b)(b + c)(c + a) \geq 8abc$ for any $a, b, c \geq 0$.

$\boxed{6}$ The point P lies on a circle with radius r which has AB as a diameter. Show that $PA \cdot PB \leq 2r^2$ and $PA + PB \leq 2\sqrt{2}r$.

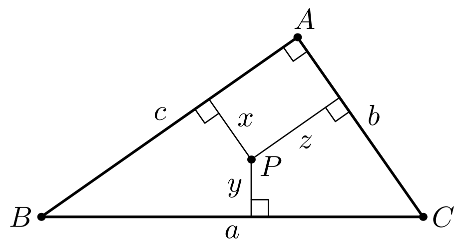


$\boxed{7}$ In a sequence, $u_1 = 1$, $u_2 = 2$ and $u_3 = 3$. For $n \geq 4$, the nth term u_n is calculated from the previous three terms as $u_n = u_{n-3} + u_{n-2} - u_{n-1}$. For example, $u_4 = u_1 + u_2 - u_3 = 0$. Write down the first 9 terms. What is the 2015th term of the sequence?

$\boxed{8}$ In the figure, a square $ABCD$ of side length 6 is given. Two circles with diameters AD and CD are drawn. Determine the combined area of two shaded regions.



9 In $\triangle ABC$, $\angle A = 90°$. The point P lies inside $\triangle ABC$ with distances from AB, BC and CA equal to x, y and z respectively. If we denote $a = BC$, $b = CA$ and $c = AB$, show that $z = \frac{bc-cx-ay}{b}$.



10 An integer is chosen from the set $\{1, 2, 3, \ldots, 100\}$. Find the probability that the integer is divisible by 3 or 7.

11 Aung Aung says to Bo Bo, "I am 5 times what you were when I was your age". The sum of their current ages is 64. Find their ages.

12 The sum of squares of three consecutive positive integers is 2 more than 100 times the sum of the numbers itself. Find the largest of the three numbers.

13 P and Q are two points on AB and AC respectively, of $\triangle ABC$. If PQ is parallel to BC, and bisects $\triangle ABC$, find $AP : PB$.

14 Find the remainder when $(x+1)^{2016} + (x+2)^{2016}$ is divided by $x^2 + 3x + 2$.

15 If a, b, c, d are in harmonic progression, prove that $ab + bc + cd = 3ad$.

16 Using mathematical induction, prove that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

17 Show that $n^7 - n$ is divisible by 42, for all positive integers n.

$\boxed{18}$ If α, β and γ are roots of the equation $x^3+px^2+qx+k=0$, show that $\alpha^2+\beta^2+\gamma^2=p^2-2q$.

2015 National Round

1. If $x^n + py^n + qz^n$ is divisible by $x^2 - (ay + bz)x + abyz$ and $y, z \neq 0$, show that

$$\frac{p}{a^n} + \frac{q}{b^n} + 1 = 0.$$

2. A sequence is defined by $u_1 = 1$, $u_{n+1} = u_n^2 - ku_n$, where $k \neq 0$ is a constant. If $u_3 = 1$, calculate the value of k and find the value of

$$u_1 + u_2 + u_3 + \cdots + u_{100}.$$

3. A rectangular room has a width of x yards. The length of the room is 4 yards longer than its width. Given that the perimeter of the room is greater than 19.2 yards and the area of the room is less than 21 square yards, find the set of possible values of x.

4. Two dice are thrown. Event A is that the sum of the numbers on the dice is 7. Event B is that at least one number on the die is 6. Find

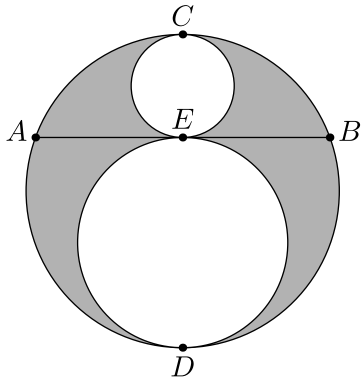
   1. $\mathbb{P}(A)$,
   2. $\mathbb{P}(B)$,
   3. $\mathbb{P}(A \cap B)$,
   4. $\mathbb{P}(A) \cdot \mathbb{P}(B)$.

   Are A and B independent?

5. In $\triangle ABC$, $\angle A = 30°$, $AB = 8$ cm and $BC = x$ cm. If $\angle C > 30°$, determine the set of all possible values of x.

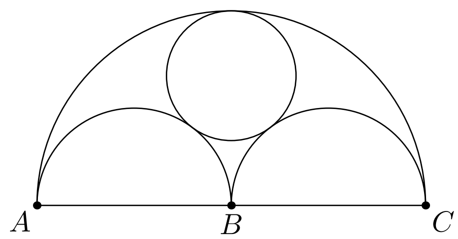
6. Let $f(x)$ be a polynomial with real coefficients. When $f(x)$ is divided by both $x - a$ and $x - b$, where a and b are distinct real numbers, the remainder is a real constant r. Prove that $f(x)$ has also the remainder r when it is divided by $x^2 - (a + b)x + ab$.

7. Three circles are tangent to each other as shown. The two smaller circles are tangent to chord AB which has length 12 at its midpoint. What is the area of the shaded region?



8 In a sequence, $u_1 = 1$, $u_2 = 2$ and $u_3 = 3$. For $n \geq 4$, the nth term u_n is calculated from the previous three terms as $u_n = u_{n-3} + u_{n-2} - u_{n-1}$. For example, $u_4 = u_1 + u_2 - u_3 = 0$. By using mathematical induction, prove that $u_{2n+1} = 2n + 1$ for all integers $n \geq 0$.

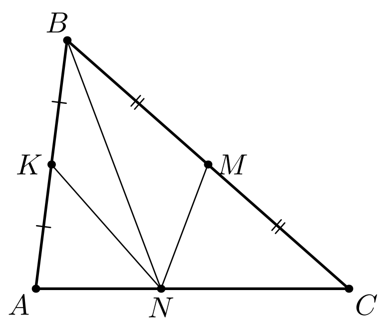
9 In the diagram, $AB = BC = 1$ and ABC is the diameter of the larger semicircle. AB and BC are diameters of the smaller semicircles. What is the diameter of the circle tangent to all three semicircles?



10 A six-digit number is of the form $abcabc$, where all the digits are nonzero. Find three different prime factors of that number.

11 Real numbers a and b are such that $a > b > 0$, $a \neq 1$ and $a^{2016} + b^{2016} = a^{2014} + b^{2014}$. Prove that $a^2 + b^2 < 2$.

12 In $\triangle ABC$, $AC = \frac{1}{2}(AB + BC)$ and BN is the bisector of $\angle ABC$. K and M are the midpoints of AB and BC respectively. If $\angle ABC = \beta$, prove that $\angle KNM = 90^\circ - \frac{1}{2}\beta$.



13 If the $(m-n)$th and $(m+n)$th terms of a geometric progression are the arithmetic mean and harmonic mean of $x > 0$ and $y > 0$, prove that the mth term is their geometric mean.

$\boxed{14}$ Show that the sum of the squares of the lengths of the sides of a parallelogram equals the sum of the squares of the lengths of the diagonals.

$\boxed{15}$ If n is a positive even integer, prove by mathematical induction that $x^n - y^n$ is divisible by $x + y$.

$\boxed{16}$ Prime numbers p, q and positive integers m, n satisfy the following conditions:

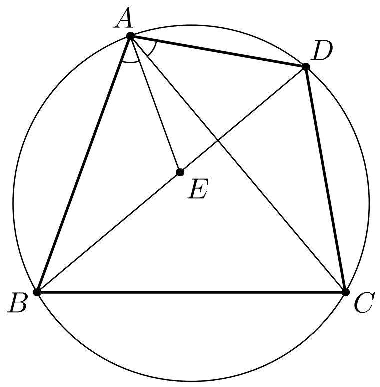
$$m < p, \quad n < q \quad \text{and} \quad \frac{p}{m} + \frac{q}{n} \text{ is an integer.}$$

Prove that $m = n$.

$\boxed{17}$ A, B and C are three points on the circumference of a circle, and the tangent at A meets BC produced at T. Prove that $AB^2 : AC^2 = TB : TC$.

$\boxed{18}$ $ABCD$ is a cyclic quadrilateral. AE is drawn to meet BD at E such that $\angle BAE = \angle CAD$. Prove that

1. $\triangle ABE \sim \triangle ACD$,
2. $\triangle AED \sim \triangle ABC$,
3. $AB \cdot CD + AD \cdot BC = AC \cdot BD$.



2016 Regional Round

$\boxed{1}$ In the sequence u_1, u_2, …, u_n, …, the nth term is defined by $u_n = 1 - \frac{1}{u_{n-1}}$ for $n \geq 2$. If $u_1 = 3$, compute u_2, u_3 and u_4. Write down u_{2016}.

$\boxed{2}$ If x and y are positive integers such that $56 \leq x + y \leq 59$ and $0.9 \leq \frac{x}{y} \leq 0.91$, find the value of $y^2 - x^2$.

$\boxed{3}$ When 15 is added to a number x, it becomes the square of an integer. When 74 is subtracted from x, the result is a square of another integer. Find the number x.

$\boxed{4}$ In a rectangle $ABCD$, the points M, N, P, Q lie on AB, BC, CD and DA respectively, such that the areas of $\triangle AQM$, $\triangle BMN$, $\triangle CNP$ and $\triangle DPQ$ are equal. Prove that $MNPQ$ is a parallelogram.

$\boxed{5}$ Draw 6 circles in the plane such that every circle passes through exactly 3 centres of other circles.

$\boxed{6}$ There are 2016 students in a secondary school. Every student writes a new year card. The cards are mixed up and randomly distributed to students. Suppose each student gets one and only one card. Find the expected number of students who get back their own cards.

$\boxed{7}$ The points P, A, B lie in that order on a circle with center O such that $\angle POB < 180°$. The point Q lies inside the circle such that $\angle PAQ = 90°$ and $PQ = BQ$. If $\angle AQB > \angle AQP$, prove that $\angle AQB - \angle AQP = \angle AOB$.

$\boxed{8}$ A fair die is thrown three times. The results of the first, second and third throw are recorded as x, y and z respectively. Suppose that $x + y = z$. What is the probability that at least one of x, y and z is 2?

$\boxed{9}$ An arithmetic progression and a harmonic progression have a and b for the first two terms. If their nth terms are x and y respectively, show that $(x - a) : (y - a) = b : y$.

$\boxed{10}$ Mr. Game owns 200 custom-made dice. Each die has four sides showing the number 2 and two sides showing the number 5. Mr. Game is about to throw all 200 dice together and find out the sum of all 200 results. How many possible values of this sum are there?

$\boxed{11}$ U Tet Toe wants to repair all 4 walls of his room. He has red paint, yellow paint and blue paint (which he cannot mix), and wants to paint his room so that adjacent walls are never of the same colour. In how many ways can U Tet Toe paint his room?

$\boxed{12}$ Given an equilateral triangle, what is the ratio of the area of its circumscribed circle to the area of the inscribed circle?

$\boxed{13}$ Solve the equation $\sqrt{3x^2 - 8x + 1} + \sqrt{9x^2 - 24x - 8} = 3$.

$\boxed{14}$ Prove by mathematical induction that

$$1^2 + 3^2 + 5^2 + 7^2 + \cdots + (2n-1)^2 = \frac{1}{3}n(4n^2 - 1).$$

$\boxed{15}$ In how many ways can 7 identical T-shirts be divided among 4 students, subject to the condition that each is to get at least 1 T-shirt.

$\boxed{16}$ $\triangle ABC$ has $AB > AC$. A line DEF, equally inclined to AB and AC, is drawn, meeting AB at F, AC at E and BC produced at D. Prove that $BD : DC = BF : CE$.

(*Hint:* Draw $BG \parallel CE$ meeting DF produced at G.)

$\boxed{17}$ The sum of the two smallest positive divisors of a positive integer N is 6, while the sum of the largest positive divisors of N is 1122. Find N.

$\boxed{18}$ If p, q and r are prime numbers such that $p < q < r$ and their product is 19 times their sum, find $p(q + r)$.

2016 National Round

$\boxed{1}$ The sequence $\log_{12} 162$, $\log_{12} x$, $\log_{12} y$, $\log_{12} z$, $\log_{12} 1250$ is an arithmetic progression. What is x?

$\boxed{2}$ A triangular array of 2016 coins has 1 coin in the first row, 2 coins in the second row and 3 coins in the third row, and so on up to N coins in the Nth row. What is the value of N?

$\boxed{3}$ Tu Tu cuts a circular paper disk of radius 12 cm along two radii to form two sectors, the smaller having a central angle of 120°. He makes two circular cones, using each sector to form the lateral surface of a cone. What is the ratio of the height of the smaller cone to that of the larger?

$\boxed{4}$ The sequence u_1, u_2, u_3, ... has the property that every term beginning with the third is the sum of the previous two terms. That is,

$$u_n = u_{n-2} + u_{n-1} \text{ for } n \geq 3.$$

Suppose that $u_9 = 110$ and $u_7 = 42$. What is u_4?

$\boxed{5}$ Line ℓ_1 has equation $3x - 2y = 1$ and goes through $A = (-1, -2)$. Line ℓ_2 has equation $y = 1$ and meets line ℓ_1 at point B. Line ℓ_3 has positive slope, goes through point A and meets ℓ_2 at point C. The area of $\triangle ABC$ is 3. What is the slope of line ℓ_3?

$\boxed{6}$ In $\triangle ABC$, $AB = 13$, $BC = 14$ and $CA = 15$. Distinct points D and E lie on segments BC and CA respectively such that $AD \perp BC$ and $DE \perp AC$. Find the exact length of segment DE.

$\boxed{7}$ In $\triangle ABC$, $AB = 6$, $BC = 7$ and $CA = 8$. Point D lies on BC and AD bisects $\angle BAC$. Point E lies on AC and BE bisects $\angle ABC$. The bisectors intersect at F. Find the ratios $AF : FD$ and $BF : FE$.

$\boxed{8}$ The sum of an infinite geometric series is a positive number S, and the second term in the series is 1. What is the smallest possible value of S?

$\boxed{9}$ Prove that $\sqrt{ab} + \sqrt[3]{abc} \leq \frac{1}{3}(a + 4b + 4c)$ for all positive numbers a, b and c.

$\boxed{10}$ Eight people are sitting around a circular table, each holding a fair coin. All eight people flip their coins and those who flip heads stand while those who flip tails remain seated. What is the number of ways that no two adjacent people will stand?

$\boxed{11}$ Let $ABCD$ be a cyclic quadrilateral and let the lines CD and BA meet at E. The line through D which is tangent to the circle ADE meets the line CB at F. Prove that $\triangle CDF$ is isosceles.

$\boxed{12}$ Prove that $n^3 - n$ is divisible by 24, for all odd integers n.

$\boxed{13}$ In how many ways can 7 paintings be posted on a wall,

1. if 3 of the paintings are always to appear altogether,
2. if 3 of the paintings are never to appear altogether?

$\boxed{14}$ A box contains 2 red marbles, 2 green marbles, and 2 yellow marbles, Mg Gyi takes 2 marbles from the box at random; then Mg Latt takes 2 of the remaining marbles at random; and then Mg Nge takes the last 2 marbles. What is the probability that Mg Nge gets two marbles of the same colour?

$\boxed{15}$ Prove by mathematical induction that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$

for all positive integers n.

$\boxed{16}$ If one of the roots of the equation:

$$ax^3 + bx^2 + cx + d = 0$$

is the geometric mean of the other two roots, prove that $ac^3 = b^3 d$.

$\boxed{17}$ Let f be a function such that

$$x^2 f(x) + f\left(\frac{x-1}{x}\right) = 2x^2,$$

for all real numbers $x \neq 0$ and $x \neq 1$. Find the value of $f\left(\frac{1}{2}\right)$.

2017 Regional Round

$\boxed{1}$ The four digit number $\overline{ABCD}$ is such that $\overline{ABCD} = A \times \overline{BCD} + \overline{ABC} \times D$. Find the smallest possible value of $\overline{ABCD}$. (Here $\overline{ABCD}$ means $1000 \cdot A + 100 \cdot B + 10 \cdot C + D$.)

$\boxed{2}$ Find the remainder when 20172017201720172017201720172017 is divided by 72.

$\boxed{3}$ Show that for all real numbers x, y, z and w,

$$\sin(x-w)\sin(y-z) + \sin(y-w)\sin(z-x) + \sin(z-w)\sin(x-y) = 0.$$

$\boxed{4}$ Two circles intersect at A and B. A common tangent to the circles touches the circles at P and Q. A circle is drawn through P, Q and A, and the line BA produced meets this circle again at C. Join CP and CQ, and extend both to meet the given circles at E and F respectively. Prove that P, Q, E, F are concyclic.

$\boxed{5}$ If $x \geq 1$, prove that $x^3 - 5x^2 + 8x - 4 \geq 0$.

$\boxed{6}$ Let $u_1, u_2, u_3, \ldots$ be a sequence of real numbers such that $u_1 > 2$ and

$$u_{n+1} = 1 + \frac{2}{u_n}$$

for $n \geq 1$. Prove that $u_{2n-1} + u_{2n} > 4$ for all $n \geq 1$.

$\boxed{7}$ Let a, b, c be positive numbers such that $ab + bc + ca + abc = 4$. Prove that

$$\frac{1}{a+2} + \frac{1}{b+2} + \frac{1}{c+2} = 1$$

and

$$a + b + c \geq 3.$$

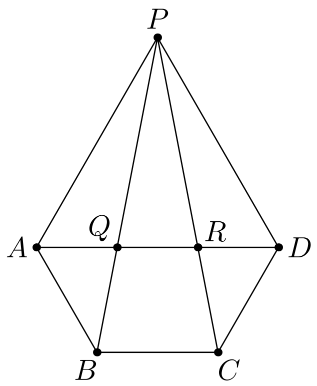
(You can use the following result: If x, y, z are positive numbers, then $\frac{x+y+z}{3} \geq \sqrt[3]{xyz}$.)

$\boxed{8}$ On each side of a triangle, there are 4 distinct points other than triangle's vertices. Determine the number of triangles having the vertices at 3 of these 12 points.

$\boxed{9}$ Two red dice and one blue die are thrown. What is the probability that the sum of the scores on the red dice is equal to the score on the blue die?

$\boxed{10}$ N is a two-digit number and $2N$ also has two digits. If N equals 2 times the sum of digits of $2N$, find all possible values of N.

$\boxed{11}$ In the figure, $AD \parallel BC$, $AB = BC = CD = \frac{1}{2}AD$ and $\triangle APD$ is equilateral. BP and CP cut AD at Q and R respectively. If the area of $\triangle APD$ is 12, find the area of the trapezium $BQRC$.



$\boxed{12}$

1. Show that $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{(n-1)n} = \frac{n-1}{n}$ for $n \geq 2$.
2. Show that for every integer $n \geq 2$, there exist positive integers $x_1, x_2, x_3, \ldots, x_n$ so that

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \cdots + \frac{1}{x_n} = 1.$$

$\boxed{13}$ Let a, b, c, d, e be five prime numbers forming an arithmetic progression with a common difference of 6. Find the smallest possible value of $a + b + c + d + e$.

$\boxed{14}$ Function $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$(a-b)f(a+b) + (b-c)f(b+c) + (c-a)f(c+a) = 0,$$

for all $a, b, c \in \mathbb{R}$. For $x \in \mathbb{R}$, letting $a = \frac{1}{2}(x-1)$, $b = \frac{1}{2}(x+1)$ and $c = \frac{1}{2}(1-x)$, show that $f(x) = Ax + B$ where $A = f(1) - f(0)$ and $B = f(0)$. Conversely, show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = Ax + B$, where A and B are constants, satisfies the given equation.

$\boxed{15}$ In a trapezium $ABCD$, with $AB \parallel CD$, there are two circles with diameter AD and BC respectively. Two circles do not intersect each other. Let X and Y which do not lie in $ABCD$ be two points on each of the circles. Show that

$$XY \leq \frac{1}{2}(AD + AB + DC + BC).$$

$\boxed{16}$ If α, β and γ are roots of the equation $x^3 + ax^2 + bx + c = 0$, evaluate $\alpha^2 + \beta^2 + \gamma^2$ and $\alpha^3 + \beta^3 + \gamma^3$.

$\boxed{17}$ Prove by mathematical induction, that $3^{2n} + 7$ is divisible by 8 for all $n \geq 1$.

$\boxed{18}$ Each vertex of convex polygon $ABCDE$ is to be assigned a colour. There are 6 colours and the ends of each diagonal must have different colours. How many different colourings are possible?

2017 National Round

1. $x = \overline{ABCDE}$ is a five digit number. If $(A+C+E)-(B+D) = 11k$ where $k = -1, 0, 1, 2$, prove that x is divisible by 11.

2. A palindrome is a number that remains the same when its digits are reversed. For example, 252 is a three-digit palindrome and 3663 is a four-digit palindrome. If the numbers $x - 22$ and x are three-digit and four-digit palindromes, respectively, find the value of x.

3.
    1. To find the exact value of $\sqrt{4+2\sqrt{3}}$, let $\sqrt{4+2\sqrt{3}} = a + b\sqrt{3}$, where a and b are integers and $a > b\sqrt{3} > 0$, and compute the exact values of a and b.[1]
    2. Length of a side of an equilateral triangle ABC is 2. If $AD \perp BC$, and the angle bisector of $\angle BAD$ meets BC at E, show that $\angle AED = 75°$. Using the angle bisector theorem, find the exact length of ED. Using the diagram, compute exact values of $\sin 75°$ and $\cos 75°$.

4. Find the nth term of a harmonic progression whose first two terms are a and b.

5. Find the relationship between a, b and c if the system

$$\begin{aligned} x + y &= a \\ x^2 + y^2 &= b \\ x^3 + y^3 &= c \end{aligned}$$

has solutions.

6. A function f is defined on the positive integers, $f(1) = 1009$ and

$$f(1) + f(2) + \cdots + f(n) = n^2 f(n).$$

    1. By expressing $f(1) + f(2) + \cdots + f(n-1)$ in two ways, find $\frac{f(n)}{f(n-1)}$.
    2. By using the result in part(a), find the formula for $f(n)$. Calculate $f(2018)$.

7.
    1. $f(x) = \frac{1}{(2x-1)(2x-3)}$ can be expressed as $f(x) = \frac{A}{2x-1} + \frac{B}{2x-3}$, where A and B are constants. Find the values of A and B.
    2. If $f(3) + f(4) + f(5) + \cdots + f(n) = c - g(n)$, where c is a constant and $g(n)$ is a function, determine c and $g(n)$.

8. Find the number of ways in which 5 men, 3 women and 2 children can sit at a round table, if

[1] The original problem asked to find the value of $\sqrt{24 - 2\sqrt{3}}$ as $a - b\sqrt{3}$, but there are no such integers.

1. there are no restrictions,
2. each child is seated between 2 women.

$\boxed{9}$ $ABCD$ is a rectangle with $AB = x$, $AD = y$ and $y > x$, and $AXYZ$ is a square. If the area of $AXYZ$ is the same as the area of $ABCD$, show that $\sqrt{xy} - x \leq BX \leq \sqrt{xy} + x$ and $y - \sqrt{xy} \leq DZ \leq y + \sqrt{xy}$.

$\boxed{10}$ The average of the numbers $1, 2, 3, \ldots, 99$ and x is $100x$. Find the value of x.

$\boxed{11}$ Each of 240 boxes in a line contains a single red marble, and for $1 \leq k \leq 240$, the box in the kth position also contains k white marbles. Phyu Phyu begins at the first box and successively draws a single marble at random from each box. She stops when she first draws a red marble. Let $\mathbb{P}(n)$ be the probability that Phyu Phyu stops after drawing exactly n marbles. What is the smallest value of n for which $\mathbb{P}(n) < \frac{1}{240}$?

$\boxed{12}$ For all positive integers n, define

$$f(n) = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2n-1} - \frac{1}{2n},$$

$$g(n) = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}.$$

By using mathematical induction, prove that $f(n) = g(n)$ for all positive integers n.

$\boxed{13}$ $ABCD$ is a cyclic quadrilateral with $AB = AC$. The line PQ is tangent to the circle at the point C, and is parallel to BD. Diagonals BD and AC intersect at E. If $AB = 18$ and $BC = 12$, find the length of AE.

$\boxed{14}$ Let $a > b > 0$. Define two sequences a_n and b_n as follows:

$$a_1 = a,\ b_1 = b,\ a_{n+1} = \frac{a_n + b_n}{2},\ b_{n+1} = \sqrt{a_n b_n}.$$

1. Prove that $a_{n+1} < a_n$ and $b_{n+1} > b_n$ for $n > 1$.
2. Prove that $a_{n+1} - b_{n+1} = \frac{(a_n - b_n)^2}{8a_{n+2}}$.
3. If $a = 4$ and $b = 1$, find the first four terms of each sequence of a_n and b_n.

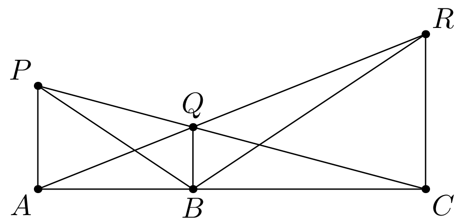
$\boxed{15}$ If $\cos\alpha, \cos\beta$ and $\cos\gamma$ are the roots of the equation $x^3 + ax^2 + bx + c = 0$, where α, β and γ are angles of a triangle, prove that $a^2 = 2b + 2c + 1$.

$\boxed{16}$ The two circles C_1 and C_2 intersect at the points A and B. The tangent to C_1 at A intersects C_2 at P and the line PB intersects C_1 at Q. The tangent to C_2 drawn from Q intersects C_1 and C_2 at the points X and Y respectively. The points A and Y lie on the different sides of PQ. Show that AY bisects $\angle XAP$.

2018 Regional Round

1 In the figure, AP, BQ, CR are perpendicular to the straight line ABC. Prove that

1. $\triangle PAB \sim \triangle RCB$
2. $\frac{1}{BQ} = \frac{1}{AP} + \frac{1}{CR}$.



2 PA and PB are the tangent segments at A and B to a circle whose center is O. AB and OP are intercept at Q. Prove that $AB \perp OP$. Hence show that $OQ : QP = AO^2 : AP^2$. Hence also show that $\alpha(\triangle OAB) : \alpha(\triangle PAB) = AO^2 : AP^2$.

3 If T_1, T_2, T_3 are the sums of n terms of three series in AP, the first term of each being a and the respective common differences being d, $2d$, $3d$, then show that $T_1 + T_3 = 2T_2$.

4 The positive difference between the zeros of the quadratic expression $x^2 + kx + 3$ is $\sqrt{69}$. Find the possible values of k.

5 In $\triangle PQR$, $\angle Q = 90°$ and S is a point on PR such that $QS \perp PR$. If $PR = kQR$, then show that $PS = (k^2 - 1)RS$.

6 Prove the following theorem:

If a ray from the vertex of an angle of a triangle divides the opposite side into segments that have the same ratio as the other two sides, then it bisects the angle.

7 The sum to k terms of an AP is 21. The sum to $2k$ terms is 78. The kth term is 11. Find the first term and the common difference.

8 Prove that if a, b, c and d are positive, the equation $x^4 + bx^2 + cx - d = 0$ has one positive, one negative and two imaginary roots.

9 Show that the sum of the squares of the first n odd numbers is $\frac{1}{3}n(4n^2 - 1)$.

$\boxed{10}$ If 100! is divisible by 7^n, find the maximum value of n.

$\boxed{11}$ Show that $x = 10°$ is a solution of $2\sin x = \frac{1+\tan^2 x}{3-\tan^2 x}$.

$\boxed{12}$ Show that $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$ for all $n > 1$.

$\boxed{13}$

1. Prove the following Cauchy inequality:

   For any real numbers $a_1, \dots, a_n$ and $b_1, \dots, b_n$,

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \le (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2).$$

2. For a set of positive real numbers $x_1, \dots, x_n$, the Root-Mean Square RMS is defined by the formula

$$\text{RMS} = \sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}}$$

   and the Arithmetic Mean AM is defined by the formula

$$\text{AM} = \frac{x_1 + \cdots + x_n}{n}.$$

   Prove that RMS $\ge$ AM.

$\boxed{14}$

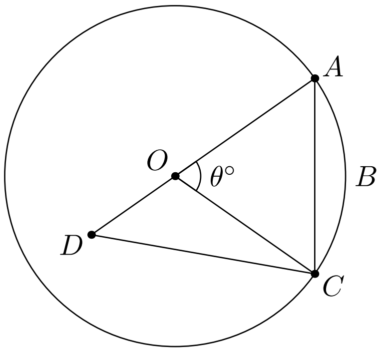
1. Prove that if p is a prime number, then the coefficient of every term in the expansion of $(a+b)^p$ except the first and last is divisible by p.
2. Hence show that if p is a prime number and N is a positive integer, then $N^p - N$ is a multiple of p.
3. Hence also show that if p is a prime number, then $10^p - 7^p - 3$ is divisible by p.

2018 National Round

$\boxed{1}$ To construct a triangle ABC, only given that $AB = 10$ and $\angle ABC$ is $30°$. Find all values of AC for which

1. there are two possible triangles ABC,
2. there is only one triangle ABC,
3. there is no triangle ABC.

$\boxed{2}$ In the given figure, O is the center of the circle. OA and OC are radii with $OA = OC = 5$ units. Find the area of region $ABCD$, bounded by arc ABC, line segments CD and DA, in terms of θ, if the line segment $DA = 8$ units and $\angle AOC$ is $\theta°$.



$\boxed{3}$ Prove that

1. $\cos 20°$,
2. $\log 21$, and
3. $\sqrt{3}$

are irrational.

$\boxed{4}$ Triangle ABC with $AB = c$, $BC = a$ and $CA = b$ is inscribed in a circle. Find the radius of the circle in terms of a, b and c.

$\boxed{5}$ Prove the following Ptolemy's theorem:

In a cyclic quadrilateral $PQRS$,

$$PQ \cdot SR + PS \cdot QR = PR \cdot SQ,$$

That is, the sum of the products of the opposite sides is equal to the product of the diagonals.

$\boxed{6}$ $f(x)$ is a real-valued function defined on $a < x < b$ such that

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{1}{2}(f(x_1) + f(x_2)),$$

for all $a < x_1, x_2 < b$. Prove that

$$f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) \le \frac{1}{n}(f(x_1) + f(x_2) + \cdots + f(x_n))$$

for all $a < x_1, x_2, \ldots, x_n < b$.

$\boxed{7}$ Point A has position vector $\overrightarrow{OA}$ and point B has position vector $\overrightarrow{OB}$. Prove that for $0 \le \lambda \le 1$, the vector $\lambda \cdot \overrightarrow{OA} + (1 - \lambda) \cdot \overrightarrow{OB}$ is a position vector of a point on the line segment AB. Prove also that any point on the line segment AB has position vector $\lambda \cdot \overrightarrow{OA} + (1 - \lambda) \cdot \overrightarrow{OB}$ for some $0 \le \lambda \le 1$.

$\boxed{8}$ $f(x)$ is a real-valued function defined on the interval $a < x < b$ such that

$$f(tx_1 + (1 - t)x_2) \le tf(x_1) + (1 - t)f(x_2)$$

for all $a < x_1, x_2 < b$ and for all $0 \le t \le 1$. Prove that for each triple x_1, x_2, x_3 of distinct numbers in the interval,

$$\frac{(x_3 - x_2)f(x_1) + (x_2 - x_1)f(x_3) + (x_1 - x_3)f(x_2)}{(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)} \ge 0.$$

$\boxed{9}$ Equilateral triangle ABC is inscribed in a circle. P is a point on arc AB. Prove that $PA + PB = PC$.

$\boxed{10}$ $\triangle ABC$ is inscribed in a circle. The tangent at C and the line through B parallel to AC meet at D. The tangent at B and the line through C parallel to AB meet at E. Prove that $BC^2 = BE \cdot CD$.

$\boxed{11}$ $\triangle ABC$ is isosceles with $AB = AC$. D is the midpoint of BC. AB and AC are produced to X and Y respectively such that $\angle XDY = \angle DCY$. Prove that $\triangle XBD$, $\triangle XDY$ and $\triangle DCY$ are similar triangles.

$\boxed{12}$ a_1, b_1, c_1, a_2, b_2, c_2 are positive real numbers with $a_1c_1 - b_1^2 \ge 0$ and $a_2c_2 - b_2^2 \ge 0$. Show that $(a_1 + a_2)(c_1 + c_2) - (b_1 + b_2)^2 \ge 0$.

$\boxed{13}$ How many permutations of the word 'TRIANGLE' have none of the vowels together?

$\boxed{14}$ From the group of $2n + 1$ people, how many ways to choose a group of n people or less?

$\boxed{15}$ Show that the product of any positive integer and its $k - 1$ successors is divisible by $k!$.

$\boxed{16}$ How many integers between 1 and 1,000,000 have the sum of digits equal to 10?

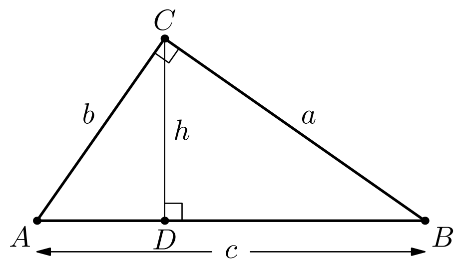
$\boxed{17}$ Each side and diagonal of a regular hexagon is coloured either red or blue. Show that there is a triangle with all three sides of the same colour.

$\boxed{18}$ Find the value of $2018^2 - 2017^2 + 2016^2 - 2015^2 + \cdots + 2^2 - 1$.

2019 Regional Round

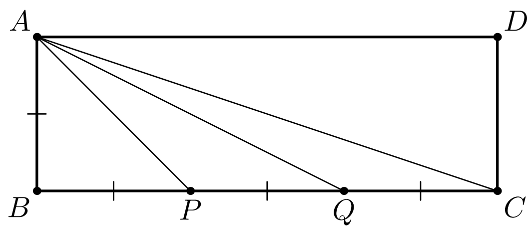
$\boxed{1}$

1. Show that $x^2 + y^2 \geq 2xy$ for any two real numbers x and y, and that the sign of the equality holds if and only if $x = y$.
2. By using similar triangles, prove that $h = \frac{ab}{c}$ for the given figure.
3. By using (1) and (2), show that $h \leq \frac{c}{2}$, and that the sign of equality only holds if and only if $a = b$.
4. Show that of all right triangles having the same length of hypotenuse, the isosceles right triangle maximizes the area.



$\boxed{2}$ Find all functions over the reals such that $f(x) + 2f(1-x) = x(1-x)$.

$\boxed{3}$ In the given figure, $ABCD$ is a rectangle. P and Q are points on BC such that $AB = BP = PQ = QC$. Find two similar but not congruent triangles and prove their similarity.



$\boxed{4}$ Let $f(x) = ax^2 + bx + c$, $a \neq 0$.

1. Fill in the following steps ($\square$) to show that $f(x) = a(x + \frac{b}{2a})^2 - \frac{b^2-4ac}{2a}$.

$$\begin{aligned}
f(x) &= a(x^2 + \square x) + c \\
&= a\left(x^2 + \square x + \square - \left(\frac{b}{2a}\right)^2\right) + c \\
&= a(x^2 + \square x + \square) - \frac{b^2}{\square} + c \\
&= a(x + \square)^2 - \frac{b^2}{\square} + c \\
&= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}
\end{aligned}$$

2. Hence show that if $a > 0$ and $b^2 - 4ac < 0$, then $f(x) > 0$ for all $x \in \mathbb{R}$.

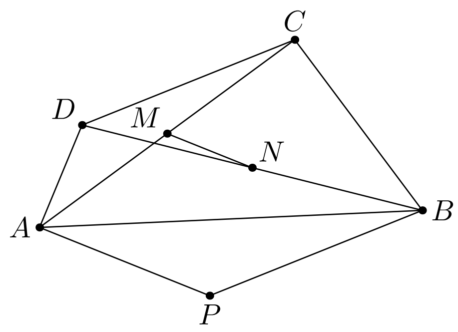
3. Show that if $a < 0$ and $b^2 - 4ac < 0$, then $f(x) < 0$ for all $x \in \mathbb{R}$.

4. If $b^2 - 4ac > 0$, find the roots of the equation $f(x) = 0$ in terms of a, b and c.

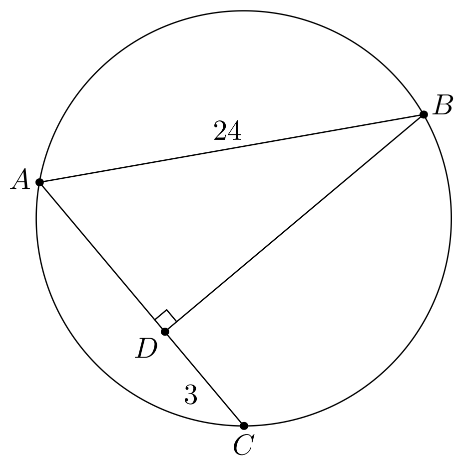
$\boxed{5}$ In the given figure, $DC = PB$ and $DC \parallel PB$. M and N are midpoints of AC and BD respectively.

1. Prove that the points P, N, C are collinear.

2. Prove that $AP \parallel MN$ and $AP = 2MN$.

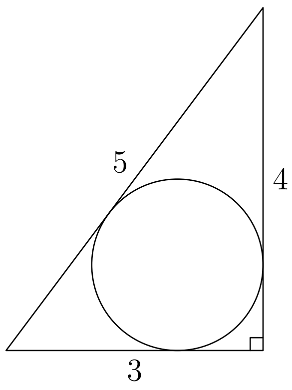


$\boxed{6}$ In the given figure, $\angle BAC = 60°$, $AB = 24$ cm, $BD \perp AC$ and $DC = 3$ cm. Find the diameter of the circle.



$\boxed{7}$ In an AP, the kth term is 11. The sum of the first k terms is 26. The sum of the next k terms is 74. Find the first term and the common difference.

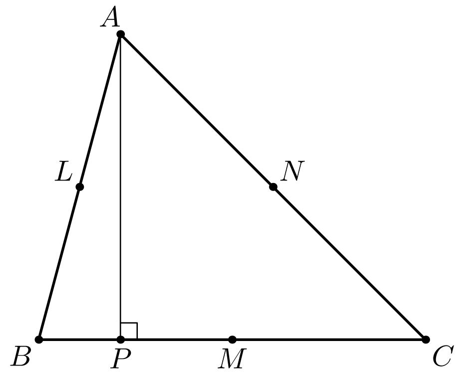
$\boxed{8}$ What is the radius of the inscribed circle of a 3-4-5 right triangle?



$\boxed{9}$ A bag contains 3 red balls and 2 green balls. Balls are drawn at random, one at a time but not replaced, until all 3 of red balls are drawn or until both green balls are drawn. What is the probability that the 3 reds are drawn?

$\boxed{10}$ If 75! is divisible by 5^n, find the maximum value of n.

$\boxed{11}$ If L, M, N are the midpoints of the sides of $\triangle ABC$, and P is the foot of perpendicular from A to BC, prove that L, P, M, N are concyclic.



$\boxed{12}$ In a GP, the kth term is 864. The sum of the first k terms is 2080. The sum of the first $2k$ terms is 12610. Find the first term and the common ratio.

$\boxed{13}$ Prove that $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ if all variables are integers and $n \geq m \geq k \geq 0$.

$\boxed{14}$

1. Prove that $n \leq -k^2 + nk + k \leq \frac{(n+1)^2}{4}$ for $1 \leq k \leq n$.
2. Consider $k(n+1-k) = -k^2+nk+k$ and by using the inequalities in (1), prove that

$$n^n \leq (n!)^2 \leq \frac{(n+1)^{2n}}{4^n}.$$

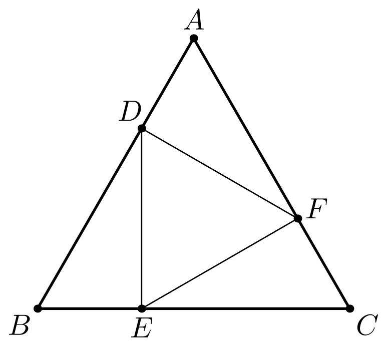
3. Hence prove that

$$n^{\frac{n}{2}} \leq n! \leq \frac{(n+1)^n}{2^n}.$$

2019 National Round

$\boxed{1}$ $\triangle ABC$ is an equilateral triangle.

1. Prove that there are points D, E and F on AB, BC and CA respectively such that $DE \perp BC$, $EF \perp CA$ and $FD \perp AB$.
2. Prove that $\triangle DEF$ is also an equilateral triangle.
3. Find the ratio of the perimeters of $\triangle DEF$ and $\triangle ABC$.
4. Find the ratio of the areas of $\triangle DEF$ and $\triangle ABC$.



$\boxed{2}$

1. If $abc = 2ab + 2bc + 2ca$ where a, b and c are integers and $1 \leq a \leq b \leq c$, then show that $3 \leq a \leq 6$.
2. Find all possible ordered triples (a, b, c) such that

$$abc = 2ab + 2bc + 2ca$$

where a, b and c are integers and $1 \leq a \leq b \leq c$.

$\boxed{3}$ Let $\mathbb{Z}$ be the set of integers. Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function from $\mathbb{Z}$ to $\mathbb{Z}$ such that

$$f(x+y) + f(x-y) = 2f(x) + 2f(y),$$

for all integers x and y.

1. Show that $f(0) = 0$ and $f(2x) = 4f(x)$.
2. Show that $f(nx) = n^2 f(x)$ for every positive integer n.
3. Show that $f(y) = f(-y)$.
4. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all integers x and y.

$\boxed{4}$ Seven integers are written around a circle in a way that no two or three adjacent numbers have a sum divisible by 3. How many of these seven numbers are divisible by 3?

$\boxed{5}$ 1. Find an ordered pair (x, y) such that

$$2019x + 2021y = 1$$

where x and y are integers.

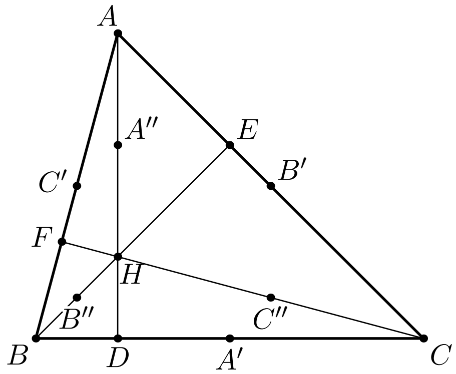
2. By using the ordered pair obtained in question 1, find all solutions of

$$2019x + 2021y = 1$$

where x and y are integers.

$\boxed{6}$ In $\triangle ABC$, altitudes AD, BE and CF pass through the point H. Points A', B' and C' are midpoints of BC, CA, AB respectively. Points A'', B'' and C'' are midpoints of AH, BH, CH respectively.

1. Prove that $B'C'B''C''$ is a rectangle.
2. Prove that $C'A'C''A''$ is a rectangle.
3. Prove that the six points A', B', C',A'', B'', C'' are concyclic.
4. Prove that the nine points $A', B', C', A'', B'', C'', D, E, F$ are concyclic.



$\boxed{7}$ Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$f(x+y) + f(x-y) = 2f(x)\cos y, \ \ x, y \in \mathbb{R}.$$

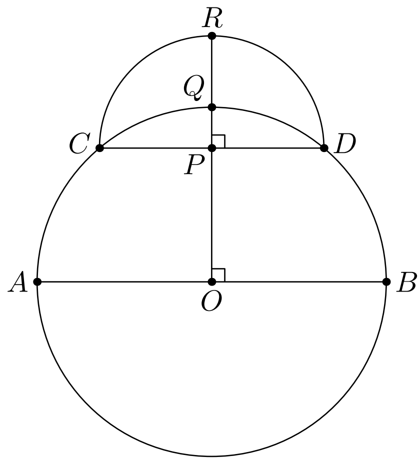
1. Show that $f(\theta) + f(-\theta) = 2a\cos\theta$, where $a = f(0)$.
2. Show that $f(\theta + \pi) + f(\theta) = 0$.
3. Show that $f(\theta + \pi) + f(-\theta) = -2b\sin\theta$, where $b = f\left(\frac{\pi}{2}\right)$.
4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) + f(x-y) = 2f(x)\cos y$$

where $x, y \in \mathbb{R}$.

$\boxed{8}$ A student council must select a two-person welcoming committee and a three-person planning committee from its members. There are exactly 15 ways to select a two-person team for the welcoming committee. It is possible for students to serve on both committees. In how many different ways can a three-person planning committee be selected?

$\boxed{9}$ AB is a diameter of a circle O with radius 10 cm. OQ is a radius of a circle O such that $QO \perp AB$. A point P is on OQ. Draw a semicircle centered at P with diameter CD where CD is the chord of circle O and $CD \perp PQ$. PQ produced meets the semicircle at R. Find the maximum possible length of QR.



$\boxed{10}$ Find all positive integers s such that $\left\lceil \frac{s}{3} \right\rceil - 21 = \left\lceil \frac{s}{5} \right\rceil$ where $\lceil x \rceil$ is the smallest integer greater than or equal to x. For example, $\lceil 3.7 \rceil = 4$, $\lceil 3 \rceil = 3$ and $\lceil 3.2 \rceil = 4$.

$\boxed{11}$ In $\triangle ABC$, $AB = 94$ and $AC = 107$. A circle with center A and radius AB intersects BC at points B and X. Moreover, BX and CX have integer lengths. What is BC?

$\boxed{12}$ Seven people are sitting around a circular table, each holding a fair coin. All seven people flip their coins and those who flip heads stand while those flip tails seated. What is the probability that no two people adjacent will stand?

$\boxed{13}$

1. Let a and b be positive integers. If there are integers x_0, y_0 such that $ax_0 + by_0 = 1$, then prove that the greatest common divisor of a and b is 1.

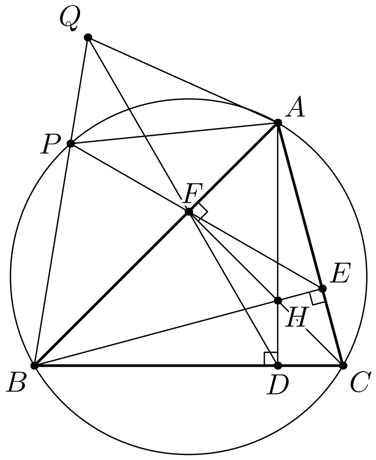
2. Prove that the fraction $\frac{12n+5}{14n+6}$ is in lowest terms for every positive integer n.

$\boxed{14}$ $\triangle ABC$ is inscribed in a circle. Altitudes AD, BE and CF pass through the point H. EF produced meets the circle at P. BP produced and DF produced meet at the point Q.

1. Show that $\angle ACF = \angle ADF = \angle ABE$.

2. Show that $\angle AFQ = \angle ACD$.

3. Show that $AP = AQ$.



15 A car drives from town A to B at the average speed of 30 km/h, from town B to town C at average speed of 60 km/h; and on the way back, the car drives from C to B at average speed of 30 km/h[1], from town B to A at average speed of 60 km/h. The whole trip takes 6 hours. What is the total distance of the round trip?

16
1. Prove that the square of an odd number gives the remainder 1 upon dividing by 8.
2. Prove that if k is odd and n is a positive integer, then $k^{2^n} - 1$ is divisible by 2^{n+2}.

[1]The original problem set this to be 20 km/h, but in that case the total distance is not unique anymore.

Part III.

Solutions

Sample Problems

Problem 1. Given the sequence $u_n = n^2 - 6n + 13$, what is the smallest term in the sequence?

Solution. By completing the square, we have

$$n^2 - 6n + 13 = n^2 - 6n + 9 + 4 = (n-3)^2 + 4.$$

Since squares are non-negative, this means that $(n-3)^2 + 4 \geq 0 + 4 = 4$. Hence the minimum is $u_3 = 4$, achieved when $n = 3$. $\square$

Problem 2. A function $f(x) = \frac{p}{x-q}$, $q > 0$, $q \neq x$ is such that $f(p) = p$ and $f(2q) = 2q$. Find the values of p and q. If $(f \circ f)(x) = x$, show that $x^2 - x - 2 = 0$.

Solution. We have the following two equations:

$$p = f(p) = \frac{p}{p-q} \implies p^2 - pq = p \tag{1}$$

$$2q = f(2q) = \frac{p}{q} \implies p = 2q^2 \tag{2}$$

Substituting the value of p in (1), we get $4q^4 - 2q^3 - 2q^2 = 0$. Since $q > 0$, we can cancel out $2q^2$ from both sides, which leaves us with

$$2q^2 - q - 1 = 0 \implies (q-1)(2q+1) = 0.$$

Solving this shows that $q = 1$ or $q = -\frac{1}{2}$. Since q is non-negative, we can discard the latter, and hence $q = 1$. This gives us $p = 2$. Therefore, the original function becomes $f(x) = \frac{2}{x-1}$. Finally,

$$x = (f \circ f)(x) = \frac{2}{\frac{2}{x-1} - 1} \implies x^2 - x - 2 = 0. \qquad \square$$

Problem 3. Let $f(x) = 1 + x + x^2 + \cdots + x^n$. The remainder when $f(x)$ is divided by $2x - 1$ is $\frac{341}{256}$ more than the remainder when $f(x)$ is divided by $2x + 1$. Find the value of n.

Solution. Since f is a polynomial, by the remainder theorem, the remainder when $f(x)$ is divided by $2x - 1$ is $f(\frac{1}{2})$ and the remainder when $f(x)$ is divided by $2x + 1$ is $f(-\frac{1}{2})$. Therefore, the given condition becomes

$$f\left(\frac{1}{2}\right) - f\left(-\frac{1}{2}\right) = \frac{341}{256}$$

$$\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}\right) - \left(1 + \frac{1}{(-2)} + \frac{1}{(-2)^2} + \cdots + \frac{1}{(-2)^n}\right) = \frac{341}{256}$$

Notice that all the fractions with even powers cancel out. Therefore, if we let t be the largest odd number less than or equal to n, we have

$$\begin{aligned} 2\left(\frac{1}{2}+\frac{1}{2^3}+\cdots+\frac{1}{2^t}\right) &= \frac{341}{256} \\ 1+\frac{1}{2^2}+\frac{1}{2^4}+\cdots+\frac{1}{2^{t-1}} &= \frac{341}{256} \end{aligned}$$

This is a GP with starting term $a = 1$ and common ratio $r = \frac{1}{4}$. Since $ar^{n-1} = u_n = \frac{1}{2^{t-1}}$, it follows that there are $n = \frac{t+1}{2}$ terms, and so the GP summation formula gives us

$$\begin{aligned} \frac{1-\frac{1}{2^{t+1}}}{\frac{3}{4}} &= \frac{341}{256} \\ \frac{1}{2^{t+1}} &= 1-\frac{341\times 3}{256\times 4} \\ \frac{1}{2^{t+1}} &= \frac{1}{1024} \end{aligned}$$

which shows that $t = 9$. Since t is the largest odd number less than or equal to n, this means that n must be either 9 or 10, and these are the only solutions. □

Problem 5. In the binomial expansion of $(1+x)^n$, the coefficients of 5th, 6th and 7th terms are consecutive terms of an AP. Find the first three terms of the binomial expansion.

Solution. By the Binomial theorem, the coefficients of 5th, 6th and 7th terms are $\binom{n}{4}$, $\binom{n}{5}$ and $\binom{n}{6}$ respectively. Since these three numbers are in AP,

$$\begin{aligned} \binom{n}{4}+\binom{n}{6} &= 2\binom{n}{5} \\ \frac{n(n-1)(n-2)(n-3)}{1\times 2\times 3\times 4} + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1\times 2\times 3\times 4\times 5\times 6} &= \frac{2n(n-1)(n-2)(n-3)(n-4)}{1\times 2\times 3\times 4\times 5} \\ 1+\frac{(n-4)(n-5)}{5\times 6} &= \frac{2(n-4)}{5} \\ 30+(n-4)(n-5) &= 12(n-4) \\ n^2-21n+98 &= 0 \\ (n-14)(n-7) &= 0 \end{aligned}$$

and so it follows that n is either 7 or 14. In the first case, the first three terms are 1, $7x$, $21x^2$, and in the second case the first three terms are 1, $14x$, $91x^2$. □

Problem 6. Find the number of integers between 100 and 999 such that the sum of the three digits is 12.

Solution. The numbers we want are of the form $100 < \overline{abc} < 999$ with $a+b+c = 12$. Notice that after we choose a and b, the value of c is automatically determined.

When $a = 1$, $b+c = 11$. This means that b cannot be less than 2, as that would make c greater than 9. Since $b \in \{2, 3, \cdots, 9\}$, there are 8 choices of b here.

When $a = 2$, $b+c = 10$. Similarly to the above case, b cannot be less than 1. Therefore, there are 9 choices of b for this case.

When $3 \leq a \leq 9$, $b+c \leq 9$, so we don't need to worry about the size of c exceeding 9 anymore. For each choice of a, b can range from 0 to $12-a$ so there are $12-a+1 = 13-a$ choices of b. Therefore, the total number of choices of b for $3 \leq a \leq 9$ is

$$10+9+8+7+6+5+4 = 49.$$

Hence, the total number is $8+9+49 = 66$. □

Problem 7. Determine the solution set for which $\frac{(x+1)(x-2)}{1-2x} > 0$.

Solution. For a rational function to be positive, both the numerator and the denominator must have the same sign. Therefore, we can consider two cases as follows.

Case 1: $(x+1)(x-2)$ and $1-2x$ are both positive.

In this case, since $(x+1)(x-2)$ is positive, we must have either $x > -1$ and $x > 2$, or $x < -1$ and $x < 2$. The former is equivalent to $x > 2$ and the latter is equivalent to $x < -1$, so we must have $x > 2$ or $x < -1$. Now since $1-2x > 0$, we see that $x < -\frac{1}{2}$. Combining all of these shows that $x < -1$.

Case 2: $(x+1)(x-2)$ and $1-2x$ are both negative.

In thise case, since $(x+1)(x-2)$ is negative, we must have either $x > -1$ and $x < 2$, or $x < -1$ and $x > 2$. The latter is impossible, so $-1 < x < 2$. Since $1-2x < 0$, we also have $x > \frac{1}{2}$. Combining these two shows that $\frac{1}{2} < x < 2$ for this case.

Hence the solution set is $\{x \in \mathbb{R} \mid x < -1 \text{ or } \frac{1}{2} < x < 2\}$. □

Problem 8. If a_1, a_2, a_3 are in AP, a_2, a_3, a_4 are in GP and a_3, a_4, a_5 are in HP, then prove a_1, a_3, a_5 are in GP.

Solution. Write $a_1 = a-d$, $a_2 = a$ and $a_3 = a+d$. Then since a_2, a_3, a_4 are in GP,

$$a_2a_4 = a_3^2 \implies a_4 = \frac{a_3^2}{a_2} = \frac{(a+d)^2}{a}.$$

Now since a_3, a_4, a_5 are in HP, $\frac{1}{a_3}$, $\frac{1}{a_4}$, $\frac{1}{a_5}$ are in AP. Therefore,

$$\begin{aligned}
\frac{1}{a_3} + \frac{1}{a_5} &= \frac{2}{a_4} \\
\frac{1}{a_5} &= \frac{2a}{(a+d)^2} - \frac{1}{a+d} \\
&= \frac{2a-a-d}{(a+d)^2} \\
&= \frac{a-d}{(a+d)^2} \\
a_5 &= \frac{(a+d)^2}{a-d}.
\end{aligned}$$

Therefore,

$$\frac{a_1}{a_3} = \frac{a-d}{a+d} = \frac{a_3}{a_5}$$

and hence a_1, a_3, a_5 are in GP. □

Problem 9. Prove by mathematical induction that

$$\sum_{r=1}^{n} r^4 = \frac{n(n+1)(6n^3+9n^2+n-1)}{30}.$$

Solution. For the base case $n = 1$, it is easy to check that both the left hand side and right hand side are equal to 1. Now suppose that this identity is true for $n = k$. We have to show that it is also true for $n = k+1$. i.e.

$$\sum_{r=1}^{k+1} r^4 = \frac{(k+1)(k+2)(6(k+1)^3+9(k+1)^2+k)}{30}.$$

But $\sum_{r=1}^{k+1} r^4 = \sum_{r=1}^{k} r^4 + (k+1)^4$, and

$$\begin{aligned}
\sum_{r=1}^{k} r^4 + (k+1)^4 &= \frac{k(k+1)(6k^3+9k^2+k-1)}{30} + (k+1)^4 \\
&= \frac{(k+1)(6k^4+9k^3+k^2-k+30k^3+90k^2+90k+30)}{30} \\
&= \frac{(k+1)(6k^4+39k^3+91k^2+89k+30)}{30} \\
&= \frac{(k+1)(k+2)(6k^3+27k^2+37k+15)}{30} \\
&= \frac{(k+1)(k+2)(6k^3+18k^2+18k+6+9k^2+18k+9+k)}{30} \\
&= \frac{(k+1)(k+2)(6(k+1)^3+9(k+1)^2+k)}{30}
\end{aligned}$$

so by mathematical induction, this identity is true for all $n \in \mathbb{N}$. □

Problem 11. If the base BC of $\triangle ABC$ is trisected at P and Q, show that

$$AB^2 + AC^2 = AP^2 + AQ^2 + 4PQ^2.$$

Solution. Let $\angle A$, $\angle B$ and $\angle C$ be α, β and γ. By the law of cosines in $\triangle ABP$ and $\triangle ACQ$,

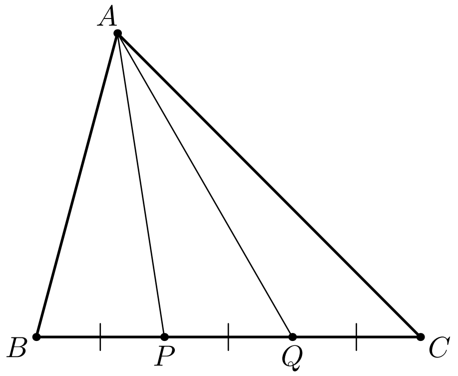
$$AP^2 = AB^2 + BP^2 - 2AB \cdot BP \cos\beta,$$

$$AQ^2 = AC^2 + CQ^2 - 2AC \cdot CQ \cos\gamma.$$

By law of cosines again in $\triangle ABC$, we also have

$$2AB \cdot BC \cos\beta = AB^2 + BC^2 - AC^2,$$

$$2AC \cdot BC \cos\gamma = AC^2 + BC^2 - AB^2.$$



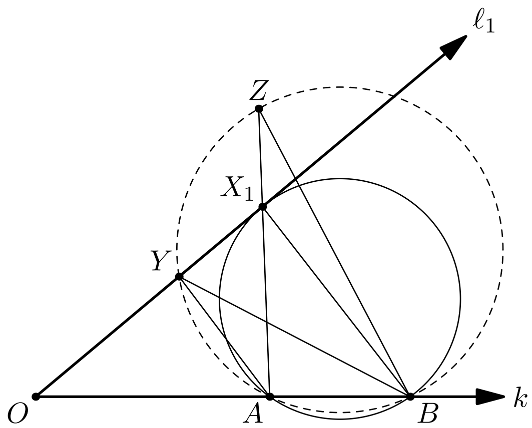
Therefore,

$$\begin{aligned}
AB^2 - AP^2 + AC^2 - AQ^2 &= 2AB \cdot BP \cos\beta + 2AC \cdot CQ \cos\gamma - BP^2 - CQ^2 \\
&= \frac{1}{3}(2AB \cdot BC \cos\beta + 2AC \cdot BC \cos\gamma) - 2PQ^2 \\
&= \frac{1}{3}(AB^2 + BC^2 - AC^2 + AC^2 + BC^2 - AB^2) - 2PQ^2 \\
&= \frac{2BC^2}{3} - 2PQ^2 \\
&= 6PQ^2 - 2PQ^2 \\
&= 4PQ^2.
\end{aligned}$$

□

Problem 12. From a point O, two straight lines are at any angle. On one of these lines, points A and B are taken such that $OA = \frac{5}{2}$ inch and $AB = \frac{3}{2}$ inch. Find the point on the other line at which AB subtends the greatest angle.

Solution. We will only describe how to construct the desired point when A lies between O and B, as the other case can be handled in the same way. Since $OA > AB$, we see that O lies outside of segment AB. Let the ray on which those two points lie be k. We can split the other line, say ℓ, into two rays ℓ_1 and ℓ_2, both originating from O and pointing at opposite directions. Let X_1 be a point on ℓ_1 such that (X_1AB) is tangent to ℓ_1. We claim that $\angle AX_1B$ is the maximal angle among any point on ray ℓ_1.



First, since

$$OX_1^2 = OA \cdot OB = 10 \Longrightarrow OX_1 = \sqrt{10},$$

it follows that there is only one such X_1. Now take any point Y other than X_1 on ℓ_1. We must show that $\angle AX_1B > \angle AYB$. Let line $\overline{AX_1}$ meet (AYB) at Z. Since (AYB) is not tangent to

ℓ_2, it follows that X lies inside (AYB). Therefore,

$$\angle AX_1B = \angle AZB + \angle X_1BZ > \angle AZB = \angle AYB$$

and hence it follows that $\angle AX_1B$ is indeed maximal.

Similarly, we can also choose a point X_2 on ray ℓ_2, such that $\angle AX_2B$ is maximal among any point on ℓ_2. Now take a point $X \in \{X_1, X_2\}$ such that $\angle AXB$ is maximal. We claim that X is our desired point. Take any point P on ℓ other than X; it must lie either on ℓ_1 or ℓ_2. Suppose WLOG that it lies on ℓ_1. Then

$$\angle APB < \angle AX_1B \leq AXB,$$

and we are done. □

Remark. In the case when $k \perp \ell$, we have $\angle AX_1B = \angle AX_2B$, so there are two points on ℓ that satisfy the given condition.

Problem 13. Given any seven distinct real numbers x_1, x_2, …, x_7, prove that we can always find the numbers x_i, $1 \leq i \leq 7$ and x_j, $1 \leq j \leq 7$ such that $0 < \frac{x_i - x_j}{1 + x_i x_j} < \frac{1}{\sqrt{3}}$.

Solution. Choose $0 \leq a_1, a_2, \ldots, a_7 < \pi$ such that $\tan a_i = x_i$ for each $i = 1, 2, \ldots, 7$. Divide the interval $[0, \pi)$ into six equally sized intervals; i.e. consider intervals $I_1, I_2, \ldots, I_6$ such that $I_k = [\frac{(k-1)\pi}{6}, \frac{k\pi}{6})$. Since there are 7 elements in the sequence $a_1, a_2, \ldots, a_7$, it follows that two of them must be in the same interval. Let the larger one be a_i and the smaller one be a_j. Then $0 < a_i - a_j < \frac{\pi}{6}$, so it follows that

$$\frac{x_i - x_j}{1 + x_i x_j} = \frac{\tan a_i - \tan a_j}{1 + \tan a_i \tan a_j} = \tan(a_i - a_j) < \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}.$$

Since $a_i > a_j$, $\tan(a_i - a_j) > 0$, so we are done. □

Problem 14. Prove that $6 \mid n^3 - n$ for all integers n.

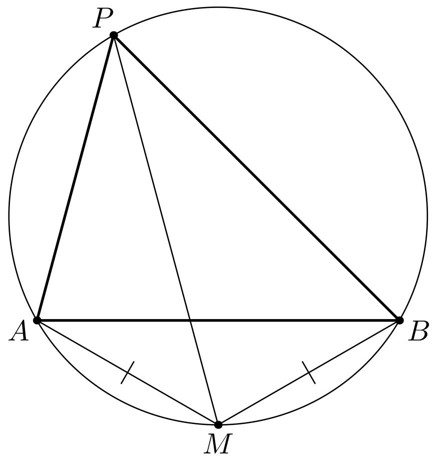
Solution. Notice that $n^3 - n = (n-1)n(n+1)$. Since $n-1$, n and $n+1$ are three consecutive integers, it follows that one of them must be divisible by 3. Similarly, since $n-1$ and n are two consecutive integers, one of them must be divisible by 2, so $n^3 - n$ is also divisible by 2. Since $n^3 - n$ is divisible by both 2 and 3, it must be divisible by their least common multiple which is 6. □

Problem 16. The 5-digit number $\overline{A986B}$ is divisible by 72. What is the value of $A + B$?

Solution. Let the number in the problem be S. Since S is divisible by 72, it is divisible by 8, which means that $\overline{86B}$ is divisible by 8. It follows that $B = 4$. Now S is also divisible by 9, so the sum of its digits must be divisible by 9. Hence $A + 9 + 8 + 6 + 4 = A + 27$ is divisible by 9, so A must be divisible by 9. However, A is non-zero since it is the first digit, A must be equal to 9. Therefore, $A + B = 9 + 4 = 13$. □

Problem 17. AB is a chord of a circle and P is any point on the arc of one of the segments cut off. Prove that the bisection of the $\angle APB$ passes through a fixed point on the circumference.

Solution. Let the angle bisector of $\angle APB$ intersect the circle at M.



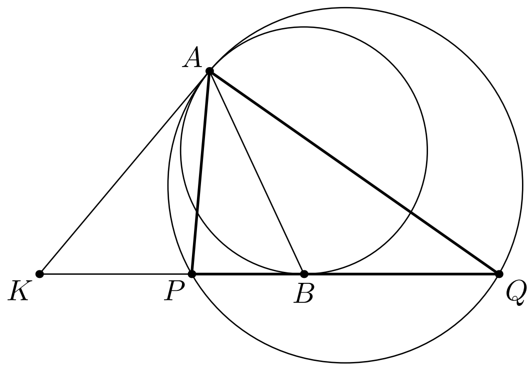
Then

$$\angle MAB = \angle MPB = \angle MPA = \angle MBA,$$

which implies that $MA = MB$. Therefore, M is the midpoint of arc AB that does not contain P. Since M does not depend on the position of P, we are done. □

Problem 18. Two circles touch each other internally at A. Through B, a point on the circumference of the inner circle, a tangent is drawn which meets the circumference of the outer at P and Q. Show that $AP : AQ = BP : BQ$.

Solution. Let the common tangent to the two circles at A meet line PQ at K. It is easy to see that $KA = KB$.



Then by the tangent-chord theorem, $\angle PAK = \angle AQB$, so

$$\angle PAB = \angle BAK - \angle PAK = \angle ABK - \angle AQB = \angle BAQ,$$

so it follows that ray AB internally bisects $\angle PAQ$. Therefore, by the angle bisector theorem, we see that $PA : AQ = PB : BQ$. □

2015 Regional Round

Problem 1. The operation $\triangle$ is defined as $a \triangle b = ab + 2a + b$. Find x, if $\frac{12}{6-x} \triangle 2 = 3 \triangle 5$.

Solution. This is just linear equation solving.

$$\begin{aligned} \frac{12}{6-x} \triangle 2 &= 3 \triangle 5 \\ \frac{24}{6-x} + \frac{24}{6-x} + 2 &= 15 + 6 + 5 \\ \frac{48}{6-x} &= 24 \\ x &= 4. \end{aligned}$$

□

Problem 2. If $x + \frac{1}{x} = 7$, find the value of $x^3 + \frac{1}{x^3}$.

Solution. Cubing the given expression gives us

$$\begin{aligned} 343 &= \left(x + \frac{1}{x}\right)^3 \\ &= x^3 + 3x^2\left(\frac{1}{x}\right) + 3x\left(\frac{1}{x}\right)^2 + \frac{1}{x^3} \\ &= x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right) \\ &= x^3 + \frac{1}{x^3} + 3 \cdot 7 \\ &= x^3 + \frac{1}{x^3} + 21 \\ x^3 + \frac{1}{x^3} &= 322. \end{aligned}$$

□

Problem 3. Find the value of

$$\frac{\frac{1}{2} - \frac{1}{3}}{\frac{1}{3} - \frac{1}{4}} \cdot \frac{\frac{1}{4} - \frac{1}{5}}{\frac{1}{5} - \frac{1}{6}} \cdot \frac{\frac{1}{6} - \frac{1}{7}}{\frac{1}{7} - \frac{1}{8}} \cdot \dots \cdot \frac{\frac{1}{2014} - \frac{1}{2015}}{\frac{1}{2015} - \frac{1}{2016}}.$$

Solution. It is easy to see that the ith term is equal to

$$\begin{aligned} \frac{\frac{1}{2i} - \frac{1}{2i+1}}{\frac{1}{2i+1} - \frac{1}{2i+2}} &= \frac{\frac{1}{2i(2i+1)}}{\frac{1}{(2i+1)(2i+2)}} \\ &= \frac{2i+2}{2i} \\ &= \frac{i+1}{i} \end{aligned}$$

Therefore, the whole product is equal to

$$\frac{\frac{1}{2}-\frac{1}{3}}{\frac{1}{3}-\frac{1}{4}}\cdot\frac{\frac{1}{4}-\frac{1}{5}}{\frac{1}{5}-\frac{1}{6}}\cdot\dots\cdot\frac{\frac{1}{2014}-\frac{1}{2015}}{\frac{1}{2015}-\frac{1}{2016}}=\frac{2}{1}\cdot\frac{3}{2}\cdot\dots\cdot\frac{1008}{1007}=1008. \qquad \square$$

Problem 4. Let p and q be the remainders when the polynomials $f(x)=x^3+2x^2-5ax-7$ and $g(x)=x^3+ax^2-12x+6$ are divided by $x+1$ and $x-2$ respecively. If $2p+q=6$, find the value of a.

Solution. By the remainder theorem, the remainders are $f(-1)$ and $f(2)$ respectively, so

$$p=f(-1)=(-1)^3+2(-1)^2-5a(-1)-7=-1+2+5a-7=5a-6,$$

and

$$q=g(2)=2^3+a(2)^2-12(2)+6=8+4a-24+6=4a-10.$$

Therefore,

$$\begin{aligned} 2p+q&=6\\ 10a-12+4a-10&=6\\ 14a&=28\\ a&=2. \end{aligned} \qquad \square$$

Problem 5. Prove that $(a+b)(b+c)(c+a)\geq 8abc$ for any $a,b,c\geq 0$.

Solution. By the AM-GM inequality,

$$(a+b)(b+c)(c+a)\geq 2\sqrt{ab}\cdot 2\sqrt{bc}\cdot 2\sqrt{ca}=8abc. \qquad \square$$

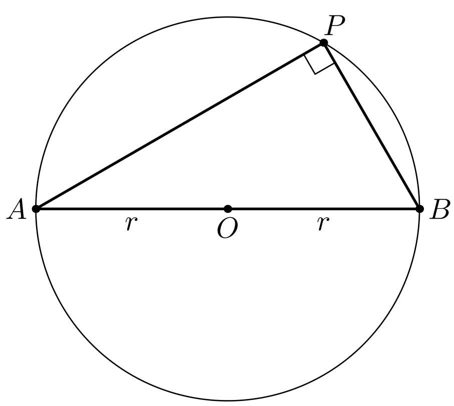
Problem 6. The point P lies on a circle with radius r which has AB as a diameter. Show that $PA\cdot PB\leq 2r^2$ and $PA+PB\leq 2\sqrt{2}r$.

Solution. By the AM-GM inequality,

$$PA\cdot PB\leq\frac{PA^2+PB^2}{2}=\frac{AB^2}{2}=\frac{4r^2}{2}=2r^2.$$

Also, by the RMS-AM inequality,

$$PA+PB\leq 2\sqrt{\frac{PA^2+PB^2}{2}}=2\sqrt{\frac{AB^2}{2}}=2\sqrt{\frac{4r^2}{2}}=2\sqrt{2}r. \qquad \square$$



Problem 7. In a sequence, $u_1 = 1$, $u_2 = 2$ and $u_3 = 3$. For $n \geq 4$, the nth term u_n is calculated from the previous three terms as $u_n = u_{n-3} + u_{n-2} - u_{n-1}$. For example, $u_4 = u_1 + u_2 - u_3 = 0$. Write down the first 9 terms. What is the 2015th term of the sequence?

Solution 1. The first 9 terms of this sequence are

$$\begin{aligned} u_1 &= 1 \\ u_2 &= 2 \\ u_3 &= 3 \\ u_4 &= 1 + 2 - 3 = 0 \\ u_5 &= 2 + 3 - 0 = 5 \\ u_6 &= 3 + 0 - 5 = -2 \\ u_7 &= 0 + 5 + 2 = 7 \\ u_8 &= 5 - 2 - 7 = -4 \\ u_9 &= -2 + 7 + 4 = 11 \end{aligned}$$

Obviously, we cannot keep doing this 2015 times, so let's try to find a pattern for general n. A closer inspection shows that $u_n = n$ for odd n, while it is an AP that decreases by 2 for even n. This lets us conjecture that

$$u_n = \begin{cases} n, & \text{if } n \text{ is odd,} \\ 4 - n & \text{if } n \text{ is even.} \end{cases}$$

We will prove this using strong induction. The base case $n = 1$ and $n = 2$ are easy to check. Now suppose that this is true for all positive integers less than or equal to $2k$. We will show that this also holds for $2k + 1$ and $2k + 2$. It is easy to see that

$$u_{2k+1} = u_{2k-2} + u_{2k-1} - u_{2k} = (4 - (2k - 2)) + (2k - 1) - (4 - 2k) = 2k + 1,$$

and

$$u_{2k+2} = u_{2k-1} + u_{2k} - u_{2k+1} = (2k - 1) + (4 - 2k) - (2k + 1) = 2 - 2k = 4 - (2k + 2).$$

Therefore, by strong induction, it follows that this pattern holds for all $n \in \mathbb{N}$. Now since 2015 is odd, we finally have $u_{2015} = 2015$. □

Solution 2. After writing down the first 9 terms as in the first solution, let $a_n = u_n - u_{n-2}$. The given condition can then be rewritten as

$$u_n - u_{n-2} = u_{n-3} - u_{n-1} \implies a_n = -a_{n-1}.$$

Therefore, $a_{n+2} = -a_{n+1} = a_n$. Since $a_3 = u_3 - u_1 = 2$, this means that $2 = a_3 = a_5 = \cdots$, or equivalently, for all odd i, $a_i = 2$. We can now find u_{2015} as follows.

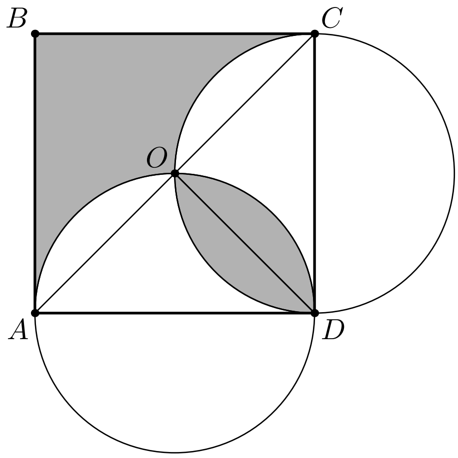
$$u_{2015} - u_1 = (u_{2015} - u_{2013}) + (u_{2013} - u_{2011}) + \cdots + (u_3 - u_1) = a_{2015} + a_{2013} + \cdots + a_3.$$

There are 1007 terms in the above expression and all of them are equal to 2, so

$$u_{2015} - u_1 = 1007 \cdot 2 \implies u_{2015} = 2015.$$ □

Problem 8. In the figure, a square $ABCD$ of side length 6 is given. Two circles with diameters AD and CD are drawn. Determine the combined area of two shaded regions.

Solution. Let O be the center of the square.



Since $\angle AOD = \angle DOC = 90°$, it follows that O lies on both circles, and hence must be the second intersection of the two circles. Also, O is the midpoint of segment AC. Now as $OA = OD$, the region bounded by OD and minor arc OD is congruent to the region bounded by OA and minor arc OA. This similarly holds for OC and OD as well, so the area of the shaded region is equal to the area of $\triangle ABC$, which turns out to be $\frac{6^2}{2} = 18$. □

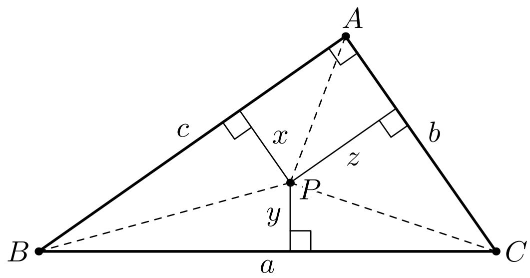
Problem 9. In $\triangle ABC$, $\angle A = 90°$. The point P lies inside $\triangle ABC$ with distances from AB, BC and CA equal to x, y and z respectively. If we denote $a = BC$, $b = CA$ and $c = AB$, show that $z = \frac{bc-cx-ay}{b}$.

Solution. Since $\angle A = 90°$, the area of $\triangle ABC$ is $\frac{bc}{2}$, so

$$ay + bz + cx = 2([BPC] + [CPA] + [APB]) = 2[ABC] = bc.$$

This means that

$$bz = bc - cx - ay \iff z = \frac{bc - cx - ay}{b}.$$ □



Problem 10. An integer is chosen from the set $\{1, 2, 3, \ldots, 100\}$. Find the probability that the integer is divisible by 3 or 7.

Solution. Let S and T be the sets of numbers less than or equal to 100 which are divisible by 3 and 7 respectively. It is easy to see that $|S| = 33$ and $|T| = 14$. Now the number of integers less than or equal to 100 which are divisible by 3 or 7 is $|S \cup T|$. By the inclusion-exclusion principle,

$$|S \cup T| = |S| + |T| - |S \cap T|.$$

Therefore, we need to find $|S \cap T|$, or the number of numbers below 100 which are divisible by both 3 *and* 7, and hence divisible by 21. There are four such numbers. Therefore,

$$|S \cup T| = 33 + 14 - 4 = 43.$$

Hence the probability that a chosen integer is divisible by 3 or 7 is $\frac{43}{100}$. $\square$

Problem 11. Aung Aung says to Bo Bo, "I am 5 times what you were when I was your age". The sum of their current ages is 64. Find their ages.

Solution. Let A and B be Aung Aung's and Bo Bo's age respectively. Then when Aung Aung was Bo Bo's current age, Bo Bo's age was $B - (A - B)$, so the problem conditions give

$$A = 5(B - (A - B))$$
$$A + B = 64$$

Solving these two equations show that $A = 40$ and $B = 24$. $\square$

Problem 12. The sum of squares of three consecutive positive integers is 2 more than 100 times the sum of the numbers itself. Find the largest of the three numbers.

Solution. Let the three consecutive integers be $n - 1$, n and $n + 1$. Then

$$(n-1)^2 + n^2 + (n+1)^2 = 100(n - 1 + n + n + 1) + 2 \implies n = 100.$$

Therefore, the largest number is $n + 1 = 101$. $\square$

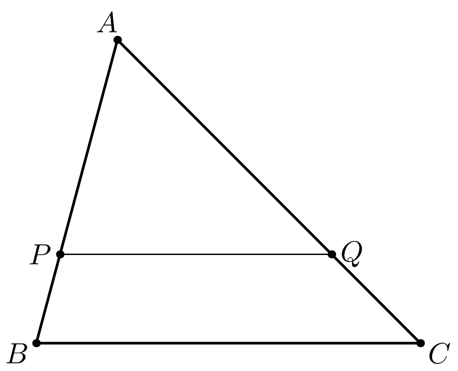
Problem 13. P and Q are two points on AB and AC respectively, of $\triangle ABC$. If PQ is parallel to BC, and bisects $\triangle ABC$, find $AP : PB$.

Solution. Since $\angle APQ = \angle ABC$ and $\angle AQP = \angle ACB$, it follows that $\triangle APQ$ and $\triangle ABC$ are similar. Since PQ bisects $\triangle ABC$,

$$\frac{AB^2}{AP^2} = \frac{[ABC]}{[APQ]} = 2 \implies \frac{AB}{AP} = \sqrt{2}.$$

Finally,

$$\sqrt{2} = \frac{AB}{AP} = \frac{AP + BP}{AP} = 1 + \frac{BP}{AP} \implies \frac{AP}{BP} = \frac{1}{\sqrt{2} - 1}.$$ $\square$



Problem 14. Find the remainder when $(x+1)^{2016}+(x+2)^{2016}$ is divided by x^2+3x+2.

Solution. By the remainder theorem, $(x+1)^{2015}$ leaves a remainder of

$$(-2+1)^{2015} = (-1)^{2015} = -1$$

when divided by $x+2$. Therefore,

$$(x+1)^{2015} = (x+2)g(x) - 1$$

for some polynomial $g(x)$. Similarly, $(x+2)^{2015}$ leaves a remainder of $(-1+2)^{2015} = 1$ when divided by $x+1$. Hence we also have

$$(x+2)^{2015} = (x+1)h(x) + 1$$

for some polynomial $h(x)$. Thus

$$\begin{aligned}
(x+1)^{2016}+(x+2)^{2016} &= (x+1)(x+1)^{2015} + (x+2)(x+2)^{2015} \\
&= (x+1)((x+2)g(x)-1) + (x+2)((x+1)h(x)+1) \\
&= (x+1)(x+2)g(x) - x - 1 + (x+2)(x+1)h(x) + x + 2 \\
&= (x+1)(x+2)(g(x)+h(x)) + 1 \\
&= (x^2+3x+2)(g(x)+h(x)) + 1.
\end{aligned}$$

Therefore, it follows that the remainder is 1. □

Remark. We can also solve this problem using modular arithmetic and noticing the fact that $\gcd(x+1, x+2) = 1$.

Problem 15. If a, b, c, d are in harmonic progression, prove that $ab+bc+cd = 3ad$.

Solution. Let $p = \frac{1}{a}$, $q = \frac{1}{b}$, $r = \frac{1}{c}$ and $s = \frac{1}{d}$. Dividing the expression by $abcd$, we just need to show that

$$\frac{1}{cd} + \frac{1}{ad} + \frac{1}{ab} = \frac{3}{bc} \iff rs + sp + pq = 3qr.$$

Since a, b, c, d are in HP, p, q, r, s are in AP. Let t be the common difference. Then

$$\begin{aligned}
\frac{rs+sp+pq}{qr} &= \frac{s(r+p)+pq}{qr} \\
&= \frac{2sq+pq}{qr} \\
&= \frac{2s+p}{r} \\
&= \frac{2(p+3t)+p}{p+2t} \\
&= \frac{3p+6t}{p+2t} \\
&= 3
\end{aligned}$$

Therefore, $rs+sp+pq = 3qr$ as desired. □

Problem 16. Using mathematical induction, prove that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution. When $n = 1$, it is easy to see that both the left hand side and the right hand side are equal to 1. Now suppose that this holds for $n = k$. Then

$$\begin{aligned}
1^2 + 2^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= (k+1)\left(\frac{k(2k+1)}{6} + k + 1\right) \\
&= \frac{(k+1)(2k^2+7k+6)}{6} \\
&= \frac{(k+1)(k+2)(2(k+1)+1)}{6},
\end{aligned}$$

so the identity also holds for $n = k+1$. Therefore, by mathematical induction, it follows that the identity holds for all $n \in \mathbb{N}$. □

Problem 17. Show that $n^7 - n$ is divisible by 42, for all positive integers n.

Solution. Since 42 is the least common multiple of 2, 3 and 7, by lemma 4.2, it suffices to show that $n^7 - n = n(n^6 - 1)$ is divisible by 2, 3 and 7. If n is even, it is obvious that $n^7 - n$ is divisible by 2. If n is odd, $n^6 - 1$ is even so in that case $n^7 - n$ is also divisible by 2. Now by Fermat's little theorem, $n^3 \equiv n \pmod 3$, so

$$n^7 \equiv n(n^2)^3 \equiv n^3 \equiv n \pmod 3 \implies n^7 - n \equiv 0 \pmod 3$$

Finally, by Fermat's little theorem again, $n^7 - n \equiv 0 \pmod 7$, so we are done. □

Problem 18. If α, β and γ are roots of the equation $x^3 + px^2 + qx + k = 0$, show that $\alpha^2 + \beta^2 + \gamma^2 = p^2 - 2q$.

Solution. By Vieta's formulas, $\alpha + \beta + \gamma = -p$ and $\alpha\beta + \beta\gamma + \gamma\alpha = q$. Therefore,

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) = p^2 - 2q.$$ □

2015 National Round

Problem 1. If $x^n + py^n + qz^n$ is divisible by $x^2 - (ay + bz)x + abyz$ and $y, z \neq 0$, show that
$$\frac{p}{a^n} + \frac{q}{b^n} + 1 = 0.$$

Solution. Since $x^2 - (ay+bz)x + abyz = (x - ay)(x - bz)$, it follows that both $x - ay$ and $x - bz$ divide $f(x) = x^n + py^n + qz^n$. Therefore, by the factor theorem,

$$f(ay) = a^n y^n + py^n + qz^n = 0,$$

and

$$f(bz) = b^n z^n + py^n + qz^n = 0.$$

From the above two equations, we have

$$a^n y^n = b^n z^n = -py^n - qz^n.$$

Then the identity we want to prove is just

$$\begin{aligned} \frac{p}{a^n} + \frac{q}{b^n} + 1 &= \frac{py^n}{a^n y^n} + \frac{qz^n}{b^n z^n} + 1 \\ &= \frac{py^n + qz^n}{-(py^n + qz^n)} + 1 \\ &= -1 + 1 = 0. \end{aligned}$$

□

Problem 2. A sequence is defined by $u_1 = 1$, $u_{n+1} = u_n^2 - ku_n$, where $k \neq 0$ is a constant. If $u_3 = 1$, calculate the value of k and find the value of

$$u_1 + u_2 + u_3 + \cdots + u_{100}.$$

Solution. First, let's find the value of k.

$$u_1 = 1, \quad u_2 = 1 - k \quad \text{and} \quad u_3 = (1-k)^2 - k(1-k) = 1 - 3k + 2k^2 = 1.$$

Since $k \neq 0$, this implies that $k = \frac{3}{2}$, and hence $u_2 = 1 - \frac{3}{2} = -\frac{1}{2}$. Now comes the key fact. Notice that the value of u_{n+1} is determined entirely by u_n, i.e. if $u_a = u_b$, $u_{a+1} = u_{b+1}$ as well (Check why this is true.) Therefore, we can deduce that $u_4 = -\frac{1}{2}$, $u_5 = 1$, and so on. In general, terms with odd indices are equal to 1 and those with even indices are equal to $-\frac{1}{2}$. Therefore,

$$u_1 + u_2 + u_3 + \cdots + u_{100} = 1 - \frac{1}{2} + 1 - \cdots - \frac{1}{2} = 50 - \frac{50}{2} = \frac{50}{2} = 25.$$

□

Problem 3. A rectangular room has a width of x yards. The length of the room is 4 yards longer than its width. Given that the perimeter of the room is greater than 19.2 yards and the area of the room is less than 21 square yards, find the set of possible values of x.

Solution. The length of the room is $x+4$ yards, so the problem conditions give

$$2x + 2(x+4) > 19.2 \quad \text{and} \quad x(x+4) < 21.$$

Solving the first inequality gives $x > 2.8$. Now let's solve the second inequality. It is equivalent to

$$x^2 + 4x - 21 < 0 \iff (x+7)(x-3) < 0.$$

Therefore, $x > 3$ and $x < -7$ or $x < 3$ and $x > -7$. The former is absurd, so it follows that $-7 < x < 3$. Since $x > 2.8$ by the first inequality, the solution set is $\{x \text{ yards} \mid x \in \mathbb{R} \text{ and } 2.8 < x < 3\}$. ☐

Problem 4. Two dice are thrown. Event A is that the sum of the numbers on the dice is 7. Event B is that at least one number on the die is 6. Find

1. $\mathbb{P}(A)$,
2. $\mathbb{P}(B)$,
3. $\mathbb{P}(A \cap B)$,
4. $\mathbb{P}(A) \cdot \mathbb{P}(B)$.

Are A and B independent?

Solution. Let $S = \{(x, y) \mid 1 \leq x, y \leq 6\}$ be the set where x is the number on the first die and y is the number on the second die.

1. First, let's calculate the number of elements of S where $x + y = 7$. As x ranges from 1 to 6, the value of y must be $7 - x$, so there are 6 such pairs. Meanwhile, the total number of pairs in S is $6 \times 6 = 36$. Therefore, $\mathbb{P}(A) = \frac{6}{36} = \frac{1}{6}$.

2. Now let's calculate the number of elements of S such that at least one of x or y is 6. When $x \in \{1, 2, 3, 4, 5\}$, y must be 6 so there are 5 such pairs. Now when $x = 6$, y can be any number so there are 6 such pairs in this case. Therefore, the total number of pairs where at least one of x and y is 6 is $5 + 6 = 11$. Hence $\mathbb{P}(B) = \frac{11}{36}$.

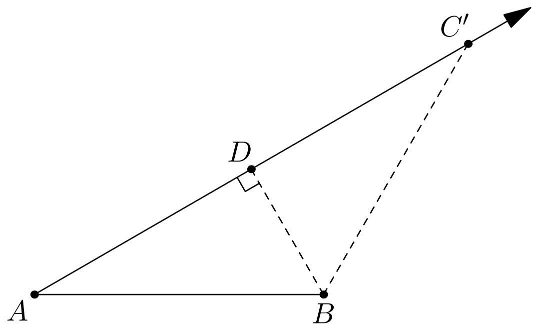
3. Out of 6 pairs that satisfy A, it is easy to see that only 2 of them, namely $(1, 6)$ and $(6, 1)$ also satisfy B. Therefore, $\mathbb{P}(A \cap B) = \frac{2}{36} = \frac{1}{18}$.

4. This one is a straightforward computation; $\mathbb{P}(A) \cdot \mathbb{P}(B) = \frac{1}{6} \cdot \frac{11}{36} = \frac{11}{216}$.

If A and B are independent, then $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$. But obviously this is not true in our case, so we conclude that A and B are not independent. ☐

Problem 5. In $\triangle ABC$, $\angle A = 30^\circ$, $AB = 8$ cm and $BC = x$ cm. If $\angle C > 30^\circ$, determine the set of all possible values of x.

Solution. Let ℓ be a ray originating at A inclined to line AB at a 30° angle. Let C' be the point different from A on ℓ such that $\angle BC'A = 30^\circ$.



Since $\angle BCA < 30°$, it follows that C must lie inside segment AC'. Therefore, $x < BA = 8$ cm. It remains to find the minimum value of x. Let D be the foot of perpendicular from B to l. As x is minimum when $C = D$,

$$x \geq BD = AB \sin 30° = \frac{AB}{2} = 4 \text{ cm}.$$

Therefore, the set of possible values of x is $\{x \text{ cm} \mid x \in \mathbb{R} \text{ and } 4 \leq x < 8\}$. □

Problem 6. Let $f(x)$ be a polynomial with real coefficients. When $f(x)$ is divided by both $x - a$ and $x - b$, where a and b are distinct real numbers, the remainder is a real constant r. Prove that $f(x)$ has also the remainder r when it is divided by $x^2 - (a + b)x + ab$.

Solution. Let $g(x) = f(x) - r$. By the remainder theorem, $f(a) = f(b) = r$, so $g(a) = g(b) = 0$. This means that $x - a$ and $x - b$ are factors of $g(x)$. Therefore, there exists a polynomial $h(x)$ such that

$$g(x) = (x - a)h(x) \iff 0 = g(b) = (b - a)h(b).$$

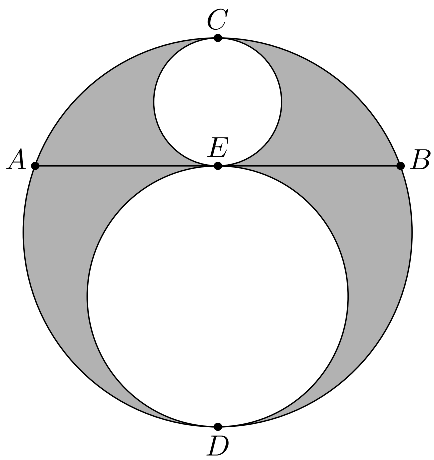
Since $a \neq b$, $h(b) = 0$ so $x - b$ is also a factor of $h(x)$. Consequently, there exists a polynomial $k(x)$ such that $h(x) = (x - b)k(x)$. Substituting into the above expression gives

$$g(x) = (x - a)(x - b)k(x) \iff f(x) = (x^2 - (a + b)x + ab)k(x) + r.$$

Hence $f(x)$ also leaves the remainder r when divided by $x^2 - (a + b)x + ab$. □

Problem 7. Three circles are tangent to each other as shown. The two smaller circles are tangent to chord AB which has length 12 at its midpoint. What is the area of the shaded region?

Solution. Let the outer circle be ω, and the inner two circles be ω_1 and ω_2. Let C, D and E be points where (ω, ω_1), (ω, ω_2) and (ω_1, ω_2) touch each other.



Then the area of the shaded region is

$$\begin{aligned}[\omega]-[\omega_1]-[\omega_2] &= \frac{\pi(CD^2-CE^2-DE^2)}{4}\\ &= \frac{\pi((CE+DE)^2-CE^2-DE^2)}{4}\\ &= \frac{2\pi\cdot CE\cdot DE}{4}\\ &= \frac{\pi\cdot AE\cdot BE}{2}\\ &= \frac{36\pi}{2}\\ &= 18\pi,\end{aligned}$$

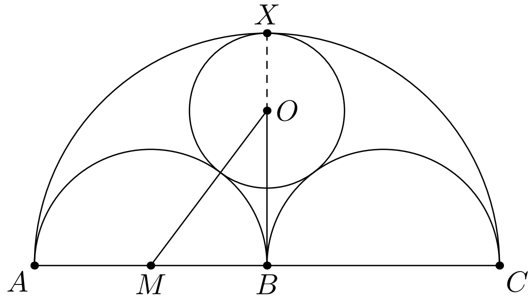
where we used the fact that $AE\cdot BE = CE\cdot DE$ since A, B, C, D are concyclic. □

Problem 8. In a sequence, $u_1=1$, $u_2=2$ and $u_3=3$. For $n\geq 4$, the nth term u_n is calculated from the previous three terms as $u_n = u_{n-3}+u_{n-2}-u_{n-1}$. For example, $u_4=u_1+u_2-u_3=0$. By using mathematical induction, prove that $u_{2n+1}=2n+1$ for all integers $n\geq 0$.

Solution. This problem has the same solution as 2015 Regional Round P7. □

Problem 9. In the diagram, $AB=BC=1$ and ABC is the diameter of the larger semicircle. AB and BC are diameters of the smaller semicircles. What is the diameter of the circle tangent to all three semicircles?

Solution. Let M be the midpoint of AB. Let the center of the small circle be O and let it touch the outer semicircle at X. Finally let its radius be r.



Then by the Pythagoras's theorem,

$$r+\frac{1}{2}=OM=\sqrt{BO^2+BM^2}=\sqrt{(1-r)^2+\frac{1}{4}}.$$

Squaring both sides and solving for r gives $r=\frac{1}{3}$, so the diameter of the small circle is $\frac{2}{3}$. □

Problem 10. A six-digit number is of the form *abcabc*, where all the digits are nonzero. Find three different prime factors of that number.

Solution. Let $N = \overline{abcabc}$. Then

$$N = 100000a + 10000b + 1000c + 100a + 10b + c = 1001(100a + 10b + c).$$

Since 7, 11 and 13 all divide 1001, they must also be prime factors of N. $\square$

Problem 11. Real numbers a and b are such that $a > b > 0$, $a \neq 1$ and

$$a^{2016} + b^{2016} = a^{2014} + b^{2014}.$$

Prove that $a^2 + b^2 < 2$.

Solution. Rearranging the given equation gives

$$a^{2014}(a^2 - 1) = b^{2014}(1 - b^2).$$

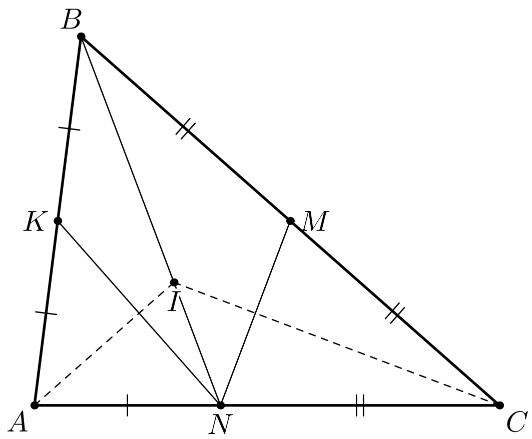
Since $a \neq 1$,

$$\frac{1 - b^2}{a^2 - 1} = \left(\frac{a}{b}\right)^{2014} > 1 \implies 1 - b^2 > a^2 - 1 \implies a^2 + b^2 < 2.$$ $\square$

Problem 12. In $\triangle ABC$, $AC = \frac{1}{2}(AB + BC)$ and BN is the bisector of $\angle ABC$. K and M are the midpoints of AB and BC respectively. If $\angle ABC = \beta$, prove that $\angle KNM = 90^\circ - \frac{1}{2}\beta$.

Solution. Let I be the incenter of $\triangle ABC$. By the angle bisector theorem,

$$\frac{AB}{BC} = \frac{AN}{NC} \implies \frac{AB}{AN} = \frac{CB}{CN} = \frac{AB + CB}{AN + CN} = \frac{AB + BC}{AC} = 2.$$



Thus $AN = \frac{AB}{2} = AK$ and similarly $CM = CN$. Since AI bisects $\angle A$, this implies that $AI \perp KN$ and similarly $CI \perp MN$. Therefore,

$$\angle KNM = 180^\circ - \angle ANK - \angle MNC = 180^\circ - \left(90^\circ - \frac{\alpha}{2}\right) - \left(90^\circ - \frac{\gamma}{2}\right) = \frac{\alpha}{2} + \frac{\gamma}{2} = 90^\circ - \frac{\beta}{2},$$

where $\alpha = \angle A$ and $\gamma = \angle C$. $\square$

Problem 13. If the $(m - n)$th and $(m + n)$th terms of a geometric progression are the arithmetic mean and harmonic mean of $x > 0$ and $y > 0$, prove that the mth term is their geometric mean.

Solution. Let the common ratio of the geometric progression be r. It is easy to see that u_m is obtained by multiplying u_{m-n} with r for n times, and similarly for u_{m+n} and u_m. Hence,

$$\frac{u_m}{u_{m-n}} = r^n = \frac{u_{m+n}}{u_m}.$$

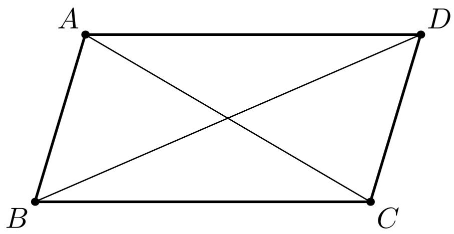
Therefore,

$$u_m = \sqrt{u_{m-n} \cdot u_{m+n}} = \sqrt{\left(\frac{x+y}{2}\right)\left(\frac{2xy}{x+y}\right)} = \sqrt{xy}$$

so u_m is the geometric mean of x and y as desired. □

Problem 14. Show that the sum of the squares of the lengths of the sides of a parallelogram equals the sum of the squares of the lengths of the diagonals.

Solution. Let $\angle ABC = \theta$, and let $AB = CD = x$ and $BC = AD = y$.



Since $DA \parallel BC$, $\angle BAD = 180^\circ - \theta$. Therefore, by law of cosines in $\triangle ABC$ and $\triangle ABD$, we have

$$AC^2 = x^2 + y^2 - 2xy\cos\theta,$$

and

$$BD^2 = x^2 + y^2 - 2xy\cos(180^\circ - \theta).$$

Adding them gives

$$\begin{aligned} AC^2 + BD^2 &= 2x^2 + 2y^2 - 2xy(\cos\theta + \cos(180^\circ - \theta)) \\ &= 2x^2 + 2y^2 - 2xy(\cos\theta - \cos\theta) \\ &= 2x^2 + 2y^2 \\ &= AB^2 + BC^2 + CD^2 + DA^2. \end{aligned}$$

□

Problem 15. If n is a positive even integer, prove by mathematical induction that $x^n - y^n$ is divisible by $x + y$.

Solution. The base case $n = 2$ is true since $x^2 - y^2 = (x+y)(x-y)$. Now suppose that $x^{2k} - y^{2k}$ is divisible by $x - y$. We must show that $x^{2k+2} - y^{2k+2}$ is also divisible by $x - y$. Observe that

$$x^{2k+2} - y^{2k+2} = x^{2k+2} - x^2y^{2k} + x^2y^{2k} - y^{2k+2} = x^2(x^{2k} - y^{2k}) - y^{2k}(x^2 - y^2).$$

But both $x^{2k} - y^{2k}$ and $x^2 - y^2$ are divisible by $x + y$, so it follows that $x^{2k+2} - y^{2k+2}$ is also divisible by $x + y$. Hence by mathematical induction, $x^n - y^n$ is divisible by $x + y$ for all even n. □

Problem 16. Prime numbers p, q and positive integers m, n satisfy the following conditions:

$$m < p, n < q \text{ and } \frac{p}{m} + \frac{q}{n} \text{ is an integer.}$$

Prove that $m = n$.

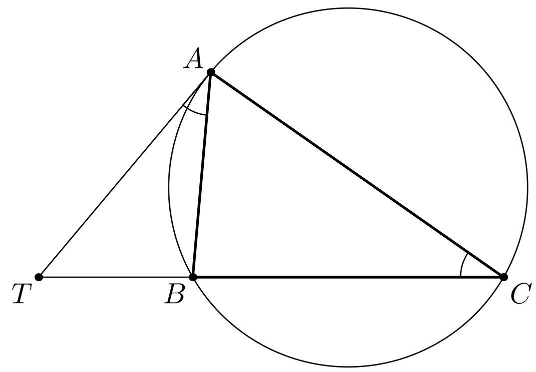
Solution. Since $m < p$, it follows that m and p are relatively prime. Similarly, n and p are also relatively prime. Now since $p/m + q/n$ is an integer,

$$\frac{pn}{m} = n\left(\frac{p}{m} + \frac{q}{n}\right) - q$$

is also an integer. Since $\gcd(m, p) = 1$, by Euclid's lemma, it follows that m divides n, and since they're positive, $m \leq n$. Similarly, $n \leq m$, so it must be the case that $m = n$. $\square$

Problem 17. A, B and C are three points on the circumference of a circle, and the tangent at A meets BC produced at T. Prove that $AB^2 : AC^2 = TB : TC$.

Solution. Notice that $\angle TAB = \angle TCA$, so $\triangle TAB \sim \triangle TCA$.



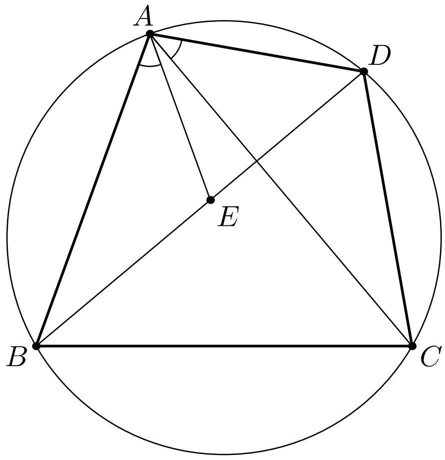
Hence

$$\frac{TB}{AB} \cdot \frac{AC}{TC} = \frac{TA}{AC} \cdot \frac{AB}{TA} = \frac{AB}{AC} \iff \frac{TB}{TC} = \frac{AB^2}{AC^2}.$$ $\square$

Problem 18. $ABCD$ is a cyclic quadrilateral. AE is drawn to meet BD at E such that $\angle BAE = \angle CAD$. Prove that

1. $\triangle ABE \sim \triangle ACD$,
2. $\triangle AED \sim \triangle ABC$,
3. $AB \cdot CD + AD \cdot BC = AC \cdot BD$.

Solution. This is known as Ptolemy's theorem, or more generally, Ptolemy's inequality.



1. Since $ABCD$ is cyclic, $\angle ABE = \angle ACD$. Combined with the fact that $\angle BAE = \angle CAD$, this lets us deduce that $\triangle ABE \sim \triangle ACD$. Hence

$$\frac{AB}{BE} = \frac{AC}{CD} \Longleftrightarrow AB \cdot CD = AC \cdot BE.$$

2. Observe that

$$\angle BAC = \angle BAE + \angle EAC = \angle CAD + \angle EAC = \angle EAD,$$

and $\angle ADE = \angle ACB$. Therefore, $\triangle AED \sim \triangle ABC$. Similarly to above, $AD \cdot BC = AC \cdot ED$.

3. From the similarities,

$$AC \cdot BD = AC(BE + ED) = AC \cdot BE + AC \cdot ED = AB \cdot CD + AD \cdot BC. \qquad \square$$

2016 Regional Round

Problem 1. In the sequence $u_1, u_2, \ldots, u_n, \ldots$, the nth term is defined by $u_n = 1 - \frac{1}{u_{n-1}}$ for $n \geq 2$. If $u_1 = 3$, compute u_2, u_3 and u_4. Write down u_{2016}.

Solution. If $u_1 = 3$,

$$\begin{aligned} u_2 &= 1 - \frac{1}{3} = \frac{2}{3} \\ u_3 &= 1 - \frac{3}{2} = -\frac{1}{2} \\ u_4 &= 1 - (-2) = 3. \end{aligned}$$

Here we see that $u_1 = u_4$. Moreover, since each term depends entirely upon the previous one, it must be the case that if $u_a = u_b$, $u_{a+1} = u_{b+1}$ as well. Therefore, $u_5 = u_2 = \frac{2}{3}$, $u_6 = u_3 = -\frac{1}{2}$, and so on. In general, the value of u_n is

$$u_n = \begin{cases} 3 & \text{if } n = 3k+1, \\ \frac{2}{3} & \text{if } n = 3k+2, \\ -\frac{1}{2} & \text{if } n = 3k. \end{cases}$$

for any positive integer k. In particular, since 2016 is divisible by 3, $u_{2016} = -\frac{1}{2}$. □

Problem 2. If x and y are positive integers such that $56 \leq x+y \leq 59$ and $0.9 \leq \frac{x}{y} \leq 0.91$, find the value of $y^2 - x^2$.

Solution. This is a straightforward inequality manipulation. From the problem, we have the following inequalities.

$$56 \leq x + y \leq 59 \quad \text{and} \quad 0.9y \leq x \leq 0.91y.$$

This allows us to bound y.

$$56 \leq x + y \leq 0.91y + y = 1.91y \implies y \geq 29.31,$$

and

$$59 \geq x + y \geq 0.9y + y = 1.9y \implies y \leq 31.06.$$

Since y is a positive integer, the only possible values are 30 and 31. So we will consider two cases.

Case 1: $y = 30$.

In this case, $27 \leq x \leq 27.3$, so it follows that $x = 27$. This satifies the first relation, and thus $(x, y) = (27, 30)$ is a solution pair.

Case 2: $y = 31$.

In this case, $27.9 \leq x \leq 28.21$ and so $x = 28$. This also satisifies the first relation, and so $(x, y) = (28, 31)$ is also a solution pair.

Finally, the possible values of $y^2 - x^2$ are 171 and 177. □

Problem 3. When 15 is added to a number x, it becomes the square of an integer. When 74 is subtracted from x, the result is a square of another integer. Find the number x.

Solution. By the problem statement, there exist positive integers y and z such that

$$x + 15 = y^2 \quad \text{and} \quad x - 74 = z^2.$$

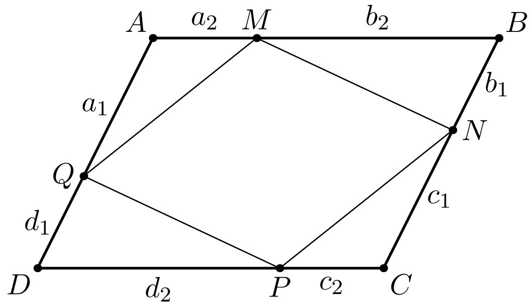
Subtracting the two equations gives

$$y^2 - z^2 = (y+z)(y-z) = 89.$$

But since 89 is a prime, the smaller one of the divisors on the left, i.e. $y - z$, must be equal to 1. Therefore, $y - z = 1$ and $y + z = 89$. Solving those two equations give $y = 45$ and $z = 44$. Substituting these into the first equation shows that $x = 2010$. $\square$

Problem 4. In a rectangle $ABCD$, the points M, N, P, Q lie on AB, BC, CD and DA respectively, such that the areas of $\triangle AQM$, $\triangle BMN$, $\triangle CNP$ and $\triangle DPQ$ are equal. Prove that $MNPQ$ is a parallelogram.

Solution. Let $[\triangle AQM] = [\triangle BMN] = [\triangle CNP] = [\triangle DPQ] = A$. Let $AQ = a_1$, $BN = b_1$, $CN = c_1$, $DQ = d_1$, and $AM = a_2$, $BM = b_2$, $CP = c_2$, $DP = d_2$.



Then we can rewrite the area equality as

$$a_1a_2 = b_1b_2 = c_1c_2 = d_1d_2 = 2A.$$

Since the opposide sides of a rectangle are equal, we also have

$$a_1 + d_1 = b_1 + c_1 \quad \text{and} \quad a_2 + b_2 = c_2 + d_2.$$

We can transform the second equation as follows:

$$\begin{aligned}
\frac{2A}{a_1} + \frac{2A}{b_1} &= \frac{2A}{c_1} + \frac{2A}{d_1} \\
\frac{1}{a_1} + \frac{1}{b_1} &= \frac{1}{c_1} + \frac{1}{d_1} \\
\frac{1}{a_1} - \frac{1}{c_1} &= \frac{1}{d_1} - \frac{1}{b_1} \\
\frac{c_1 - a_1}{a_1c_1} &= \frac{b_1 - d_1}{b_1d_1}
\end{aligned}$$

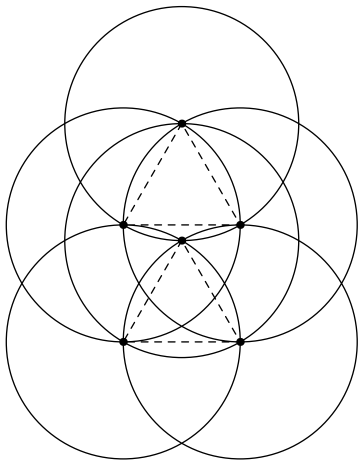
From the first equation, we have $a_1 - c_1 = b_1 - d_1$. Therefore,

$$\begin{aligned}
\frac{c_1 - a_1}{a_1c_1} &= -\frac{c_1 - a_1}{b_1d_1} \\
(c_1 - a_1)\left(\frac{1}{a_1c_1} + \frac{1}{b_1d_1}\right) &= 0.
\end{aligned}$$

Since the latter bracket is positive, it follows that $a_1 = c_1$. Similarly, we also have $b_1 = d_1$, $a_2 = c_2$ and $b_2 = d_2$. Therefore, $\triangle AQM \cong \triangle CNP$ and $\triangle BNM \cong \triangle DQP$. Thus $QM = NP$ and $NM = QP$ implying that $MNPQ$ is a parallelogram as desired. □

Problem 5. Draw 6 circles in the plane such that every circle passes through exactly 3 centres of other circles.

Solution. Consider two unit equilateral triangles, separated unit distance apart. Next, we draw 6 circles with unit radius centered at each of the vertices. It is obvious that each circle passes through exactly three other vertices. □



Problem 6. There are 2016 students in a secondary school. Every student writes a new year card. The cards are mixed up and randomly distributed to students. Suppose each student gets one and only one card. Find the expected number of students who get back their own cards.

Solution. Label each of the students and their corresponding cards from 1 to 2016, and call a student *fixed point* if they receive their own card. By the definition of expected value,

$$\begin{aligned}
\text{E[number of fixed points]} &= \sum_{i=1}^{2016} i \cdot \Pr(\text{number of fixed points is } i) \\
&= \sum_{i=1}^{2016} \frac{i \cdot \text{number of distributions with } i \text{ fixed points}}{\text{total number of distributions}} \\
&= \frac{\text{total number of fixed points over all distributions}}{\text{total number of distributions}}.
\end{aligned}$$

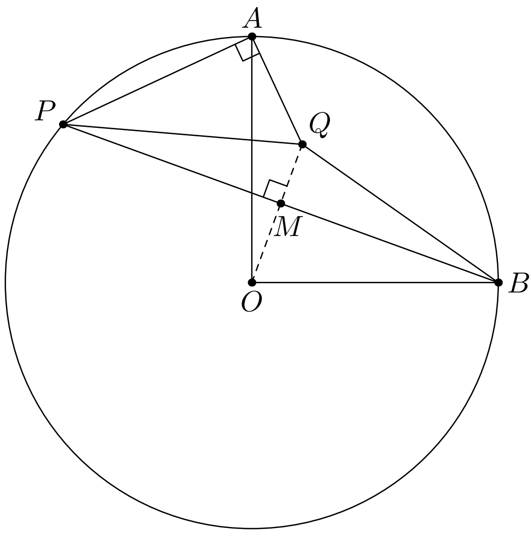
Consider a table, consisting of a row for each distribution of cards (An example for 3 students is provided below.) The total number of fixed points in this whole table can be found by finding the number of fixed points in each column, and then summing them. For each student i, there are 2015! ways to permute the other 2015 students. Therefore, the number of distributions where student i is a fixed point is 2015!. Since there are 2016 columns, the total number of fixed points in the whole table is 2016!.

Students	**1**	**2**	**3**
	1	2	3
	1	3	2
	2	1	3
	2	3	1
	3	1	2
	3	2	1
Number of fixed points	2	2	2

Finally, since the total number of distributions is also 2016!, it follows that the expected value is 1. $\square$

Problem 7. The points P, A, B lie in that order on a circle with center O such that $\angle POB < 180°$. The point Q lies inside the circle such that $\angle PAQ = 90°$ and $PQ = BQ$. If $\angle AQB > \angle AQP$, prove that $\angle AQB - \angle AQP = \angle AOB$.

Solution. Let M be the midpoint of BP.



Since $\triangle PQB$ is an isosceles triangle, $\angle QMP = 90° = \angle QAP$ so $QAPM$ is cyclic. Moreover, $\angle PQM = \angle MQB$. Therefore,

$$\angle AQB - \angle AQP = 360° - 2\angle PQM - 2\angle AQP = 2(180° - \angle AQM) = 2\angle APB = \angle AOB. \quad \square$$

Remark. There are configuration issues depending on Q being inside or outside of $\triangle OPB$, but the proof is more or less the same so we will not mention them here.

Problem 8. A fair die is thrown three times. The results of the first, second and third throw are recorded as x, y and z respectively. Suppose that $x + y = z$. What is the probability that at least one of x, y and z is 2?

Solution. Since the value of z is automatically determined by those of x and y, we only need to consider the possible outcomes of x and y. The following table summarizes the possible triples of (x, y, z) satisfying $x + y = z$. In addition, those that doesn't satisfy $x, y, z \leq 6$ are greyed out.

$x \backslash y$	1	2	3	4	5	6
1	(1, 1, 2)	(1, 2, 3)	(1, 3, 4)	(1, 4, 5)	(1, 5, 6)	(1, 6, 7)
2	(2, 1, 3)	(2, 2, 4)	(2, 3, 5)	(2, 4, 6)	(2, 5, 7)	(2, 6, 8)
3	(3, 1, 4)	(3, 2, 5)	(3, 3, 6)	(3, 4, 7)	(3, 5, 8)	(3, 6, 9)
4	(4, 1 ,5)	(4, 2, 6)	(4, 3, 7)	(4, 4, 8)	(4, 5, 9)	(4, 6, 10)
5	(5, 1, 6)	(5, 2, 7)	(5, 3, 8)	(5, 4, 9)	(5, 5, 10)	(5, 6, 11)
6	(6, 1, 7)	(6, 2, 8)	(6, 3, 9)	(6, 4, 10)	(6, 5, 11)	(6, 6, 12)

It is easy to see that the number of desired triples (those in which at least one component is 2) is 8 and the total is 15. Thus the probability is $\frac{8}{15}$. □

Remark. This problem asks for the probability that at least one of x, y, z is 2, *given that* $x + y = z$. This is not the same with the probability that $x + y = z$ *and* at least one of x, y, z is 2, which would be $\frac{8}{6^3}$.

Problem 9. An arithmetic progression and a harmonic progression have a and b for the first two terms. If their nth terms are x and y respectively, show that $(x - a) : (y - a) = b : y$.

Solution. Since a, b are the first two terms of a HP, $\frac{1}{a}$, $\frac{1}{b}$ are the first two terms of an AP. The first term of this AP is $\frac{1}{a}$, and the common difference is $\frac{1}{b} - \frac{1}{a}$, so the nth term of this AP is

$$u_n = \frac{1}{a} + (n-1)\left(\frac{1}{b} - \frac{1}{a}\right) = \frac{b + (n-1)(a-b)}{ab},$$

which means that the nth term of the HP is

$$y = \frac{ab}{b + (n-1)(a-b)}.$$

Meanwhile, the nth term of the AP is $x = a + (n-1)(b-a)$. Therefore,

$$\begin{aligned}\frac{x-a}{y-a} &= (n-1)(b-a) \div \left(\frac{ab}{b+(n-1)(a-b)} - a\right) \\ &= (n-1)(b-a) \cdot \frac{b+(n-1)(a-b)}{a(n-1)(b-a)} \\ &= \frac{b+(n-1)(a-b)}{a} \\ &= \frac{b}{y}.\end{aligned}$$

□

Problem 10. Mr. Game owns 200 custom-made dice. Each die has four sides showing the number 2 and two sides showing the number 5. Mr. Game is about to throw all 200 dice together and find out the sum of all 200 results. How many possible values of this sum are there?

Solution. Suppose that in a throw, there are n dice that show 2, and $200 - n$ dice that show 5. Then the sum of their values is $2n + (200 - n)5 = 1000 - 3n$. Obviously, for different values of n, the sum is also different. Since n can range from 0 to 200, this means that the sum can also have 201 different values. □

Problem 11. U Tet Toe wants to repair all 4 walls of his room. He has red paint, yellow paint and blue paint (which he cannot mix), and wants to paint his room so that adjacent walls are never of the same colour. In how many ways can U Tet Toe paint his room?

Solution 1. Let R be the number of colourings where the top wall is red. Since there are three possible colours for the top wall, by symmetry, the total number of colourings is $3R$. Now let R_1 be the number of colourings where the top wall is red and the right wall is yellow. Since the right wall can be yellow or green, again by symmetry, $R = 2R_1$. Now let's count R_1. The left wall can either be yellow or green. If it is yellow, then the bottom wall can be either red or green, which gives 2 colourings. If it is green, then the bottom wall can only be red, which gives 1 colouring. Hence, in total, $R_1 = 3$, so the total number of colourings is $3R = 6R_1 = 18$. □

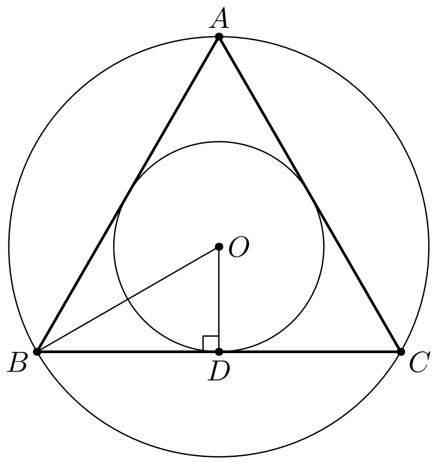
Solution 2. We just need to find the chromatic polynomial of the cycle graph of 4 vertices, C_4. By deletion-contraction,

$$P(C_4, 3) = P(P_4, 3) - P(C_3, 3),$$

where P_4 denotes the path graph with 4 vertices. We will first find $P(P_4, 3)$. There are 3 choices for the first vertex and 2 choices each for the remaining vertices. Therefore, $P(P_3, 3) = 3\cdot2\cdot2\cdot2 = 24$. Next, let's find the value of $P(C_3, 3)$. There are 3 choices for the first vertex, 2 choices for the second vertex, and only 1 choice for the final vertex. Therefore, $P(C_3, 3) = 3 \cdot 2 \cdot 1 = 6$. Hence there are $24 - 6 = 18$ ways in which U Tet Toe can paint his room. □

Problem 12. Given an equilateral triangle, what is the ratio of the area of its circumscribed circle to the area of the inscribed circle?

Solution. Let O be the circumcenter of the equilateral triangle ABC and let the incircle touch side BC at D.



Due to the symmetry, O is also the incenter of $\triangle ABC$. Since $\angle OBD = 30°$, $\triangle OBD$ is a $30°$-$60°$ right triangle, and $OB : OD = 2 : 1$. Therefore, if we let the area of the circumcircle by A_1 and that of the incircle by A_2,

$$\frac{A_1}{A_2} = \frac{\pi OB^2}{\pi OD^2} = 4.$$ □

Problem 13. Solve the equation $\sqrt{3x^2 - 8x + 1} + \sqrt{9x^2 - 24x - 8} = 3$.

Solution. Let $u = 3x^2 - 8x + 1$. Then we can rewrite the equation as

$$\sqrt{u} + \sqrt{3u - 11} = 3$$

Squaring both sides to remove the square roots gives

$$\begin{aligned} u + 3u - 11 + 2u\sqrt{3u - 11} &= 9 \\ 2u - 10 &= -\sqrt{u(3u - 11)} \end{aligned}$$

Squaring again, we have

$$\begin{aligned} 4u^2 - 4u + 100 &= 3u^2 - 11u \\ u^2 - 29u + 100 &= 0 \\ (u - 25)(u - 4) &= 0 \end{aligned}$$

which shows that $u = 4$ or $u = 25$. It is easy to check that $u = 25$ does not satisfy the equation, so $u = 4$ must be the only solution. Finally,

$$3x^2 - 8x + 1 = 4 \implies (x - 3)(3x + 1) = 0.$$

and hence $x = 3$ or $x = -\frac{1}{3}$. □

Problem 14. Prove by mathematical induction that

$$1^2 + 3^2 + 5^2 + 7^2 + \cdots + (2n - 1)^2 = \frac{1}{3}n(4n^2 - 1).$$

Solution. We will show this by induction. For the base case, when $n = 1$, both the left hand side and the right hand side are equal to 1, so the identity is true. Now suppose that for $n = k$,

$$1^2 + 3^2 + \cdots + (2k - 1)^2 = \frac{k(4k^2 - 1)}{3}.$$

Then when $n = k + 1$,

$$\begin{aligned} 1^2 + 3^2 + \cdots + (2k + 1)^2 &= \frac{1}{3}k(4k^2 - 1) + (2k + 1)^2 \\ &= \frac{1}{3}(k(2k + 1)(2k - 1) + 3(2k + 1)^2) \\ &= \frac{1}{3}(2k + 1)(2k^2 - k + 6k + 3) \\ &= \frac{1}{3}(2k + 1)(2k + 3)(k + 1) \\ &= \frac{1}{3}(k + 1)(4(k + 1)^2 - 1) \end{aligned}$$

and so the identity is true for $n = k + 1$. Hence by mathematical induction it follows that the identity is true for all $n \in \mathbb{N}$. □

Problem 15. In how many ways can 7 identical T-shirts be divided among 4 students, subject to the condition that each is to get at least 1 T-shirt.

Solution. This is a classic example of stars and bars.

Label the students from 1 to 4. Suppose that we have all 7 T-shirts lying in a line, and we want to divide them up into 4 sections corresponding to each student. One way to do this would be to use 3 dividers, and insert them in the gaps between the shirts, so that the first section corresponds to student 1, the second section corresponds to student 2, and so on. Since each student gets at least one T-shirt, this means that there can only be at least one divider in each gap. The number of ways to insert 3 dividers in 6 gaps is $\binom{6}{3} = 20$. Therefore, there are 20 ways to distribute the T-shirts. □

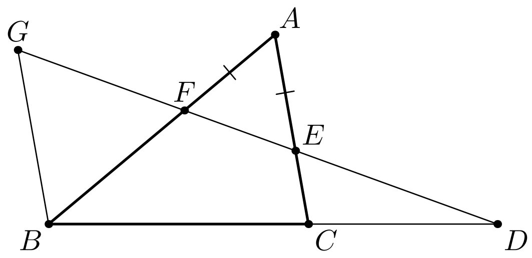
Problem 16. $\triangle ABC$ has $AB > AC$. A line DEF, equally inclined to AB and AC, is drawn, meeting AB at F, AC at E and BC produced at D. Prove that $BD : DC = BF : CE$.
(Hint: Draw $BG \parallel CE$ meeting DF produced at G.)

Solution. Some angle chasing gives us

$$\angle BFG = \angle AFE = \angle AEF = \angle BGF,$$

from which we get that $BF = BG$. Since $CE \parallel BG$, we see that

$$\frac{DB}{DC} = \frac{BG}{CE} = \frac{BF}{CE}.$$ □



Problem 17. The sum of the two smallest positive divisors of a positive integer N is 6, while the sum of the largest positive divisors of N is 1122. Find N.

Solution. Let the two smallest and largest positive divisors be d_1, d_2, d_3, d_4, with $d_1 < d_2 < d_3 < d_4$. Then since d_1 is the smallest divisor of N, it must be 1. Similarly, since d_4 is the larget divisor of N, it must be N itself. Now the problem gives us $d_1 + d_2 = 6$ and $d_3 + d_4 = 1122$, which shows that

$$d_2 = 5 \quad \text{and} \quad d_3 = 1122 - N.$$

But since d_2 is the second smallest divisor and d_3 is the second biggest divisor, $d_2 d_3 = N$. Hence

$$5(1122 - N) = N \implies N = 935.$$ □

Problem 18. If p, q and r are prime numbers such that $p < q < r$ and their product is 19 times their sum, find $p(q + r)$.

Solution. Let x, y and z be primes such that $xyz = 19(x + y + z)$. Then since $19 \mid xyz$ and x, y, z are primes, one of them must be 19. WLOG, suppose that $x = 19$. Then,

$$yz = 19 + y + z \implies (y - 1)(z - 1) = 20.$$

Remember that y and z are both positive integers. This means that $y-1$ and $z-1$ are both positive divisors of 20. Hence

$y-1$	$z-1$
1	20
2	10
4	5
5	4
10	2
20	1

Checking these cases, we see that $(y, z) = (11, 3)$ or $(3, 11)$. Hence (x, y, z) can be any permutation of $(19, 11, 3)$. Adding the size condition gives $(p, q, r) = (3, 11, 19)$. Consequently, $p(q + r) = 90$. $\square$

2016 National Round

Problem 1. The sequence $\log_{12} 162$, $\log_{12} x$, $\log_{12} y$, $\log_{12} z$, $\log_{12} 1250$ is an arithmetic progression for real numbers x, y and z. What is x?

Solution. The first term of this AP is $\log_{12} 162$, and the common difference is $\log_{12} x - \log_{12} 162 = \log_{12}\left(\frac{x}{162}\right)$. Then the nth term u_n is

$$u_n = \log_{12} 162 + (n-1)\log_{12}\left(\frac{x}{162}\right) = \log_{12} 162 + \log_{12}\left(\frac{x^{n-1}}{162^{n-1}}\right) = \log_{12}\left(\frac{x^{n-1}}{162^{n-2}}\right).$$

Since the 5th term is $\log_{12} 1250$, we have

$$\log_{12}\left(\frac{x^4}{162^3}\right) = \log_{12} 1250 \implies x^4 = 1250 \cdot 162^3 = \frac{5^4 \cdot 162^4}{3^4} \implies x = 270. \qquad \square$$

Problem 2. A triangular array of 2016 coins has 1 coin in the first row, 2 coins in the second row and 3 coins in the third row, and so on up to N coins in the Nth row. What is the value of N?

Solution. The number of coins up to Nth row is

$$1 + 2 + 3 + \cdots + N = \frac{N(N+1)}{2}.$$

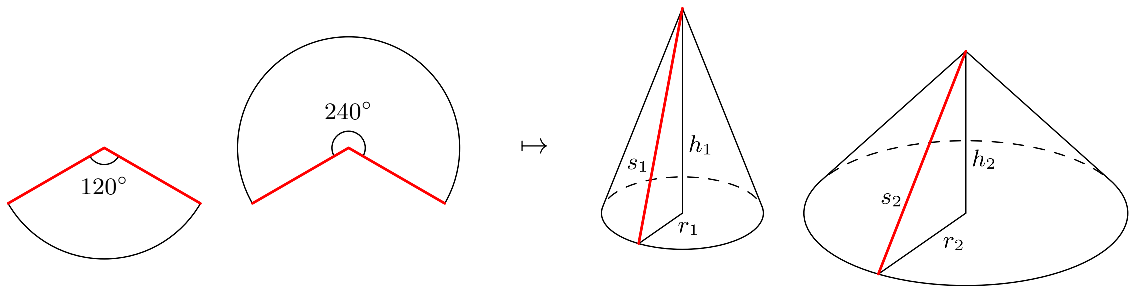
Since this is equal to 2016, we see that

$$N^2 + N = 4032 \iff (N+64)(N-63) = 0.$$

As N is positive, it follows that $N = 63$. $\square$

Problem 3. Tu Tu cuts a circular paper disk of radius 12 cm along two radii to form two sectors, the smaller having a central angle of 120°. He makes two circular cones, using each sector to form the lateral surface of a cone. What is the ratio of the height of the smaller cone to that of the larger?

Solution. Let r_1, s_1 and h_1 be the radius, slant height and height of the smaller cone. Define r_2, s_2 and h_2 for the bigger cone respectively.



First, we need to know the base circumference c_1 of the smaller cone. This is the same as the circumference of the smaller sector. Since its central angle is $120°$ and radius is 12 cm, it follows that the circumference is

$$c_1 = \frac{120°}{360°}(2)(\pi)(12) = 8\pi \text{ cm}.$$

Therefore,

$$r_1 = \frac{c_1}{2\pi} = 4 \text{ cm}.$$

Meanwhile, the slant height of the smaller cone is the same as the radius of the smaller sector which is equal to 12 cm. Therefore, by Pythagoras's theorem, the height of the smaller cone is

$$h_1 = \sqrt{s_1^2 - r_1^2} = \sqrt{12^2 - 4^2} = \sqrt{128} \text{ cm}.$$

Similarly, we can compute that $h_2 = \sqrt{80}$ cm. Therefore,

$$\frac{h_1}{h_2} = \sqrt{\frac{128}{80}} = \sqrt{\frac{8}{5}}.$$

□

Problem 4. The sequence u_1, u_2, u_3, ... has the property that every term beginning with the third is the sum of the previous two terms. That is,

$$u_n = u_{n-2} + u_{n-1} \text{ for } n \geq 3.$$

Suppose that $u_9 = 110$ and $u_7 = 42$. What is u_4?

Solution. First, we have $u_8 = u_9 - u_7 = 110 - 42 = 68$. Now we can calculate all the terms backwards as follows:

$$\begin{aligned} u_6 &= u_8 - u_7 = 68 - 42 = 26 \\ u_5 &= u_7 - u_6 = 42 - 26 = 16 \\ u_4 &= u_6 - u_5 = 26 - 16 = 10. \end{aligned}$$

□

Problem 5. Line ℓ_1 has equation $3x - 2y = 1$ and goes through $A = (-1, -2)$. Line ℓ_2 has equation $y = 1$ and meets line ℓ_1 at point B. Line ℓ_3 has positive slope, goes through point A and meets ℓ_2 at point C. The area of $\triangle ABC$ is 3. What is the slope of line ℓ_3?

Solution. Since B is the intersection of ℓ_1 and ℓ_2, its x-coordinate satisfies

$$3x - 2(1) = 1 \implies x = 1.$$

Therefore, $B = (1, 1)$. Notice that the perpendicular distance from A to ℓ_2 is $1 - (-2) = 3$. As the area of triangle ABC is 3, this means that

$$BC = \frac{2[ABC]}{3} = 2.$$

Since C also lies on ℓ_2, there are only two possible positions that C can be, namely $(-1, 1)$ or $(3, 1)$. When $C = (-1, 1)$, the line ℓ_3 is actually vertical, so the slope is undefined. Therefore, $C = (3, 1)$. Then the slope of ℓ_3 is

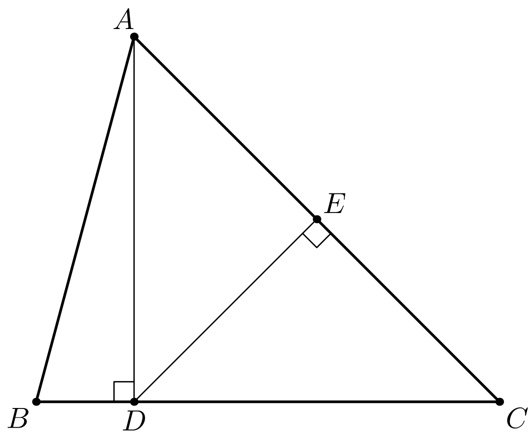
$$m = \frac{1 - (-2)}{3 - (-1)} = \frac{3}{4}.$$

□

Problem 6. In $\triangle ABC$, $AB = 13$, $BC = 14$ and $CA = 15$. Distinct points D and E lie on segments BC and CA respectively such that $AD \perp BC$ and $DE \perp AC$. Find the exact length of segment DE.

Solution. By law of cosines in $\triangle ABC$,

$$\cos \angle C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{14^2 + 15^2 - 13^2}{2 \cdot 14 \cdot 15} = \frac{3}{5}.$$



Since $\cos \angle C > 0$, this also shows that $\angle C$ is acute. Therefore,

$$\sin \angle C = \sqrt{1 - \frac{3^2}{5^2}} = \frac{4}{5}.$$

Now we can compute CD.

$$CD = AC \cos \angle C = 15 \cdot \frac{3}{5} = 9.$$

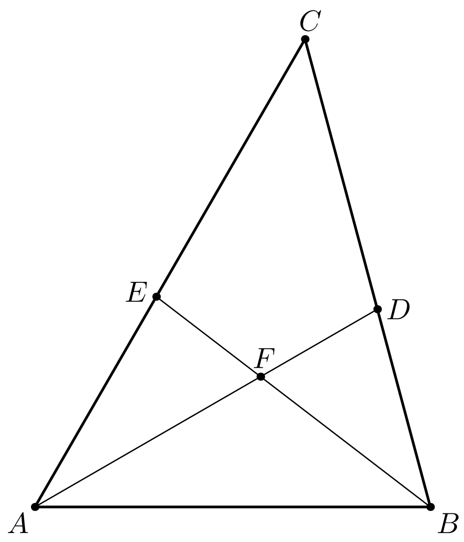
Therefore, finally,

$$DE = CD \sin \angle C = \frac{9 \cdot 4}{5} = \frac{36}{5}. \qquad \square$$

Problem 7. In $\triangle ABC$, $AB = 6$, $BC = 7$ and $CA = 8$. Point D lies on BC and AD bisects $\angle BAC$. Point E lies on AC and BE bisects $\angle ABC$. The bisectors intersect at F. Find the ratios $AF : FD$ and $BF : FE$.

Solution. By the angle bisector theorem, $BD : DC = BA : BC$, so

$$\frac{AB}{BD} = \frac{AC}{CD} = \frac{AB + AC}{BD + CD} = \frac{6 + 8}{7} = 2.$$



By angle bisector theorem again, $AF : FD = AB : BD = 2$. In a similar way, it is easy to see that

$$\frac{BF}{FE} = \frac{BA}{AE} = \frac{BA + BC}{AC} = \frac{13}{8}. \qquad \square$$

Problem 8. The sum of an infinite geometric series is a positive number S, and the second term in the series is 1. What is the smallest possible value of S?

Solution. Let a and r be the starting term and common ratio of the geometric series. Then $ar = 1$ so $a = \frac{1}{r}$. Then,

$$S = \frac{a}{1-r} = \frac{1}{r(1-r)}.$$

Now observe that

$$\left(r - \frac{1}{2}\right)^2 \geq 0 \Longrightarrow r(1-r) \leq \frac{1}{4} \Longrightarrow S = \frac{1}{r(1-r)} \geq 4.$$

Therefore, the minimum value of S is 4, attained when $r = \frac{1}{2}$ and $a = 2$. $\square$

Problem 9. Prove that $\sqrt{ab} + \sqrt[3]{abc} \leq \frac{1}{3}(a + 4b + 4c)$ for all positive numbers a, b and c.

Solution. By the AM - GM inequality,

$$\sqrt{ab} + \sqrt[3]{abc} = \sqrt{\frac{a}{2} \cdot 2b} + \sqrt[3]{\frac{a}{4} \cdot b \cdot 4c} \leq \frac{1}{2}\left(\frac{a}{2} + 2b\right) + \frac{1}{3}\left(\frac{a}{4} + b + 4c\right) = \frac{1}{3}(a + 4b + 4c),$$

and the equality occurs if and only if $a = 4b = 16c$. $\square$

Remark. The solution above may seem like it came out of nowhere at first glance, so I will try to give a motivation. It is pretty tempting to use the AM - GM inequality, since both $\sqrt{ab}$ and $\sqrt[3]{abc}$ appear on the smaller side. But straightup using AM - GM turns out to be not strong enough, so we partition the larger side more carefully. This is done by considering variables such that

$$p + q = \frac{1}{3}, \quad r + s = \frac{4}{3} \quad \text{and} \quad t = \frac{4}{3}$$

and applying AM-GM to the two terms $pa + rb$ and $qa + sb + tc$. Doing so yields

$$\frac{1}{3}(a + 4b + 4c) = (pa + rb) + (qa + sb + tc) \geq 2\sqrt{pr} \cdot \sqrt{ab} + 3\sqrt[3]{qst}\sqrt[3]{abc}.$$

We want $2\sqrt{pr} = 3\sqrt[3]{qst} = 1$, and this gives us four equations.

$$p + q = \frac{1}{3} \quad (1)$$

$$r + s = \frac{4}{3} \quad (2)$$

$$pr = \frac{1}{4} \quad (3)$$

$$qs = \frac{1}{36}. \quad (4)$$

From now on it's just traditional equation solving. But I actually failed at this part and had to look up the solution online because I'm absolutely terrible at solving equations[1]. Anyway, substituting $r = \frac{1}{4p}$ and $s = \frac{1}{36q}$ into equation (2) gives us

$$\frac{1}{4p} + \frac{1}{36q} = \frac{4}{3} \implies 27q + 3p = 144pq.$$

Substituting $3p = 1 - 3q$ gives us the following quadratic,

$$144q^2 - 24q + 1 = 0 \implies (12q - 1)^2 = 0$$

which gives us $q = \frac{1}{12}$. Consequently we obtain $(p, q, r, s) = (\frac{1}{4}, \frac{1}{12}, 1, \frac{1}{3})$.

Problem 10. Eight people are sitting around a circular table, each holding a fair coin. All eight people flip their coins and those who flip heads stand while those who flip tails remain seated. What is the number of ways that no two adjacent people will stand?

Solution. Note that this number is the same as the number of ways to choose a set of people from the table such that no two people in the set are adjacent to each other[2]. In fact, we will solve the following more general problem. The original problem is the special case when G is the cycle graph C_8.

Reformulation. Let G be a graph. What is the number of ways to choose a set of vertices of G, such that none of them are connected by an edge?

Let $f(G)$ be the number of ways. We will use a deletion-contraction style argument here to find a recursion for f.

Claim. Let v be any vertex of G, and let $N[v]$ be the set of vertices connected to v, including v. (This is also called the *closed* neighbourhood of v.) Then

$$f(G) = f(G - v) + f(G - N[v]).$$

Proof. For a vertex v, we can interpret $f(G)$ as the sum of the following two values: the number of ways where v is not included, and those where v is included. The first one is easy, this is the number of ways to choose vertices from G excluding v, or equivalently $f(G - v)$. For the second one, notice that once we choose v, we can't choose any of the vertices from $N[v]$. Therefore, the number of ways to do this is $f(G - N[v])$. Combining these gives us our claim. $\square$

[1] That's what happens when you avoid solving any algebra problem for a whole year.
[2] Such a set is called an *independent set*.

Now going back to our problem, take a vertex v of C_8. Then applying our claim gives us

$$f(C_8) = f(P_7) + f(P_5)$$

where P_n is the path graph with n vertices. We can do the same thing repeatedly to find the values of f for P_5 and P_7, so I'll only show how to find $f(P_5)$. The main idea remains the same, break down the graph until it becomes feasible to find the values of f manually. In the following, we always remove the left most vertex.

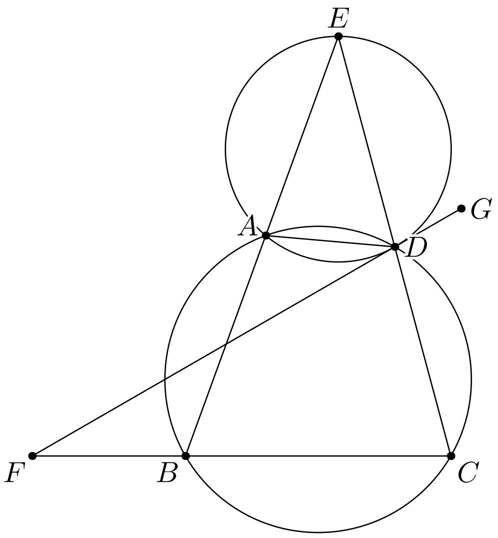
$$\begin{aligned} f(P_5) &= f(P_4) + f(P_3) \\ &= f(P_3) + f(P_2) + f(P_3) \\ &= 5 + 3 + 5 \\ &= 13. \end{aligned}$$

Similarly, we have $f(P_7) = 34$. Hence $f(C_8) = 47$. □

Remark. From the recurrence we got, it is not hard to prove that $f(P_n) = F_{n+1}$ where F_n is the nth Fibonacci number.

Problem 11. Let $ABCD$ be a cyclic quadrilateral and let the lines CD and BA meet at E. The line through D which is tangent to the circle ADE meets the line CB at F. Prove that $\triangle CDF$ is isosceles.

Solution. Let G be a point on line FD such that D is between F and G.



Then since DG is tangent to (ADE),

$$\angle FDC = \angle GDE = \angle DAE = \angle DCF.$$ □

Problem 12. Prove that $n^3 - n$ is divisible by 24, for all odd integers n.

Solution. Let $n = 2k + 1$ for some integer k. Then the expression can be factorized as

$$n^3 - n = n(n^2 - 1) = n(n + 1)(n - 1) = 2k(2k + 1)(2k + 2) = 4k(k + 1)(2k + 1).$$

Since $2k$, $2k + 1$ and $2k + 2$ are three consecutive integers, one of them must be divisible by 3. Therefore, their product, $n^3 - n$, is divisible by 3. Similarly, k and $k + 1$ are two consecutive integers, so one of them must be divisible by 2. Therefore, $4k(k+1)$ is divisible by 8 and hence $n^3 - n$ is also divisible by 8. Since $n^3 - n$ is divisible by both 8 and 3, it is divisible by their least common multiple which is 24. □

Problem 13. In how many ways can 7 paintings be posted on a wall,

1. if 3 of the paintings are always to appear altogether,
2. if 3 of the paintings are never to appear altogether?

Solution. The second part can be a bit ambiguous, but here I will also count the arrangements where 2 of the 3 paintings are together.

1. Consider 3 paintings that are to be together as a block. Then there are $5! = 120$ ways to permute the 4 remaining paintings and the block. In addition, there are $3! = 6$ ways to permute the 3 paintings in the block. Therefore, there are a total of $120 \cdot 6 = 720$ ways to permute 7 paintings such that 3 of the painting always appear together.
2. This is just the total number of arrangements minus the number of permutations where those three paintings are together, which is equal to $7! - 720 = 5040 - 720 = 4320$. □

Problem 14. A box contains 2 red marbles, 2 green marbles, and 2 yellow marbles, Mg Gyi takes 2 marbles from the box at random; then Mg Latt takes 2 of the remaining marbles at random; and then Mg Nge takes the last 2 marbles. What is the probability that Mg Nge gets two marbles of the same colour?

Solution. Let a *draw* be an arrangement of 6 marbles, where the first 2 are the ones chosen by Mg Gyi, the 3rd and 4th are the ones chosen by Mg Latt, and the last 2 are the ones chosen by Mg Nge.

We will first find the number of all possible draws. Since there are 2 identical marbles for each colour and there are 3 colours, this is equal to

$$\frac{6!}{2!2!2!} = 90.$$

Now let's find the number of draws whose last two marbles have the same colour. WLOG, suppose that this colour is red. Then there are 2 green marbles and 2 yellow marbles remaining, so the number of ways to permute them in the first four places is

$$\frac{4!}{2!2!} = 6.$$

Similarly, the number of draws where the last two marbles are green and that where the last two marbles are yellow are also 6. Therefore, the number of draws where the last two marbles are of the same colour is 18.

Hence the probability is $\frac{18}{90} = \frac{1}{5}$. □

Problem 15. Prove by mathematical induction that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$

for all positive integers n.

Solution. For the base case $n = 1$, both the left and right hand sides are equal to 1 so the identity is true. Suppose that it is true for $n = k$. Then when $n = k + 1$,

$$\begin{aligned}
1^3 + 2^3 + \cdots + k^3 + (k+1)^3 &= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \\
&= \frac{(k+1)^2(k^2+4k+4)}{4} \\
&= \left(\frac{(k+1)(k+2)}{2}\right)^2 \\
&= (1 + 2 + \cdots + k + (k+1))^2
\end{aligned}$$

so the identity is also true in this case. Hence the identity holds for all positive integers n. □

Problem 16. If one of the roots of the equation:

$$ax^3 + bx^2 + cx + d = 0$$

is the geometric mean of the other two roots, prove that $ac^3 = b^3 d$.

Solution. Let p, q, r be the roots of the equation, and suppose that $q^2 = pr$. Then by Vieta's formulas,

$$\begin{aligned}
p + q + r &= -\frac{b}{a}, \\
pq + qr + rp &= \frac{c}{a}, \\
pqr &= -\frac{d}{a}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{c}{b} &= \frac{c}{a} \cdot \frac{a}{b} \\
&= -\frac{pq + qr + rp}{p + q + r} \\
&= -\frac{pq + qr + q^2}{p + q + r} \\
&= -\frac{q(p + q + r)}{p + q + r} \\
&= -q.
\end{aligned}$$

Finally,

$$\frac{c^3}{b^3} = -q^3 = -pqr = \frac{d}{a} \implies ac^3 = b^3 d.$$

□

Problem 17. Let f be a function such that

$$x^2 f(x) + f\left(\frac{x-1}{x}\right) = 2x^2,$$

for all real numbers $x \neq 0$ and $x \neq 1$. Find the value of $f\left(\frac{1}{2}\right)$.

Solution. Substituting $x = 2$, $x = \frac{1}{2}$ and $x = -1$ gives the following 3 equations.

$$4f(2) + f\left(\frac{1}{2}\right) = 8, \tag{1}$$

$$\frac{1}{4}f\left(\frac{1}{2}\right) + f(-1) = \frac{1}{2}, \tag{2}$$

$$f(-1) + f(2) = 2. \tag{3}$$

Solving them gives

$$(1) - 4 \times (3) \Longrightarrow \qquad f\left(\frac{1}{2}\right) - 4f(-1) = 0 \tag{4}$$

$$4 \times (2) \Longrightarrow \qquad f\left(\frac{1}{2}\right) + 4f(-1) = 2 \tag{5}$$

$$(4) + (5) \Longrightarrow \qquad 2f\left(\frac{1}{2}\right) = 2 \tag{6}$$

$$f\left(\frac{1}{2}\right) = 1.$$

□

2017 Regional Round

Problem 1. The four digit number $\overline{ABCD}$ is such that $\overline{ABCD} = A \times \overline{BCD} + \overline{ABC} \times D$. Find the smallest possible value of $\overline{ABCD}$. (Here $\overline{ABCD}$ means $1000 \times A + 100 \times B + 10 \times C + D$.)

Solution. The key here is that we only need to find the smallest value of $\overline{ABCD}$. So let $A = 1$. Then

$$\overline{1BCD} = \overline{BCD} + \overline{1BC} \cdot D \Longrightarrow 1000 = \overline{1BC} \cdot D.$$

Since $D < 10$, D can only be 2, 4, 5 or 8. In these 4 cases, it is easy to see that only 8 gives solutions $B = 2$ and $C = 5$. Therefore, 1258 is the only number starting with 1 that satisfies the given equation. Since the next number that also satifies the equation must start with 2, this must be the smallest value of $\overline{ABCD}$. □

Problem 2. Find the remainder when 20172017201720172017201720172017 is divided by 72.

Solution. Consider the number $N = 20172017201720172017201720172000$. Since the last three digits of this number are all zeroes, N must be divisible by 8. Also since the sum of digits of N is divisible by 9, N must be divisible by 9. Since 8 and 9 are relatively prime, by lemma 4.2, it follows that N is divisible by their least common multiple which is 72. Therefore, $N + 17$ leaves a remainder of 17 when divided by 72. □

Problem 3. Show that for all real numbers x, y, z and w,

$$\sin(x - w)\sin(y - z) + \sin(y - w)\sin(z - x) + \sin(z - w)\sin(x - y) = 0.$$

Solution. Remember the indentity

$$\sin a \sin b = \frac{1}{2}(\cos(a - b) - \cos(a + b)).$$

Applying this to the left hand side gives

$$\begin{aligned}
&\sin(x - w)\sin(y - z) + \sin(y - w)\sin(z - x) + \sin(z - w)\sin(x - y) \\
&\quad = \frac{1}{2}(\cos(x + z - y - w) - \cos(x + y - z - w) + \cos(x + y - z - w) \\
&\qquad\quad - \cos(y + z - x - w) + \cos(y + z - x - w) - \cos(x + z - y - w)) \\
&\quad = 0.
\end{aligned}$$

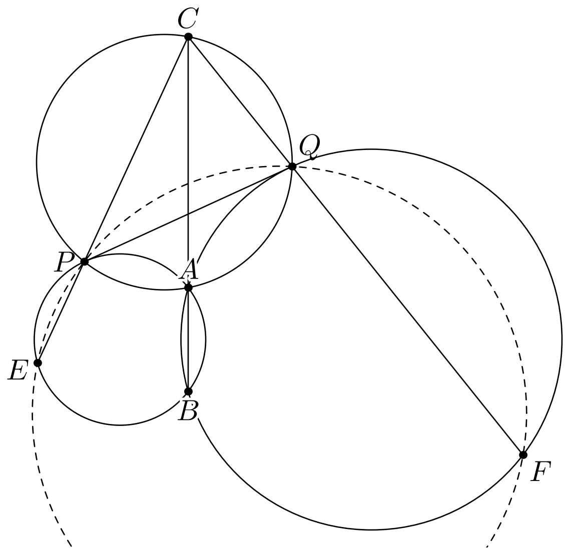
□

Problem 4. Two circles intersect at A and B. A common tangent to the circles touches the circles at P and Q. A circle is drawn through P, Q and A, and the line BA produced meets this circle again at C. Join CP and CQ, and extend both to meet the given circles at E and F respectively. Prove that P, Q, E, F are concyclic.

Solution. Most of the conditions in this problem are unnecessary. Since $PABE$ and $QABF$ are cyclic,

$$CP \cdot CE = CA \cdot CB = CQ \cdot CF,$$

and so P, Q, E, F are concyclic. □



Problem 5. If $x \geq 1$, prove that $x^3 - 5x^2 + 8x - 4 \geq 0$.

Solution. Since $x - 1 \geq 0$,

$$\begin{aligned} x^3 - 5x^2 + 8x - 4 &= x^3 - x^2 - 4x^2 + 8x - 4 \\ &= x^2(x-1) - 4(x-1)^2 \\ &= (x-1)(x^2 - 4x + 4) \\ &= (x-1)(x-2)^2 \\ &\geq 0. \end{aligned}$$

□

Problem 6. Let $u_1, u_2, u_3, \ldots$ be a sequence of real numbers such that $u_1 > 2$ and

$$u_{n+1} = 1 + \frac{2}{u_n}$$

for $n \geq 1$. Prove that $u_{2n-1} + u_{2n} > 4$ for all $n \geq 1$.

Solution. We will first prove by induction that $u_{2n-1} > 2$ for all n. The base case $n = 1$ is given to be true, so suppose that this is true for $n = k$. Then

$$\begin{aligned} u_{2k} &= 1 + \frac{2}{u_{2k-1}} < 1 + 1 = 2 \\ u_{2k+1} &= 1 + \frac{2}{u_{2k}} > 1 + 1 = 2. \end{aligned}$$

so this is also true for $n = k + 1$. Hence by induction, $u_{2n-1} > 2$ for all $n \in \mathbb{N}$.

Now the given inequality is equivalent to

$$\begin{aligned}
\iff &\quad u_{2n-1} + u_n > 4 \\
\iff &\quad u_{2n-1} + 1 + \frac{2}{u_{2n-1}} > 4 \\
\iff &\quad u_{2n-1}^2 + 2 > 3u_{2n-1} \\
\iff &\quad (u_{2n-1} - 2)(u_{2n-1} - 1) > 0
\end{aligned}$$

which is true as $u_{2n-1} > 2$. □

> **Problem 7.** Let a, b, c be positive numbers such that $ab + bc + ca + abc = 4$. Prove that
>
> $$\frac{1}{a+2} + \frac{1}{b+2} + \frac{1}{c+2} = 1$$
>
> and
>
> $$a + b + c \geq 3.$$
>
> (You can use the following result: If x, y, z are positive numbers, then $\frac{x+y+z}{3} \geq \sqrt[3]{xyz}$.)

Solution. First, add $a + b + c + 1$ to both sides to get

$$1 + a + b + c + ab + bc + ca + abc = a + b + c + 5.$$

We can factorize the left hand side as follows:

$$(a + 1)(b + 1)(c + 1) = a + b + c + 5.$$

Now let $u = a + 2$, $v = b + 2$, and $w = c + 2$. Then this can be rewritten as

$$\begin{aligned}
\iff &\quad (u - 1)(v - 1)(w - 1) = u + v + w - 1 \\
\iff &\quad uvw - uv - vw - uw + u + v + w - 1 = u + v + w - 1 \\
\iff &\quad uvw = uv + vw + uv \\
\iff &\quad \frac{1}{u} + \frac{1}{v} + \frac{1}{w} = 1 \\
\iff &\quad \frac{1}{a+2} + \frac{1}{b+2} + \frac{1}{c+2} = 1
\end{aligned}$$

which is what we wanted. Now applying the AM-HM inequality gives us

$$\frac{u+v+w}{3} \geq \frac{3}{\frac{1}{u} + \frac{1}{v} + \frac{1}{w}} = \frac{3}{1} = 3.$$

Hence,

$$u + v + w \geq 9 \iff a + b + c \geq 3.$$ □

> **Problem 8.** On each side of a triangle, there are 4 distinct points other than triangle's vertices. Determine the number of triangles having the vertices at 3 of these 12 points.

Solution. There are two different types of triangles we have to count: (1) triangles with all three vertices on different edges, and (2) triangles with exactly two vertices on the same edge.

Let's count the number of triangles of the first type. There are 4 ways to choose a vertex on each edge of the given triangle, and there are 3 edges so the number of triangles of type 1 is $4 \times 4 \times 4 = 64$.

Now let's count the number of triangles of the second type. There are three edges for which those two vertices can lie on. Out of 4 vertices on that edge, there are 6 ways to choose the 2 vertices of the triangle. Finally the remaining vertex of the triangle can be chosen from the remaining 8 points of the given triangle. Hence the total number of triangles of type 2 is $3 \times 6 \times 8 = 144$.

Therefore, the total number of triangles is $64 + 144 = 208$. □

Problem 9. Two red dice and one blue die are thrown. What is the probability that the sum of the scores on the red dice is equal to the score on the blue die?

Solution. Let the ordered pair (x, y, z) denote the dice throw where x, y and z are numbers on the first red die, second red die and the blue die respectively. Now there are 6 possibilties for each of x, y and z, so the total number of such ordered pairs is $6^3 = 216$. We need to find the number of ordered pairs (x, y, z) such that $x+y = z$. If $z = k$, then x can range from 1 to $k-1$. Notice that the value of y is determined once we choose x and z, so there are $k - 1$ pairs such that $x + y = k$. Since z can range from 1 to 6, this means that there are a total of

$$0 + 1 + 2 + 3 + 4 + 5 = 15$$

ordered pairs such that $x + y = z$. Therefore, the probability that $x + y = z$ is $\frac{15}{216} = \frac{5}{72}$. □

Problem 10. N is a two-digit number and $2N$ also has two digits. If N equals 2 times the sum of digits of $2N$, find all possible values of N.

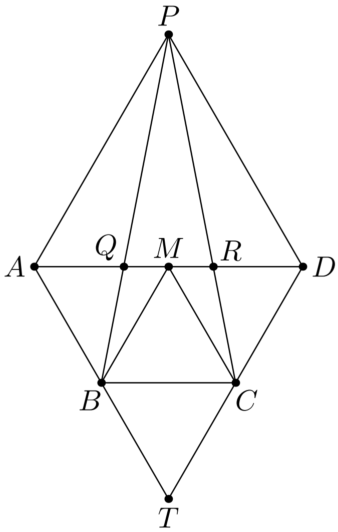
Solution. Suppose that $2N = 10a + b$. Then the problem condition gives

$$2(a + b) = N = \frac{1}{2}(10a + b) \Longrightarrow 2a = b.$$

Since b is an even integer, b can only be 2, 4, 6 or 8. Checking these values, we see that only $b = 2$ causes N to be a one-digit number so we can discard it. Thus the possible values of N are 12, 18 and 24. □

Problem 11. In the figure, $AD \parallel BC$, $AB = BC = CD = \frac{1}{2}AD$ and $\triangle APD$ is equilateral. BP and CP cut AD at Q and R respectively. If the area of $\triangle APD$ is 12, find the area of the trapezium $BQRC$.

Solution. Let M be the midpoint of segment AD. Then $AM = BC$, so $AMCB$ is a parallelogram. Therefore, $MD = CD = AB = MC$, so $\triangle MCD$ is equilateral. Similarly, $\triangle MAB$ is also equilateral.



Let lines AB and CD intersect at T. Then as $\angle TAD = \angle TDA = 60°$, it follows that $\triangle TAD$ is equilateral too. This means that $\triangle APD \cong \triangle ATD$. It is also easy to see that B and C are midpoints of AT and BT.

Now notice that $\triangle ABP$ and $\triangle TBP$ have the same area because B is the midpoint of AT. Similarly, $\triangle DCP$ and $\triangle TCP$ have the same area. Hence the area of quadrilateral $PBTC$ is half the area of quadrilateral $PATD$, which is equal to 12. Meanwhile the area of $\triangle BCT$ is a quarter of the area of $\triangle TAD$, so it is equal to 3. Hence the area of $\triangle PBC$ is $12 - 3 = 9$.

Since $QR \parallel BC$, $\triangle PQR \sim \triangle PBC$. As the ratio of their heights is $2 : 3$, the ratios of their areas must be $4 : 9$, and hence $\triangle PQR$ has area 4. Finally, the area of quadrilateral $BCRQ$ is $9 - 4 = 5$. □

Problem 12.

1. Show that $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{(n-1)n} = \frac{n-1}{n}$ for $n \geq 2$.
2. Show that for every integer $n \geq 2$, there exist positive integers $x_1, x_2, x_3, \ldots, x_n$ so that
$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \cdots + \frac{1}{x_n} = 1.$$

Solution. It is easy to see that in general the nth term has the form $\frac{1}{n(n+1)}$. This can be written as
$$\frac{1}{n(n+1)} = \frac{n+1-n}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$
Therefore, the sum is
$$\begin{aligned}\frac{1}{1\times 2} + \frac{1}{2\times 3} + \cdots + \frac{1}{(n-1)n} &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n-1} - \frac{1}{n} \\ &= 1 - \frac{1}{n} \\ &= \frac{n-1}{n}.\end{aligned}$$
Now take $x_i = \frac{1}{i(i+1)}$ for $1 \leq i \leq n-1$. Finally, take $x_n = \frac{1}{n}$. Then from the above, we have
$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = 1$$
as desired. □

Remark. You can also take $x_1 = x_2 = \cdots = x_n = n$ for the second part. Unfortunately, there isn't a good way to fix the problem so that it does not include trivial solutions.

Problem 13. Let a, b, c, d, e be five prime numbers forming an arithmetic progression with a common difference of 6. Find the smallest possible value of $a + b + c + d + e$.

Solution. It is easy to see that 5, 11, 17, 23, 29 is such an arithmetic progression. In this case, $a+b+c+d+e = 85$. Now suppose that a is lower than 5. Then it must be either 2 or 3. But when $a = 2$, $b = 8$ is not a prime, and when $a = 3$, $b = 9$ is also not a prime, Hence a is at least 5, which means that 85 is indeed the minimum value of $a + b + c + d + e$. □

Remark. In fact, we can even show that 5, 11, 17, 23, 29 is the only such AP. Observe that $a, b, c, d, e \equiv a, a+1, a+2, a+3, a+4 \pmod 5$. This means that one of them must be divisible by 5 and since all of them are primes, this means that it must be equal to 5. However, b, c, d and e are all greater than 5, so $a = 5$.

Problem 14. Function $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$(a-b)f(a+b) + (b-c)f(b+c) + (c-a)f(c+a) = 0,$$

for all $a, b, c \in \mathbb{R}$. For $x \in \mathbb{R}$, letting $a = \frac{1}{2}(x-1)$, $b = \frac{1}{2}(x+1)$ and $c = \frac{1}{2}(1-x)$, show that $f(x) = Ax + B$ where $A = f(1) - f(0)$ and $B = f(0)$. Conversely, show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = Ax + B$, where A and B are constants, satisfies the given equation.

Solution. Letting $a = \frac{1}{2}(x-1)$, $b = \frac{1}{2}(x+1)$ and $c = \frac{1}{2}(1-x)$, we see that

$$\begin{aligned}
(a-b)f(a+b) + (b-c)f(b+c) + (c-a)f(c+a) &= 0 \\
-f(x) + xf(1) + (1-x)f(0) &= 0 \\
f(x) &= x(f(1) - f(0)) + f(0) \\
&= Ax + B.
\end{aligned}$$

where $A = f(1) - f(0)$ and $B = f(0)$. Now suppose that $f(x) = Ax + B$ for some constants A and B. Then

$$\begin{aligned}
&(a-b)f(a+b) + (b-c)f(b+c) + (c-a)f(c+a) \\
&= (a-b)(A(a+b)+B) + (b-c)(A(b+c)+B) + (c-a)(A(c+a)+B) \\
&= A(a^2-b^2) + Ba - Bb + A(b^2-c^2) + Bb - Bc + A(c^2-a^2) + Bc - Ba \\
&= 0.
\end{aligned}$$

□

Problem 15. In a trapezium $ABCD$, with $AB \parallel CD$, there are two circles with diameter AD and BC respectively. Two circles do not intersect each other. Let X and Y which do not lie in $ABCD$ be two points on each of the circles. Show that

$$XY \le \frac{1}{2}(AD + AB + DC + BC).$$

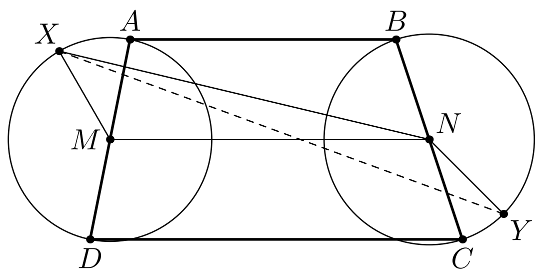
Solution. Let M and N be the midpoints of AD and BC. Then by triangle inequality,

$$XY \le XN + NY \le XM + MN + NY.$$

We also have to include equality cases too, as triangles can be degenerate. Anyway, since $MN = \frac{1}{2}(AB + DC)$, the right hand side is equivalent to

$$XM + MN + NY = \frac{1}{2}(AD + AB + DC + BC),$$

which is exactly what we want. □



Problem 16. If α, β and γ are roots of the equation $x^3 + ax^2 + bx + c = 0$, evaluate $\alpha^2 + \beta^2 + \gamma^2$ and $\alpha^3 + \beta^3 + \gamma^3$.

Solution. By Vieta's formulas,

$$\begin{aligned} \alpha + \beta + \gamma &= -a, \\ \alpha\beta + \beta\gamma + \gamma\alpha &= b, \\ \alpha\beta\gamma &= -c. \end{aligned}$$

We can now calculate both of the quantities. First,

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) = a^2 - 2b.$$

Next,

$$\begin{aligned} \alpha^3 + \beta^3 + \gamma^3 &= (\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha) + 3\alpha\beta\gamma \\ &= (-a)(a^2 - 2b - b) - 3c \\ &= -a^3 + 3ab - 3c. \end{aligned}$$

□

Problem 17. Prove by mathematical induction, that $3^{2n} + 7$ is divisible by 8 for all $n \geq 1$.

Solution. For the base case $n = 1$, it is easy to see that $3^2 + 7$ is divisible by 8. Now assume that $3^{2k} + 7$ is divisible by 8. Then

$$\begin{aligned} 3^{2(k+1)} + 7 &= 9(3^{2k}) + 7 \\ &= 9(3^{2k}) + 63 + 7 - 63 \\ &= 9(3^{2k} + 7) - 56. \end{aligned}$$

Since both $3^{2k} + 7$ and 56 are both divisible by 8, it follows that $3^{2(k+1)} + 7$ is also divisible by 8. Therefore, by mathematical induction, $3^{2n} + 7$ is divisible by 8 for all natural numbers n. □

Problem 18. Each vertex of convex polygon $ABCDE$ is to be assigned a colour. There are 6 colours and the ends of each diagonal must have different colours. How many different colourings are possible?

Solution. Consider the graph G formed by the diagonals of the pentagon. We just need to find the chromatic polynomial of this graph. Notice that this graph can be 'unfolded'[1] to form a regular pentagon C_5. Of course, the chromatic polynomial of this graph does not change in this process. By deletion-contraction,

$$P(C_5, 6) = P(P_5, 6) - P(C_4, 6).$$

We can use deletion-contraction again on $P(C_4, 6)$ to get

$$P(C_5, 6) = P(P_5, 6) - P(P_4, 6) + P(C_3, 6).$$

At this point, we can just calculate each term manually. For $P(P_5, 6)$, there are 6 colour choices for the first vertex, and 5 choices each for the other vertices. A similar thing is true for $P(P_4, 6)$. Finally, for $P(C_3, 6)$, there are 6 choices for the first vertex, 5 choices for the second vertex and 4 choices for the third vertex. Therefore,

$$P(C_5, 6) = 6 \cdot 5^4 - 6 \cdot 5^2 + 6 \cdot 5 \cdot 4 = 3120. \qquad \square$$

[1] In technical jargon, we say that these two graphs are isomorphic.

2017 National Round

Problem 1. $x = \overline{ABCDE}$ is a five digit number. If $(A+C+E)-(B+D) = 11k$ where $k = -1, 0, 1, 2$, prove that x is divisible by 11.

Solution. Notice that x can be rewritten as

$$\begin{aligned} x &= 10000A + 1000B + 100C + 10D + E \\ &= 9999A + 1001B + 99C + 11D + (A - B + C - D + E) \\ &= 11(909A + 91B + 9C + D + k). \end{aligned}$$

and so x is divisible by 11. □

Problem 2. A palindrome is a number that remains the same when its digits are reversed. For example, 252 is a three-digit palindrome and 3663 is a four-digit palindrome. If the numbers $x - 22$ and x are three-digit and four-digit palindromes, respectively, find the value of x.

Solution. Let's look at the sequence of all palindromes in ascending order, near 1000.

$$\ldots, 969, 979, 989, 999, 1001, 1111, 1221, \ldots$$

Notice that x cannot be greater than 1001 or otherwise $x - 22$ will not be a three digit number. Therefore, x can only be 1001. It is also easy to see that $x - 22 = 979$ is also a three-digit palindrome, so $x = 1001$ must be the only solution. □

Problem 3.

1. To find the exact value of $\sqrt{4 + 2\sqrt{3}}$, let $\sqrt{4 + 2\sqrt{3}} = a + b\sqrt{3}$, where a and b are integers and $a > b\sqrt{3} > 0$, and compute the exact values of a and b.

2. Length of a side of an equilateral triangle ABC is 2. If $AD \perp BC$, and the angle bisector of $\angle BAD$ meets BC at E, show that $\angle AED = 75°$. Using the angle bisector theorem, find the exact length of ED. Using the diagram, compute exact values of $\sin 75°$ and $\cos 75°$.

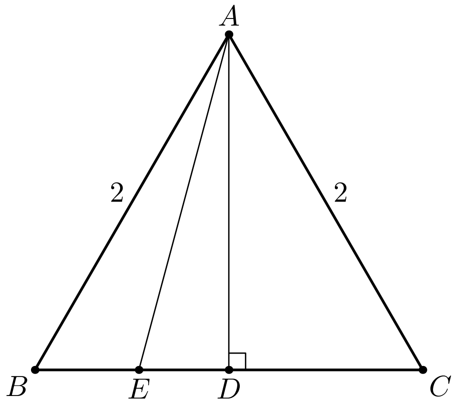
Solution. 1. Let $\sqrt{4 + 2\sqrt{3}} = a + b\sqrt{3}$. Then

$$4 + 2\sqrt{3} = a^2 + 3b^2 + 2ab\sqrt{3}.$$

Therefore, we will be done if we can find integers such that $a^2 + 3b^2 = 4$ and $ab = 1$. Substituting $b = \frac{1}{a}$ into the first equation,

$$a^2 - 4a^2 + 3 = 0 \Longrightarrow (a^2 - 1)(a^2 - 3) = 0.$$

and hence we have the following four solutions pairs (a, b): $(1, 1)$, $(-1, -1)$, $(\sqrt{3}, \frac{1}{\sqrt{3}})$, and $(-\sqrt{3}, -\frac{1}{\sqrt{3}})$. Out of all these four pairs, we see that only one pair $(1, 1)$ satisfies $a > b\sqrt{3} > 0$. Hence $a = 1$ and $b = 1$.



2. Since $\triangle ABC$ is equilateral, D must be the midpoint of BC. By Pythagoras's theorem,

$$AD = \sqrt{AB^2 - BD^2} = \sqrt{4-1} = \sqrt{3}.$$

Therefore, by the angle bisector theorem,

$$\frac{ED}{DA} = \frac{EB}{BA} = \frac{ED+EB}{DA+BA} = \frac{1}{\sqrt{3}+2} \implies ED = \frac{\sqrt{3}}{\sqrt{3}+2}.$$

Now we can calculate the length of AE as follows:

$$AE = \sqrt{DE^2 + DA^2} = \sqrt{\left(\frac{\sqrt{3}}{\sqrt{3}+2}\right)^2 + 3} = \sqrt{\frac{12(2+\sqrt{3})}{(2+\sqrt{3})^2}} = \sqrt{\frac{12}{2+\sqrt{3}}}.$$

The right hand side can be written as

$$AE = \sqrt{\frac{12}{2+\sqrt{3}}} = \sqrt{\frac{24}{4+2\sqrt{3}}} = \frac{\sqrt{24}}{1+\sqrt{3}}.$$

Therefore, finally,

$$\sin 75^\circ = \frac{AD}{AE} = \frac{1+\sqrt{3}}{2\sqrt{2}},$$

and

$$\cos 75^\circ = \frac{ED}{AE} = \frac{\sqrt{3}(1+\sqrt{3})}{\sqrt{24}(2+\sqrt{3})} = \frac{1+\sqrt{3}}{\sqrt{2}(4+2\sqrt{3})} = \frac{1+\sqrt{3}}{\sqrt{2}(1+\sqrt{3})^2} = \frac{\sqrt{3}-1}{2\sqrt{2}}. \qquad \square$$

Remark. It is way easier to calculate these using law of sines or angle addition formulas.

Problem 4. Find the nth term of a harmonic progression whose first two terms are a and b.

Solution. Suppose that $a, b, u_3, u_4, \dots$ is a harmonic progression. Then $\frac{1}{a}, \frac{1}{b}, \frac{1}{u_3}, \frac{1}{u_4}, \dots$ is an arithmetic progression. Since the first term of this arithmetic progression is $\frac{1}{a}$ and the common difference is $\frac{1}{b} - \frac{1}{a}$, the nth term of this arithmetic progression is

$$\frac{1}{u_n} = \frac{1}{a} + (n-1)\left(\frac{1}{b} - \frac{1}{a}\right) = \frac{(n-1)(a-b)+b}{ab}.$$

Hence the nth term of the harmonic progression must be

$$u_n = \frac{ab}{(n-1)(a-b)+b}. \qquad \square$$

Problem 5. Find the relationship between a, b and c if the system

$$\begin{aligned} x+y &= a \\ x^2+y^2 &= b \\ x^3+y^3 &= c \end{aligned}$$

has solutions.

Solution. By the sum of cubes identity,

$$c = x^3+y^3 = (x+y)(x^2-xy+y^2) = a(b-xy).$$

Therefore, we just need to find the value of xy in terms of a, b and c. This can be done as follows:

$$xy = \frac{(x+y)^2-x^2-y^2}{2} = \frac{a^2-b}{2}.$$

Hence the relationship between a, b and c is

$$c = a\left(b+\frac{b-a^2}{2}\right) = \frac{a(3b-a^2)}{2} \implies a^3-3ab+2c=0.$$

□

Problem 6. A function f is defined on the positive integers, $f(1)=1009$ and

$$f(1)+f(2)+\cdots+f(n) = n^2 f(n).$$

1. By expressing $f(1)+f(2)+\cdots+f(n-1)$ in two ways, find $\frac{f(n)}{f(n-1)}$.
2. By using the result in part(a), find the formula for $f(n)$. Calculate $f(2018)$.

Solution. Observe that

$$\begin{aligned} n^2 f(n) &= f(1)+f(2)+\cdots+f(n-1)+f(n) \\ &= (n-1)^2 f(n-1)+f(n) \\ (n^2-1)f(n) &= (n-1)^2 f(n-1) \\ \frac{f(n)}{f(n-1)} &= \frac{n-1}{n+1}. \end{aligned}$$

Therefore,

$$f(n) = f(1)\cdot\frac{f(2)}{f(1)}\cdot\frac{f(3)}{f(2)}\cdot\frac{f(4)}{f(3)}\cdot\cdots\cdot\frac{f(n)}{f(n-1)} = 1009\cdot\frac{1}{3}\cdot\frac{2}{4}\cdot\frac{3}{5}\cdot\cdots\cdot\frac{n-1}{n+1} = \frac{2018}{n(n+1)}.$$

Finally, $f(2018) = \frac{2018}{2018\cdot 2019} = \frac{1}{2019}$. □

Problem 7.

1. $f(x) = \frac{1}{(2x-1)(2x-3)}$ can be expressed as $f(x) = \frac{A}{2x-1}+\frac{B}{2x-3}$, where A and B are constants. Find the values of A and B.
2. If $f(3)+f(4)+f(5)+\cdots+f(n) = c-g(n)$, where c is a constant and $g(n)$ is a function, determine c and $g(n)$.

Solution. Since

$$\frac{1}{(2x-1)(2x-3)} = \frac{A}{2x-1} + \frac{B}{2x-3} = \frac{A(2x-3)+B(2x-1)}{(2x-1)(2x-3)},$$

Clearing the denominators gives[1]

$$A(2x-3)+B(2x-1) = 1 \implies (2A+2B)x - 3A - B = 1,$$

By equating the coefficients, we see that $2A + 2B = 0$ and $-3A - B = 1$. Solving these two equations give $A = -\frac{1}{2}$ and $B = \frac{1}{2}$. Therefore, $f(x)$ can be represented as

$$f(x) = \frac{1}{2}\left(\frac{1}{2x-3} - \frac{1}{2x-1}\right).$$

Consequently, the sum in the second part is

$$\begin{aligned} f(3)+f(4)+\cdots+f(n) &= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{1}{2n-3} - \frac{1}{2n-1}\right) \\ &= \frac{1}{2}\left(\frac{1}{3} - \frac{1}{2n-1}\right) \\ &= \frac{1}{6} - \frac{1}{4n-2}. \end{aligned}$$

Therefore, $c = \frac{1}{6}$ and $g(n) = \frac{1}{4n-2}$. □

Problem 8. Find the number of ways in which 5 men, 3 women and 2 children can sit at a round table, if

1. there are no restrictions,
2. each child is seated between 2 women.

Solution.

1. The number of circular permutations of n different people is $(n-1)!$. Since there are $5+3+2 = 10$ people in total, the number of circular permutations is $9! = 362880$.

2. Since each child is seated between 2 women, this means that the 3 women and 2 children form a block like so:

$$W - C - W - C - W$$

There are $(6-1)! = 120$ ways to permute the block and 5 men around the table. Inside the block, there are $3! = 6$ ways to permute the women, and 2 ways to permute the children. Therefore, the total number of permutations is $120 \cdot 6 \cdot 2 = 1440$. □

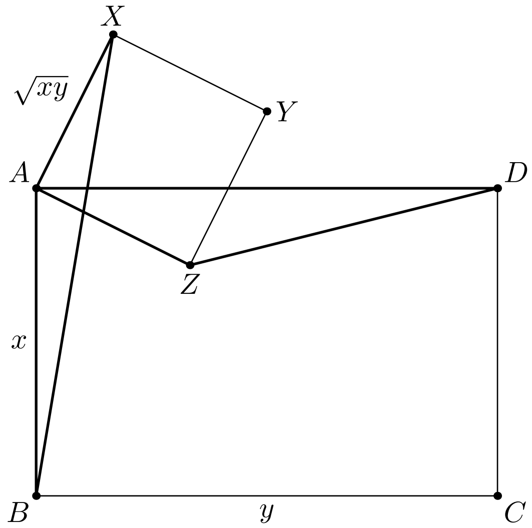
Problem 9. $ABCD$ is a rectangle with $AB = x$, $AD = y$ and $y > x$, and $AXYZ$ is a square. If the area of $AXYZ$ is the same as the area of $ABCD$, show that $\sqrt{xy} - x \leq BX \leq \sqrt{xy} + x$ and $y - \sqrt{xy} \leq DZ \leq y + \sqrt{xy}$.

Solution. Let the side length of the square be z. Then since $ABCD$ and $AXYZ$ have the same area, we have $z = \sqrt{xy}$. By the triangle inequality in $\triangle AXB$,

$$AX \leq AB + BX \implies \sqrt{xy} - x \leq BX \quad \text{and} \quad BX \leq BA + AX = x + \sqrt{xy}$$

which gives the first inequality. We can obtain the second inequality similarly by applying triangle inequality on $\triangle AZD$. □

[1] We can do this since the domain of f does not include $\frac{1}{2}$ or $\frac{3}{3}$.



Problem 10. The average of the numbers $1, 2, 3, \ldots, 99$ and x is $100x$. Find the value of x.

Solution. By the AP summation formula,

$$1 + 2 + \cdots + 99 = \frac{99 \cdot 100}{2} = 4950.$$

Therefore,

$$\frac{4950 + x}{100} = 100x \Longrightarrow x = \frac{4950}{9999} = \frac{50}{101}.$$

□

Problem 11. Each of 240 boxes in a line contains a single red marble, and for $1 \leq k \leq 240$, the box in the kth position also contains k white marbles. Phyu Phyu begins at the first box and successively draws a single marble at random from each box. She stops when she first draws a red marble. Let $\mathbb{P}(n)$ be the probability that Phyu Phyu stops after drawing exactly n marbles. What is the smallest value of n for which $\mathbb{P}(n) < \frac{1}{240}$?

Solution. Let's first calculate $\mathbb{P}(n)$ for general n. It is easy to see that the kth box contain a total of $k + 1$ marbles.

$$\begin{aligned}
\mathbb{P}(n) &= \mathbb{P}(\text{First marble is white}) \cdot \mathbb{P}(\text{Second marble is white}) \cdot \cdots \cdot \mathbb{P}(n\text{th marble is red}) \\
&= \frac{1}{2} \cdot \frac{2}{3} \cdot \cdots \cdot \frac{n-1}{n} \cdot \frac{1}{n+1} \\
&= \frac{1}{n(n+1)}.
\end{aligned}$$

Therefore,

$$\mathbb{P}(n) = \frac{1}{n(n+1)} < \frac{1}{240} \Longrightarrow (n - 15)(n + 16) > 0.$$

This shows that $n > 15$ or $n < -16$. As n is positive, the smallest value of n is 16. □

Problem 12. For all positive integers n, define

$$f(n) = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2n-1} - \frac{1}{2n},$$

$$g(n) = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n}.$$

By using mathematical induction, prove that $f(n) = g(n)$ for all positive integers n.

Solution. For the base case $n = 1$, both $f(1)$ and $g(1)$ are equal to $\frac{1}{2}$. Now suppose that $f(k) = g(k)$ for $n = k$. Then $f(k+1)$ is equal to

$$\begin{aligned} f(k+1) &= 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2k-1} - \frac{1}{2k} + \frac{1}{2k+1} - \frac{1}{2k+2} \\ &= f(k) + \frac{1}{2k+1} - \frac{1}{2k+2} \\ &= -\frac{1}{k+1} + g(k) + \frac{1}{2k+1} + \frac{1}{2k+2} \\ &= \frac{1}{k+2} + \frac{1}{k+3} + \cdots + \frac{1}{2k+2} \\ &= g(k+1) \end{aligned}$$

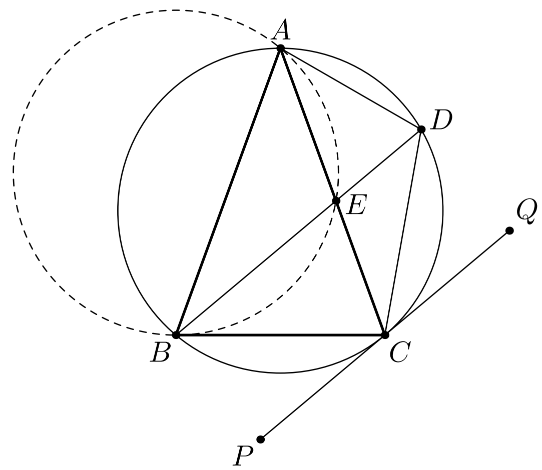
and hence by mathematical induction, $f(n) = g(n)$ for all natural numbers n. □

Problem 13. $ABCD$ is a cyclic quadrilateral with $AB = AC$. The line PQ is tangent to the circle at the point C, and is parallel to BD. Diagonals BD and AC intersect at E. If $AB = 18$ and $BC = 12$, find the length of AE.

Solution. Since $PQ \parallel BD$,

$$\angle EAB = \angle CAB = \angle PCB = \angle DBC = \angle EBC,$$

and hence by tangent-chord theorem, segment BC is tangent to (ABE).



Therefore,

$$CE \cdot CA = CB^2 \implies CE = 8.$$

Finally, $AE = AC - CE = 18 - 8 = 10$. □

Problem 14. Let $a > b > 0$. Define two sequences a_n and b_n as follows:

$$a_1 = a,\ b_1 = b,\ a_{n+1} = \frac{a_n + b_n}{2},\ b_{n+1} = \sqrt{a_n b_n}.$$

1. Prove that $a_{n+1} < a_n$ and $b_{n+1} > b_n$ for $n > 1$.
2. Prove that $a_{n+1} - b_{n+1} = \frac{(a_n - b_n)^2}{8a_{n+2}}$.
3. If $a = 4$ and $b = 1$, find the first four terms of each sequence of a_n and b_n.

Solution. When $n = 1$, $a_1 = a > b = b_1$. Now suppose that $a_k > b_k$ for some $k \geq 1$. Then by the AM-GM inequality we have

$$a_{k+1} = \frac{a_k + b_k}{2} < \sqrt{a_k b_k} = b_{k+1}.$$

Hence by induction, $a_n > b_n$ for all $n \in \mathbb{N}$. Therefore, for $n > 1$,

$$a_{n+1} = \frac{a_n + b_n}{2} < \frac{a_n + a_n}{2} = a_n.$$

Similarly, we can show that $b_{n+1} > b_n$. Now observe that

$$\begin{aligned}
a_{n+2}(a_{n+1} - b_{n+1}) &= \frac{1}{2}(a_{n+1} + b_{n+1})(a_{n+1} - b_{n+1}) \\
&= \frac{1}{2}(a_{n+1}^2 - b_{n+1}^2) \\
&= \frac{1}{8}(a_n^2 + b_n^2 + 2a_n b_n - 4a_n b_n) \\
&= \frac{(a_n - b_n)^2}{8} \\
a_{n+1} - b_{n+1} &= \frac{(a_n - b_n)^2}{8a_{n+2}}
\end{aligned}$$

Finally when $a = 4$ and $b = 1$, the first four terms of a_n are $4, \frac{5}{2}, \frac{9}{4}, \frac{9+4\sqrt{5}}{8}$ and those of b_n are $1, 2, \sqrt{5}, \frac{3\sqrt[4]{5}}{2}$. □

Problem 15. If $\cos\alpha, \cos\beta$ and $\cos\gamma$ are the roots of the equation $x^3 + ax^2 + bx + c = 0$, where α, β and γ are angles of a triangle, prove that $a^2 = 2b + 2c + 1$.

Solution. By Vieta's formulas,

$$\begin{aligned}
-a &= \cos\alpha + \cos\beta + \cos\gamma \\
b &= \cos\alpha\cos\beta + \cos\beta\cos\gamma + \cos\gamma\cos\alpha \\
-c &= \cos\alpha\cos\beta\cos\gamma
\end{aligned}$$

Therefore, we just need to show that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = a^2 - 2b = 1 + 2c = 1 - 2\cos\alpha\cos\beta\cos\gamma.$$

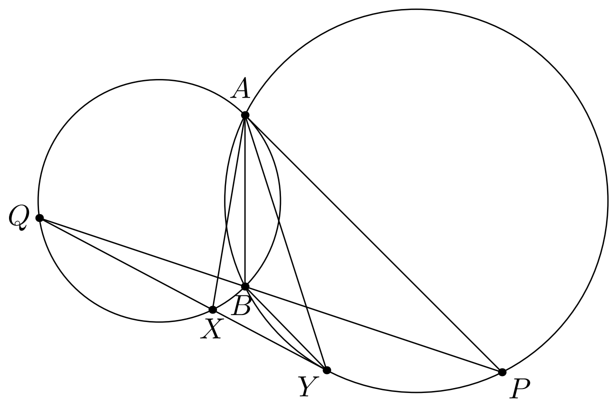
Remember that we have $\cos 2\theta = 2\cos^2\theta - 1$. Therefore,

$$\begin{aligned}
\cos^2\alpha + \cos^2\beta + \cos^2\gamma &= \frac{1}{2}(2 + \cos 2\alpha + \cos 2\beta + 2\cos^2\gamma) \\
&= \frac{1}{2}(2 + 2\cos(\alpha+\beta)\cos(\alpha-\beta) + 2\cos^2\gamma) \\
&= 1 - \cos\gamma\cos(\alpha-\beta) + \cos^2\gamma \\
&= 1 - \cos\gamma(\cos(\alpha-\beta) - \cos\gamma) \\
&= 1 - \cos\gamma(\cos(\alpha-\beta) + \cos(\alpha+\beta)) \\
&= 1 - 2\cos\alpha\cos\beta\cos\gamma. \qquad \square
\end{aligned}$$

Problem 16. The two circles C_1 and C_2 intersect at the points A and B. The tangent to C_1 at A intersects C_2 at P and the line PB intersects C_1 at Q. The tangent to C_2 drawn from Q intersects C_1 and C_2 at the points X and Y respectively. The points A and Y lie on the different sides of PQ. Show that AY bisects $\angle XAP$.

Solution. In fact, we don't even need the fact that segment AP is tangent to C_1. Note that

$$\angle XAY = \angle XAB + \angle BAY = \angle XQB + \angle BYQ = \angle YBP = \angle YAP. \qquad \square$$



2018 Regional Round

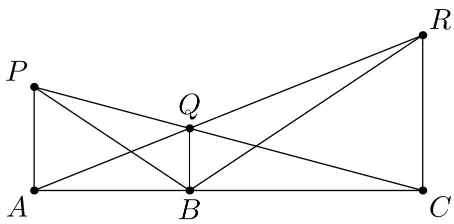
Problem 1. In the figure, AP, BQ, CR are perpendicular to the straight line ABC. Prove that

1. $\triangle PAB \sim \triangle RCB$
2. $\frac{1}{BQ} = \frac{1}{AP} + \frac{1}{CR}$.

Solution. Notice that $\angle PAQ = \angle CRQ$ and $\angle APQ = \angle RCQ$, so $\triangle PAQ \sim \triangle CRQ$. Since $BQ \parallel PA$, we have

$$\frac{AB}{BC} = \frac{PQ}{QC} = \frac{PA}{RC} \Longrightarrow \frac{PA}{AB} = \frac{RC}{CB}.$$

As $\angle PAB = \angle RCB = 90°$, we see that $\triangle PAB \sim \triangle RCB$.



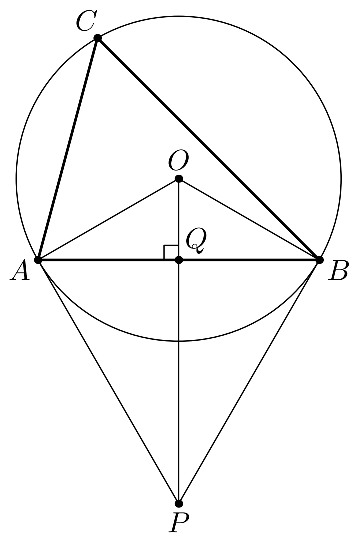
Finally,

$$\frac{BQ}{AP} + \frac{BQ}{CR} = \frac{CB}{CA} + \frac{AB}{CA} = 1 \Longrightarrow \frac{1}{AP} + \frac{1}{CR} = \frac{1}{BQ}. \qquad \square$$

Problem 2. PA and PB are the tangent segments at A and B to a circle whose center is O. AB and OP are intercept at Q. Prove that $AB \perp OP$. Hence show that $OQ : QP = AO^2 : AP^2$. Hence also show that $\alpha(\triangle OAB) : \alpha(\triangle PAB) = AO^2 : AP^2$.

Solution. Since $PA = PB$ and $OA = OB$, $OAPB$ is a kite, and hence $AB \perp OP$. Now, Q is the foot of perpendicular from A in right triangle OAP. Therefore,

$$\frac{AO^2}{AP^2} = \frac{OQ \cdot OP}{PQ \cdot PO} = \frac{OQ}{QP}.$$



Since $\triangle OAB$ and $\triangle PAB$ share the same base AB, the ratio of their areas is equal to the ratio of their heights. Hence

$$\frac{[\triangle OAB]}{[\triangle PAB]} = \frac{OQ}{QP} = \frac{AO^2}{AP^2}.$$

□

Problem 3. If T_1, T_2, T_3 are the sums of n terms of three series in AP, the first term of each being a and the respective common differences being d, $2d$, $3d$, then show that $T_1 + T_3 = 2T_2$.

Solution. Let (x_k), (y_k) and (z_k) denote the three APs. First, for any $i \in \mathbb{N}$, we have

$$x_i + z_i = a + (i-1)d + a + 3(i-1)d = 2a + 4(i-1)d = 2(a + 2(i-1)d) = 2y_i.$$

Therefore,

$$T_1 + T_3 = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} z_i = \sum_{i=1}^{n}(x_i + z_i) = \sum_{i=1}^{n} 2y_i = 2\sum_{i=1}^{n} y_i = 2T_2.$$

□

Problem 4. The positive difference between the zeros of the quadratic expression $x^2 + kx + 3$ is $\sqrt{69}$. Find the possible values of k.

Solution. Let the roots of this polynomial be p and q and WLOG assume that $p \geq q$. Then by Vieta's formulas, $p + q = -k$ and $pq = 3$. We also have $p - q = \sqrt{69}$. Therefore,

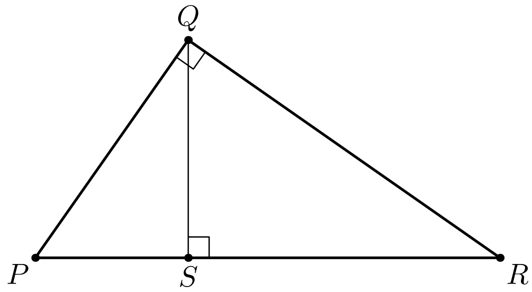
$$3 = pq = \frac{(p+q)^2 - (p-q)^2}{4} = \frac{k^2 - 69}{4} \implies k^2 = 81.$$

Hence $k = \pm 9$. □

Problem 5. In $\triangle PQR$, $\angle Q = 90°$ and S is a point on PR such that $QS \perp PR$. If $PR = kQR$, then show that $PS = (k^2 - 1)RS$.

Solution. Since R is the foot of perpendicular from Q in right triangle PQR,

$$\frac{PS}{RS} = \frac{PS \cdot PR}{RS \cdot PR} = \frac{PQ^2}{QR^2}.$$



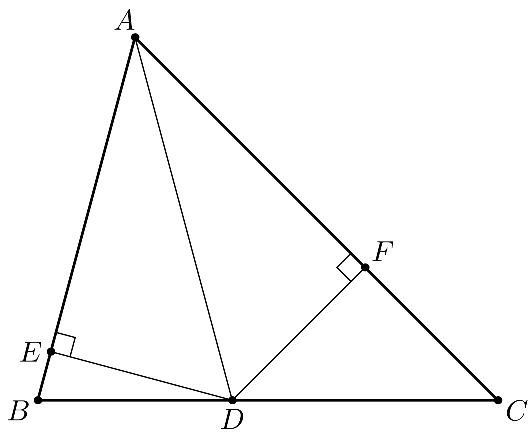
Now by Pythagoras's theorem,

$$PQ^2 = PR^2 - QR^2 = k^2QR^2 - QR^2 = (k^2-1)QR^2 \Longrightarrow \frac{PQ^2}{QR^2} = k^2 - 1.$$

Therefore, $PS = (k^2-1)RS$. □

Problem 6. Prove the following theorem:
If a ray from the vertex of an angle of a triangle divides the opposite side into segments that have the same ratio as the other two sides, then it bisects the angle.

Solution. Let the triangle be ABC. Let the ray originate at A and let it intersect side BC at D. Finally, let E and F be the foots of perpendiculars from D onto sides AB and AC.



The problem condition then gives us $BD : DC = BA : AC$. Since $\triangle ABD$ and $\triangle ACD$ share the same height, the ratio of their areas is the ratio of the bases. Therefore,

$$\frac{BD}{DC} = \frac{[ABD]}{[ACD]} = \frac{BA \cdot DE}{AC \cdot DF} \Longrightarrow DE = DF.$$

In right triangles ADE and ADF, we have $DE = DF$ and hypotenuse AD, so they are congruent. Thus $\angle BAD = \angle CAD$ as desired. □

Remark. This is commonly known as the angle bisector theorem.

Problem 7. The sum to k terms of an AP is 21. The sum to $2k$ terms is 78. The kth term is 11. Find the first term and the common difference.

Solution. Let u_1 and d be the first term and common difference of the AP. Then we have the following three equations:

$$\begin{aligned} 21 &= \frac{k(u_1+u_k)}{2}, \\ 78 &= k(u_1+u_{2k}), \\ 11 &= u_k. \end{aligned}$$

Notice that u_{2k} is obtained by adding d to u_k for k more times. Therefore, $u_{2k} = u_k + kd$. From the first equation, we have $k(u_1 + u_k) = 42$. Therefore, the second equation gives

$$78 = k(u_1 + u_k) + k^2d = 42 + k^2d \implies k^2d = 36.$$

We also have $u_k = u_1 + (k-1)d$, so $u_1 = u_k - (k-1)d$. Substituting this into the second equation gives

$$\begin{aligned} k(u_1 + u_{2k}) &= 78 \\ k(u_k - (k-1)d + u_k + kd) &= 78 \\ k(2u_k + d) &= 78 \\ 22k + kd &= 78 \\ 22k^2 + k^2d &= 78k \\ 22k^2 - 78k + 36 &= 0 \\ 11k^2 - 39k + 18 &= 0 \\ (11k-6)(k-3) &= 0. \end{aligned}$$

Since k is a positive integer, it follows that $k = 3$ and hence $d = 4$. Therefore,

$$u_1 = u_k - (k-1)d = 11 - 2 \cdot 4 = 3.$$ □

Problem 8. Prove that if a, b, c and d are positive, the equation $x^4 + bx^2 + cx - d = 0$ has one positive, one negative and two imaginary roots.

Solution. Let $f(x) = x^4 + bx^2 + cx - d$. This polynomial has exactly one sign change, so by Descartes' rule of signs, it has exactly one positive root. Now $f(-x) = x^4 + bx^2 - cx - d$ also has one sign change, so f must also have exactly one negative root. Since f is a polynomial of degree 4, by the fundamental theorem of algebra, it must have exactly 4 complex roots. Since we know that it only has 2 real roots, the rest two roots must be imaginary and we are done. □

Problem 9. Show that the sum of the squares of the first n odd numbers is $\frac{1}{3}n(4n^2-1)$.

Solution. This is the same problem as 2016 Regional Round problem 14. □

Problem 10. If 100! is divisible by 7^n, find the maximum value of n.

Solution. Since $14 \cdot 7 < 100 < 15 \cdot 7$, there are 14 numbers less than 100 which are divisible by 7. Out of these 14 numbers, there are 2 numbers, namely 49 and 98 which are divisible by 7^2. Therefore, the power of 7 in the prime factorization of 100! is $14 + 2 = 16$ and so $n = 16$. □

Remark. The argument in this problem can be generalized to get what is known as Legendre's formula.

Problem 11. Show that $x = 10^\circ$ is a solution of $2\sin x = \frac{1+\tan^2 x}{3-\tan^2 x}$.

Solution. We will first show the identity

$$\sin 3x = 3\sin x - 4\sin^3 x$$

for any real x. This can be done as follows:

$$\begin{aligned}\sin 3x &= \sin(2x + x)\\ &= \sin 2x \cos x + \cos 2x \sin x\\ &= 2\sin x \cos^2 x + (1 - 2\sin^2 x)\sin x\\ &= \sin x(2 - 2\sin^2 x + 1 - 2\sin^2 x)\\ &= 3\sin x - 4\sin^3 x\end{aligned}$$

In particular, when $x = 10^\circ$,

$$\begin{aligned}\frac{1}{2} &= 3\sin 10^\circ - 4\sin^3 10^\circ\\ \frac{1}{2} &= \sin 10^\circ(3 - 4\sin^2 10^\circ)\\ 2\sin 10^\circ &= \frac{1}{3 - 4\sin^2 10^\circ}\\ &= \frac{1}{4\cos^2 10^\circ - 1}\\ &= \frac{\sec^2 10^\circ}{4 - \sec^2 10^\circ}\\ &= \frac{1 + \tan^2 10^\circ}{3 - \tan^2 10^\circ}\end{aligned}$$

so $x = 10^\circ$ is a solution to the given equation. □

Problem 12. Show that $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$ for all $n > 1$.

Solution. For $1 \le i < n$, we have $\frac{1}{\sqrt{i}} > \frac{1}{\sqrt{n}}$. Therefore,

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \cdots + \frac{1}{\sqrt{n}} = \frac{n}{\sqrt{n}} = \sqrt{n}.$$ □

Problem 13.

1. Prove the following Cauchy inequality:

   For any real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$,

   $$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \leq (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2).$$

2. For a set of positive real numbers $x_1, \ldots, x_n$, the Root-Mean Square RMS is defined by the formula

   $$\text{RMS} = \sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}}$$

   and the Arithmetic Mean AM is defined by the formula

   $$\text{AM} = \frac{x_1 + \cdots + x_n}{n}.$$

   Prove that RMS $\geq$ AM.

Solution. Consider the following quadratic polynomial in x:

$$P(x) = (a_1x + b_1)^2 + (a_2x + b_2)^2 + \cdots + (a_nx + b_n)^2 \geq 0.$$

Letting $A = a_1^2 + \cdots + a_n^2$, $B = 2(a_1b_1 + \cdots + a_nb_n)$ and $C = b_1^2 + \cdots + b_n^2$, the quadratic can be rewritten as

$$P(x) = Ax^2 + Bx + C \geq 0.$$

Since this quadratic is non-negative, it can only have at most one real root (i.e. the parabola cannot intersect the x axis at two distinct points.) Therefore, its discriminant is less than or equal to zero; which yields the desired inequality.

$$B^2 - 4AC \leq 0 \Longrightarrow (a_1b_1 + \cdots + a_nb_n)^2 \leq (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2).$$

Now letting $a_i = x_i$ and $b_i = 1$ for all $1 \leq i \leq n$, by the Cauchy-Schwarz inequality,

$$(x_1 + x_2 + \cdots + x_n)^2 \leq (x_1^2 + x_2^2 + \cdots + x_n^2)n.$$

Rearranging the equation gives

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \leq \sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}}.$$ □

Problem 14.

1. Prove that if p is a prime number, then the coefficient of every term in the expansion of $(a + b)^p$ except the first and last is divisible by p.

2. Hence show that if p is a prime number and N is a positive integer, then $N^p - N$ is a multiple of p.

3. Hence also show that if p is a prime number, then $10^p - 7^p - 3$ is divisible by p.

Solution. This problem is a step-by-step proof of Fermat's little theorem.

1. By the binomial theorem,

$$(a+b)^p = a^p + \binom{p}{1}a^{p-1}b + \cdots + \binom{p}{p-1}ab^{p-1} + b^p.$$

For $1 \leq r \leq p-1$,

$$\binom{p}{r} = \frac{p(p-1)\cdots(p-r+1)}{r!}.$$

Since $r < p$, there are no factors of p in the denominator. However, p obviously divides the numerator. Since $\binom{p}{r}$ is an integer, during the cancellation, there was no number in the denominator that could cancel out the p in the denominator. Therefore, p divides $\binom{p}{r}$.

2. We will show that $n^p - n$ is divisible by p, by induction on n. For the base case, when $n = 1$, this is equal to 0 so it is divisible by p. Now suppose that for $n = k$, $k^p - k$ is divisible by p. Then when $n = k+1$,

$$(k+1)^p - (k+1) = (k^p - k) + \left(\binom{p}{1}k^{p-1} + \binom{p}{2}k^{p-2} + \cdots + \binom{p}{p-1}k\right).$$

Since p divides the first bracket by the induction hypothesis and the second bracket by the first part, it follows that p also divides $(k+1)^p - (k+1)$. Therefore, by induction it follows that p divides $n^p - n$ for all positive integers n.

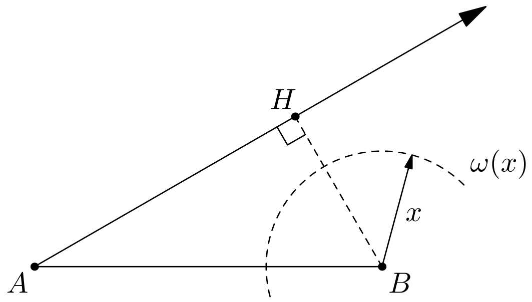
3. Notice that $10^p - 7^p - 3 = (10^p - 10) - (7^p - 7)$. Since both brackets are divisible by p, we are done. □

2018 National Round

Problem 1. To construct a triangle ABC, only given that $AB = 10$ and $\angle ABC$ is $30°$. Find all values of AC for which

1. there are two possible triangles ABC.
2. there is only one triangle ABC.
3. there is no triangle ABC.

Solution. Let ℓ be the ray originating from B which is inclined to segment AB at a $30°$ angle. Obviously C must lie on this ray. Now let H be the foot of perpendicular from A onto ℓ. Let $x = AC$ be the length of segment AC, and let $\omega(x)$ be the circle centered at A with radius x. It is easy to see that $AH = AB \sin 30° = 5$.



Case 1: $x < AH$

In this case, $\omega(x)$ does not intersect ℓ at all, so there is no triangle ABC.

Case 2: $x = AH$

In this case, $\omega(x)$ is tangent to ℓ at H. Therefore, there is only one place C can be, namely H and hence there is only one triangle ABC.

Case 3: $AH < x < AB$

In this case, $\omega(x)$ intersects the ray ℓ at two points. Therefore, C can be at any of those two points, which gives two possible triangles ABC.

Case 4: $AB \leq x$

Finally in this case, $\omega(x)$ intersects ray ℓ at only one point, so similarly to case 2 there is only one triangle ABC. Therefore, the final answers are

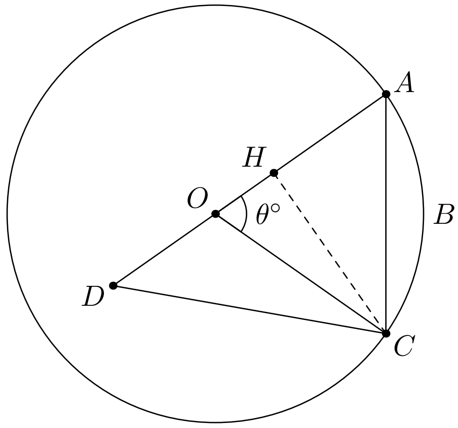
1. $\{x \in \mathbb{R} \mid 5 < x < 10\}$.
2. $\{x \in \mathbb{R} \mid x = 5 \text{ or } x \geq 10\}$.
3. $\{x \in \mathbb{R} \mid 0 < x < 5\}$. □

Problem 2. In the given figure, O is the center of the circle. OA and OC are radii with $OA = OC = 5$ units. Find the area of region $ABCD$, bounded by arc ABC, line segments CD and DA, in terms of θ, if the line segment $DA = 8$ units and $\angle AOC$ is $\theta°$.

Solution. First, let's calculate the area of sector AOC. It subtends an angle of θ degrees at the center, so its area A_1 is

$$A_1 = \frac{\theta}{360} \cdot \pi(5)^2 = \frac{5\pi\theta}{72}.$$

Therefore we just need to calculate the area of $\triangle DOC$. The length of OD is $8 - 5 = 3$ units.



Let H be the foot of perpendicular from C onto line AD. Then the area A_2 of $\triangle DOC$ is

$$A_2 = \frac{CH \cdot OD}{2} = \frac{CO \cdot \sin\angle HOC \cdot OD}{2} = \frac{CO \cdot \sin\angle AOC \cdot OD}{2} = \frac{5 \cdot \sin\theta \cdot 3}{2} = \frac{15\sin\theta}{2}.$$

Therefore, the total area of the region is

$$A_1 + A_2 = \frac{5\pi\theta}{72} + \frac{15\sin\theta}{2}. \qquad \square$$

Problem 3. Prove that

1. $\cos 20°$,
2. $\log 21$, and
3. $\sqrt{3}$

are irrational.

Solution. We will use proof by contradiction in all the following subproblems.

1. Suppose in contrary that $\cos 20°$ is rational. Then $\cos 20° = \frac{a}{b}$ for some relatively prime integers a and b. Then since $\cos(3x) = 4\cos^3 x - 3\cos x$, we see that

$$\frac{1}{2} = \cos 60° = 4\cos^3 20° - 3\cos 20° = \frac{4a^3}{b^3} - \frac{3a}{b}.$$

Multiplying both sides of the equation by $2b^3$ gives $b^3 = 8a^3 - 6ab^2$, and since a divides the right hand side it follows that $a \mid b^3$. However, since we assumed that a and b are relatively prime, this is a contradiction. Therefore, $\cos 20°$ must be irrational.

2. Similarly, assume that $\log 21$ is rational. Then $\log 21 = \frac{a}{b}$ for some relatively prime integers a and b. Hence

$$\log 21 = \frac{a}{b} \implies 21 = 10^{\frac{a}{b}} \implies 21^b = 10^a.$$

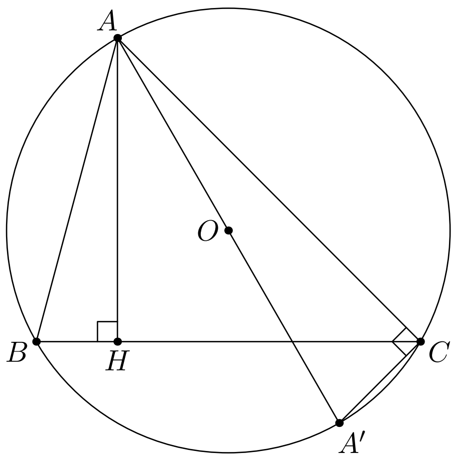
However, the left hand side is even and the right hand side is odd which is clearly not possible unless $a = b = 0$. Therefore, $\log 21$ must be irrational.

3. Finally, suppose that $\sqrt{3}$ is irrational. Then $\sqrt{3} = \frac{a}{b}$ for some relatively prime integers a and b. Then
$$\sqrt{3} = \frac{a}{b} \implies 3b^2 = a^2.$$
Since 3 divides the left hand side, $3 \mid a^2 \Rightarrow 3 \mid a$. Therefore, $a = 3k$ for some k. Substituting gives
$$3b^2 = 9k^2 \implies 3k^2 = b^2.$$
Similarly to above, this means that 3 also divides b. However, since we assumed that a and b are relatively prime, this is not possible. Hence $\sqrt{3}$ is irrational. □

Problem 4. Triangle ABC with $AB = c$, $BC = a$ and $CA = b$ is inscribed in a circle. Find the radius of the circle in terms of a, b and c.

Solution. Let H be the foot of perpendicular from A onto side BC, and let A' be the diametrically opposide point of A on circle (ABC). Then $\angle AHB = \angle ACA' = 90°$. We also have $\angle ABH = \angle ABC = \angle AA'C$ so $\triangle ABH \sim \triangle AA'C$. Let S be the area of $\triangle ABC$ and R be the radius of (ABC). Then

$$2S = AH \cdot BC = \frac{AB \cdot AC \cdot BC}{AA'} = \frac{abc}{2R}.$$



Meanwhile, by Heron's formula,

$$S = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = \frac{a+b+c}{2}$ is the semiperimeter of $\triangle ABC$. Therefore,

$$R = \frac{abc}{4S} = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}.$$ □

Problem 5. Prove the following Ptolemy's theorem:
In a cyclic quadrilateral $PQRS$,

$$PQ \cdot SR + PS \cdot QR = PR \cdot SQ,$$

That is, the sum of the products of the opposite sides is equal to the product of the diagonals.

Solution. This is Ptolemy's theorem which has appeared before as 2015 National Round P18. □

Problem 6. $f(x)$ is a real-valued function defined on $a < x < b$ such that

$$f\left(\frac{x_1+x_2}{2}\right) \leq \frac{1}{2}(f(x_1)+f(x_2)),$$

for all $a < x_1, x_2 < b$. Prove that

$$f\left(\frac{x_1+x_2+\cdots+x_n}{n}\right) \leq \frac{1}{n}(f(x_1)+f(x_2)+\cdots+f(x_n))$$

for all $a < x_1, x_2, \ldots, x_n < b$.

Solution. This is one of the rare problems which need forward-backward induction to be solved. The base case $n = 2$ is true. Now suppose that the inequality is true for $n = k$. We must first show that it is true for $n = 2k$:

$$\begin{aligned}
&f\left(\frac{x_1+x_2+\cdots+x_k+x_{k+1}+\cdots+x_{2k}}{2k}\right)\\
&\quad= f\left(\frac{1}{2}\left(\frac{x_1+x_2+\cdots+x_k}{k}+\frac{x_{k+1}+x_{k+2}+\cdots+x_{2k}}{k}\right)\right)\\
&\quad\leq \frac{1}{2}\left(f\left(\frac{x_1+x_2+\cdots+x_k}{k}\right)+f\left(\frac{x_{k+1}+x_{k+2}+\cdots+x_{2k}}{k}\right)\right)\\
&\quad\leq \frac{1}{2}\left(\frac{1}{k}\left(f(x_1)+f(x_2)+\cdots+f(x_k)\right)+\frac{1}{k}\left(f(x_{k+1})+f(x_{k+2})+\cdots+f(x_{2k})\right)\right)\\
&\quad= \frac{f(x_1)+f(x_2)+\cdots+f(x_{2k})}{2k}
\end{aligned}$$

where the third inequality is because of the base case and the fourth is because of the induction hypothesis. Hence the inequality is true for $n = 2k$. Now observe that

$$f\left(\frac{1}{k}\left(x_1+x_2+\cdots+x_{k-1}+\frac{x_1+x_2+\cdots+x_{k-1}}{k-1}\right)\right) = f\left(\frac{x_1+x_2+\cdots+x_{k-1}}{k-1}\right)$$

Since the inequality is true for any k real numbers between a and b, it is also true when $x_k = \frac{x_1+x_2+\cdots+x_{k-1}}{k-1}$. As $a < x_1, x_2, \ldots, x_{k-1} < b$, it is guaranteed that x_k lies between a and b as well. Therefore, we have

$$\begin{aligned}
f\left(\frac{x_1+x_2+\cdots+x_{k-1}}{k-1}\right) &\leq \frac{1}{k}\left(f(x_1)+f(x_2)+\cdots+f\left(\frac{x_1+x_2+\cdots+x_{k-1}}{k-1}\right)\right)\\
(k-1)f\left(\frac{x_1+x_2+\cdots+x_{k-1}}{k-1}\right) &\leq f(x_1)+f(x_2)+\cdots+f(x_{k-1})\\
f\left(\frac{x_1+x_2+\cdots+x_{k-1}}{k-1}\right) &\leq \frac{f(x_1)+f(x_2)+\cdots+f(x_{k-1})}{k-1}.
\end{aligned}$$

Therefore, we have proved that this inequality is also true for $n = k - 1$, and we are done by forward-backward induction. □

Problem 7. Point A has position vector $\overrightarrow{OA}$ and point B has position vector $\overrightarrow{OB}$. Prove that for $0 \le \lambda \le 1$, the vector $\lambda\overrightarrow{OA} + (1-\lambda)\overrightarrow{OB}$ is a position vector of a point on the line segment AB. Prove also that any point on the line segment AB has position vector $\lambda\overrightarrow{OA} + (1-\lambda)\overrightarrow{OB}$ for some $0 \le \lambda \le 1$.

Solution. Let P be a point such that P has position vector $\lambda\overrightarrow{OA}+(1-\lambda)\overrightarrow{OB}$. This means that

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \lambda\overrightarrow{OA} + (1-\lambda)\overrightarrow{OB} - \overrightarrow{OA} = (1-\lambda)(\overrightarrow{OB} - \overrightarrow{OA}) = (1-\lambda)\overrightarrow{AB},$$

which shows that A, P and B are collinear. Also, as $AP = (1-\lambda)AB$, it follows that $AP \le AB$ and hence P lies on segment AB.

Now suppose that P is a point on segment AB. We will show that this point P has the position vector $\lambda\overrightarrow{OA} + (1-\lambda)\overrightarrow{OB}$ for some $0 \le \lambda \le 1$. Since $\overrightarrow{BP}$ and $\overrightarrow{BA}$ have the same direction, we must have $\overrightarrow{BP} = \lambda\overrightarrow{BA}$ for some positive constant λ. As P lies on segment AB, we also have $0 \le \lambda \le 1$. Therefore,

$$\overrightarrow{OP} = \overrightarrow{OB} + \overrightarrow{BP} = \overrightarrow{OB} + \lambda\overrightarrow{BA} = \overrightarrow{OB} + \lambda\overrightarrow{OA} - \lambda\overrightarrow{OB} = \lambda\overrightarrow{OA} + (1-\lambda)\overrightarrow{OB}. \qquad \square$$

Problem 8. $f(x)$ is a real-valued function defined on the interval $a < x < b$ such that

$$f(tx_1 + (1-t)x_2) \le tf(x_1) + (1-t)f(x_2)$$

for all $a < x_1, x_2 < b$ and for all $0 \le t \le 1$. Prove that for each triple x_1, x_2, x_3 of distinct numbers in the interval,

$$\frac{(x_3 - x_2)f(x_1) + (x_2 - x_1)f(x_3) + (x_1 - x_3)f(x_2)}{(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)} \ge 0.$$

Solution. WLOG, we can assume that $x_1 < x_2 < x_3$. Then $(x_1 - x_2)$ and $(x_2 - x_3)$ are negative while $(x_3 - x_1)$ is positive, so $(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)$ must be positive. Therefore, we just need to show the numerator is also positive, i.e.,

$$(x_3 - x_2)f(x_1) + (x_2 - x_1)f(x_3) + (x_1 - x_3)f(x_2) \ge 0.$$

Let $P_1 = (x_1, f(x_1))$ and $P_3 = (x_3, f(x_3))$ be points in the plane. Since x_2 lies between x_1 and x_3, by the above problem we must have

$$x_2 = tx_1 + (1-t)x_3$$

for some real number $0 < t < 1$. Now define P_2 to be the point which has position vector

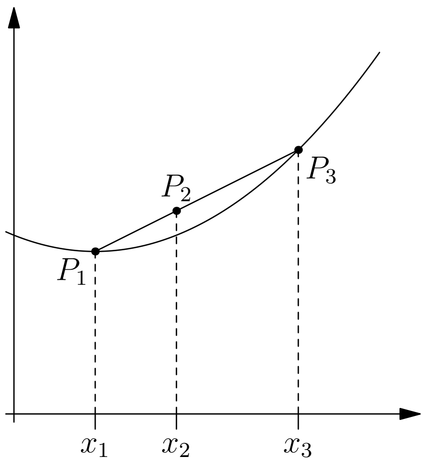
$$\vec{P_2} = t\vec{P_1} + (1-t)\vec{P_3}.$$

From the above problem, P_2 lies on segment P_1P_2. The coordinates of this point must be $(x_2, y_2) = (tx_1 + (1-t)x_3, tf(x_1) + (1-t)f(x_3))$. From the given condition, we have

$$f(x_2) = f(tx_1 + (1-t)x_3) \le tf(x_1) + (1-t)f(x_3) = y_2.$$

Therefore,

$$\begin{aligned}
&(x_3 - x_2)f(x_1) + (x_2 - x_1)f(x_3) + (x_1 - x_3)f(x_2) \\
&\quad \ge (x_3 - x_2)f(x_1) + (x_2 - x_1)f(x_3) + (x_1 - x_3)y_2 \\
&\quad = (x_3 - x_2)f(x_1) + (x_2 - x_1)f(x_3) - ((x_3 - x_2) + (x_2 - x_1))y_2 \\
&\quad = (x_3 - x_2)(f(x_1) - y_2) + (x_2 - x_1)(f(x_3) - y_2).
\end{aligned}$$



However, since P_1, P_2 and P_3 are collinear, lines P_1P_2 and P_2P_3 have the same slope. Therefore,

$$\frac{y_2 - f(x_1)}{x_2 - x_1} = \frac{f(x_3) - y_2}{x_3 - x_2} \Longrightarrow (x_3 - x_2)(f(x_1) - y_2) + (x_2 - x_1)(f(x_3) - y_2) = 0.$$

and we are done. □

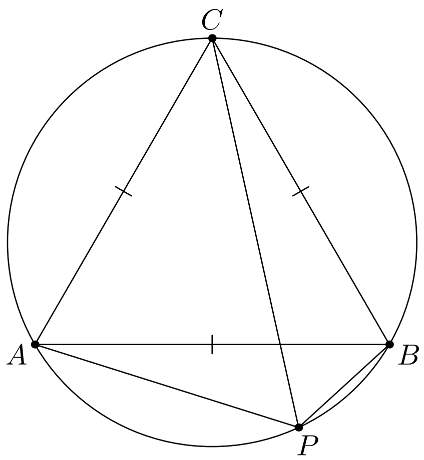
Remark. The reason for constructing the point P_2 is due to the following observation. Let A_1 be the area of the rectangle with base length $x_3 - x_2$ and height $f(x_1)$. Similarly define A_2 and A_3 with base lengths $x_3 - x_1$ and $x_2 - x_1$. Then the complicated inequality is equivalent to much simpler $A_1 + A_3 \geq A_2$. However, it can be shown that the sum on the left is the same as the area of a rectangle with length $x_3 - x_1$, and height equal to that of P_2. Therefore, the inequality amounts to showing that the height of P_2 is greater than or equal to $f(x_2)$ which follows readily from the given convexity condition. The above proof is just an algebraic translation of this geometric argument.

Problem 9. Equilateral triangle ABC is inscribed in a circle. P is a point on arc AB. Prove that $PA + PB = PC$.

Solution 1. Let p be the side length of the equilateral triangle ABC. Then by Ptolemy's theorem on quadrilateral $ACBP$,

$$PA \cdot p + PB \cdot p = PC \cdot p \Longrightarrow PA + PB = PC.$$

□



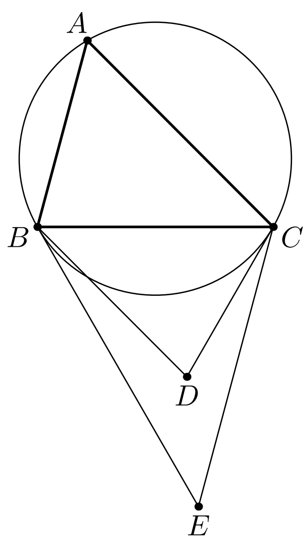
Solution 2. Let segment PC and segment AB intersect at E. Then $\angle PCA = \angle PBA = \angle PBE$, and $\angle CPA = \angle CBA = \angle CAB = \angle CPB = \angle BPE$. Therefore, $\triangle PCA \sim \triangle PBE$. Similarly, $\triangle PCB \sim \triangle PAE$. Consequently,

$$\frac{PA}{PC} + \frac{PB}{PC} = \frac{AE}{BC} + \frac{EB}{AC} = \frac{AE + EB}{AB} = \frac{AB}{AB} = 1.$$

Rearranging gives our desired identity. $\square$

Problem 10. $\triangle ABC$ is inscribed in a circle. The tangent at C and the line through B parallel to AC meet at D. The tangent at B and the line through C parallel to AB meet at E. Prove that $BC^2 = BE \cdot CD$.

Solution. Since $AB \parallel CE$ and BE is tangent to (ABC), we have $\angle ABC = \angle BCE$ and $\angle CAB = \angle EBC$, so $\triangle ABC \sim \triangle BCE$. Analogously, as $AC \parallel BD$ and CD is tangent to (ABC), we have $\angle BCA = \angle DBC$ and $\angle CAB = \angle BCD$. Hence $\triangle BCE \sim \triangle ABC \sim \triangle CDB$.



This shows that

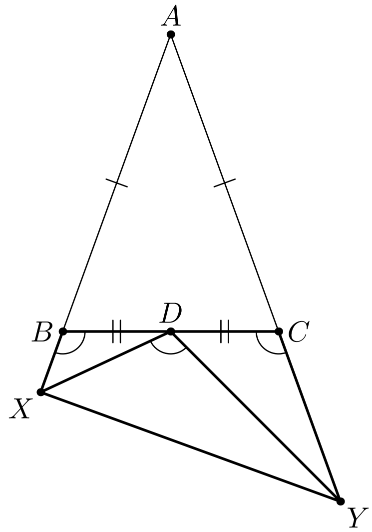
$$\frac{BC}{CD} = \frac{EB}{BC} \Longrightarrow BC^2 = BE \cdot CD.$$ $\square$

Problem 11. $\triangle ABC$ is isosceles with $AB = AC$. D is the midpoint of BC. AB and AC are produced to X and Y respectively such that $\angle XDY = \angle DCY$. Prove that $\triangle XBD$, $\triangle XDY$ and $\triangle DCY$ are similar triangles.

Solution. Since $AB = AC$, we must have $\angle XBD = \angle DCY = \angle XDY$. Also,

$$\angle BDX = 180^\circ - \angle XDY - \angle YDC = 180^\circ - \angle DCY - \angle YDC = \angle CYD,$$

so it follows that $\triangle XBD \sim \triangle DCY$. This means that $\frac{XD}{DY} = \frac{XB}{DC} = \frac{XB}{BD}$. As $\angle XDY = \angle XBD$, we also see that $\triangle XBD \sim \triangle XDY$ as desired. $\square$



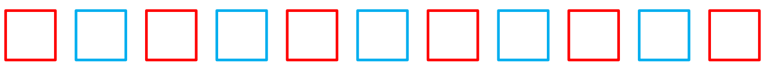
> **Problem 12.** a_1, b_1, c_1, a_2, b_2, c_2 are positive real numbers with $a_1c_1 - b_1^2 \geq 0$ and $a_2c_2 - b_2^2 \geq 0$. Show that $(a_1 + a_2)(c_1 + c_2) - (b_1 + b_2)^2 \geq 0$.

Solution. From the given conditions we have $a_1c_1 \geq b_1^2$ and $a_2c_2 \geq b_2^2$. By Cauchy-Schwarz inequality,

$$(a_1 + a_2)(c_1 + c_2) = ((\sqrt{a_1})^2 + (\sqrt{a_2})^2)((\sqrt{c_1})^2 + (\sqrt{c_2})^2) \geq (\sqrt{a_1c_1} + \sqrt{a_2c_2})^2 = (b_1 + b_2)^2. \quad \square$$

> **Problem 13.** How many permutations of the word 'TRIANGLE' have none of the vowels together?

Solution. First notice that there are 5 consonants and 3 vowels. Consider the following template; blue boxes are for consonants and red boxes are for vowels.



Since no two vowels can be together, this means that there must be at most 1 vowel in each red box. There are 6 red boxes, so the number of ways to arrange 3 vowels in 6 red boxes is $6 \times 5 \times 4 = 120$. Now the number of ways to arrange the 5 consonants in 5 blue boxes is $5! = 120$. Therefore, the total number of words is $120^2 = 14400$. $\square$

> **Problem 14.** From the group of $2n + 1$ people, how many ways to choose a group of n people or less?

Solution 1. Label the people from 1 to $2n + 1$, and let S be the set $\{1, 2, \ldots, 2n + 1\}$. Let A be the set of subsets of S which have n elements or less, and let B be the set of subsets of S which has $n + 1$ or more elements. It is easy to see that for each element X in A, we can pair it up with $S \setminus X$, which belongs in B. This means that A and B have the same number of elements. Meanwhile, the number of subsets of a set of $2n + 1$ elements is 2^{2n+1}, so

$$|A| + |B| = 2^{2n+1} \implies |A| = 2^{2n} = 4^n.$$

One thing to be careful is that we also included the empty set in A. Subtracting it from the total, we see that the number of ways to choose a group of n people or less from a group of $2n + 1$ people is $4^n - 1$. $\square$

Solution 2. The number of ways to choose a group of i people from a group of $2n+1$ people is $\binom{2n+1}{i}$. Therefore, the quantity we seek is

$$N = \binom{2n+1}{1} + \binom{2n+1}{2} + \cdots + \binom{2n+1}{n}$$

where N is the total number of ways. Now by the binomial theorem, we have

$$2^{2n+1} = (1+1)^{2n+1} = \binom{2n+1}{0} + \binom{2n+1}{1} + \cdots + \binom{2n+1}{2n+1}.$$

However, remember that $\binom{2n+1}{i} = \binom{2n+1}{2n+1-i}$ for $0 \leq i \leq 2n+1$. Therefore, the quantity on the right hand side can be rewritten as

$$\begin{aligned}
&\binom{2n+1}{0} + \binom{2n+1}{1} + \cdots + \binom{2n+1}{n} + \binom{2n+1}{n+1} + \cdots + \binom{2n+1}{2n+1} \\
&= 2\left(\binom{2n+1}{0} + \binom{2n+1}{1} + \cdots + \binom{2n+1}{n}\right) \\
&= 2\left(N + \binom{2n+1}{0}\right) \\
&= 2N + 2.
\end{aligned}$$

Therefore, we finally have

$$N = \frac{2^{2n+1} - 2}{2} = 4^n - 1.$$

□

Problem 15. Show that the product of any positive integer and its $k-1$ successors is divisible by $k!$.

Solution. Let the k consecutive integers be $n+1, n+2, \ldots, n+k$. Then

$$N = \frac{(n+1)(n+2)\cdots(n+k)}{k!} = \frac{n!(n+1)(n+2)\cdots(n+k)}{n!k!} = \frac{(n+k)!}{n!(n+k-n)!} = \binom{n+k}{n}.$$

On the other hand, the right hand side is the number of ways to choose n objects from $n+k$ objects, so this is clearly an integer. Hence N is an integer and we are done. □

Problem 16. How many integers between 1 and 1,000,000 have the sum of digits equal to 10?

Solution. Note that this number is just the number of 6-tuples $(x_1, x_2, \ldots, x_6)$ such that $x_1 + x_2 + \cdots + x_6 = 10$, with the restriction that $0 \leq x_i \leq 9$ for all $1 \leq i \leq 6$ since they are supposed to represent digits. If we remove this restriction, then by stars and bars, the number of such tuples is $\binom{15}{5} = 72072$. Since such a tuple already satisfies $0 \leq x_i \leq 10$ for all $1 \leq i \leq 6$, the only tuples that violate the restriction are $(10, 0, 0, 0, 0, 0)$ and its variants. There are 6 of them, so the total number of integers between 1 and 1,000,000 with digit sum equal to 10 is $72072 - 6 = 72066$. □

Problem 17. Each side and diagonal of a regular hexagon is coloured either red or blue. Show that there is a triangle with all three sides of the same colour.

Solution. Take any vertex v of the hexagon. Since there are 5 edges connected to v, by the pigeonhole principle, there are at least three edges of the same colour. WLOG, suppose that these three edges are blue, and suppose that they connect v to three other vertices x, y and z. Then if one of the edges (x, y), (y, z) or (z, x) are blue, we will have a completely blue triangle, so suppose that none of them are blue. This means that those three edges form a completely red triangle, and hence we have proved that there is always a triangle with all three sides of the same colour. $\square$

Problem 18. Find the value of $2018^2 - 2017^2 + 2016^2 - 2015^2 + \cdots + 2^2 - 1$.

Solution. Differences of squares!

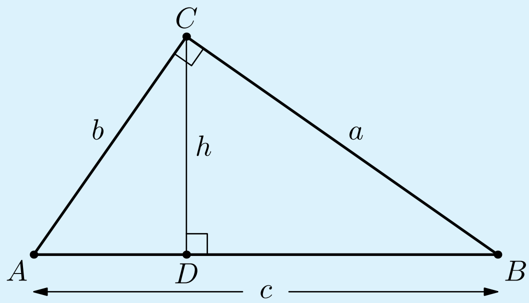
$$\begin{aligned}
&2018^2 - 2017^2 + 2016^2 - 2015^2 + \cdots + 2^2 - 1 \\
&\qquad = (2018 + 2017)(2018 - 2017) + (2016 + 2015)(2016 - 2015) + \cdots + (2 + 1)(2 - 1) \\
&\qquad = 2018 + 2017 + 2016 + 2015 + \cdots + 2 + 1 \\
&\qquad = \frac{2018 \cdot 2019}{2} \\
&\qquad = 2037171.
\end{aligned}$$

$\square$

2019 Regional Round

Problem 1.

1. Show that $x^2 + y^2 \geq 2xy$ for any two real numbers x and y, and that the sign of the equality holds if and only if $x = y$.
2. By using similar triangles, prove that $h = \frac{ab}{c}$ for the given figure.



3. By using (1) and (2), show that $h \leq \frac{c}{2}$, and that the sign of equality only holds if and only if $a = b$.
4. Show that of all right triangles having the same length of hypotenuse, the isosceles right triangle maximizes the area.

Solution. Since the square of a real number is always non-negative,

$$(x-y)^2 \geq 0 \Longrightarrow x^2 + y^2 \geq 2xy,$$

and the equality holds if and only if $x - y = 0 \Longleftrightarrow x = y$.

In $\triangle BDC$ and $\triangle BCA$, we have $\angle BDC = \angle BCA = 90°$ and $\angle B$ as a common angle, so these two triangles are similar. Therefore,

$$\frac{BC}{CD} = \frac{BA}{AC} \Longrightarrow \frac{a}{h} = \frac{c}{b} \Longrightarrow h = \frac{ab}{c}.$$

Now by the first part and the Pythagoras's theorem,

$$h = \frac{1}{c} \cdot ab \leq \frac{1}{c} \cdot \frac{a^2 + b^2}{2} = \frac{c^2}{2c} = \frac{c}{2}.$$

Finally, as the equality holds if and only if $a = b$, we see that the h is maximum when the right triangle is isosceles. Since the area of $\triangle ABC$ is $hc/2$ and c is constant, it is maximized when h is maximized, so we're done. □

Problem 2. Find all functions over the reals such that $f(x) + 2f(1-x) = x(1-x)$.

Solution. Substituing x with $1 - x$ in the equation, we have

$$f(1-x) + 2f(x) = (1-x)x.$$

Multiplying this equation by 2 and subtracting the original equation gives us

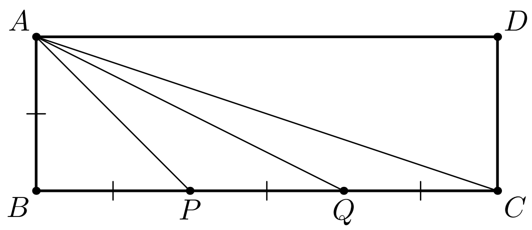
$$3f(x) = x(1-x) \implies f(x) = \frac{x(1-x)}{3}.$$

It is easy to check that this function satisfies the given equation, so $f(x) = \frac{x(1-x)}{3}$ is the only solution to the given equation. $\square$

Problem 3. In the given figure, $ABCD$ is a rectangle. P and Q are points on BC such that $AB = BP = PQ = QC$. Find two similar but not congruent triangles and prove their similarity.

Solution. Let $x = AB = BP = PQ = QC$. We claim that $\triangle PAQ \sim \triangle PCA$. First by Pythagoras's theorem, we have

$$PA^2 = 2x^2 = x(2x) = PQ \cdot PC \implies \frac{PA}{PQ} = \frac{PC}{PA}.$$



We also have $\angle APQ = \angle CPA$, so $\triangle PAQ \sim \triangle PCA$. $\square$

Problem 4. Let $f(x) = ax^2 + bx + c$, $a \neq 0$.

1. Fill in the following steps ($\square$) to show that $f(x) = a(x + \frac{b}{2a})^2 - \frac{b^2-4ac}{2a}$.

$$\begin{aligned} f(x) &= a(x^2 + \square x) + c \\ &= a\left(x^2 + \square x + \square - \left(\frac{b}{2a}\right)^2\right) + c \\ &= a(x^2 + \square x + \square) - \frac{b^2}{\square} + c \\ &= a(x + \square)^2 - \frac{b^2}{\square} + c \\ &= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a} \end{aligned}$$

2. Hence show that if $a > 0$ and $b^2 - 4ac < 0$, then $f(x) > 0$ for all $x \in \mathbb{R}$.
3. Show that if $a < 0$ and $b^2 - 4ac < 0$, then $f(x) < 0$ for all $x \in \mathbb{R}$.
4. If $b^2 - 4ac > 0$, find the roots of the equation $f(x) = 0$ in terms of a, b and c.

Solution. This is just completing the square.

$$\begin{aligned}
f(x) &= a\left(x^2 + \frac{b}{a}\cdot x\right) + c \\
&= a\left(x^2 + \frac{b}{a}\cdot x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + c \\
&= a\left(x^2 + \frac{b}{a}\cdot x + \left(\frac{b}{2a}\right)^2\right) - \frac{b^2}{4a} + c \\
&= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c \\
&= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a}.
\end{aligned}$$

If $a > 0$ and $b^2 - 4ac < 0$, we must have $\frac{b^2-4ac}{4a} < 0$. Since squares are non-negative,

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a} \geq -\frac{b^2-4ac}{4a} > 0.$$

Now if $a < 0$ and $b^2 - 4ac < 0$, we must have $\frac{b^2-4ac}{4a} > 0$. Therefore,

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a} \leq -\frac{b^2-4ac}{4a} < 0.$$

Finally, let's find the roots of $f(x) = 0$.

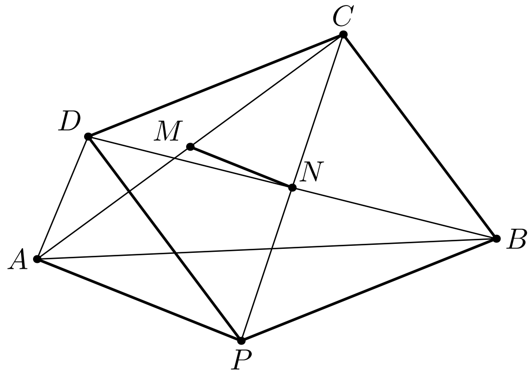
$$\begin{aligned}
& & f(x) &= 0 \\
\iff & & a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a} &= 0 \\
\iff & & \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2-4ac}{4a^2} \\
\iff & & x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2-4ac}}{2a} \\
\iff & & x = &= \frac{-b \pm \sqrt{b^2-4ac}}{2a}.
\end{aligned}$$

Since $b^2 - 4ac > 0$, the square root will evaluate to a positive real number, and we will have two distinct real roots. □

Problem 5. In the given figure, $DC = PB$ and $DC \parallel PB$. M and N are midpoints of AC and BD respectively.

1. Prove that the points P, N, C are collinear.
2. Prove that $AP \parallel MN$ and $AP = 2MN$.

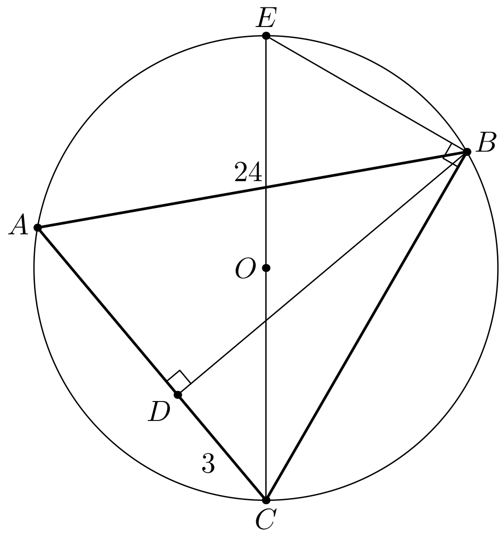
Solution. As $DC = PB$ and $DC \parallel PB$, it follows that $DCBP$ is a parallelogram. Therefore, the diagonals bisect each other, and hence the midpoint N of segment BD must also be the midpoint of segment PC, which means that P, N, C are collinear.



Now M is also the midpoint of AC, so $MN \parallel AP$ and

$$\frac{MN}{AP} = \frac{CN}{CP} = \frac{1}{2} \implies AP = 2MN. \qquad \square$$

Problem 6. In the given figure, $\angle BAC = 60°$, $AB = 24$ cm, $BD \perp AC$ and $DC = 3$ cm. Find the diameter of the circle.



Solution. Let O be the center of the circle and let line CO meet the circle again at E. Since $\triangle BAD$ is a 30-60 right triangle, we have $AD = 12$. Therefore, $AC = AD + DC = 15$. Now by law of cosines in $\triangle ABC$,

$$BC^2 = AC^2 + AB^2 - 2AB \cdot AC \cdot \cos 60° = 15^2 + 24^2 - 15 \cdot 24 = 441 \implies BC = 21.$$

Now notice that $\angle CEB = \angle CAB = 60°$ and $\angle CBE = 90°$. Therefore,

$$CE = \frac{BC}{\sin 60°} = \frac{42}{\sqrt{3}} = 14\sqrt{3},$$

and hence the diameter of the circle is $14\sqrt{3}$ cm. $\square$

Problem 7. In an AP, the kth term is 11. The sum of the first k terms is 26. The sum of the next k terms is 74. Find the first term and the common difference.

Solution. Let u_1 and d be the first term and common difference of the sequence. It is easy to see that the sum to $2k$ terms is $26+74=100$. Therefore, we have the following three equations:

$$\begin{aligned} 26 &= \frac{k(u_1+u_k)}{2}, \\ 100 &= k(u_1+u_{2k}), \\ 11 &= u_k. \end{aligned}$$

Since u_{2k} is obtained by adding d to u_k for k more times, $u_{2k}=u_k+kd$. From the first equation, we also have $k(u_1+u_k)=52$. Substituting all of this into the second equation gives

$$100 = k(u_1+u_{2k}) = k(u_1+u_k)+k^2d = 52+k^2d \implies k^2d = 48.$$

Now from the second equation, $u_k = u_1+(k-1)d$, so $u_1 = u_k-(k-1)d$. Substituting again into the second equation,

$$\begin{aligned} 100 &= k(u_k-(k-1)d+u_k+kd) \\ &= k(2u_k+d) \\ &= 22k+kd \\ 100k &= 22k^2+k^2d \\ 100k &= 22k^2+48 \\ 22k^2-100k+48 &= 0 \\ 11k^2-50k+24 &= 0 \\ (11k-6)(k-4) &= 0 \end{aligned}$$

Since k is a positive integer, k must be 4 and so $d=3$. Finally,

$$u_1 = u_k-(k-1)d = 11-3\cdot 3 = 2.$$ □

Remark. This problem is basically the same as 2018 Regional Round Problem 7.

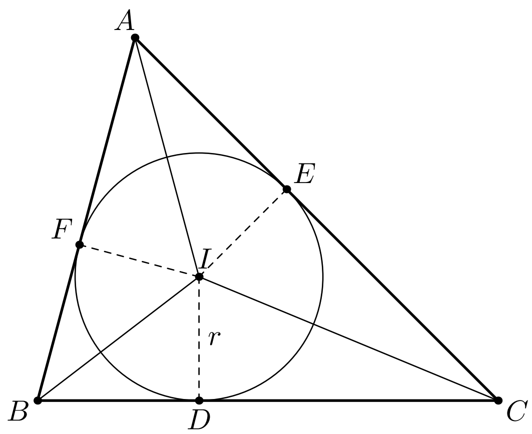
Problem 8. What is the radius of the inscribed circle of a 3-4-5 right triangle?

Solution. We will first derive a formula to find the inradius of a general triangle. Suppose that we have $\triangle ABC$ with incenter I. Let its inradius be r, then the area of $\triangle IBC$ is

$$[\triangle IBC] = \frac{r\cdot BC}{2}.$$

Similarly, we can also compute the areas of $\triangle ICA$ and $\triangle IAB$. Therefore summing all of them gives

$$[\triangle ABC] = [\triangle IBC]+[\triangle ICA]+[\triangle IAB] = \frac{r(AB+BC+CA)}{2} \implies r = \frac{2[\triangle ABC]}{AB+BC+CA}.$$



Now let's apply this formula to our 3-4-5 right triangle. Its area is 10, so its inradius is

$$r = \frac{20}{3+4+5} = \frac{20}{12} = \frac{5}{3}. \qquad \square$$

Problem 9. A bag contains 3 red balls and 2 green balls. Balls are drawn at random, one at a time but not replaced, until all 3 of red balls are drawn or until both green balls are drawn. What is the probability that the 3 reds are drawn?

Solution. Suppose that we continue drawing after we are supposed to stop until there are no balls left. Note that 3 reds will be drawn before both greens are drawn if and only if the fifth ball is green. Therefore, we just need to calculate the probability that the fifth ball is green. Note that by symmetry, the number of draws where the fifth ball is green is exactly the same as the number of draws where the first ball is green. Therefore, the probability is just that of drawing a green ball at the start, which is $2/5$. $\square$

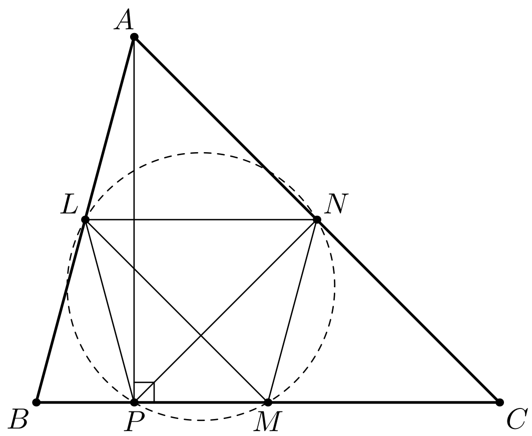
Problem 10. If $75!$ is divisible by 5^n, find the maximum value of n.

Solution. Since $75 = 5 \cdot 15$, there are 15 numbers less than or equal to 75 which are divisible by 5. Out of those 15 numbers, there are 3 numbers which are divisible by $5^2 = 25$. Therefore, the total power of 5 in the prime factorization of $75!$ is $15 + 3 = 18$. Hence $n = 18$. $\square$

Remark. This problem uses the same idea as 2018 Regional Round Problem 10.

Problem 11. If L, M, N are the midpoints of the sides of $\triangle ABC$, and P is the foot of perpendicular from A to BC, prove that L, P, M, N are concyclic.

Solution. First observe that $ML \parallel AC$ and $MN \parallel AB$ due to the midpoints. Therefore, $ALMN$ is a parallelogram, and so $\angle LMN = \angle LAN$. Now since L is the midpoint of the hypotenuse in right triangle ABP, we see that $LA = LP$. Similarly, $NA = NP$. Therefore, $\triangle LAN \cong \triangle LPN$. This implies that $\angle LMN = \angle LAN = \angle LPN$ so L, M, N, P are concyclic as desired. $\square$



Problem 12. In a GP, the kth term is 864. The sum of the first k terms is 2080. The sum of the first $2k$ terms is 12610. Find the first term and the common ratio.

Solution. Let the first term of the GP be a and common ratio be r. We can view S_{2k} in the following way.

$$\begin{aligned}
S_{2k} &= a + ar + ar^2 + \cdots + ar^{k-1} + ar^k + \cdots + ar^{2k-1} \\
&= (a + ar + ar^2 + \cdots + ar^{k-1}) + r^k(a + ar + ar^2 + \cdots + ar^{k-1}) \\
&= S_k + r^k S_k \\
&= S_k(1 + r^k) \\
1 + r^k &= \frac{97}{16} \\
r^k &= \frac{81}{16}.
\end{aligned}$$

We will now use the condition $u_k = ar^{k-1} = 864$.

$$\begin{aligned}
S_k &= \frac{a - ar^k}{1 - r} \\
(1 - r)2080 &= r\left(\frac{a}{r} - ar^{k-1}\right) \\
&= r\left(\frac{ar^{k-1}}{r^k} - ar^{k-1}\right) \\
&= r\left(\frac{864 \cdot 16}{81} - 864\right) \\
(r - 1)2080 &= \frac{56160r}{81} \\
r - 1 &= \frac{r}{3} \\
r &= \frac{3}{2}.
\end{aligned}$$

Therefore, $k = 4$. Finally,

$$a\left(\frac{3}{2}\right)^3 = 864 \Longrightarrow a = 256.$$

□

Problem 13. Prove that $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$ if all variables are integers and $n \geq m \geq k \geq 0$.

Solution. Suppose that there are $n - m$ blue balls, $m - k$ green balls, and k red balls. The total number balls is $n - m + m - k + k = n$. We will count the number of ways to arrange these balls in two different ways. First suppose that we place the blue balls first. There are $\binom{n}{n-m}$ ways to choose $n - m$ positions out of n positions. After this, there will be m places left for red and green balls. The number of ways to choose k places from these m places for red balls is $\binom{m}{k}$. The green balls are placed in the places left, so there is no choice. Therefore, the total number of arrangements is

$$\binom{n}{n-m}\binom{m}{k} = \binom{n}{m}\binom{m}{k}.$$

Now suppose that we place the red balls first. There are $\binom{n}{k}$ ways to choose k positions out of n positions. Then there will be $n - k$ positions remaining for blue and green balls. Out of those $n - k$ positions, there are $\binom{n-k}{m-k}$ ways to choose $m - k$ positions for the green balls. Again, the remaining blue balls are placed in the places left, so there is no choice. Therefore, the total number of arrangements counted this way is

$$\binom{n}{k}\binom{n-k}{m-k}.$$

Since the total number of arrangements is the same, we have our desired identity. □

Problem 14.

1. Prove that $n \le -k^2 + nk + k \le \frac{(n+1)^2}{4}$ for $1 \le k \le n$.
2. Consider $k(n+1-k) = -k^2 + nk + k$ and by using the inequalities in (1), prove that
$$n^n \le (n!)^2 \le \frac{(n+1)^{2n}}{4^n}.$$
3. Hence prove that
$$n^{\frac{n}{2}} \le n! \le \frac{(n+1)^n}{2^n}.$$

Solution. Let's show the first inequality.

$$\begin{aligned} & k \le n \\ \iff\quad & k(k-1) \le n(k-1) \\ \iff\quad & k^2 - k \le nk - n \\ \iff\quad & n \le -k^2 + nk + k. \end{aligned}$$

The second inequality follows from the AM-GM inequality.

$$\frac{(n+1)^2}{4} = \left(\frac{(n+1-k)+k}{2}\right)^2 \ge k(n+1-k) = -k^2 + nk + k.$$

Now consider the following n inequalities, obtained when k ranges from 1 to n.

$$\begin{aligned} n &\le 1 \cdot n \le \frac{(n+1)^2}{4} \\ n &\le 2 \cdot (n-1) \le \frac{(n+1)^2}{4} \\ &\vdots \\ n &\le n \cdot 1 \le \frac{(n+1)^2}{4}. \end{aligned}$$

Multiplying all of them gives
$$n^n \le (n!)^2 \le \frac{(n+1)^{2n}}{4^n}.$$

Taking the square roots of every term, we finally have
$$n^{\frac{n}{2}} \le n! \le \frac{(n+1)^n}{2^n}.$$ □

2019 National Round

Problem 1. $\triangle ABC$ is an equilateral triangle.

1. Prove that there are points D, E and F on AB, BC and CA respectively such that $DE \perp BC$, $EF \perp CA$ and $FD \perp AB$.

2. Prove that $\triangle DEF$ is also an equilateral triangle.

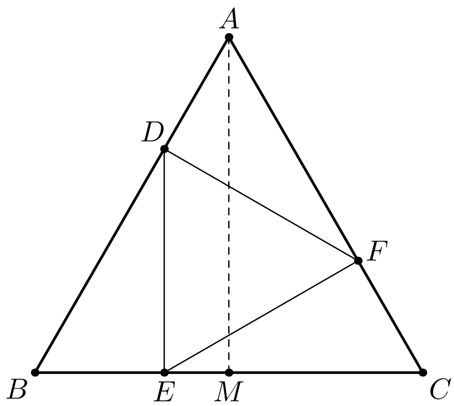
3. Find the ratio of the perimeters of $\triangle DEF$ and $\triangle ABC$.

4. Find the ratio of the areas of $\triangle DEF$ and $\triangle ABC$.

Solution. Let D, E and F be points on sides AB, BC and CA such that $3AD = AB$, $3BE = BC$ and $3CF = CA$. We will first show that $DE \perp BC$. Let M be the midpoint of BC. Then since $\triangle ABC$ is equilateral, $AM \perp BC$. Moreover,

$$\frac{BE}{BM} = \frac{\frac{1}{3}BC}{\frac{1}{2}BC} = \frac{2}{3} = \frac{BD}{BA}.$$

Therefore, $DE \parallel AM$ and hence $DE \perp BC$. Moreover, it is easy to see that $\triangle AFD \cong \triangle BDE \cong \triangle CEF$, and so $DE = EF = FD$ which implies that $\triangle DEF$ is equilateral.



Now remember that $\triangle BAM \sim \triangle BDE$, and $\triangle BAM$ is a 30-60 right triangle. Since $BA : BD = 3 : 2$,

$$\frac{BD}{DE} = \frac{BA}{AM} = \frac{2}{\sqrt{3}} \Longrightarrow \frac{BA}{DE} = \frac{BA}{BD} \cdot \frac{BD}{DE} = \frac{3}{2} \cdot \frac{2}{\sqrt{3}} = \sqrt{3}.$$

Hence the ratio of their perimeters is also $\sqrt{3}$. Finally, since $\triangle ABC$ and $\triangle DEF$ are both equilateral triangles, they are similar. Therefore the ratio of their areas is the square of ratio of their sides, which is equal to $(\sqrt{3})^2 = 3$. □

Problem 2.

1. If $abc = 2ab + 2bc + 2ca$ where a, b and c are integers and $1 \le a \le b \le c$, then show that $3 \le a \le 6$.
2. Find all possible ordered triples (a, b, c) such that
$$abc = 2ab + 2bc + 2ca$$
where a, b and c are integers and $1 \le a \le b \le c$.

Solution. We can divide the both sides of our given equation by abc to get

$$1 = \frac{2}{a} + \frac{2}{b} + \frac{2}{c}.$$

Suppose that $6 < a$. Then as $a \le b \le c$, it follows that both b and c are also greater than 6, so

$$1 = \frac{2}{a} + \frac{2}{b} + \frac{2}{c} < \frac{2}{6} + \frac{2}{6} + \frac{2}{6} = 1,$$

which is a contradiction. Therefore, $a \le 6$. Now suppose that $a < 3$. Then $a \le 2$ so

$$1 = \frac{2}{a} + \frac{2}{b} + \frac{2}{c} \ge 1 + \frac{2}{b} + \frac{2}{c} > 1,$$

which is also a contradiction. Therefore, $3 \le a \le 6$ as desired.

Now we can plug in each value of a and solve for b and c. We will only show how to solve the equation for $a = 3$, as the other cases can be done the exact same way. When $a = 3$, the equation becomes

$$\frac{1}{6} = \frac{1}{b} + \frac{1}{c} \iff (b-6)(c-6) = 36.$$

Notice that b cannot be less than 6 or otherwise c will be negative. This means that both $b - 6$ and $c - 6$ are positive divisors of 36. This gives us the following set of solutions pairs:

$$(b, c) \in \{(7, 42), (8, 24), (9, 18), (10, 15), (12, 12)\}.$$

When $a = 4$, the equation becomes

$$\frac{1}{4} = \frac{1}{b} + \frac{1}{c} \iff (b-4)(c-4) = 16.$$

In this case, the solution pairs are $(b, c) \in \{(5, 20), (6, 12), (8, 8)\}$.

When $a = 5$, the equation is

$$\frac{1}{10} = \frac{1}{3b} + \frac{1}{3c} \iff (3b-10)(3c-10) = 100.$$

In this case, the solution pairs are $(b, c) \in \{(4, 20), (5, 10)\}$.

Finally, when $a = 6$, the equation is

$$\frac{1}{3} = \frac{1}{b} + \frac{1}{c} \iff (b-3)(c-3) = 9.$$

In this case, the solution pairs are $(b, c) \in \{(4, 12), (6, 6)\}$.

The table below illustrates all the ordered pairs (a, b, c) that satify the given equation. $\square$

a	Solution triples
3	(3, 7, 42), (3, 8, 24), (3, 9, 18), (3, 10, 15), (3, 12, 12)
4	(4, 5, 20), (4, 6, 20), (4, 8, 8)
5	(5, 4, 20), (5, 5, 10)
6	(6, 4, 12), (6, 6, 6)

Problem 3. Let $\mathbb{Z}$ be the set of integers. Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function from $\mathbb{Z}$ to $\mathbb{Z}$ such that

$$f(x+y) + f(x-y) = 2f(x) + 2f(y),$$

for all integers x and y.

1. Show that $f(0) = 0$ and $f(2x) = 4f(x)$.
2. Show that $f(nx) = n^2 f(x)$ for every positive integer n.
3. Show that $f(y) = f(-y)$.
4. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all integers x and y.

Solution. Let $P(x, y)$ be the assertion $f(x+y) + f(x-y) = 2f(x) + 2f(y)$.

$$\begin{aligned}
P(0,0) \implies \qquad & f(0) + f(0) = 2f(0) + 2f(0) \\
& 2f(0) = 4f(0) \\
& f(0) = 0 \\
P(x,x) \implies \qquad & f(2x) + f(0) = 2f(x) + 2f(x) \\
& f(2x) = 4x
\end{aligned}$$

We will prove the second part using strong induction. For the base case $n = 1$, we have to show $f(x) = f(x)$ and this is obviously true. So suppose that this is true for all $n \leq k$. We need to show that this is also true for $n = k+1$. $P(kx, x)$ then gives

$$\begin{aligned}
P(kx,x) \implies \qquad f((k+1)x) + f((k-1)x) &= 2f(kx) + 2f(x) \\
f((k+1)x) &= 2f(kx) + 2f(x) - f((k-1)x) \\
&= 2k^2 f(x) + 2f(x) - (k-1)^2 f(x) \\
&= (k^2 + 2k + 1) f(x) \\
&= (k+1)^2 f(x)
\end{aligned}$$

Therefore, by induction it follows that $f(nx) = n^2 f(x)$ for all positive integers n. In particular if n is a positive integer, $f(n) = n^2 f(1)$.

$$\begin{aligned}
P(0,y) \implies \qquad f(y) + f(-y) &= 2f(0) + 2f(y) \\
f(-y) &= f(y)
\end{aligned}$$

Now from the second part, letting $x = -1$ gives $f(-n) = n^2 f(-1) = (-n)^2 f(1)$. Hence, we see that $f(x) = x^2 f(1) = cx^2$ where c is a constant, for all integers x. On the other hand, it is easy to check that all functions of the form $f(x) = cx^2$ satisfy the given equation. Therefore, it follows that they must be the only solutions. $\square$

Problem 4. Seven integers are written around a circle in a way that no two or three adjacent numbers have a sum divisible by 3. How many of these seven numbers are divisible by 3?

Solution. There are exactly 3 numbers which are divisible by 3.

Call a number *good* if it is divisible by 3, and *bad* otherwise. Let N be the number of good numbers. First, suppose that $N > 3$. We claim that there are two adjacent good numbers. Otherwise, there has to be a bad number between each of those N good numbers, which is not possible as there are only 7 numbers. Therefore, there are 2 adjacent good numbers, so their sum is divisible by 3 which is a contradiction to the problem statement. Hence $N \leq 3$.

Now suppose that $N < 3$. Then we claim that there are 3 consecutive bad numbers. Otherwise, there can only be atmost 2 bad numbers between any 2 good numbers. This means that the total number of integers can only be at most 6, which is a contradiction as well. Therefore, there must be three consecutive bad integers. Since an integer can leave a remainder of 0, 1 or 2 when divided by 3, a bad integer can only leave a remainder of 1 or 2 when divided by 3. If all of those 3 consecutive bad numbers leave the same remainder when divided by 3, their sum is divisible by 3 which is a contradiction. If they leave different remainders when divided by 3, this means that there are 2 consecutive bad numbers which leave different remainders when divided by 3. Then their sum is divisible by 3, which is also a contradiction. Hence $N \geq 3$.

These two parts show that $N = 3$ and we are done. □

Remark. One trick that can help a lot in this problem is that we only need to care about the residue modulo 3 of each of the 7 numbers. That is, we can basically assume that each of those 7 integers are 0, 1 or 2.

Problem 5.

1. Find an ordered pair (x, y) such that
$$2019x + 2021y = 1$$
where x and y are integers.
2. By using the ordered pair obtained in question 1, find all solutions of
$$2019x + 2021y = 1$$
where x and y are integers.

Solution. Using the Euclidean algorithm, we can find a solution pair as follows:

$$\begin{aligned}
2021 &= 2019 + 2 \\
2019 &= 2 \cdot 1009 + 1 \\
&= (2021 - 2019)(1009) + 1 \\
2019 &= 2021 \cdot 1009 - 2019 \cdot 1009 + 1 \\
1010 \cdot 2019 - 1009 \cdot 2021 &= 1.
\end{aligned}$$

Therefore, $(x, y) = (1010, -1009)$ is a solution pair. Now suppose that (x_0, y_0) is another pair that satisfies the given equation. i.e.

$$2019x_0 + 2021y_0 = 1 = 1010 \cdot 2019 - 1009 \cdot 2021 \iff 2019(1010 - x_0) = 2021(y_0 + 1009).$$

Now notice that $\gcd(2019, 2021) = 1$. Therefore, by Euclid's lemma, $2021 \mid 1010 - x_0 \Rightarrow x_0 = 1010 - 2021k$ for some integer k. Similarly, $2019 \mid y_0 + 1009 \Rightarrow y_0 = 2019j - 1009$ for some integer j. Substituting these back into the equation again gives

$$2019 \cdot 2021k = 2021 \cdot 2019j \implies k = j.$$

Hence $(x_0, y_0) = (-2021k + 1010, 2019k - 1009)$ for some integer k. Now it is easy to check that all pairs of the form $(x, y) = (-2021k + 1010, 2019k - 1009)$ satisfy the given equation. Therefore, it follows that they are the only solutions. $\square$

Problem 6. In $\triangle ABC$, altitudes AD, BE and CF pass through the point H. Points A', B' and C' are midpoints of BC, CA, AB respectively. Points A'', B'' and C'' are midpoints of AH, BH, CH respectively.

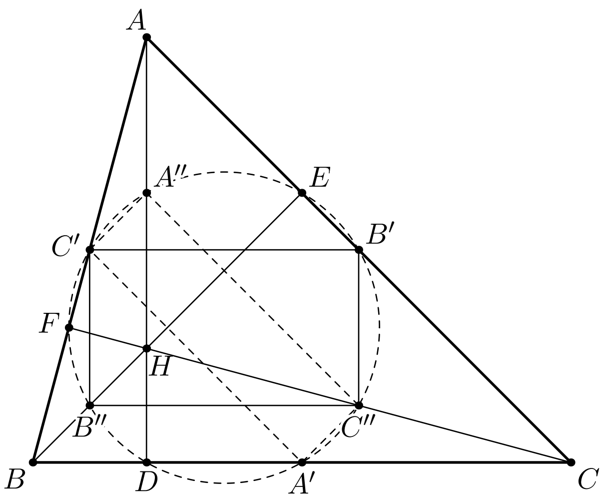
1. Prove that $B'C'B''C''$ is a rectangle.
2. Prove that $C'A'C''A''$ is a rectangle.
3. Prove that the six points A', B', C',A'', B'', C'' are concyclic.
4. Prove that the nine points $A', B', C', A'', B'', C'', D, E, F$ are concyclic.

Solution. Since C' and B'' are midpoints of sides BA and BH in $\triangle BAH$, it follows that $C'B'' \parallel AH$. Now C' and B' are midpoints of sides AB and AC, so $B'C' \parallel BC$. Since $BC \perp AH$, $B'C' \perp B''C'$. Similarly, we can show that other three angles are also right angles, so $B'C'B''C''$ is a rectangle.

Similarly to above, we can show that $C'A'C''A''$ is also a rectangle. We now have

$$\angle CA'C'' = \angle CA''C'' = \angle CB'C'' = \angle CB''C'' = 90°,$$

so $A', B', C', A'', B'', C''$ all lie on the circle with diameter CC''. Call this circle ω.



By 2019 Regional Round problem 11, we see that D lies on $(A'B'C')$, which is ω. Similarly, E and F also lie on ω, so combined with above we see that $D, E, F, A', B', C', A'', B'', C''$ all lie on ω as desired. $\square$

Problem 7. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

$$f(x+y) + f(x-y) = 2f(x)\cos y,\ \ x, y \in \mathbb{R}.$$

1. Show that $f(\theta) + f(-\theta) = 2a\cos\theta$, where $a = f(0)$.
2. Show that $f(\theta+\pi) + f(\theta) = 0$.
3. Show that $f(\theta+\pi) + f(-\theta) = -2b\sin\theta$, where $b = f\left(\frac{\pi}{2}\right)$.
4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+y) + f(x-y) = 2f(x)\cos y$$

where $x, y \in \mathbb{R}$.

Solution. Let $P(x, y)$ be the assertion $f(x+y) + f(x-y) = 2f(x)\cos y,\ x, y \in \mathbb{R}$. Then

$$\begin{aligned}
P(0,\theta) \implies \quad & f(\theta) + f(-\theta) = 2f(0)\cos\theta \\
& \qquad\qquad\qquad\ = 2a\cos\theta. \\
P\left(\theta + \frac{\pi}{2}, \frac{\pi}{2}\right) \implies \quad & f(\theta+\pi) + f(\theta) = 2f\left(\theta + \frac{\pi}{2}\right)\cos\frac{\pi}{2} \\
& \qquad\qquad\qquad\ = 0. \\
P\left(\frac{\pi}{2}, \theta + \frac{\pi}{2}\right) \implies \quad & f(\theta+\pi) + f(-\theta) = 2f\left(\frac{\pi}{2}\right)\cos\left(\theta + \frac{\pi}{2}\right) \\
& \qquad\qquad\qquad\ = -2b\sin\theta.
\end{aligned}$$

Now subtracting the third equation from the second equation gives

$$f(\theta) - f(-\theta) = 2b\sin\theta.$$

Adding this to the first equation gives

$$\begin{aligned}
2f(\theta) &= 2a\cos\theta + 2b\sin\theta \\
f(\theta) &= a\cos\theta + b\sin\theta.
\end{aligned}$$

Hence if f satisfies the given equation, then $f(x) = a\cos x + b\sin x$ for some real constants a and b. On the other hand, it is easy to check that all functions of the above form satisfy the given equation. Therefore, it follows that functions of the form $f(x) = a\cos x + b\sin x$ are the only solutions to the given functional equation. □

Problem 8. A student council must select a two-person welcoming committee and a three-person planning committee from its members. There are exactly 15 ways to select a two-person team for the welcoming committee. It is possible for students to serve on both committees. In how many different ways can a three-person planning committee be selected?

Solution. Let the number of students be n. Then

$$15 = \binom{n}{2} = \frac{n(n-1)}{2} \iff (n-6)(n+5) = 0.$$

Since n is a positive integer, $n = 6$. Therefore, the number of ways to choose a three-person planning committee is

$$\binom{6}{3} = \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} = 20.$$

□

Problem 9. AB is a diameter of a circle O with radius 10 cm. OQ is a radius of a circle O such that $QO \perp AB$. A point P is on OQ. Draw a semicircle centered at P with diameter CD where CD is the chord of circle O and $CD \perp PQ$. PQ produced meets the semicircle at R. Find the maximum possible length of QR.

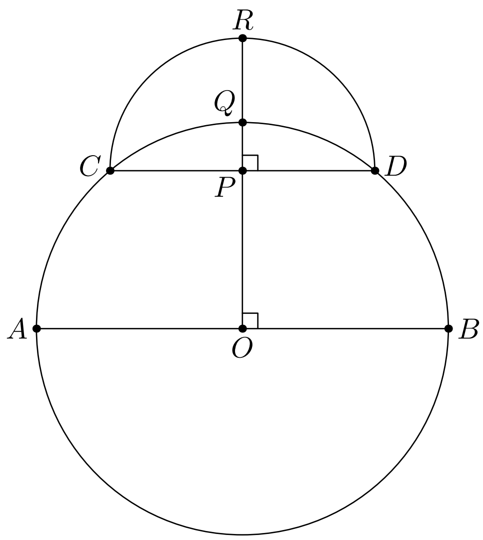
Solution. The length of QR is

$$QR = PR - PQ = PC - (OQ - OP) = PC + OP - 10.$$

By the RMS-AM inequality,

$$CP + OP - 10 \le 2\sqrt{\frac{CP^2 + OP^2}{2}} - 10 = 2\sqrt{\frac{100}{2}} - 10 = 10(\sqrt{2} - 1).$$

Hence the maximum value of QR is $10(\sqrt{2} - 1)$, achieved when $CP = OP = 5\sqrt{2}$. □



Problem 10. Find all positive integers s such that $\left\lceil \frac{s}{3} \right\rceil - 21 = \left\lceil \frac{s}{5} \right\rceil$ where $\lceil x \rceil$ is the smallest integer greater than or equal to x. For example, $\lceil 3.7 \rceil = 4$, $\lceil 3 \rceil = 3$ and $\lceil 3.2 \rceil = 4$.

Solution. It is easy to see that for any real number x,

$$x \le \lceil x \rceil < x + 1.$$

Therefore, from the given equation, we have

$$\frac{s}{3} - \frac{s}{5} - 1 < \left\lceil \frac{s}{3} \right\rceil - \left\lceil \frac{s}{5} \right\rceil = 21 < \frac{s}{3} + 1 - \frac{s}{5}.$$

Simplifying the first part of the inequality gives us

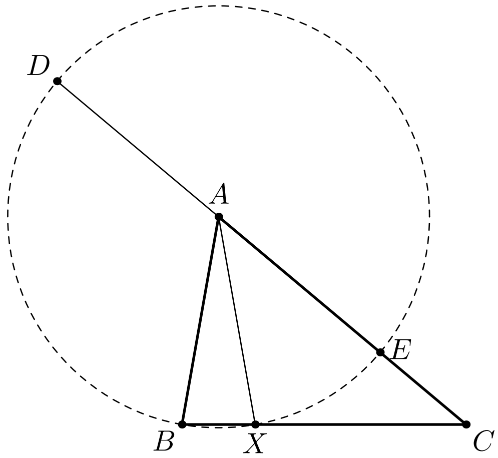
$$5s - 3s - 15 < 315 \implies s < 165.$$

Similarly, the second part of the inequality gives us $s > 150$. Therefore, s must be in the set $\{151, 152, \ldots, 164\}$. By checking each element in the set, we see that the only possible values of s are 154, 155, 157, 158, 159, 161, 162. □

Problem 11. In $\triangle ABC$, $AB = 94$ and $AC = 107$. A circle with center A and radius AB intersects BC at points B and X. Moreover, BX and CX have integer lengths. What is BC?

Solution. Since BX and CX are integers, BC must also be an integer too. Let the circle intersect line AC at points D and E. Then since B, X, D, E are concyclic,

$$CX \cdot CB = CD \cdot CE = (CA + AB)(CA - AB) = CA^2 - AB^2 = 107^2 - 94^2 = 2613.$$

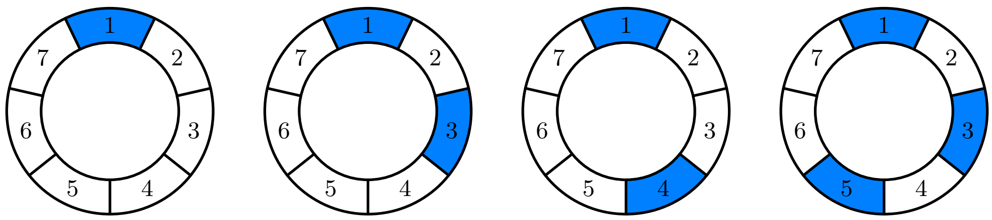


By the triangle inequality, $CB < CA + AB = 94 + 107 = 201$. We also have $CB > CX$ since X lies on the side BC. Since the factors of 2613 are 1, 3, 13, 39, 67, 201, 871 and 2613, we see that BC must be 67. $\square$

Problem 12. Seven people are sitting around a circular table, each holding a fair coin. All seven people flip their coins and those who flip heads stand while those flip tails seated. What is the probability that no two people adjacent will stand?

Solution. Label the people from 1 to 7. First let's compute the total number of configurations. Each person has 2 possible states, either seated or standing. Since there are 7 people, the total number of configurations is $2^7 = 128$.

Now let's calculate the number of configurations where no two people standing are adjacent. If the number of people standing is greater than or equal to 4, there will be at least two of them which are adjacent. Therefore, there can only be at most 3 people who are standing. Then it is easy to see that the following are all the possible configurations, unique up to rotation[1]. The blue blocks denote the people who are standing.



Since there are 7 ways to rotate each of those configurations, and since there are 4 of them, the number of configurations is 28. Also counting the configuration where no person is standing, the final number of configurations is 29.

Therefore, the probability required is $\frac{29}{128}$. $\square$

[1]This means that any other configuration can be obtained by rotating one of these configurations.

Remark. You can also use the same technique as in 2016 National Round problem 10 to show that $f(C_7) = 29$.

Problem 13.

1. Let a and b be positive integers. If there are integers x_0, y_0 such that $ax_0 + by_0 = 1$, then prove that the greatest common divisor of a and b is 1.

2. Prove that the fraction $\frac{12n+5}{14n+6}$ is in lowest terms for every positive integer n.

Solution. Let the $\gcd(a, b) = d$. Since d divides both a and b, d also divides ax_0 and by_0 and hence

$$d \mid (ax_0 + by_0) = 1.$$

Since the only positive divisor of 1 is 1 itself, it follows that $d = 1$. Now notice that

$$6(14n + 6) + (-7)(12n + 5) = 1$$

for any positive integer n. Therefore, by the first part, $\gcd(14n + 6, 12n + 5) = 1$, and so $\frac{12n+5}{14n+6}$ is in lowest terms for every positive integer n. $\square$

Problem 14. $\triangle ABC$ is inscribed in a circle. Altitudes AD, BE and CF pass through the point H. EF produced meets the circle at P. BP produced and DF produced meet at the point Q.

1. Show that $\angle ACF = \angle ADF = \angle ABE$.

2. Show that $\angle AFQ = \angle ACD$.

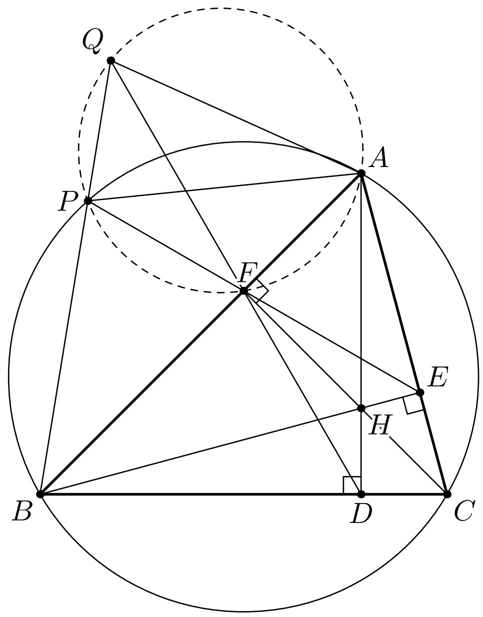
3. Show that $AP = AQ$.

Solution. Since $\angle AFC = \angle ADC = 90°$, $AFDC$ is cyclic, and so

$$\angle ACF = \angle ADF.$$

Also $\angle HDB + \angle HFB = 90° + 90° = 180°$, so $HFBD$ is also cyclic and

$$\angle ADF = \angle HDF = \angle HBF = \angle ABE.$$



As $AFDC$ is cyclic, we also have $\angle AFQ = \angle ACD$. But since $APBC$ is cyclic, $\angle ACD = \angle APQ$ so $\angle AFQ = \angle APQ$ and hence $AQPF$ is cyclic. Since we also have $\angle BFC = \angle BEC = 90°$, $BFEC$ is also cyclic. Therefore, finally,

$$\angle AQP = \angle AFE = \angle ACB = \angle APQ,$$

and so $AP = AQ$ as desired. $\square$

Problem 15. A car drives from town A to B at the average speed of 30 km/h, from town B to town C at average speed of 60 km/h; and on the way back, the car drives from C to B at average speed of 30 km/h, from town B to A at average speed of 60 km/h. The whole trip takes 6 hours. What is the total distance of the round trip?

Solution. Let x and y be the distances in kilometres between town A and town B, and town B and town C. Then from the given conditions we have

$$\frac{x}{30} + \frac{y}{60} + \frac{y}{30} + \frac{x}{60} = 6 \Longrightarrow (x+y)\left(\frac{1}{30} + \frac{1}{60}\right) = 6.$$

Solving this gives $x + y = 120$, so the total distance of the round trip is $2(x + y) = 240$ km. $\square$

Problem 16.

1. Prove that the square of an odd number gives the remainder 1 upon dividing by 8.
2. Prove that if k is odd and n is a positive integer, then $k^{2^n} - 1$ is divisible by 2^{n+2}.

Solution. Let the odd number be $2m + 1$. Then

$$(2m + 1)^2 - 1 = 4m(m + 1).$$

Since m and $m + 1$ are consecutive integers, one must be even and the other must be odd. Therefore, their product must be even, and hence $4m(m+1) = (2m+1)^2 - 1$ must be divisible by 8. Thus $(2m + 1)^2$ leaves a remainder of 1 when divided by 8.

Now we will prove the general statement by induction on n. The base case $n = 1$ is just the first part. Now suppose that this is true for $n = p$. When $n = p + 1$,

$$k^{2^{p+1}} - 1 = k^{2^p \cdot 2} - 1 = (k^{2^p})^2 - 1 = (k^{2^p} + 1)(k^{2^p} - 1).$$

Since k is odd, k^x is odd for any positive integer x. Therefore, k^{2^p} is odd implying $k^{2^p} + 1$ is even. Hence the left bracket in the above expression is divisible by 2. Meanwhile, the right bracket is divisible by 2^{p+2} by the induction hypothesis. Therefore, the whole expression is divisible by $2^{(p+1)+2}$, and so by induction it follows that $k^{2^n} - 1$ is divisible by 2^{n+2} for all positive integers n. $\square$