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Chapter 2

Convolutions on graph domains

Introduction

Defining a convolution of signals over graph domains is a challenging problem. Obviously, if the graph is not a grid graph there exists no natural definition. We first analyse the reasons why the euclidean convolution operator is useful in deep learning, and give a characterization. Then we will search for domains onto which a convolution with these properties can be naturally obtained. This will lead us to put our interest on representation theory and convolutions defined on groups. As the euclidean convolution is just a particular case of the group convolution, it makes perfect sense to steer our construction in this direction. Hence, we will aim at transferring its representation on the vertex domain. First we will do this construction agnostically of the edge set. Then, we will introduce the role of the edge set and see how it should influence it. This will provide us with some particular classes of graphs for which we will obtain a natural construction with the wanted characteristics that we exposed in the first place. Finally, we can relax some aspect of the construction to adapt it to graphs that are not order-regular. The obtained construction is a set of general expressions that describes convolutions on graph domains, which preserve some key properties.

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2.1 Analysis of the classical convolution

In this section, we are exposing a few properties of the classical convolution that a generalization to graphs would likely try to preserve. For now let's consider a graph G agnostically of its edges *i.e.* $G \cong V$ is just the set of its vertices.

2.1.1 Properties of the convolution

Consider an edge-less grid graph *i.e.* $G \cong \mathbb{Z}^2$. By restriction to compactly supported signals, this case encompass the case of images.

Definition 1. Convolution on $\mathcal{S}(\mathbb{Z}^2)$

Recall that the (discrete) convolution between two signals s_1 and s_2 over $\mathbb{Z}^2$ is a binary operation in $\mathcal{S}(\mathbb{Z}^2)$ defined as:

$$\forall (a,b) \in \mathbb{Z}^2, (s_1 * s_2)[a,b] = \sum_i \sum_j s_1[i,j] \, s_2[a-i, b-j]$$

Definition 2. Convolution operator

A *convolution operator* is a function of the form $f_w : x \mapsto x * w$, where x and w are signals of domains for which the convolution $*$ is defined. When $*$ is not commutative, we differentiate the *right-action* operator $x \mapsto x * w$ from the *left-action* one $x \mapsto w * x$.

The following properties of the convolution on $\mathbb{Z}^2$ are of particular interest for our study.

Linearity

Operators produced by the convolution are linear. So they can be used as linear parts of layers of neural networks.

84 **Locality and weight sharing**

85 When w is compactly supported on K, an impulse response $f_w(x)[a, b]$ amounts

86 to a w-weighted aggregation of entries of x in a neighbourhood of (a, b), called

87 the *local receptive field*.

88 **Commutativity**

89 The convolution is commutative. However, it won't necessarily be the case

90 on other domains.

91 **Equivariance to translations**

92 Convolution operators are equivariant to translations. Below, we show that

93 the converse of this result also holds with Proposition 6.

94 ## 2.1.2 Characterization on grid graphs

95 Let's recall first what is a transformation, and how it acts on signals.

96 **Definition 3. Transformation**

97 A *transformation* $f : V \to V$ is a function with same domain and codomain.

98 The set of transformations is denoted $\Phi(V)$. The set of bijective transforma-

99 tions is denoted $\Phi^*(V) \subset \Phi(V)$.

100 In particular, $\Phi^*(V)$ forms the symmetric group of V and can move signals

101 of $\mathcal{S}(V)$ by linear extension of its group action.

102 **Lemma 4. Extension to $\mathcal{S}(V)$ by group action**

103 An transformation $f \in \Phi^*(V)$ can be extended linearly to the signal space

104 $\mathcal{S}(V)$, and we have:

$$\forall s \in \mathcal{S}(V), \forall v \in V, f(s)[v] := L_f(s)[v] = s[f^{-1}(v)]$$

105 *Proof.* Let $s \in \mathcal{S}(V), f \in \Phi^*(V), L_f \in \mathcal{L}(\mathcal{S}(V))$ *s.t.* $\forall v \in V, L_f(\delta_v) = \delta_{f(v)}$.

106 Then, we have:

$$L_f(s) = \sum_{v \in V} s[v] \, L_f(\delta_v)$$

$$= \sum_{v \in V} s[v] \, \delta_{f(v)}$$

$$\text{So, } \forall v \in V, L_f(s)[v] = s[f^{-1}(v)]$$

107   □

108 We also recall the formalism of translations.

109 **Definition 5. Translation on $\mathcal{S}(\mathbb{Z}^2)$**

110 A translation on $\mathbb{Z}^2$ is defined as a transformation $t \in \Phi^*(\mathbb{Z}^2)$ such that

$$\exists (a, b) \in \mathbb{Z}^2, \forall (x, y) \in \mathbb{Z}^2, t(x, y) = (x + a, y + b)$$

111 It also acts on $\mathcal{S}(\mathbb{Z}^2)$ with the notation $t_{a,b}$ *i.e.*

$$\forall s \in \mathcal{S}(\mathbb{Z}^2), \forall (x, y) \in \mathbb{Z}^2, t_{a,b}(s)[x, y] = s[x - a, y - b]$$

112 For any set E, we denote by $\mathcal{T}(E)$ its translations if they are defined.

113 The next proposition fully characterizes convolution operators with their
114 translational equivariance property. This can be seen as a discretization of a
115 classic result from the theory of distributions (Schwartz, 1957).

116 **Proposition 6. Characterization of convolution operators on $\mathcal{S}(\mathbb{Z}^2)$**
117 On real-valued signals over $\mathbb{Z}^2$, the class of linear transformations that are
118 equivariant to translations is exactly the class of convolutive operations *i.e.*

$$\exists w \in \mathcal{S}(\mathbb{Z}^2), f = . * w \Leftrightarrow \begin{cases} f \in \mathcal{L}(\mathcal{S}(\mathbb{Z}^2)) \\ \forall t \in \mathcal{T}(\mathcal{S}(\mathbb{Z}^2)), f \circ t = t \circ f \end{cases}$$

119

120   *Proof.* The result from left to right is a direct consequence of the definitions:

$$\forall s \in \mathcal{S}(\mathbb{Z}^2), \forall s' \in \mathcal{S}(\mathbb{Z}^2), \forall (\alpha, \beta) \in \mathbb{R}^2, \forall (a, b) \in \mathbb{Z}^2,$$

$$f_w(\alpha s + \beta s')[a, b] = \sum_i \sum_j (\alpha s + \beta s')[i, j] \, w[a - i, b - j]$$

$$= \alpha f_w(s)[a, b] + \beta f_w(s')[a, b] \qquad \text{(linearity)}$$

$$\forall s \in \mathcal{S}(\mathbb{Z}^2), \forall (\alpha, \beta) \in \mathbb{Z}^2, \forall (a, b) \in \mathbb{Z}^2,$$

$$f_w \circ t_{\alpha, \beta}(s)[a, b] = \sum_i \sum_j t_{\alpha, \beta}(s)[i, j] \, w[a - i, b - j]$$

$$= \sum_i \sum_j s[i - \alpha, j - \beta] \, w[a - i, b - j]$$

$$= \sum_{i'} \sum_{j'} s[i', j'] \, w[a - i' - \alpha, b - j' - \beta] \qquad (1)$$

$$= f_w(s)[a - \alpha, b - \beta]$$

$$= t_{\alpha, \beta} \circ f_w(s)[a, b] \qquad \text{(equivariance)}$$

121   Now let's prove the result from right to left.

122   Let $f \in \mathcal{L}(\mathcal{S}(\mathbb{Z}^2))$, $s \in \mathcal{S}(\mathbb{Z}^2)$. We suppose that f commutes with trans-
123   lations. Recall that s can be linearly decomposed on the infinite family of
124   dirac signals:

$$s = \sum_i \sum_j s[i, j] \, \delta_{i,j}, \text{ where } \delta_{i,j}[x, y] = \begin{cases} 1 & \text{if } (x, y) = (i, j) \\ 0 & \text{otherwise} \end{cases}$$

125   By linearity of f and then equivariance to translations:

$$f(s) = \sum_i \sum_j s[i, j] \, f(\delta_{i,j})$$

$$= \sum_i \sum_j s[i, j] \, f \circ t_{i,j}(\delta_{0,0})$$

$$= \sum_i \sum_j s[i,j] \, t_{i,j} \circ f(\delta_{0,0})$$

By denoting $w = f(\delta_{0,0}) \in \mathcal{S}(\mathbb{Z}^2)$, we obtain:

$$\forall (a,b) \in \mathbb{Z}^2, f(s)[a,b] = \sum_i \sum_j s[i,j] \, t_{i,j}(w)[a,b] \tag{2}$$

$$= \sum_i \sum_j s[i,j] \, w[a-i, b-j]$$

$$\textit{i.e. } f(s) = s * w$$

$\square$

2.1.3 Usefulness of convolutions in deep learning

Equivariance property of CNNs

In deep learning, an important argument in favor of CNNs is that convolutional layers are equivariant to translations. Intuitively, that means that a detail of an object in an image should produce the same features independently of its position in the image.

Lossless superiority of CNNs over MLPs

The converse result, as a consequence of Proposition 6, is never mentioned in deep learning literature. However it is also a strong one. For example, let's consider a linear function that is equivariant to translations. Thanks to the converse result, we know that this function is a convolution operator parameterized by a weight vector w, $f_w : . * w$. If the domain is compactly supported, as in the case of images, we can break down the information of w in a finite number n_q of kernels w_q with small compact supports of same size (for instance of size 2×2), such that we have $f_w = \sum_{q \in \{1,2,...,n_q\}} f_{w_q}$. The convolution operators f_{w_q} are all in the search space of 2×2 convolutional layers. In other words, every translational equivariant linear function can

145 have its information parameterized by these layers. So that means that the
146 reduction of parameters from an MLP to a CNN is done with strictly no loss of
147 expressivity (provided the objective function is known to bear this property).
148 Besides, it also helps the training to search in a much more confined space.

149 **Methodology for extending to general graphs**
150 Hence, in our construction, we will try to preserve the characterization from
151 Proposition 6 as it is mostly the reason why they are successful in deep
152 learning. Note that the reduction of parameters compared to a dense layer
153 is also a consequence of this characterization.

2.2 Construction from the vertex set

As Proposition 6 is a complete characterization of convolutions, it can be used to define them *i.e.* convolution operators can be constructed as the set of linear transformations that are equivariant to translations. However, in the general case where G is not a grid graph, translations are not defined, so that construction needs to be generalized beyond translational equivariances. In mathematics, convolutions are more generally defined for signals defined over a group structure. The classical convolution that is used in deep learning is just a narrow case where the domain group is an euclidean space. Therefore, constructing a convolution on graphs should start from the more general definition of convolution on groups rather than convolution on euclidean domains.

Our construction is motivated by the following questions:

- Does the equivariance property holds ? Does the characterization from Proposition 6 still holds ?

- Is it possible to extend the construction on non-group domains, or at least on mixed domains ? (*i.e.* one signal is defined over a set, and the other is defined over a subgroup of the transformations of this set).

- Can a group domain draw an underlying graph structure ? Is the group convolution naturally defined on this class of graphs ?

We first recall the notion of group and group convolution.

Definition 7. Group

A group Γ is a set equipped with a closed, associative and invertible composition law that admits a unique left-right identity element.

The group convolution extends the notion of the classical discrete convolution.

Definition 8. Group convolution I

Let a group Γ, the group convolution I between two signals s_1 and $s_2 \in \mathcal{S}(\Gamma)$
is defined as:

$$\forall h \in \Gamma, (s_1 *_{\text{\tiny I}} s_2)[h] = \sum_{g \in \Gamma} s_1[g]\, s_2[g^{-1}h]$$

provided at least one of the signals has finite support if Γ is not finite.

2.2.1   Steered construction from groups

For a graph $G = \langle V, E \rangle$ and a subgroup $\Gamma \subset \Phi^*(V)$ or its invertible transfor-
mations, Definition 8 is applicable for $\mathcal{S}(\Gamma)$, but not for $\mathcal{S}(V)$ as V is not a
group. Nonetheless, our point here is that we will use the group convolution
on $\mathcal{S}(\Gamma)$ to construct the convolutions on $\mathcal{S}(V)$.
For now, let's assume Γ is in one-to-one correspondence with V, and let's
define a bijective map φ from Γ to V. We denote $\Gamma \overset{\varphi}{\cong} V$ and $g_v \overset{\varphi}{\mapsto} v$.
Then, the linear morphism $\widetilde{\varphi}$ from $\mathcal{S}(\Gamma)$ to $\mathcal{S}(V)$ defined on the Dirac bases
by $\widetilde{\varphi}(\delta_g) = \delta_{\varphi(g)}$ is a linear isomorphism. Hence, $\mathcal{S}(V)$ would inherit the same
inherent structural properties as $\mathcal{S}(\Gamma)$. For the sake of notational simplicity,
we will use the same symbol φ for both φ and $\widetilde{\varphi}$ (as done between f and
L_f). A commutative diagram between the sets is depicted on Figure 2.1.

$$
\begin{array}{ccc}
\Gamma & \overset{\varphi}{\longrightarrow} & V \\
{\scriptstyle \mathcal{S}}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{S}} \\
\mathcal{S}(\Gamma) & \underset{\varphi}{\longrightarrow} & \mathcal{S}(V)
\end{array}
$$

Figure 2.1: Commutative diagram between sets

We naturally obtain the following relation, which put in simpler words means
that signals on $\mathcal{S}(\Gamma)$ are mapped to $\mathcal{S}(V)$ when φ is simultaneously applied
on both the signal space and its domain.

199 **Lemma 9. Relation between $\mathcal{S}(\Gamma)$ and $\mathcal{S}(V)$**

200 $\forall s \in \mathcal{S}(\Gamma), \forall u \in V, \varphi(s)[u] = s[\varphi^{-1}(u)] = s[g_u]$

Proof.

$$\forall s \in \mathcal{S}(\Gamma), \varphi(s) = \varphi\Big(\sum_{g \in \Gamma} s[g]\,\delta_g\Big) = \sum_{g \in \Gamma} s[g]\,\varphi(\delta_g) = \sum_{g \in \Gamma} s[g]\,\delta_{\varphi(g)}$$
$$= \sum_{v \in V} s[g_v]\,\delta_v$$

So $\forall v \in V, \varphi(s)[u] = s[g_u]$

201 $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

202 Hence, we can steer the definition of the group convolution from $\mathcal{S}(\Gamma)$ to

203 $\mathcal{S}(V)$ as follows:

204 **Definition 10. Group convolution II**

205 Let a subgroup $\Gamma \subset \Phi^*(V)$ such that $\Gamma \overset{\varphi}{\cong} V$. The group convolution II

206 between two signals s_1 and $s_2 \in \mathcal{S}(V)$ is defined as:

$$\forall u \in V, (s_1 *_{\mathrm{II}} s_2)[u] = \sum_{v \in V} s_1[v]\,s_2[\varphi(g_v^{-1}g_u)]$$

207

208 **Lemma 11. Relation between group convolution I and II**

209 Let a subgroup $\Gamma \subset \Phi^*(V)$ such that $\Gamma \overset{\varphi}{\cong} V$,

$$\forall s_1, s_2 \in \mathcal{S}(\Gamma), \forall u \in V, (\varphi(s_1) *_{\mathrm{II}} \varphi(s_2))[u] = (s_1 *_{\mathrm{I}} s_2)[g_u]$$

210

211   *Proof.* Using Lemma 9,

$$
\begin{aligned}
(\varphi(s_1) *_{\mathrm{II}} \varphi(s_2))[u] &= \sum_{v \in V} \varphi(s_1)[v]\, \varphi(s_2)[\varphi(g_v^{-1} g_u)] \\
&= \sum_{v \in V} s_1[g_v]\, s_2[g_v^{-1} g_u] \\
&= \sum_{g \in \Gamma} s_1[g]\, s_2[g^{-1} g_u] \\
&= (s_1 *_{\mathrm{I}} s_2)[g_u]
\end{aligned}
$$

212   □

213   For convolution II, we only obtain a weak version of Proposition 6.

214   **Proposition 12. Equivariance to $\varphi(\Gamma)$**

215   If φ is a homomorphism, convolution operators acting on the right of $\mathcal{S}(V)$

216   are equivariant to $\varphi(\Gamma)$ *i.e.*

$$
\begin{aligned}
&\text{if } \varphi \in \mathrm{ISO}(\Gamma, V), \\
&\exists w \in \mathcal{S}(V), f = . *_{\mathrm{II}} w \Rightarrow \forall v \in V, f \circ \varphi(g_v) = \varphi(g_v) \circ f
\end{aligned}
$$

217

    Proof.

$$
\begin{aligned}
&\forall s \in \mathcal{S}(V), \forall u \in V, \forall v \in V, \\
(f_w \circ \varphi(g_u))(s)[v] &= \sum_{v \in V} \varphi(g_u)(s)[v]\, w[\varphi(g_v^{-1} g_u)] \\
&= \sum_{\substack{(a,b) \in V^2 \\ s.t.\ g_a g_b = g_v}} \varphi(g_u)(s)[a]\, w[b] \\
&= \sum_{\substack{(a,b) \in V^2 \\ s.t.\ g_a g_b = g_v}} s[\varphi(g_u)^{-1}(a)]\, w[b]
\end{aligned}
$$

$$= \sum_{\substack{(a,b)\in V^2 \\ s.t.\ g_{\varphi(g_u)(a)}g_b=g_v}} s[a]\,w[b]$$

Because φ is an isomorphism, its inverse $c \mapsto g_c$ is also an isomorphism and so $g_{\varphi(g_u)(a)}g_b = g_v \Leftrightarrow g_a g_b = g_{\varphi(g_u)^{-1}(v)}$. So we have both:

$$(f_w \circ \varphi(g_u))(s)[v] = \sum_{\substack{(a,b)\in V^2 \\ s.t.\ g_a g_b=g_{\varphi(g_u)^{-1}(v)}}} s[a]\,w[b]$$
$$= s *_{\text{II}} w[\varphi(g_u)^{-1}(v)]$$
$$= (\varphi(g_u) \circ f_w)(s)[v]$$

$\square$

Remark. Note that convolution operators of the form $f_w = .\ *_{\text{I}} w$ are also equivariant to Γ, but the proposition and the proof are omitted as they are similar to the latter.

In fact, both group convolutions are the same as the latter one borrows the algebraic structure of the first one. Thus we only obtain equivariance to $\varphi(\Gamma)$ when φ also transfer the group structure from Γ to V, and the converse don't hold. To obtain equivariance to Γ (and its converse), we will drop the direct homomorphism condition, and instead we will take into account the fact that it contains invertible transformations of V.

2.2.2 Construction under group actions

Definition 13. Group action

An *action* of a group Γ on a set V, is a function $L : \Gamma \times V \to V, (g, v) \mapsto L_g(v)$, such that the map $g \mapsto L_g$ is a homomorphism.

Given $g \in \Gamma$, the transformation L_g is called the action of g by L on V.

Remark. When there is no ambiguity, we use the same symbol for g and L_g.

Hence, note that $g \in \Gamma$ can act on both Γ through the left multiplication and on V as being an object of $\Phi^*(V)$. This ambivalence can be seen on a commutative diagram, see Figure 2.2.

$$
\begin{array}{ccc}
g_u & \xrightarrow{\ g_v\ } & g_v g_u \\
\varphi \downarrow & & \downarrow \varphi \\
u & \xrightarrow[g_v]{(P)} & \varphi(g_v g_u)
\end{array}
$$

Figure 2.2: Commutative diagram. All arrows except for the one labeled with (P) are always True.

For (P) to be true means that φ is an equivariant map *i.e.* whether the mapping is done before or after the action of Γ has no impact on the result. When such φ exists, Γ and V are said to be equivalent and we denote $\Gamma \equiv V$.

Definition 14. Equivariant map

A map φ from a group Γ acting on the destination set V is said to be an *equivariant map* if

$$\forall g, h \in \Gamma, g(\varphi(h)) = \varphi(gh)$$

In our case we have $\Gamma \overset{\varphi}{\cong} V$. If we also have that $\Gamma \equiv V$, we are interested to know if then φ exhibits the equivalence.

248 **Definition 15. φ-Equivalence**

249 A subgroup $\Gamma \subset \Phi^*(V)$ such that $\Gamma \overset{\varphi}{\cong} V$, is said to be φ-*equivalent* if φ is a

250 bijective equivariant map *i.e.* if it verifies the property:

$$\forall v, u \in V, g_v(u) = \varphi(g_v g_u) \tag{P}$$

251 In that case we denote $\Gamma \overset{\varphi}{\equiv} V$.

252 *Remark.* For example, translations on the grid graph, with $\varphi(t_{i,j}) = (i,j)$,

253 are φ-equivalent as $t_{i,j}(a,b) = \varphi(t_{i,j} \circ t_{a,b})$. However, with $\varphi(t_{i,j}) = (-i, -j)$,

254 they would not be φ-equivalent.

255 **Definition 16. Group convolution III**

256 Let a subgroup $\Gamma \subset \Phi^*(V)$ such that $\Gamma \overset{\varphi}{\cong} V$. The group convolution III

257 between two signals s_1 and $s_2 \in \mathcal{S}(V)$ is defined as:

$$s_1 *_{\text{III}} s_2 = \sum_{v \in V} s_1[v] \, g_v(s_2) \tag{3}$$

$$= \sum_{g \in \Gamma} s_1[\varphi(g)] \, g(s_2) \tag{4}$$

258

259 The two expressions differ on the domain upon which the summation is done.

260 The expression (3) put the emphasis on each vertex and its action, whereas

261 the expression (4) emphasizes on each object of Γ.

262 **Lemma 17. Relation with group convolution II**

263 $\Gamma \overset{\varphi}{\equiv} V \Leftrightarrow *_{\text{II}} = *_{\text{III}}$

Proof.

$$\forall s_1, s_2 \in \mathcal{S}(V),$$

$$s_1 *_{\text{II}} s_2 = s_1 *_{\text{III}} s_2$$

$$\Leftrightarrow \forall u \in V, \sum_{v \in V} s_1[v] \, s_2[\varphi(g_v^{-1} g_u)] = \sum_{v \in V} s_1[v] \, s_2[g_v^{-1}(u)] \tag{5}$$

264   Hence, the direct sense is obtained by applying (P).

265   For the converse, given $u, v \in V$, we first realize (5) for $s_1 := \delta_v$, obtaining

266   $s_2[\varphi(g_v^{-1} g_u)] = s_2[g_v^{-1}(u)]$, which we then realize for a real signal s_2 having no

267   two equal entries, obtaining $\varphi(g_v^{-1} g_u) = g_v^{-1}(u)$. From the latter we finally

268   obtain (P) with the one-to-one correspondence $g_{v'} := g_v^{-1}$. $\square$

269   We can then coin the term φ-convolution.

270   **Definition 18. φ-convolution**

271   Let $\Gamma \overset{\varphi}{\equiv} V$, the φ-*convolution* between two signals s_1 and $s_2 \in \mathcal{S}(V)$ is

272   defined as:

$$s_1 *_\varphi s_2 = s_1 *_\text{II} s_2 = s_1 *_\text{III} s_2$$

273

274   This time, we do obtain equivariance to Γ as expected, and the full charac-

275   terization as well.

276   **Proposition 19. Characterization by right-action equivariance to Γ**

277   If Γ is φ-equivalent, the class of linear transformations of $\mathcal{S}(V)$ that are

278   equivariant to Γ is exactly the class of φ-convolution operators acting on the

279   right of $\mathcal{S}(V)$ *i.e.*

$$\text{If } \Gamma \overset{\varphi}{\equiv} V,$$

$$\exists w \in \mathcal{S}(V), f = \,.\, *_\varphi w \Leftrightarrow \begin{cases} f \in \mathcal{L}(\mathcal{S}(V)) \\ \forall g \in \Gamma, f \circ g = g \circ f \end{cases}$$

280

281   *Proof.*   1. From left to right:

282   In the following equations, (6) is obtained by definition, (7) is obtained

283   because left multiplication in a group is bijective, and (8) is obtained

284 because of (P).

$$\forall g \in \Gamma, \forall s \in \mathcal{S}(V),$$

$$f_w \circ g(s) = \sum_{h \in \Gamma} g(s)[\varphi(h)]\, h(w) \tag{6}$$

$$= \sum_{h \in \Gamma} g(s)[\varphi(gh)]\, gh(w) \tag{7}$$

$$= \sum_{h \in \Gamma} g(s)[g(\varphi(h))]\, gh(w) \tag{8}$$

$$= \sum_{h \in \Gamma} s[\varphi(h)]\, gh(w)$$

$$= \sum_{h \in \Gamma} s[\varphi(h)]\, h(w)[g^{-1}(.)]$$

$$= f_w(s)[g^{-1}(.)]$$

$$= g \circ f_w(s)$$

285 Of course, we also have that f_w is linear.

286 2. From right to left:

287 Let $f \in \mathcal{L}(\mathcal{S}(V)), s \in \mathcal{S}(V)$. By linearity of f, we distribute $f(s)$ on

288 the family of dirac signals:

$$f(s) = \sum_{v \in V} s[v] f(\delta_v) \tag{9}$$

289 Thanks to (P), we have that:

$$g_v(\varphi(\mathrm{Id})) = \varphi(g_v\,\mathrm{Id}) = v$$

$$\text{So, } v = u \Leftrightarrow \varphi(\mathrm{Id}) = g_v^{-1}(u)$$

$$\text{So, } \delta_v = g_v(\delta_{\varphi(\mathrm{Id})})$$

290 By denoting $w = f(\delta_{\varphi(\mathrm{Id})})$, and using the hypothesis of equivariance,

we obtain from (9) that:

$$
\begin{aligned}
f(s) &= \sum_{v \in V} s[v]\, f \circ g_v(\delta_{\varphi(\mathrm{Id})}) \\
&= \sum_{v \in V} s[v]\, g_v \circ f(\delta_{\varphi(\mathrm{Id})}) \\
&= \sum_{v \in V} s[v]\, g_v(w) \\
&= s *_\varphi w
\end{aligned}
$$

$\square$

Construction of φ-convolutions on vertex domains

Proposition 19 tells us that in order to define a convolution on the vertex domain of a graph $G = \langle V, E \rangle$, all we need is a subgroup Γ of invertible transformations of V, that is equivalent to V. The choice of Γ can be done with respect to E. This is discussed in more details in Section 2.3, where we will see that in fact, we only need a generating set of Γ.

Exposure of φ

This construction relies on exposing a bijective equivariant map φ between Γ and V. In the next subsection, we show that in cases where Γ is abelian, we even need not expose φ and the characterization still holds.

2.2.3   Mixed domain formulation

From (4), we can define a mixed domain convolution *i.e.* that is defined for $r \in \mathcal{S}(\Gamma)$ and $s \in \mathcal{S}(V)$, without the need of expliciting φ.

306 **Definition 20. Mixed domain convolution**

307 Let a subgroup $\Gamma \subset \Phi^*(V)$ such that $V \cong \Gamma$. The *mixed domain convolution*

308 between two signals $r \in \mathcal{S}(\Gamma)$ and $s \in \mathcal{S}(V)$ results in a signal $r *_{\mathrm{M}} s \in \mathcal{S}(V)$

309 and is defined as:

$$r *_{\mathrm{M}} s = \sum_{g \in \Gamma} r[g]\, g(s)$$

310

311 We coin it M-convolution. From a practical point of view, this expression of

312 the convolution is useful because it relegates φ as an underpinning object.

313 **Lemma 21. Relation with group convolution III**

314 $\forall \varphi \in \mathrm{BIJ}(\Gamma, V), \forall (r, s) \in \mathcal{S}(\Gamma) \times \mathcal{S}(V),$

315
$$r *_{\mathrm{M}} s = \varphi(r) *_{\mathrm{III}} s$$

316

317 *Proof.* Let $\varphi \in \mathrm{BIJ}(\Gamma, V), (r, s) \in \mathcal{S}(\Gamma) \times \mathcal{S}(V),$

$$r *_{\mathrm{M}} s = \sum_{g \in \Gamma} r[g]\, g(s) = \sum_{v \in V} r[g_v]\, g_v(s) \stackrel{(\diamond)}{=} \sum_{v \in V} \varphi(r)[v]\, g_v(s)$$
$$= \varphi(r) *_{\mathrm{III}} s$$

318 Where $\stackrel{(\diamond)}{=}$ comes from Lemma 9. $\qquad\qquad\square$

319 In other words, $*_{\mathrm{M}}$ is a convenient reformulation of $*_{\mathrm{III}}$ which does not depend

320 on a particular φ.

321 **Lemma 22. Relation with group convolution I, II and φ-convolution**

322 Let $\varphi \in \mathrm{BIJ}(\Gamma, V), (r, s) \in \mathcal{S}(\Gamma) \times \mathcal{S}(V)$, we have:

$$\Gamma \stackrel{\varphi}{\equiv} V \Leftrightarrow \forall v \in V, (r *_{\mathrm{M}} s)[v] = (r *_{\mathrm{I}} \varphi^{-1}(s))[g_v]$$
$$\Leftrightarrow r *_{\mathrm{M}} s = \varphi(r) *_{\mathrm{II}} s$$
$$\Leftrightarrow r *_{\mathrm{M}} s = \varphi(r) *_{\varphi} s$$

323

Proof. On one hand, Lemma 21 gives $r *_{\text{M}} s = \varphi(r) *_{\text{III}} s$. On the other hand, Lemma 11 gives $\forall v \in V, (r *_{\text{I}} \varphi^{-1}(s))[g_v] = (\varphi(r) *_{\text{II}} s)[v]$. Then Lemma 17 concludes. $\square$

Remark. The converse sense is meaningful because it justifies that when the M-convolution is employed, the property $\Gamma \equiv V$ underlies, without the need of expliciting φ.

From M-convolution, we can derive operators acting on the left of $\mathcal{S}(V)$, of the form $s \mapsto w *_{\text{M}} s$, parameterized by $w \in \mathcal{S}(\Gamma)$. In particular, these operators would be relevant as layers of neural networks. On the contrary, derived operators acting on the right such as $r \mapsto r *_{\text{M}} w$ wouldn't make sense with this formulation as they would make φ resurface. However, the equivariance to Γ incurring from Lemma 21 and Proposition 19 only holds for operators acting on the right. So we need to intertwine an abelian condition as follows. This is also a good excuse to see the influence of abelianity.

Proposition 23. Equivariance to Γ through left action

Let a subgroup $\Gamma \subset \Phi^*(V)$ such that $\Gamma \cong V$. Γ is abelian, if and only if, M-convolution operators acting on the left of $\mathcal{S}(V)$ are equivariant to it *i.e.*

$$\forall g, h \in \Gamma, gh = hg \Leftrightarrow \forall w, g \in \Gamma, w *_{\text{M}} g(.) = g \circ (w *_{\text{M}} .)$$

Proof. Let $w, g \in \Gamma$, and define $f_w : s \mapsto w *_{\text{M}} s$. In the following expressions, Γ is abelian if and only if (10) and (11) are equal (the converse is obtained

343 by particularizing on well chosen signals):

$$f_w \circ g(s) = \sum_{h \in \Gamma} w[h] \, hg(s) \tag{10}$$

$$= \sum_{h \in \Gamma} w[h] \, gh(s) \tag{11}$$

$$= \sum_{h \in \Gamma} w[h] \, h(s)[g^{-1}(.)]$$

$$= (w *_{\mathrm{M}} s)[g^{-1}(.)]$$

$$= g \circ f_w(s)$$

344   □

345 *Remark.* Similarly, $*_\varphi$ is also equivariant to Γ through left action if and only
346 if Γ is abelian, as a consequence of being commutative if and only if Γ is
347 abelian. On the contrary, note that commutativity of $*_{\mathrm{M}}$ doesn't make sense.

348 **Corrolary 24. Characterization by left-action equivariance to Γ**
349 Let $\Gamma \cong V$. If Γ is abelian, the class of linear transformations of $\mathcal{S}(V)$ that
350 are equivariant to Γ is exactly the class of M-convolution operators acting on
351 the left of $\mathcal{S}(V)$ *i.e.*

If $\Gamma \cong V$ and Γ is abelian,

$$\exists w \in \mathcal{S}(\Gamma), f = w *_{\mathrm{M}} . \Leftrightarrow \begin{cases} f \in \mathcal{L}(\mathcal{S}(V)) \\ \forall g \in \Gamma, f \circ g = g \circ f \end{cases}$$

352

353 *Proof.* By picking φ such that $\Gamma \overset{\varphi}{\cong} V$ with Lemma 22 and using the relation
354 between $*_{\mathrm{M}}$ and $*_\varphi$.   □

355 Depending on the applications, we will build upon either $*_\varphi$ or $*_{\mathrm{M}}$ when the
356 abelian condition is satisfied.

2.3   Inclusion of the edge set in the construction

The constructions from the previous section involve the vertex set V and depend on Γ, a subgroup of the set of invertible transformations on V. Therefore, it looks natural to try to relate the edge set and Γ.

There are two approaches. Either Γ describes an underlying graph structure $G = \langle V, E \rangle$, either G can be used to define a relevant subgroup Γ to which the produced convolutive operators will be equivariant. Both approaches will help characterize classes of graphs that can support natural definitions of convolutions.

2.3.1   Edge-constrained convolutions

In this subsection, we are trying to answswer the following question:

- What graphs admit a φ-convolution, or an M-convolution (in the sense that they can be defined with the characterization), under the condition that Γ is generated by a set of edge-constrained transformations ?

Definition 25. Edge-constrained transformation

An *edge-constrained* (EC) transformation on a graph $G = \langle V, E \rangle$ is a transformation $f : V \mapsto V$ such that

$$\forall u, v \in V, f(u) = v \Rightarrow u \stackrel{E}{\sim} v$$

We denote $\Phi_{\mathrm{EC}}(G)$ and $\Phi_{\mathrm{EC}}^{*}(G)$ the sets of (EC) and invertible (EC) transformations. When a convolution is defined as a sum over a set that is in one-to-one correspondence with a group that is generated from a set of (EC) transformations, we call it an (EC) convolution.

378   *Remark.* Note that $\Phi^*_{\mathrm{EC}}(G)$ is not a group, thus why we are interested in
379   groups and their generating sets.

380   This leads us to consider Cayley graphs (Cayley, 1878; Wikipedia, 2018).

381 Definition 26. Cayley graph

382   Let a group Γ and one of its generating set $\mathcal{U}$. The *Cayley graph* generated
383   by $\mathcal{U}$, is the digraph $\vec{G} = \langle V, E \rangle$ such that $V = \Gamma$ and E is such that:

$$u \to v \Leftrightarrow \exists g \in \mathcal{U}, ga = b$$

384   Also, if Γ is abelian, we call it an *abelian Cayley graph*. We call *Cayley*
385   *subgraph*, a subgraph that is isomorph to a Cayley graph.

386   *Remark.* Note that for compatibility with the functional notation that we
387   use, we define Cayley graphs with $ga = b$ instead of $ag = b$.

388 Convolution on Cayley graphs

389   In the case of Cayley graphs, it is clear that $\mathcal{U} \subseteq \Phi^*_{\mathrm{EC}}$ and $\Phi^* \supseteq \langle \mathcal{U} \rangle \equiv V$.
390   So that they admit (EC) φ-convolutions, and (EC) M-convolutions in the
391   abelian case.
392   More precisely, we obtain the following characterization:

393 Proposition 27. Characterization by Cayley subgraph isomorphism

394   Let a graph $G = \langle V, E \rangle$, then:

395   (i) G admits an (EC) φ-convolution if and only if it contains a subgraph
396   isomorph to a Cayley graph

397   (ii) G admits an (EC) M-convolution if and only if it contains a subgraph
398   isomorph to an abelian Cayley graph

399   *Proof.* We show the result only in the general case as the proof for the abelian
400   case is similar.

1. From left to right: as a direct application of the definitions.

2. From right to left:

   Let a graph $G = \langle V, E \rangle$. We suppose it contains a subgraph $\vec{G_s} = \langle V_s, E_s \rangle$ that is graph-isomorph to a Cayley graph $\vec{G_c} = \langle V_c, E_c \rangle$, generated by $\mathcal{U}$. Let ψ be a graph isomorphism from G_s to G_c. To obtain the proof, we need to find a group of invertible transformations Γ of V_s generated by a set of (EC) transformations, such that $\Gamma \equiv V_s$.

   Let's define the group action $L : V_c \times V_s \to V_s$ inductively as follows:

   (a) $\forall g \in \mathcal{U}, L_g(u) = v \Leftrightarrow g\psi(u) = \psi(v)$

   (b) Whenever L_g and L_h are defined, the action of gh is defined by homomorphism as $L_{gh} = L_g \circ L_h$

   (c) Whenever L_g is defined, the action of g^{-1} is defined by homomorphism as $L_{g^{-1}} = L_g^{-1}$ i.e. $L_{g^{-1}}(u) = v \Leftrightarrow \psi(u) = g\psi(v)$

   Note that the induction transfers the property (a) to all $g \in V_c$ in a transitive manner because

   $$L_{gh}(u) = L_g(L_h(u)) = w \Leftrightarrow \exists v \in V_s \begin{cases} L_h(u) = v \\ L_g(v) = w \end{cases}$$

   and

   $$\exists v \in V_s \begin{cases} h\psi(u) = \psi(v) \\ g\psi(v) = \psi(w) \end{cases} \Leftrightarrow gh\psi(u) = \psi(w)$$

   We must also verify that this construction is well-defined, *i.e.* whenever we define an action with (b) or (c), if the action was already defined, then they must be equal. This is the case because the homomorphism

420     $g \mapsto L_g$ on V_c is in fact an isomorphism as

$$L_g = L_h \Leftrightarrow \forall u \in V, L_g(u) = L_h(u)$$
$$\Leftrightarrow \forall u \in V, g\psi(u) = h\psi(u)$$
$$\Leftrightarrow g = h$$

421   Also note that (c) is needed only in case that V_c is infinite.

422   Denote the set $L_{\mathcal{U}} = \{L_g, g \in \mathcal{U}\}$ and $\Gamma = \langle L_{\mathcal{U}} \rangle \cong V_c$. Let's define the

423   map φ as:

$$\Gamma \to V_s$$
$$\varphi : L_g \mapsto L_g(\psi^{-1}(\mathrm{Id}))$$

424   φ is bijective because $\forall g \in V_c, \varphi(L_g) = \psi^{-1}(g)$ thanks to (a).

425   Additionally, we have:

$$L_h(\varphi(L_g) = L_h(L_g(\psi^{-1}(\mathrm{Id})))$$
$$= L_h \circ L_g(\psi^{-1}(\mathrm{Id}))$$
$$= L_{hg}(\psi^{-1}(\mathrm{Id}))$$
$$= \varphi(L_{hg})$$
$$= \varphi(L_h \circ L_g)$$

426   That is, φ is a bijective equivariant map and $\langle L_{\mathcal{U}} \rangle = \Gamma \overset{\varphi}{\equiv} V_s$. Moreover,

427     $L_{\mathcal{U}}$ is a set of (EC) transformations thanks to (a). Therefore, G admits

428   an (EC) φ-convolution.

429     $\square$

430 **Corrolary 28. Characterization by φ**

431 Let a graph $G = \langle V, E \rangle$, and a set $\mathcal{U} \subset \Phi^*_{\text{EC}}(G)$ *s.t.*

$$\langle \mathcal{U} \rangle \cong \Gamma \equiv V' \subset V$$

432 G admits an (EC) φ-convolution, if and only if, φ is a graph isomorphism
433 between the Cayley graph generated by $\mathcal{U}$ and the subgraph induced by V'.

434 The proof is omitted as it would be highly similar to the previous one.

435 2.3.2   Intrinsic properties

436   • Obviously the constructed convolutions are linear. But do they also
437   preserve the locality and weight sharing properties ?

438 Let $\vec{G} = \langle V, E \rangle$ be a Cayley subgraph, generated by $\mathcal{U}$, of some graph G.
439 Recall that its (EC) φ-convolution operator is a right operator, and can be
440 expressed as

$$\forall s \in \mathcal{S}(V), \forall u \in V,$$
$$f_w(s)[u] = (s *_\varphi w)[u]$$
$$= \sum_{v \in V} s[v]\, w[g_v^{-1}(u)] \tag{12}$$

441 From this expression, it is not obvious that f_w is a local operator. To see
442 this, we can show for example the following proposition.

443 **Proposition 29. Locality**
444 When the support of w is a compact (in the sense that its induced subgraph
445 in G is connected), of diameter d, the same holds for the support of the
446 sum Σ in (12). More precisely, the subgraph induced by the support of Σ is
447 isomorphic to the transpose of the subgraph induced by the support of w.

448 *Proof.* Without loss of generality subject to growing $\mathcal{U}$, let's suppose that
449 w has a support $\mathcal{M} = \varphi(\mathcal{N})$, such that $\mathcal{N} \subset \mathcal{U}$. $\mathcal{N}$ and $\mathcal{M}$ are obviously
450 compacts of diameter 2. Thanks to (P), we have

$$
\begin{aligned}
g_v^{-1}(u) \in \mathcal{M} &\Leftrightarrow u \in g_v(\mathcal{M}) = g_v(\varphi(\mathcal{N})) = \varphi(g_v\mathcal{N}) \\
&\Leftrightarrow g_u \in g_v\mathcal{N} \\
&\Leftrightarrow g_v^{-1} \in \mathcal{N}g_u^{-1} \\
&\Leftrightarrow g_v \in g_u\mathcal{N}^{-1} \\
&\Leftrightarrow v \in g_u(\varphi(\mathcal{N}^{-1}))
\end{aligned}
$$

451 where $\mathcal{N}^{-1}$ reverses the edges of $\mathcal{N}$. Let's denote $\mathcal{K}_u = g_u(\varphi(\mathcal{N}^{-1})) \subset V$.
452 By composing edge reversal and graph isomorphisms (as φ and its inverse are
453 graph isomorphisms by Proposition 28), the compactness and diameter of $\mathcal{M}$
454 is preserved for $\mathcal{K}_u$. More preceisely, the transposed subgraph structure is
455 also preserved.   □

456 Let's define $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{K}_u$ as in the previous proof.

457 **Definition 30. Supporting set**
458 The *supporting set* of an (EC) convolution operator f_w, is a set $\mathcal{N} \subset \Phi^*_{\mathrm{EC}}$,
459 such that

460   (i) when $*$ is $*_\varphi$: $0 \notin w[\mathcal{M}]$, where $\mathcal{M} = \varphi(\mathcal{N})$

461   (ii) when $*$ is $*_{\mathrm{M}}$: $0 \notin w[\mathcal{N}]$

462 **Definition 31. Local patch for $*_\varphi$**
463 The *local patch* at $u \in V$ of an (EC) φ-convolution operator f_w is defined as
464 $\mathcal{K}_u = g_u(\varphi(\mathcal{N}^{-1}))$.

465 *Remark.* In other terms, $\mathcal{K}_{\mathrm{Id}} = \varphi(\mathcal{N}^{-1})$ is the *initial local patch*, which is
466 composed of all vertices that are connected in direction to $\varphi(\mathrm{Id})$; and $\mathcal{K}_u$ is
467 obtained by moving $\mathcal{K}_{\mathrm{Id}}$ on the Cayley subgraph via the edges corresponding
468 to the decomposition of g_u on the generating set $\mathcal{U}$.

469 To see that the weights are tied in the general case (i), we can show the
470 following proposition.

471 **Proposition 32. Weight sharing**
472 $\forall a, \alpha \in V, \forall b \in \mathcal{K}_a : \exists \beta \in \mathcal{K}_\alpha \Leftrightarrow g_\beta^{-1}(\alpha) = g_b^{-1}(a)$

473 *Proof.* By using (P),

$$g_{\mathcal{K}_\alpha}^{-1}(\alpha) = g_{\mathcal{K}_a}^{-1}(a) \Leftrightarrow g_\alpha^{-1} g_{\mathcal{K}_\alpha} = g_a^{-1} g_{\mathcal{K}_a}$$
$$\Leftrightarrow \mathcal{K}_\alpha = g_\alpha g_a^{-1}(\mathcal{K}_a) = g_\alpha g_a^{-1} g_a(\varphi(\mathcal{N}^{-1}))$$
$$\Leftrightarrow \mathcal{K}_\alpha = g_\alpha(\varphi(\mathcal{N}^{-1}))$$

474 $\square$

475 2.3.3   Stricly edge-constrained convolutions

476 We make the disctinction between general (EC) convolution operators and
477 those for which the weight kernel w is smaller and is supported only on (EC)
478 transformations of $\mathcal{U}$.

479 **Definition 33. Strictly (EC) convolution operator**
480 A *strictly* edge-constrained (EC*) convolution operator f_w, is an (EC) con-
481 volution operator such that its supporting set $\mathcal{N} \subset \mathcal{U}$.

482 Let f_w be an (EC*) convolutional operator. In the general case (i), $w \in \mathcal{S}(V)$,
483 so its support is $\mathcal{M} = \varphi(\mathcal{N})$ such that $\mathcal{N} \subseteq \mathcal{U}$. In the abelian case (ii), we
484 use instead $w \in \mathcal{S}(\Gamma)$, and thus its support is directly $\mathcal{N}$. Therefore, we can
485 rewrite the expressions of the convolution operator as:

486   (i) $\forall s \in \mathcal{S}(V), \forall u \in V, f_w(s)[u] \overset{(\varphi)}{=} \sum_{v \in \mathcal{K}_u} s[v]\, w[g_v^{-1}(u)]$

487   (ii) $\forall s \in \mathcal{S}(V), f_w(s) \overset{(\text{M})}{=} \sum_{g \in \mathcal{N}} w[g]\, g(s)$

488   *Remark.* Note that in the abelian case, we can see from (ii) that a definition
489   of a local patch would coincide with the supporting set, so that locality and
490   weight sharing is straightforward.

491   From these expressions, it is clear that Γ needs not to be fully determined
492   to calculate $f_w(s)[u]$. The case (ii) is the simplest as the only requirement
493   is a supporting set $\mathcal{N}$ of (EC) invertible transformations. In the case (i), we
494   only need to determine $\mathcal{K}_u$.

2.4   From groups to groupoids

2.4.1   Motivation

One possible limitation coming from searching for Cayley subgraphs is that they are order-regular *i.e.* the in- and the out-degree $d = |\mathcal{U}|$ of each vertex is the same. That is, for a general graph G, the size of the weight kernel w of an (EC*) convolution operator f_w supported on $\mathcal{U}$ is bounded by d, which in turn is bounded by twice the mininimal degree of G (twice because G is undirected and $\mathcal{U}$ can contain every inverse).

There are a lot of possible strategies to overcome this limitation. For example:

1. connecting each vertex with its k-hop neighbors, with $k > 1$,

2. artificially creating new connections for less connected vertices,

3. allowing the supporting set $\mathcal{N}$ to exceed $\mathcal{U}$ *i.e.* dropping * in (EC*).

These strategies require to concede that the topological structure supported by G is not the best one to support an (EC*) convolution on it, which breeds the following question:

- What can we relax in the previous (EC*) contruction in order to unbound the supporting set, and still preserve the equivariance characterization?

The latter constraint is a consequence that every vertex of the Cayley subgraph $\vec{G}$ must be composable with every generator from $\mathcal{U}$. Therefore, an answer consists in considering groupoids (Brandt, 1927) instead of groups. Roughly speaking, a groupoid is almost a group except that its composition law needs not be defined everywhere. Weinstein, 1996, unveiled the benefits to base convolutions on groupoids instead of groups in order to exploit partial symmetries.

2.4.2 Definition of notions related to groupoids

Definition 34. Groupoid

A *groupoid* Υ is a set equipped with a partial composition law with domain $\mathcal{D} \subset \Upsilon \times \Upsilon$, called *composition rule*, that is

1. closed into Υ *i.e.* $\forall (g, h) \in \mathcal{D}, gh \in \Upsilon$

2. associative *i.e.* $\forall f, g, h \in \Upsilon,$
$$\begin{cases} (f, g), (g, h) \in \mathcal{D} \Leftrightarrow (fg, h), (f, gh) \in \mathcal{D} \\ (f, g), (fg, h) \in \mathcal{D} \Leftrightarrow (g, h), (f, gh) \in \mathcal{D} \\ \text{when defined, } (fg)h = f(gh) \end{cases}$$

3. invertible *i.e.* $\forall g \in \Upsilon, \exists! g^{-1} \in \Upsilon \quad s.t. \begin{cases} (g, g^{-1}), (g^{-1}, g) \in \mathcal{D} \\ (g, h) \in \mathcal{D} \Rightarrow g^{-1}gh = h \\ (h, g) \in \mathcal{D} \Rightarrow hgg^{-1} = h \end{cases}$

Optionally, it can be *domain-symmetric i.e.* $(g, h) \in \mathcal{D} \Leftrightarrow (h, g) \in \mathcal{D}$, and *abelian i.e.* domain-symmetric with $gh = hg$.

Remark. Note that left and right inverses are necessarily equal (because $(gg^{-1})g = g(g^{-1}g)$). Also note we can define a right identity element $e_g^r = g^{-1}g$, and a left one $e_g^l = gg^{-1}$, but they are not necessarily equal and depend on g.

Most definitions related to groups can be adapted to groupoids. In particular, let's adapt a few notions.

Definition 35. Groupoid partial action

A partial *action* of a groupoid Υ on a set V, is a function L, with domain $\mathcal{D}_L \subset \Upsilon \times V$ and valued in V, such that the map $g \mapsto L_g$ is a groupoid homomorphism.

₅₄₀ *Remark.* As usual, we will confound L_g and g when there is no possible
₅₄₁ confusion, and we denote $\mathcal{D}_{L_g} = \mathcal{D}_g = \{v \in V, (g, v) \in \mathcal{D}_L\}$.

Definition 36. Partial equivariant map

₅₄₃ A map φ from a groupoid Υ partially acting on the destination set V is said
₅₄₄ to be a *partial equivariant map* if

$$\forall g, h \in \Upsilon, \begin{cases} \varphi(h) \in \mathcal{D}_g \Leftrightarrow (g, h) \in \mathcal{D} \\ g(\varphi(h)) = \varphi(gh) \end{cases}$$

₅₄₅ Also, φ-equivalence between a subgroupoid and a set is defined similarly with
₅₄₆ φ being a bijective *partial* equivariant map between them.

Definition 37. Partial transformations groupoid

₅₄₈ The *partial transformations groupoid* $\Psi^*(V)$, is the set of invertible par-
₅₄₉ tial transformations, equipped with the functional composition law with do-
₅₅₀ main $\mathcal{D}$ such that

$$\begin{cases} \mathcal{D}_{gh} = h(\mathcal{D}_h) \cap \mathcal{D}_g \\ (g, h) \in \mathcal{D} \Leftrightarrow \mathcal{D}_{gh} \neq \emptyset \end{cases}$$

₅₅₁ *Remark.* Note that a subgroupoid $\Upsilon \subset \Psi^*(V)$ is domain-symmetric when
₅₅₂ $\exists v \in V, g(v) \in \mathcal{D}_h \Leftrightarrow \exists u \in V, h(u) \in \mathcal{D}_g$

2.4.3   Construction of partial convolutions

₅₅₄ The expression of the convolution we constructed in the previous section
₅₅₅ cannot be applied as is. We first need to extend the algebraic objects we
₅₅₆ work with. Extending a partial transformation g on the signal space $\mathcal{S}(V)$
₅₅₇ (and thus the convolutions) is a bit tricky, because only the signal entries
₅₅₈ corresponding to $\mathcal{D}_g$ are moved. A convenient way to do this is to consider
₅₅₉ the groupoid closure obtained with the addition of an absorbing element.

Definition 38. Zero-closure

The *zero-closure* of a groupoid Υ, denoted Υ^0, is the set $\Upsilon \cup 0$, such that the groupoid axioms 1, 2 and 3, and the domain $\mathcal{D}$ are left unchanged, and

    4. the composition law is extended to $\Upsilon^0 \times \Upsilon^0$ with $\forall (g, h) \notin \mathcal{D}, gh = 0$

Remark. Note that this is coherent as the properties 2 and 3 are still partially defined on the original domain $\mathcal{D}$.

Now, we will also extend every other algebraic object used in the expression of the φ-convolution and the M-convolution, so that we can directly apply our previous constructions.

Lemma 39. Extension of φ on V^0

Let a partial equivariant map $\varphi : \Upsilon \to V$. It can be extended to a (total) equivariant map $\varphi : \Upsilon^0 \to V^0 = V \cup \varphi(0)$, such that $\varphi(0) \notin V$, that we denote $0_V = \varphi(0)$, and such that

$$\forall g \in \Upsilon^0, \forall v \in V^0, g(v) = \begin{cases} \varphi(gg_v) & \text{if } g_v \in \mathcal{D}_g \\ 0_V & \text{else} \end{cases}$$

Proof. We have $\varphi(0) \notin V$ because φ is bijective. Additionaly, we must have $\forall (g, h) \notin \mathcal{D}, g(\varphi(h)) = \varphi(gh) = \varphi(0) = 0_V$. $\qquad\square$

Remark. Note that for notational conveniency, we may use the same symbol 0 for 0_Υ, 0_V and $0_\mathbb{R}$.

Similarly to $\Phi^*(V)$, $\Psi^*(V)$ can also move signals of $\mathcal{S}(V)$.

Lemma 40. Extension of injective partial transformations to $\mathcal{S}(V)$

Let $g \in \Psi^*(V)$. Its extension is done in two steps:

    1. g is extended to $V^0 = V \cup \{0_V\}$ as $g(v) = 0_V \Leftrightarrow v \notin \mathcal{D}_g$.

581   2. Under the convention $\forall s \in \mathcal{S}(V), s[0_V] = 0_{\mathbb{R}}$, g is extended via linear

582   extension to $\mathcal{S}(V)$, and we have

$$\forall s \in \mathcal{S}(V), \forall v \in V, g(s)[v] = s[g^{-1}(v)]$$

583   *Proof.* Straightforward.   □

584   With these extensions, we can obtain the partial φ- and M-convolutions re-

585   lated to Υ almost by substituting Υ^0 to Γ in Definition 18 and Definition 20.

586   **Definition 41. Partial convolution**

587   Let a subgroupoid $\Upsilon \subset \Psi^*(V)$, such that $\Upsilon \overset{\varphi}{\Bumpeq} V$. The partial φ- and

588   M-convolutions, based on Υ, are defined on its zero-closure, with the same

589   expression as if Υ^0 were a subgroup, and by extension of φ and of the groupoid

590   partial actions *i.e.*

591   (i) $\forall s, w \in \mathcal{S}(V), s *_\varphi w = \sum_{v \in V} s[v] \, g_v(w) = \sum_{g \in \Upsilon} s[\varphi(g)] \, g(w)$

592   (ii) $\forall (w, s) \in \mathcal{S}(\Upsilon) \times \mathcal{S}(V), w *_M s = \sum_{g \in \Upsilon} w[g] \, g(s)$

593   where (ii) applies in the abelian case.

594   **Symmetrical expressions**

595   Note that, as $\forall r, r[0] = 0$, the partial convolutions can also be expressed on

596   the domain $\mathcal{D}$ with a convenient symmetrical expression:

597   (i) $\forall u \in V, (s *_\varphi w)[u] = \sum_{\substack{(g_a, g_b) \in \mathcal{D} \\ s.t. \ g_a g_b = g_u}} s[a] \, w[b]$

598   (ii) $\forall u \in V, (w *_M s)[u] = \sum_{\substack{v \in \mathcal{D}_g \\ s.t. \ g(v) = u}} w[g] \, s[v]$

599   We obtain an equivariance characterization similar to Proposition 19 and

600   Corrolary 24.

601 **Proposition 42. Characterization by equivariance to Υ**

602 Let a subgroupoid $\Upsilon \subset \Psi^*(V)$, such that $\Upsilon \overset{\varphi}{\equiv} V$. Then,

603   (i) right partial φ-convolution operators are equivariant to Υ,

604   (ii) if Υ is abelian, left partial M-convolution operators are equivariant to Υ.

605 Conversely,

606   (i) if Υ is domain-symmetric, linear transformations of $\mathcal{S}(V)$ that are

607   equivariant to Υ are right partial φ-convolution operators,

608   (ii) if Υ is abelian, linear transformations of $\mathcal{S}(V)$ that are equivariant to Υ

609   are also left partial M-convolution operators.

610 *Proof.*   (i)   (a) Direct sense:

611   Using the symmetrical expressions, and the fact that $\forall r, r[0] = 0$,

612   we have

$$
(f_w \circ g(s))[u] = \sum_{\substack{(g_a, g_b) \in \mathcal{D} \\ \text{s.t. } g_a g_b = g_u}} g(s)[a]\, w[b]
$$

$$
= \sum_{\substack{(g_a, g_b) \in \mathcal{D} \\ \text{s.t. } g_a g_b = g_u}} s[g^{-1}(a)]\, w[b]
$$

$$
= \sum_{\substack{(g_a, g_b) \in \mathcal{D} \\ \text{s.t. } (g, g_a) \in \mathcal{D} \\ \text{s.t. } g g_a g_b = g_u}} s[a]\, w[b]
$$

$$
= \sum_{\substack{(g_a, g_b) \in \mathcal{D} \\ \text{s.t. } (g, g_a) \in \mathcal{D} \\ \text{s.t. } g_a g_b = g^{-1} g_u = g_{\varphi(g^{-1} g_u)} = g_{g^{-1}(u)}}} s[a]\, w[b]
$$

$$
= f_w(s)[g^{-1}(u)]
$$

$$
= (g \circ f_w(s))[u]
$$

613   (b) Converse:

614   Let $v \in V$. Denote $e^r_{g_v} = g_v^{-1} g_v$ the right identity element of g_v,

615   and $e_v^r = \varphi(e_{g_v}^r)$. We have that

$$g_v(e_v^r) = v$$
$$\text{So, } \delta_v = g_v(\delta_{e_v^r})$$

616   Let $f \in \mathcal{L}(\mathcal{S}(V))$ that is equivariant to Υ, and $s \in \mathcal{S}(V)$. Thanks

617   to the previous remark we obtain that

$$
\begin{aligned}
f(s) &= \sum_{v \in V} s[v]\, f(\delta_v) \\
&= \sum_{v \in V} s[v]\, f(g_v(\delta_{e_v^r})) \\
&= \sum_{v \in V} s[v]\, g_v(f(\delta_{e_v^r})) \\
&= \sum_{v \in V} s[v]\, g_v(w_v) \qquad\qquad (13)
\end{aligned}
$$

618   where $w_v = f(\delta_{e_v^r})$. In order to finish the proof, we need to find w

619   such that $\forall v \in V, g_v(w) = g_v(w_v)$.

620   Let's consider the equivalence relation $\mathcal{R}$ defined on $V \times V$ such

621   that:

$$
\begin{aligned}
a\mathcal{R}b &\Leftrightarrow w_a = w_b \\
&\Leftrightarrow e_a^r = e_b^r \\
&\Leftrightarrow g_a^{-1} g_a = g_b^{-1} g_b \\
&\Leftrightarrow (g_b, g_a^{-1}) \in \mathcal{D} \\
&\Leftrightarrow (g_a^{-1}, g_b) \in \mathcal{D} \qquad\qquad (14)
\end{aligned}
$$

622   with (14) owing to the fact that Υ is domain-symmetric.

623   Given $x \in V$, denote its equivalence class $\mathcal{R}(x)$. Under the hy-

624   pothesis of the axiom of choice (Zermelo, 1904) (if V is infinite),

625   define the set $\aleph$ that contains exactly one representative per equiv-
626   alence class. Let $w = \sum_{n \in \aleph} w_n$. Then V is the disjoint union
627   $V = \cup_{n \in \aleph} \mathcal{R}(n)$ and (13) rewrites:

$$\forall u \in V, f(s)[u] = \sum_{n \in \aleph} \sum_{v \in \mathcal{R}(n)} s[v] \, g_v(w_n)[u]$$

$$= \sum_{n \in \aleph} \sum_{v \in \mathcal{R}(n)} s[v] \, w_n[g_v^{-1}(u)]$$

$$= \sum_{n \in \aleph} \sum_{v \in \mathcal{R}(n)} s[v] \, w[g_v^{-1}(u)] \tag{15}$$

$$= (s *_\varphi w)[u]$$

628   where (15) is obtained thanks to (14).

629   (ii) With symmetrical expressions, it is clear that the convolution is abelian,
630   if and only if, Υ is abelian. Then (i) concludes.

631   $\square$

632   **Inclusion of (EC)**
633   Similarly to the construction in Section 2.3, partial convolutions can define
634   (EC) and (EC*) counterparts with a characterization of admissibility by
635   groupoid Cayley subgraph isomorphism.

636   **Limitation of partial convolutions**
637   However, because of the groupoid associativity, if $g \in \Psi_{\text{EC}}^*(G)$, then, any
638   $v \in V$ *s.t.* $g(u) = v$ would be constrained to allow to be acted by every
639   h *s.t.* $(h, g) \in \mathcal{D}$, which fails at unbounding the supporting set of a partial
640   (EC*) convolutions.

2.4.4   Construction of path convolutions

To answer the limitation of partial convolutions, given $g \in \langle \mathcal{U} \rangle$ where $\mathcal{U} \subset \Psi^*_{\mathrm{EC}}(G)$, the idea is to proceed with a foliation of g into pieces, each corresponding to an edge $e \in E$, and together generating another groupoid with a different associativity law, as follows.

TODO: This subsection is still work in progress.

Definition 43. Path groupoid

Let $\mathcal{U} \subset \Psi^*_{\mathrm{EC}}(G)$. The *path groupoid* generated from $\mathcal{U}$, denoted $\mathcal{U} \ltimes V$, with composition rule $\mathcal{D}_\ltimes$, is the groupoid obtained inductively as:

1. $\mathcal{U} \ltimes_0 V = \{(g, v) \in \mathcal{U} \times V, v \in \mathcal{D}_g\} \subset \mathcal{U} \ltimes V$

2. $((g_1, v_1) \cdots (g_n, v_n), (h_1, u_1) \cdots (h_m, u_m)) \in \mathcal{D}_\ltimes \Leftrightarrow g_n(v_n) = u_1$

3. $(g_1, v_1) \cdots (g_n, v_n) \in \mathcal{U} \ltimes V \Rightarrow (g_n^{-1}, g_n(v_n)) \cdots (g_1^{-1}, g_1(v_1)) \in \mathcal{U} \ltimes V$

Remark. This groupoid construction is inspired from the field of operator algebra where partial action groupoids have been extensively studied, *e.g.* Nica, 1994; Exel, 1998; Li, 2016.

Definition 44. Source, target, path, and length maps

Let a path groupoid $\mathcal{U} \ltimes V$. We define on it the *source map* α, the *target map* β, the *path map* γ, and the *length map* λ as:

$$\begin{cases} \alpha : (g_1, v_1) \cdots (g_n, v_n) \mapsto v_1 \in V \\ \beta : (g_1, v_1) \cdots (g_n, v_n) \mapsto g_n(v_n) \in V \\ \gamma : (g_1, v_1) \cdots (g_n, v_n) \mapsto g_n g_{n-1} \ldots g_1 \in \Psi^*_{\mathrm{EC}}(G) \\ \lambda : (g_1, v_1) \cdots (g_n, v_n) \mapsto n \in \mathbb{N}^* \end{cases}$$

Remark. Note that the path groupoid can also be obtained by discrete derivation of the partial transformation groupoid (*i.e.* $p \in \mathcal{U} \ltimes V$ can be expressed as a derivative of $\gamma(p)$ *w.r.t.* $\alpha(p)$), and can thus be seen as the local structure of it.

Lemma 45. Useful properties of α, β, and γ

1. $(p, q) \in \mathcal{D}_{\ltimes} \Leftrightarrow \beta(p) = \alpha(q)$.

2. γ is a groupoid partial action. Denote $p(v) := \gamma(p)(v)$.

3. β is a partial equivariant map for the groupoid partial action γ on V.

We can now define the convolution based on a path groupoid, as an equivalent of a φ-convolution where β takes the role of φ.

Definition 46. Path convolution

The *path convolution* $*$, based on a path groupoid $\mathcal{U} \ltimes V$, is defined for signals $s_1, s_2 \in \mathcal{S}(V)$, or with a mixed expression $*_{\mathrm{M}}$ for signals $\widetilde{s}_1 \in \mathcal{U} \ltimes V$ and $s_2 \in \mathcal{S}(V)$ as:

(i) $\forall u \in V, (s_1 * s_2)[u] = \displaystyle\sum_{\substack{p \in \mathcal{U} \ltimes V \\ s.t. \ \beta(p)=u}} s_1[\beta(\gamma(p))] \, s_2[\alpha(p)]$

(ii) $\forall u \in V, (\widetilde{s}_1 *_{\mathrm{M}} s_2)[u] = \displaystyle\sum_{\substack{p \in \mathcal{U} \ltimes V \\ s.t. \ \beta(p)=u}} \widetilde{s}_1[\gamma(p)] \, s_2[\alpha(p)]$

Indeed, we will see that corresponding operators are equivariant to $\mathcal{U} \ltimes V$. To address the converse, let's introduce the following notions.

Definition 47. Tree-covering set

Let a graph $G = \langle V, E \rangle$ that is connected. A *tree-covering set* is a set $\mathcal{U} \subset \Psi^*_{\mathrm{EC}}(G)$ such that

1. An edge can only correspond to a unique $g \in \mathcal{U}$,
   i.e. $\forall g, h \in \mathcal{U} : \exists v \in V, g(v) = h(v) \Rightarrow g = h$

2. The graph $G_{\mathcal{U}} = \langle V, E_{\mathcal{U}} \rangle$ is a covering tree graph of G, where

$$E_{\mathcal{U}} = \{\{v, g(v)\} \in E, (g, v) \in \mathcal{U} \times V\}$$

If $\mathcal{U}$ is a tree-covering set, then the path convolution based on $\mathcal{U} \ltimes V$ is said to be *rooted*. Also, denote by r the root of $G_{\mathcal{U}}$.

Remark. The assumption that the graph G is connected has been made. This doesn't lose generality as the construction can be replicated to each connected component in the general case.

Proposition 48. Characterization by equivariance to $\mathcal{U} \ltimes V$

Let a set $\mathcal{U} \subset \Psi^*_{\mathrm{EC}}(G)$, and let's base the path convolutions on $\mathcal{U} \ltimes V$. Then, path convolution right-operators are equivariant to $\mathcal{U} \ltimes V$. Conversely, if $\mathcal{U} \ltimes V$ is tree-covering, linear transformations of $\mathcal{S}(V)$ that are equivariant to $\mathcal{U} \ltimes V$ are path convolution right-operators with (EC*) sum support.

Proof. TODO: I mixed up p and $\gamma(p)$, and lemma45.3 is wrong (should rephrase). Maybe covering tree comes before and introduces a φ.

$\square$

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