

# Contents

<b>1</b>	<b>Presentation of the field</b>	<b>5</b>
1.1	Tensors . . . . .	7
1.1.1	Definition . . . . .	7
1.1.2	Manipulation . . . . .	9
1.1.3	Binary operations . . . . .	11
1.2	Neural Networks . . . . .	15
1.2.1	Simple formalization . . . . .	15
1.2.2	Generic formalization . . . . .	18
1.2.3	Interpretation . . . . .	20
1.2.4	Parameterization and training . . . . .	21
1.2.5	Examples of layer . . . . .	25
1.2.6	Examples of regularization . . . . .	30
1.2.7	Examples of architecture . . . . .	30
1.3	Graphs and signals . . . . .	31
1.3.1	Basic definitions . . . . .	31
1.3.1.1	Graphs . . . . .	31
1.3.1.2	(Real-valued) Signals . . . . .	34
1.3.2	Graphs in deep learning . . . . .	35
1.3.2.1	Connectivity graph . . . . .	36
1.3.2.2	Computation graph . . . . .	37
1.3.2.3	Underlying graph structure and signals . . . . .	37
1.3.2.4	Graph-structured dataset . . . . .	37

<b>2</b>	<b>Convolutions on graph domains</b>	<b>39</b>
2.1	Analysis of the classical convolution . . . . .	41
2.1.1	Properties of the convolution . . . . .	41
2.1.2	Characterization on grid graphs . . . . .	42
2.1.3	Usefulness of convolutions in deep learning . . . . .	45
2.2	Construction from the vertex set . . . . .	47
2.2.1	Steered construction from groups . . . . .	48
2.2.2	Construction under group actions . . . . .	52
2.2.3	Mixed domain formulation . . . . .	57
2.3	Inclusion of the edge set in the construction . . . . .	61
2.3.1	Edge-constrained convolutions . . . . .	61
2.3.2	Intrinsic properties . . . . .	65
2.3.3	Stricly edge-constrained convolutions . . . . .	67
2.4	From groups to groupoids . . . . .	70
2.4.1	Motivation . . . . .	70
2.4.2	Definition of notions related to groupoids . . . . .	71
2.4.3	Construction of partial convolutions . . . . .	72
2.4.4	Construction of path convolutions . . . . .	78
2.5	Conclusion . . . . .	83
<b>3</b>	<b>Application to neural networks</b>	<b>85</b>
3.1	Layer representations . . . . .	87
3.1.1	Neural interpretation of tensor spaces . . . . .	87
3.1.2	Propagational interpretation . . . . .	88
3.1.3	Graph representation of the input space . . . . .	89
3.1.4	General representation with weight sharing . . . . .	91
3.2	Study of the ternary representation . . . . .	94
3.2.1	Genericity . . . . .	94
3.2.2	Computationally efficient representation . . . . .	94
3.2.3	Robustness in extending the convolution . . . . .	97
3.2.4	Learning the weight sharing scheme . . . . .	97

<i>CONTENTS</i>	3
3.3 Extending CNNs . . . . .	98
3.3.1 Introduction . . . . .	98
3.3.2 Related works . . . . .	98
3.3.3 Convolution . . . . .	98
3.3.4 Subsampling . . . . .	98
3.3.5 Experiments . . . . .	98
<b>Bibliography</b>	<b>99</b>



# Chapter 1

## Presentation of the field

### Contents

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<b>1.1</b>	<b>Tensors . . . . .</b>	<b>7</b>
1.1.1	Definition . . . . .	7
1.1.2	Manipulation . . . . .	9
1.1.3	Binary operations . . . . .	11
<b>1.2</b>	<b>Neural Networks . . . . .</b>	<b>15</b>
1.2.1	Simple formalization . . . . .	15
1.2.2	Generic formalization . . . . .	18
1.2.3	Interpretation . . . . .	20
1.2.4	Parameterization and training . . . . .	21
1.2.5	Examples of layer . . . . .	25
1.2.6	Examples of regularization . . . . .	30
1.2.7	Examples of architecture . . . . .	30
<b>1.3</b>	<b>Graphs and signals . . . . .</b>	<b>31</b>
1.3.1	Basic definitions . . . . .	31
1.3.2	Graphs in deep learning . . . . .	35

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## Introduction

In this chapter, we present notions related to our domains of interest. In particular, for tensors we give original definitions that are more appropriate for our study. In the neural network's section, we present the concepts necessary to understand the evolution of the state of the art research in this field. In the last section, we present graphs for their usage in deep learning.

Vector spaces considered in what follows are assumed to be finite-dimensional and over the field of real numbers  $\mathbb{R}$ .

## 1.1 Tensors

Intuitively, tensors in the field of deep learning are defined as a generalization of vectors and matrices, as if vectors were tensors of rank 1 and matrices were tensors of rank 2. That is, they are objects in a vector space and their dimensions are indexed using as many indices as their rank, so that they can be represented by multidimensional arrays. In mathematics, a tensor can be defined as a special type of multilinear function (Bass, 1968; Marcus, 1975; Williamson, 2015), which image of a basis can be represented by a multidimensional array. Alternatively, Hackbush propose a mathematical construction of a tensor space as a quotient set of the span of an appropriately defined tensor product (Hackbusch, 2012), which coordinates in a basis can also be represented by a multidimensional array. In particular in the field of mathematics, tensors enjoy an intrinsic definition that neither depend on a representation nor would change the underlying object after a change of basis, whereas in our domain, tensors are confounded with their representation.

### 1.1.1 Definition

Our definition of tensors is such that they are a bit more than multidimensional arrays but not as much as mathematical tensors, for that they are embedded in a vector space so that deep learning objects can be later defined rigorously.

Given canonical bases, we first define a tensor space, then we relate it to the definition of the tensor product of vector spaces.

#### Definition 1. Tensor space

We define a *tensor space*  $\mathbb{T}$  of rank  $r$  as a vector space such that its canonical basis is a cartesian product of the canonical bases of  $r$  other vector spaces. Its shape is denoted  $n_1 \times n_2 \times \cdots \times n_r$ , where the  $\{n_k\}$  are the dimensions of the vector spaces.

**Definition 2. Tensor product of vector spaces**

Given  $r$  vector spaces  $\mathbb{V}_1, \mathbb{V}_2, \dots, \mathbb{V}_r$ , their *tensor product* is the tensor space  $\mathbb{T}$  spanned by the cartesian product of their canonical bases under coordinate-wise sum and outer product.

We use the notation  $\mathbb{T} = \bigotimes_{k=1}^r \mathbb{V}_k$ .

*Remark.* This simpler definition is indeed equivalent with the definition of the tensor product given in (Hackbusch, 2012, p. 51). The drawback of our definition is that it depends on the canonical bases, which at first can seem limiting as being canon implies that they are bounded to a certain system of coordinates. However this is not a concern in our domain as we need not distinguish tensors from their representation.

**Naming convention**

For naming convenience, from now on, we will distinguish between the terms *linear space* and *vector space* *i.e.* we will abusively use the term *vector space* only to refer to a linear space that is seen as a tensor space of rank 1. If we don't know its rank, we rather use the term *linear space*. We also make a clear distinction between the terms *dimension* (that is, for a tensor space it is equal to  $\prod_{k=1}^r n_k$ ) and the term *rank* (equal to  $r$ ). Note that some authors use the term *order* instead of *rank* (*e.g.* Hackbusch, 2012) as the latter is affected to another notion.

**Definition 3. Tensor**

A *tensor*  $t$  is an object of a tensor space. The *shape* of  $t$ , which is the same as the shape of the tensor space it belongs to, is denoted  $n_1^{(t)} \times n_2^{(t)} \times \dots \times n_r^{(t)}$ .



### 1.1.2 Manipulation

In this subsection, we describe notations and operators used to manipulate data stored in tensors.

**Definition 4. Indexing**

An *entry* of a tensor  $t \in \mathbb{T}$  is one of its scalar coordinates in the canonical basis, denoted  $t[i_1, i_2, \dots, i_r]$ .

More precisely, if  $\mathbb{T} = \bigotimes_{k=1}^r \mathbb{V}_k$ , with bases  $((e_k^i)_{i=1, \dots, n_k})_{k=1, \dots, r}$ , then we have

$$t = \sum_{i_1=1}^{n_1} \cdots \sum_{i_r=1}^{n_r} t[i_1, i_2, \dots, i_r] (e_1^{i_1}, \dots, e_r^{i_r})$$

The cartesian product  $\mathbb{I} = \prod_{k=1}^r \llbracket 1, n_k \rrbracket$  is called the *index space* of  $\mathbb{T}$

*Remark.* When using an index  $i_k$  for an entry of a tensor  $t$ , we implicitly assume that  $i_k \in \llbracket 1, n_k^{(t)} \rrbracket$  if nothing is specified.

**Definition 5. Subtensor**

A *subtensor*  $t'$  is a tensor of same rank composed of entries of  $t$  that are contiguous in the indexing, with at least one entry per rank. We denote  $t' = t[l_1:u_1, l_2:u_2, \dots, l_r:u_r]$ , where the  $\{l_k\}$  and the  $\{u_k\}$  are the lower and upper bounds of the indices used by the entries that compose  $t'$ .

*Remark.* We don't necessarily write the lower bound index if it is equal to 1, neither the upper bound index if it is equal to  $n_k^{(t)}$ .

**Definition 6. Slicing**

A *slice* operation, along the last ranks  $\{r_1, r_2, \dots, r_s\}$ , and indexed by  $(i_{r_1}, i_{r_2}, \dots, i_{r_s})$ ,

is a morphism  $s : \mathbb{T} = \bigotimes_{k=1}^r \mathbb{V}_k \rightarrow \bigotimes_{k=1}^{r-s} \mathbb{V}_k$ , such that:

$$s(t)[i'_1, i'_2, \dots, i'_{r-s}] = t[i'_1, i'_2, \dots, i'_{r-s}, i_{r_1}, i_{r_2}, \dots, i_{r_s}]$$

$$i.e. \quad s(t) := t[:, :, \dots, :, i_{r_1}, i_{r_2}, \dots, i_{r_s}]$$

where  $:=$  means that entries of the right operand are assigned to the left operand. We denote  $t_{i_{r_1}, i_{r_2}, \dots, i_{r_s}}$  and call it the *slice* of  $t$ . Slicing along a subset of ranks that are not the lasts is defined similarly.  $s(\mathbb{T})$  is called a *slice subspace*.

**Definition 7. Flattening**

A *flatten* operation is an isomorphism  $f : \mathbb{T} \rightarrow \mathbb{V}$ , between a tensor space  $\mathbb{T}$  of rank  $r$  and an  $n$ -dimensional vector space  $\mathbb{V}$ , where  $n = \prod_{k=1}^r n_k$ . It is

characterized by a bijection in the index spaces  $g : \prod_{k=1}^r \llbracket 1, n_k \rrbracket \rightarrow \llbracket 1, n \rrbracket$  such that

$$\forall t \in \mathbb{T}, f(t)[g(i_1, i_2, \dots, i_r)] = f(t[i_1, i_2, \dots, i_r])$$

We call an inverse operation a *de-flatten* operation.

**Row major ordering**

The choice of  $g$  determines in which order the indexing is made.  $g$  is reminiscent of how data of multidimensional arrays or tensors are stored internally by programming languages. In most tensor manipulation languages, incrementing the memory address (*i.e.* the output of  $g$ ) will first increment the last index  $i_r$  if  $i_r < n_r$  (and if else  $i_r = n_r$ , then  $i_r := 1$  and ranks are ordered in reverse lexicographic order to decide what indices are incremented). This

is called *row major ordering*, as opposed to *column major ordering*. That is, in row major,  $g$  is defined as

$$g(i_1, i_2, \dots, i_r) = \sum_{p=1}^r \left( \prod_{k=p+1}^r n_k \right) i_p \quad (1)$$

### Definition 8. Reshaping

A *reshape* operation is an isomorphism defined on a tensor space  $\mathbb{T} = \bigotimes_{k=1}^r \mathbb{V}_k$  such that some of its basis vector spaces  $\{\mathbb{V}_k\}$  are de-flattened and some of its slice subspaces are flattened.

### 1.1.3 Binary operations

We define binary operations on tensors that we'll later have use for. In particular, we define *tensor contraction* which is sometimes called *tensor multiplication*, *tensor product* or *tensor dotproduct* by other sources. We also define *convolution* and *pooling* which serve as the common building blocks of convolution neural network architectures (see Section 1.2.7).

### Definition 9. Contraction

A *tensor contraction* between two tensors, along ranks of same dimensions, is defined by natural extension of the dot product operation to tensors.

More precisely, let  $\mathbb{T}_1$  a tensor space of shape  $n_1^{(1)} \times n_2^{(1)} \times \dots \times n_{r_1}^{(1)}$ , and  $\mathbb{T}_2$  a tensor space of shape  $n_1^{(2)} \times n_2^{(2)} \times \dots \times n_{r_2}^{(2)}$ , such that  $\forall k \in \llbracket 1, s \rrbracket, n_{r_1-(s-k)}^{(1)} = n_k^{(2)}$ , then the tensor contraction between  $t_1 \in \mathbb{T}_1$  and  $t_2 \in \mathbb{T}_2$  is defined as:

$$\left\{ \begin{array}{l} t_1 \otimes t_2 = t_3 \in \mathbb{T}_3 \text{ of shape } n_1^{(1)} \times \dots \times n_{r_1-s}^{(1)} \times n_{s+1}^{(2)} \times \dots \times n_{r_2}^{(2)} \text{ where} \\ t_3[i_1^{(1)}, \dots, i_{r_1-s}^{(1)}, i_{s+1}^{(2)}, \dots, i_{r_2}^{(2)}] = \\ \sum_{k_1=1}^{n_1^{(2)}} \dots \sum_{k_s=1}^{n_s^{(2)}} t_1[i_1^{(1)}, \dots, i_{r_1-s}^{(1)}, k_1, \dots, k_s] t_2[k_1, \dots, k_s, i_{s+1}^{(2)}, \dots, i_{r_2}^{(2)}] \end{array} \right.$$

For the sake of simplicity, we omit the case where the contracted ranks are not the last ones for  $t_1$  and the first ones for  $t_2$ . But this definition still holds in the general case subject to a permutation of the indices.

**Definition 10. Covariant and contravariant indices**

Given a tensor contraction  $t_1 \otimes t_2$ , indices of the left hand operand  $t_1$  that are not contracted are called *covariant* indices. Those that are contracted are called *contravariant* indices. For the right operand  $t_2$ , the naming convention is the opposite. The set of covariant and contravariant indices of both operands are called the *transformation laws* of the tensor contraction.

*Remark.* Contrary to most mathematical definitions, tensors in deep learning are independent of any transformation law, so that they must be specified for tensor contractions.

**Einstein summation convention**

Using subscript notation for covariant indices and superscript notation for contravariant indices, the previous tensor contraction can be written using the Einstein summation convention as:

$$t_{1,i_1^{(1)} \dots i_{r_1-s}^{(1)}}{}^{k_1 \dots k_s} t_{2,k_1 \dots k_s}{}^{i_{s+1}^{(2)} \dots i_{r_2}^{(2)}} = t_{3,i_1^{(1)} \dots i_{r_1-s}^{(1)}}{}^{i_{s+1}^{(2)} \dots i_{r_2}^{(2)}} \quad (2)$$

Dot product  $u_k v^k = \lambda$  and matrix product  $A_i{}^k B_k{}^j = C_i{}^j$  are common examples of tensor contractions.

**Proposition 11.** A contraction can be rewritten as a matrix product.

*Proof.* Using notation of (2), with the reshapings  $t_1 \mapsto T_1$ ,  $t_2 \mapsto T_2$  and  $t_3 \mapsto T_3$  defined by grouping all covariant indices into a single index and all contravariant indices into another single index, we can rewrite

$$T_{1,g_i(i_1^{(1)}, \dots, i_{r_1-s}^{(1)})}{}^{g_k(k_1, \dots, k_s)} T_{2,g_k(k_1, \dots, k_s)}{}^{g_j(i_{s+1}^{(2)}, \dots, i_{r_2}^{(2)})} = T_{3,g_i(i_1^{(1)}, \dots, i_{r_1-s}^{(1)})}{}^{g_j(i_{s+1}^{(2)}, \dots, i_{r_2}^{(2)})}$$

where  $g_i$ ,  $g_k$  and  $g_j$  are bijections defined similarly as in (1).  $\square$

**Definition 12. Convolution**

The  $n$ -dimensional convolution, denoted  $*^n$ , between  $t_1 \in \mathbb{T}_1$  and  $t_2 \in \mathbb{T}_2$ , where  $\mathbb{T}_1$  and  $\mathbb{T}_2$  are of the same rank  $n$  such that  $\forall p \in \llbracket 1, n \rrbracket, n_p^{(1)} \geq n_p^{(2)}$ , is defined as:

$$\left\{ \begin{array}{l} t_1 *^n t_2 = t_3 \in \mathbb{T}_3 \text{ of shape } n_1^{(3)} \times \cdots \times n_n^{(3)} \text{ where} \\ \forall p \in \llbracket 1, n \rrbracket, n_p^{(3)} = n_p^{(1)} - n_p^{(2)} + 1 \\ t_3[i_1, \dots, i_n] = \sum_{k_1=1}^{n_1^{(2)}} \cdots \sum_{k_n=1}^{n_n^{(2)}} t_1[i_1 + n_1^{(2)} - k_1, \dots, i_n + n_n^{(2)} - k_n] t_2[k_1, \dots, k_n] \end{array} \right.$$

**Proposition 13.** A convolution can be rewritten as a matrix product.

*Proof.* Let  $t_1 *^n t_2 = t_3$  defined as previously with  $\mathbb{T}_1 = \bigotimes_{k=1}^r \mathbb{V}_k^{(1)}$ ,  $\mathbb{T}_2 = \bigotimes_{k=1}^r \mathbb{V}_k^{(2)}$ .

Let  $t'_1 \in \bigotimes_{k=1}^r \mathbb{V}_k^{(1)} \otimes \bigotimes_{k=1}^r \mathbb{V}_k^{(2)}$  such that  $t'_1[i_1, \dots, i_n, k_1, \dots, k_n] = t_1[i_1 + n_1^{(2)} - k_1, \dots, i_n + n_n^{(2)} - k_n]$ , then

$$t_3[i_1, \dots, i_n] = \sum_{k_1=1}^{n_1^{(2)}} \cdots \sum_{k_n=1}^{n_n^{(2)}} t'_1[i_1, \dots, i_n, k_1, \dots, k_n] t_2[k_1, \dots, k_n]$$

where we recognize a tensor contraction. Proposition 11 concludes.  $\square$

The two following operations are meant to further decrease the shape of the resulting output.

**Definition 14. Strided convolution**

The  $n$ -dimensional *strided* convolution, with strides  $s = (s_1, s_2, \dots, s_n)$ , denoted  $*_s^n$ , between  $t_1 \in \mathbb{T}_1$  and  $t_2 \in \mathbb{T}_2$ , where  $\mathbb{T}_1$  and  $\mathbb{T}_2$  are of the same rank

$n$  such that  $\forall p \in \llbracket 1, n \rrbracket, n_p^{(1)} \geq n_p^{(2)}$ , is defined as:

$$\left\{ \begin{array}{l} t_1 *_s^n t_2 = t_4 \in \mathbb{T}_4 \text{ of shape } n_1^{(4)} \times \cdots \times n_n^{(4)} \text{ where} \\ \forall p \in \llbracket 1, n \rrbracket, n_p^{(4)} = \lfloor \frac{n_p^{(1)} - n_p^{(2)} + 1}{s_p} \rfloor \\ t_4[i_1, \dots, i_n] = (t_1 *_s^n t_2)[(i_1 - 1)s_n + 1, \dots, (i_n - 1)s_n + 1] \end{array} \right.$$

*Remark.* Informally, a strided convolution is defined as if it were a regular subsampling of a convolution. They match if  $s = (1, 1, \dots, 1)$ .

### Definition 15. Pooling

Let a real-valued function  $f$  defined on all tensor spaces of any shape, *e.g.* the *max* or *average* function. An  $f$ -pooling operation is a mapping  $t \mapsto t'$  such that each entry of  $t'$  is an image by  $f$  of a subtensor of  $t$ .

*Remark.* Usually, the set of subtensors that are reduced by  $f$  into entries of  $t'$  are defined by a regular partition of the entries of  $t$ .

## 1.2 Neural Networks

A feed-forward neural network could originally be formalized as a composite function chaining linear and non-linear functions (Rumelhart et al., 1985; LeCun et al., 1989; LeCun, Bengio, et al., 1995), even up until the important breakthroughs that generated a surge of interest in the field (Hinton et al., 2012; Krizhevsky et al., 2012; Simonyan and Zisserman, 2014). However, in more recent advances, more complex architectures have emerged (Szegedy et al., 2015; He et al., 2016; Zoph and Le, 2016; Huang et al., 2017), such that the former formalization does not suffice. We provide a definition for the first kind of neural networks (Definition 16) and use it to present its related concepts. Then we give a more generic definition (Definition 20).

Note that in this manuscript, we only consider neural networks that are *feed-forward* (Zell, 1994; Wikipedia, 2018).

### 1.2.1 Simple formalization

We denote by  $I_f$  the *domain of definition* of a function  $f$  ("I" stands for "input") and by  $O_f = f(I_f)$  its *image* ("O" stands for "output"), and we represent it as  $I_f \xrightarrow{f} O_f$ .

#### Definition 16. Neural network (simply connected)

Let  $f$  be a function such that  $I_f$  and  $O_f$  are vector or tensor spaces.

$f$  is a (*simply connected*) *neural network function* if there are a series of affine functions  $(g_k)_{k=1,2,\dots,L}$  and a series of non-linear derivable univariate functions  $(h_k)_{k=1,2,\dots,L}$  such that:

$$\left\{ \begin{array}{l} \forall k \in \llbracket 1, L \rrbracket, f_k = h_k \circ g_k, \\ I_f = I_{f_1} \xrightarrow{f_1} O_{f_1} \cong I_{f_2} \xrightarrow{f_2} \dots \xrightarrow{f_L} O_{f_L} = O_f, \\ f = f_L \circ \dots \circ f_2 \circ f_1 \end{array} \right.$$

The couple  $(g_k, h_k)$  is called the *k-th layer* of the neural network.  $L$  is its

depth. For  $x \in I_f$ , we denote by  $x_k = f_k \circ \dots \circ f_2 \circ f_1(x)$  the *activations* of the  $k$ -th layer. We denote by  $\mathcal{N}$  the set of neural network functions.

**Definition 17. Activation function**

An *activation function*  $h$  is a real-valued univariate function that is non-linear and derivable, that is also defined by extension on any linear space with the functional notation  $h(v)[i] = h(v[i])$ .

**Definition 18. Layer**

A couple  $(g, h)$ , where  $g$  is an affine or linear function, and  $h$  is an activation function is called a *layer*. The set of layers is denoted  $\mathcal{L}$ .

**Adoption of ReLU activations**

Historically, sigmoidal and tanh activations were mostly used (Cybenko, 1989; LeCun et al., 1989). However in recent practice, the *rectified linear unit* (ReLU), which implements the *rectifier* function  $h : x \mapsto \max(0, x)$  with convention  $h'(0) = 0$  (first introduced as the *positive part*, Jarrett et al., 2009), is the most used activation, as it was demonstrated to be faster and to obtain better results (Glorot et al., 2011). ReLU originated numerous variants *e.g.* *leaky rectified linear unit* (Maas et al., 2013), *parametric rectified linear unit* (PReLU, He et al., 2015), *exponential linear unit* (ELU, Clevert et al., 2015), *scaled exponential linear unit* (SELU, Klambauer et al., 2017).

**Universal approximation**

Early researches have shown that neural networks with one level of depth can approximate any real-valued function defined on a compact subset of  $\mathbb{R}^n$ . This result was first proved for sigmoidal activations (Cybenko, 1989), and then it was shown it did not depend on the sigmoidal activations (Hornik et al., 1989; Hornik, 1991).

For example, for the application of supervised learning when a neural network is trained from data (see Section 1.2.4), this result is quite important because



it brings theoretical justification that the objective exists (even though it doesn't inform whether an algorithm to approach it exists or is efficient).

### Computational difficulty

However, reaching such objective is a computationally difficult problem, which drove back interest from the field. Thanks to better hardware and to using better initialization schemes that speed up learning, researchers started to report more successes with deep neural networks (Hinton et al., 2006; Glorot and Bengio, 2010) ; see (Bengio, 2009) for a review of this period. It ultimately came to a surge of interest in the field after a significant breakthrough on the imagenet dataset (Deng et al., 2009) with a deep convolutional architecture (Krizhevsky et al., 2012), see Section 1.2.7. The use of the fast ReLU activation function (Glorot et al., 2011) as well as leveraging graphical processing units with CUDA (Nickolls et al., 2008) were also key factors in overcoming this computational difficulty.

### Expressivity and expressive efficiency

The study of the *expressivity* (also called *representational power*) of families of neural networks is the field that is interested in the range of functions that can be realized or approximated by this family (Håstad and Goldmann, 1991; Pascanu et al., 2013). In general, given a maximal error  $\epsilon$  and an objective  $F$ , the more expressive is a family  $N \subset \mathcal{N}$ , the more likely it is to contain an approximation  $f \in N$  such that  $d(f, F) < \epsilon$ . However, if we consider the approximation  $f_{min} \in N$  that have the lowest number of neurons, it is possible that  $f_{min}$  is still too large and may be unpractical. For this reason, expressivity is often studied along the related notion of *expressive efficiency* (Delalleau and Bengio, 2011; Cohen et al., 2018).

### Rectifier neural networks

Of particular interest for the intuition is a result stating that a simply connected neural networks with only ReLU activations (a rectifier neural net-

work) is a piecewise linear function (Pascanu et al., 2013; Montufar et al., 2014), and that conversely any piecewise linear function is also a rectifier neural network such that an upper bound of its depth is logarithmically related to the input dimension (Arora et al., 2018, th. 2.1.). Their expressive efficiency have also been demonstrated compared to neural networks using threshold or sigmoid activations (Pan and Srikumar, 2016).

### Benefits of depth

Expressive efficiency analysis have demonstrated the benefits of depth, *i.e.* a shallow neural network would need an unfeasible large number of neurons to approximate the function of a deep neural network (*e.g.* Delalleau and Bengio, 2011; Bianchini and Scarselli, 2014; Poggio et al., 2015; Eldan and Shamir, 2016; Poole et al., 2016; Raghu et al., 2016; Cohen and Shashua, 2016; Mhaskar et al., 2016; Lin et al., 2017; Arora et al., 2018).

### Bias

Note that affine functions  $\tilde{g}$  can be written as a sum between a linear function  $g$  and a constant vector  $b$  which is called the *bias*. It augments the expressivity of the neural network's family of functions. For notational convenience, we will often omit to write down the biases in the layer's equations.

## 1.2.2 Generic formalization

The former neural networks are said to be *simply connected* because each layer only takes as input the output of the previous one. We'll give a more general definition after first defining branching operations.

### Definition 19. Branching

A *binary branching operation* between two tensors,  $x_{k_1} \bowtie x_{k_2}$ , outputs, subject to shape compatibility, either their addition, either their concatenation along a rank, or their concatenation as a list.

A *branching operation* between  $n$  tensors,  $x_{k_1} \bowtie x_{k_2} \bowtie \cdots \bowtie x_{k_n}$ , is a composition of binary branching operations, or is the identity function  $Id$  if  $n = 1$ . Branching operations are also naturally defined on tensor-valued functions through their realizations.

**Definition 20. Neural network (generic definition)**

The set of *neural network* functions  $\mathcal{N}$  is defined inductively as follows

1.  $Id \in \mathcal{N}$
2.  $f \in \mathcal{N} \wedge (g, h) \in \mathcal{L} \wedge O_f \subset I_g \Rightarrow h \circ g \circ f \in \mathcal{N}$
3. for all shape compatible branching operations:  
 $f_1, f_2, \dots, f_n \in \mathcal{N} \Rightarrow f_1 \bowtie f_2 \bowtie \cdots \bowtie f_n \in \mathcal{N}$

**Examples**

The neural network proposed in (Szegedy et al., 2015), called *Inception*, use depth-wise concatenation of feature maps. Residual networks (ResNets, He et al., 2016) make use of *residual connections*, also called *skip connections*, *i.e.* an activation that is used as input in a lower level is added to another activation at an upper level. Densely connected networks (DenseNets, Huang et al., 2017) have their activations concatenated with all lower level activations. These neural networks had demonstrated state of the art performances on the imagenet classification challenge (Deng et al., 2009), outperforming simply connected neural networks.

**Benefits of branching operations**

Recent works have provided rationales supporting benefits of using branching operations, thus giving justifications for architectures obtained with the generic formalization. In particular, (Cohen et al., 2018) have analyzed the impact of residual connections used in Wavenet-like architectures (Van Den Oord et al., 2016) in terms of expressive efficiency (see related paragraph

in Section 1.2.1) using tools from the field of tensor analysis ; (Orhan and Pitkow, 2018) have empirically demonstrated that skip connections can resolve some inefficiency problems inherent of fully-connected networks (dead activations, activations that are always equal, linearly dependent sets of activations).

*Remark.* For layer indexing convenience, we still use the simple formalization in the subsequent subsections, even though the presentation would be similar with the generic formalization.

### 1.2.3 Interpretation

Until now, we have formally introduced a neural network as a mathematical function. As its name suggests, such function can be interpreted from a connectivity viewpoint (LeCun, 1987).

#### Definition 21. Connectivity matrix

Let  $g$  a linear function. Without loss of generality subject to a flattening, let's suppose  $I_g$  and  $O_g$  are vector spaces. Then there exists a *connectivity matrix*  $W_g$ , such that:

$$\forall x \in I_g, g(x) = W_g x$$

We denote  $W_k$  the connectivity matrix of the  $k$ -th layer.

#### Biological inspiration

A (*computational*) *neuron* is a computational unit that is biologically inspired (McCulloch and Pitts, 1943). Each neuron is capable of:

1. receiving modulated signals from other neurons and aggregate them,
2. applying to the result a derivable activation,
3. passing the signal to other neurons.

That is to say, each domain  $\{I_{f_k}\}$  and  $O_f$  can be interpreted as a layer of neurons, with one neuron for each dimension. The connectivity matrices  $\{W_k\}$  describe the connections between each successive layers. A neuron is illustrated on Figure 1.

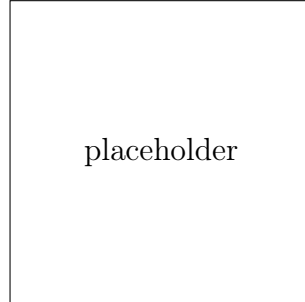


Figure 1: A neuron

#### 1.2.4 Parameterization and training

TODO: pass on this section

Given an objective function  $F$ , training is the process of incrementally modifying a neural network  $f$  upon obtaining a better approximation of  $F$ . The most used training algorithms are based on gradient descent, as proposed in (Widrow and Hoff, 1960). These algorithms became popular since (Rumelhart et al., 1985). Informally,  $f$  is parameterized with initial weights that characterize its linear parts. These weights are modified step by step in the opposite direction of their gradient with respect to a loss. All possible realizations of  $f$  through its weights draw a family  $N$  which expressivity is crucial for the success of the training. The common points between  $f$  and other objects of  $N$  define what is called a neural network *architecture*. That is

We present gradient based learning more formally in what follows.

**Definition 22. Architecture** Let  $f \in \mathcal{N}$  with weights  $(\theta_k)_k \in$ .

### Gradient descent

The most used training algorithms are based on gradient descent, as proposed in (Widrow and Hoff, 1960). These algorithms became popular since (Rumelhart et al., 1985). In order to be trained,  $f$  is parameterized with initial weights that characterize its linear parts. These weights are modified step by step in the opposite direction of their gradient with respect to a loss.

### Architecture

All possible realizations of  $f$  through its weights draw a family  $N$  which expressivity is crucial for the success of the training. The common points between  $f$  and other objects of  $N$  define what is called a neural network *architecture*. **TODO: refactor**

### Definition 23. Weights

Let consider the  $k$ -th layer of a neural network  $f$ . We define its weights as coordinates of a vector  $\theta_k$ , called the *weight kernel*, such that:

$$\forall(i, j), \begin{cases} \exists p, W_k[i, j] := \theta_k[p] \\ \text{or } W_k[i, j] = 0 \end{cases}$$

A weight  $p$  that appears multiple times in  $W_k$  is said to be *shared*. Two parameters of  $W_k$  that share a same weight  $p$  are said to be *tied*. The number of weights of the  $k$ -th layer is  $n_1^{(\theta_k)}$ .

### Learning

A *loss* function  $\mathcal{L}$  penalizes the output  $x_L = f(x)$  relatively to the approximation error  $|f(x) - F(x)|$ . Gradient w.r.t.  $\theta_k$ , denoted  $\vec{\nabla}_{\theta_k}$ , is used to update the weights via an optimization algorithm based on gradient descent and a

learning rate  $\alpha$ , that is:

$$\theta_k^{(\text{new})} = \theta_k^{(\text{old})} - \alpha \cdot \vec{\nabla}_{\theta_k} \left( \mathcal{L}(x_L, \theta_k^{(\text{old})}) + R(\theta_k^{(\text{old})}) \right) \quad (3)$$

where  $\alpha$  can be a scalar or a vector,  $\cdot$  can denote outer or pointwise product, and  $R$  is a regularizer. They depend on the optimization algorithm.

**TODO: examples of optimization**

### Linear complexity

Without loss of generality, we assume that the neural network is simply connected. Thanks to the chain rule,  $\vec{\nabla}_{\theta_k}$  can be computed using gradients that are w.r.t.  $x_k$ , denoted  $\vec{\nabla}_{x_k}$ , which in turn can be computed using gradients w.r.t. outputs of the next layer  $k+1$ , up to the gradients given on the output layer.

That is:

$$\vec{\nabla}_{\theta_k} = J_{\theta_k}(x_k) \vec{\nabla}_{x_k} \quad (4)$$

$$\begin{aligned} \vec{\nabla}_{x_k} &= J_{x_k}(x_{k+1}) \vec{\nabla}_{x_{k+1}} \\ \vec{\nabla}_{x_{k+1}} &= J_{x_{k+1}}(x_{k+2}) \vec{\nabla}_{x_{k+2}} \end{aligned} \quad (5)$$

...

$$\vec{\nabla}_{x_{L-1}} = J_{x_{L-1}}(x_L) \vec{\nabla}_{x_L}$$

Obtaining,

$$\vec{\nabla}_{\theta_k} = J_{\theta_k}(x_k) \left( \prod_{p=k}^{L-1} J_{x_p}(x_{p+1}) \right) \vec{\nabla}_{x_L} \quad (6)$$

where  $J_{\text{wrt}}(\cdot)$  are the respective jacobians which can be determined with the

layer's expressions and the  $\{x_k\}$ ; and  $\vec{\nabla}_{x_L}$  can be determined using  $\mathcal{L}$ ,  $R$  and  $x_L$ . This allows to compute the gradients with a complexity that is linear with the number of weights (only one computation of the activations), instead of being quadratic if it were done with the difference quotient expression of the derivatives (one more computation of the activations for each weight).

### Backpropagation

We can remark that (5) rewrites as

$$\begin{aligned}\vec{\nabla}_{x_k} &= J_{x_k}(x_{k+1}) \vec{\nabla}_{x_{k+1}} \\ &= J_{x'_k}(h(x'_k)) J_{x_k}(W_k x_k) \vec{\nabla}_{x_{k+1}}\end{aligned}\tag{7}$$

where  $x'_k = W_k x_k$ , and these jacobians can be expressed as:

$$J_{x'_k}(h(x'_k))[i, j] = \delta_i^j h'(x'_k[i])\tag{8}$$

$$J_{x'_k}(h(x'_k)) = I h'(x'_k)$$

$$J_{x_k}(W_k x_k) = W_k^T\tag{9}$$

That means that we can write  $\vec{\nabla}_{x_k} = (\tilde{h}_k \circ \tilde{g}_k)(\vec{\nabla}_{x_{k+1}})$  such that the connectivity matrix  $\widetilde{W}_k$  is obtained by transposition. This can be interpreted as gradient calculation being a *back-propagation* on the same neural network, in opposition of the *forward-propagation* done to compute the output.

**TODO: Overfitting remark**



### 1.2.5 Examples of layer

**Definition 24. Connections**

The set of *connections* of a layer  $(g, h)$ , denoted  $C_g$ , is defined as:

$$C_g = \{(i, j), \exists p, W_g[i, j] := \theta_g[p]\}$$

We have  $0 \leq |C_g| \leq n_1^{(W_g)} n_2^{(W_g)}$ .

**Definition 25. Dense layer**

A *dense layer*  $(g, h)$  is a layer such that  $|C_g| = n_1^{(W_g)} n_2^{(W_g)}$ , *i.e.* all possible connections exist. The map  $(i, j) \mapsto p$  is usually a bijection, meaning that there is no weight sharing.

**Definition 26. Partially connected layer**

A *partially connected layer*  $(g, h)$  is a layer such that  $|C_g| < n_1^{(W_g)} n_2^{(W_g)}$ .  
A *sparsely connected layer*  $(g, h)$  is a layer such that  $|C_g| \ll n_1^{(W_g)} n_2^{(W_g)}$ .

**Definition 27. Convolutional layer**

A *n-dimensional convolutional layer*  $(g, h)$  is such that the weight kernel  $\theta_g$  can be reshaped into a tensor  $w$  of rank  $n + 2$ , and such that

$$\begin{cases} I_g \text{ and } O_g \text{ are tensor spaces of rank } n + 1 \\ \forall x \in I_g, g(x) = (g(x)_q = \sum_p x_p *^n w_{p,q})_{\forall q} \end{cases}$$

where  $p$  and  $q$  index slices along the last ranks.

**Definition 28. Feature maps and input channels**

The slices  $g(x)_q$  are typically called *feature maps*, and the slices  $x_p$  are called *input channels*. Let's denote by  $n_o = n_{n+1}^{(O_g)}$  and  $n_i = n_{n+1}^{(I_g)}$  the number of feature maps and input channels. In other words, Definition 27 means that for each feature maps, a convolution layer computes  $n_i$  convolutions and sums them, computing a total of  $n_i \times n_o$  convolutions.

*Remark.* Note that because they are simply summed, entries of two different input channels that have the same coordinates are assumed to share some sort of relationship. For instance on images, entries of each input channel (typically corresponding to Red, Green and Blue channels) that have the same coordinates share the same pixel location.

**Affect on tensor shape**

Given a tensor input  $x$ , the  $n$ -dimensional convolutions between the inputs channels  $x_p$  and slices of a weight tensor  $w_{p,q}$  would result in outputs  $y_q$  of shape  $n_1^{(x)} - n_1^{(w)} + 1 \times \dots \times n_n^{(x)} - n_n^{(w)} + 1$ . So, in order to preserve shapes, a padding operation must pad  $x$  with  $n_1^{(w)} - 1 \times \dots \times n_n^{(w)} - 1$  zeros beforehand. For example, the padding function of the library *tensorflow* (Abadi et al., 2015) pads each rank with a balanced number of zeros on the left and right indices (except if  $n_t^{(w)} - 1$  is odd then there is one more zero on the left).

**Definition 29. Padding**

A convolutional layer with *padding*  $(g, h)$  is such that  $g$  can be decomposed as  $g = g_{\text{pad}} \circ g'$ , where  $g'$  is the linear part of a convolution layer as in Definition 27, and  $g_{\text{pad}}$  is an operation that pads zeros to its inputs such that  $g$  preserves tensor shapes.

*Remark.* One asset of padding operations is that they limit the possible loss of information on the borders of the subsequent convolutions, as well as preventing a decrease in size. Moreover, preserving shape is needed to build some neural network architectures, especially for ones with branching operations *e.g.* examples in Section 1.2.2. On the other hand, they increase memory and computational footprints.

**Proposition 30. Connectivity matrix of a convolution with padding**

A convolutional layer with padding  $(g, h)$  is equivalently defined as  $W_g$  being a  $n_i \times n_o$  block matrix such that its blocks are Toeplitz matrices.

*Proof.* Let's consider the slices indexed by  $p$  and  $q$ , and to simplify the notations, let's drop the subscripts  $p, q$ . We recall from Definition 12 that

$$\begin{aligned}
y &= (x *^n w)[j_1, \dots, j_n] \\
&= \sum_{k_1=1}^{n_1^{(w)}} \cdots \sum_{k_n=1}^{n_n^{(w)}} x[j_1 + n_1^{(w)} - k_1, \dots, j_n + n_n^{(w)} - k_n] w[k_1, \dots, k_n] \\
&= \sum_{i_1=j_1}^{j_1+n_1^{(w)}-1} \cdots \sum_{i_n=j_n}^{j_n+n_n^{(w)}-1} x[i_1, \dots, i_n] w[j_1 + n_1^{(w)} - i_1, \dots, j_n + n_n^{(w)} - i_n] \\
&= \sum_{i_1=1}^{n_1^{(x)}} \cdots \sum_{i_n=1}^{n_n^{(x)}} x[i_1, \dots, i_n] \tilde{w}[i_1, j_1, \dots, i_n, j_n] \\
&\text{where } \tilde{w}[i_1, j_1, \dots, i_n, j_n] = \\
&\quad \begin{cases} w[j_1 + n_1^{(w)} - i_1, \dots, j_n + n_n^{(w)} - i_n] & \text{if } \forall t, 0 \leq i_t - j_t \leq n_t^{(w)} - 1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Using Einstein summation convention as in (2) and permuting indices, we recognize the following tensor contraction

$$y_{j_1 \dots j_n} = x_{i_1 \dots i_n} \tilde{w}^{i_1 \dots i_n}_{j_1 \dots j_n} \quad (10)$$

Following Proposition 11, we reshape (10) as a matrix product. To reshape  $y \mapsto Y$ , we use the row major order bijections  $g_j$  as in (1) defined onto  $\{(j_1, \dots, j_n), \forall t, 1 \leq j_t \leq n_t^{(y)}\}$ . To reshape  $x \mapsto X$ , we use the same row major order bijection  $g_j$ , however defined on the indices that support non zero-padded values, so that zero-padded values are lost after reshaping. That is, we use a bijection  $g_i$  such that  $g_i(i_1, i_2, \dots, i_n) = g_j(i_1 - o_1, i_2 - o_2, \dots, i_n - o_n)$  defined if and only if  $\forall t, 1 + o_t \leq i_t \leq n_t^{(y)}$ , where the  $\{o_t\}$  are the starting offsets of the non zero-padded values.  $\tilde{w} \mapsto W$  is reshaped by using  $g_j$  for its covariant indices, and  $g_i$  for its contravariant indices. The entries lost by using  $g_i$  do not matter because they would have been nullified by the resulting matrix product. We remark that  $W$  is exactly the block  $(p, q)$  of  $W_g$  (and not of  $W_{g'}$ ). Now let's prove that it is a Toeplitz matrix. Thanks to the linearity of the expression (1) of  $g_j$ , by denoting  $i'_t = i_t - o_t$ , we obtain

$$g_i(i_1, i_2, \dots, i_n) - g_j(j_1, j_2, \dots, j_n) = g_j(i'_1 - j_1, i'_2 - j_2, \dots, i'_n - j_n) \quad (11)$$

To simplify the notations, let's drop the arguments of  $g_i$  and  $g_j$ . By bijectivity of  $g_j$ , (11) tells us that  $g_i - g_j$  remains constant if and only if  $i'_t - j_t$  remains constant for all  $t$ . Recall that

$$W[g_i, g_j] = \begin{cases} w[j_1 + n_1^{(w)} - i'_1, \dots, j_n + n_n^{(w)} - i'_n] & \text{if } \forall t, 0 \leq i'_t - j_t \leq n_t^{(w)} - 1 \\ 0 & \text{otherwise} \end{cases} \quad (12)$$

Hence, on a diagonal of  $W$ ,  $g_i - g_j$  remaining constant means that  $W[g_i, g_j]$  also remains constants. So  $W$  is a Toeplitz matrix.

The converse is also true as we used invertible functions in the index spaces through the proof.  $\square$

*Remark.* The former proof makes clear that the result doesn't hold in case there is no padding. This is due to border effects when the index of the  $n^{\text{th}}$  rank resets in the definition of the row-major ordering function  $g_j$  that would be used. Indeed, under appropriate definitions, the matrices could be seen as almost Toeplitz.

### Benefits of convolutional layers

Comparatively with dense layers, convolution layers enjoy a significant decrease in the number of weights. For example, an input  $2 \times 2$  convolution on images with 3-color input channels, would breed only 12 weights per feature maps, independently of the numbers of input neurons. On image datasets, their usage also breeds a significant boost in performance compared with dense layers (Krizhevsky et al., 2012), for they allow to take advantage of the topology of the inputs while dense layers don't (LeCun, Bengio, et al., 1995). A more thorough comparison and explanation of their assets will be discussed in Section 2.1.3.

#### Definition 31. Stride

A convolutional layer with *stride* is a convolutional layer that computes strided convolutions (with  $\text{stride} > 1$ ) instead of convolutions.

#### Definition 32. Pooling

A layer with *pooling* ( $g, h$ ) is such that  $g$  can be decomposed as  $g = g' \circ g_{\text{pool}}$ , where  $g_{\text{pool}}$  is a pooling operation.

### Downscaling

Layers with stride or pooling downscale the signals that passes through the layer. These types of layers allows to compute features at a coarser level, giving the intuition that the deeper a layer is in the network, the more abstract is the information captured by the weights of the layer.

TODO: below

## 1.2.6 Examples of regularization

### Overfitting

TODO:

A layer with *dropout*  $(g, h)$  is such that  $h = h_1 \circ h_2$ , where  $(g, h_2)$  is a layer and  $h_1$  is a dropout operation (Srivastava et al., 2014). When dropout is used, a certain number of neurons are randomly set to zero during the training phase, compensated at test time by scaling down the whole layer. This is done to prevent overfitting.

## 1.2.7 Examples of architecture

TODO: rephrase

A multilayer perceptron (MLP) (Hornik et al., 1989) is a neural network composed of only dense layers. A convolutional neural network (CNN) (LeCun et al., 1998) is a neural network composed of convolutional layers.

Neural networks are commonly used for machine learning tasks. For example, to perform supervised classification, we usually add a dense output layer  $s = (g_{L+1}, h_{L+1})$  with as many neurons as classes. We measure the error between an output and its expected output with a discriminative loss function  $\mathcal{L}$ . During the training phase, the weights of the network are adapted for the classification task based on the errors that are back-propagated (Hornik et al., 1989) via the chain rule and according to a chosen optimization algorithm (*e.g.* Bottou, 2010).

## 1.3 Graphs and signals

### 1.3.1 Basic definitions

#### 1.3.1.1 Graphs

##### **Definition 33. Graph**

A *graph*  $G$  is defined as a couple of sets  $\langle V, E \rangle$  where  $V$  is the set of *vertices*, also called *nodes*, and  $E \subseteq \binom{V}{2}$  is the set of *edges*. For all  $u, v \in V$  we define the relation  $u \sim v \Leftrightarrow \{u, v\} \in E$ . Unless stated otherwise, we will consider only *weighted* graphs *i.e.* each graph  $G$  is associated with a weight mapping  $w : E \rightarrow \mathbb{R}^*$ .

Figure 2 illustrates an example of a graph. Note that we employ interchangeably the terms *vertex* and *node*.

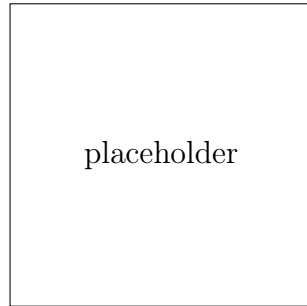


Figure 2: Example of a graph

*Remark.* According to this definition, we consider simple graphs *i.e.* no two edges share the same set of vertices and there is no self-loop.

**Definition 34. Path**

A *path* of length  $n \in \mathbb{N}$  in a graph  $G = \langle V, E \rangle$  is a sequence  $(v_1 v_2 \cdots v_n)$  in  $V$  such that  $\forall i, v_i \sim v_{i+1}$ .

*Remark.* Our definition of graphs admit no self-loop so  $\forall i, v_i \neq v_{i+1}$

**Definition 35. Order**

The order of a graph  $G = \langle V, E \rangle$  is define as  $\text{order}(G) = |V| \in \mathbb{N} \cup \{+\infty\}$

**Definition 36. Adjacency matrix**

The *adjacency matrix* of a finite graph  $G = \langle V, E \rangle$  of order  $n$ , is a  $n \times n$  real-valued matrix  $A$  associated to an indexing of  $V = \{v_1, v_2, \dots, v_n\}$ , such that

$$A[i, j] = \begin{cases} w(\{v_i, v_j\}) & \text{if } v_i \sim v_j \\ 0 & \text{otherwise} \end{cases}$$

**Definition 37. Degree**

The *degree* of a vertex  $v \in V$  of a graph  $G = \langle V, E \rangle$  is defined as  $\deg(v) = |\{u \in V, u \sim v\}| \in \mathbb{N} \cup \{+\infty\}$ .

The *degree* of the graph  $G$  is defined as  $\deg(G) = \max_{v \in V} \{\deg(v)\}$ .

A graph is said to be *regular* if  $\deg$  is constant on the vertices.

**Definition 38. Degree matrix**

The *degree matrix* of a finite graph  $G = \langle V, E \rangle$  of order  $n$ , is the diagonal matrix  $D$ , associated to an indexing of  $V = \{v_1, v_2, \dots, v_n\}$ , such that  $D = \text{diag}(\deg(v_1), \deg(v_2), \dots, \deg(v_n))$ .



**Definition 39. Laplacian matrix**

The *laplacian matrix* of a graph  $G = \langle V, E \rangle$  of order  $n$ , associated to an indexing of  $V = \{v_1, v_2, \dots, v_n\}$ , is defined as  $L = D - A$ , where  $D$  is the degree matrix and  $A$  is the adjacency matrix.

**Definition 40. Digraph**

A digraph is an oriented graph *i.e.*  $E \subseteq V \times V - \{(v, v), v \in V\}$ . Contrary to a graph, the weight mapping  $w$ , the relation  $\sim$ , the adjacency matrix  $A$ , and the laplacian matrix  $L$  are not symmetric. Notions defined on graphs naturally extends to digraphs where possible.

**Definition 41. Bipartite graph**

A *bipartite graph* is a triplet of sets  $\langle V^{(1)}, V^{(2)}, E \rangle$ , where  $V^{(1)}$  and  $V^{(2)}$  are sets of vertices,  $V^{(1)} \cap V^{(2)} \neq \emptyset$ , and  $E \subseteq V^{(1)} \times V^{(2)}$ . It is associated with a weight mapping  $w : E \rightarrow \mathbb{R}^*$ . Its adjacency matrix  $A$  is associated to indexings of  $V^{(1)} = \{v_1^{(1)}, v_2^{(1)}, \dots, v_n^{(1)}\}$  and  $V^{(2)} = \{v_1^{(2)}, v_2^{(2)}, \dots, v_n^{(2)}\}$ , such that

$$A[i, j] = \begin{cases} w\left((v_i^{(1)}, v_j^{(2)})\right) & \text{if } (v_i^{(1)}, v_j^{(2)}) \in E \\ 0 & \text{otherwise} \end{cases}$$

**Definition 42. Induced subgraph**

The *subgraph*  $\tilde{G} = \langle \tilde{V}, \tilde{E} \rangle$  of a graph  $G = \langle V, E \rangle$ , *induced* by  $\tilde{V} \subseteq V$ , is such that  $\forall (u, v) \in \tilde{V}^2, u \tilde{\sim} v \Leftrightarrow u \stackrel{G}{\sim} v$ .

TODO: subgraph (including direct or undirected)

**Definition 43. Grid graph**

A *grid graph*  $G = \langle V, E \rangle$  is such that  $V \cong \mathbb{Z}^2$ ,  $v_1 \sim v_2 \Rightarrow \|v_2 - v_1\|_\infty \in \{0, 1\}$  and either one of the following is true:

$$\begin{cases} (i_1, j_1) \sim (i_2, j_2) \Leftrightarrow |i_2 - i_1| \text{ XOR } |j_2 - j_1| & (4 \text{ neighbours}) \\ (i_1, j_1) \sim (i_2, j_2) \Leftrightarrow |i_2 - i_1| \text{ AND } |j_2 - j_1| & (4 \text{ neighbours}) \\ (i_1, j_1) \sim (i_2, j_2) \Leftrightarrow |i_2 - i_1| \text{ OR } |j_2 - j_1| & (8 \text{ neighbours}) \end{cases}$$

A *(rectangular) grid graph* of size  $n \times m$  is the subgraph of a grid graph induced by  $\llbracket 1, n \rrbracket \times \llbracket 1, m \rrbracket$ .

A *square grid graph* is a rectangular grid graph of square size.

**1.3.1.2 (Real-valued) Signals****Definition 44. Signal space**

The *signal space*  $\mathcal{S}(V)$ , over the set  $V$ , is the linear space of real-valued functions defined on  $V$ .

We have  $\dim(\mathcal{S}(V)) = |V| \in \mathbb{N} \cup \{+\infty\}$ .

*Remark.* In particular, a vector space, and more generally a tensor space, are finite-dimensional signal spaces over any of their bases.

**Definition 45. Signal**

A *signal* over  $V$ ,  $s \in \mathcal{S}(V)$ , is a function  $s : V \rightarrow \mathbb{R}$ .

An *entry* of a signal  $s$  is an image by  $s$  of some  $v \in V$  and we denote  $s[v]$ .

If  $v$  is represented by a  $n$ -tuple, we can also write  $s[v_1, v_2, \dots, v_n]$ .

The *support* of a signal  $s \in \mathcal{S}(V)$  is  $\text{supp}(s) = \{v \in V, s[v] \neq 0\}$ .

**Definition 46. Graph signal**

A *graph signal* over  $G$  is a signal over its vertex set. We denote by  $\mathcal{S}(G)$  the graph signal space.

We have  $\dim(\mathcal{S}(G)) = \text{order}(G) \in \mathbb{N} \cup \{+\infty\}$ .

**Definition 47. Underlying structure**

An (*underlying*) *structure* of a signal  $s$  over a set  $V$ , is a graph  $G$  with vertex set  $V$ .

**Examples**

When there is a unique clear underlying structure, we say that it is *the* underlying structure. For example, images are compactly supported signals over  $\mathbb{Z}^2$  and their underlying structure is a rectangular grid graph. Time series are signals over  $\mathbb{N}$  and their underlying structure is a line graph. The underlying structure of a graph signal is obviously the graph itself.

**1.3.2 Graphs in deep learning**

TODO: below

We come across the notion of graphs several times in deep learning:

- Connections between two layers of a deep learning model can be represented as a bipartite graph, the *connectivity graph*. It encodes how the information is propagated through a layer to another. See Section 1.3.2.1.
- Neural architectures can be represented by a graph. In particular, a computation graph is used by deep learning programming languages to keep track of the dependencies between layers of a deep learning model, in order to compute forward and back-propagation. See Section 1.3.2.2.

- A graph can represent the underlying structure of an object (often a vector or a signal). The nodes represent its features, and the edges represent some structural property. See Section 1.3.2.3.
- Datasets can also be graph-structured. The nodes represent the objects of the dataset, and its edge represent some sort of relation between them. See Section 1.3.2.4.

### 1.3.2.1 Connectivity graph

A Connectivity graph is the bipartite graph whose adjacency matrix is the connectivity matrix of a layer of neurons. Formally, given a linear part of a layer, let  $\mathbf{x}$  and  $\mathbf{y}$  be the input and output signals,  $n$  the size of the set of input neurons  $N = \{u_1, u_2, \dots, u_n\}$ , and  $m$  the size of the set of output neurons  $M = \{v_1, v_2, \dots, v_m\}$ . This layer implements the equation  $y = \Theta x$  where  $\Theta$  is a  $n \times m$  matrix.

**Definition 48.** The *connectivity graph*  $G = (V, E)$  is defined such that  $V = N \cup M$  and  $E = \{(u_i, v_j) \in N \times M, \Theta_{ij} \neq 0\}$ .

I.e. the connectivity graph is obtained by drawing an edge between neurons for which  $\Theta_{ij} \neq 0$ . For instance, in the special case of a complete bipartite graph, we would obtain a dense layer. Connectivity graphs are especially useful to represent partially connected layers, for which most of the  $\Theta_{ij}$  are 0. For example, in the case of layers characterized by a small local receptive field, the connectivity graph would be sparse, and output neurons would be connected to a set of input neurons that corresponds to features that are close together in the input space. Figure 3 depicts some examples.

**TODO:** Figure 3. It's just a placeholder right now

Connectivity graphs also allow to graphically modelize how weights are tied in a neural layer. Let's suppose the  $\Theta_{ij}$  are taking their values only into the finite set  $K = \{w_1, w_2, \dots, w_\kappa\}$  of size  $\kappa$ , which we will refer to as the *kernel*

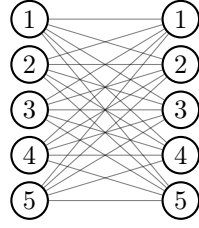


Figure 3: Examples

of *weights*. Then we can define a labelling of the edges  $s : E \rightarrow K$ .  $s$  is called the *weight sharing scheme* of the layer. This layer can then be formulated as  $\forall v \in M, y_v = \sum_{u \in N, (u,v) \in E} w_{s(u,v)} x_u$ . Figure 4 depicts the connectivity graph of a 1-d convolution layer and its weight sharing scheme.

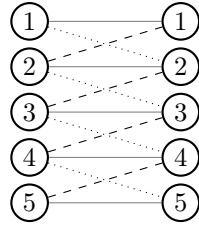


Figure 4: Depiction of a 1D-convolutional layer and its weight sharing scheme.

TODO: Add weight sharing scheme in Figure 4

#### 1.3.2.2 Computation graph

#### 1.3.2.3 Underlying graph structure and signals

#### 1.3.2.4 Graph-structured dataset



# Chapter 2

## Convolutions on graph domains

### Introduction

Defining a convolution of signals over graph domains is a challenging problem. If the graph is not a grid graph, there exists no natural extension of the euclidean convolution.

In Section 2.1, we analyze the reasons why the euclidean convolution operator is useful in deep learning, and give a characterization. Then we will search for domains onto which a convolution with these properties can be naturally obtained.

This will lead us to put our interest on representation theory and convolutions defined on groups in Section 2.2. As the euclidean convolution is just a particular case of the group convolution, it makes perfect sense to steer our construction in this direction. Hence, we will aim at transferring its representation on the vertex domain.

Then, in Section 2.3, we will introduce the role of the edge set and see how it should influence it. This will provide us with some particular classes of graphs for which we will obtain a natural construction with the wanted characteristics that we exposed in the first place.

Finally, we will relax some aspect of the construction to adapt it to general graphs in Section 2.4. The obtained construction is a set of general expressions that describes convolutions on graph domains and preserves some key properties.

We summarize our constructions in a conclusive Section 2.5.

## Contents

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<b>2.1</b>	<b>Analysis of the classical convolution . . . . .</b>	<b>41</b>
2.1.1	Properties of the convolution . . . . .	41
2.1.2	Characterization on grid graphs . . . . .	42
2.1.3	Usefulness of convolutions in deep learning . . . . .	45
<b>2.2</b>	<b>Construction from the vertex set . . . . .</b>	<b>47</b>
2.2.1	Steered construction from groups . . . . .	48
2.2.2	Construction under group actions . . . . .	52
2.2.3	Mixed domain formulation . . . . .	57
<b>2.3</b>	<b>Inclusion of the edge set in the construction . . . .</b>	<b>61</b>
2.3.1	Edge-constrained convolutions . . . . .	61
2.3.2	Intrinsic properties . . . . .	65
2.3.3	Stricly edge-constrained convolutions . . . . .	67
<b>2.4</b>	<b>From groups to groupoids . . . . .</b>	<b>70</b>
2.4.1	Motivation . . . . .	70
2.4.2	Definition of notions related to groupoids . . . . .	71
2.4.3	Construction of partial convolutions . . . . .	72
2.4.4	Construction of path convolutions . . . . .	78
<b>2.5</b>	<b>Conclusion . . . . .</b>	<b>83</b>

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## 2.1 Analysis of the classical convolution

In this section, we are exposing a few properties of the classical convolution that a generalization to graphs would likely try to preserve. For now let's consider a graph  $G$  agnostically of its edges *i.e.*  $G \cong V$  is just the set of its vertices.

### 2.1.1 Properties of the convolution

Consider an edge-less grid graph *i.e.*  $G \cong \mathbb{Z}^2$ . By restriction to compactly supported signals, this case encompass the case of images.

**Definition 49. Convolution on  $\mathcal{S}(\mathbb{Z}^2)$**

Recall that the (discrete) convolution between two signals  $s_1$  and  $s_2$  over  $\mathbb{Z}^2$  is a binary operation in  $\mathcal{S}(\mathbb{Z}^2)$  defined as:

$$\forall (a, b) \in \mathbb{Z}^2, (s_1 * s_2)[a, b] = \sum_i \sum_j s_1[i, j] s_2[a - i, b - j]$$

**Definition 50. Convolution operator**

A *convolution operator* is a function of the form  $f_w : x \mapsto x * w$ , where  $x$  and  $w$  are signals of domains for which the convolution  $*$  is defined. When  $*$  is not commutative, we differentiate the *right-action* operator  $x \mapsto x * w$  from the *left-action* one  $x \mapsto w * x$ .

The following properties of the convolution on  $\mathbb{Z}^2$  are of particular interest for our study.

### Linearity

Operators produced by the convolution are linear. So they can be used as linear parts of layers of neural networks.

**Locality and weight sharing**

When  $w$  is compactly supported on  $K$ , an impulse response  $f_w(x)[a, b]$  amounts to a  $w$ -weighted aggregation of entries of  $x$  in a neighbourhood of  $(a, b)$ , called the *local receptive field*.

**Commutativity**

The convolution is commutative. However, it won't necessarily be the case on other domains.

**Equivariance to translations**

Convolution operators are equivariant to translations. Below, we show that the converse of this result also holds with Proposition 54.

**2.1.2 Characterization on grid graphs**

Let's recall first what is a transformation, and how it acts on signals.

**Definition 51. Transformation**

A *transformation*  $f : V \rightarrow V$  is a function with same domain and codomain. The set of transformations is denoted  $\Phi(V)$ . The set of bijective transformations is denoted  $\Phi^*(V) \subset \Phi(V)$ .

In particular,  $\Phi^*(V)$  forms the symmetric group of  $V$  and can move signals of  $\mathcal{S}(V)$  by linear extension of its group action.

**Lemma 52. Extension to  $\mathcal{S}(V)$  by group action**

A bijective transformation  $f \in \Phi^*(V)$  can be extended linearly to the signal space  $\mathcal{S}(V)$ , and we have:

$$\forall s \in \mathcal{S}(V), \forall v \in V, f(s)[v] = s[f^{-1}(v)]$$

*Proof.* Let  $s \in \mathcal{S}(V)$ ,  $f \in \Phi^*(V)$ ,  $L_f \in \mathcal{L}(\mathcal{S}(V))$  s.t.  $\forall v \in V, L_f(\delta_v) = \delta_{f(v)}$ . Then, we have:

$$\begin{aligned} L_f(s) &= \sum_{v \in V} s[v] L_f(\delta_v) \\ &= \sum_{v \in V} s[v] \delta_{f(v)} \end{aligned}$$

$$\text{So, } \forall v \in V, L_f(s)[v] = s[f^{-1}(v)]$$

□

We also recall the formalism of translations.

**Definition 53. Translation on  $\mathcal{S}(\mathbb{Z}^2)$**

A translation on  $\mathbb{Z}^2$  is defined as a transformation  $t \in \Phi^*(\mathbb{Z}^2)$  such that

$$\exists(a, b) \in \mathbb{Z}^2, \forall(x, y) \in \mathbb{Z}^2, t(x, y) = (x + a, y + b)$$

It also acts on  $\mathcal{S}(\mathbb{Z}^2)$  with the notation  $t_{a,b}$  i.e.

$$\forall s \in \mathcal{S}(\mathbb{Z}^2), \forall(x, y) \in \mathbb{Z}^2, t_{a,b}(s)[x, y] = s[x - a, y - b]$$

For any set  $E$ , we denote by  $\mathcal{T}(E)$  its translations if they are defined.

The next proposition fully characterizes convolution operators with their translational equivariance property. This can be seen as a discretization of a classic result from the theory of distributions (Schwartz, 1957).

**Proposition 54. Characterization of convolution operators on  $\mathcal{S}(\mathbb{Z}^2)$**

On real-valued signals over  $\mathbb{Z}^2$ , the class of linear transformations that are equivariant to translations is exactly the class of convolutive operations i.e.

$$\exists w \in \mathcal{S}(\mathbb{Z}^2), f = . * w \Leftrightarrow \begin{cases} f \in \mathcal{L}(\mathcal{S}(\mathbb{Z}^2)) \\ \forall t \in \mathcal{T}(\mathcal{S}(\mathbb{Z}^2)), f \circ t = t \circ f \end{cases}$$

*Proof.* The result from left to right is a direct consequence of the definitions:

$$\begin{aligned}
& \forall s \in \mathcal{S}(\mathbb{Z}^2), \forall s' \in \mathcal{S}(\mathbb{Z}^2), \forall (\alpha, \beta) \in \mathbb{R}^2, \forall (a, b) \in \mathbb{Z}^2, \\
& f_w(\alpha s + \beta s')[a, b] = \sum_i \sum_j (\alpha s + \beta s')[i, j] w[a - i, b - j] \\
& \quad = \alpha f_w(s)[a, b] + \beta f_w(s')[a, b] \quad (\text{linearity}) \\
& \forall s \in \mathcal{S}(\mathbb{Z}^2), \forall (\alpha, \beta) \in \mathbb{Z}^2, \forall (a, b) \in \mathbb{Z}^2, \\
& f_w \circ t_{\alpha, \beta}(s)[a, b] = \sum_i \sum_j t_{\alpha, \beta}(s)[i, j] w[a - i, b - j] \\
& \quad = \sum_i \sum_j s[i - \alpha, j - \beta] w[a - i, b - j] \\
& \quad = \sum_{i'} \sum_{j'} s[i', j'] w[a - i' - \alpha, b - j' - \beta] \quad (13) \\
& \quad = f_w(s)[a - \alpha, b - \beta] \\
& \quad = t_{\alpha, \beta} \circ f_w(s)[a, b] \quad (\text{equivariance})
\end{aligned}$$

Now let's prove the result from right to left.

Let  $f \in \mathcal{L}(\mathcal{S}(\mathbb{Z}^2))$ ,  $s \in \mathcal{S}(\mathbb{Z}^2)$ . We suppose that  $f$  commutes with translations. Recall that  $s$  can be linearly decomposed on the infinite family of dirac signals:

$$s = \sum_i \sum_j s[i, j] \delta_{i, j}, \text{ where } \delta_{i, j}[x, y] = \begin{cases} 1 & \text{if } (x, y) = (i, j) \\ 0 & \text{otherwise} \end{cases}$$

By linearity of  $f$  and then equivariance to translations:

$$\begin{aligned}
f(s) &= \sum_i \sum_j s[i, j] f(\delta_{i, j}) \\
&= \sum_i \sum_j s[i, j] f \circ t_{i, j}(\delta_{0, 0})
\end{aligned}$$

$$= \sum_i \sum_j s[i, j] t_{i,j} \circ f(\delta_{0,0})$$

By denoting  $w = f(\delta_{0,0}) \in \mathcal{S}(\mathbb{Z}^2)$ , we obtain:

$$\begin{aligned} \forall (a, b) \in \mathbb{Z}^2, f(s)[a, b] &= \sum_i \sum_j s[i, j] t_{i,j}(w)[a, b] \\ &= \sum_i \sum_j s[i, j] w[a - i, b - j] \\ \text{i.e. } f(s) &= s * w \end{aligned} \tag{14}$$

□

### 2.1.3 Usefulness of convolutions in deep learning

#### Equivariance property of CNNs

In deep learning, an important argument in favor of CNNs is that convolutional layers are equivariant to translations. Intuitively, that means that a detail of an object in an image should produce the same features independently of its position in the image.

#### Lossless superiority of CNNs over MLPs

The converse result, as a consequence of Proposition 54, is never mentioned in deep learning literature. However it is also a strong one. For example, let's consider a linear function that is equivariant to translations. Thanks to the converse result, we know that this function is a convolution operator parameterized by a weight vector  $w$ ,  $f_w : \cdot * w$ . If the domain is compactly supported, as in the case of images, we can break down the information of  $w$  in a finite number  $n_q$  of kernels  $w_q$  with small compact supports of same size (for instance of size  $2 \times 2$ ), such that we have  $f_w = \sum_{q \in \{1, 2, \dots, n_q\}} f_{w_q}$ . The convolution operators  $f_{w_q}$  are all in the search space of  $2 \times 2$  convolutional layers. In other words, every translational equivariant linear function can

have its information parameterized by these layers. So that means that the reduction of parameters from an MLP to a CNN is done with strictly no loss of expressivity (provided the objective function is known to bear this property). Besides, it also helps the training to search in a much more confined space.

**Methodology for extending to general graphs**

Hence, in our construction, we will try to preserve the characterization from Proposition 54 as it is mostly the reason why they are successful in deep learning. Note that the reduction of parameters compared to a dense layer is also a consequence of this characterization.

## 2.2 Construction from the vertex set

As Proposition 54 is a complete characterization of convolutions, it can be used to define them *i.e.* convolution operators can be constructed as the set of linear transformations that are equivariant to translations. However, in the general case where  $G$  is not a grid graph, translations are not defined, so that construction needs to be generalized beyond translational equivariances. In mathematics, convolutions are more generally defined for signals defined over a group structure. The classical convolution that is used in deep learning is just a narrow case where the domain group is an euclidean space. Therefore, constructing a convolution on graphs should start from the more general definition of convolution on groups rather than convolution on euclidean domains.

Our construction is motivated by the following questions:

- Does the equivariance property holds ? Does the characterization from Proposition 54 still holds ?
- Is it possible to extend the construction on non-group domains, or at least on mixed domains ? (*i.e.* one signal is defined over a set, and the other is defined over a subgroup of the transformations of this set).
- Can a group domain draw an underlying graph structure ? Is the group convolution naturally defined on this class of graphs ?

We first recall the notion of group and group convolution.

### Definition 55. Group

A group  $\Gamma$  is a set equipped with a closed, associative and invertible composition law that admits a unique left-right identity element.

The group convolution extends the notion of the classical discrete convolution.

**Definition 56. Group convolution I**

Let a group  $\Gamma$ , the group convolution I between two signals  $s_1$  and  $s_2 \in \mathcal{S}(\Gamma)$  is defined as:

$$\forall h \in \Gamma, (s_1 *_I s_2)[h] = \sum_{g \in \Gamma} s_1[g] s_2[g^{-1}h]$$

provided at least one of the signals has finite support if  $\Gamma$  is not finite.

**2.2.1 Steered construction from groups**

For a graph  $G = \langle V, E \rangle$  and a subgroup  $\Gamma \subset \Phi^*(V)$  or its invertible transformations, Definition 56 is applicable for  $\mathcal{S}(\Gamma)$ , but not for  $\mathcal{S}(V)$  as  $V$  is not a group. Nonetheless, our point here is that we will use the group convolution on  $\mathcal{S}(\Gamma)$  to construct the convolutions on  $\mathcal{S}(V)$ .

For now, let's assume  $\Gamma$  is in one-to-one correspondence with  $V$ , and let's define a bijective map  $\varphi$  from  $\Gamma$  to  $V$ . We denote  $\Gamma \xrightarrow{\varphi} V$  and  $g_v \xrightarrow{\varphi} v$ .

Then, the linear morphism  $\tilde{\varphi}$  from  $\mathcal{S}(\Gamma)$  to  $\mathcal{S}(V)$  defined on the Dirac bases by  $\tilde{\varphi}(\delta_g) = \delta_{\varphi(g)}$  is a linear isomorphism. Hence,  $\mathcal{S}(V)$  would inherit the same inherent structural properties as  $\mathcal{S}(\Gamma)$ . For the sake of notational simplicity, we will use the same symbol  $\varphi$  for both  $\varphi$  and  $\tilde{\varphi}$  (as done between  $f$  and  $L_f$ ). A commutative diagram between the sets is depicted on Figure 5.

$$\begin{array}{ccc} \Gamma & \xrightarrow{\varphi} & V \\ s \downarrow & & \downarrow s \\ \mathcal{S}(\Gamma) & \xrightarrow{\varphi} & \mathcal{S}(V) \end{array}$$

Figure 5: Commutative diagram between sets

We naturally obtain the following relation, which put in simpler words means that signals on  $\mathcal{S}(\Gamma)$  are mapped to  $\mathcal{S}(V)$  when  $\varphi$  is simultaneously applied on both the signal space and its domain.



**Lemma 57. Relation between  $\mathcal{S}(\Gamma)$  and  $\mathcal{S}(V)$** 

$$\forall s \in \mathcal{S}(\Gamma), \forall u \in V, \varphi(s)[u] = s[\varphi^{-1}(u)] = s[g_u]$$

*Proof.*

$$\begin{aligned} \forall s \in \mathcal{S}(\Gamma), \varphi(s) &= \varphi\left(\sum_{g \in \Gamma} s[g] \delta_g\right) = \sum_{g \in \Gamma} s[g] \varphi(\delta_g) = \sum_{g \in \Gamma} s[g] \delta_{\varphi(g)} \\ &= \sum_{v \in V} s[g_v] \delta_v \end{aligned}$$

$$\text{So } \forall v \in V, \varphi(s)[u] = s[g_u]$$

□

Hence, we can steer the definition of the group convolution from  $\mathcal{S}(\Gamma)$  to  $\mathcal{S}(V)$  as follows:

**Definition 58. Group convolution II**

Let a subgroup  $\Gamma \subset \Phi^*(V)$  such that  $\Gamma \stackrel{\varphi}{\cong} V$ . The group convolution II between two signals  $s_1$  and  $s_2 \in \mathcal{S}(V)$  is defined as:

$$\forall u \in V, (s_1 *_{\text{II}} s_2)[u] = \sum_{v \in V} s_1[v] s_2[\varphi(g_v^{-1} g_u)]$$

**Lemma 59. Relation between group convolution I and II**

Let a subgroup  $\Gamma \subset \Phi^*(V)$  such that  $\Gamma \stackrel{\varphi}{\cong} V$ ,

$$\forall s_1, s_2 \in \mathcal{S}(\Gamma), \forall u \in V, (\varphi(s_1) *_{\text{II}} \varphi(s_2))[u] = (s_1 *_{\text{I}} s_2)[g_u]$$

*Proof.* Using Lemma 57,

$$\begin{aligned}
(\varphi(s_1) *_{\text{II}} \varphi(s_2))[u] &= \sum_{v \in V} \varphi(s_1)[v] \varphi(s_2)[\varphi(g_v^{-1} g_u)] \\
&= \sum_{v \in V} s_1[g_v] s_2[g_v^{-1} g_u] \\
&= \sum_{g \in \Gamma} s_1[g] s_2[g^{-1} g_u] \\
&= (s_1 *_{\text{I}} s_2)[g_u]
\end{aligned}$$

□

For convolution II, we only obtain a weak version of Proposition 54.

**Proposition 60. Equivariance to  $\varphi(\Gamma)$**

If  $\varphi$  is a homomorphism, convolution operators acting on the right of  $\mathcal{S}(V)$  are equivariant to  $\varphi(\Gamma)$  i.e.

if  $\varphi \in \text{ISO}(\Gamma, V)$ ,

$$\exists w \in \mathcal{S}(V), f = . *_{\text{II}} w \Rightarrow \forall v \in V, f \circ \varphi(g_v) = \varphi(g_v) \circ f$$

*Proof.*

$$\forall s \in \mathcal{S}(V), \forall u \in V, \forall v \in V,$$

$$\begin{aligned}
(f_w \circ \varphi(g_u))(s)[v] &= \sum_{v \in V} \varphi(g_u)(s)[v] w[\varphi(g_v^{-1} g_u)] \\
&= \sum_{\substack{(a,b) \in V^2 \\ \text{s.t. } g_a g_b = g_v}} \varphi(g_u)(s)[a] w[b] \\
&= \sum_{\substack{(a,b) \in V^2 \\ \text{s.t. } g_a g_b = g_v}} s[\varphi(g_u)^{-1}(a)] w[b]
\end{aligned}$$

$$= \sum_{\substack{(a,b) \in V^2 \\ s.t. \ g_{\varphi(g_u)(a)} g_b = g_v}} s[a] w[b]$$

Because  $\varphi$  is an isomorphism, its inverse  $c \mapsto g_c$  is also an isomorphism and so  $g_{\varphi(g_u)(a)} g_b = g_v \Leftrightarrow g_a g_b = g_{\varphi(g_u)^{-1}(v)}$ . So we have both:

$$\begin{aligned} (f_w \circ \varphi(g_u))(s)[v] &= \sum_{\substack{(a,b) \in V^2 \\ s.t. \ g_a g_b = g_{\varphi(g_u)^{-1}(v)}}} s[a] w[b] \\ &= s *_\Pi w[\varphi(g_u)^{-1}(v)] \\ &= (\varphi(g_u) \circ f_w)(s)[v] \end{aligned}$$

□

*Remark.* Note that convolution operators of the form  $f_w = . *_\Pi w$  are also equivariant to  $\Gamma$ , but the proposition and the proof are omitted as they are similar to the latter.

In fact, both group convolutions are the same as the latter one borrows the algebraic structure of the first one. Thus we only obtain equivariance to  $\varphi(\Gamma)$  when  $\varphi$  also transfer the group structure from  $\Gamma$  to  $V$ , and the converse does not hold. To obtain equivariance to  $\Gamma$  (and its converse), we will drop the direct homomorphism condition, and instead we will take into account the fact that it contains invertible transformations of  $V$ .

### 2.2.2 Construction under group actions

#### Definition 61. Group action

An *action* of a group  $\Gamma$  on a set  $V$  is a function  $L : \Gamma \times V \rightarrow V, (g, v) \mapsto L_g(v)$ , such that the map  $g \mapsto L_g$  is a homomorphism.

Given  $g \in \Gamma$ , the transformation  $L_g$  is called the action of  $g$  by  $L$  on  $V$ .

*Remark.* When there is no ambiguity, we use the same symbol for  $g$  and  $L_g$ .

Hence, note that  $g \in \Gamma$  can act on both  $\Gamma$  through the left multiplication and on  $V$  as being an object of  $\Phi^*(V)$ . This ambivalence can be seen on a commutative diagram, see Figure 6.

$$\begin{array}{ccc} g_u & \xrightarrow{g_v} & g_v g_u \\ \varphi \downarrow & & \downarrow \varphi \\ u & \xrightarrow[g_v]{(P)} & \varphi(g_v g_u) \end{array}$$

Figure 6: Commutative diagram. All arrows except for the one labeled with (P) are always True.

For (P) to be true means that  $\varphi$  is an equivariant map *i.e.* whether the mapping is done before or after the action of  $\Gamma$  has no impact on the result. When such  $\varphi$  exists,  $\Gamma$  and  $V$  are said to be equivalent and we denote  $\Gamma \equiv V$ .

#### Definition 62. Equivariant map

A map  $\varphi$  from a group  $\Gamma$  acting on the destination set  $V$  and itself, is said to be an *equivariant map* if

$$\forall g, h \in \Gamma, g(\varphi(h)) = \varphi(g(h))$$

*Remark.* Here  $g$  acts on  $\Gamma$  through left multiplication so  $g(h) = gh$ .

Suppose we have  $\Gamma \stackrel{\varphi}{\cong} V$ . If we also have that  $\Gamma \equiv V$ , we are interested to know if then  $\varphi$  exhibits the equivalence.

**Definition 63.  $\varphi$ -Equivalence**

A subgroup  $\Gamma \subset \Phi^*(V)$  such that  $\Gamma \stackrel{\varphi}{\cong} V$ , is said to be  $\varphi$ -equivalent if  $\varphi$  is a bijective equivariant map *i.e.* if it verifies the property:

$$\forall v, u \in V, g_v(u) = \varphi(g_v g_u) \quad (\text{P})$$

In that case we denote  $\Gamma \stackrel{\varphi}{\equiv} V$ .

*Remark.* For example, translations on the grid graph, with  $\varphi(t_{i,j}) = (i, j)$ , are  $\varphi$ -equivalent as  $t_{i,j}(a, b) = \varphi(t_{i,j} \circ t_{a,b})$ . However, with  $\varphi(t_{i,j}) = (-i, -j)$ , they would not be  $\varphi$ -equivalent.

**Definition 64. Group convolution III**

Let a subgroup  $\Gamma \subset \Phi^*(V)$  such that  $\Gamma \stackrel{\varphi}{\cong} V$ . The group convolution III between two signals  $s_1$  and  $s_2 \in \mathcal{S}(V)$  is defined as:

$$s_1 *_{\text{III}} s_2 = \sum_{v \in V} s_1[v] g_v(s_2) \quad (15)$$

$$= \sum_{g \in \Gamma} s_1[\varphi(g)] g(s_2) \quad (16)$$

The two expressions differ on the domain upon which the summation is done. The expression (15) put the emphasis on each vertex and its action, whereas the expression (16) emphasizes on each object of  $\Gamma$ .

**Lemma 65. Relation with group convolution II**

$$\Gamma \stackrel{\varphi}{\equiv} V \Leftrightarrow *_{\text{II}} = *_{\text{III}}$$

*Proof.*

$$\begin{aligned} \forall s_1, s_2 \in \mathcal{S}(V), \\ s_1 *_{\text{II}} s_2 &= s_1 *_{\text{III}} s_2 \\ \Leftrightarrow \forall u \in V, \sum_{v \in V} s_1[v] s_2[\varphi(g_v^{-1} g_u)] &= \sum_{v \in V} s_1[v] s_2[g_v^{-1}(u)] \end{aligned} \quad (17)$$

Hence, the direct sense is obtained by applying (P).

For the converse, given  $u, v \in V$ , we first realize (17) for  $s_1 := \delta_v$ , obtaining  $s_2[\varphi(g_v^{-1} g_u)] = s_2[g_v^{-1}(u)]$ , which we then realize for a real signal  $s_2$  having no two equal entries, obtaining  $\varphi(g_v^{-1} g_u) = g_v^{-1}(u)$ . From the latter we finally obtain (P) with the one-to-one correspondence  $g_{v'} := g_v^{-1}$ .  $\square$

We can then coin the term  $\varphi$ -convolution.

**Definition 66.  $\varphi$ -convolution**

Let  $\Gamma \stackrel{\varphi}{\equiv} V$ , the  $\varphi$ -convolution between two signals  $s_1$  and  $s_2 \in \mathcal{S}(V)$  is defined as:

$$s_1 *_{\varphi} s_2 = s_1 *_{\text{II}} s_2 = s_1 *_{\text{III}} s_2$$

This time, we do obtain equivariance to  $\Gamma$  as expected, and the full characterization as well.

**Proposition 67. Characterization by right-action equivariance to  $\Gamma$** 

If  $\Gamma$  is  $\varphi$ -equivalent, the class of linear transformations of  $\mathcal{S}(V)$  that are equivariant to  $\Gamma$  is exactly the class of  $\varphi$ -convolution operators acting on the

right of  $\mathcal{S}(V)$  i.e.

If  $\Gamma \stackrel{\varphi}{\cong} V$ ,

$$\exists w \in \mathcal{S}(V), f = . *_{\varphi} w \Leftrightarrow \begin{cases} f \in \mathcal{L}(\mathcal{S}(V)) \\ \forall g \in \Gamma, f \circ g = g \circ f \end{cases}$$

*Proof.* 1. From left to right:

In the following equations, (18) is obtained by definition, (19) is obtained because left multiplication in a group is bijective, and (20) is obtained because of (P).

$$\forall g \in \Gamma, \forall s \in \mathcal{S}(V),$$

$$f_w \circ g(s) = \sum_{h \in \Gamma} g(s)[\varphi(h)] h(w) \quad (18)$$

$$= \sum_{h \in \Gamma} g(s)[\varphi(gh)] gh(w) \quad (19)$$

$$= \sum_{h \in \Gamma} g(s)[g(\varphi(h))] gh(w) \quad (20)$$

$$= \sum_{h \in \Gamma} s[\varphi(h)] gh(w)$$

$$= \sum_{h \in \Gamma} s[\varphi(h)] h(w)[g^{-1}(.)]$$

$$= f_w(s)[g^{-1}(.)]$$

$$= g \circ f_w(s)$$

Of course, we also have that  $f_w$  is linear.

2. From right to left:

Let  $f \in \mathcal{L}(\mathcal{S}(V))$ ,  $s \in \mathcal{S}(V)$ . By linearity of  $f$ , we distribute  $f(s)$  on

the family of dirac signals:

$$f(s) = \sum_{v \in V} s[v] f(\delta_v) \quad (21)$$

Thanks to (P), we have that:

$$\begin{aligned} g_v(\varphi(\text{Id})) &= \varphi(g_v \text{Id}) = v \\ \text{So, } v = u &\Leftrightarrow \varphi(\text{Id}) = g_v^{-1}(u) \\ \text{So, } \delta_v &= g_v(\delta_{\varphi(\text{Id})}) \end{aligned}$$

By denoting  $w = f(\delta_{\varphi(\text{Id})})$ , and using the hypothesis of equivariance, we obtain from (21) that:

$$\begin{aligned} f(s) &= \sum_{v \in V} s[v] f \circ g_v(\delta_{\varphi(\text{Id})}) \\ &= \sum_{v \in V} s[v] g_v \circ f(\delta_{\varphi(\text{Id})}) \\ &= \sum_{v \in V} s[v] g_v(w) \\ &= s *_{\varphi} w \end{aligned}$$

□

### Construction of $\varphi$ -convolutions on vertex domains

Proposition 67 tells us that in order to define a convolution on the vertex domain of a graph  $G = \langle V, E \rangle$ , all we need is a subgroup  $\Gamma$  of invertible transformations of  $V$ , that is equivalent to  $V$ . The choice of  $\Gamma$  can be done with respect to  $E$ . This is discussed in more details in Section 2.3, where we will see that in fact, we only need a generating set of  $\Gamma$ .



**Exposure of  $\varphi$** 

This construction relies on exposing a bijective equivariant map  $\varphi$  between  $\Gamma$  and  $V$ . In the next subsection, we show that in cases where  $\Gamma$  is abelian, we even need not expose  $\varphi$  and the characterization still holds.

**2.2.3 Mixed domain formulation**

From (16), we can define a mixed domain convolution *i.e.* that is defined for  $r \in \mathcal{S}(\Gamma)$  and  $s \in \mathcal{S}(V)$ , without the need of expliciting  $\varphi$ .

**Definition 68. Mixed domain convolution**

Let a subgroup  $\Gamma \subset \Phi^*(V)$  such that  $V \cong \Gamma$ . The *mixed domain convolution* between two signals  $r \in \mathcal{S}(\Gamma)$  and  $s \in \mathcal{S}(V)$  results in a signal  $r *_{\text{M}} s \in \mathcal{S}(V)$  and is defined as:

$$r *_{\text{M}} s = \sum_{g \in \Gamma} r[g] g(s)$$

We coin it M-convolution. From a practical point of view, this expression of the convolution is useful because it relegates  $\varphi$  as an underpinning object.

**Lemma 69. Relation with group convolution III**

$\forall \varphi \in \text{BIJ}(\Gamma, V), \forall (r, s) \in \mathcal{S}(\Gamma) \times \mathcal{S}(V),$

$$r *_{\text{M}} s = \varphi(r) *_{\text{III}} s$$

*Proof.* Let  $\varphi \in \text{BIJ}(\Gamma, V), (r, s) \in \mathcal{S}(\Gamma) \times \mathcal{S}(V),$

$$\begin{aligned} r *_{\text{M}} s &= \sum_{g \in \Gamma} r[g] g(s) = \sum_{v \in V} r[g_v] g_v(s) \stackrel{(\diamond)}{=} \sum_{v \in V} \varphi(r)[v] g_v(s) \\ &= \varphi(r) *_{\text{III}} s \end{aligned}$$

Where  $\stackrel{(\diamond)}{=}$  comes from Lemma 57. □

In other words,  $*_{\text{M}}$  is a convenient reformulation of  $*_{\text{III}}$  which does not depend on a particular  $\varphi$ .

**Lemma 70. Relation with group convolution I, II and  $\varphi$ -convolution**

Let  $\varphi \in \text{BIJ}(\Gamma, V)$ ,  $(r, s) \in \mathcal{S}(\Gamma) \times \mathcal{S}(V)$ , we have:

$$\begin{aligned} \Gamma \stackrel{\varphi}{\equiv} V &\Leftrightarrow \forall v \in V, (r *_{\text{M}} s)[v] = (r *_{\text{I}} \varphi^{-1}(s))[g_v] \\ &\Leftrightarrow r *_{\text{M}} s = \varphi(r) *_{\text{II}} s \\ &\Leftrightarrow r *_{\text{M}} s = \varphi(r) *_{\varphi} s \end{aligned}$$

*Proof.* On one hand, Lemma 69 gives  $r *_{\text{M}} s = \varphi(r) *_{\text{III}} s$ . On the other hand, Lemma 59 gives  $\forall v \in V, (r *_{\text{I}} \varphi^{-1}(s))[g_v] = (\varphi(r) *_{\text{II}} s)[v]$ . Then Lemma 65 concludes.  $\square$

*Remark.* The converse sense is meaningful because it justifies that when the M-convolution is employed, the property  $\Gamma \equiv V$  underlies, without the need of expliciting  $\varphi$ .

From M-convolution, we can derive operators acting on the left of  $\mathcal{S}(V)$ , of the form  $s \mapsto w *_{\text{M}} s$ , parameterized by  $w \in \mathcal{S}(\Gamma)$ . In particular, these operators would be relevant as layers of neural networks. On the contrary, derived operators acting on the right such as  $r \mapsto r *_{\text{M}} w$  wouldn't make sense with this formulation as they would make  $\varphi$  resurface. However, the equivariance to  $\Gamma$  incurring from Lemma 69 and Proposition 67 only holds for operators acting on the right. So we need to intertwine an abelian condition as follows. This is also a good excuse to see the influence of abelianity.

**Proposition 71. Equivariance to  $\Gamma$  through left action**

Let a subgroup  $\Gamma \subset \Phi^*(V)$  such that  $\Gamma \cong V$ .  $\Gamma$  is abelian, if and only if, M-convolution operators acting on the left of  $\mathcal{S}(V)$  are equivariant to it *i.e.*

$$\forall g, h \in \Gamma, gh = hg \Leftrightarrow \forall w, g \in \Gamma, w *_M g(.) = g \circ (w *_M .)$$

*Proof.* Let  $w, g \in \Gamma$ , and define  $f_w : s \mapsto w *_M s$ . In the following expressions,  $\Gamma$  is abelian if and only if (22) and (23) are equal (the converse is obtained by particularizing on well chosen signals):

$$f_w \circ g(s) = \sum_{h \in \Gamma} w[h] hg(s) \tag{22}$$

$$= \sum_{h \in \Gamma} w[h] gh(s) \tag{23}$$

$$= \sum_{h \in \Gamma} w[h] h(s)[g^{-1}(.)]$$

$$= (w *_M s)[g^{-1}(.)]$$

$$= g \circ f_w(s)$$

□

*Remark.* Similarly,  $*_\varphi$  is also equivariant to  $\Gamma$  through left action if and only if  $\Gamma$  is abelian, as a consequence of being commutative if and only if  $\Gamma$  is abelian. On the contrary, note that commutativity of  $*_M$  doesn't make sense.

**Corrolary 72. Characterization by left-action equivariance to  $\Gamma$** 

Let  $\Gamma \cong V$ . If  $\Gamma$  is abelian, the class of linear transformations of  $\mathcal{S}(V)$  that are equivariant to  $\Gamma$  is exactly the class of M-convolution operators acting on

the left of  $\mathcal{S}(V)$  *i.e.*

If  $\Gamma \cong V$  and  $\Gamma$  is abelian,

$$\exists w \in \mathcal{S}(\Gamma), f = w *_{\mathbf{M}} \cdot \Leftrightarrow \begin{cases} f \in \mathcal{L}(\mathcal{S}(V)) \\ \forall g \in \Gamma, f \circ g = g \circ f \end{cases}$$

*Proof.* By picking  $\varphi$  such that  $\Gamma \stackrel{\varphi}{\cong} V$  with Lemma 70 and using the relation between  $*_{\mathbf{M}}$  and  $*_{\varphi}$ .  $\square$

Depending on the applications, we will build upon either  $*_{\varphi}$  or  $*_{\mathbf{M}}$  when the abelian condition is satisfied.

## 2.3 Inclusion of the edge set in the construction

The constructions from the previous section involve the vertex set  $V$  and depend on  $\Gamma$ , a subgroup of the set of invertible transformations on  $V$ . Therefore, it looks natural to try to relate the edge set and  $\Gamma$ .

There are two approaches. Either  $\Gamma$  describes an underlying graph structure  $G = \langle V, E \rangle$ , either  $G$  can be used to define a relevant subgroup  $\Gamma$  to which the produced convolutive operators will be equivariant. Both approaches will help characterize classes of graphs that can support natural definitions of convolutions.

### 2.3.1 Edge-constrained convolutions

In this subsection, we are trying to answer the following question:

- What graphs admit a  $\varphi$ -convolution, or an M-convolution (in the sense that they can be defined with the characterization), under the condition that  $\Gamma$  is generated by a set of edge-constrained transformations ?

#### **Definition 73. Edge-constrained transformation**

An *edge-constrained* (EC) transformation on a graph  $G = \langle V, E \rangle$  is a transformation  $f : V \mapsto V$  such that

$$\forall u, v \in V, f(u) = v \Rightarrow u \stackrel{E}{\sim} v$$

We denote  $\Phi_{\text{EC}}(G)$  and  $\Phi_{\text{EC}}^*(G)$  the sets of (EC) and invertible (EC) transformations. When a convolution is defined as a sum over a set that is in one-to-one correspondence with a group that is generated from a set of (EC) transformations, we call it an (EC) convolution.

*Remark.* Note that  $\Phi_{\text{EC}}^*(G)$  is not a group, thus why we are interested in groups and their generating sets.

This leads us to consider Cayley graphs (Cayley, 1878).

**Definition 74. Cayley graph**

Let a group  $\Gamma$  and one of its generating set  $\mathcal{U}$ . The *Cayley graph* generated by  $\mathcal{U}$ , is the digraph  $\vec{G} = \langle V, E \rangle$  such that  $V = \Gamma$  and  $E$  is such that:

$$a \rightarrow b \Leftrightarrow \exists g \in \mathcal{U}, ga = b$$

Also, if  $\Gamma$  is abelian, we call it an *abelian Cayley graph*. We call *Cayley subgraph*, a subgraph that is isomorph to a Cayley graph.

*Remark.* Note that for compatibility with transformations that are left actions, we define Cayley graphs with  $ga = b$  instead of  $ag = b$ .

**Convolution on Cayley graphs**

In the case of Cayley graphs, it is clear that  $\mathcal{U} \subseteq \Phi_{\text{EC}}^*$  and  $\Phi^* \supseteq \langle \mathcal{U} \rangle \equiv V$ . So that they admit (EC)  $\varphi$ -convolutions, and (EC) M-convolutions in the abelian case.

More precisely, we obtain the following characterization:

**Proposition 75. Characterization by Cayley subgraph isomorphism**

Let a graph  $G = \langle V, E \rangle$ , then:

- (i)  $G$  admits an (EC)  $\varphi$ -convolution if and only if it contains a subgraph isomorph to a Cayley graph
- (ii)  $G$  admits an (EC) M-convolution if and only if it contains a subgraph isomorph to an abelian Cayley graph

*Proof.* We show the result only in the general case as the proof for the abelian case is similar.

1. From left to right: as a direct application of the definitions.

2. From right to left:

Let a graph  $G = \langle V, E \rangle$ . We suppose it contains a subgraph  $\vec{G}_s = \langle V_s, E_s \rangle$  that is graph-isomorph to a Cayley graph  $\vec{G}_c = \langle V_c, E_c \rangle$ , generated by  $\mathcal{U}$ . Let  $\psi$  be a graph isomorphism from  $G_s$  to  $G_c$ . To obtain the proof, we need to find a group of invertible transformations  $\Gamma$  of  $V_s$  generated by a set of (EC) transformations, such that  $\Gamma \equiv V_s$ .

Let's define the group action  $L : V_c \times V_s \rightarrow V_s$  inductively as follows:

- (a)  $\forall g \in \mathcal{U}, L_g(u) = v \Leftrightarrow g\psi(u) = \psi(v)$
- (b) Whenever  $L_g$  and  $L_h$  are defined, the action of  $gh$  is defined by homomorphism as  $L_{gh} = L_g \circ L_h$
- (c) Whenever  $L_g$  is defined, the action of  $g^{-1}$  is defined by homomorphism as  $L_{g^{-1}} = L_g^{-1}$  *i.e.*  $L_{g^{-1}}(u) = v \Leftrightarrow \psi(u) = g\psi(v)$

Note that the induction transfers the property (a) to all  $g \in V_c$  in a transitive manner because

$$L_{gh}(u) = L_g(L_h(u)) = w \Leftrightarrow \exists v \in V_s \begin{cases} L_h(u) = v \\ L_g(v) = w \end{cases}$$

and

$$\exists v \in V_s \begin{cases} h\psi(u) = \psi(v) \\ g\psi(v) = \psi(w) \end{cases} \Leftrightarrow gh\psi(u) = \psi(w)$$

We must also verify that this construction is well-defined, *i.e.* whenever we define an action with (b) or (c), if the action was already defined, then they must be equal. This is the case because the homomorphism

$g \mapsto L_g$  on  $V_c$  is in fact an isomorphism as

$$\begin{aligned} L_g = L_h &\Leftrightarrow \forall u \in V, L_g(u) = L_h(u) \\ &\Leftrightarrow \forall u \in V, g\psi(u) = h\psi(u) \\ &\Leftrightarrow g = h \end{aligned}$$

Also note that (c) is needed only in case that  $V_c$  is infinite.

Denote the set  $L_{\mathcal{U}} = \{L_g, g \in \mathcal{U}\}$  and  $\Gamma = \langle L_{\mathcal{U}} \rangle \cong V_c$ . Let's define the map  $\varphi$  as:

$$\begin{aligned} \Gamma &\rightarrow V_s \\ \varphi : L_g &\mapsto L_g(\psi^{-1}(\text{Id})) \end{aligned}$$

$\varphi$  is bijective because  $\forall g \in V_c, \varphi(L_g) = \psi^{-1}(g)$  thanks to (a).

Additionally, we have:

$$\begin{aligned} L_h(\varphi(L_g)) &= L_h(L_g(\psi^{-1}(\text{Id}))) \\ &= L_h \circ L_g(\psi^{-1}(\text{Id})) \\ &= L_{hg}(\psi^{-1}(\text{Id})) \\ &= \varphi(L_{hg}) \\ &= \varphi(L_h \circ L_g) \end{aligned}$$

That is,  $\varphi$  is a bijective equivariant map and  $\langle L_{\mathcal{U}} \rangle = \Gamma \stackrel{\varphi}{\cong} V_s$ . Moreover,  $L_{\mathcal{U}}$  is a set of (EC) transformations thanks to (a). Therefore,  $G$  admits an (EC)  $\varphi$ -convolution.

□



**Corrolary 76. Characterization by  $\varphi$** 

Let a graph  $G = \langle V, E \rangle$ , and a set  $\mathcal{U} \subset \Phi_{\text{EC}}^*(G)$  s.t.

$$\langle \mathcal{U} \rangle \cong \Gamma \equiv V' \subset V$$

$G$  admits an (EC)  $\varphi$ -convolution, if and only if,  $\varphi$  is a graph isomorphism between the Cayley graph generated by  $\mathcal{U}$  and the subgraph induced by  $V'$ .

The proof is omitted as it would be highly similar to the previous one.

**2.3.2 Intrinsic properties**

- Obviously the constructed convolutions are linear. But do they also preserve the locality and weight sharing properties ?

Let  $\vec{G} = \langle V, E \rangle$  be a Cayley subgraph, generated by  $\mathcal{U}$ , of some graph  $G$ . Recall that its (EC)  $\varphi$ -convolution operator is a right operator, and can be expressed as

$$\begin{aligned} \forall s \in \mathcal{S}(V), \forall u \in V, \\ f_w(s)[u] &= (s *_{\varphi} w)[u] \\ &= \sum_{v \in V} s[v] w[g_v^{-1}(u)] \end{aligned} \tag{24}$$

From this expression, it is not obvious that  $f_w$  is a local operator. To see this, we can show for example the following proposition.

**Proposition 77. Locality**

When the support of  $w$  is a compact (in the sense that its induced subgraph in  $G$  is connected), of diameter  $d$ , the same holds for the support of the sum  $\Sigma$  in (24). More precisely, the subgraph induced by the support of  $\Sigma$  is isomorphic to the transpose of the subgraph induced by the support of  $w$ .

*Proof.* Without loss of generality subject to growing  $\mathcal{U}$ , let's suppose that  $w$  has a support  $\mathcal{M} = \varphi(\mathcal{N})$ , such that  $\mathcal{N} \subset \mathcal{U}$ .  $\mathcal{N}$  and  $\mathcal{M}$  are obviously compacts of diameter 2. Thanks to (P), we have

$$\begin{aligned}
 g_v^{-1}(u) \in \mathcal{M} &\Leftrightarrow u \in g_v(\mathcal{M}) = g_v(\varphi(\mathcal{N})) = \varphi(g_v\mathcal{N}) \\
 &\Leftrightarrow g_u \in g_v\mathcal{N} \\
 &\Leftrightarrow g_v^{-1} \in \mathcal{N}g_u^{-1} \\
 &\Leftrightarrow g_v \in g_u\mathcal{N}^{-1} \\
 &\Leftrightarrow v \in g_u(\varphi(\mathcal{N}^{-1}))
 \end{aligned}$$

where  $\mathcal{N}^{-1}$  reverses the edges of  $\mathcal{N}$ . Let's denote  $\mathcal{K}_u = g_u(\varphi(\mathcal{N}^{-1})) \subset V$ .

TODO: FALSE, consider  $g_u(v) = g_v(u)$

TODO: In reality we have: local conv iff SNP iff simple transitive Aut iff Cayley graph

TODO: EC local conv iff Abelian Cayley graph

By composing edge reversal and graph isomorphisms (as  $\varphi$  and its inverse are graph isomorphisms by Proposition 76), the compactness and diameter of  $\mathcal{M}$  is preserved for  $\mathcal{K}_u$ . More precisely, the transposed subgraph structure is also preserved.  $\square$

Let's define  $\mathcal{M}$ ,  $\mathcal{N}$  and  $\mathcal{K}_u$  as in the previous proof.

**Definition 78. Supporting set**

The *supporting set* of an (EC) convolution operator  $f_w$ , is a set  $\mathcal{N} \subset \Phi_{\text{EC}}^*$ , such that

- (i) when  $*$  is  $*_{\varphi}$ :  $0 \notin w[\mathcal{M}]$ , where  $\mathcal{M} = \varphi(\mathcal{N})$
- (ii) when  $*$  is  $*_{\text{M}}$ :  $0 \notin w[\mathcal{N}]$

**Definition 79. Local patch for  $*_\varphi$** 

The *local patch* at  $u \in V$  of an (EC)  $\varphi$ -convolution operator  $f_w$  is defined as  $\mathcal{K}_u = g_u(\varphi(\mathcal{N}^{-1}))$ .

*Remark.* In other terms,  $\mathcal{K}_{\text{Id}} = \varphi(\mathcal{N}^{-1})$  is the *initial local patch*, which is composed of all vertices that are connected in direction to  $\varphi(\text{Id})$ ; and  $\mathcal{K}_u$  is obtained by moving  $\mathcal{K}_{\text{Id}}$  on the Cayley subgraph via the edges corresponding to the decomposition of  $g_u$  on the generating set  $\mathcal{U}$ .

To see that the weights are tied in the general case (i), we can show the following proposition.

**Proposition 80. Weight sharing**

$$\forall a, \alpha \in V, \forall b \in \mathcal{K}_a : \exists \beta \in \mathcal{K}_\alpha \Leftrightarrow g_\beta^{-1}(\alpha) = g_b^{-1}(a)$$

*Proof.* By using (P),

$$\begin{aligned} g_{\mathcal{K}_\alpha}^{-1}(\alpha) = g_{\mathcal{K}_a}^{-1}(a) &\Leftrightarrow g_\alpha^{-1}g_{\mathcal{K}_\alpha} = g_a^{-1}g_{\mathcal{K}_a} \\ &\Leftrightarrow \mathcal{K}_\alpha = g_\alpha g_a^{-1}(\mathcal{K}_a) = g_\alpha g_a^{-1}g_a(\varphi(\mathcal{N}^{-1})) \\ &\Leftrightarrow \mathcal{K}_\alpha = g_\alpha(\varphi(\mathcal{N}^{-1})) \end{aligned}$$

□

**2.3.3 Stricly edge-constrained convolutions**

We make the distinction between general (EC) convolution operators and those for which the weight kernel  $w$  is smaller and is supported only on (EC) transformations of  $\mathcal{U}$ .

**Definition 81. Strictly (EC) convolution operator**

A *strictly* edge-constrained (EC\*) convolution operator  $f_w$ , is an (EC) convolution operator such that its supporting set  $\mathcal{N} \subset \mathcal{U}$ .

*Remark.* (EC\*) convolution operators are simpler to obtain as we can construct them just with  $\mathcal{U} \subset \Phi_{\text{EC}}^*(G)$  without composing the transformations.

Let  $f_w$  be an (EC\*) convolutional operator. In the general case (i),  $w \in \mathcal{S}(V)$ , so its support is  $\mathcal{M} = \varphi(\mathcal{N})$  such that  $\mathcal{N} \subseteq \mathcal{U}$ . In the abelian case (ii), we use instead  $w \in \mathcal{S}(\Gamma)$ , and thus its support is directly  $\mathcal{N}$ . Therefore, we can rewrite the expressions of the convolution operator as:

$$\begin{aligned} \text{(i)} \quad & \forall s \in \mathcal{S}(V), \forall u \in V, f_w(s)[u] \stackrel{(\varphi)}{=} \sum_{v \in \mathcal{K}_u} s[v] w[g_v^{-1}(u)] \\ \text{(ii)} \quad & \forall s \in \mathcal{S}(V), f_w(s) \stackrel{(\text{M})}{=} \sum_{g \in \mathcal{N}} w[g] g(s) \end{aligned}$$

*Remark.* Note that in the abelian case, we can see from (ii) that a definition of a local patch would coincide with the supporting set, so that locality and weight sharing is straightforward.

**Construction**

From these expressions, it is clear that  $\Gamma$  needs not to be fully determined to calculate  $f_w(s)[u]$ . The case (ii) is the simplest as the only requirement is a supporting set  $\mathcal{N}$  of (EC) invertible transformations. In the case (i), we also need to determine  $\mathcal{K}_u$ .

**Strict locality**

Note that  $f_w(s)[u]$  is a weighted aggregation of entries  $s[v]$  for  $v \in \mathcal{K}_u$ . As  $\mathcal{K}_{\text{Id}} = \varphi(\mathcal{N}^{-1}) = \mathcal{N}^{-1}(\varphi(\text{Id}))$ ,  $\mathcal{K}_{\text{Id}}$  contains only neighbors of  $\varphi(\text{Id})$ , and so  $\mathcal{K}_u = g_u(\mathcal{K}_{\text{Id}})$  contains only neighbors of  $u$ . Therefore, in both cases  $f_w(s)[u]$  is a weighted aggregation of entries located in the neighborhood of  $u$ .

**Complexity**

Another merit is that (EC\*) convolutions have a complexity of  $\mathcal{O}(kn)$ , where  $n = |V|$  is the degree of the graph, and  $k = |\mathcal{N}|$  is the size of the weight kernel. In comparison, (EC) convolutions have complexity up to  $\mathcal{O}(n^2)$ .

## 2.4 From groups to groupoids

### 2.4.1 Motivation

One possible limitation coming from searching for Cayley subgraphs is that they are degree-regular *i.e.* the in- and the out-degree  $d = |\mathcal{U}|$  of each vertex is the same. That is, for a general graph  $G$ , the size of the weight kernel  $w$  of an (EC\*) convolution operator  $f_w$  supported on  $\mathcal{U}$  is bounded by  $d$ , which in turn is bounded by twice the minimal degree of  $G$  (twice because  $G$  is undirected and  $\mathcal{U}$  can contain every inverse).

There are a lot of possible strategies to overcome this limitation. For example:

1. connecting each vertex with its  $k$ -hop neighbors, with  $k > 1$ ,
2. artificially creating new connections for less connected vertices,
3. ignoring less connected vertices,
4. allowing the supporting set  $\mathcal{N}$  to exceed  $\mathcal{U}$  *i.e.* dropping  $*$  in (EC\*).

These strategies require to concede that the topological structure supported by  $G$  is not the best one to support an (EC\*) convolution on it, which breeds the following question:

- What can we relax in the previous (EC\*) construction in order to unbound the supporting set, and still preserve the equivariance characterization?

The latter constraint is a consequence that every vertex of the Cayley subgraph  $\vec{G}$  must be composable with every generator from  $\mathcal{U}$ . Therefore, an answer consists in considering groupoids (Brandt, 1927) instead of groups. Roughly speaking, a groupoid is almost a group except that its composition law needs not be defined everywhere. Weinstein, 1996, unveiled the benefits to base convolutions on groupoids instead of groups in order to exploit partial symmetries.

### 2.4.2 Definition of notions related to groupoids

#### Definition 82. Groupoid

A *groupoid*  $\Upsilon$  is a set equipped with a partial composition law with domain  $\mathcal{D} \subset \Upsilon \times \Upsilon$ , called *composition rule*, that is

1. closed into  $\Upsilon$  i.e.  $\forall (g, h) \in \mathcal{D}, gh \in \Upsilon$
2. associative i.e.  $\forall f, g, h \in \Upsilon, \begin{cases} (f, g), (g, h) \in \mathcal{D} \Leftrightarrow (fg, h), (f, gh) \in \mathcal{D} \\ (f, g), (fg, h) \in \mathcal{D} \Leftrightarrow (g, h), (f, gh) \in \mathcal{D} \\ \text{when defined, } (fg)h = f(gh) \end{cases}$
3. invertible i.e.  $\forall g \in \Upsilon, \exists ! g^{-1} \in \Upsilon$  s.t.  $\begin{cases} (g, g^{-1}), (g^{-1}, g) \in \mathcal{D} \\ (g, h) \in \mathcal{D} \Rightarrow g^{-1}gh = h \\ (h, g) \in \mathcal{D} \Rightarrow hgg^{-1} = h \end{cases}$

Optionally, it can be *domain-symmetric* i.e.  $(g, h) \in \mathcal{D} \Leftrightarrow (h, g) \in \mathcal{D}$ , and *abelian* i.e. domain-symmetric with  $gh = hg$ .

*Remark.* Note that left and right inverses are necessarily equal (because  $(gg^{-1})g = g(g^{-1}g)$ ). Also note we can define a right identity element  $e_g^r = g^{-1}g$ , and a left one  $e_g^l = gg^{-1}$ , but they are not necessarily equal and depend on  $g$ .

Most definitions related to groups can be adapted to groupoids. In particular, let's adapt a few notions.

#### Definition 83. Groupoid partial action

A partial *action* of a groupoid  $\Upsilon$  on a set  $V$ , is a function  $L$ , with domain  $\mathcal{D}_L \subset \Upsilon \times V$  and valued in  $V$ , such that the map  $g \mapsto L_g$  is a groupoid homomorphism.

*Remark.* As usual, we will confound  $L_g$  and  $g$  when there is no possible confusion, and we denote  $\mathcal{D}_{L_g} = \mathcal{D}_g = \{v \in V, (g, v) \in \mathcal{D}_L\}$ .

**Definition 84. Partial equivariant map**

A map  $\varphi$  from a groupoid  $\Upsilon$  partially acting on the destination set  $V$  is said to be a *partial equivariant map* if

$$\forall g, h \in \Upsilon, \begin{cases} \varphi(h) \in \mathcal{D}_g \Leftrightarrow (g, h) \in \mathcal{D} \\ g(\varphi(h)) = \varphi(gh) \end{cases}$$

Also,  $\varphi$ -equivalence between a subgroupoid and a set is defined similarly with  $\varphi$  being a bijective *partial equivariant map* between them.

**Definition 85. Partial transformations groupoid**

The *partial transformations groupoid*  $\Psi^*(V)$ , is the set of invertible partial transformations, equipped with the functional composition law with domain  $\mathcal{D}$  such that

$$\begin{cases} \mathcal{D}_{gh} = h(\mathcal{D}_h) \cap \mathcal{D}_g \\ (g, h) \in \mathcal{D} \Leftrightarrow \mathcal{D}_{gh} \neq \emptyset \end{cases}$$

*Remark.* Note that a subgroupoid  $\Upsilon \subset \Psi^*(V)$  is domain-symmetric when  $\exists v \in V, g(v) \in \mathcal{D}_h \Leftrightarrow \exists u \in V, h(u) \in \mathcal{D}_g$

### 2.4.3 Construction of partial convolutions

The expression of the convolution we constructed in the previous section cannot be applied as is. We first need to extend the algebraic objects we work with. Extending a partial transformation  $g$  on the signal space  $\mathcal{S}(V)$  (and thus the convolutions) is a bit tricky, because only the signal entries corresponding to  $\mathcal{D}_g$  are moved. A convenient way to do this is to consider the groupoid closure obtained with the addition of an absorbing element.



**Definition 86. Zero-closure**

The *zero-closure* of a groupoid  $\Upsilon$ , denoted  $\Upsilon^0$ , is the set  $\Upsilon \cup 0$ , such that the groupoid axioms 1, 2 and 3, and the domain  $\mathcal{D}$  are left unchanged, and

4. the composition law is extended to  $\Upsilon^0 \times \Upsilon^0$  with  $\forall (g, h) \notin \mathcal{D}, gh = 0$

*Remark.* Note that this is coherent as the properties 2 and 3 are still partially defined on the original domain  $\mathcal{D}$ .

Now, we will also extend every other algebraic object used in the expression of the  $\varphi$ -convolution and the M-convolution, so that we can directly apply our previous constructions.

**Lemma 87. Extension of  $\varphi$  on  $V^0$** 

Let a partial equivariant map  $\varphi : \Upsilon \rightarrow V$ . It can be extended to a (total) equivariant map  $\varphi : \Upsilon^0 \rightarrow V^0 = V \cup \varphi(0)$ , such that  $\varphi(0) \notin V$ , that we denote  $0_V = \varphi(0)$ , and such that

$$\forall g \in \Upsilon^0, \forall v \in V^0, g(v) = \begin{cases} \varphi(gg_v) & \text{if } g_v \in \mathcal{D}_g \\ 0_V & \text{else} \end{cases}$$

*Proof.* We have  $\varphi(0) \notin V$  because  $\varphi$  is bijective. Additionally, we must have  $\forall (g, h) \notin \mathcal{D}, g(\varphi(h)) = \varphi(gh) = \varphi(0) = 0_V$ .  $\square$

*Remark.* Note that for notational conveniency, we may use the same symbol 0 for  $0_\Upsilon$ ,  $0_V$  and  $0_{\mathbb{R}}$ .

Similarly to  $\Phi^*(V)$ ,  $\Psi^*(V)$  can also move signals of  $\mathcal{S}(V)$ .

**Lemma 88. Extension of injective partial transformations to  $\mathcal{S}(V)$** 

Let  $g \in \Psi^*(V)$ . Its extension is done in two steps:

1.  $g$  is extended to  $V^0 = V \cup \{0_V\}$  as  $g(v) = 0_V \Leftrightarrow v \notin \mathcal{D}_g$ .

2. Under the convention  $\forall s \in \mathcal{S}(V), s[0_V] = 0_{\mathbb{R}}$ ,  $g$  is extended via linear extension to  $\mathcal{S}(V)$ , and we have

$$\forall s \in \mathcal{S}(V), \forall v \in V, g(s)[v] = s[g^{-1}(v)]$$

*Proof.* Straightforward. □

With these extensions, we can obtain the partial  $\varphi$ - and M-convolutions related to  $\Upsilon$  almost by substituting  $\Upsilon^0$  to  $\Gamma$  in Definition 66 and Definition 68.

**Definition 89. Partial convolution**

Let a subgroupoid  $\Upsilon \subset \Psi^*(V)$ , such that  $\Upsilon \stackrel{\varphi}{=} V$ . The partial  $\varphi$ - and M-convolutions, based on  $\Upsilon$ , are defined on its zero-closure, with the same expression as if  $\Upsilon^0$  were a subgroup, and by extension of  $\varphi$  and of the groupoid partial actions *i.e.*

- (i)  $\forall s, w \in \mathcal{S}(V), s *_{\varphi} w = \sum_{v \in V} s[v] g_v(w) = \sum_{g \in \Upsilon} s[\varphi(g)] g(w)$
- (ii)  $\forall (w, s) \in \mathcal{S}(\Upsilon) \times \mathcal{S}(V), w *_{\text{M}} s = \sum_{g \in \Upsilon} w[g] g(s)$

**Symmetrical expressions**

Note that, as  $\forall r, r[0] = 0$ , the partial convolutions can also be expressed on the domain  $\mathcal{D}$  with a convenient symmetrical expression:

- (i)  $\forall u \in V, (s *_{\varphi} w)[u] = \sum_{\substack{(g_a, g_b) \in \mathcal{D} \\ \text{s.t. } g_a g_b = g_u}} s[a] w[b]$
- (ii)  $\forall u \in V, (w *_{\text{M}} s)[u] = \sum_{\substack{v \in \mathcal{D}_g \\ \text{s.t. } g(v) = u}} w[g] s[v]$

We obtain an equivariance characterization similar to Proposition 67 and Corrolary 72.

**Proposition 90. Characterization by equivariance to  $\Upsilon$** 

Let a subgroupoid  $\Upsilon \subset \Psi^*(V)$ , such that  $\Upsilon \stackrel{\varphi}{\equiv} V$ , with  $*$  based on  $\Upsilon$ .

1. Then,

- (i) partial  $\varphi$ -convolution right-operators are equivariant to  $\Upsilon$ ,
- (ii) if  $\Upsilon$  is abelian, partial M-convolution left-operators are equiv to  $\Upsilon$ .

2. Conversely,

- (i) if  $\Upsilon$  is domain-symmetric, linear transformations of  $\mathcal{S}(V)$  that are equivariant to  $\Upsilon$  are partial  $\varphi$ -convolution right-operators,
- (ii) if  $\Upsilon$  is abelian, they are also partial M-convolution left-operators.

*Proof.* (i) (a) Direct sense:

Using the symmetrical expressions, and the fact that  $\forall r, r[0] = 0$ , we have

$$\begin{aligned}
 (f_w \circ g(s))[u] &= \sum_{\substack{(g_a, g_b) \in \mathcal{D} \\ s.t. \ g_a g_b = g_u}} g(s)[a] w[b] \\
 &= \sum_{\substack{(g_a, g_b) \in \mathcal{D} \\ s.t. \ g_a g_b = g_u}} s[g^{-1}(a)] w[b] \\
 &= \sum_{\substack{(g_a, g_b) \in \mathcal{D} \\ s.t. \ (g, g_a) \in \mathcal{D} \\ s.t. \ g g_a g_b = g_u}} s[a] w[b] \\
 &= \sum_{\substack{(g_a, g_b) \in \mathcal{D} \\ s.t. \ (g, g_a) \in \mathcal{D} \\ s.t. \ g_a g_b = g^{-1} g_u = g_{\varphi(g^{-1} g_u)} = g_{g^{-1}(u)}}} s[a] w[b] \\
 &= f_w(s)[g^{-1}(u)] \\
 &= (g \circ f_w(s))[u]
 \end{aligned}$$

(b) Converse:

Let  $v \in V$ . Denote  $e_{g_v}^r = g_v^{-1}g_v$  the right identity element of  $g_v$ , and  $e_v^r = \varphi(e_{g_v}^r)$ . We have that

$$\begin{aligned} g_v(e_v^r) &= v \\ \text{So, } \delta_v &= g_v(\delta_{e_v^r}) \end{aligned}$$

Let  $f \in \mathcal{L}(\mathcal{S}(V))$  that is equivariant to  $\Upsilon$ , and  $s \in \mathcal{S}(V)$ . Thanks to the previous remark we obtain that

$$\begin{aligned} f(s) &= \sum_{v \in V} s[v] f(\delta_v) \\ &= \sum_{v \in V} s[v] f(g_v(\delta_{e_v^r})) \\ &= \sum_{v \in V} s[v] g_v(f(\delta_{e_v^r})) \\ &= \sum_{v \in V} s[v] g_v(w_v) \end{aligned} \tag{25}$$

where  $w_v = f(\delta_{e_v^r})$ . In order to finish the proof, we need to find  $w$  such that  $\forall v \in V, g_v(w) = g_v(w_v)$ .

Let's consider the equivalence relation  $\mathcal{R}$  defined on  $V \times V$  such that:

$$\begin{aligned} a\mathcal{R}b &\Leftrightarrow w_a = w_b \\ &\Leftrightarrow e_a^r = e_b^r \\ &\Leftrightarrow g_a^{-1}g_a = g_b^{-1}g_b \\ &\Leftrightarrow (g_b, g_a^{-1}) \in \mathcal{D} \\ &\Leftrightarrow (g_a^{-1}, g_b) \in \mathcal{D} \end{aligned} \tag{26}$$

with (26) owing to the fact that  $\Upsilon$  is domain-symmetric.

Given  $x \in V$ , denote its equivalence class  $\mathcal{R}(x)$ . Under the hypothesis of the axiom of choice (Zermelo, 1904) (if  $V$  is infinite), define the set  $\aleph$  that contains exactly one representative per equivalence class. Let  $w = \sum_{n \in \aleph} w_n$ . Then  $V$  is the disjoint union  $V = \cup_{n \in \aleph} \mathcal{R}(n)$  and (25) rewrites:

$$\begin{aligned}
 \forall u \in V, f(s)[u] &= \sum_{n \in \aleph} \sum_{v \in \mathcal{R}(n)} s[v] g_v(w_n)[u] \\
 &= \sum_{n \in \aleph} \sum_{v \in \mathcal{R}(n)} s[v] w_n[g_v^{-1}(u)] \\
 &= \sum_{n \in \aleph} \sum_{v \in \mathcal{R}(n)} s[v] w[g_v^{-1}(u)] \quad (27) \\
 &= (s *_{\varphi} w)[u]
 \end{aligned}$$

where (27) is obtained thanks to (26).

- (ii) With symmetrical expressions, it is clear that the convolution is abelian, if and only if,  $\Upsilon$  is abelian. Then (i) concludes.

□

### Inclusion of (EC)

Similarly to the construction in Section 2.3, partial convolutions can define (EC) and (EC\*) counterparts with a characterization of admissibility by groupoid Cayley subgraph isomorphism, and similar intrinsic properties.

### Limitation of partial convolutions

However, because of the groupoid associativity, if  $g \in \Psi_{\text{EC}}^*(G)$ , then, any  $v \in V$  s.t.  $g(u) = v$  would be constrained to allow to be acted by every  $h$  s.t.  $(h, g) \in \mathcal{D}$ , which fails at unbounding the supporting set of a partial (EC\*) convolutions.

### 2.4.4 Construction of path convolutions

To answer the limitation of partial convolutions, given  $g \in \langle \mathcal{U} \rangle$  where  $\mathcal{U} \subset \Psi_{\text{EC}}^*(G)$ , the idea is to proceed with a foliation of  $g$  into pieces, each corresponding to an edge  $e \in E$ , and together generating another groupoid with a different associativity law, as follows.

**Definition 91. Path groupoid**

Let  $\mathcal{U} \subset \Psi_{\text{EC}}^*(G)$ . The *path groupoid* generated from  $\mathcal{U}$ , denoted  $\mathcal{U} \ltimes G$ , with composition rule  $\mathcal{D}_{\ltimes}$ , is the groupoid obtained inductively with:

1.  $\mathcal{U} \ltimes_1 G = \{(g, v) \in \mathcal{U} \times V, v \in \mathcal{D}_g\} \subset \mathcal{U} \ltimes G$
2.  $((g_n, v_n) \cdots (g_1, v_1), (h_m, u_m) \cdots (h_1, u_1)) \in \mathcal{D}_{\ltimes} \Leftrightarrow h_m(u_m) = v_1$
3.  $((g_n, v_n) \cdots (g_1, v_1))^{-1} = (g_1^{-1}, g_1(v_1)) \cdots (g_n^{-1}, g_n(v_n))$

Call path its objects. Given a length  $l \in \mathbb{N}$ , denote  $\mathcal{U} \ltimes_l G$  the subset composed of the paths that are the composition of exactly  $l$  paths of  $\mathcal{U} \ltimes_1 G$ .

*Remark.* This groupoid construction is inspired from the field of operator algebra where partial action groupoids have been extensively studied, *e.g.* Nica, 1994; Exel, 1998; Li, 2016.

Such groupoids usually come equipped with source and target maps. We also define the path map.

**Definition 92. Source, target and path maps**

Let a path groupoid  $\mathcal{U} \ltimes G$ . We define on it the *source map*  $\alpha$  the *target map*  $\beta$  and the *path map*  $\gamma$  as:

$$\begin{cases} \alpha : (g_n, v_n) \cdots (g_1, v_1) \mapsto v_1 \in V \\ \beta : (g_n, v_n) \cdots (g_1, v_1) \mapsto g_n(v_n) \in V \\ \gamma : (g_n, v_n) \cdots (g_1, v_1) \mapsto g_n g_{n-1} \cdots g_1 \in \Psi^*(V^0) \end{cases}$$

*Remark.* Note that the path groupoid can also be obtained by derivation of the partial transformation groupoid (*i.e.*  $p \in \mathcal{U} \ltimes G$  can be seen as a derivative of  $\gamma(p)$  *w.r.t.*  $\alpha(p)$ ), and can thus be seen as the local structure of it.

**Lemma 93.**

Note the following properties:

1.  $(p, q) \in \mathcal{D}_\ltimes \Leftrightarrow \alpha(p) = \beta(q)$
2.  $\alpha(p) = \beta(p^{-1})$
3.  $e_p^l = pp^{-1} = (\text{Id}, \beta(p))$  and  $e_p^r = p^{-1}p = (\text{Id}, \alpha(p))$
4.  $\gamma$  is a groupoid partial action. We will denote  $\gamma_p$  instead of  $\gamma(p)$ .

*Remark.* Note that this time we won't use the notation  $p(v)$  for  $\gamma_p(v)$  for clarity.

One of the key object of our contruction is the use of  $\varphi$ -equivalence in order to transform a sum over a group(oid) of (partial) transformations, into a sum over the vertex set. With the current notion of path groupoid, searching for something similar amounts to searching for a graph traversal.

**Definition 94. Traversal set**

Let a graph  $G = \langle V, E \rangle$  that is connected. A *traversal set* is a pair  $(\mathcal{U}, \mathcal{T})$  of (EC) partial transformations subsets  $\subset \Psi_{\text{EC}}^*(G)$ , such that

1.  $\mathcal{U}$  is *edge-deterministic*, in the sense that an edge can only correspond to a unique  $g$ , *i.e.*  $\forall g, h \in \mathcal{U} : \exists v \in V, g(v) = h(v) \Rightarrow g = h$
2. The (EC) partial transformations of  $\mathcal{T}$  are restrictions of those of  $\mathcal{U}$ ,  
*i.e.*  $\forall g \in \mathcal{U}, \exists! h \in \mathcal{T}, \begin{cases} \mathcal{D}_h \subset \mathcal{D}_g \\ \forall v \in \mathcal{D}_h, h(v) = g(v) \end{cases}$   
(equivalently,  $\mathcal{T} \ltimes G$  is a path subgroupoid of  $\mathcal{U} \ltimes G$  *s.t.*  $|\mathcal{T}| = |\mathcal{U}|$ )
3. The subgraph  $G_{\mathcal{T}} = \langle V, \mathcal{T} \ltimes_1 G \rangle$  is a spanning tree of  $G$ .

We denote  $(\mathcal{U}, \mathcal{T}) \in \text{trav}(G)$ , and denote by  $r$  the root of  $G_{\mathcal{T}}$ .

For  $p \in \mathcal{T} \ltimes G \subset \mathcal{U} \ltimes G$ , we denote  $\gamma_p^{\mathcal{T} \ltimes G}$  and  $\gamma_p^{\mathcal{U} \ltimes G}$  its path maps.

*Remark.* The assumption that the graph  $G$  is connected doesn't lose generality as the construction can be replicated to each connected component in the general case.

A traversal set  $(\mathcal{U}, \mathcal{T})$  defines a  $\varphi$ -equivalence between the  $\alpha$ -fiber of the root  $r$  and the vertex set  $V$  as follows.

**Lemma 95. Path  $\varphi$ -Equivalence**

Let  $(\mathcal{U}, \mathcal{T}) \in \text{trav}(G)$ . Given  $v \in V$ , there exists a unique  $p_v \in \mathcal{T} \ltimes G$  such that  $\alpha(p_v) = r$  and  $\beta(p_v) = v$ . Denote  $\mathcal{T} \ltimes^r G = \alpha_{\mathcal{T} \ltimes G}^{-1}\{r\}$ . We can do the following construction:

1. Define  $\varphi : p_v \mapsto v$ .
2. Define  $(p_v, p_u) \mapsto p_v^u \in \mathcal{U}^0 \ltimes^r G$  such that the sequence of partial transformations of  $p_v^u$  and  $p_v$  are the same (*i.e.*  $\gamma_{p_v^u}^{\mathcal{U}^0 \ltimes G} = \gamma_{p_v}^{\mathcal{U} \ltimes G}$ ), and the source of  $p_v^u$  is the target of  $p_u$  (*i.e.*  $\alpha(p_v^u) = \beta(p_u) = u$ ).
3. Define the external composition  $p_v p_u = p_v^u p_u \in \mathcal{U}^0 \ltimes^r G$ .

Then  $\varphi : \alpha_{\mathcal{T} \ltimes G}^{-1}\{r\} \rightarrow V$  is a bijective partial equivariant map.

**TODO: check domain of  $\varphi$  and bijectivity**

*Proof.* Bijectivity is a consequence of the spanning tree structure of  $\mathcal{T}$ . Equivariance because  $\gamma_{p_v}(u) = \gamma_{p_v} \gamma_{p_u}(r) = \gamma_{p_v p_u}(r) = \varphi(p_v p_u)$ .  $\square$

We can now define the convolution that is based on a path groupoid.



**Definition 96. Path convolution**

Let  $(\mathcal{U}, \mathcal{T}) \in \text{trav}(G)$ . The *path convolution* is the partial convolution based on the path subgroupoid  $\mathcal{T} \ltimes G$ , which uses the groupoid partial action  $\gamma := \gamma^{\mathcal{U}^0 \ltimes G}$  of the embedding groupoid zero-closure  $\mathcal{U}^0 \ltimes G$ .

- (i) In what follows are the three expressions of the path  $\varphi$ -convolution for signals  $s_1, s_2 \in \mathcal{S}(V)$ , and  $u \in V$ :

$$\begin{aligned} (s *_{\varphi} w) &= \sum_{v \in V} s[v] \gamma_{p_v}(w) \\ &= \sum_{\substack{p \in \mathcal{T} \ltimes G \\ \text{s.t. } \alpha(p)=r}} s[\varphi(p)] \gamma_p(w) \\ (s *_{\varphi} w)[u] &= \sum_{\substack{(a,b) \in V \\ \text{s.t. } \gamma_{p_a}(b)=u}} s[a] w[b] \end{aligned}$$

- (ii) The mixed formulations with  $w \in \mathcal{S}(\mathcal{T} \ltimes G)$  are:

$$\begin{aligned} (w *_{\text{M}} s) &= \sum_{\substack{p \in \mathcal{T} \ltimes G \\ \text{s.t. } \alpha(p)=r}} w[p] \gamma_p(s) \\ (w *_{\text{M}} s)[u] &= \sum_{\substack{(p,v) \in \mathcal{T} \ltimes G \times V \\ \text{s.t. } \alpha(p)=r \\ \text{s.t. } \gamma_p(v)=u}} w[p] s[v] \end{aligned}$$

*Remark.* The role of  $\mathcal{T}$  is to provide a  $\varphi$ -equivalence. The role of  $\mathcal{U}$  is to extend every partial transformation  $\gamma_g^{\mathcal{T} \ltimes G}$  to the domain of its unrestricted counterpart  $\gamma_g^{\mathcal{U} \ltimes G}$ .

Proposition 90 also holds for path groupoids, except that the domain-symmetric condition of 2.(i) is not needed.

**Proposition 97. Characterization by equivariance to  $\mathcal{U} \ltimes G$ 's action**

Let  $(\mathcal{U}, \mathcal{T}) \in \text{trav}(G)$ .

- (i) The class of linear transformations of  $\mathcal{S}(V)$  that are equivariant to the path actions of  $\mathcal{U} \ltimes G$  is exactly the path  $\varphi$ -convolution right-operators;
- (ii) in the abelian case, they are also exactly the M-convolution left-operators.

*Proof.* Instead of the domain-symmetric condition that was used in the proof of the converse of Proposition 90 (2.(i)), we use the fact that any vertex can be reached with an action from the root of the spanning tree of the traversal set. Indeed, given  $v \in V$ , as we have  $\gamma_{p_v}(r) = v$ , then  $\gamma_{p_v}(\delta_r) = \delta_v$ . Therefore, by developping a linear transformation  $f(s)$  on the dirac family, and commuting  $f$  with  $\gamma_{p_v}$ , we obtain that  $f(s) = s *_{\varphi} w$ , where  $w = f(\delta_r)$ . The rest of the proof is similar to that of Proposition 90.  $\square$

*Remark.* Note that  $\mathcal{U} \ltimes V$ 's action is almost the same as the groupoid partial action of  $\Upsilon = \langle \mathcal{U} \rangle$  (only "almost" because not all combinations of partial transformations might exist in the paths). However  $\mathcal{U} \ltimes V$  associativity law doesn't have the limitation of  $\Upsilon$ 's.

**Edge convolution operators**

The counterparts of strictly edge-constrained (EC\*) convolution operators for path convolutions, are indeed path convolution operators obtained with supporting set  $\mathcal{N} \subset \mathcal{T} \ltimes_1 G$  which any graph can admit. By extrapolation, we can coin them *edge convolution operators*. As shown by this section, to construct one, all we need is a traversal set of partial transformations  $(\mathcal{U}, \mathcal{T})$ .

## 2.5 Conclusion

In this chapter, we constructed the convolution on graph domains.

1. We first saw that classical convolutions are in fact the class of linear transformations of the signal space that are equivariant to translations. For signals defined on graph domains, there is no natural definition of translations.
2. Therefore, we adopted a more abstract standpoint and considered in the first place any kind of transformation of the vertex set  $V$ . Hence, given a subgroup of transformation  $\Gamma$ , we constructed the class of linear transformations of the signal space that are equivariant to it. This provided us with an expression of a convolution based on this subgroup, and a bijective equivariant map between  $\Gamma$  and  $V$ , in order to transport a sum over  $\Gamma$  into a sum over  $V$ . We also proposed a simpler expression in the abelian case.
3. Then, we introduced the role of the edge set  $E$ , and we constrained  $\Gamma$  by it. This allows us to obtain a characterization of admissibility of convolutions by Cayley subgraph isomorphism, and to analyze intrinsic properties of the constructed convolution operator, namely locality and weight sharing. We also discussed operators with a smaller kernel, in particular those that are strictly edge-constrained (EC\*), as they are simpler to construct.
4. Finally, we overcame the limitation that some graphs only have trivials or low degree Cayley subgraphs. In this case, we rebased our construction on groupoids of partial transformations  $\Upsilon$  as a first iteration, but this one didn't overcome fully the above-mentioned limitation. As a last iteration, we broke down the previous construction into elementary partial actions onto the edges, recomposed into path groupoids  $\mathcal{U} \ltimes G$ .

Similarly, equivariance characterization and intrinsic properties hold, and the simpler (EC\*) construction is also possible.

### Summary of practical (EC\*) convolution operators

3. For graphs that are quite regular, in the sense that they contain an above-low-degree Cayley subgraph (degree  $d \geq 4$ ), we saw in Section 2.3.3 that all we need to construct an (EC\*) convolution operator is a generating set  $\mathcal{U}$  of transformations, without the need of composing its elements, and optionally (in the non-abelian case) to move a local patch  $\mathcal{K}_{\text{Id}}$  over the graph domain.
4. For a general graph, we saw in Section 2.4.4 that all we need to construct an (EC\*) path convolution operator is a traversal set  $(\mathcal{U}, \mathcal{T})$  of partial transformations, without the need to compose the paths.

In the next chapter, we will encounter examples of (EC) and (EC\*) convolution operators defined on graphs, that can be expressed under group representations or under path groupoid representations.

# Chapter 3

## Application to neural networks

### Contents

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<b>3.1</b>	<b>Layer representations . . . . .</b>	<b>87</b>
3.1.1	Neural interpretation of tensor spaces . . . . .	87
3.1.2	Propagational interpretation . . . . .	88
3.1.3	Graph representation of the input space . . . . .	89
3.1.4	General representation with weight sharing . . . . .	91
<b>3.2</b>	<b>Study of the ternary representation . . . . .</b>	<b>94</b>
3.2.1	Genericity . . . . .	94
3.2.2	Computationally efficient representation . . . . .	94
3.2.3	Robustness in extending the convolution . . . . .	97
3.2.4	Learning the weight sharing scheme . . . . .	97
<b>3.3</b>	<b>Extending CNNs . . . . .</b>	<b>98</b>
3.3.1	Introduction . . . . .	98
3.3.2	Related works . . . . .	98
3.3.3	Convolution . . . . .	98
3.3.4	Subsampling . . . . .	98
3.3.5	Experiments . . . . .	98

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## Introduction

TODO:

### 3.1 Layer representations

Let  $\mathcal{L} = (g, h)$  a neural network layer, where  $g : I \rightarrow O$  is its linear part,  $h : O \rightarrow O$  is its activation function,  $I$  and  $O$  are its input and output spaces, which are tensor spaces.

#### 3.1.1 Neural interpretation of tensor spaces

Recall from Definition 1 that a tensor space has been defined such that its canonical basis is a cartesian product of canonical bases of vector spaces. Let  $I = \bigotimes_{k=1}^p \mathbb{V}_k$  and  $O = \bigotimes_{l=1}^q \mathbb{U}_l$ . Their canonical bases are denoted  $\mathbf{v}_k = (\mathbf{v}_k^1, \dots, \mathbf{v}_k^{n_k})$  and  $\mathbf{u}_l = (\mathbf{u}_l^1, \dots, \mathbf{u}_l^{n_l})$ .

*Remark.* Note that a tensor space is isomorph to the signal space defined over its canonical basis.

More precisely, we have the following relation.

**Lemma 98. Relation between tensor and signal spaces**

Let  $\mathbb{V}, \mathbb{U}$  vector spaces, and  $\mathbf{v}, \mathbf{u}$  their canonical bases. Let  $\mathbb{T}$  a tensor space.  $\otimes$  and  $\times$  denote tensor and cartesian products. Then,

- (i)  $\mathbb{V} \cong \mathcal{S}(\mathbf{v})$
- (ii)  $\mathbb{V} \otimes \mathbb{U} \cong \mathcal{S}(\mathbf{v} \times \mathbf{u})$
- (iii)  $\mathbb{V} \otimes \mathbb{T} \cong \mathcal{S}_{\mathbb{T}}(\mathbf{v})$

where  $\mathcal{S}_{\mathbb{U}}$  are signals taking values in  $\mathbb{U}$  (and  $\mathcal{S}$  are real-valued signals).

*Proof.* (i) Given  $x \in \mathbb{V}$ , define  $\tilde{x} \in \mathcal{S}(\mathbf{v})$  such that  $\forall i, \tilde{x}[\mathbf{v}^i] = x[i]$ . The mapping  $x \mapsto \tilde{x}$  is a linear isomorphism.

- (ii)  $\tilde{x}[\mathbf{v}^i, \mathbf{u}^j] = x[i, j]$
- (iii)  $\tilde{x}[\mathbf{v}^i] = x[i, :, \dots, :]$

□

Let  $d \leq n_k$  and  $e \leq n_l$ . Define  $V$  and  $U$  as the cartesian products  $V = \times_{k=1}^d \mathbb{V}_k$  and  $U = \times_{l=1}^e \mathbb{U}_l$ . Thanks to Lemma 98, we can identify the input and output spaces as  $I = \mathcal{S}(V) \otimes \bigotimes_{k=d+1}^p \mathbb{V}_k$  and  $O = \mathcal{S}(U) \otimes \bigotimes_{l=e+1}^q \mathbb{U}_l$ . As  $\mathcal{S}(\mathbb{V}) \otimes \mathbb{T} = \mathcal{S}_{\mathbb{T}}(\mathbb{V})$ , an object of  $V$  or  $U$  can be interpreted as the representation of a *neuron* which can take multiple values.

In what follows, without loss of generality, we will make the simplification that a neuron can only take a single value (we don't consider input channels and feature maps yet). We'll thus consider that  $I = \mathcal{S}(V)$  and  $O = \mathcal{S}(U)$ , where  $V$  is the set of *input neurons*, and  $U$  is the set of *output neurons*.

### 3.1.2 Propagational interpretation

Let  $\mathcal{L} = (g, h)$ , recall that  $g : \mathcal{S}(V) \rightarrow \mathcal{S}(U)$  is characterized by a connectivity matrix  $W$  such that,  $g(x) = Wx$ .

*Remark.* Using the mapping defined in the proof of Lemma 98, for notational conveniency, we'll abusively consider  $x$  as a vector (eventually reshaped from a tensor), and  $W$  as an object of a binary tensor product for its indexing (*i.e.*  $W[u, v] := W[i, j]$  where  $u = \mathbb{U}^i$  and  $v = \mathbb{V}^j$ ).

#### Definition 99. Propagation graph

The *propagation graph*  $P = \langle (V, U), E_P \rangle$  of a layer  $\mathcal{L} = (W, h)$  is the bipartite graph that has the connectivity matrix  $W$  for bipartite adjacency matrix.

An example is depicted on Figure 7.



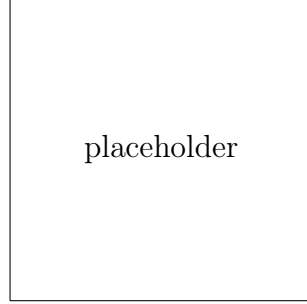


Figure 7: A propagation graph

The propagation graph defines an input topological space  $\mathcal{T}_V$ , and an output topological space  $\mathcal{T}_U$ .

**Definition 100. Topological space**

A *topological space* is a pair  $\mathcal{T} = (X, \mathcal{O})$ , where  $X$  is a set of points,  $\mathcal{O}$  is a set of sets that is closed under intersection (the *open sets*), and such that every point  $x \in X$  is associated with a set  $\mathcal{N}_x \in \mathcal{O}$ , called its *neighborhood*.

Hence, the *neural topologies*  $\mathcal{T}_V$  and  $\mathcal{T}_U$  are defined as

1.  $\mathcal{T}_V = (V, \mathcal{O}(U))$ , with  $\forall v \in V, \mathcal{N}_v = \{u \in U, v \stackrel{P}{\sim} u\}$
2.  $\mathcal{T}_U = (U, \mathcal{O}(V))$ , with  $\forall u \in U, \mathcal{N}_u = \{v \in V, v \stackrel{P}{\sim} u\}$

In particular, given an output neuron  $u \in U$ , a neighborhood  $\mathcal{N}_u$  is also called a *receptive field*, that we denote  $\mathcal{R}_u$ .

### 3.1.3 Graph representation of the input space

Let's consider that the input neurons  $V$  have a (possibly edge-less) graph structure  $G = \langle V, E \rangle$ . We define an edge-constrained layer as follows.

**Definition 101. Edge-constrained layer**

A layer  $\mathcal{L} : G = \langle V, E \rangle \rightarrow U$ , is said to be *edge-constrained* (EC) if:

1. There is a one-to-one correspondence  $\pi : V_\pi \rightarrow U$ , where  $V_\pi \subset V$ .
2. Given an output neuron  $u$ , an input neuron  $v$  is in its receptive field, if and only if,  $v$  and the  $\pi$ -fiber of  $u$  are connected in  $G$ ,  
*i.e.*  $\forall u \in U, v \in \mathcal{R}_u \Leftrightarrow v \stackrel{E}{\sim} \pi^{-1}(u)$

Note that (EC) convolutions are (EC) layers. We have the following characterization.

**Proposition 102. (EC) Characterization with receptive fields**

Let a layer  $\mathcal{L} : V \rightarrow U$ ,  $V_\pi \subset V$ , and a one-to-one correspondence  $\pi : V_\pi \rightarrow U$ . There exists a graph  $G = \langle V, E \rangle$  for which  $\mathcal{L}$  is (EC), if and only if, the receptive fields are *intertwined* (*i.e.*  $\forall a, b \in V_\pi, a \in \mathcal{R}_{\pi(b)} \Leftrightarrow b \in \mathcal{R}_{\pi(a)}$ ).

*Proof.*  $\Rightarrow$  : Thanks to  $a \in \mathcal{R}_{\pi(b)} \Leftrightarrow a \stackrel{E}{\sim} b \Leftrightarrow b \in \mathcal{R}_{\pi(a)}$

$\Leftarrow$  : If the receptive fields are intertwined, then the relation defined as  $a \sim b \Leftrightarrow a \in \mathcal{R}_{\pi(b)}$  is symmetric, and thus can define an edge set.

□

Therefore, any layer that has its receptive fields intertwined, admits an *underlying* graph structure. For example, a 2-d convolution operator can be rewritten as an (EC\*) convolution on a lattice graph, and as an (EC) convolution on a grid graph.

Figure 8 depicts an underlying graph and its corresponding propagation graph.

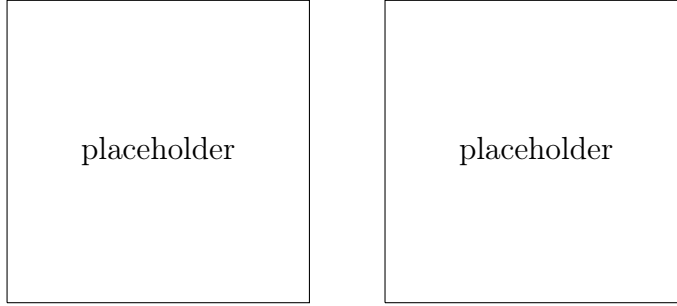


Figure 8: Underlying graph Vs Prop graph

### 3.1.4 General representation with weight sharing

Weight sharing refers to the fact that some parameters of the connectivity matrix  $W$  are equal, and stay equal after each learning iteration. In other words they are tied together. From a propagational point of view, this amounts to label the edges of the propagation graph  $P$  with weights, where weights can be used multiple times to label these edges. Supposed  $W$  is of shape  $m \times n$  and there are  $\omega$  weights in the kernel used to label the edges. Given a input neuron  $i$  and an output neuron  $j$ , the edge labelling can be expressed as:

$$W[j, i] = \theta[h] = \theta^T a \quad (28)$$

where  $\theta$ , the weight kernel, is a vector of size  $\omega$  and  $a$  is a one-hot vector (full of 0s except for one 1) of size  $\omega$  with a 1 at the index  $h$  corresponding to the weight that labels the edge  $i \sim j$ ; or  $a$  is the *zero* vector in case  $i \not\sim j$ .

This equation (28) can be rewritten as a tensor contraction under Einstein summation convention, by noticing that  $a$  depends on  $i, j$  and by defining a tensor  $S$  such that  $a = S[:, i, j]$ , as follows:

$$W_{ij} = \theta_k S^k_{ij} \quad (29)$$

Therefore, the linear part of  $\mathcal{L}$  can be rewritten as:

$$g(x)_j = \theta_k S_j^k x_i \quad (30)$$

If we consider that the layer  $\mathcal{L}$  is duplicated with input channels and feature maps, then  $\theta$ ,  $x$  and  $g(x)$  become tensors, denoted  $\Theta$ ,  $X$  and  $g(X)$ . Usually, for stochastic gradient descent,  $X$  and  $g(X)$  are also expanded with a further rank corresponding to the batch size and we obtain:

$$g(X) = \Theta S X \text{ where } \begin{cases} W_{pq}^{ij} = \Theta_{pq}^k S_k^{ij} \\ g(X)_{jq}^b = W_{jq}^{ip} X_{ip}^b \end{cases} \quad (31)$$

index	size	description
$i$	$n$	input neuron
$j$	$m$	output neuron
$p$	$N$	input channel
$q$	$M$	feature map
$k$	$\omega$	kernel weight
$b$	$B$	batch instance

Table 1: Table of indices

*Remark.* Note that the expression  $\Theta S X$  is written regardless of the ordering of the tensors ranks and is defined by index juggling. Therefore, it is an associative and commutative ternary operation between  $\Theta$ ,  $S$  and  $X$ , which contracts them along the input ranks (neurons and channels) and the weight rank.

We call  $S$  the *weight sharing scheme* of the layer  $\mathcal{L}$ .

**Definition 103. Ternary representation**

The *ternary representation* of a layer  $\mathcal{L} : X \mapsto Y$ , with activation function  $h$ , is the equation  $Y = h(\Theta SX)$ , as defined in (31), where  $\Theta$  is the *weight kernel*, and  $S$  is called the *weight sharing scheme*.

*Remark.* In (28), we defined  $a = S[:, i, j]$  as a one-hot vector when  $i \sim j$ , as its role is to select a weight in  $\theta = \Theta[p, q, :]$ . However,  $a$  can also do this selection *linearly*, so in fact it is not necessarily a one-hot vector.

Figure 9 depicts an example of how the equation (31) labels the edges of  $P$ .

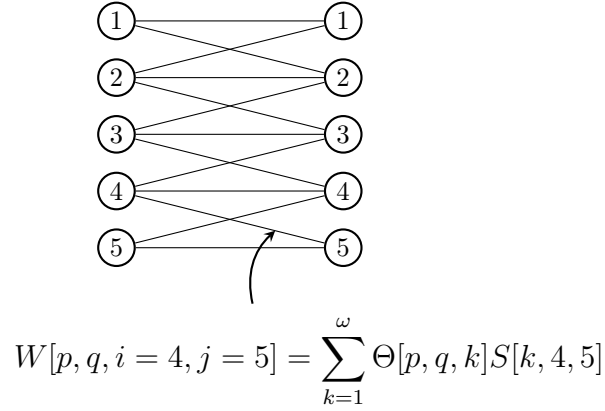


Figure 9: Example of a propagation graph  $P$  for a given input channel  $p$  and feature map  $q$ . The edge  $4 \sim 5$  is labelled with a linear combination of kernel weights from  $\Theta[p, q, :]$ . In the usual case,  $S[k, 4, 5]$  is a one-hot vector that selects a single kernel weight:  $\exists h, W[p, q, 4, 5] = \Theta[p, q, h]$ .

The ternary representation uncouples the roles of  $\Theta$  and  $S$  in  $W$ , and is the most general way of representing any kind of partially connected layer with weight sharing.

## 3.2 Study of the ternary representation

### 3.2.1 Genericity

The ternary representation can represent any kind of layer. For example,

- To obtain a fully connected layer, one can choose  $\omega$  to be of size  $nm$  and  $S$  the matrix of vectors that contains all possible one-hot vectors.
- To obtain a convolutional layer, one can choose  $\omega$  to be the size of the kernel.  $S$  would contain one-hot vectors. A stride  $> 1$  can be obtained by removing the corresponding dimensions. If the convolution is a classical convolution, or is supported by a Cayley subgraph (see Chapter 2), then  $S$  would be circulant along the input neurons rank in the canonical basis.
- Any partially connected layer with (or without) weight sharing can be obtained with appropriate construction of  $S$ .

### 3.2.2 Computationally efficient representation

**What is the fastest way to compute  $\Theta SX$  ?**

As the equation (31) is associative and commutative, there are three ways to start to calculate it: with  $\Theta S$ ,  $SX$ , or  $\Theta X$ , which we will call *middle-stage tensors*. The computation of a middle-stage tensor is the bottleneck to compute (31) as it imposes to compute significantly more entries than for the second tensor contraction. In Table 2, we compare their shapes. We refer the reader to Table 1 for the denomination of the indices.

tensor	shape
$\Theta$	$\omega \times P \times Q$
$S$	$\omega \times n \times m$
$X$	$n \times P \times B$
$\Theta S$	$n \times m \times P \times Q$
$SX$	$\omega \times m \times P \times B$
$\Theta X$	$\omega \times n \times Q \times B$
$\Theta SX$	$m \times Q \times B$

Table 2: Table of shapes

In usual settings, we want to have  $\omega \ll n$  and  $\omega \ll m$ , which means that we have weight kernels of small sizes (for example in the case of images, convolutional kernel are of size significantly smaller than that of the images). Also, the number of input channels  $P$  and of feature maps  $Q$  are roughly in the same order, with  $P < Q$  more often than the contrary. It turns out that in practice, the size of  $\Theta S$  is significantly bigger than the size of  $SX$  and of  $\Theta X$ , and the size of  $SX$  is usually the smallest.

### How to exploit $S$ sparsity ?

Also, in usual settings,  $S$  is sparse as  $S[:, i, j]$  are one-hot vectors. So computing  $SX$  should be faster than computing  $\Theta X$ , provided we exploit the sparsity. Although  $S$  is very sparse as it contains at most a fraction  $\frac{1}{w}$ -th of non-zero values, it is only sparse along the first rank, which makes implementation with sparse classes of common deep learning libraries not optimized. So we proceed differently. The idea is to use a non-sparse tensor  $X_{\text{LRF}}$  that has a rank that indexes local receptive fields (LRF), and another rank that indexes elements of these LRF, in order to lower the computation to a dense matrix multiplication (or dense tensor contraction) which is already well optimized. This approach, proposed in Chellapilla et al., 2006, is also exploited in the cudnn primitives (Chetlur et al., 2014) to efficiently implement the classical convolution.

### The LRF representation

In our case, it turns out that  $X_{\text{LRF}}$  can be exactly  $SX$ , as given fixed  $b$ ,  $p$ , and  $j$ ,  $SX[:, j, p, b]$  corresponds to entries of the input signal  $X[:, p, b]$  restrained to a LRF  $\mathcal{R}_j$  of size  $\omega$ . Therefore,

$$\exists \text{LRF}_j = [i_1, \dots, i_\omega] \text{ s.t. } SX[:, j, p, b] = X[\text{LRF}_j, p, b] \quad (32)$$

The elements of  $\text{LRF}_j$  can be found by doing a lookup in the one-hot vectors of  $S$ , provided a weight occur exactly once in each LRF. We have:

$$R_j[k] = i_k \text{ s.t. } S[:, i_k, j][k] = 1 \quad (33)$$

This lookup need not be computed each time and can be done beforehand. Finally, if we define  $\text{LRF} = [\text{LRF}_1, \dots, \text{LRF}_m]$ , (32) gives:

$$SX = X[\text{LRF}, :, :] \quad (34)$$

The equation (34) is computed with only  $\omega \times m$  assignments and can be simply implemented with automatic differentiation in commonly used deep learning libraries.

### Benchmarks

To support our theoretical analysis, we benchmark three methods for computing the tensor contraction  $SX$ :

- naively using dense multiplication,
- using sparse classes of deep learning libraries,
- using the LRF based method we described above.

We run the benchmarks under the assumptions that  $S[:, i, j]$  are one-hot vectors, and that a weight occur exactly once in each LRF (as it is the case for convolutions supported by a Cayley subgraph). For each method,



we make 100 runs of computations of  $SX$ , with  $S$  and  $X$  being randomly generated according to the assumptions. In Table 3, we report the mean time and standard deviation. The value of the hyperparameters were each time  $n = m = N = M = B = 100$ , and  $\omega = 10$ . The computations were done on graphical processing units (GPU).

Method	Time
Naive	<i>todo</i> $\mu s \pm$ <i>todo</i>
Sparse	<i>todo</i> $\mu s \pm$ <i>todo</i>
LRF	<i>todo</i> $\mu s \pm$ <i>todo</i>

Table 3: Benchmark results

As expected, the LRF method is faster.

### 3.2.3 Robustness in extending the convolution

### 3.2.4 Learning the weight sharing scheme

## 3.3 Extending CNNs

### 3.3.1 Introduction

### 3.3.2 Related works

### 3.3.3 Convolution

### 3.3.4 Subsampling

### 3.3.5 Experiments

# Bibliography

- Abadi, Martín, Ashish Agarwal, Paul Barham, Eugene Brevdo, Zhifeng Chen, Craig Citro, Greg S. Corrado, Andy Davis, Jeffrey Dean, Matthieu Devin, Sanjay Ghemawat, Ian Goodfellow, Andrew Harp, Geoffrey Irving, Michael Isard, Yangqing Jia, Rafal Jozefowicz, Lukasz Kaiser, Manjunath Kudlur, Josh Levenberg, Dan Mané, Rajat Monga, Sherry Moore, Derek Murray, Chris Olah, Mike Schuster, Jonathon Shlens, Benoit Steiner, Ilya Sutskever, Kunal Talwar, Paul Tucker, Vincent Vanhoucke, Vijay Vasudevan, Fernanda Viégas, Oriol Vinyals, Pete Warden, Martin Wattenberg, Martin Wicke, Yuan Yu, and Xiaoqiang Zheng (2015). *TensorFlow: Large-Scale Machine Learning on Heterogeneous Systems*. Software available from tensorflow.org. URL: <http://tensorflow.org/> (cit. on p. 26).
- Arora, Raman, Amitabh Basu, Poorya Mianjy, and Anirbit Mukherjee (2018). “Understanding Deep Neural Networks with Rectified Linear Units”. In: *International Conference on Learning Representations*. URL: [https://openreview.net/forum?id=B1J\\_rgWRW](https://openreview.net/forum?id=B1J_rgWRW) (cit. on p. 18).
- Bass, Jean (1968). “Cours de mathématiques”. In: (cit. on p. 7).
- Bengio, Yoshua (2009). “Learning deep architectures for AI”. In: *Foundations and trends® in Machine Learning* 2.1, pp. 1–127 (cit. on p. 17).
- Bianchini, Monica and Franco Scarselli (2014). “On the complexity of neural network classifiers: A comparison between shallow and deep architectures”. In: *IEEE transactions on neural networks and learning systems* 25.8, pp. 1553–1565 (cit. on p. 18).

- Bottou, Léon (2010). “Large-scale machine learning with stochastic gradient descent”. In: *Proceedings of COMPSTAT’2010*. Springer, pp. 177–186 (cit. on p. 30).
- Brandt, Heinrich (1927). “Über eine Verallgemeinerung des Gruppenbegriffes”. In: *Mathematische Annalen* 96.1, pp. 360–366 (cit. on p. 70).
- Cayley, Professor (1878). “Desiderata and Suggestions: No. 2. The Theory of Groups: Graphical Representation”. In: *American Journal of Mathematics* 1.2, pp. 174–176. ISSN: 00029327, 10806377. URL: <http://www.jstor.org/stable/2369306> (cit. on p. 62).
- Chellapilla, Kumar, Sidd Puri, and Patrice Simard (2006). “High performance convolutional neural networks for document processing”. In: *Tenth International Workshop on Frontiers in Handwriting Recognition*. Suvisoft (cit. on p. 95).
- Chetlur, Sharan, Cliff Woolley, Philippe Vandermersch, Jonathan Cohen, John Tran, Bryan Catanzaro, and Evan Shelhamer (2014). “cudnn: Efficient primitives for deep learning”. In: *arXiv preprint arXiv:1410.0759* (cit. on p. 95).
- Clevert, Djork-Arné, Thomas Unterthiner, and Sepp Hochreiter (2015). “Fast and accurate deep network learning by exponential linear units (elus)”. In: *arXiv preprint arXiv:1511.07289* (cit. on p. 16).
- Cohen, Nadav and Amnon Shashua (2016). “Convolutional rectifier networks as generalized tensor decompositions”. In: *International Conference on Machine Learning*, pp. 955–963 (cit. on p. 18).
- Cohen, Nadav, Ronen Tamari, and Amnon Shashua (2018). “Boosting Dilated Convolutional Networks with Mixed Tensor Decompositions”. In: *International Conference on Learning Representations*. URL: <https://openreview.net/forum?id=S1JHhv6TW> (cit. on pp. 17, 19).
- Cybenko, George (1989). “Approximation by superpositions of a sigmoidal function”. In: *Mathematics of control, signals and systems* 2.4, pp. 303–314 (cit. on p. 16).

- Delalleau, Olivier and Yoshua Bengio (2011). “Shallow vs. deep sum-product networks”. In: *Advances in Neural Information Processing Systems*, pp. 666–674 (cit. on pp. 17, 18).
- Deng, Jia, Wei Dong, Richard Socher, Li-Jia Li, Kai Li, and Li Fei-Fei (2009). “Imagenet: A large-scale hierarchical image database”. In: *Computer Vision and Pattern Recognition, 2009. CVPR 2009. IEEE Conference on*. IEEE, pp. 248–255 (cit. on pp. 17, 19).
- Eldan, Ronen and Ohad Shamir (2016). “The power of depth for feedforward neural networks”. In: *Conference on Learning Theory*, pp. 907–940 (cit. on p. 18).
- Exel, Ruy (1998). “Partial actions of groups and actions of inverse semi-groups”. In: *Proceedings of the American Mathematical Society* 126.12, pp. 3481–3494 (cit. on p. 78).
- Glorot, Xavier and Yoshua Bengio (2010). “Understanding the difficulty of training deep feedforward neural networks”. In: *Proceedings of the thirteenth international conference on artificial intelligence and statistics*, pp. 249–256 (cit. on p. 17).
- Glorot, Xavier, Antoine Bordes, and Yoshua Bengio (2011). “Deep sparse rectifier neural networks”. In: *International Conference on Artificial Intelligence and Statistics*, pp. 315–323 (cit. on pp. 16, 17).
- Hackbusch, Wolfgang (2012). *Tensor spaces and numerical tensor calculus*. Vol. 42. Springer Science & Business Media (cit. on pp. 7, 8).
- Håstad, Johan and Mikael Goldmann (1991). “On the power of small-depth threshold circuits”. In: *Computational Complexity* 1.2, pp. 113–129 (cit. on p. 17).
- He, Kaiming, Xiangyu Zhang, Shaoqing Ren, and Jian Sun (2015). “Delving deep into rectifiers: Surpassing human-level performance on imagenet classification”. In: *Proceedings of the IEEE international conference on computer vision*, pp. 1026–1034 (cit. on p. 16).

- (2016). “Deep residual learning for image recognition”. In: *Proceedings of the IEEE conference on computer vision and pattern recognition*, pp. 770–778 (cit. on pp. 15, 19).
- Hinton, Geoffrey, Li Deng, Dong Yu, George E Dahl, Abdel-rahman Mohamed, Navdeep Jaitly, Andrew Senior, Vincent Vanhoucke, Patrick Nguyen, Tara N Sainath, et al. (2012). “Deep neural networks for acoustic modeling in speech recognition: The shared views of four research groups”. In: *IEEE Signal Processing Magazine* 29.6, pp. 82–97 (cit. on p. 15).
- Hinton, Geoffrey E, Simon Osindero, and Yee-Whye Teh (2006). “A fast learning algorithm for deep belief nets”. In: *Neural computation* 18.7, pp. 1527–1554 (cit. on p. 17).
- Hornik, Kurt (1991). “Approximation capabilities of multilayer feedforward networks”. In: *Neural networks* 4.2, pp. 251–257 (cit. on p. 16).
- Hornik, Kurt, Maxwell Stinchcombe, and Halbert White (1989). “Multilayer feedforward networks are universal approximators”. In: *Neural Networks* 2.5, pp. 359–366 (cit. on pp. 16, 30).
- Huang, Gao, Zhuang Liu, Kilian Q Weinberger, and Laurens van der Maaten (2017). “Densely connected convolutional networks”. In: *Proceedings of the IEEE conference on computer vision and pattern recognition*. Vol. 1. 2, p. 3 (cit. on pp. 15, 19).
- Jarrett, Kevin, Koray Kavukcuoglu, Yann LeCun, et al. (2009). “What is the best multi-stage architecture for object recognition?” In: *Computer Vision, 2009 IEEE 12th International Conference on*. IEEE, pp. 2146–2153 (cit. on p. 16).
- Klambauer, Günter, Thomas Unterthiner, Andreas Mayr, and Sepp Hochreiter (2017). “Self-Normalizing Neural Networks”. In: *Advances in Neural Information Processing Systems 30*. Ed. by I. Guyon, U. V. Luxburg, S. Bengio, H. Wallach, R. Fergus, S. Vishwanathan, and R. Garnett. Curran Associates, Inc., pp. 971–980. URL: <http://papers.nips.cc/paper/6698-self-normalizing-neural-networks.pdf> (cit. on p. 16).

- Krizhevsky, Alex, Ilya Sutskever, and Geoffrey E Hinton (2012). “Imagenet classification with deep convolutional neural networks”. In: *Advances in Neural Information Processing Systems*, pp. 1097–1105 (cit. on pp. 15, 17, 29).
- LeCun, Y. (1987). “Modeles connexionnistes de l’apprentissage (connectionist learning models)”. PhD thesis. Université P. et M. Curie (Paris 6) (cit. on p. 20).
- LeCun, Yann, Yoshua Bengio, et al. (1995). “Convolutional networks for images, speech, and time series”. In: *The handbook of brain theory and neural networks* 3361.10, p. 1995 (cit. on pp. 15, 29).
- LeCun, Yann, Bernhard Boser, John S Denker, Donnie Henderson, Richard E Howard, Wayne Hubbard, and Lawrence D Jackel (1989). “Backpropagation applied to handwritten zip code recognition”. In: *Neural computation* 1.4, pp. 541–551 (cit. on pp. 15, 16).
- LeCun, Yann, Léon Bottou, Yoshua Bengio, and Patrick Haffner (1998). “Gradient-based learning applied to document recognition”. In: *Proceedings of the IEEE* 86.11, pp. 2278–2324 (cit. on p. 30).
- Li, Xin (2016). “Partial transformation groupoids attached to graphs and semigroups”. In: *International Mathematics Research Notices* 2017.17, pp. 5233–5259 (cit. on p. 78).
- Lin, Henry W, Max Tegmark, and David Rolnick (2017). “Why does deep and cheap learning work so well?” In: *Journal of Statistical Physics* 168.6, pp. 1223–1247 (cit. on p. 18).
- Maas, Andrew L, Awni Y Hannun, and Andrew Y Ng (2013). “Rectifier nonlinearities improve neural network acoustic models”. In: *Proceedings of the 30th international conference on machine learning* (cit. on p. 16).
- Marcus, Marvin (1975). “Finite dimensional multilinear algebra”. In: (cit. on p. 7).

- McCulloch, Warren S and Walter Pitts (1943). “A logical calculus of the ideas immanent in nervous activity”. In: *The bulletin of mathematical biophysics* 5.4, pp. 115–133 (cit. on p. 20).
- Mhaskar, Hrushikesh, Qianli Liao, and Tomaso Poggio (2016). “Learning functions: when is deep better than shallow”. In: *arXiv preprint arXiv:1603.00988* (cit. on p. 18).
- Montufar, Guido F, Razvan Pascanu, Kyunghyun Cho, and Yoshua Bengio (2014). “On the number of linear regions of deep neural networks”. In: *Advances in neural information processing systems*, pp. 2924–2932 (cit. on p. 18).
- Nica, Alexandru (1994). “On a groupoid construction for actions of certain inverse semigroups”. In: *International Journal of Mathematics* 5.03, pp. 349–372 (cit. on p. 78).
- Nickolls, John, Ian Buck, Michael Garland, and Kevin Skadron (2008). “Scalable parallel programming with CUDA”. In: *ACM SIGGRAPH 2008 classes*. ACM, p. 16 (cit. on p. 17).
- Orhan, Emin and Xaq Pitkow (2018). “Skip Connections Eliminate Singularities”. In: *International Conference on Learning Representations*. URL: <https://openreview.net/forum?id=HkwBEMWCZ> (cit. on p. 20).
- Pan, Xingyuan and Vivek Srikumar (2016). “Expressiveness of rectifier networks”. In: *International Conference on Machine Learning*, pp. 2427–2435 (cit. on p. 18).
- Pascanu, Razvan, Guido Montufar, and Yoshua Bengio (2013). “On the number of response regions of deep feed forward networks with piece-wise linear activations”. In: *arXiv preprint arXiv:1312.6098* (cit. on pp. 17, 18).
- Poggio, Tomaso, Fabio Anselmi, and Lorenzo Rosasco (2015). *I-theory on depth vs width: hierarchical function composition*. Tech. rep. Center for Brains, Minds and Machines (CBMM) (cit. on p. 18).



- Poole, Ben, Subhaneil Lahiri, Maithra Raghu, Jascha Sohl-Dickstein, and Surya Ganguli (2016). “Exponential expressivity in deep neural networks through transient chaos”. In: *Advances in neural information processing systems*, pp. 3360–3368 (cit. on p. 18).
- Raghu, Maithra, Ben Poole, Jon Kleinberg, Surya Ganguli, and Jascha Sohl-Dickstein (2016). “On the expressive power of deep neural networks”. In: *arXiv preprint arXiv:1606.05336* (cit. on p. 18).
- Rumelhart, David E, Geoffrey E Hinton, and Ronald J Williams (1985). *Learning internal representations by error propagation*. Tech. rep. California Univ San Diego La Jolla Inst for Cognitive Science (cit. on pp. 15, 21, 22).
- Schwartz, Laurent (1957). *Théorie des distributions*. Vol. 2. Hermann Paris (cit. on p. 43).
- Simonyan, Karen and Andrew Zisserman (2014). “Very deep convolutional networks for large-scale image recognition”. In: *arXiv preprint arXiv:1409.1556* (cit. on p. 15).
- Srivastava, Nitish, Geoffrey E Hinton, Alex Krizhevsky, Ilya Sutskever, and Ruslan Salakhutdinov (2014). “Dropout: a simple way to prevent neural networks from overfitting.” In: *Journal of Machine Learning Research* 15.1, pp. 1929–1958 (cit. on p. 30).
- Szegedy, Christian, Wei Liu, Yangqing Jia, Pierre Sermanet, Scott Reed, Dragomir Anguelov, Dumitru Erhan, Vincent Vanhoucke, Andrew Rabinovich, et al. (2015). “Going deeper with convolutions”. In: *Conference on Computer Vision and Pattern Recognition* (cit. on pp. 15, 19).
- Van Den Oord, Aaron, Sander Dieleman, Heiga Zen, Karen Simonyan, Oriol Vinyals, Alex Graves, Nal Kalchbrenner, Andrew Senior, and Koray Kavukcuoglu (2016). “Wavenet: A generative model for raw audio”. In: *arXiv preprint arXiv:1609.03499* (cit. on p. 19).
- Weinstein, Alan (1996). “Groupoids: unifying internal and external symmetry”. In: *Notices of the AMS* 43.7, pp. 744–752 (cit. on p. 70).

- Widrow, Bernard and Marcian E Hoff (1960). *Adaptive switching circuits*. Tech. rep. STANFORD UNIV CA STANFORD ELECTRONICS LABS (cit. on pp. 21, 22).
- Wikipedia, contributors (2018). *Feedforward neural network* — *Wikipedia, The Free Encyclopedia*. [Online; accessed April-2018]. URL: [https://en.wikipedia.org/wiki/Feedforward\\_neural\\_network](https://en.wikipedia.org/wiki/Feedforward_neural_network) (cit. on p. 15).
- Williamson, S Gill (2015). “Tensor spaces-the basics”. In: *arXiv preprint arXiv:1510.02428* (cit. on p. 7).
- Zell, Andreas (1994). *Simulation neuronaler netze*. Vol. 1. Addison-Wesley Bonn (cit. on p. 15).
- Zermelo, Ernst (1904). “Beweis, daß jede Menge wohlgeordnet werden kann”. In: *Mathematische Annalen* 59.4, pp. 514–516 (cit. on p. 77).
- Zoph, Barret and Quoc V Le (2016). “Neural architecture search with reinforcement learning”. In: *arXiv preprint arXiv:1611.01578* (cit. on p. 15).