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Chapter 2

Convolutions on graph domains

Introduction

Defining a convolution of signals over graph domains is a challenging problem. Obviously, if the graph is not a grid graph there exists no natural definition. We first analyse the reasons why the euclidean convolution operator is useful in deep learning, and give a characterization. Then we will search for domains onto which a convolution with these properties can be naturally obtained. This will lead us to put our interest on representation theory and convolutions defined on groups. As the euclidean convolution is just a particular case of the group convolution, it makes perfect sense to steer our construction in this direction. Hence, we will aim at transferring its representation on the vertex domain. First we will do this construction agnostically of the edge set. Then, we will introduce the role of the edge set and see how it should influence it. This will provide us with some particular classes of graphs for which we will obtain a natural construction with the wanted characteristics that we exposed in the first place. Finally, we can relax some aspect of the construction to adapt it to graphs that are not order-regular. The obtained construction is a general expression for convolutions on graph domains, which preserve some key properties.

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2.1 Analysis of the classical convolution

In this section, we are exposing a few properties of the classical convolution that a generalization to graphs would likely try to preserve. For now let's consider a graph G agnostically of its edges *i.e.* $G \cong V$ is just the set of its vertices.

2.1.1 Properties of the convolution

Consider an edge-less grid graph *i.e.* $G \cong \mathbb{Z}^2$. By restriction to compactly supported signals, this case encompass the case of images.

Definition 1. Convolution on $\mathcal{S}(\mathbb{Z}^2)$

Recall that the (discrete) convolution between two signals s_1 and s_2 over $\mathbb{Z}^2$ is a binary operation in $\mathcal{S}(\mathbb{Z}^2)$ defined as:

$$\forall (a,b) \in \mathbb{Z}^2, (s_1 * s_2)[a,b] = \sum_i \sum_j s_1[i,j] \, s_2[a-i, b-j]$$

Definition 2. Convolution operator

A *convolution operator* is a function of the form $f_w : x \mapsto x * w$, where x and w are signals of domains for which the convolution $*$ is defined. When $*$ is not commutative, we differentiate the *right-action* operator $x \mapsto x * w$ from the *left-action* one $x \mapsto w * x$.

The following properties of the convolution on $\mathbb{Z}^2$ are of particular interest for our study.

Linearity

Operators produced by the convolution are linear. So they can be used as linear parts of layers of neural networks.

Locality and weight sharing

When w is compactly supported on K, an impulse response $f_w(x)[a, b]$ amounts to a w-weighted aggregation of entries of x in a neighbourhood of (a, b), called the *local receptive field*.

Commutativity

The convolution is commutative. However, it won't necessarily be the case on other domains.

Equivariance to translations

Convolution operators are equivariant to translations. Below, we show that the converse of this result also holds with Proposition 6.

2.1.2   Characterization on grid graphs

Let's recall first what is a transformation, and how it acts on signals.

Definition 3. Transformation

A *transformation* $f : V \to V$ is a function with same domain and codomain. The set of transformations is denoted $\Phi(V)$. The set of bijective transformations is denoted $\Phi^*(V) \subset \Phi(V)$.

In particular, $\Phi^*(V)$ forms the symmetric group of V and can move signals of $\mathcal{S}(V)$ by linear extension of its group action.

Lemma 4. Extension to $\mathcal{S}(V)$ by group action

An transformation $f \in \Phi^*(V)$ can be extended linearly to the signal space $\mathcal{S}(V)$, and we have:

$$\forall s \in \mathcal{S}(V), \forall v \in V, f(s)[v] := L_f(s)[v] = s[f^{-1}(v)]$$

105 *Proof.* Let $s \in \mathcal{S}(V), f \in \Phi^*(V), L_f \in \mathcal{L}(\mathcal{S}(V))$ *s.t.* $\forall v \in V, L_f(\delta_v) = \delta_{f(v)}$.

106 Then, we have:

$$L_f(s) = \sum_{v \in V} s[v] \, L_f(\delta_v)$$
$$= \sum_{v \in V} s[v] \, \delta_{f(v)}$$
$$\text{So, } \forall v \in V, L_f(s)[v] = s[f^{-1}(v)]$$

107 □

108 We also recall the formalism of translations.

109 **Definition 5. Translation on $\mathcal{S}(\mathbb{Z}^2)$**

110 A translation on $\mathbb{Z}^2$ is defined as a transformation $t \in \Phi^*(\mathbb{Z}^2)$ such that

$$\exists (a, b) \in \mathbb{Z}^2, \forall (x, y) \in \mathbb{Z}^2, t(x, y) = (x + a, y + b)$$

111 It also acts on $\mathcal{S}(\mathbb{Z}^2)$ with the notation $t_{a,b}$ *i.e.*

$$\forall s \in \mathcal{S}(\mathbb{Z}^2), \forall (x, y) \in \mathbb{Z}^2, t_{a,b}(s)[x, y] = s[x - a, y - b]$$

112 For any set E, we denote by $\mathcal{T}(E)$ its translations if they are defined.

113 The next proposition fully characterizes convolution operators with their
114 translational equivariance property. This can be seen as a discretization of a
115 classic result from the theory of distributions (Schwartz, 1957).

116 **Proposition 6. Characterization of convolution operators on $\mathcal{S}(\mathbb{Z}^2)$**
117 On real-valued signals over $\mathbb{Z}^2$, the class of linear transformations that are
118 equivariant to translations is exactly the class of convolutive operations *i.e.*

$$\exists w \in \mathcal{S}(\mathbb{Z}^2), f = . * w \Leftrightarrow \begin{cases} f \in \mathcal{L}(\mathcal{S}(\mathbb{Z}^2)) \\ \forall t \in \mathcal{T}(\mathcal{S}(\mathbb{Z}^2)), f \circ t = t \circ f \end{cases}$$

119

120 *Proof.* The result from left to right is a direct consequence of the definitions:

$$\forall s \in \mathcal{S}(\mathbb{Z}^2), \forall s' \in \mathcal{S}(\mathbb{Z}^2), \forall(\alpha, \beta) \in \mathbb{R}^2, \forall(a,b) \in \mathbb{Z}^2,$$

$$f_w(\alpha s + \beta s')[a,b] = \sum_i \sum_j (\alpha s + \beta s')[i,j]\, w[a-i, b-j]$$

$$= \alpha f_w(s)[a,b] + \beta f_w(s')[a,b] \qquad \text{(linearity)}$$

$$\forall s \in \mathcal{S}(\mathbb{Z}^2), \forall(\alpha, \beta) \in \mathbb{Z}^2, \forall(a,b) \in \mathbb{Z}^2,$$

$$f_w \circ t_{\alpha,\beta}(s)[a,b] = \sum_i \sum_j t_{\alpha,\beta}(s)[i,j]\, w[a-i,b-j]$$

$$= \sum_i \sum_j s[i-\alpha, j-\beta]\, w[a-i, b-j]$$

$$= \sum_{i'} \sum_{j'} s[i', j']\, w[a-i'-\alpha, b-j'-\beta] \qquad (1)$$

$$= f_w(s)[a-\alpha, b-\beta]$$

$$= t_{\alpha,\beta} \circ f_w(s)[a,b] \qquad \text{(equivariance)}$$

121 Now let's prove the result from right to left.

122 Let $f \in \mathcal{L}(\mathcal{S}(\mathbb{Z}^2))$, $s \in \mathcal{S}(\mathbb{Z}^2)$. We suppose that f commutes with trans-
123 lations. Recall that s can be linearly decomposed on the infinite family of
124 dirac signals:

$$s = \sum_i \sum_j s[i,j]\, \delta_{i,j}, \text{ where } \delta_{i,j}[x,y] = \begin{cases} 1 & \text{if } (x,y) = (i,j) \\ 0 & \text{otherwise} \end{cases}$$

125 By linearity of f and then equivariance to translations:

$$f(s) = \sum_i \sum_j s[i,j]\, f(\delta_{i,j})$$

$$= \sum_i \sum_j s[i,j]\, f \circ t_{i,j}(\delta_{0,0})$$

$$= \sum_i \sum_j s[i,j]\, t_{i,j} \circ f(\delta_{0,0})$$

By denoting $w = f(\delta_{0,0}) \in \mathcal{S}(\mathbb{Z}^2)$, we obtain:

$$\forall (a,b) \in \mathbb{Z}^2, f(s)[a,b] = \sum_i \sum_j s[i,j]\, t_{i,j}(w)[a,b] \tag{2}$$

$$= \sum_i \sum_j s[i,j]\, w[a-i, b-j]$$

$$i.e.\ f(s) = s * w$$

$\square$

2.1.3   Usefulness of convolutions in deep learning

Equivariance property of CNNs

In deep learning, an important argument in favor of CNNs is that convolutional layers are equivariant to translations. Intuitively, that means that a detail of an object in an image should produce the same features independently of its position in the image.

Lossless superiority of CNNs over MLPs

The converse result, as a consequence of Proposition 6, is never mentioned in deep learning literature. However it is also a strong one. For example, let's consider a linear function that is equivariant to translations. Thanks to the converse result, we know that this function is a convolution operator parameterized by a weight vector w, $f_w : .\,*\,w$. If the domain is compactly supported, as in the case of images, we can break down the information of w in a finite number n_q of kernels w_q with small compact supports of same size (for instance of size 2×2), such that we have $f_w = \sum_{q \in \{1,2,...,n_q\}} f_{w_q}$. The convolution operators f_{w_q} are all in the search space of 2×2 convolutional layers. In other words, every translational equivariant linear function can

145 have its information parameterized by these layers. So that means that the
146 reduction of parameters from an MLP to a CNN is done with strictly no loss of
147 expressivity (provided the objective function is known to bear this property).
148 Besides, it also helps the training to search in a much more confined space.

Methodology for extending to general graphs
150 Hence, in our construction, we will try to preserve the characterization from
151 Proposition 6 as it is mostly the reason why they are successful in deep
152 learning. Note that the reduction of parameters compared to a dense layer
153 is also a consequence of this characterization.

2.2 Construction from the vertex set

As Proposition 6 is a complete characterization of convolutions, it can be used to define them *i.e.* convolution operators can be constructed as the set of linear transformations that are equivariant to translations. However, in the general case where G is not a grid graph, translations are not defined, so that construction needs to be generalized beyond translational equivariances. In mathematics, convolutions are more generally defined for signals defined over a group structure. The classical convolution that is used in deep learning is just a narrow case where the domain group is an euclidean space. Therefore, constructing a convolution on graphs should start from the more general definition of convolution on groups rather than convolution on euclidean domains.

Our construction is motivated by the following questions:

- Does the equivariance property holds ? Does the characterization from Proposition 6 still holds ?

- Is it possible to extend the construction on non-group domains, or at least on mixed domains ? (*i.e.* one signal is defined over a set, and the other is defined over a subgroup of the transformations of this set).

- Can a group domain draw an underlying graph structure ? Is the group convolution naturally defined on this class of graphs ?

We first recall the notion of group and group convolution.

Definition 7. Group

A group Γ is a set equipped with a closed, associative and invertible composition law that admits a unique left-right identity element.

The group convolution extends the notion of the classical discrete convolution.

Definition 8. Group convolution I

Let a group Γ, the group convolution I between two signals s_1 and $s_2 \in \mathcal{S}(\Gamma)$ is defined as:

$$\forall h \in \Gamma, (s_1 *_\mathrm{I} s_2)[h] = \sum_{g \in \Gamma} s_1[g]\, s_2[g^{-1}h]$$

provided at least one of the signals has finite support if Γ is not finite.

2.2.1 Steered construction from groups

For a graph $G = \langle V, E \rangle$ and a subgroup $\Gamma \subset \Phi^*(V)$ or its invertible transformations, Definition 8 is applicable for $\mathcal{S}(\Gamma)$, but not for $\mathcal{S}(V)$ as V is not a group. Nonetheless, our point here is that we will use the group convolution on $\mathcal{S}(\Gamma)$ to construct the convolutions on $\mathcal{S}(V)$.

For now, let's assume Γ is in one-to-one correspondence with V, and let's define a bijective map φ from Γ to V. We denote $\Gamma \overset{\varphi}{\cong} V$ and $g_v \overset{\varphi}{\mapsto} v$.
Then, the linear morphism $\widetilde{\varphi}$ from $\mathcal{S}(\Gamma)$ to $\mathcal{S}(V)$ defined on the Dirac bases by $\widetilde{\varphi}(\delta_g) = \delta_{\varphi(g)}$ is a linear isomorphism. Hence, $\mathcal{S}(V)$ would inherit the same inherent structural properties as $\mathcal{S}(\Gamma)$. For the sake of notational simplicity, we will use the same symbol φ for both φ and $\widetilde{\varphi}$ (as done between f and L_f). A commutative diagram between the sets is depicted on Figure 2.1.

$$
\begin{array}{ccc}
\Gamma & \xrightarrow{\ \varphi\ } & V \\
{\scriptstyle \mathcal{S}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathcal{S}} \\
\mathcal{S}(\Gamma) & \xrightarrow[\varphi]{} & \mathcal{S}(V)
\end{array}
$$

Figure 2.1: Commutative diagram between sets

We naturally obtain the following relation, which put in simpler words means that signals on $\mathcal{S}(\Gamma)$ are mapped to $\mathcal{S}(V)$ when φ is simultaneously applied on both the signal space and its domain.

199 **Lemma 9. Relation between $\mathcal{S}(\Gamma)$ and $\mathcal{S}(V)$**

200 $\forall s \in \mathcal{S}(\Gamma), \forall u \in V, \varphi(s)[u] = s[\varphi^{-1}(u)] = s[g_u]$

Proof.

$$\forall s \in \mathcal{S}(\Gamma), \varphi(s) = \varphi\left(\sum_{g \in \Gamma} s[g]\,\delta_g\right) = \sum_{g \in \Gamma} s[g]\,\varphi(\delta_g) = \sum_{g \in \Gamma} s[g]\,\delta_{\varphi(g)}$$
$$= \sum_{v \in V} s[g_v]\,\delta_v$$

So $\forall v \in V, \varphi(s)[u] = s[g_u]$

201 $\square$

202 Hence, we can steer the definition of the group convolution from $\mathcal{S}(\Gamma)$ to
203 $\mathcal{S}(V)$ as follows:

204 **Definition 10. Group convolution II**

205 Let a subgroup $\Gamma \subset \Phi^*(V)$ such that $\Gamma \overset{\varphi}{\cong} V$. The group convolution II
206 between two signals s_1 and $s_2 \in \mathcal{S}(V)$ is defined as:

$$\forall u \in V, (s_1 *_{\text{II}} s_2)[u] = \sum_{v \in V} s_1[v]\, s_2[\varphi(g_v^{-1} g_u)]$$

207

208 **Lemma 11. Relation between group convolution I and II**

209 Let a subgroup $\Gamma \subset \Phi^*(V)$ such that $\Gamma \overset{\varphi}{\cong} V$,

$$\forall s_1, s_2 \in \mathcal{S}(\Gamma), \forall u \in V, (\varphi(s_1) *_{\text{II}} \varphi(s_2))[u] = (s_1 *_{\text{I}} s_2)[g_u]$$

210

211   *Proof.* Using Lemma 9,

$$
\begin{aligned}
(\varphi(s_1) *_{\mathrm{II}} \varphi(s_2))[u] &= \sum_{v \in V} \varphi(s_1)[v]\, \varphi(s_2)[\varphi(g_v^{-1} g_u)] \\
&= \sum_{v \in V} s_1[g_v]\, s_2[g_v^{-1} g_u] \\
&= \sum_{g \in \Gamma} s_1[g]\, s_2[g^{-1} g_u] \\
&= (s_1 *_{\mathrm{I}} s_2)[g_u]
\end{aligned}
$$

212   □

213   For convolution II, we only obtain a weak version of Proposition 6.

214   **Proposition 12. Equivariance to $\varphi(\Gamma)$**

215   If φ is a homomorphism, convolution operators acting on the right of $\mathcal{S}(V)$

216   are equivariant to $\varphi(\Gamma)$ *i.e.*

$$
\begin{aligned}
&\text{if } \varphi \in \mathrm{ISO}(\Gamma, V), \\
&\exists w \in \mathcal{S}(V), f = .\, *_{\mathrm{II}} w \Rightarrow \forall v \in V, f \circ \varphi(g_v) = \varphi(g_v) \circ f
\end{aligned}
$$

217

Proof.

$$
\begin{aligned}
&\forall s \in \mathcal{S}(V), \forall u \in V, \forall v \in V, \\
&(f_w \circ \varphi(g_u))(s)[v] = \sum_{v \in V} \varphi(g_u)(s)[v]\, w[\varphi(g_v^{-1} g_u)] \\
&\qquad\qquad\qquad = \sum_{\substack{(a,b) \in V^2 \\ s.t.\ g_a g_b = g_v}} \varphi(g_u)(s)[a]\, w[b] \\
&\qquad\qquad\qquad = \sum_{\substack{(a,b) \in V^2 \\ s.t.\ g_a g_b = g_v}} s[\varphi(g_u)^{-1}(a)]\, w[b]
\end{aligned}
$$

$$= \sum_{\substack{(a,b)\in V^2 \\ s.t. \ g_{\varphi(g_u)(a)}g_b=g_v}} s[a]\, w[b]$$

218 Because φ is an isomorphism, its inverse $c \mapsto g_c$ is also an isomorphism and
219 so $g_{\varphi(g_u)(a)}g_b = g_v \Leftrightarrow g_a g_b = g_{\varphi(g_u)^{-1}(v)}$. So we have both:

$$(f_w \circ \varphi(g_u))(s)[v] = \sum_{\substack{(a,b)\in V^2 \\ s.t. \ g_a g_b=g_{\varphi(g_u)^{-1}(v)}}} s[a]\, w[b]$$

$$= s *_{\mathrm{II}} w[\varphi(g_u)^{-1}(v)]$$

$$= (\varphi(g_u) \circ f_w)(s)[v]$$

220 $\square$

221 *Remark.* Note that convolution operators of the form $f_w = . *_{\mathrm{I}} w$ are also
222 equivariant to Γ, but the proposition and the proof are omitted as they are
223 similar to the latter.

224 In fact, both group convolutions are the same as the latter one borrows the
225 algebraic structure of the first one. Thus we only obtain equivariance to $\varphi(\Gamma)$
226 when φ also transfer the group structure from Γ to V, and the converse don't
227 hold. To obtain equivariance to Γ (and its converse), we will drop the direct
228 homomorphism condition, and instead we will take into account the fact that
229 it contains invertible transformations of V.

230 ## 2.2.2 Construction under group actions

231 Definition 13. Group action

232 An *action* of a group Γ on a set V, is a function $L : \Gamma \times V \to V, (g, v) \mapsto L_g(v)$,

233 such that the map $g \mapsto L_g$ is a homomorphism.

234 Given $g \in \Gamma$, the transformation L_g is called the action of g by L on V.

235 *Remark.* When there is no ambiguity, we use the same symbol for g and L_g.

236 Hence, note that $g \in \Gamma$ can act on both Γ through the left multiplication

237 and on V as being an object of $\Phi^*(V)$. This ambivalence can be seen on a

238 commutative diagram, see Figure 2.2.

$$
\begin{array}{ccc}
g_u & \xrightarrow{\ g_v\ } & g_v g_u \\
\varphi \downarrow & & \downarrow \varphi \\
u & \xrightarrow[\ g_v\ ]{(P)} & \varphi(g_v g_u)
\end{array}
$$

Figure 2.2: Commutative diagram. All arrows except for the one labeled with (P) are always True.

239 For (P) to be true means that φ is an equivariant map *i.e.* whether the

240 mapping is done before or after the action of Γ has no impact on the result.

241 When such φ exists, Γ and V are said to be equivalent and we denote $\Gamma \equiv V$.

242 Definition 14. Equivariant map

243 A map φ from a group Γ acting on the destination set V is said to be an

244 *equivariant map* if

$$ \forall g, h \in \Gamma, g(\varphi(h)) = \varphi(gh) $$

245

246 In our case we have $\Gamma \overset{\varphi}{\cong} V$. If we also have that $\Gamma \equiv V$, we are interested to

247 know if then φ exhibits the equivalence.

Definition 15. φ-Equivalence

A subgroup $\Gamma \subset \Phi^*(V)$ such that $\Gamma \overset{\varphi}{\cong} V$, is said to be φ-*equivalent* if φ is a bijective equivariant map *i.e.* if it verifies the property:

$$\forall v, u \in V, g_v(u) = \varphi(g_v g_u) \tag{P}$$

In that case we denote $\Gamma \overset{\varphi}{\equiv} V$.

Remark. For example, translations on the grid graph, with $\varphi(t_{i,j}) = (i, j)$, are φ-equivalent as $t_{i,j}(a, b) = \varphi(t_{i,j} \circ t_{a,b})$. However, with $\varphi(t_{i,j}) = (-i, -j)$, they would not be φ-equivalent.

Definition 16. Group convolution III

Let a subgroup $\Gamma \subset \Phi^*(V)$ such that $\Gamma \overset{\varphi}{\cong} V$. The group convolution III between two signals s_1 and $s_2 \in \mathcal{S}(V)$ is defined as:

$$s_1 *_{\text{III}} s_2 = \sum_{v \in V} s_1[v] \, g_v(s_2) \tag{3}$$

$$= \sum_{g \in \Gamma} s_1[\varphi(g)] \, g(s_2) \tag{4}$$

The two expressions differ on the domain upon which the summation is done. The expression (3) put the emphasis on each vertex and its action, whereas the expression (4) emphasizes on each object of Γ.

Lemma 17. Relation with group convolution II

$\Gamma \overset{\varphi}{\equiv} V \Leftrightarrow *_{\text{II}} = *_{\text{III}}$

Proof.

$$\forall s_1, s_2 \in \mathcal{S}(V),$$

$$s_1 *_{\text{II}} s_2 = s_1 *_{\text{III}} s_2$$

$$\Leftrightarrow \forall u \in V, \sum_{v \in V} s_1[v] \, s_2[\varphi(g_v^{-1} g_u)] = \sum_{v \in V} s_1[v] \, s_2[g_v^{-1}(u)] \tag{5}$$

264   Hence, the direct sense is obtained by applying (P).

265   For the converse, given $u, v \in V$, we first realize (5) for $s_1 := \delta_v$, obtaining

266   $s_2[\varphi(g_v^{-1} g_u)] = s_2[g_v^{-1}(u)]$, which we then realize for a real signal s_2 having no

267   two equal entries, obtaining $\varphi(g_v^{-1} g_u) = g_v^{-1}(u)$. From the latter we finally

268   obtain (P) with the one-to-one correspondence $g_{v'} := g_v^{-1}$. $\square$

269   We can then coin the term φ-convolution.

270   **Definition 18. φ-convolution**

271   Let $\Gamma \overset{\varphi}{\equiv} V$, the φ-*convolution* between two signals s_1 and $s_2 \in \mathcal{S}(V)$ is

272   defined as:

$$s_1 *_\varphi s_2 = s_1 *_{\text{II}} s_2 = s_1 *_{\text{III}} s_2$$

273

274   This time, we do obtain equivariance to Γ as expected, and the full charac-

275   terization as well.

276   **Proposition 19. Characterization by right-action equivariance to Γ**

277   If Γ is φ-equivalent, the class of linear transformations of $\mathcal{S}(V)$ that are

278   equivariant to Γ is exactly the class of φ-convolution operators acting on the

279   right of $\mathcal{S}(V)$ *i.e.*

If $\Gamma \overset{\varphi}{\equiv} V$,

$$\exists w \in \mathcal{S}(V), f = . *_\varphi w \Leftrightarrow \begin{cases} f \in \mathcal{L}(\mathcal{S}(V)) \\ \forall g \in \Gamma, f \circ g = g \circ f \end{cases}$$

280

281   *Proof.*   1. From left to right:

282   In the following equations, (6) is obtained by definition, (7) is obtained

283   because left multiplication in a group is bijective, and (8) is obtained

284   because of (P).

$$\forall g \in \Gamma, \forall s \in \mathcal{S}(V),$$

$$f_w \circ g(s) = \sum_{h \in \Gamma} g(s)[\varphi(h)]\, h(w) \tag{6}$$

$$= \sum_{h \in \Gamma} g(s)[\varphi(gh)]\, gh(w) \tag{7}$$

$$= \sum_{h \in \Gamma} g(s)[g(\varphi(h))]\, gh(w) \tag{8}$$

$$= \sum_{h \in \Gamma} s[\varphi(h)]\, gh(w)$$

$$= \sum_{h \in \Gamma} s[\varphi(h)]\, h(w)[g^{-1}(.)]$$

$$= f_w(s)[g^{-1}(.)]$$

$$= g \circ f_w(s)$$

285   Of course, we also have that f_w is linear.

286   2. From right to left:

287   Let $f \in \mathcal{L}(\mathcal{S}(V)), s \in \mathcal{S}(V)$. By linearity of f, we distribute $f(s)$ on

288   the family of dirac signals:

$$f(s) = \sum_{v \in V} s[v] f(\delta_v) \tag{9}$$

289   Thanks to (P), we have that:

$$g_v(\varphi(\mathrm{Id})) = \varphi(g_v\, \mathrm{Id}) = v$$

$$\text{So, } v = u \Leftrightarrow \varphi(\mathrm{Id}) = g_v^{-1}(u)$$

$$\text{So, } \delta_v = g_v(\delta_{\varphi(\mathrm{Id})})$$

290   By denoting $w = f(\delta_{\varphi(\mathrm{Id})})$, and using the hypothesis of equivariance,

291   we obtain from (9) that:

$$\begin{aligned}
f(s) &= \sum_{v \in V} s[v]\, f \circ g_v(\delta_{\varphi(\mathrm{Id})}) \\
&= \sum_{v \in V} s[v]\, g_v \circ f(\delta_{\varphi(\mathrm{Id})}) \\
&= \sum_{v \in V} s[v]\, g_v(w) \\
&= s *_\varphi w
\end{aligned}$$

292   □

Construction of φ-convolutions on vertex domains

294 Proposition 19 tells us that in order to define a convolution on the vertex
295 domain of a graph $G = \langle V, E \rangle$, all we need is a subgroup Γ of invertible
296 transformations of V, that is equivalent to V. The choice of Γ can be done
297 with respect to E. This is discussed in more details in Section 2.3, where we
298 will see that in fact, we only need a generating set of Γ.

Exposure of φ

300 This construction relies on exposing a bijective equivariant map φ between
301 Γ and V. In the next subsection, we show that in cases where Γ is abelian,
302 we even need not expose φ and the characterization still holds.

2.2.3   Mixed domain formulation

304 From (4), we can define a mixed domain convolution *i.e.* that is defined for
305 $r \in \mathcal{S}(\Gamma)$ and $s \in \mathcal{S}(V)$, without the need of expliciting φ.

306 Definition 20. Mixed domain convolution

307 Let a subgroup $\Gamma \subset \Phi^*(V)$ such that $V \cong \Gamma$. The *mixed domain convolution*

308 between two signals $r \in \mathcal{S}(\Gamma)$ and $s \in \mathcal{S}(V)$ results in a signal $r *_{\mathrm{M}} s \in \mathcal{S}(V)$

309 and is defined as:

$$r *_{\mathrm{M}} s = \sum_{g \in \Gamma} r[g] \, g(s)$$

310

311 We coin it M-convolution. From a practical point of view, this expression of

312 the convolution is useful because it relegates φ as an underpinning object.

313 Lemma 21. Relation with group convolution III

314 $\forall \varphi \in \mathrm{BIJ}(\Gamma, V), \forall (r, s) \in \mathcal{S}(\Gamma) \times \mathcal{S}(V),$

315
$$r *_{\mathrm{M}} s = \varphi(r) *_{\mathrm{III}} s$$

316

317 *Proof.* Let $\varphi \in \mathrm{BIJ}(\Gamma, V), (r, s) \in \mathcal{S}(\Gamma) \times \mathcal{S}(V),$

$$r *_{\mathrm{M}} s = \sum_{g \in \Gamma} r[g] \, g(s) = \sum_{v \in V} r[g_v] \, g_v(s) \overset{(\diamond)}{=} \sum_{v \in V} \varphi(r)[v] \, g_v(s)$$
$$= \varphi(r) *_{\mathrm{III}} s$$

318 Where $\overset{(\diamond)}{=}$ comes from Lemma 9. $\qquad\square$

319 In other words, $*_{\mathrm{M}}$ is a convenient reformulation of $*_{\mathrm{III}}$ which does not depend

320 on a particular φ.

321 Lemma 22. Relation with group convolution I, II and φ-convolution

322 Let $\varphi \in \mathrm{BIJ}(\Gamma, V), (r, s) \in \mathcal{S}(\Gamma) \times \mathcal{S}(V),$ we have:

$$\Gamma \overset{\varphi}{\equiv} V \Leftrightarrow \forall v \in V, (r *_{\mathrm{M}} s)[v] = (r *_{\mathrm{I}} \varphi^{-1}(s))[g_v]$$
$$\Leftrightarrow r *_{\mathrm{M}} s = \varphi(r) *_{\mathrm{II}} s$$
$$\Leftrightarrow r *_{\mathrm{M}} s = \varphi(r) *_{\varphi} s$$

323

324 *Proof.* On one hand, Lemma 21 gives $r *_{\text{M}} s = \varphi(r) *_{\text{III}} s$. On the other hand,
325 Lemma 11 gives $\forall v \in V, (r *_{\text{I}} \varphi^{-1}(s))[g_v] = (\varphi(r) *_{\text{II}} s)[v]$. Then Lemma 17
326 concludes. $\square$

327 *Remark.* The converse sense is meaningful because it justifies that when the
328 M-convolution is employed, the property $\Gamma \equiv V$ underlies, without the need
329 of expliciting φ.

330 From M-convolution, we can derive operators acting on the left of $\mathcal{S}(V)$, of
331 the form $s \mapsto w *_{\text{M}} s$, parameterized by $w \in \mathcal{S}(\Gamma)$. In particular, these
332 operators would be relevant as layers of neural networks. On the contrary,
333 derived operators acting on the right such as $r \mapsto r *_{\text{M}} w$ wouldn't make
334 sense with this formulation as they would make φ resurface. However, the
335 equivariance to Γ incurring from Lemma 21 and Proposition 19 only holds for
336 operators acting on the right. So we need to intertwine an abelian condition
337 as follows. This is also a good excuse to see the influence of abelianity.

338 **Proposition 23. Equivariance to Γ through left action**
339 Let a subgroup $\Gamma \subset \Phi^*(V)$ such that $\Gamma \cong V$. Γ is abelian, if and only if,
340 M-convolution operators acting on the left of $\mathcal{S}(V)$ are equivariant to it *i.e.*

$$\forall g, h \in \Gamma, gh = hg \Leftrightarrow \forall w, g \in \Gamma, w *_{\text{M}} g(.) = g \circ (w *_{\text{M}} .)$$

341 *Proof.* Let $w, g \in \Gamma$, and define $f_w : s \mapsto w *_{\text{M}} s$. In the following expressions,
342 Γ is abelian if and only if (10) and (11) are equal (the converse is obtained

343 by particularizing on well chosen signals):

$$f_w \circ g(s) = \sum_{h \in \Gamma} w[h] \, hg(s) \tag{10}$$

$$= \sum_{h \in \Gamma} w[h] \, gh(s) \tag{11}$$

$$= \sum_{h \in \Gamma} w[h] \, h(s)[g^{-1}(.)]$$

$$= (w *_{\mathrm{M}} s)[g^{-1}(.)]$$

$$= g \circ f_w(s)$$

344 $\square$

345 *Remark.* Similarly, $*_\varphi$ is also equivariant to Γ through left action if and only

346 if Γ is abelian, as a consequence of being commutative if and only if Γ is

347 abelian. On the contrary, note that commutativity of $*_{\mathrm{M}}$ doesn't make sense.

348 **Corrolary 24. Characterization by left-action equivariance to Γ**

349 Let $\Gamma \cong V$. If Γ is abelian, the class of linear transformations of $\mathcal{S}(V)$ that

350 are equivariant to Γ is exactly the class of M-convolution operators acting on

351 the left of $\mathcal{S}(V)$ *i.e.*

If $\Gamma \cong V$ and Γ is abelian,

$$\exists w \in \mathcal{S}(\Gamma), f = w *_{\mathrm{M}} . \Leftrightarrow \begin{cases} f \in \mathcal{L}(\mathcal{S}(V)) \\ \forall g \in \Gamma, f \circ g = g \circ f \end{cases}$$

352

353 *Proof.* By picking φ such that $\Gamma \overset{\varphi}{\equiv} V$ with Lemma 22 and using the relation

354 between $*_{\mathrm{M}}$ and $*_\varphi$. $\square$

355 Depending on the applications, we will build upon either $*_\varphi$ or $*_{\mathrm{M}}$ when the

356 abelian condition is satisfied.

2.3   Inclusion of the edge set in the construction

The constructions from the previous section involve the vertex set V and depend on Γ, a subgroup of the set of invertible transformations on V. Therefore, it looks natural to try to relate the edge set and Γ.

There are two approaches. Either Γ describes an underlying graph structure $G = \langle V, E \rangle$, either G can be used to define a relevant subgroup Γ to which the produced convolutive operators will be equivariant. Both approaches will help characterize classes of graphs that can support natural definitions of convolutions.

TODO: introduction to be developped more

2.3.1   Convolution on Cayley subgraphs

In this subsection, we are trying to answser the following question:

- What graphs admit a φ-convolution, or an M-convolution (in the sense that they can be defined with the characterization), under the condition that Γ is generated by a set of edge-constrained transformations ?

Definition 25. Edge-constrained transformation

An *edge-constrained* (EC) transformation on a graph $G = \langle V, E \rangle$ is a transformation $f : V \mapsto V$ such that

$$\forall u, v \in V, f(u) = v \Rightarrow u \overset{E}{\sim} v$$

We denote $\Phi_{\text{EC}}(G)$ and $\Phi^*_{\text{EC}}(G)$ the sets of (EC) and invertible (EC) transformations. When a convolution is defined as a sum over a set that is in one-to-one correspondence with a group that is generated from a set of (EC) transformations, we call it an (EC) convolution.

380 *Remark.* Note that $\Phi^*_{\text{EC}}(G)$ is not a group, thus why we are interested in
381 groups and their generating sets.

382 This leads us to consider Cayley graphs (Cayley, 1878; Wikipedia, 2018).

383 Definition 26. Cayley graph

384 Let a group Γ and one of its generating set $\mathcal{U}$. The *Cayley graph* generated
385 by $\mathcal{U}$, is the digraph $\vec{G} = \langle V, E \rangle$ such that $V = \Gamma$ and E is such that:

$$u \to v \Leftrightarrow \exists g \in \mathcal{U}, ga = b$$

386 Also, if Γ is abelian, we call it an *abelian Cayley graph*. We call *Cayley*
387 *subgraph*, a subgraph that is isomorph to a Cayley graph.

388 *Remark.* Note that for compatibility with the functional notation that we
389 use, we define Cayley graphs with $ga = b$ instead of $ag = b$.

390 Convolution on Cayley graphs

391 In the case of Cayley graphs, it is clear that $\mathcal{U} \subseteq \Phi^*_{\text{EC}}$ and $\Phi^* \supseteq \langle \mathcal{U} \rangle \equiv V$.
392 So that they admit (EC) φ-convolutions, and (EC) M-convolutions in the
393 abelian case.
394 More precisely, we obtain the following characterization:

395 Proposition 27. Characterization by Cayley subgraph isomorphism

396 Let a graph $G = \langle V, E \rangle$, then:

397 (i) G admits an (EC) φ-convolution if and only if it contains a subgraph
398 isomorph to a Cayley graph

399 (ii) G admits an (EC) M-convolution if and only if it contains a subgraph
400 isomorph to an abelian Cayley graph

401 *Proof.* We show the result only in the general case as the proof for the abelian
402 case is similar.

403   1. From left to right: as a direct application of the definitions.

404   2. From right to left:

405   Let a graph $G = \langle V, E \rangle$. We suppose it contains a subgraph $\vec{G}_s =$

406    $\langle V_s, E_s \rangle$ that is graph-isomorph to a Cayley graph $\vec{G}_c = \langle V_c, E_c \rangle$, gen-

407   erated by $\mathcal{U}$. Let ψ be a graph isomorphism from G_s to G_c. To obtain

408   the proof, we need to find a group of invertible transformations Γ of V_s

409   generated by a set of (EC) transformations, such that $\Gamma \equiv V_s$.

410   Let's define the group action $L : V_c \times V_s \rightarrow V_s$ inductively as follows:

411   (a) $\forall g \in \mathcal{U}, L_g(u) = v \Leftrightarrow g\psi(u) = \psi(v)$

412   (b) Whenever L_g and L_h are defined, the action of gh is defined by

413   homomorphism as $L_{gh} = L_g \circ L_h$

414   (c) Whenever L_g is defined, the action of g^{-1} is defined by homomor-

415   phism as $L_{g^{-1}} = L_g^{-1}$ *i.e.* $L_{g^{-1}}(u) = v \Leftrightarrow \psi(u) = g\psi(v)$

416   Note that the induction transfers the property (a) to all $g \in V_c$ in a

417   transitive manner because

$$L_{gh}(u) = L_g(L_h(u)) = w \Leftrightarrow \exists v \in V_s \begin{cases} L_h(u) = v \\ L_g(v) = w \end{cases}$$

418   and

$$\exists v \in V_s \begin{cases} h\psi(u) = \psi(v) \\ g\psi(v) = \psi(w) \end{cases} \Leftrightarrow gh\psi(u) = \psi(w)$$

419   We must also verify that this construction is well-defined, *i.e.* whenever

420   we define an action with (b) or (c), if the action was already defined,

421   then they must be equal. This is the case because the homomorphism

$g \mapsto L_g$ on V_c is in fact an isomorphism as

$$L_g = L_h \Leftrightarrow \forall u \in V, L_g(u) = L_h(u)$$
$$\Leftrightarrow \forall u \in V, g\psi(u) = h\psi(u)$$
$$\Leftrightarrow g = h$$

Also note that (c) is needed only in case that V_c is infinite.

Denote the set $L_{\mathcal{U}} = \{L_g, g \in \mathcal{U}\}$ and $\Gamma = \langle L_{\mathcal{U}} \rangle \cong V_c$. Let's define the map φ as:

$$\Gamma \rightarrow V_s$$
$$\varphi : L_g \mapsto L_g(\psi^{-1}(\mathrm{Id}))$$

φ is bijective because $\forall g \in V_c, \varphi(L_g) = \psi^{-1}(g)$ thanks to (a).

Additionally, we have:

$$L_h(\varphi(L_g) = L_h(L_g(\psi^{-1}(\mathrm{Id})))$$
$$= L_h \circ L_g(\psi^{-1}(\mathrm{Id}))$$
$$= L_{hg}(\psi^{-1}(\mathrm{Id}))$$
$$= \varphi(L_{hg})$$
$$= \varphi(L_h \circ L_g)$$

That is, φ is a bijective equivariant map and $\langle L_{\mathcal{U}} \rangle = \Gamma \overset{\varphi}{\equiv} V_s$. Moreover, $L_{\mathcal{U}}$ is a set of (EC) transformations thanks to (a). Therefore, G admits an (EC) φ-convolution.

$\square$

Corrolary 28. Characterization by φ

Let a graph $G = \langle V, E \rangle$, and a set $\mathcal{U} \subset \Phi^*_{\text{EC}}(G)$ *s.t.*

$$\langle \mathcal{U} \rangle \cong \Gamma \equiv V' \subset V$$

G admits an (EC) φ-convolution, if and only if, φ is a graph isomorphism between the Cayley graph generated by $\mathcal{U}$ and the subgraph induced by V'.

The proof is omitted as it would be highly similar to the previous one.

2.3.2   Intrinsic properties

- Obviously the contructed convolutions are linear. But do they also preserve the locality and weight sharing properties ?

Let $\vec{G} = \langle V, E \rangle$ be a Cayley subgraph, generated by $\mathcal{U}$, of some graph G. Recall that its (EC) φ-convolution operator is a right operator, and can be expressed as

$$\forall s \in \mathcal{S}(V), \forall u \in V,$$
$$f_w(s)[u] = (s *_\varphi w)[u]$$
$$= \sum_{v \in V} s[v]\, w[g_v^{-1}(u)] \tag{12}$$

From this expression, it is not obvious that f_w is a local operator. To see this, we can show for example the following proposition.

Proposition 29. Locality

When the support of w is a compact (in the sense that its induced subgraph in G is connected), of diameter d, the same holds for the support of the sum Σ in (12). More precisely, the subgraph induced by the support of Σ is isomorphic to the transpose of the subgraph induced by the support of w.

450 *Proof.* Without loss of generality subject to growing $\mathcal{U}$, let's suppose that
451 w has a support $\mathcal{M} = \varphi(\mathcal{N})$, such that $\mathcal{N} \subset \mathcal{U}$. $\mathcal{N}$ and $\mathcal{M}$ are obviously
452 compacts of diameter 2. Thanks to (P), we have

$$
\begin{aligned}
g_v^{-1}(u) \in \mathcal{M} &\Leftrightarrow u \in g_v(\mathcal{M}) = g_v(\varphi(\mathcal{N})) = \varphi(g_v\mathcal{N}) \\
&\Leftrightarrow g_u \in g_v\mathcal{N} \\
&\Leftrightarrow g_v^{-1} \in \mathcal{N}g_u^{-1} \\
&\Leftrightarrow g_v \in g_u\mathcal{N}^{-1} \\
&\Leftrightarrow v \in g_u(\varphi(\mathcal{N}^{-1}))
\end{aligned}
$$

453 where $\mathcal{N}^{-1}$ reverses the edges of $\mathcal{N}$. Let's denote $\mathcal{K}_u = g_u(\varphi(\mathcal{N}^{-1})) \subset V$.
454 By composing edge reversal and graph isomorphisms (as φ and its inverse are
455 graph isomorphisms by Proposition 28), the compactness and diameter of $\mathcal{M}$
456 is preserved for $\mathcal{K}_u$. More preceisely, the transposed subgraph structure is
457 also preserved. $\qquad\square$

458 Let's define $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{K}_u$ as in the previous proof.

459 **Definition 30. Supporting set**
460 The *supporting set* of an (EC) convolution operator f_w, is a set $\mathcal{N} \subset \Phi_{\mathrm{EC}}^*$,
461 such that

462   (i) when $*$ is $*_\varphi$: $0 \notin w[\mathcal{M}]$, where $\mathcal{M} = \varphi(\mathcal{N})$

463   (ii) when $*$ is $*_{\mathrm{M}}$: $0 \notin w[\mathcal{N}]$

464 **Definition 31. Local patch for $*_\varphi$**
465 The *local patch* at $u \in V$ of an (EC) φ-convolution operator f_w is defined as
466 $\mathcal{K}_u = g_u(\varphi(\mathcal{N}^{-1}))$.

467 To see that the weights are tied in the general case (i), we can show the
468 following proposition.

469 **Proposition 32. Weight sharing**

470 $\forall a, \alpha \in V, \forall b \in \mathcal{K}_a : \exists \beta \in \mathcal{K}_\alpha \Leftrightarrow g_\beta^{-1}(\alpha) = g_b^{-1}(a)$

471 *Proof.* By using (P),

$$
\begin{aligned}
g_{\mathcal{K}_\alpha}^{-1}(\alpha) = g_{\mathcal{K}_a}^{-1}(a) &\Leftrightarrow g_\alpha^{-1} g_{\mathcal{K}_\alpha} = g_a^{-1} g_{\mathcal{K}_a} \\
&\Leftrightarrow \mathcal{K}_\alpha = g_\alpha g_a^{-1}(\mathcal{K}_a) = g_\alpha g_a^{-1} g_a(\varphi(\mathcal{N}^{-1})) \\
&\Leftrightarrow \mathcal{K}_\alpha = g_\alpha(\varphi(\mathcal{N}^{-1}))
\end{aligned}
$$

472 $\square$

473 2.3.3   Stricly edge-constrained convolutions

474 We make the disctinction between general (EC) convolution operators and
475 those for which the weight kernel w is smaller and is supported only on (EC)
476 transformations of $\mathcal{U}$.

477 **Definition 33. Strictly (EC) convolution operator**

478 A *strictly* edge-constrained (EC*) convolution operator f_w, is an (EC) con-
479 volution operator such that its supporting set $\mathcal{N} \subset \mathcal{U}$.

480 Let f_w be an (EC*) convolutional operator. In the general case (i), $w \in \mathcal{S}(V)$,
481 so its support is $\mathcal{M} = \varphi(\mathcal{N})$ such that $\mathcal{N} \subseteq \mathcal{U}$. In the abelian case (ii), we
482 use instead $w \in \mathcal{S}(\Gamma)$, and thus its support is directly $\mathcal{N}$. Therefore, we can
483 rewrite the expressions of the convolution operator as:

484   (i) $\forall s \in \mathcal{S}(V), \forall u \in V, f_w(s)[u] \overset{(\varphi)}{=} \sum_{v \in \mathcal{K}_u} s[v]\, w[g_v^{-1}(u)]$

485   (ii) $\forall s \in \mathcal{S}(V), f_w(s) \overset{(\text{M})}{=} \sum_{g \in \mathcal{N}} w[g]\, g(s)$

Remark. Note that in the abelian case, we can see from (ii) that a definition of a local patch would coincide with the supporting set, so that locality and weight sharing is straightforward.

From these expressions, it is clear that Γ needs not to be fully determined to calculate $f_w(s)[u]$. The case (ii) is the simplest as the only requirement is a supporting set $\mathcal{N}$ of (EC) invertible transformations. In the case (i), we only need to determine $\mathcal{K}_u$.

What exactly is the local patch $\mathcal{K}_u$?

TODO: or not todo?

2.4   From groups to groupoids

2.4.1   Motivation

One possible limitation coming from searching for Cayley subgraphs is that they are order-regular *i.e.* the in- and the out-degree $d = |\mathcal{U}|$ of each vertex is the same. That is, for a general graph G, the size of the weight kernel w of an (EC*) convolution operator f_w supported on $\mathcal{U}$ is bounded by d, which in turn is bounded by twice the mininimal degree of G (twice because G is undirected and $\mathcal{U}$ can contain every inverse).

There are a lot of possible strategies to overcome this limitation. For example:

1. connecting each vertex with its k-hop neighbors, with $k > 1$,

2. artificially creating new connections for less connected vertices,

3. allowing the supporting set $\mathcal{N}$ to exceed $\mathcal{U}$ *i.e.* dropping * in (EC*).

These strategies require to concede that the topological structure supported by G is not the best one to support an (EC*) convolution on it, which breeds the following question:

- What can we relax in the previous (EC*) contruction in order to unbound the supporting set, and still preserve the equivariance characterization?

The latter constraint is a consequence that every vertex of the Cayley subgraph $\vec{G}$ must be composable with every generator from $\mathcal{U}$. Therefore, the answer consists in considering groupoids (Brandt, 1927) instead of groups. Roughly speaking, a groupoid is almost a group except that its composition law needs not be defined everywhere. Weinstein, 1996, unveiled the benefits to base convolutions on groupoids instead of groups in order to exploit partial symmetries.

2.4.2   Definitions of some groupoid related notions

Definition 34. Groupoid

A *groupoid* Υ is a set equipped with a partial composition law with domain $\mathcal{D} \subset \Upsilon \times \Upsilon$ that is

1. closed into Υ *i.e.* $\forall (g, h) \in \mathcal{D}, gh \in \Upsilon$

2. associative *i.e.* $\forall f, g, h \in \Upsilon,$ $\begin{cases} (f,g),(g,h) \in \mathcal{D} \Leftrightarrow (fg,h),(f,gh) \in \mathcal{D} \\ (f,g),(fg,h) \in \mathcal{D} \Leftrightarrow (g,h),(f,gh) \in \mathcal{D} \\ \text{when defined, } (fg)h = f(gh) \end{cases}$

3. invertible *i.e.* $\forall g \in \Upsilon, \exists! g^{-1} \in \Upsilon \quad s.t. \begin{cases} (g,g^{-1}),(g^{-1},g) \in \mathcal{D} \\ (g,h) \in \mathcal{D} \Rightarrow g^{-1}gh = h \\ (h,g) \in \mathcal{D} \Rightarrow hgg^{-1} = h \end{cases}$

Optionally, it can be *domain-symmetric i.e.* $(g,h) \in \mathcal{D} \Leftrightarrow (h,g) \in \mathcal{D}$, and *abelian i.e.* domain-symmetric with $gh = hg$.

Remark. Note that left and right inverses are necessarily equal (because $(gg^{-1})g = g(g^{-1}g)$). Also note we can define a right identity element $e_g^r = g^{-1}g$, and a left one $e_g^l = gg^{-1}$, but they are not necessarily equal and depend on g.

Most definitions related to groups can be adapted to groupoids. In particular, let's adapt a few notions.

Definition 35. Groupoid partial action

A partial *action* of a groupoid Υ on a set V, is a function L, with domain $\mathcal{D}_L \subset \Upsilon \times V$ and valued in V, such that the map $g \mapsto L_g$ is a groupoid homomorphism.

541 *Remark.* As usual, we will confound L_g and g when there is no possible
542 confusion, and we denote $\mathcal{D}_{L_g} = \mathcal{D}_g = \{v \in V, (g, v) \in \mathcal{D}_L\}$.

543 **Definition 36. Partial equivariant map**

544 A map φ from a groupoid Υ partially acting on the destination set V is said
545 to be a *partial equivariant map* if

$$\forall g, h \in \Upsilon, \begin{cases} \varphi(h) \in \mathcal{D}_g \Leftrightarrow (g, h) \in \mathcal{D} \\ g(\varphi(h)) = \varphi(gh) \end{cases}$$

546 Also, φ-equivalence between a subgroupoid and a set is defined similarly with
547 φ being a bijective *partial* equivariant map between them.

548 **Partial transformations groupoid**

549 We denote by $\Psi(V)$ the set of partially defined transformations on v, and
550 $\Psi^*(V)$ the groupoid of its invertible ones (*i.e.* injective ones).

551 ## 2.4.3   Construction of the groupoid convolution

552 The expression of the convolution we constructed in the previous section
553 cannot be applied as is. We first need to extend the algebraic objects we
554 work with. Extending a partial transformation g on the signal space $\mathcal{S}(V)$
555 (and thus the convolutions) is a bit tricky, because only the signal entries
556 corresponding to $\mathcal{D}_g$ are moved. A convenient way to do this is to consider
557 the groupoid closure obtained with the addition of an absorbing element

558 **Definition 37. Zero-closure**

559 The *zero-closure* of a groupoid Υ, denoted Υ^0, is the set $\Upsilon \cup 0$, such that the
560 groupoid axioms 1, 2 and 3, and the domain $\mathcal{D}$ are left unchanged, and

561   4. the composition law is extended to $\Upsilon^0 \times \Upsilon^0$ with $\forall (g, h) \notin \mathcal{D}, gh = 0$

562 *Remark.* Note that this is coherent as the properties 2 and 3 are still partially
563 defined on the original domain $\mathcal{D}$.

564 Now, we will also extend every other algebraic object used in the expression
565 of the φ-convolution and the M-convolution, so that we can directly apply
566 our previous constructions.

567 **Lemma 38. Extension of φ on V^0**
568 Let a partial equivariant map $\varphi : \Upsilon \to V$. It can be extended to a (total)
569 equivariant map $\varphi : \Upsilon^0 \to V^0 = V \cup \varphi(0)$, such that $\varphi(0) \notin V$, that we
570 denote $0_V = \varphi(0)$, and such that

$$\forall g \in \Upsilon^0, \forall v \in V^0, g(v) = \begin{cases} \varphi(gg_v) & \text{if } g_v \in \mathcal{D}_g \\ 0_V & \text{else} \end{cases}$$

571 *Proof.* We have $\varphi(0) \notin V$ because φ is bijective. Additionaly, we must have
572 $\forall (g, h) \notin \mathcal{D}, g(\varphi(h)) = \varphi(gh) = \varphi(0) = 0_V$. $\qquad\square$

573 *Remark.* Note that for notational conveniency, we may use the same symbol 0
574 for 0_Υ, 0_V and $0_\mathbb{R}$.

575 **Lemma 39. Extension of the subgroupoid actions on $\mathcal{S}(V)$**
576 Let a subgroupoid $\Upsilon \subset \Psi^*(V)$ such that $\Upsilon \overset{\varphi}{\equiv} V$. Under the convention that
577 $s[0_V] = 0_\mathbb{R}$, a transformation $g \in \Upsilon^0$ can be extended to the signal space
578 $\mathcal{S}(V)$ as usual:

$$\forall s \in \mathcal{S}(V), \forall v \in V, g(s)[v] = \begin{cases} s[g^{-1}(v)] & \text{if } g \neq 0_\Upsilon \text{ and } v \in \mathcal{D}_{g^{-1}} \\ 0_\mathbb{R} & \text{else} \end{cases}$$

579 *Proof.* Straightforward. $\qquad\square$

580 With these extensions, we can obtain the φ- and M-convolutions related to Υ
581 almost by substituting Υ^0 to Γ in Definition 18 and Definition 20.

Definition 40. Groupoid convolution

Let a subgroupoid $\Upsilon \subset \Psi^*(V)$, such that $\Upsilon \overset{\varphi}{\equiv} V$. The φ- and M-convolutions, based on Υ, are defined on its zero-closure, with the same expression as if Υ^0 were a subgroup, and by extension of φ and of the groupoid actions *i.e.*

(i) $\forall s, w \in \mathcal{S}(V), s *_\varphi w = \displaystyle\sum_{v \in V} s[v]\, g_v(w) = \sum_{g \in \Upsilon} s[\varphi(g)]\, g(w)$

(ii) $\forall (w, s) \in \mathcal{S}(\Upsilon) \times \mathcal{S}(V), w *_{\text{M}} s = \displaystyle\sum_{g \in \Upsilon} w[g]\, g(s)$

where (ii) applies in the abelian case.

Symmetrical expressions

Note that, as $\forall r, r[0] = 0$, the groupoid convolutions can also be expressed on the domain $\mathcal{D}$ with a convenient symmetrical expression:

(i) $\forall u \in V, (s *_\varphi w)[u] = \displaystyle\sum_{\substack{(g_a, g_b) \in \mathcal{D} \\ s.t.\ g_a g_b = g_u}} s[a]\, w[b]$

(ii) $\forall u \in V, (w *_{\text{M}} s)[u] = \displaystyle\sum_{\substack{v \in \mathcal{D}_g \\ s.t.\ g(v) = u}} w[g]\, s[v]$

We obtain an equivariance characterization, similarly to Proposition 19 and Corrolary 24.

Proposition 41. Characterization by equivariance to Υ

Let a subgroupoid $\Upsilon \subset \Psi^*(V)$, such that $\Upsilon \overset{\varphi}{\equiv} V$. Then,

(i) right φ-convolution operators are equivariant to Υ,

(ii) if Υ is abelian, left M-convolution operators are equivariant to Υ.

Conversely,

(i) if Υ is domain-symmetric, linear transformations of $\mathcal{S}(V)$ that are equivariant to Υ are right φ-convolution operators,

603 (ii) if Υ is abelian, linear transformations of $\mathcal{S}(V)$ that are equivariant to Υ

604 are also left M-convolution operators.

605 *Proof.* (i) (a) Direct sense:

606 Using the symmetrical expressions, and the fact that $\forall r, r[0] = 0$,

607 we have

$$(f_w \circ g(s))[u] = \sum_{\substack{(g_a,g_b)\in\mathcal{D} \\ s.t.\ g_a g_b = g_u}} g(s)[a]\,w[b]$$

$$= \sum_{\substack{(g_a,g_b)\in\mathcal{D} \\ s.t.\ g_a g_b = g_u}} s[g^{-1}(a)]\,w[b]$$

$$= \sum_{\substack{(g_a,g_b)\in\mathcal{D} \\ s.t.\ (g,g_a)\in\mathcal{D} \\ s.t.\ g g_a g_b = g_u}} s[a]\,w[b]$$

$$= \sum_{\substack{(g_a,g_b)\in\mathcal{D} \\ s.t.\ (g,g_a)\in\mathcal{D} \\ s.t.\ g_a g_b = g^{-1}g_u = g_{\varphi(g^{-1}g_u)} = g_{g^{-1}(u)}}} s[a]\,w[b]$$

$$= f_w(s)[g^{-1}(u)]$$

$$= (g \circ f_w(s))[u]$$

608 (b) Converse:

609 Let $v \in V$. Denote $e^r_{g_v} = g_v^{-1}g_v$ the right identity element of g_v,

610 and $e^r_v = \varphi(e^r_{g_v})$. We have that

$$g_v(e^r_v) = v$$
$$\text{So, } \delta_v = g_v(\delta_{e^r_v})$$

611 Let $f \in \mathcal{L}(\mathcal{S}(V))$ that is equivariant to Υ, and $s \in \mathcal{S}(V)$. Thanks

612   to the previous remark we obtain that

$$
\begin{aligned}
f(s) &= \sum_{v \in V} s[v] \, f(\delta_v) \\
&= \sum_{v \in V} s[v] \, f(g_v(\delta_{e_v^r})) \\
&= \sum_{v \in V} s[v] \, g_v(f(\delta_{e_v^r})) \\
&= \sum_{v \in V} s[v] \, g_v(w_v) \qquad (13)
\end{aligned}
$$

613   where $w_v = f(\delta_{e_v^r})$. In order to finish the proof, we need to find w
614   such that $\forall v \in V, g_v(w) = g_v(w_v)$.

615   Let's consider the equivalence relation $\mathcal{R}$ defined on $V \times V$ such
616   that:

$$
\begin{aligned}
a\mathcal{R}b &\Leftrightarrow w_a = w_b \\
&\Leftrightarrow e_a^r = e_b^r \\
&\Leftrightarrow g_a^{-1} g_a = g_b^{-1} g_b \\
&\Leftrightarrow (g_b, g_a^{-1}) \in \mathcal{D} \\
&\Leftrightarrow (g_a^{-1}, g_b) \in \mathcal{D} \qquad (14)
\end{aligned}
$$

617   with (14) owing to the fact that Υ is domain-symmetric.

618   Given $x \in V$, denote its equivalence class $\mathcal{R}(x)$. Under the hy-
619   pothesis of the axiom of choice (Zermelo, 1904) (if V is infinite),
620   define the set $\aleph$ that contains exactly one representative per equiv-
621   alence class. Let $w = \sum_{n \in \aleph} w_n$. Then V is the disjoint union
622          $V = \cup_{n \in \aleph} \mathcal{R}(n)$ and (13) rewrites:

$$
\forall u \in V, f(s)[u] = \sum_{n \in \aleph} \sum_{v \in \mathcal{R}(n)} s[v] \, g_v(w_n)[u]
$$

$$= \sum_{n \in \aleph} \sum_{v \in \mathcal{R}(n)} s[v] \, w_n[g_v^{-1}(u)]$$

$$= \sum_{n \in \aleph} \sum_{v \in \mathcal{R}(n)} s[v] \, w[g_v^{-1}(u)] \qquad (15)$$

$$= (s *_\varphi w)[u]$$

where (15) is obtained thanks to (14).

(ii) With symmetrical expressions, it is clear that the convolution is abelian, if and only if, Υ is abelian. Then (i) concludes.

$\square$

2.4.4 Edge-constrained groupoid convolutions

Similarly to the construction of 2.3.1, we start by adapting the related notions.

Definition 42. Groupoid generating set

A set $\mathcal{U}$, equipped with a partial composition law of domain $\mathcal{D}_0$, is a *generating set* of a groupoid Υ, if every object of Υ can be expressed as a composition of objects of $\mathcal{U}$ and their inverses. In this case, the domain $\mathcal{D}$ of the partial composition law of Υ can be deduced inductively from $\mathcal{D}_0$ with the associativity and invertibility axioms of Υ (and eventually also with the domain-symmetric axiom if Υ is domain-symmetric).

A partial Cayley graph is defined similarly as the (total) Cayley graph.

Definition 43. Partial Cayley graph

Let a groupoid $\Upsilon = \langle \mathcal{U}, \mathcal{D}_0 \rangle$. The *partial Cayley graph* generated by $\mathcal{U}$ and $\mathcal{D}_0$, is the digraph $\vec{G} = \langle V, E \rangle$ such that $V = \Upsilon$ and E is such that:

$$u \to v \Leftrightarrow \exists g \in \mathcal{U}, ga = b$$

641 We call *partial Cayley subgraph*, a subgraph that is isomorph to a partial

642 Cayley graph.

643 *Remark.* The characterization by a partial Cayley subgraph, of graphs ad-

644 mitting an (EC) convolution based on a groupoid, similar to Proposition 27

645 also holds for the groupoid representation. However, it is somewhat trivial

646 as every graph admits a partial Cayley subgraph (for example by labelling

647 every edge in a neighbour with a transformation).

648 In the case of the groupoid representation, what is more interesting is that

649 an (EC) groupoid convolution can be characterized by a generating set of

650 the groupoid it is based on.

651 Proposition 44. Characterization by generating set

652 Let a graph $G = \langle V, E \rangle$ such that it admits an (EC) convolution based on

653 a groupoid Υ. Then, G contains a sugraph $\vec{G}$ that is isomorph to a partial

654 Cayley subgraph $\vec{G}_c$, of generating set $\mathcal{U} \subset \Psi_{\text{EC}}^*(G)$, such that the underly-

655 ing equivariant map φ of the (EC) convolution is also a graph isomorphism

656 between $\vec{G}_c$ and $\vec{G}$.

657 The proof is omitted because it would be highly similar with the one of

658 Corrolary 28.

659 Strictly edge-constrained groupoid convolution

660 Thanks to the previous construction, in order to define a meaningful con-

661 volution operator on a general graph $G = \langle V, E \rangle$, all we need is a set $\mathcal{U}$

662 of (EC) partial transformations *that need not to be defined on every vertex,*

663 and a composition rule $\mathcal{D}_0$. From the set $\mathcal{U}$, we can choose a supporting set

664 $\mathcal{N} \subset \mathcal{U}$ (from which we eventually need to derive the local patches $\mathcal{K}_u$), and

665 obtain the (EC*) convolution operator f_w, without the need of fully deter-

666 mining $\Upsilon = \langle \mathcal{U} \rangle$. We can choose between three levels of constraints, which

667 are inherited from $\mathcal{U}$ and $\mathcal{D}_0$:

1. Υ is unconstrained, then f_w is an (EC*) φ-convolution right-operator that is equivariant to Υ, but the converse doesn't hold,

2. Υ is domain-symmetric, then f_w is an (EC*) φ-convolution right operator and the equivariance characterization holds,

3. Υ is abelian, then f_w is an (EC*) M-convolution operator, the equivariance characterization holds, and we don't need to compute the local patches $\mathcal{K}_u$.

In the next chapter, we will encounter examples of convolutions defined on graphs, that can be expressed under group representations or under groupoid representations.

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