Probabilistic Inference on Bayesian Network and d-Separation Algorithm

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#1. Consider a joint distribution of five random variables, a, b, c, d, and e. Show that

$$P(a, b, c|d, e) = P(a, b|c, d, e)P(c|d, e)$$

Solution: Using product rule $\Rightarrow P(a, b) = P(a|b)P(b)$, we can write,

$$\begin{aligned} P(a, b, c|d, e) &= \frac{P(a, b, c, d, e)}{P(d, e)} \\ &= \frac{P(a, b|c, d, e)P(c, d, e)}{P(d, e)} \\ &= \frac{P(a, b|c, d, e)P(c|d, e)P(d, e)}{P(d, e)} \\ &= P(a, b|c, d, e)P(c|d, e) \end{aligned}$$

#2. Consider the joint distribution of three random variables a, b, and c. Show that

$$P(a|b, c) = \frac{P(b|a, c)P(a|c)}{P(b|c)}$$

Solution: Using product rule $\Rightarrow P(a, b) = P(a|b)P(b)$, we can write,

$$\begin{aligned} P(a|b, c) &= \frac{P(a, b, c)}{P(b, c)} \\ &= \frac{P(b|a, c)P(a, c)}{P(c)P(b|c)} \\ &= \frac{P(b|a, c)P(a|c)P(c)}{P(c)P(b|c)} \\ &= \frac{P(b|a, c)P(a|c)}{P(b|c)} \end{aligned}$$

#3. Consider the full joint distribution table for the three random variables below. Compute the following conditional probability (cavity = True and False): i.e. $P(cavity|\neg catch)$.

	toothache		$\neg$*toothache*	
	catch	$\neg$*catch*	*catch*	$\neg$*catch*
cavity	0.108	0.012	0.072	0.008
$\neg$*cavity*	0.016	0.064	0.144	0.576

Figure 1:

Solution: Using conditional probability distribution (in the above table),

$$\begin{aligned} P(cavity|\neg catch) &= 0.012 + 0.008 \\ &= 0.020 \end{aligned}$$

and,

$$\begin{aligned} P(\neg cavity|\neg catch) &= 0.064 + 0.576 \\ &= 0.64 \end{aligned}$$

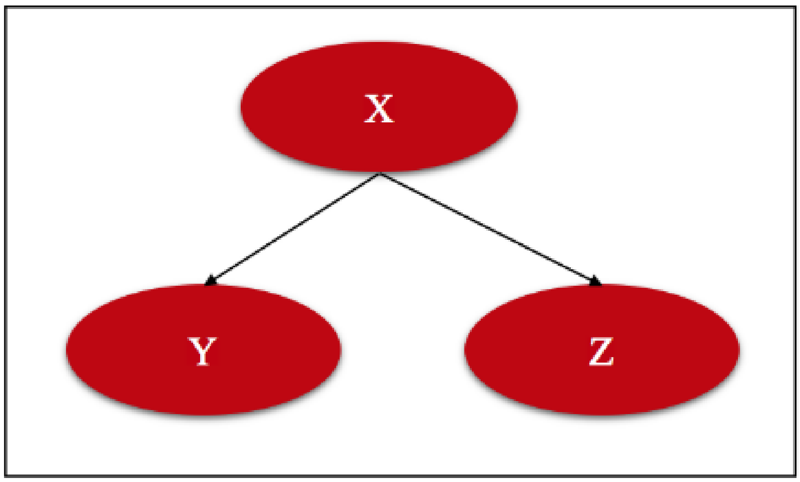

Figure 2:

#4. Consider the following Bayesian network, see Fig.(2), and prove the that,

$$P(Y|X, Z) = P(Y|X)$$

Solution: The given Bayesian network in Fig.(2) is a tail-to-tail connection DAG. In the graph, the events Y and Z are independent, i.e. $P(Y, Z) = P(Y)P(Z)$. Using product rule (or conditional probability distribution) and conditional independence, we can write,

$$\begin{aligned} P(Y|X, Z) &= \frac{P(X, Y, Z)}{P(X, Z)} \\ &= \frac{P(X)P(Y|X)P(Z|X)}{P(X)P(Z|X)} \\ &= P(Y|X) \end{aligned}$$

#5. Consider the following Bayesian network, see Fig.(3), and compute the following conditional probability:

$$P(W|R)$$

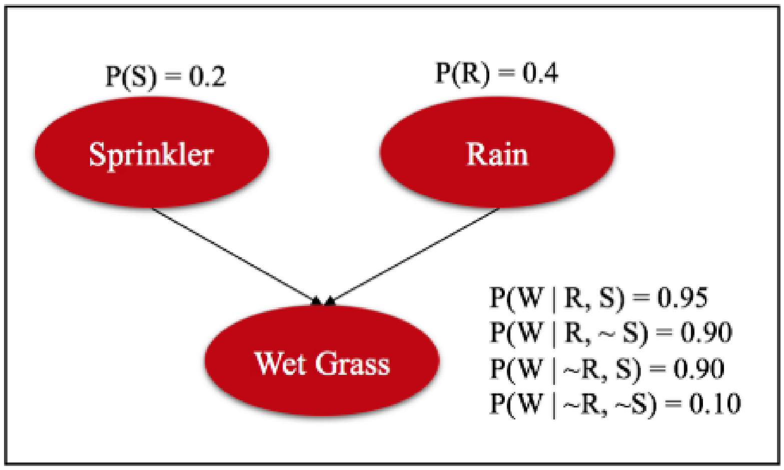

Figure 3:

Solution: Using marginal probability,

$$\begin{aligned} P(W|R) &= \sum_S P(W,S|R) \\ &= P(W,S|R) + P(W,\neg S|R) \\ &= P(W|S,R)P(S|R) + P(W|\neg S,R)P(\neg S|R) \\ &= P(W|S,R)P(S) + P(W|\neg S,R)P(\neg S) \quad \text{[when W is not given]} \\ &= 0.95 \times 0.20 + 0.90 \times (1 - 0.2) \\ &= 0.91 \end{aligned} \tag{1}$$

#6. Consider the following Bayesian network, see Fig.(4), and compute the following conditional probabilities:

$$\begin{aligned} &(a) \quad P(W|\neg C, R) \\ &(b) \quad P(W|C, R) \end{aligned}$$

Solution: (a). Here, $P(W|\neg C, R)$ can be written as below using the marginal probability,

$$\begin{aligned} P(W|\neg C,R) &= \sum_S P(W,S|\neg C,R) \\ &= P(W,S|\neg C,R) + P(W,\neg S|\neg C,R) \end{aligned} \tag{2}$$

Using chain/product rules, we find $P(W,S|\neg C,R)$ below starting from the identity,

$$\begin{aligned} &P(W,S,\neg C,R) = P(W,S,\neg C,R) \\ &\Leftrightarrow P(W|S,\neg C,R)P(S,\neg C,R) = P(W,S|\neg C,R)P(\neg C,R) \\ &\Leftrightarrow P(W|S,\neg C,R)P(S,R|\neg C)P(\neg C) = P(W,S|\neg C,R)P(R|\neg C)P(\neg C) \\ &\Leftrightarrow P(W|S,\neg C,R)P(S,R|\neg C) = P(W,S|\neg C,R)P(R|\neg C) \\ &\Leftrightarrow P(W,S|\neg C,R) = \frac{P(W|S,\neg C,R)P(S,R|\neg C)}{P(R|\neg C)} \end{aligned} \tag{3}$$

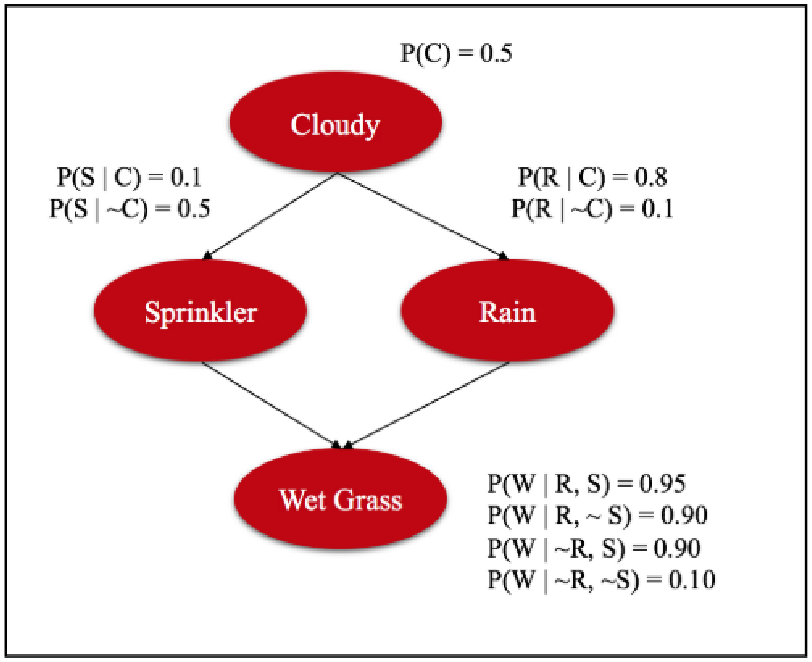

Figure 4:

In the similar fashion, $P(W, \neg S|\neg C, R)$ can be expressed as,

$$\Leftrightarrow P(W, \neg S|\neg C, R) = \frac{P(W|\neg S, \neg C, R)P(\neg S, R|\neg C)}{P(R|\neg C)} \tag{4}$$

Now, using independence condition for head-to-head connection:

$$\begin{aligned} P(W|S, \neg C, R) &= P(W|S, R) \\ P(W|\neg S, \neg C, R) &= P(W|\neg S, R). \end{aligned}$$

And, using independence condition for tail-to-tail connection:

$$\begin{aligned} P(\neg S, R|\neg C) &= P(\neg S|\neg C)P(R|\neg C) \\ P(S, R|\neg C) &= P(S|\neg C)P(R|\neg C). \end{aligned}$$

From Eqs. (2), (3) and (4), we obtain,

$$\begin{aligned} P(W|\neg C, R) &= P(W|S, R)P(S|\neg C) + P(W|\neg S, R)P(\neg S|\neg C) \\ &= 0.95 \times 0.5 + 0.9 \times (1 - 0.5) \\ &= 0.925 \end{aligned}$$

(b). Similarly, we can write,

$$\begin{aligned} P(W|C, R) &= P(W|S, R)P(S|C) + P(W|\neg S, R)P(\neg S|C) \\ &= 0.95 \times 0.1 + 0.9 \times (1 - 0.1) \\ &= 0.905 \end{aligned}$$

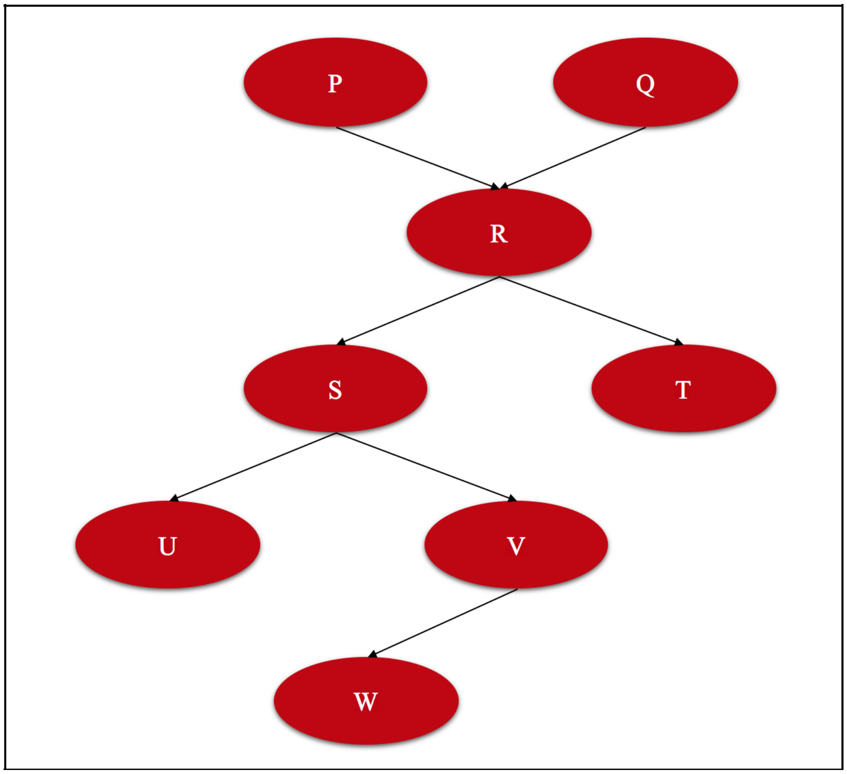

Figure 5:

#7. Consider the following Bayesian network and answer the following questions, see Fig.(5). For justification you must use the d-separation algorithm and specify which canonical case of conditional independence is in effect.

<u>*d-separation theorem:*</u> *If the path between two vertices A and B are blocked given the vertex C, then vertices A and B are d-separated. Given that vertices A and B are d-separated given vertex C, they are conditionally independent.*

(a) Are P and Q conditionally independent given S? Justify.
Solution: No, P and Q are not conditionally independent given S because P and Q are in head-to-head connection with R but S (where there is the condition on) is the descendent of R (implying that the path between P and Q is not blocked, and hence P and Q are not d-separated).

b) Are P and Q conditionally independent given V? Justify.
Solution: No, P and Q are not conditionally independent given V because P and Q are in head-to-head connection with R but V (where there is condition on) is the descendent of R (implying that the path between P and Q is not blocked, and hence P and Q are not d-separated).

(c) Are P and W conditionally independent given R? Justify.
Solution: Yes. The path between P and W given R is blocked because the connection PRS is head-to-tail and hence block the path - which is the only path via which P and W are connected with.

(d) Are S and T conditionally independent given P? Justify.

Solution: No, S and T are not conditionally independent given P because S and T are in tail-to-tail connection with R and P is not in the path between S and T.

(e) Are U and W conditionally independent given V? Justify.
Solution: Yes. The path USVW is blocked because S is a tail-to-tail node and V is in the path between U and W. Since the path is between U and W is blocked, U and W are conditionally independent given V.