1 Introduction

As said in the title, this handout is going to be about spiral similarity and Miquel points. You'll see how useful they are. Just wait.
P.S. I suggest you guys to prove the following lemmas by yourself, and I will only provide proofs for lemmas which are difficult. I'm *definitely not* bored to type them, I just want you to discover the proofs by yourself so that they will last longer in your memory.

2 Spiral Similarities

First of all, what is a spiral similarity? Simply put, a spiral similarity is a combination of a rotation and a dilation centered at a point. Notice that this preserves a lot of stuff, most importantly the ratios and angles. At first you might not see how these can be useful in Olympiad Geometry problems but trust me they appear everywhere. This is because of the following lemma.

> **(Center of a spiral similarity)** Let A, B, C and D be points in the plane such that $ABCD$ is not a parallelogram. Let X be the intersection of AB and CD and let Y be the intersection of (XAC) and (XBD). Then Y is the unique spiral similarity center which carries AB to CD. Conversely, if Y is the center of spiral similarity which sends AB to CD, and they intersect at X, it follows that $XACY$ and $XBDY$ are cyclic.

The proof is just simple angle chasing so I'll leave this as an exercise for you, although the uniqueness part requires a tiny bit of complex numbers. One thing to notice is that spiral similarities come in pairs. This becomes obvious by the Miquel point lemma for quadrilaterals in the next section. Just remember that whenever you see two circles intersecting, there is a similarity hiding in the corner.

3 Miquel Points

> **(Miquel's Theorem)** Let D, E, and F be points on lines BC, CA and AB of $\triangle ABC$. Then (AFE), (BFD) and (CDE) are concurrent at a point P.

Proof. Trivial. □

Now we can use this to prove the following more general theorem.

> **(Miquel point of a quadrilateral)** Let l_1, l_2, l_3 and l_4 be distinct lines such that no three are parallel, and let Γ be the quadrilateral they define. Let ω_{xyz} denote the circumcircle of the triangle defined by l_x, l_y and l_z. Then $\omega_{234}, \omega_{341}, \omega_{412}$ and ω_{123} are concurrent at a point M, and this point is called the Miquel point of Γ.

Notice that Γ doesn't need to be convex. In fact, the Miquel points of self intersecting quadrilaterals are just as useful. Most of the time, however, we are interested in the case where Γ is cyclic. This time, we have a nice characterization of the Miquel point.

(Miquel Point of a cyclic quadrilateral) Let A, B, C and D be point on a circle, not necessarily in this order. P is the intersection of diagonals in the quadrilateral $ABCD$. Then the Miquel point of quadrilateral $ABCD$ is the inverse of P with respect to $(ABCD)$.

Proof. Let $E = AB \cap CD$, $F = AD \cap BC$, O be the center of $(ABCD)$ and M be the point defined previously. Then as M and P are inverses, $\measuredangle OMA = -\measuredangle OAC$ and $\measuredangle OMD = -\measuredangle ODB$. Therefore,

$$\begin{aligned}\measuredangle AMD &= \measuredangle AMO + \measuredangle OMD \\ &= \measuredangle OAC - \measuredangle ODB \\ &= (90^\circ - \measuredangle CBA) - (90^\circ - \measuredangle BCD) \\ &= \measuredangle ABC + \measuredangle BCD \\ &= \measuredangle AED\end{aligned}$$

and hence M lies on (EAD). Similarly, M lies on (FAB) and so M must be the Miquel point of $(ABCD)$. □

Also notice that due to the inversion, it immediately follows that M also lies on (OAC) and (OBD). Since M lies on so many circles, most of the time when it appears, it is usually the intersection of just two circles, and the other ones are hidden.

3.1 Properties of Miquel points

Now we will show you even more ways to identify a Miquel point. We will denote by M the Miquel point of quadrilateral $ABCD$ in the following.

Property 1: Let AB and CD intersect at E and BC and AD at F. Then M lies on EF.

By the previous section, it is easy to deduce that M is the center of spiral similarity which sends AB to DC and BC to AD. Using this fact, we can prove the following property 2.

Property 2: OM is perpendicular to EF.

Proof. Let M_1 be the midpoint of AB and M_2 be the midpoint of DC. Since M sends AB to DC, it also sends AM_1 to DM_2. Hence by the spiral similarity lemma it follows that M lies on (M_1M_2E). However (M_1M_2E) is the circle with diameter OE and the conclusion follows. This is enough of properties (Although there are still stuff like Brocard's Theorem which you will see after reading projective geometry.) so let's dive into problems!

4 Examples

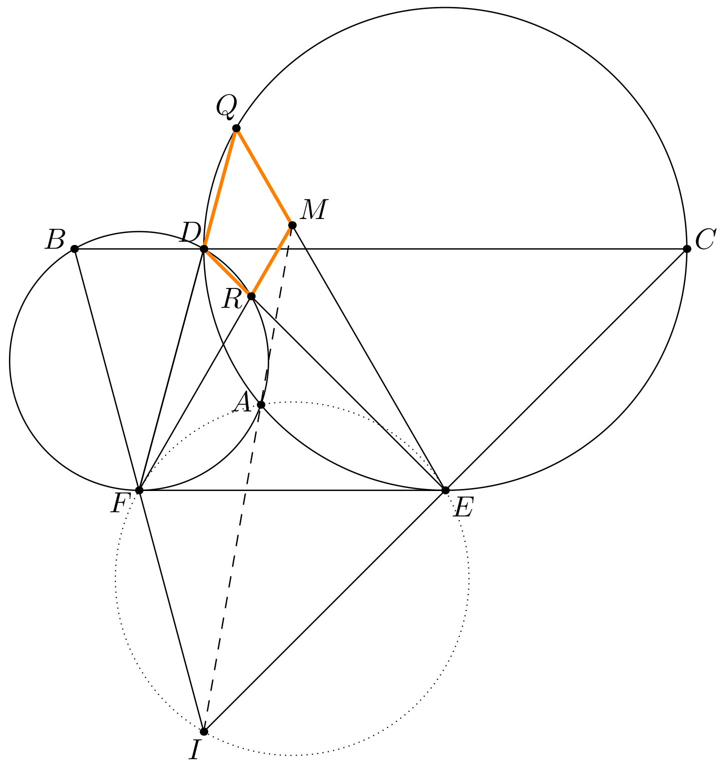
Before all that, let's take a look at a Miquel point which appears quite a lot.

An extension of the orthocenter lemma Let H and D, E, F be the orthocenter and foots of perpendiculars in $\triangle ABC$. Let (AFE) cut (ABC) at P. Let M be the midpoint of BC. Then M, H, P are collinear.

Proof. Notice that P is the Miquel point of $BCEF$, whose center is M and H is the intersection of diagonals in that quadrilateral so this is obvious. □

As a bonus fact, we see that P is the second intersection of (BMF) and (CME). Ok now let's solve problems for real.

Example 1 (Taiwan TST Round 1 Mock IMO P4) Two line BC and EF are parallel. Let D be a point on segment BC different from B,C. Let I be the intersection of BF and CE. Denote the circumcircle of $\triangle CDE$ and $\triangle BDF$ as K,L. Circle K,L are tangent with EF at E,F,respectively. Let A be the other intersection of circle K and L. Let DF and circle K intersect again at Q, and DE and circle L intersect again at R. Let EQ and FR intersect at M. Prove that I, A, M are collinear.

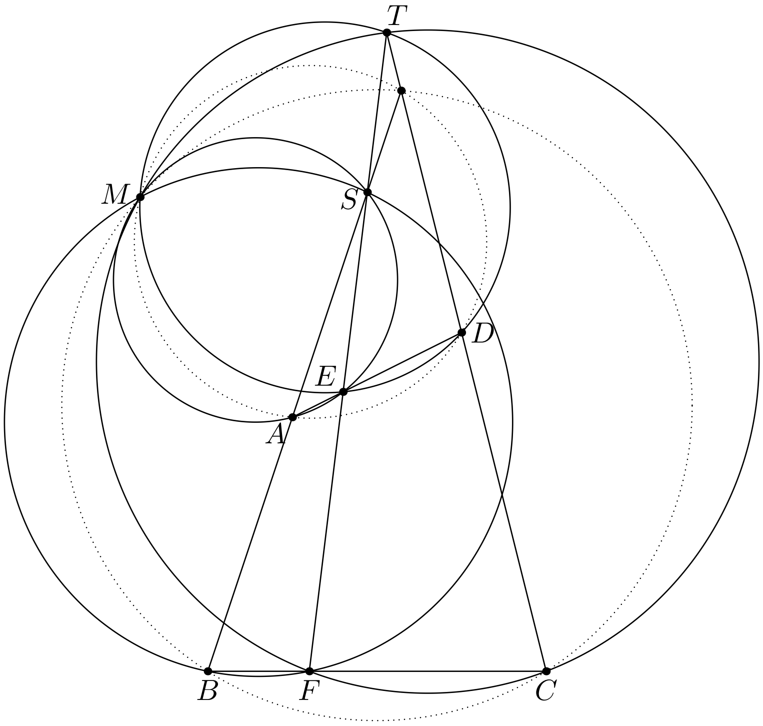


Solution. One tip is that in problems where the definitions are symmetric, the Miquel points are probably symmetric too. After inspecting the figure, we see that A is just the Miquel point of quadrilateral $DRMQ$. But this quadrilateral doesn't appear to be concyclic, at least in our figure, so we suspect that we will only use the concyclic properties (LoL no contradictions intended). Indeed, a tiny bit of angle chasing gives us the result.

$$\measuredangle FAI = \measuredangle FEI = \measuredangle BCI = \measuredangle DCE = \measuredangle DQE = \measuredangle FAM.$$

Notice that we didn't even use the tangencies. This is because they're extraneous and this problem is a gyinn.

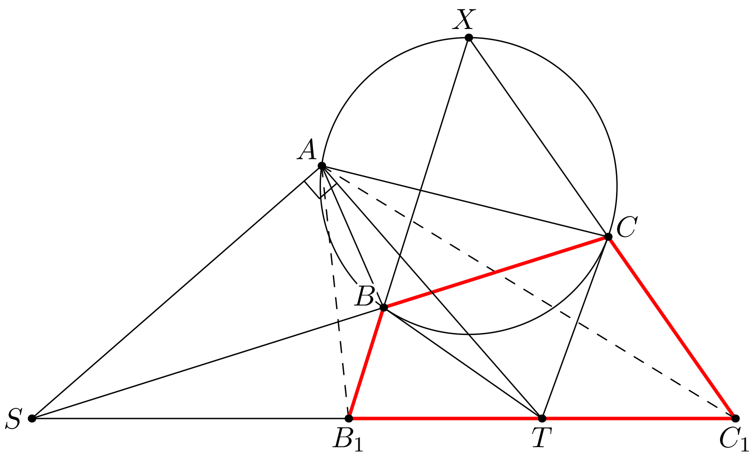
Example 2 (USAMO 2006 P6) Let $ABCD$ be a quadrilateral, and let E and F be points on sides AD and BC, respectively, such that $\frac{AE}{ED} = \frac{BF}{FC}$. Ray FE meets rays BA and CD at S and T, respectively. Prove that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through a common point.



Seeing a lot of circles intersecting at one point, we remember our Miquel point. The ratios give a big hint too as spiral similarities preserve these ratios. So we draw in the Miquel point of $ABCD$. Now can you see why the problem dies instantly?

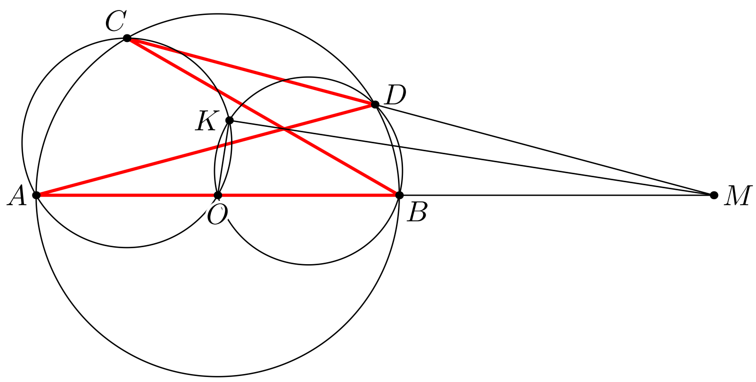
Solution. Let M be the Miquel point of $ABCD$. Then as M is the center of spiral similarity which sends AD to BC and since $\frac{AE}{ED} = \frac{BF}{FC}$, it follows that M also sends AE to BF. This immediately show that both (SAE) and (SBD) passes through M. Similarly, (TED) and (TFC) passes through M so we're done. □

Example 3 (USA TST 2007 P5) Triangle ABC is inscribed in circle ω. The tangent lines to ω at B and C meet at T. Point S lies on ray BC such that $AS \perp AT$. Points B_1 and C_1 lie on ray ST (with C_1 in between B_1 and S) such that $B_1T = BT = C_1T$. Prove that triangles ABC and AB_1C_1 are similar to each other.



Solution. Basically, the problem is asking us to show that A is the spiral similarity center carrying BC to B_1C_1, or equivalently, to show that A is the Miquel point of quadrilateral BCC_1B_1. Notice that this quadrilateral is cyclic, and T is its center. S is the point where the two opposite sides meet. Therefore, we have one property of a Miquel point satisfied, but obviously this is not enough. Now the only plausible other option is to prove that BB_1 and CC_1 intersect on ω but this is straightforward angle chasing. Now is this enough to conclude that A is the Miquel point? Unfortunately no as there are two points on ω which satisfy those two properties. But we can deal with it by WLOG assuming that A and T lie on the opposite sides of BC. The other case can be dealt with similarly so we're finally done. □

Example 4 (All Russian Olympiad 1995 Grade 10 P6) Let be given a semicircle with diameter AB and center O, and a line intersecting the semicircle at C and D and the line AB at M ($MB < MA$, $MD < MC$). The circumcircles of the triangles AOC and DOB meet again at K. Prove that $\angle MKO$ is right.



Solution. Seeing two circles passing through point O motivates us to invert with respect to circle O. Doing so, we see that K is mapped to $AC \cap BD$. This means that K is the Miquel point of self intersecting quadrilateral $ADCB$. Notice that CD and AB are opposite sides of this quadrilateral so by property (2), it follows that $\angle MKO = 90°$ as desired. □

I hope these are enough examples. Spiral similarities pop up literally everywhere. (If you guys are interested I might make a collection of such problems.) Remember that the ability to find similar triangles is something you can only get by doing a lot of problems and getting the feel for it.

4.1 Problems for Practice

These are the handouts that I used and referenced to write this. There are a lot of problems at the ends and I recommend you to try them out as much as possible. (Of course except for Hein Thant who is mandatory to do *all* the exercises.)

1. Yufei Zhao's cyclic quadrilaterals
2. Jafet Baca's handout on spiral similarity