Toy model

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The Hamiltonian is the $\epsilon_i \equiv 0$ limit of the Richardson's "picket-fence" model

$$H = \sum_{i\sigma} \epsilon_i c_{i\sigma}^\dagger c_{i\sigma} - \alpha d \sum_{i,j} c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger c_{j\downarrow} c_{j\uparrow} \tag{1}$$

where $d = 2D/L$, $D \equiv 1$ sets the scale and L is the number of levels. Thus

$$H = -\frac{g}{L} \sum_{i,j} c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger c_{j\downarrow} c_{j\uparrow} = -\frac{g}{L} \sum_{i,j} A_i^\dagger A_j. \tag{2}$$

where $g = 2\alpha$ is the strength of the pairing interaction, $i, j = 1, \ldots, L$, and $A_j = c_{j\downarrow} c_{j\uparrow}$ are hard-core bosonic operators with the following commutation rule,

$$[A_i, A_j^\dagger] = \delta_{ij}(1 - \hat{n}_i), \tag{3}$$

where $\hat{n}_i = \sum_\sigma c_{i\sigma}^\dagger c_{i\sigma}$.

Let U be the set of "unblocked levels" (those occupied by either 0 or 2 electrons). Blocked levels (with occupancy 1) do not participate in pairing, thus all the sums in the following are constrained to the set of unblocked levels. We will also use the same symbol U to denote the number of unblocked levels. In the absence of quasiparticles, we have $U = L$.

We introduce

$$B = \frac{1}{\sqrt{U}} \sum_i^U A_i, \tag{4}$$

so that

$$H = -g\frac{U}{L} B^\dagger B = -h B^\dagger B, \tag{5}$$

where we defined the rescaled coupling $h = g\frac{U}{L}$.

Summing (3) over i, j in the set U we obtain

$$[B, B^\dagger] = 1 - \frac{1}{U}\hat{N}, \tag{6}$$

where $\hat{N} = \sum_i^U \hat{n}_i$.

The eigenstates for M Cooper pairs in the U levels are

$$|M\rangle = \mathcal{N}_M (B^\dagger)^M |0\rangle, \tag{7}$$

where $\mathcal{N}_M$ is a normalization factor.

We have $H|0\rangle = 0$ and $H|1\rangle = -h|1\rangle$. General eigenvalues E_M can be computed by recursion:

$$\begin{aligned}
H|M\rangle &= \left(-hB^\dagger B\right)\left(\mathcal{N}_M (B^\dagger)^M\right)|0\rangle \\
&= -h\mathcal{N}_M \left[B^\dagger B B^\dagger (B^\dagger)^{M-1}\right]|0\rangle \\
&= -h\mathcal{N}_M \left[B^\dagger \left(B^\dagger B + 1 - \frac{1}{U}\hat{N}\right)(B^\dagger)^{M-1}\right]|0\rangle \\
&= -h\mathcal{N}_M \left[(B^\dagger)^2 B (B^\dagger)^{M-1} + \left(1 - \frac{2(M-1)}{U}\right)(B^\dagger)^M\right]|0\rangle \\
&= -h\mathcal{N}_M \left[(B^\dagger)^3 B (B^\dagger)^{M-2} + \left(1 - \frac{2(M-2)}{U}\right)(B^\dagger)^M + \left(1 - \frac{2(M-1)}{U}\right)(B^\dagger)^M\right]|0\rangle \\
&= -h\mathcal{N}_M \sum_{m=1}^M \left(1 - \frac{2(m-1)}{U}\right)|0\rangle \\
&= -h\frac{(1+U-M)M}{U}|M\rangle.
\end{aligned} \tag{8}$$

Thus

$$E_M^{L,U} = -h\frac{(1+U-M)M}{U} = -2\alpha\frac{(1+U-M)M}{L}. \tag{9}$$

In DMRG, we use a constant offset of $\tilde{\epsilon} = \alpha d/2 = \alpha/L$ to restore the p-h symmetry of the problem. For M pairs, the total shift is $2M\alpha/L$. The shifted eigenvalues are

$$\tilde{E}_M^{L,U} = -\alpha\frac{(U-M)M}{L}. \tag{10}$$

For half-filling, $N = L$ or $M = L/2$, and one has for $U = L$

$$E_{L/2}^{L,L} = -\frac{g}{2}\left(1+\frac{L}{2}\right) = -\alpha\left(1+\frac{L}{2}\right), \tag{11}$$

or

$$\tilde{E}_{L/2}^{L,L} = -\alpha\frac{L}{2} \equiv \mathcal{E}, \tag{12}$$

Also, we have

$$\begin{aligned}\tilde{E}_{L/2+1}^{L,L} &= \mathcal{E} + \frac{2\alpha}{L},\\ \tilde{E}_{L/2+2}^{L,L} &= \mathcal{E} + \frac{8\alpha}{L},\end{aligned} \tag{13}$$

thus the additional Cooper pairs have an energy cost of order $1/L$.

We now consider a single quasiparticle in the system, i.e., we make one of the L levels singly occupied, so that $U = L-1$. The energy cost in the $N = L+1$ charge sector is thus

$$\left(\tilde{E}_{L/2}^{L,L-1} + \tilde{\epsilon}\right) - \tilde{E}_{L/2}^{L,L} = \left(1+\frac{1}{L}\right)\alpha. \tag{14}$$

and in the $N = L-1$ charge sector it is

$$\left(\tilde{E}_{L/2-1}^{L,L-1} + \tilde{\epsilon}\right) - \tilde{E}_{L/2}^{L,L} = \left(1+\frac{1}{L}\right)\alpha. \tag{15}$$

These expressions are exact and the numerical DMRG results agree perfectly with them.