Let's first define what a homothety is.

Definition. A homothety Φ is a map from $\mathbb{R}^2 \to \mathbb{R}^2$, determined by a point T called its center and a non zero real constant k called the scale factor or ratio. If we denote the images with $\bullet'$, we have

$$\overrightarrow{TA'} = k \cdot \overrightarrow{TA}$$

for any point $A \in \mathbb{R}^2$.

This is the usual definition of homothety, which most people see as zooming in or out, just stated more rigorously. It is also easy to check that for any two points $A, B \in \mathbb{R}$, we always have

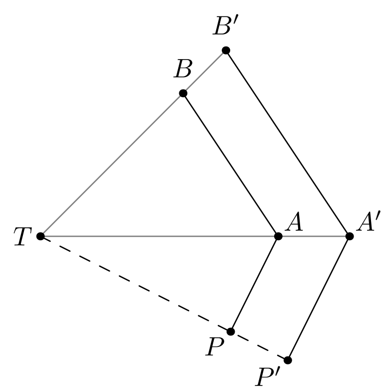
$$\overrightarrow{A'B'} = k \cdot \overrightarrow{AB}.$$

Now we will state another characterization of homotheties, perhaps more useful as it does not require a center.

Lemma. Let f be a map from $\mathbb{R}^2 \to \mathbb{R}^2$ such that

$$f(\overrightarrow{AB}) = k \cdot \overrightarrow{AB}$$ [1]

for all points $A, B \in \mathbb{R}^2$ and a constant $k \notin \{0, 1\}$. Then f is a homothety with ratio k.



Proof. Consider two distinct points A and B, and let their images be A' and B'. Let $T = \overline{AA'} \cap \overline{BB'}$[2]. Then as $\overrightarrow{A'B'} = k \cdot \overrightarrow{AB}$, we see that T is the center of homothety Φ with ratio k which maps segment AB to segment $A'B'$. Now consider any point P distinct from A and B and let $P_1 = f(P)$, $P_2 = \Phi(P)$. Then

$$\overrightarrow{A'P_1} = f(\overrightarrow{AP}) = k \cdot \overrightarrow{AP} = \Phi(\overrightarrow{AP}) = \overrightarrow{A'P_2}$$

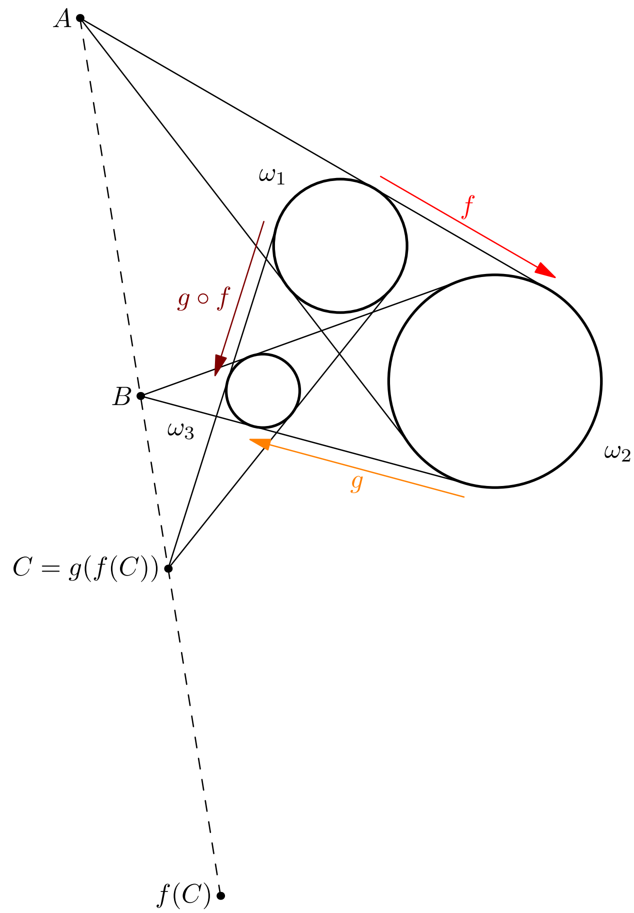
which shows that $P_1 = P_2$. Therefore, the values f and Φ coincide for all points in the plane, and thus $f = \Phi$. $\square$

This is basically saying that any map which maps segments to parallel segments scaled by a real constant are homotheties. Using this lemma, we can prove a very famous result in olympiad geometry quite effortlessly.

Theorem. Let ω_1, ω_2 and ω_3 be three disjoint circles with distinct radii. Let the common external tangents of ω_1 and ω_2 intersect at A, and define B and C similarly. Then A, B and C are collinear.

[1] Here $f(\overrightarrow{AB})$ of course denotes the vector that starts at $f(A)$ and ends at $f(B)$.

[2] This is a Euclidean point and not a point at infinity because we assumed that $k \neq 1$.



Proof. Let f be the homothety centered at A that takes ω_1 to ω_2, and let g be the homothety centered at B that takes ω_2 to ω_3. It is easy to see that the scale factors of both f and g (Call them k and l respectively.) are positive and not equal to 1.

Claim. $g \circ f$ is a homothety.

Proof. Similarly to the above proof, consider any two points X and Y in the plane. Then

$$(g \circ f)(\overrightarrow{XY}) = g(f(\overrightarrow{XY})) = g(k \cdot \overrightarrow{XY}) = kl \cdot \overrightarrow{XY}.$$

Since $kl \neq 1$ (This would imply that ω_1 and ω_3 have the same radii.)[3] and obviously $kl \neq 0$, it follows by the above lemma that $g \circ f$ is a homothety. □

Going back to the main problem, since C is the center of positive homothety which maps ω_1 to ω_3, it follows that C must be the center of $g \circ f$[4]. Due to homotheties, $A, C, f(C)$ are collinear[5], and so are $B, f(C), g(f(C))$. But as C is the center of $g \circ f$, $g(f(C)) = C$[6], so A, B, C are collinear as desired. □

[3] f maps ω_1 to ω_2 and g maps ω_2 to ω_3, so $g \circ f$ maps ω_1 to ω_3.
[4] Why must the positive homothety which maps ω_1 to ω_3 be unique?
[5] If T is the center of a homothety Φ and P is a point, $T, P, \Phi(P)$ are collinear.
[6] The center of a homothety is the only fixed point of that homothety.