THE LANGUAGE OF SCHEMES

Contents

SUMMARY

The §§1–8 do little more than develop a language, which will be used in the following. It should be noted, however, that in accordance with the general spirit of this treatise, §§7–8 will be used less than the others, and in a less essential way; we have moreover spoken of Chevalley's schemes only to make the link with the language of Chevalley [CC] and Nagata [Nag58]. The §9 gives definitions and results on quasi-coherent sheaves, some of which are no longer limited to a translation into a "geometric" language of known notions of commutative algebra, but are already of a global nature; they will be indispensable, in the following chapters, for the global study of morphisms. Finally, §10 introduces a generalization of the notion of schemes, which will be used as an intermediary in Chapter III to formulate and prove in a convenient way the fundamental results of the cohomological study of the proper morphisms; moreover, it should be noted that the notion of formal schemes seems indispensable to express certain facts of the "theory of modules" (classification problems of algebraic varieties). The results of §10 will not be used before §3 of Chapter III and it is recommended to omit reading until then.

1. AFFINE SCHEMES

1.1. **The prime spectrum of a ring.**

(1.1.1). *Notation*. Let A be a (commutative) ring, M an A-module. In this chapter and the following, we will constantly use the following notations:

- $\mathrm{Spec}(A)$ = *set of prime ideals* of A, also called the *prime spectrum* of A; for an $x \in X = \mathrm{Spec}(A)$, it will often be convenient to write $\mathfrak{j}_x$ instead of x. When Spec(A) is *empty*, it is necessary and sufficient that the ring A is reduced to 0.
- $A_x = A_{\mathfrak{j}_x}$ = *(local) ring of fractions* $S^{-1}A$, where $S = A - \mathfrak{j}_x$.
- $\mathfrak{m}_x = \mathfrak{j}_x A_{\mathfrak{j}_x}$ = *maximal ideal of* A.
- $k(x) = A_x/\mathfrak{m}_x$ = *residue field of* A_x, canonically isomorphic to the field of fractions of the integral ring $A/\mathfrak{j}_x$, to which it is identified.
- $f(x)$ = *class of* f modulo $\mathfrak{j}_x$ in $A/\mathfrak{j}_x \subset k(x)$, for $f \in A$ and $x \in X$. We still say that $f(x)$ is the *value* of f at a point $x \in \mathrm{Spec}(A)$; the relations $f(x) = 0$ and $f \in \mathfrak{j}_x$ are *equivalent*.
- $M_x = M \otimes_A A_x$ = *module of denominators of fractions in* $A - \mathfrak{j}_x$.
- $\mathfrak{r}(E)$ = *radical of the ideal of* A *generated by a subset* E *of* A.
- $V(E)$ = *set of* $x \in X$ *such that* $E \subset \mathfrak{j}_x$ (or the set of $x \in X$ such that $f(x) = 0$ for all $f \in E$), for $E \subset A$. So we have

(1.1.1.1)
$$\mathfrak{r}(E) = \bigcap_{x \in V(E)} \mathfrak{j}_x.$$

- $V(f) = V(\{f\})$ for $f \in A$.
- $D(f) = X - V(f)$ = *set of* $x \in X$ *where* $f(x) \neq 0$.

Proposition (1.1.2). — *We have the following properties:*

(i) $V(0) = X$, $V(1) = \varnothing$.
(ii) *The relation* $E \subset E'$ *implies* $V(E) \supset V(E')$.
(iii) *For each family* (E_λ) *of subsets of* A, $V(\bigcup_\lambda E_\lambda) = V(\sum_\lambda E_\lambda) = \bigcap_\lambda V(E_\lambda)$.
(iv) $V(EE') = V(E) \cup V(E')$.
(v) $V(E) = V(\mathfrak{r}(E))$.

Proof. The properties (i), (ii), (iii) are trivial, and (v) follows from (ii) and from the formula (1.1.1.1). It is evident that $V(EE') \supset V(E) \cap V(E')$; conversely, if $x \notin V(E)$ and $x \notin V(E')$, there exists $f \in E$ and $f' \in E'$ such that $f(x) \neq 0$ and $f'(x) \neq 0$ in $k(x)$, hence $f(x)f'(x) \neq 0$, i.e., $x \notin V(EE')$, which proves (iv). □

Remark. — Proposition (1.1.2) shows, among other things, that sets of the form $\mathrm{V(E)}$ (where E runs through all the subsets of A) are the *closed sets* of a topology on X, which we will call the *spectral topology*[1]; unless expressly stated otherwise, always assume $\mathrm{X} = \mathrm{Spec(A)}$ is equipped with the spectral topology. 81

(1.1.3). For each subset Y of X, we denote by $\mathfrak{j}(\mathrm{Y})$ the set of $f \in \mathrm{A}$ such that $f(y) = 0$ for all $y \in \mathrm{Y}$; equivalently, $\mathfrak{j}(\mathrm{Y})$ is the intersection of the prime ideals $\mathfrak{j}_y$ for $y \in \mathrm{Y}$. It is clear that the relation $\mathrm{Y} \subset \mathrm{Y}'$ implies that $\mathfrak{j}(\mathrm{Y}) \supset \mathfrak{j}(\mathrm{Y}')$ and that we have

$$\mathfrak{j}\Big(\bigcup_\lambda \mathrm{Y}_\lambda\Big) = \bigcap_\lambda \mathfrak{j}(\mathrm{Y}_\lambda) \tag{1.1.3.1}$$

for each family (Y_λ) of subsets of X. Finally we have

$$\mathfrak{j}(\{x\}) = \mathfrak{j}_x. \tag{1.1.3.2}$$

Proposition (1.1.4). —
(i) *For each subset* E *of* A*, we have* $\mathfrak{j}(\mathrm{V(E)}) = \mathfrak{r}(\mathrm{E})$.
(ii) *For each subset* Y *of* X*,* $\mathrm{V}(\mathfrak{j}(\mathrm{Y})) = \overline{\mathrm{Y}}$*, the closure of* Y *in* X.

Proof. (i) is an immeidate consequence of the definitions and (1.1.1.1); on the other hand, $\mathrm{V}(\mathfrak{j}(\mathrm{Y}))$ is closed and contains Y; conversely, if $\mathrm{Y} \subset \mathrm{V(E)}$, we have $f(y) = 0$ for $f \in \mathrm{E}$ and all $y \in \mathrm{Y}$, so $\mathrm{E} \subset \mathfrak{j}(\mathrm{Y})$, $\mathrm{V(E)} \supset \mathrm{V}(\mathfrak{j}(\mathrm{Y}))$, which proves (ii). ☐

Corollary (1.1.5). — *The closed subsets of* $\mathrm{X} = \mathrm{Spec(A)}$ *and the ideals of* A *equal to their radicals (otherwise the intersection of prime ideals) correspond bijectively by the descent (?) maps* $\mathrm{Y} \mapsto \mathfrak{j}(\mathrm{Y})$, $\mathfrak{a} \mapsto \mathrm{V}(\mathfrak{a})$*; the union* $\mathrm{Y}_1 \cup \mathrm{Y}_2$ *of two closed subsets corresponds to* $\mathfrak{j}(\mathrm{Y}_1) \cap \mathfrak{j}(\mathrm{Y}_2)$*, and the intersection of any family* (Y_λ) *of closed subsets corresponds to the radical of the sum of the* $\mathfrak{j}(\mathrm{Y}_\lambda)$.

Corollary (1.1.6). — *If* A *is a Noetherian ring,* $\mathrm{X} = \mathrm{Spec(A)}$ *is a Noetherian space.*

Note that the converse of this corollary is false, as shown in the example of a non-Noetherian integral ring with a single prime ideal $\neq \{0\}$, for example a non-discrete valuation ring of rank 1.

As an example of ring A whose spectrum is not a Noetherian space, one can consider the ring $\mathscr{C}(\mathrm{Y})$ of continuous real functions on an infinite compact space Y; we know that as a whole, Y corresponds with the set of maximal ideals of A, and it is easy to see that the topology induced on Y by that of $\mathrm{X} = \mathrm{Spec(A)}$ is the initial topology of Y. Since Y is not a Noetherian space, the same is true for X.

Corollary (1.1.7). — *For each* $x \in \mathrm{X}$*, the closure of* $\{x\}$ *is the set of* $y \in \mathrm{X}$ *such that* $\mathfrak{j}_x \subset \mathfrak{j}_y$*. For* $\{x\}$ *to be closed, it is necessary and sufficient that* $\mathfrak{j}_x$ *is maximal.*

Corollary (1.1.8). — *The space* $\mathrm{X} = \mathrm{Spec(A)}$ *is a Kolmogoroff space.*

Proof. If x, y are two distinct points of X, we have either $\mathfrak{j}_x \not\subset \mathfrak{j}_y$ or $\mathfrak{j}_y \not\subset \mathfrak{j}_x$, so one of the points x, y does not belong to the closure of the other. ☐

(1.1.9). According to Proposition (1.1.2), (iv), for two elements f, g of A, we have

$$\mathrm{D}(fg) = \mathrm{D}(f) \cap \mathrm{D}(g). \tag{1.1.9.1}$$

Note also that the relation $\mathrm{D}(f) = \mathrm{D}(g)$ means, according to Proposition (1.1.4), (i) and Proposition (1.1.2), (v) that $\mathfrak{r}(f) = \mathfrak{r}(g)$, or that the minimal prime ideals containing (f) and (g) are the same; in particular, when $f = ug$, where u is invertible.

Proposition (1.1.10). — 82
(i) *When* f *ranges over* A*, the sets* $\mathrm{D}(f)$ *forms a basis for the topology of* X.
(ii) *For each* $f \in \mathrm{A}$*,* $\mathrm{D}(f)$ *is quasi-compact. In particular* $\mathrm{X} = \mathrm{D}(1)$ *is quasi-compact.*

Proof.
(i) Let U be an open set in X; by definition, we have $\mathrm{U} = \mathrm{X} - \mathrm{V(E)}$ where E is a subset of A, and $\mathrm{V(E)} = \bigcap_{f \in \mathrm{E}} \mathrm{V}(f)$, hence $\mathrm{U} = \bigcup_{f \in \mathrm{E}} \mathrm{D}(f)$.
(ii) According to (i), it is sufficient to prove that if $(f_\lambda)_{\lambda \in \mathrm{L}}$ is a family of elements of A such that $\mathrm{D}(f) \subset \bigcup_{\lambda \in \mathrm{L}} \mathrm{D}(f_\lambda)$, there exists a finite subset J of L such that $\mathrm{D}(f) \subset \bigcup_{\lambda \in \mathrm{J}} \mathrm{D}(f_\lambda)$. Let $\mathfrak{a}$ be the ideal of A generated by the f_λ; we have by hypothesis that $\mathrm{V}(f) \supset \mathrm{V}(\mathfrak{a})$, so $\mathfrak{r}(f) \subset \mathfrak{r}(\mathfrak{a})$; as $f \in \mathfrak{r}(f)$, there exists an integer $n \geqslant 0$ such that $f^n \in \mathfrak{a}$. But then f^n belongs to the ideal $\mathfrak{b}$ generated by the finite subfamily $(f_\lambda)_{\lambda \in \mathrm{J}}$, and we have $\mathrm{V}(f) = \mathrm{V}(f^n) \supset \mathrm{V}(\mathfrak{b}) = \bigcap_{\lambda \in \mathrm{J}} \mathrm{V}(f_\lambda)$, that is to say, $\mathrm{D}(f) \supset \bigcup_{\lambda \in \mathrm{J}} \mathrm{D}(f_\lambda)$.

[1] The introduction of this topology in algebraic geometry is due to Zariski. So this topology is usually called the "Zariski topology" of X.

□

Proposition (1.1.11). — *For each ideal $\mathfrak{a}$ of* A*,* Spec(A/$\mathfrak{a}$) *canonically identifies with the closed subspace* V($\mathfrak{a}$) *of* Spec(A)*.*

Proof. We know there is a canonical bijective correspondence (respecting the inclusion order structure) between ideals (resp. prime ideals) of $A/\mathfrak{a}$ and ideals (resp. prime ideals) of A containing $\mathfrak{a}$. □

Recall that the set $\mathfrak{N}$ of nilpotent elements of A (the *nilradical* of A) is an ideal equal to $\mathfrak{r}(0)$, the intersection of all the prime ideals of A (**0**, 1.1.1).

Corollary (1.1.12). — *The topological spaces* Spec(A) *and* Spec(A/$\mathfrak{N}$) *are canonically homeomorphic.*

Proposition (1.1.13). — *For* $X = \mathrm{Spec}(A)$ *to be irreducible (**0**, 2.1.1), it is necessary and sufficient that the ring* $A/\mathfrak{N}$ *is integral (or, equivalently, that the ideal* $\mathfrak{N}$ *is prime).*

Proof. By virtue of Corollary (1.1.12), we can restrict to the case where $\mathfrak{N} = 0$. If X is reducible, then there exist two distinct closed subsets Y_1, Y_2 of X such that $X = Y_1 \cup Y_2$, so $\mathfrak{j}(X) = \mathfrak{j}(Y_1) \cap \mathfrak{j}(Y_2) = 0$, the ideals $\mathfrak{j}(Y_1)$ and $\mathfrak{j}(Y_2)$ being distinct from (0) (1.1.5); so A is not integral. Conversely, if in A there are elements $f \neq 0$, $g \neq 0$ such that $fg = 0$, we have $V(f) \neq X$, $V(g) \neq X$ (since the intersection of the prime ideals of A is (0)), and $X = V(fg) = V(f) \cup V(g)$. □

Corollary (1.1.14). —

(i) *In the bijective correspondence between closed subsets of* $X = \mathrm{Spec}(A)$ *and ideals of* A *equal to their roots, the irreducible closed subsets of* X *correspond to the prime ideals of* A*. In particular, the irreducible components of* X *correspond to the minimal prime ideals of* A*.*

(ii) *The map* $x \mapsto \overline{\{x\}}$ *establishes a bijective correspondence between* X *and the set of closed irreducible subsets of* X *(said otherwise, all closed irreducible subsets of* X *containing only one generic point).*

Proof. (i) follows immediately from (1.1.13) and (1.1.11); and for proving (ii), we can, by virtue of (1.1.11), we restrict to the case where X is irreducible; then, according to Proposition (1.1.13), there exists in A a smaller prime ideal $\mathfrak{N}$, which corresponds to the generic point of X; in addition, X does not admit only one generic point since it is a Kolmogoroff space ((1.1.8) and (**0**, 2.1.3)). □

Proposition (1.1.15). — *If* $\mathfrak{J}$ *is an ideal in* A *containing the radical* $\mathfrak{N}(A)$*, the only neighborhood of* $V(\mathfrak{J})$ *in* $X = \mathrm{Spec}(A)$ *is the whole space* X*.*

Proof. Each maximal ideal of A belongs by definition of $V(\mathfrak{J})$. As each ideal $\mathfrak{a}$ of A is contained in a maximal ideal, we have $V(\mathfrak{a}) \cap V(\mathfrak{J}) \neq 0$, hence the proposition. □

1.2. **Functorial properties of prime spectra of rings.**

(1.2.1). Let A, A′ be two rings,

$$\phi : A' \longrightarrow A$$

a homomorphism of rings. For each prime ideal $x = \mathfrak{j}_x \in \mathrm{Spec}(A) = X$, the ring $A'/\phi^{-1}(\mathfrak{j}_x)$ is canonically isomorphic to a subring of $A/\mathfrak{j}_x$, so it is integral, otherwise we say $\phi^{-1}(\mathfrak{j}_x)$ is a prime ideal of A′; we denote it by ${}^a\phi(x)$, and we have also defined a map

$${}^a\phi : X = \mathrm{Spec}(A) \longrightarrow X' = \mathrm{Spec}(A')$$

(also denoted Spec(ϕ)) we call this the map *associated* to the homomorphism ϕ. We denote by ϕ^x the injective homomorphism of $A'/\phi^{-1}(\mathfrak{j}_x)$ to $A/\mathfrak{j}_x$ induced by ϕ by passing to quotients, so the canonical extention is a monomorphism of fields

$$\phi^x : k({}^a\phi(x)) \longrightarrow k(x);$$

for each $f' \in A'$, we therefore have by definition

$$\phi^x(f'({}^a\phi(x))) = (\phi(f'))(x) \quad (x \in X). \tag{1.2.1.1}$$

Proposition (1.2.2). —

(i) *For each subset* E′ *of* A′*, we have*

$${}^a\phi^{-1}(V(E')) = V(\phi(E')), \tag{1.2.2.1}$$

and in particular, for each $f' \in A'$*,*

$${}^a\phi^{-1}(D(f')) = D(\phi(f')). \tag{1.2.2.2}$$

(ii) *For each ideal* $\mathfrak{a}$ *of* A*, we have*

$$\overline{{}^a\phi(V(\mathfrak{a}))} = V(\phi^{-1}(\mathfrak{a})). \tag{1.2.2.3}$$

Proof. Indeed, the relation ${}^a\phi(x) \in \mathrm{V}(\mathrm{E}')$ is by definition equivalent to $\mathrm{E}' \subset \phi^{-1}(\mathfrak{j}_x)$, so $\phi(\mathrm{E}') \subset \mathfrak{j}_x$, and finally $x \in \mathrm{V}(\phi(\mathrm{E}'))$, hence (i). To prove (ii), we can suppose that $\mathfrak{a}$ is equal to its radical, since $\mathrm{V}(\mathfrak{r}(\mathfrak{a})) = \mathrm{V}(\mathfrak{a})$ ((1.1.2), (v)) and $\phi^{-1}(\mathfrak{r}(\mathfrak{a})) = \mathfrak{r}(\phi^{-1}(\mathfrak{a}))$; the relation $f' \in \mathfrak{a}'$ is by definition equivalent to $f'(x') = 0$ for each $x \in {}^a\phi(\mathrm{Y})$, so, by virtue of the formula (1.2.1.1), it is equivalent as well to $\phi(f')(x) = 0$ for each $x \in \mathrm{Y}$, or $\phi(f') \in \mathfrak{j}(\mathrm{Y}) = \mathfrak{a}$, since $\mathfrak{a}$ is equal to its radical; hence (ii). □

Corollary (1.2.3). — *The map ${}^a\phi$ is continuous.*

We remark that if A″ is a third ring, ϕ' a homomorphism $\mathrm{A}'' \to \mathrm{A}'$, we have ${}^a(\phi' \circ \phi) = {}^a\phi \circ {}^a\phi'$; this result and Corollary (1.2.3) gives that Spec(A) is a *contravariant functor* in A, from the category of rings to that of topological spaces. 84

Corollary (1.2.4). — *Suppose that ϕ is such that for each $f \in \mathrm{A}$ written as $f = h\phi(f')$, where h is invertible in* A *(which is in particular the case when ϕ is surjective). Then ${}^a\phi$ is a homeomorphism from* X *to ${}^a\phi(\mathrm{X})$.*

Proof. We show that for each subset $\mathrm{E} \subset \mathrm{A}$, there exists a subset E′ of A′ such that $\mathrm{V}(\mathrm{E}) = \mathrm{V}(\phi(\mathrm{E}'))$; according to the axiom (T_0) (1.1.8) and the formula (1.2.2.1), this implies first that ${}^a\phi$ is injective, then, according to (1.2.2.1), that ${}^a\phi$ is a homeomorphism. Or, it suffices for each $f \in \mathrm{E}$ to have a $f' \in \mathrm{A}'$ such that $h\phi(f') = f$ with h invertible in A; the set E′ of these elements f' provides the answer. □

(1.2.5). In particular, when ϕ is the canonical homomorphism of A to a ring quotient $\mathrm{A}/\mathfrak{a}$, we get (1.1.12), and ${}^a\phi$ is the *canonical injection* of $\mathrm{V}(\mathfrak{a})$, identifed with $\mathrm{Spec}(\mathrm{A}/\mathfrak{a})$, in $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$.

Another particular case of (1.2.4):

Corollary (1.2.6). — *If* S *is a multiplicative subset of* A*, the spectrum* $\mathrm{Spec}(\mathrm{S}^{-1}\mathrm{A})$ *identifies canonically (with its topology) with the subspace of* $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$ *consisting of the x such that $\mathfrak{j}_x \cap \mathrm{S} = \varnothing$.*

Proof. We know by (**0**, 1.2.6) that the prime ideals of $\mathrm{S}^{-1}\mathrm{A}$ are the ideals $\mathrm{S}^{-1}\mathfrak{j}_x$ such that $\mathfrak{j}_x \cap \mathrm{S} = \varnothing$, and that we have $\mathfrak{j}_x = (i_\mathrm{A}^\mathrm{S})^{-1}(\mathrm{S}^{-1}\mathfrak{j}_x)$. It suffices to apply the i_A^S with Corollary (1.2.4). □

Corollary (1.2.7). — *For ${}^a\phi(\mathrm{X})$ to be dense in* X′*, it is necessary and sufficient that each element of the kernel* $\mathrm{Ker}\,\phi$ *is nilpotent.*

Proof. Applying the formula (1.2.2.3) to the ideal $\mathfrak{a} = (0)$, we have $\overline{{}^a\phi(\mathrm{X})} = \mathrm{V}(\mathrm{Ker}\,\phi)$, and for $\mathrm{V}(\mathrm{Ker}\,\phi) = \mathrm{X}$ to hold, it is necessary and sufficient that $\mathrm{Ker}\,\phi$ is contained in all the prime ideals of A′, that is to say in the nilradical $\mathfrak{r}'$ of A′. □

1.3. **Sheaf associated to a module.**

(1.3.1). Let A be a commutative ring, M an A-module, f an element of A, S_f the multiplicative set of the f^n, where $n \geqslant 0$. Recall that we put $\mathrm{A}_f = \mathrm{S}_f^{-1}\mathrm{A}$, $\mathrm{M}_f = \mathrm{S}_f^{-1}\mathrm{M}$. If S'_f is the saturated multiplicative subset of A consisting of the $g \in \mathrm{A}$ which divide an element of S_f, we know that A_f and M_f canonically identify with ${\mathrm{S}'_f}^{-1}\mathrm{A}$ and ${\mathrm{S}'_f}^{-1}\mathrm{M}$ (**0**, 1.4.3).

Lemma (1.3.2). — *The following conditions are equivalent:*

$$\text{(a) } g \in \mathrm{S}'_f;\ \text{(b) } \mathrm{S}'_g \subset \mathrm{S}'_f;\ \text{(c) } f \in \mathfrak{r}(g);\ \text{(d) } \mathfrak{r}(f) \subset \mathfrak{r}(g);\ \text{(e) } \mathrm{V}(g) \subset \mathrm{V}(f);\ \text{(f) } \mathrm{D}(f) \subset \mathrm{D}(g).$$

This follows immediately from the definitions and from Corollary (1.1.5).

(1.3.3). If $\mathrm{D}(f) = \mathrm{D}(g)$, then Lemma (1.3.2), (b), shows that $\mathrm{M}_f = \mathrm{M}_g$. More generally, if $\mathrm{D}(f) \supset \mathrm{D}(g)$, then $\mathrm{S}'_f \subset \mathrm{S}'_g$, and we know (**0**, 1.4.1) that there exists a canonical functorial homomorphism

$$\rho_{g,f} : \mathrm{M}_f \longrightarrow \mathrm{M}_g,$$

and if $\mathrm{D}(f) \supset \mathrm{D}(g) \supset \mathrm{D}(h)$, we have (**0**, 1.4.4)

$$\rho_{h,g} \circ \rho_{g,f} = \rho_{h,f}. \tag{1.3.3.1}$$

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When f runs over the elements of $\mathrm{A} - \mathfrak{j}_x$ (for a given x in $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$), the sets S'_f constitute an increasing filtered set of subsets of $\mathrm{A} - \mathfrak{j}_x$, since for two elements f, g of $\mathrm{A} - \mathfrak{j}_x$, S'_f and S'_g are contained in S'_{fg}; as the union of the S'_f for $f \in \mathrm{A} - \mathfrak{j}_x$ is $\mathrm{A} - \mathfrak{j}_x$, we conclude (**0**, 1.4.5) that the A_x-module M_x canonically identifies with the *inductive limit* $\varinjlim \mathrm{M}_f$, relative to the family of homomorphisms $(\rho_{g,f})$. We denote by

$$\rho_x^f : \mathrm{M}_f \longrightarrow \mathrm{M}_x$$

the canonical homomorphism for $f \in \mathrm{A} - \mathfrak{j}_x$ (or, equivalently, $x \in \mathrm{D}(f)$).

Definition (1.3.4). — We define the structure sheaf of the prime spectrum $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$ (resp. sheaf associated to an A-module M) and denote it by $\widetilde{\mathrm{A}}$ or $\mathscr{O}_\mathrm{X}$ (resp. $\widetilde{\mathrm{M}}$) as the sheaf of rings (resp. the $\widetilde{\mathrm{A}}$-module) associated to the presheaf $\mathrm{D}(f) \mapsto \mathrm{A}_f$ (resp. $\mathrm{D}(f) \mapsto \mathrm{M}_f$) over the basis $\mathfrak{B}$ of X consisting of the $\mathrm{D}(f)$ for $f \in \mathrm{A}$ ((1.1.10), (**0**, 3.2.1), and (**0**, 3.5.6)).

We saw (**0**, 3.2.4) that the stalk $\widetilde{A}_x$ (resp. $\widetilde{M}_x$) *identifies with the ring* A_x (resp. *the* A_x*-module* M_x); we denote by

$$\theta_f : A_f \longrightarrow \Gamma(D(f), \widetilde{A})$$

$$(\text{resp. } \theta_f : M_f \longrightarrow \Gamma(D(f), \widetilde{M})),$$

the canonical map, so that for each $x \in D(f)$ and each $\xi \in M_f$, we have

$$(\theta_f(\xi))_x = \rho_x^f(\xi). \tag{1.3.4.1}$$

Proposition (1.3.5). — *$\widetilde{M}$ is an exact covariant functor in M, from the category of A-modules to the category of $\widetilde{A}$-modules.*

Proof. Indeed, let M, N be two A-modules, u a homomorphism $M \to N$; for each $f \in A$, it canonically assigns to u a homomorphism u_f of the A_f-module M_f to the A_f-module N_f, and the diagram (for $D(g) \subset D(f)$)

$$\begin{array}{ccc} M_f & \xrightarrow{u_f} & N_f \\ {\scriptstyle \rho_{g,f}}\downarrow & & \downarrow{\scriptstyle \rho_{g,f}} \\ M_g & \xrightarrow{u_g} & N_g \end{array}$$

is commutative (1.4.1); these homomorphisms then define a homomorphism of $\widetilde{A}$-modules $\widetilde{u} : \widetilde{M} \to \widetilde{N}$ (**0**, 3.2.3). In addition, for each $x \in X$, $\widetilde{u}_x$ is the inductive limit of the u_f for $x \in D(f)$ ($f \in A$), and as a result (**0**, 1.4.5), if we canonically identify $\widetilde{M}_x$ and $\widetilde{N}_x$ with M_x and N_x respectively, then $\widetilde{u}_x$ identifies with the homomorphism u_x canonically induced by u. If P is a third A-module, v a homomorphism $N \to P$ and $w = v \circ u$, it is immediate that $w_x = v_x \circ u_x$, so $\widetilde{w} = \widetilde{v} \circ \widetilde{u}$. We have therefore clearly defined a *covariant functor* $\widetilde{M}$ in M, from the category of A-modules to that of $\widetilde{A}$-modules. *This functor is exact*, since for each $x \in X$, M_x is an exact functor in M (**0**, 1.3.2); in addition, we have $\mathrm{Supp}(M) = \mathrm{Supp}(\widetilde{M})$ by the definitions (**0**, 1.7.1) and (**0**, 3.1.6). □

Proposition (1.3.6). — *For each $f \in A$, the open subset $D(f) \subset X$ canonically identifies with the prime spectrum $\mathrm{Spec}(A_f)$, and the sheaf $\widetilde{M_f}$ associated to the A_f-module M_f canonically identifies with the restriction $\widetilde{M}|D(f)$.*

The first assertion is a particular case of (1.2.6). In addition, for $g \in A$ is such that $D(g) \subset D(f)$, M_g canonically identifies with the module of fractions of M_f whose denominators are the powers of the canonical image of g in A_f (**0**, 1.4.6). The canonical identification of $\widetilde{M_f}$ with $\widetilde{M}|D(f)$ then follows from the definitions.

Theorem (1.3.7). — *For each A-module M and each $f \in A$, the homomorphism*

$$\theta_f : M_f \longrightarrow \Gamma(D(f), \widetilde{M})$$

is bijective (in other words, the presheaf $D(f) \mapsto M_f$ is a sheaf). In particular, M identifies with $\Gamma(X, \widetilde{M})$ via θ_1.

Proof. We note that, if $M = A$, θ_f is a homomorphism of structure rings; Theorem (1.3.7) implies then that, if we identify the rings A_f and $\Gamma(D(f), \widetilde{A})$ by means of the θ_f, the homomorphism $\theta_f : M_f \to \Gamma(D(f), \widetilde{M})$ is an isomorphism of *modules*.

We show first that θ_f is *injective*. Indeed, if $\xi \in M_f$ is such that $\theta_f(\xi) = 0$, then this means that for each prime ideal $\mathfrak{p}$ of A_f, there exists $h \notin \mathfrak{p}$ such that $h\xi = 0$; as the annihilator of ξ is not contained in any prime ideal of A_f, each A_f integral, so $\xi = 0$.

It remains to show that θ_f is *surjective*; we can reduce to the case where $f = 1$, the general case deduced by "localizing" using (1.3.6). Now let s be a section of $\widetilde{M}$ over X; according to (1.3.4) and (1.1.10), (ii), there exists a *finite* cover $(D(f_i))_{i \in I}$ of X ($f_i \in A$) such that, for each $i \in I$, the restriction $s_i = s|D(f_i)$ is of the form $\theta_{f_i}(\xi_i)$, where $\xi_i \in M_{f_i}$. If i, j are two indices of I, and if the restrictions of s_i and s_j to $D(f_i) \cap D(f_j) = D(f_i f_j)$ are equal, then it follows by definition of M that

$$\rho_{f_i f_j, f_i}(\xi_i) = \rho_{f_i f_j, f_j}(\xi_j). \tag{1.3.7.1}$$

By definition, we can write, for each $i \in I$, $\xi_i = z_i / f_i^{n_i}$, where $z_i \in M$, and as I is finite, by multiplying each z_i by a power of f_i, we can suppose that all the n_i are equal to the same n. Then, by definition, (1.3.7.1) implies that there exists an integer $m_{ij} \geqslant 0$ such that $(f_i f_j)^{m_{ij}}(f_j^n z_i - f_i^n z_j) = 0$, and we can moreover suppose that the m_{ij} are equal to the same integer m; replacing then the z_i by $f_i^m z_i$, it remains to prove for the case where $m = 0$, in other words, the case where we have

$$f_j^n z_i = f_i^n z_j \tag{1.3.7.2}$$

for any i, j. We have $D(f_i^n) = D(f_i)$, and as the $D(f_i)$ form a cover of X, the ideal generated by the f_i^n is A; in other words, there exist elements $g_i \in A$ such that $\sum_i g_i f_i^n = 1$. Then consider the element $z = \sum_i g_i z_i$ of M; in (1.3.7.2), we have $f_i^n z = \sum_j g_j f_i^n z_j = (\sum_j g_j f_j^n) z_i = z_i$, where by definition $\xi_i = z/1$ in M_{f_i}. We conclude that the s_i are the restrictions to $D(f_i)$ of $\theta_1(z)$, which proves that $s = \theta_1(z)$ and finishes the proof. □

Corollary (1.3.8). — *Let* M, N *be two* A*-modules; the canonical homomorphism* $u \mapsto \widetilde{u}$ *from* $\mathrm{Hom}_{\mathrm{A}}(\mathrm{M},\mathrm{N})$ *to* $\mathrm{Hom}_{\widetilde{\mathrm{A}}}(\widetilde{\mathrm{M}},\widetilde{\mathrm{N}})$ *is bijective. In particular, the relations* $\mathrm{M}=0$ *and* $\widetilde{\mathrm{M}}=0$ *are equivalent.*

Proof. Consider the canonical homomorphism $v \mapsto \Gamma(v)$ from $\mathrm{Hom}_{\widetilde{\mathrm{A}}}(\widetilde{\mathrm{M}},\widetilde{\mathrm{N}})$ to $\mathrm{Hom}_{\Gamma(\widetilde{\mathrm{A}})}(\Gamma(\widetilde{\mathrm{M}}),\Gamma(\widetilde{\mathrm{N}}))$; the latter module canonically identifies with $\mathrm{Hom}_{\mathrm{A}}(\mathrm{M},\mathrm{N})$ according to Theorem (1.3.7). It remains to show that $u \mapsto \widetilde{u}$ and $v \mapsto \Gamma(v)$ are inverses of each other; it is evident that $\Gamma(\widetilde{u}) = u$ by definition of $\widetilde{u}$; on the other hand, if we put $u = \Gamma(v)$ for $v \in \mathrm{Hom}_{\widetilde{\mathrm{A}}}(\widetilde{\mathrm{M}},\widetilde{\mathrm{N}})$, the map $w : \Gamma(\mathrm{D}(f),\widetilde{\mathrm{M}}) \to \Gamma(\mathrm{D}(f),\widetilde{\mathrm{N}})$ canonically induced from v is such that the diagram

$$\begin{tikzcd} \mathrm{M} \arrow[r, "u"] \arrow[d, "\rho_{f,1}"'] & \mathrm{N} \arrow[d, "\rho_{f,1}"] \\ \mathrm{M}_f \arrow[r, "w"] & \mathrm{N}_f \end{tikzcd}$$

is commutative; so we have necessarily that $w = u_f$ for all $f \in \mathrm{A}$ (**0**, 1.2.4), which shows that $\widetilde{\Gamma(v)} = v$. □

Corollary (1.3.9). —

(i) *Let u be a homomorphism from an* A*-module* M *to an* A*-module* N*; then the sheaves associated to* $\mathrm{Ker}\,u$, $\mathrm{Im}\,u$, $\mathrm{Coker}\,u$, *are respectively* $\mathrm{Ker}\,\widetilde{u}$, $\mathrm{Im}\,\widetilde{u}$, $\mathrm{Coker}\,\widetilde{u}$. *In particular, for $\widetilde{u}$ to be injective (resp. surjective, bijective), it is necessary and sufficient that u is.*

(ii) *If* M *is an inductive limit (resp. direct sum) of a family of* A*-modules* (M_λ), $\widetilde{\mathrm{M}}$ *is the inductive limit (resp. direct sum) of the family* $(\widetilde{\mathrm{M}_\lambda})$, *via a canonical isomorphism.*

Proof.

(i) If suffices to apply the fact that $\widetilde{\mathrm{M}}$ is an exact functor in M (1.3.5) to the two exact sequences of A-modules

$$0 \longrightarrow \mathrm{Ker}\,u \longrightarrow \mathrm{M} \longrightarrow \mathrm{Im}\,u \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{Im}\,u \longrightarrow \mathrm{N} \longrightarrow \mathrm{Coker}\,u \longrightarrow 0.$$

The second assertion then follows from Theorem (1.3.7).

(ii) Let $(\mathrm{M}_\lambda, g_{\mu\lambda})$ be an inductive system of A-modules, with inductive limit M, and let g_λ be the canonical homomorphism $\mathrm{M}_\lambda \to \mathrm{M}$. As we have $\widetilde{g_{\nu\mu}} \circ \widetilde{g_{\mu\lambda}} = \widetilde{g_{\nu\lambda}}$ and $\widetilde{g_\lambda} = \widetilde{g_\mu} \circ \widetilde{g_{\mu\lambda}}$ for $\lambda \leqslant \mu \leqslant \nu$, $(\widetilde{\mathrm{M}_\lambda}, \widetilde{g_{\mu\lambda}})$ is an inductive system of sheaves on X, and if we denote by h_λ the canonical homomorphism $\widetilde{\mathrm{M}_\lambda} \to \varinjlim \widetilde{\mathrm{M}_\lambda}$, there is a unique homomorphism $v : \varinjlim \widetilde{\mathrm{M}_\lambda} \to \widetilde{\mathrm{M}}$ such that $v \circ h_\lambda = \widetilde{g_\lambda}$. To see that v is bijective, it suffices to check, for each $x \in \mathrm{X}$, that v_x is a bijection from $(\varinjlim \widetilde{\mathrm{M}_\lambda})_x$ to $\widetilde{\mathrm{M}}_x$; but $\widetilde{\mathrm{M}}_x = \mathrm{M}_x$, and

$$(\varinjlim \widetilde{\mathrm{M}_\lambda})_x = \varinjlim (\widetilde{\mathrm{M}_\lambda})_x = \varinjlim (\mathrm{M}_\lambda)_x = \mathrm{M}_x \quad (\mathbf{0},\ 1.3.3).$$

Conversely, it follows from the definitions that $(\widetilde{g_\lambda})_x$ and (h_λ) are all equal to the canonical map from $(\mathrm{M}_\lambda)_x$ to M_x; as $(\widetilde{g_\lambda})_x = v_x \circ (h_\lambda)_x$, v_x is the identity. 88

Finally, if M is the direct sum of two A-modules N, P, it is immediate that $\widetilde{\mathrm{M}} = \widetilde{\mathrm{N}} \oplus \widetilde{\mathrm{P}}$; each direct sum being the inductive limit of finite direct sums, the assertions of (ii) are proved.

□

We note that Corollary (1.3.8) proves that the sheaves isomorphic to the associated sheaves of A-modules forms an *abelian category* (T, I, 1.4).

We also note that it follows from Corollary (1.3.9) that if M is an A-module *of finite type*, that is to say there exists a surjective homomorphism $\mathrm{A}^n \to \mathrm{M}$, then there exists a surjective homomorphism $\widetilde{\mathrm{A}^n} \to \widetilde{\mathrm{M}}$, in other words, the $\widetilde{\mathrm{A}}$-module $\widetilde{\mathrm{M}}$ is *generated by a finite family of sections over* X (**0**, 5.1.1), and conversely.

(1.3.10). If N is a submodule of an A-module M, the canonical injection $j : \mathrm{N} \to \mathrm{M}$ gives by (1.3.9) an injective homomorphism $\widetilde{\mathrm{N}} \to \widetilde{\mathrm{M}}$, which allows us to canonically identify $\widetilde{\mathrm{N}}$ with a $\widetilde{\mathrm{A}}$-*submodule* of $\widetilde{\mathrm{M}}$; we will always assume we have made this identification. If N and P are two submodules of M, we then have

$$(\mathrm{N}+\mathrm{P})^{\sim} = \widetilde{\mathrm{N}} + \widetilde{\mathrm{P}}, \tag{1.3.10.1}$$

$$(\mathrm{N}\cap\mathrm{P})^{\sim} = \widetilde{\mathrm{N}} \cap \widetilde{\mathrm{P}}, \tag{1.3.10.2}$$

since $\mathrm{N}+\mathrm{P}$ and $\mathrm{N}\cap\mathrm{P}$ are respectively the images of the canonical homomorphism $\mathrm{N}\oplus\mathrm{P} \to \mathrm{M}$, and the kernel of the canonical homomorphism $\mathrm{M} \to (\mathrm{M}/\mathrm{N}) \oplus (\mathrm{M}/\mathrm{P})$, and it suffices to apply (1.3.9).

We conclude from (1.3.10.1) and (1.3.10.2) that if $\widetilde{\mathrm{N}} = \widetilde{\mathrm{P}}$, we have $\mathrm{N} = \mathrm{P}$.

Corollary (1.3.11). — *On the category of sheaves isomorphic to the associated sheaves of* A*-modules, the functor* Γ *is exact.*

Proof. Let $\widetilde{M} \xrightarrow{\widetilde{u}} \widetilde{N} \xrightarrow{\widetilde{v}} \widetilde{P}$ be an exact sequence corresponding to two homomorphisms $u : M \to N$, $v : N \to P$ of A-modules. If $Q = \operatorname{Im} u$ and $R = \operatorname{Ker} v$, we have $\widetilde{Q} = \operatorname{Im} \widetilde{u} = \operatorname{Ker} \widetilde{v} = \widetilde{R}$ (Corollary (1.3.9)), hence $Q = R$. □

Corollary (1.3.12). — *Let* M, N *be two* A*-modules.*

(i) *The sheaf associated to* $M \otimes_A N$ *canonically identifies with* $\widetilde{M} \otimes_{\widetilde{A}} \widetilde{N}$.

(ii) *If in addition* M *admits a finite presentation, the sheaf associated to* $\operatorname{Hom}_A(M, N)$ *canonically identifies with* $\mathscr{H}om_{\widetilde{A}}(\widetilde{M}, \widetilde{N})$.

Proof.

(i) The sheaf $\mathscr{F} = \widetilde{M} \otimes_{\widetilde{A}} \widetilde{N}$ is associated to the presheaf

$$U \longmapsto \mathscr{F}(U) = \Gamma(U, \widetilde{M}) \otimes_{\Gamma(U,\widetilde{A})} \Gamma(U, \widetilde{N}),$$

U varying over the basis (1.1.10), (i) of X consisting of the $D(f)$, where $f \in A$. We have that $\mathscr{F}(D(f))$ canonically identifies with $M_f \otimes_{A_f} N_f$ according to (1.3.7) and (1.3.6). Moreover, we have that the A_f-module $M_f \otimes_{A_f} N_f$ is canonically isomorphic to $(M \otimes_A N)_f$ (**0**, 1.3.4), which itself is canonically isomorphic to $\Gamma(D(f), (M \otimes_A N)^\sim)$ ((1.3.7) and (1.3.6)). In addition, we check immediately that the canonical isomorphisms

$$\mathscr{F}(D(f)) \xrightarrow{\sim} \Gamma(D(f), (M \otimes_A N)^\sim)$$

thus obtained satisfy the compatibility conditions with respect to the restriction operations (**0**, 1.4.2), so they define canonical functorial isomorphism

$$\widetilde{M} \otimes_{\widetilde{A}} \widetilde{N} \xrightarrow{\sim} (M \otimes_A N)^\sim.$$

(ii) The sheaf $\mathscr{G} = \mathscr{H}om_{\widetilde{A}}(\widetilde{M}, \widetilde{N})$ is associated to the presheaf

$$U \longmapsto \mathscr{G}(U) = \operatorname{Hom}_{\widetilde{A}|U}(\widetilde{M}|U, \widetilde{N}|U),$$

U varying over the basis of X consisting of the $D(f)$. We have that $\mathscr{G}(D(f))$ canonically identifies with $\operatorname{Hom}_{A_f}(M_f, N_f)$ (Proposition (1.3.6) and Corollary (1.3.8)), which, according to the hypotheses on M, canonically identifies with $(\operatorname{Hom}_A(M, N))_f$ (**0**, 1.3.5). Finally, $(\operatorname{Hom}_A(M, N))_f$ canonically identifies with $\Gamma(D(f), (\operatorname{Hom}_A(M, N))^\sim)$ (Proposition (1.3.6) and Theorem (1.3.7)), and the canonical isomorphisms $\mathscr{G}(D(f)) \xrightarrow{\sim} \Gamma(D(f), (\operatorname{Hom}_A(M, N))^\sim)$ thus obtained are compatible with the restriction operations (**0**, 1.4.2); they thus define a canonical isomorphism $\mathscr{H}om_{\widetilde{A}}(\widetilde{M}, \widetilde{N}) \xrightarrow{\sim} (\operatorname{Hom}_A(M, N))^\sim$.

□

(1.3.13). Now let B be a (commutative) A-algebra; this can be interpreted by saying that B is an A-module such that we are given an element $e \in B$ and an A-homomorphism $\phi : B \otimes_A B \to B$, so that: 1st the diagrams

$$\begin{tikzcd} B \otimes_A B \otimes_A B \arrow[r, "\phi\otimes 1"] \arrow[d, "1\otimes\phi"'] & B \otimes_A B \arrow[d, "\phi"] \\ B \otimes_A B \arrow[r, "\phi"] & B \end{tikzcd} \qquad \begin{tikzcd} B \otimes_A B \arrow[rr, "\sigma"] \arrow[dr, "\phi"'] & & B \otimes_A B \arrow[dl, "\phi"] \\ & B & \end{tikzcd}$$

(σ the canonical symmetry map) are commutative; 2nd $\phi(e \otimes x) = \phi(x \otimes e) = x$. According to Corollary (1.3.12), the homomorphism $\widetilde{\phi} : \widetilde{B} \otimes_{\widetilde{A}} \widetilde{B} \to \widetilde{B}$ of $\widetilde{A}$-modules satisfies the analogous conditions, thus defines an $\widetilde{A}$-*algebra* structure on $\widetilde{B}$. In a similar way, the data of a B-module N is the same as the data of an A-module N and an A-homomorphism $\psi : B \otimes_A N \to N$ such that the diagram

$$\begin{tikzcd} B \otimes_A B \otimes_A B \arrow[r, "\phi\otimes 1"] \arrow[d, "1\otimes\psi"'] & B \otimes_A N \arrow[d, "\psi"] \\ B \otimes_A N \arrow[r, "\psi"] & N \end{tikzcd}$$

is commutative and $\psi(e \otimes n) = n$; the homomorphism $\widetilde{\psi} : \widetilde{B} \otimes_{\widetilde{A}} \widetilde{N} \to \widetilde{N}$ satisfies the analogous condition, and so defines a $\widetilde{B}$-*module* structure on $\widetilde{N}$.

In a similar way, we see that if $u : B \to B'$ (resp. $v : N \to N'$) is a homomorphism of A-algebras (resp. of B-modules), $\widetilde{u}$ (resp. $\widetilde{v}$) is a homomorphism of $\widetilde{A}$-algebras (resp. of $\widetilde{B}$-modules), $\operatorname{Ker} \widetilde{u}$ is a $\widetilde{B}$-ideal (resp. $\operatorname{Ker} \widetilde{v}$, $\operatorname{Coker} \widetilde{v}$, and $\operatorname{Im} \widetilde{v}$ are $\widetilde{B}$-modules). If N is a B-module, $\widetilde{N}$ is a $\widetilde{B}$-module of finite type if and only if N is a B-module of finite type (**0**, 5.2.3).

If M, N are two B-modules, the $\widetilde{B}$-module $\widetilde{M} \otimes_{\widetilde{B}} \widetilde{N}$ canonically identifies with $(M \otimes_B N)^\sim$; similarly $\mathscr{H}om_{\widetilde{B}}(\widetilde{M}, \widetilde{N})$ canonically identifies with $(\mathrm{Hom}_B(M, N))^\sim$ when M admits a finite presentation; the proofs are similar to those in Corollary (1.3.12).

If $\mathfrak{J}$ is an ideal of B, N a B-module, then we have $(\mathfrak{J}N)^\sim = \widetilde{\mathfrak{J}} \cdot \widetilde{N}$.

Finally, if B is an A-algebra *graded* by the A-submodules B_n ($n \in \mathbf{Z}$), the $\widetilde{A}$-algebra $\widetilde{B}$, the direct sum of the $\widetilde{A}$-modules $\widetilde{B_n}$ (1.3.9), is graded by these $\widetilde{A}$-submodules, the axiom of graded algebras giving that the image of the homomorphism $B_m \otimes B_n \to B$ is contained in B_{m+n}. Similarly, if M is a B-module graded by the submodules M_n, then $\widetilde{M}$ is a $\widetilde{B}$-module graded by the $\widetilde{M_n}$.

(1.3.14). If B is an A-algebra, M a submodule of B, then the $\widetilde{A}$-subalgebra of $\widetilde{B}$ generated by $\widetilde{M}$ (**0**, 4.1.3) is the $\widetilde{A}$-subalgebra $\widetilde{C}$, where we denote by C the subalgebra of B generated by M. Indeed, C is the direct sum of the submodules of B which are the images of the homomorphisms $\bigotimes^n M \to B$ ($n \geqslant 0$), and it suffices to apply (1.3.9) and (1.3.12).

1.4. **Quasi-coherent sheaves over a prime spectrum.**

Theorem (1.4.1). — *Let* X *be the prime spectrum of a ring* A*,* V *a quasi-compact open subset of* X*, and* $\mathscr{F}$ *an* $(\mathscr{O}_X|V)$*-module. The four following conditions are equivalent:*

(a) *There exists an* A*-module* M *such that* $\mathscr{F}$ *is isomorphic to* $\widetilde{M}|V$.

(b) *There exists a finite open cover* (V_i) *of* V *by sets of the form* $D(f_i)$ *(*$f_i \in A$*) contained in* V*, such that, for each* i*,* $\mathscr{F}|V_i$ *is isomorphic to a sheaf of the form* $\widetilde{M_i}$*, where* M_i *is an* A_{f_i}*-module.*

(c) *The sheaf* $\mathscr{F}$ *is quasi-coherent (**0**, 5.1.3).*

(d) *The two following properties are satisfied:*

  (d1) *For each* $f \in A$ *such that* $D(f) \subset V$ *and for each section* $s \in \Gamma(D(f), \mathscr{F})$*, there exists an integer* $n \geqslant 0$ *such that* $f^n s$ *extends to a section of* $\mathscr{F}$ *over* V.

  (d2) *For each* $f \in A$ *such that* $D(f) \subset V$ *and for each section* $t \in \Gamma(V, \mathscr{F})$ *such that the restriction of* t *to* $D(f)$ *is* 0*, there exists an integer* $n \geqslant 0$ *such that* $f^n t = 0$.

(In the statement of the conditions (d1) and (d2), we have tacitly identified A and $\Gamma(\widetilde{A})$ according to Theorem (1.3.7)).

Proof. The fact that (a) implies (b) is an immediate consequence of Proposition (1.3.6) and the fact that the $D(f_i)$ form a basis for the topology of X (1.1.10). As each A-module is isomorphic to the cokernel of a homomorphism of the form $A^{(I)} \to A^{(J)}$, (1.3.6) proves that each sheaf associated to an A-module is quasi-coherent; so (b) implies (c). Conversely, if $\mathscr{F}$ is quasi-coherent, each $x \in V$ has a neighborhood of the form $D(f) \subset V$ such that $\mathscr{F}|D(f)$ is isomorphic to the cokernel of a homomorphism $\widetilde{A_f}^{(I)} \to \widetilde{A_f}^{(J)}$, so a sheaf $\widetilde{N}$ associated to the module N, the cokernel of the corresponding homomorphism $A_f^{(I)} \to A_f^{(J)}$ (Corollaries (1.3.8) and (1.3.9)); as V is quasi-compact, it is clear that (c) implies (b). 91

To prove that (b) implies (d1) and (d2), we first assume that $V = D(g)$ for a $g \in A$, and that $\mathscr{F}$ is isomorphic to the sheaf $\widetilde{N}$ associated to an A_g-module N; by replacing X with V and A with A_g (1.3.6), we can reduce to the case where $g = 1$. Then $\Gamma(D(f), \widetilde{N})$ and N_f are canonically identified (Proposition (1.3.6) and Theorem (1.3.7)), so a section $s \in \Gamma(D(f), \widetilde{N})$ identifies with an element of the form z/f^n, where $z \in N$; the section $f^n s$ identifies with the element $z/1$ of N_f and as a result the restriction to $D(f)$ of a section of $\widetilde{N}$ over X identifies with the element $z \in N$; hence (d1) in this case. Similarly, $t \in \Gamma(X, \widetilde{N})$ is identified with an element $z' \in N$, the restriction of t to $D(f)$ is identified with the image $z'/1$ of z' in N_f, and we say that this image is zero means that there exists an $n \geqslant 0$ such that $f^n z' = 0$ in N, or, equivalently, $f^n t = 0$.

To finish the proof that (b) implies (d1) and (d2), it suffices to establish the following lemma:

Lemma (1.4.1.1). — *Suppose that* V *is the finite union of sets of the form* $D(g_i)$*, and that each of the sheaves* $\mathscr{F}|D(g_i)$*,* $\mathscr{F}|(D(g_i) \cap D(g_j)) = \mathscr{F}|D(g_i g_j)$ *satisfy (d1) and (d2); then* $\mathscr{F}$ *has the following two properties:*

(d′1) *For each* $f \in A$ *and for each section* $s \in \Gamma(D(f) \cap V, \mathscr{F})$*, there exists an integer* $n \geqslant 0$ *such that* $f^n s$ *extends to a section of* $\mathscr{F}$ *over* V.

(d′2) *For each* $f \in A$ *and for each section* $t \in \Gamma(V, \mathscr{F})$ *such that the restriction of* t *to* $D(f) \cap V$ *is* 0*, there exists an integer* $n \geqslant 0$ *such that* $f^n t = 0$.

We first prove (d′2): as $D(f) \cap D(g_i) = D(f g_i)$, there exists for each i an integer n_i such that the restriction of $(f g_i)^{n_i} t$ to $D(g_i)$ is zero: as the image of g_i in A_{g_i} is invertible, the restriction of $f^{n_i} t$ to $D(g_i)$ is also zero; taking for n the largest of the n_i, we have proved (d′2).

To show (d′1), we apply (d1) to the sheaf $\mathscr{F}|D(g_i)$: there exists an integer $n_i \geqslant 0$ and a section s_i' of $\mathscr{F}$ over $D(g_i)$ extending the restriction of $(f g_i)^{n_i} s$ to $D(f g_i)$; as the image of g_i in A_{g_i} is invertible, there is a section s_i of $\mathscr{F}$ over $D(g_i)$ such that $s_i' = g_i^{n_i} s_i$, and s_i extends the restriction of $f^{n_i} s$ to $D(f g_i)$; in addition we can suppose that all the n_i are equal to the same integer n. By construction, the restriction of $s_i - s_j$ to $D(f) \cap D(g_i) \cap D(g_j) = D(f g_i g_j)$ is zero; according to (d2) applied to the sheaf $\mathscr{F}|D(g_i g_j)$, there exists an integer $m_{ij} \geqslant 0$ such that the restriction to $D(g_i g_j)$ of $(f g_i g_j)^{m_{ij}}(s_i - s_j)$ is zero; as the image of $g_i g_j$ in $A_{g_i g_j}$ is invertible, the restriction of $f^{m_{ij}}(s_i - s_j)$ to $D(g_i g_j)$ is zero. We can then assume that all the m_{ij} are equal to the same integer m, and so there exists a section $s' \in \Gamma(V, \mathscr{F})$ extending the $f^m s_i$; as a result, this section extends $f^{n+m}s$, hence we have proved (d′1).

It remains to prove that (d1) and (d2) imply (a). We show first that (d1) and (d2) imply that these conditions are satisfied for each sheaf $\mathscr{F}|D(g)$, where $g \in A$ is such that $D(g) \subset V$. It is evident for (d1); on the other hand, if $t \in \Gamma(D(g), \mathscr{F})$ is such that its restriction to $D(f) \subset D(g)$ is zero, there exists by (d1) an integer $m \geqslant 0$ such that $g^m t$ extends to a section s of $\mathscr{F}$ over V; applying (d2), we see that there exists an integer $n \geqslant 0$ such that $f^n g^m t = 0$, and as the image of g in A_g is invertible, $f^n t = 0$.

That being so, as V is quasi-compact, Lemma (1.4.1.1) proves that the conditions (d′1) and (d′2) are satisfied. Consider then the A-module $M = \Gamma(V, \mathscr{F})$, and define a homomorphism of $\widetilde{A}$-modules $u : \widetilde{M} \to j_*(\mathscr{F})$, where j is the canonical injection $V \to X$. As the $D(f)$ form a basis for the topology of X, it suffices, for each $f \in A$, to define a homomorphism $u_f : M_f \to \Gamma(D(f), j_*(\mathscr{F})) = \Gamma(D(f) \cap V, \mathscr{F})$, with the usual compatibility conditions (**0**, 3.2.5). As the canonical image of f in A_f is invertible, the restriction homomorphism $M = \Gamma(V, \mathscr{F}) \to \Gamma(D(f) \cap V, \mathscr{F})$ factorizes as $M \to M_f \xrightarrow{u_f} \Gamma(D(f) \cap V, \mathscr{F})$ (**0**, 1.2.4), and the verfication of these compatibility conditions for $D(g) \subset D(f)$ is immediate. This being so, we show that the condition (d′1) (resp. (d′2)) implies that each of the u_f are surjective (resp. injective), which proves that u is *bijective*, and as a result that $\mathscr{F}$ is the restriction to V of an $\widetilde{A}$-module isomorphic to $\widetilde{M}$. If $s \in \Gamma(D(f) \cap V, \mathscr{F})$, there exists according to (d′1) an integer $n \geqslant 0$ such that $f^n s$ extends to a section $z \in M$; we then have $u_f(z/f^n) = s$, so u_f is surjective. Similarly, if $z \in M$ is such that $u_f(z/1) = 0$, this means that the restriction to $D(f) \cap V$ of the section z is zero; according to (d′2), there exists an integer $n \geqslant 0$ such that $f^n z = 0$, hence $z/1 = 0$ in M_f, and therefore u_f is injective.

Q.E.D.

□

Corollary (1.4.2). — *Each quasi-coherent sheaf over a quasi-compact open subset of* X *is induced by a quasi-coherent sheaf on* X.

Corollary (1.4.3). — *Each quasi-coherent $\mathscr{O}_X$-algebra over $X = \mathrm{Spec}(A)$ is isomorphic to an $\mathscr{O}_X$-algebra of the form $\widetilde{B}$, where* B *is an algebra over* A*; each quasi-coherent $\widetilde{B}$-module is isomorphic to a $\widetilde{B}$-module of the form $\widetilde{N}$, where* N *is a* B*-module.*

Proof. Indeed, a quasi-coherent $\mathscr{O}_X$-algebra is a quasi-coherent $\mathscr{O}_X$-module, therefore of the form $\widetilde{B}$, where B is an A-module; the fact that B is an A-algebra follows from the characterization of the structure of an $\mathscr{O}_X$-algebra using the homomorphism $\widetilde{B} \otimes_{\widetilde{A}} \widetilde{B} \to \widetilde{B}$ of $\widetilde{A}$-modules, as well as Corollary (1.3.12). If $\mathscr{G}$ is a quasi-coherent $\widetilde{B}$-module, it suffices to show, in a similar way, that it is also a quasi-coherent $\widetilde{A}$-module to conclude the proof; as the question is local, we can, by restricting to an open subset of X of the form $D(f)$, assume that $\mathscr{G}$ is the cokernel of a homomorphism $\widetilde{B}^{(I)} \to \widetilde{B}^{(J)}$ of $\widetilde{B}$-modules (and *a fortiori* of $\widetilde{A}$-modules); the proposition then follows from Corollaries (1.3.8) and (1.3.9). □

1.5. **Coherent sheaves over a prime spectrum.**

Theorem (1.5.1). — *Let* A *be a Noetherian ring,* $X = \mathrm{Spec}(A)$ *its prime spectrum,* V *an open subset of* X*, and* $\mathscr{F}$ *an* $(\mathscr{O}_X|V)$*-module. The following conditions are equivalent:*

(a) *$\mathscr{F}$ is coherent.*
(b) *$\mathscr{F}$ is of finite type and quasi-coherent.*
(c) *There exists an* A*-module* M *of finite type such that $\mathscr{F}$ is isomorphic to the sheaf $\widetilde{M}|V$.*

Proof. (a) trivially implies (b). To see the (b) implies (c), we have previously seen, since V is quasi-compact (**0**, 2.2.3), that $\mathscr{F}$ is isomorphic to a sheaf $\widetilde{N}|V$, where N is an A-module (1.4.1). We have $N = \varinjlim M_\lambda$, where M_λ vary over the set of A-submodules of N of finite type, hence (1.3.9) $\mathscr{F} = \widetilde{N}|V = \varinjlim \widetilde{M_\lambda}|V$; but as $\mathscr{F}$ is of finite type, and V is quasi-compact, there exists an index λ such that $\mathscr{F} = \widetilde{M_\lambda}|V$ (**0**, 5.2.3).

Finally, we show that (c) implies (a). It is clear that $\mathscr{F}$ is then of finite type ((1.3.6) and (1.3.9)); in addition, the question being local, we can reduce to the case where $V = D(f)$, $f \in A$. As A_f is Noetherian, we see finally that it reduces to proving that the kernel of a homomorphism $\widetilde{A^n} \to \widetilde{M}$, where M is an A-module, is of finite type. Such a homomorphism is of the form $\widetilde{u}$, where u is a homomorphism $A^n \to M$ (1.3.8), and if $P = \mathrm{Ker}\, u$, we have $\widetilde{P} = \mathrm{Ker}\, \widetilde{u}$ (1.3.9). As A is Noetherian, P is of finite type, which finishes the proof. □

Corollary (1.5.2). — *Under the hypotheses of (1.5.1), the sheaf $\mathscr{O}_X$ is a quasi-coherent sheaf of rings.*

Corollary (1.5.3). — *Under the hypotheses of (1.5.1), each coherent sheaf over an open subset of* X *is induced by a coherent sheaf on* X.

Corollary (1.5.4). — *Under the hypotheses of (1.5.1), each quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ is the inductive limit of the coherent $\mathscr{O}_X$-submodules of $\mathscr{F}$.*

Proof. Indeed, $\mathscr{F} = \widetilde{M}$ where M is an A-module, and M is the inductive limit of its submodules of finite type; we conclude the proof by (1.3.9) and (1.5.1). □

1.6. **Functorial properties of quasi-coherent sheaves over a prime spectrum.**

(1.6.1). Let A, A′ be two rings,

$$\phi : A' \to A$$

a homomorphism,

$$^a\phi : X = \mathrm{Spec}(A) \longrightarrow X' = \mathrm{Spec}(A')$$

the continuous map associated to ϕ (1.2.1). We will define a *canonical homomorphism*

$$\widetilde{\phi} : \mathscr{O}_{X'} \longrightarrow {}^a\phi_*(\mathscr{O}_X)$$

of sheaves of rings. For each $f' \in A'$, we put $f = \phi(f')$; we have $^a\phi^{-1}(D(f')) = D(f)$ (1.2.2.2). The rings $\Gamma(D(f'), \widetilde{A'})$ and $\Gamma(D(f), \widetilde{A})$ identify respectively with $A'_{f'}$ and A_f ((1.3.6) and (1.3.7)). The homomorphism ϕ canonically defines a homomorphism $\phi_{f'} : A'_{f'} \to A_f$ (**0**, 1.5.1), in other words, we have a homomorphism of rings

$$\Gamma(D(f), \widetilde{A'}) \longrightarrow \Gamma({}^a\phi^{-1}(D(f')), \widetilde{A}) = \Gamma(D(f'), {}^a\phi_*(\widetilde{A}))$$

In addition, these homomorphism satisfy the usual compatibility conditions: for $D(f') \supset D(g')$, the diagram 94

$$\begin{tikzcd}
\Gamma(D(f'), \widetilde{A'}) \arrow[r] \arrow[d] & \Gamma(D(f'), {}^a\phi_*(\widetilde{A})) \arrow[d] \\
\Gamma(D(g'), \widetilde{A'}) \arrow[r] & \Gamma(D(g'), {}^a\phi_*(\widetilde{A}))
\end{tikzcd}$$

is commutative (**0**, 1.5.1); we have thus defined a homomorphism of $\mathscr{O}_{X'}$-algebras, as the $D(f')$ form a basis for the topology og X′ (**0**, 3.2.3). The pair $\Phi = ({}^a\phi, \widetilde{\phi})$ is thus a *morphism* of ringed spaces

$$\Phi : (X, \mathscr{O}_X) \longrightarrow (X', \mathscr{O}_{X'}),$$

(**0**, 4.1.1).

We note further that, if we put $x' = {}^a\phi(x)$, then the homomorphism $\widetilde{\phi}^\sharp_x$ (**0**, 3.7.1) is none other than the homomorphism

$$\phi_x : A'_{x'} \longrightarrow A_x$$

canonically induced by $\phi : A' \to A$ (**0**, 1.5.1). Indeed, each $z' \in A'_{x'}$ can be written as g'/f', where f', g' are in A′ and $f' \notin \mathfrak{j}_{x'}$; $D(f')$ is then a neighborhood of x' in X′, and the homomorphism $\Gamma(D(f'), \widetilde{A'}) \to \Gamma({}^a\phi^{-1}(D(f')), \widetilde{A})$ induced by $\widetilde{\phi}$ is none other than $\phi_{f'}$; by considering the section $s' \in \Gamma(D(f'), \widetilde{A'})$ corresponding to $g'/f' \in A'_{f'}$, we obtain $\widetilde{\phi}^\sharp_x(z') = \phi(g')/\phi(f')$ in A_x.

Example (1.6.2). — Let S be a multiplicative subset of A, ϕ the canonical homomorphism $A \to S^{-1}A$; we have already seen (1.2.6) that $^a\phi$ is a *homeomorphism* from $Y = \mathrm{Spec}(S^{-1}A)$ to the subspace of $X = \mathrm{Spec}(A)$ consisting of the x such that $\mathfrak{j}_x \cap S = \varnothing$. In addition, for each x in this subspace, thus of the form $^a\phi(y)$ with $y \in Y$, the homomorphism $\widetilde{\phi}^\sharp_y : \mathscr{O}_x \to \mathscr{O}_y$ is *bijective* (**0**, 1.2.6); in other words, $\mathscr{O}_Y$ identifies with the sheaf on Y induced by $\mathscr{O}_X$.

Proposition (1.6.3). — *For each* A*-module* M*, there exists a canonical functorial isomorphism from the $\mathscr{O}_{X'}$-module* $(M_{[\phi]})^\sim$ *to the direct image* $\Phi_*(\widetilde{M})$*.*

Proof. For purposes of abbreviation, we set $M' = M_{[\phi]}$, and for each $f' \in A'$, we put $f = \phi(f')$. The modules of sections $\Gamma(D(f'), \widetilde{M'})$ and $\Gamma(D(f), \widetilde{M})$ identify respectively with the modules $M'_{f'}$ and M_f (over $A'_{f'}$ and A_f, respectively); in addition, the $A'_{f'}$-module $(M_f)_{[\phi_{f'}]}$ is canonically isomorphic to $M'_{f'}$ (**0**, 1.5.2). We thus have a functorial isomorphism of $\Gamma(D(f'), \widetilde{A'})$-modules: $\Gamma(D(f'), \widetilde{M'}) \xrightarrow{\sim} \Gamma({}^a\phi^{-1}(D(f')), \widetilde{M})_{[\phi_{f'}]}$ and these isomorphisms satisfy the usual compatibility conditions with the restrictions (**0**, 1.5.6), thus defining the desired functorial isomorphism. We note that, in a precise way, if $u : M_1 \to M_2$ is a homomorphism of A-modules, it can be considered as a homomorphism $(M_1)_{[\phi]} \to (M_2)_{[\phi]}$ of A′-modules; if we denote by $u_{[\phi]}$ this homomorphism, $\Phi_*(\widetilde{u})$ identifies with $(u_{[\phi]})^\sim$. □

This proof also shows that for each A-*algebra* B, the canonical functorial isomorphism $(B_{[\phi]})^\sim \xrightarrow{\sim} \Phi_*(\widetilde{B})$ is an isomorphism 95
of $\mathscr{O}_{X'}$-*algebras*; if M is a B-module, the canonical functorial isomorphism $(M_{[\phi]})^\sim \xrightarrow{\sim} \Phi_*(\widetilde{M})$ is an isomorphism of $\Phi_*(\widetilde{B})$-modules.

Corollary (1.6.4). — *The direct image functor Φ_* is exact on the category of quasi-coherent $\mathscr{O}_X$-modules.*

Proof. Indeed, it is clear that $M_{[\phi]}$ is an exact functor in M and $\widetilde{M'}$ is an exact functor in M′ (1.3.5). □

Proposition (1.6.5). — *Let* N′ *be an* A′*-module,* N *the* A*-module* $N' \otimes_{A'} A_{[\phi]}$*; there exists a canonical functorial isomorphism from the $\mathscr{O}_X$-module $\Phi^*(\widetilde{N'})$ to $\widetilde{N}$.*

Proof. We first remark that $j : z' \mapsto z' \otimes 1$ is an A′-homomorphism from N′ to $N_{[\phi]}$: indeed, by definition, for $f' \in A'$, we have $(f'z') \otimes 1 = z' \otimes \phi(f') = \phi(f')(z' \otimes 1)$. We have (1.3.8) a homomorphism $\widetilde{j} : \widetilde{N'} \to (N_{[\phi]})^\sim$ of $\mathscr{O}_{X'}$-modules, and according to (1.6.3), we can consider that $\widetilde{j}$ maps $\widetilde{N'}$ to $\Phi_*(\widetilde{N})$. There canonically corresponds to this homomorphism $\widetilde{j}$ a homomorphism $h = \widetilde{j}^\sharp$ from $\Phi^*(\widetilde{N'})$ to $\widetilde{N}$ (**0**, 4.4.3); we will see that for each stalk, h_x is *bijective*. Put $x' = {}^a\phi(x)$ and let $f' \in A'$ be such that $x' \in D(f')$; let $f = \phi(f')$. The ring $\Gamma(D(f), \widetilde{A})$ identifies with A_f, the modules $\Gamma(D(f), \widetilde{N})$ and $\Gamma(D(f'), \widetilde{N'})$ with N_f and $N'_{f'}$ respectively; let $s \in \Gamma(D(f'), \widetilde{N'})$, identified with n'/f'^p ($n' \in N'$), s its image under $\widetilde{j}$ in $\Gamma(D(f), \widetilde{N})$; s identifies with $(n' \otimes 1)/f^p$. On the other hand, let $t \in \Gamma(D(f), \widetilde{A})$, identified with g/f^q ($g \in A$); then, by definition, we have $h_x(s'_x \otimes t_x) = t_x \cdot s_x$ (**0**, 4.4.3). But we can canonically identify N_f with $N'_{f'} \otimes_{A'_{f'}} (A_f)_{[\phi_{f'}]}$ (**0**, 1.5.4); s then corresponds to the element $(n'/f'^p) \otimes 1$, and the section $y \mapsto t_y \cdot s_y$ with $(n'/f'^p) \otimes (g/f^q)$. The compatibility diagram of (**0**, 1.5.6) show that h_x is none other than the canonical isomorphism

$$N'_{x'} \otimes_{A'_{x'}} (A_x)_{[\phi_{x'}]} \xrightarrow{\sim} N_x = (N' \otimes_{A'} A_{[\phi]})_x. \tag{1.6.5.1}$$

In addition, let $v : N'_1 \to N'_2$ be a homomorphism of A′-modules; as $\widetilde{v}_{x'} = v_{x'}$ for each $x' \in X'$, it follows immediately from the above that $\Phi^*(\widetilde{v})$ canonically identifies with $(v \otimes 1)^\sim$, which finishes the proof of (1.6.5). □

If B′ is an A′-algebra, the canonical isomorphism from $\Phi^*(\widetilde{B'})$ to $(B' \otimes_{A'} A_{[\phi]})^\sim$ is an isomorphism of $\mathscr{O}_X$-algebras; if in addition N′ is a B′-module, the canonical isomorphism from $\Phi^*(\widetilde{N'})$ to $(N' \otimes_{A'} A_{[\phi]})^\sim$ is an isomorphism of $\Phi^*(\widetilde{B'})$-modules.

Corollary (1.6.6). — *The sections of $\Phi^*(\widetilde{N'})$, the canonical images of the sections s', where s' varies over the* A′*-module $\Gamma(\widetilde{N'})$, generate the* A*-module $\Gamma(\Phi^*(N'))$.*

Proof. Indeed. these images identify with the elements $z' \otimes 1$ of N, when we identify N′ and N with $\Gamma(\widetilde{N'})$ and $\Gamma(\widetilde{N})$ respectively (1.3.7) and z' varies over N′. □

(1.6.7). In the proof of (1.6.5), we had proved in passing that the canonical map (**0**, 4.4.3.2) $\rho : \widetilde{N'} \to \Phi_*(\Phi^*(\widetilde{N'}))$ is none other than the homomorphism $\widetilde{j}$, where $j : N' \to N' \otimes_{A'} A_{[\phi]}$ is the homomorphism $z' \mapsto z' \otimes 1$. Similarly, the canonical map (**0**, 4.4.3.3) $\sigma : \Phi^*(\Phi_*(\widetilde{M})) \to \widetilde{M}$ is none other than $\widetilde{p}$, where $p : M_{[\phi]} \otimes_{A'} A_{[\phi]} \to M$ is the canonical homomorphism which, sends each tensor product $z \otimes a$ ($z \in M$, $a \in A$) to $a \cdot z$; this follows immediately from the definitions ((**0**, 3.7.1), (**0**, 4.4.3), and (1.3.7)).

We conclude ((**0**, 4.4.3) and (**0**, 3.5.4.4)) that if $v : N' \to M_{[\phi]}$ is an A′-homomorphism, we have $\widetilde{v}^\sharp = (v \otimes 1)^\sim$.

(1.6.8). Let N'_1, N'_2 be two A′-modules, and assume N'_1 admits a *finite presentation*; it then follows from (1.6.7) and (1.3.12), (ii) that the canonical homomorphism (**0**, 4.4.6)

$$\Phi^*(\mathscr{H}om_{\widetilde{A'}}(\widetilde{N'_1}, \widetilde{N'_2})) \longrightarrow \mathscr{H}om_{\widetilde{A}}(\Phi^*(\widetilde{N'_1}), \Phi^*(\widetilde{N'_2}))$$

is none other than $\widetilde{\gamma}$, where γ denotes the canonical homomorphism of A-modules $\mathrm{Hom}_{A'}(N'_1, N'_2) \otimes_{A'} A \to \mathrm{Hom}_A(N'_1 \otimes_{A'} A, N'_2 \otimes_{A'} A)$.

(1.6.9). Let $\mathfrak{J}'$ be an ideal of A′, M an A-module; as by definition $\widetilde{\mathfrak{J}'}\widetilde{M}$ is the image of the canonical homomorphism $\Phi^*(\widetilde{\mathfrak{J}'}) \otimes_{\widetilde{A}} \widetilde{M} \to \widetilde{M}$, it follows from Proposition (1.6.5) and Corollary (1.3.12), (i) that $\widetilde{\mathfrak{J}'}\widetilde{M}$ canonically identifies with $(\mathfrak{J}'M)^\sim$; in particular, $\Phi^*(\widetilde{\mathfrak{J}'})\widetilde{A}$ identifies with $(\mathfrak{J}'A)^\sim$, and taking into account the right exactness of the functor Φ^*, the $\widetilde{A}$-algebra $\Phi^*((A'/\mathfrak{J}')^\sim)$ identifies with $(A/\mathfrak{J}'A)^\sim$.

(1.6.10). Let A″ be a third ring, ϕ' a homomorphism $A'' \to A'$, and put $\phi'' = \phi \circ \phi'$. It follows immediately from the definitions that ${}^a\phi'' = ({}^a\phi') \circ ({}^a\phi)$, and $\widetilde{\phi''} = \widetilde{\phi} \circ \widetilde{\phi'}$ (**0**, 1.5.7). We conclude that we have $\Phi'' = \Phi' \circ \Phi$; in other words, $(\mathrm{Spec}(A), \widetilde{A})$ is a *functor* from the category of rings to that of ringed spaces.

1.7. **Characterization of morphisms of affine schemes.**

Definition (1.7.1). — We say that a ringed space $(X, \mathscr{O}_X)$ is an affine scheme if it is isomorphic to a ringed space of the form $(\mathrm{Spec}(A), \widetilde{A})$, where A is a ring; we then say that $\Gamma(X, \mathscr{O}_X)$, which canonically identifies with the ring A (1.3.7) is the ring of the affine scheme $(X, \mathscr{O}_X)$, and we denote it by A(X) when there is no chance of confusion.

By abuse of language, when we speak of an *affine scheme* Spec(A), it will always be the ringed space $(\mathrm{Spec}(A), \widetilde{A})$.

(1.7.2). Let A, B be two rings, $(X, \mathscr{O}_X)$, $(Y, \mathscr{O}_Y)$ the affine schemes corresponding to the prime spectra $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$. We have seen (1.6.1) that each ring homomorphism $\phi : B \to A$ corresponds to a morphism $\Phi = ({}^a\phi, \widetilde{\phi}) = \mathrm{Spec}(\phi) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$. We note that ϕ is entirely determined by Φ, since we have by definition $\phi = \Gamma(\widetilde{\phi}) : \Gamma(\widetilde{B}) \to \Gamma({}^a\phi_*(\widetilde{A}) = \Gamma(\widetilde{A})$.

Theorem (1.7.3). — *Let $(X, \mathscr{O}_X)$, $(Y, \mathscr{O}_Y)$ be two affine schemes. For a morphism of ringed spaces $(\psi, \theta) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ to be of the form $({}^a\phi, \widetilde{\phi})$, where ϕ is a homomorphism of rings: $A(Y) \to A(X)$, it is necessary and sufficient that, for each $x \in X$, $\theta^\sharp_x$ is a local homomorphism: $\mathscr{O}_{\psi(x)} \to \mathscr{O}_x$.*

Proof. Set $A = A(X)$, $B = A(Y)$. The condition is necessary, since we saw (1.6.1) that $\widetilde{\phi}^\sharp_x$ is the homomorphism from $B_{{}^a\phi(x)}$ 97
to A_x canonically induced by ϕ, and by definition of ${}^a\phi(x) = \phi^{-1}(\mathfrak{j}_x)$, this homomorphism is local.

We prove that the condition is sufficient. By definition, θ is a homomorphism $\mathscr{O}_Y \to \psi_*(\mathscr{O}_X)$, and we canonically obtain a ring homomorphism

$$\phi = \Gamma(\theta) : B = \Gamma(Y, \mathscr{O}_Y) \longrightarrow \Gamma(Y, \psi_*(\mathscr{O}_X)) = \Gamma(X, \mathscr{O}_X) = A.$$

The hypotheses on $\theta^\sharp_x$ allow us to deduce from this homomorphism, by passing to quotients, a momomorphism θ^x from the residue field $k(\psi(x))$ to the residue field $k(x)$, such that, for each section $f \in \Gamma(Y, \mathscr{O}_Y) = B$, we have $\theta^x(f(\psi(x))) = \phi(f)(x)$. The relation $f(\psi(x)) = 0$ is therefore equivalent to $\phi(f)(x) = 0$, which means that $\mathfrak{j}_{\psi(x)} = \mathfrak{j}_{{}^a\phi(x)}$, and we now write $\psi(x) = {}^a\phi(x)$ for each $x \in X$, or $\psi = {}^a\phi$. We also know that the diagram

$$\begin{tikzcd} B = \Gamma(Y, \mathscr{O}_Y) \arrow[r, "\phi"] \arrow[d] & \Gamma(X, \mathscr{O}_X) = A \arrow[d] \\ B_{\psi(x)} \arrow[r, "\theta^\sharp_x"] & A_x \end{tikzcd}$$

is commutative (**0**, 3.7.2), which means that $\theta^\sharp_x$ is equal to the homomorphism $\phi_x : B_{\psi(x)} \to A_x$ canonically induced by ϕ (**0**, 1.5.1). As the data of the $\theta^\sharp_x$ completely characterize $\theta^\sharp$, and as a result θ (**0**, 3.7.1), we conclude that we have $\theta = \widetilde{\phi}$, by definition of $\widetilde{\phi}$ (1.6.1). □

We say that a mormphism (ψ, θ) of ringed spaces satisfying the condition of (1.7.3) is a *morphism of affine schemes*.

Corollary (1.7.4). — *If $(X, \mathscr{O}_X)$, $(Y, \mathscr{O}_Y)$ are affine schemes, there exists a canonical isomorphism from the set of morphisms of affine schemes $\mathrm{Hom}((X, \mathscr{O}_X), (Y, \mathscr{O}_Y))$ to the set of ring homomorphisms from B to A, where $A = \Gamma(\mathscr{O}_X)$ and $B = \Gamma(\mathscr{O}_Y)$.*

Furthermore, we can say that the functors $(\mathrm{Spec}(A), \widetilde{A})$ in A and $\Gamma(X, \mathscr{O}_X)$ in $(X, \mathscr{O}_X)$ define an *equivalence* between the category of commutative rings and the opposite category of affine schemes (T, I, 1.2).

Corollary (1.7.5). — *If $\phi : B \to A$ is surjective, then the corresponding morphism $({}^a\phi, \widetilde{\phi})$ is a momomorphism of ringed spaces (cf. (4.1.7)).*

Proof. Indeed, we know that ${}^a\phi$ is injective (1.2.5), and as ϕ is surjective, for each $x \in X$, $\phi^\sharp_x : B_{{}^a\phi(x)} \to A_x$, which is induced by ϕ by passing to rings of fractions, is also surjective (**0**, 1.5.1); hence the conclusion (**0**, 4.1.1). □

1.8. **Morphisms from locally ringed spaces to affine schemes.** Due to a remark by J. Tate, the statements given in 217′
Theorem (1.7.3) and Proposition (2.2.4) can be generalized as follows:[2]

Proposition (1.8.1). — *Let $(S, \mathscr{O}_S)$ be an affine scheme, $(X, \mathscr{O}_X)$ a locally ringed space. Then there is a canonical bijection from the set of ring homomorphisms $\Gamma(S, \mathscr{O}_S) \to \Gamma(X, \mathscr{O}_X)$ to the set of morphisms of ringed spaces $(\psi, \theta) : (X, \mathscr{O}_X) \to (S, \mathscr{O}_S)$* 218′
such that, for each $x \in X$, $\theta^\sharp_x$ is a local homomorphism: $\mathscr{O}_{\psi(x)} \to \mathscr{O}_x$.

Proof. We note first that if $(X, \mathscr{O}_X)$, $(S, \mathscr{O}_S)$ are any two ringed spaces, a morphism (ψ, θ) from $(X, \mathscr{O}_X)$ to $(S, \mathscr{O}_S)$ canonically defines a ring homomorphism $\Gamma(\theta) : \Gamma(S, \mathscr{O}_S) \to \Gamma(X, \mathscr{O}_X)$, hence a first map

$$\rho : \mathrm{Hom}((X, \mathscr{O}_X), (S, \mathscr{O}_S)) \longrightarrow \mathrm{Hom}(\Gamma(S, \mathscr{O}_S), \Gamma(X, \mathscr{O}_X)). \tag{1.8.1.1}$$

Conversely, under the stated hypotheses, we set $A = \Gamma(S, \mathscr{O}_S)$, and consider a ring homomorphism $\phi : A \to \Gamma(X, \mathscr{O}_X)$. For each $x \in X$, it is clear that the set of the $f \in A$ such that $\phi(f)(x) = 0$ is a *prime ideal* of A, since $\mathscr{O}_x / \mathfrak{m}_x = k(x)$ is a field; it is therefore an element of $S = \mathrm{Spec}(A)$, which we denote ${}^a\phi(x)$. In addition, for each $f \in A$, we have by definition (**0**, 5.5.2) that ${}^a\phi(D(f)) = X_f$, which proves that ${}^a\phi$ is a *continuous map* $X \to S$. We define then a homomorphism

$$\widetilde{\phi} : \mathscr{O}_S \longrightarrow {}^a\phi_*(\mathscr{O}_X)$$

of $\mathscr{O}_S$-modules; for each $f \in A$, we have $\Gamma(D(f), \mathscr{O}_S) = A_f$ (1.3.6); for each $s \in A$, we correspond to $s/f \in A_f$ the element $(\phi(s)|X_f)(\phi(f)|X_f)^{-1}$ of $\Gamma(X_f, \mathscr{O}_X) = \Gamma(D(f), {}^a\phi(\mathscr{O}_X))$, and we check immediately (by passing from $D(f)$ to $D(fg)$)

[2] [Trans] The following section (I.1.8) was added in the errata of EGA II, hence the temporary change in page numbers, which refer to EGA II.

that this is a well-defined homomorphism of $\mathscr{O}_S$-modules, hence a morphism $({}^a\phi,\widetilde{\phi})$ of ringed spaces. In addition, with the same notations, and setting $y = {}^a\phi(x)$ for purposes of simplification, we see immediately (**0**, 3.7.1) that we have $\widetilde{\phi}_x^\sharp(s_y/f_y) = (\phi(s)_x)(\phi(f)_x)^{-1}$; as the relation $s_y \in \mathfrak{m}_y$ is by definition equivalent to $\phi(s)_x \in \mathfrak{m}_x$, we see that $\widetilde{\phi}_x^\sharp$ is a *local* homomorphism $\mathscr{O}_y \to \mathscr{O}_x$, and we have so defined a second map $\sigma : \mathrm{Hom}(\Gamma(S,\mathscr{O}_S),\Gamma(X,\mathscr{O}_X)) \to \mathfrak{L}$, where $\mathfrak{L}$ is the set of the morphisms $(\psi,\theta) : (X,\mathscr{O}_X) \to (S,\mathscr{O}_S)$ such that $\theta_x^\sharp$ is local for each $x \in X$. It remains to prove that σ and ρ (restricted to $\mathfrak{L}$) are inverses of each other; the definition of $\widetilde{\phi}$ immediately shows that $\Gamma(\widetilde{\phi}) = \phi$, and as a result $\rho\circ\sigma$ is the identity. To see that $\sigma\circ\rho$ is the identity, start with a morphism $(\psi,\theta) \in \mathfrak{L}$ and let $\phi = \Gamma(\theta)$; the hypotheses on $\theta_x^\sharp$ allows us to induce from this morphism, by passing to quotients, a monomorphism $\theta^x : k(\psi(x)) \to k(x)$ such that for each section $f \in A = \Gamma(S,\mathscr{O}_S)$, we have $\theta^x(f(\psi(x))) = \phi(f)(x)$; the relation $f(\phi(x)) = 0$ is therefore equivalent to $\phi(f)(x) = 0$, which proves that ${}^a\phi = \psi$. On the other hand, the definitions imply that the diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d] & \Gamma(X,\mathscr{O}_X) \arrow[d] \\ A_{\psi(x)} \arrow[r, "\theta_x^\sharp"] & \mathscr{O}_x \end{tikzcd}$$

is commutative, and it is the same for the analogous diagram where $\theta_x^\sharp$ is replaced by $\widetilde{\phi}_x^\sharp$, hence $\widetilde{\phi}_x^\sharp = \theta_x^\sharp$ (**0**, 1.2.4), and as a result $\widetilde{\phi} = \theta$. □

(1.8.2). When $(X,\mathscr{O}_X)$ and $(Y,\mathscr{O}_Y)$ are *locally* ringed spaces, we will consider the morphisms $(\psi,\theta) : (X,\mathscr{O}_X) \to (Y,\mathscr{O}_Y)$ such that, for each $x \in X$, $\theta_x^\sharp$ is a *local* homomorphism: $\mathscr{O}_{\psi(x)} \to \mathscr{O}_x$. Henceforth when we speak of a *morphism of locally ringed spaces*, it will always be a morphism such as the above; with this definition of morphisms, it is clear that the locally ringed spaces form a *category*; for two objects X, Y of this category, Hom(X, Y) thus denotes the set of morphisms of locally ringed spaces from X to Y (the set denoted $\mathfrak{L}$ in (1.8.1)); when we consider the set of *morphisms of ringed spaces* from X to Y, we will denote it by $\mathrm{Hom}_{\mathrm{rs}}(X,Y)$ to avoid any confusion. The map (1.8.1.1) is then written as

(1.8.2.1) $$\rho : \mathrm{Hom}_{\mathrm{rs}}(X,Y) \longrightarrow \mathrm{Hom}(\Gamma(Y,\mathscr{O}_Y),\Gamma(X,\mathscr{O}_X))$$

and its restriction

(1.8.2.2) $$\rho' : \mathrm{Hom}(X,Y) \longrightarrow \mathrm{Hom}(\Gamma(Y,\mathscr{O}_Y),\Gamma(X,\mathscr{O}_X))$$

is a *functorial* map in X and Y on the category of locally ringed spaces.

Corollary (1.8.3). — *Let $(Y,\mathscr{O}_Y)$ be a locally ringed space. For Y to be an affine scheme, it is necessary and sufficient that for each locally ringed space $(X,\mathscr{O}_X)$, the map (1.8.2.2) is bijective.*

Proof. Proposition (1.8.1) shows that the condition is necessary. Conversely, if we suppose that the condition is satisfied and if we put $A = \Gamma(Y,\mathscr{O}_Y)$, it follows from the hypotheses and from (1.8.1) that the functors $X \mapsto \mathrm{Hom}(X,Y)$ and $X \mapsto \mathrm{Hom}(X,\mathrm{Spec}(A))$, from the category of locally ringed spaces to that of sets, are *isomorphic*. We know that this implies the existence of a canonical isomorphism $X \to \mathrm{Spec}(A)$ (cf. **0**, 8). □

(1.8.4). Let $S = \mathrm{Spec}(A)$ be an affine scheme; denote by (S',A') the ringed space whose underlying space is *reduced to a point* and the structure sheaf A′ is the (necessarily simple) sheaf on S′ defined by the ring A. Let $\pi : S \to S'$ be the unique map from S to S′; on the other hand, we note that for each open subset U of S, we have a canonical map $\Gamma(S',A') = \Gamma(S,\mathscr{O}_S) \to \Gamma(U,\mathscr{O}_S)$ which thus defines a π-*morphism* $\iota : A' \to \mathscr{O}_S$ of sheaves of rings. We have thus canonically defined a *morphism of ringed spaces* $i = (\pi,\iota) : (S,\mathscr{O}_S) \to (S',A')$. For each A-module M, we denote by M′ the simple sheaf on S′ defined by M, which is evidently an A′-module. It is clear that we have $i_*(\widetilde{M}) = M'$ (1.3.7).

Lemma (1.8.5). — *With the notation of (1.8.4), for each* A*-module* M*, the canonical functorial $\mathscr{O}_S$-homomorphism (**0**, 4.4.3.3)*

(1.8.5.1) $$i^*(i_*(\widetilde{M})) \longrightarrow \widetilde{M}$$

is an isomorphism.

Proof. Indeed, the two parts of (1.8.5.1) are right exact (the functor $M \mapsto i_*(\widetilde{M})$ being evidently exact) and commute with direct sums; by considering M as the cokernel of a homomorphism $A^{(I)} \to A^{(J)}$, we reduce to proving the lemma for the case where $M = A$, and it is evident in this case. □

Corollary (1.8.6). — *Let $(X,\mathscr{O}_X)$ be a ringed space, $u : X \to S$ a morphism of ringed spaces. For each* A*-module* M*, we have (with the notation of (1.8.4)) a canonical functorial isomorphism of $\mathscr{O}_X$-modules*

(1.8.6.1) $$u^*(\widetilde{M}) \xrightarrow{\sim} u^*(i^*(M')).$$

Corollary (1.8.7). — *Under the hypotheses of (1.8.6), we have, for each* A*-module* M *and each* $\mathscr{O}_X$*-module* $\mathscr{F}$*, a canonical functorial isomorphism in* M *and* $\mathscr{F}$

$$\operatorname{Hom}_{\mathscr{O}_S}(\widetilde{M}, u_*(\mathscr{F})) \xrightarrow{\sim} \operatorname{Hom}_A(M, \Gamma(X, \mathscr{F})). \tag{1.8.7.1}$$

Proof. We have, according to (**0**, 4.4.3) and Lemma (1.8.5), a canonical isomorphism of bifunctors

$$\operatorname{Hom}_{\mathscr{O}_S}(\widetilde{M}, u_*(\mathscr{F})) \xrightarrow{\sim} \operatorname{Hom}_{A'}(M', i_*(u_*(\mathscr{F})))$$

and it is clear that the right hand side is none other than $\operatorname{Hom}_A(M, \Gamma(X, \mathscr{F}))$. We note that the canonical homomorphism (1.8.7.1) sends each $\mathscr{O}_S$-homomorphism $h : \widetilde{M} \to u_*(\mathscr{F})$ (in other words, each u-morphism $\widetilde{M} \to \mathscr{F}$) to the A-homomorphism $\Gamma(h) : M \to \Gamma(S, u_*(\mathscr{F})) = \Gamma(X, \mathscr{F})$. □

(1.8.8). With the notation of (1.8.4), it is clear (**0**, 4.1.1) that each morphism of ringed spaces $(\psi, \theta) : X \to S'$ is equivalent to the data of a ring homomorphism $A \to \Gamma(X, \mathscr{O}_X)$. We can thus interpret Proposition (1.8.1) as defining a canonical bijection $\operatorname{Hom}(X, S) \xrightarrow{\sim} \operatorname{Hom}(X, S')$ (where we understand that the right hand side are morphisms of ringed spaces, since in general A is not a local ring). More generally, if X, Y are two locally ringed spaces and if (Y', A') is the ringed space whose underlying space is reduced to a point and whose sheaf of rings A' is the simple sheaf defined by the ring $\Gamma(Y, \mathscr{O}_Y)$, we can interpret (1.8.2.1) as a map

$$\rho : \operatorname{Hom}_{rs}(X, Y) \longrightarrow \operatorname{Hom}(X, Y'). \tag{1.8.8.1}$$

The result of Corollary (1.8.3) is interepreted by saying that affine schemes are characterized among locally ringed spaces as those for which the restriction o ρ to $\operatorname{Hom}(X, Y)$:

$$\rho' : \operatorname{Hom}(X, Y) \longrightarrow \operatorname{Hom}(X, Y') \tag{1.8.8.2}$$

is *bijective* for *each* locally ringed space X. In the following chapter, we generalize this definition, which allows us to associate to *any* ringed space Z (and not only to a ringed space whose underlying space is reduced to a point) a locally ringed space which we will call Spec(Z); this will be the starting point for a "relative" theory of preschemes over any ringed space, extending the results of Chapter I.

(1.8.9). We can consider the pairs $(X, \mathscr{F})$ consisting of a locally ringed space X and an $\mathscr{O}_X$-module $\mathscr{F}$ as forming a category, a *morphism* of this category being a pair (u, h) consisting of a morphism of locally ringed spaces $u : X \to Y$ and a u-morphism $h : \mathscr{G} \to \mathscr{F}$ of modules; these morphisms (for $(X, \mathscr{F})$ and $(Y, \mathscr{G})$ fixed) form a set which we denote by $\operatorname{Hom}((X, \mathscr{F}), (Y, \mathscr{G}))$; the map $(u, h) \mapsto (\rho'(u), \Gamma(h))$ is a canonical map 221′

$$\operatorname{Hom}((X, \mathscr{F}), (Y, \mathscr{G})) \longrightarrow \operatorname{Hom}((\Gamma(Y, \mathscr{O}_Y), \Gamma(Y, \mathscr{G})), (\Gamma(X, \mathscr{O}_X), \Gamma(X, \mathscr{F}))) \tag{1.8.9.1}$$

functorial in $(X, \mathscr{F})$ and $(Y, \mathscr{G})$, the right hand side being the set of di-homomorphisms corresponding to the rings and modules considered (**0**, 1.0.2).

Corollary (1.8.10). — *Let* Y *be a locally ringed space,* $\mathscr{G}$ *an* $\mathscr{O}_Y$*-module. For* Y *to be an affine scheme and* $\mathscr{G}$ *to be a quasi-coherent* $\mathscr{O}_Y$*-module, it is necessary and sufficient that for each pair* $(X, \mathscr{F})$ *consisting of a locally ringed space* X *and an* $\mathscr{O}_X$*-module* $\mathscr{F}$*, the canonical map (1.8.9.1) is bijective.*

We leave the reader to give the proof, which is modeled on that of (1.8.3), and using (1.8.1) and (1.8.7).

Remark (1.8.11). — The statements (1.7.3), (1.7.4), and (2.2.4) are particular cases of (1.8.1), as well as the definition in (1.6.1); similarly, (2.2.5) follows from (1.8.7). Corollary (1.8.7) also implies (1.6.3) (and as a result (1.6.4)) as a particular case, since if X is an affine scheme and $\Gamma(X, \mathscr{F}) = N$, the functors $M \mapsto \operatorname{Hom}_{\mathscr{O}_S}(\widetilde{M}, u_*(\widetilde{N}))$ and $M \mapsto \operatorname{Hom}_{\mathscr{O}_S}(\widetilde{M}, (N_{[\phi]})^{\sim})$ (where $\phi : A \to \Gamma(X, \mathscr{O}_X)$ corresponds to u) are isomorphic by Corollaries (1.8.7) and (1.3.8). Finally, (1.6.5) (and as a result (1.6.6)) follow from (1.8.6) and the fact that for each $f \in A$, the A_f-modules $N' \otimes_{A'} A_f$ and $(N' \otimes_{A'} A)_f$ (notations of (1.6.5)) are canonically isomorphic.

2. Preschemes and morphisms of preschemes

2.1. **Definition of preschemes.**

(2.1.1). Given a ringed space $(X, \mathscr{O}_X)$, we say that an open subset V of X is an *affine open* if the ringed space $(V, \mathscr{O}_X|V)$ is an affine scheme (1.7.1).

Definition (2.1.2). — We define a prescheme to be a ringed space $(X, \mathscr{O}_X)$ such that every point of X admits an affine open neighbourhood.

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Proposition (2.1.3). — *If* $(X, \mathscr{O}_X)$ *is a prescheme then the affine opens give a basis for the topology of* X.

Proof. If V is an arbitrary open neighbourhood of $x \in X$, then there exists by hypothesis an open neighbourhood W of x such that $(W, \mathcal{O}_X|W)$ is an affine scheme; we write A to mean its ring. In the space W, $V \cap W$ is an open neighbourhood of x; thus there exists $f \in A$ such that $D(f)$ is an open neighbourhood of x contained inside $V \cap W$ (1.1.10), (i). The ringed space $(D(f), \mathcal{O}_X|D(f))$ is thus an affine scheme, isomorphic to A_f (1.3.6), whence the proposition. □

Proposition (2.1.4). — *The underlying space of a prescheme is a Kolmogoroff space.*

Proof. If x, y are two distinct points of a prescheme X then it is clear that there exists an open neighbourhood of one of these points that does not contain the other if x and y are not in the same affine open; and if they are in the same affine open, this is a result of (1.1.8). □

Proposition (2.1.5). — *If* $(X, \mathcal{O}_X)$ *is a prescheme then every closed irreducible subset of* X *admits exactly one generic point, and the map* $x \mapsto \overline{\{x\}}$ *is thus a bijection of* X *onto its set of closed irreducible subsets.*

Proof. If Y is a closed irreducible subset of X and $y \in Y$, and if U is an open affine neighbourhood of y in X, then $U \cap Y$ is everywhere dense in Y, as well as irreducible ((**0**, 2.1.1) and (**0**, 2.1.4)); thus by Corollary (1.1.14), $U \cap Y$ is the closure in U of a point x, and then $Y \subset \overline{U}$ is the closure of x in X. The uniqueness of the generic point of X is a result of Proposition (2.1.4) and of (**0**, 2.1.3). □

(2.1.6). If Y is a closed irreducible subset of X and y its generic point then the local ring $\mathcal{O}_y$, also written $\mathcal{O}_{X/Y}$, is called the *local ring of* X *along* Y, or the *local ring of* Y *in* X.

If X itself is irreducible and x its generic point then we say that $\mathcal{O}_x$ is the *ring of rational functions on* X (cf. §7).

Proposition (2.1.7). — *If* $(X, \mathcal{O}_X)$ *is a prescheme then the ringed space* $(U, \mathcal{O}_X|U)$ *is a prescheme for every open subset* U.

Proof. This follows directly from Definition (2.1.2) and Proposition (2.1.3). □

We say that $(U, \mathcal{O}_X|U)$ is the prescheme *induced* on U by $(X, \mathcal{O}_X)$, or the *restriction* of $(X, \mathcal{O}_X)$ to U.

(2.1.8). We say that a prescheme $(X, \mathcal{O}_X)$ is *irreducible* (resp. *connected*) if the underlying space X is irreducible (resp. connected). We say that a prescheme is *integral* if it is *irreducible and reduced* (cf. (5.1.4)). We say that a prescheme $(X, \mathcal{O}_X)$ is *locally integral* if each $x \in X$ admits an open neighbourhood U such that the prescheme induced on U by $(X, \mathcal{O}_X)$ is integral.

2.2. **Morphisms of preschemes.**

Definition (2.2.1). — Given two preschemes $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$, we define a morphism (of preschemes) of $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ to be a morphism of ringed spaces (ψ, θ) such that, for all $x \in X$, $\theta_x^\sharp$ is a local homomorphism $\mathcal{O}_{\psi(x)} \to \mathcal{O}_x$.

By passing to quotients, the map $\mathcal{O}_{\psi(x)} \to \mathcal{O}_x$ gives us a monomorphism $\theta^x : k(\psi(x)) \to k(x)$, which lets us consider $k(x)$ as an *extension* of the field $k(\psi(x))$.

(2.2.2). The composition (ψ'', θ'') of two morphisms (ψ, θ), (ψ', θ') of preschemes is also a morphism of preschemes, since it is given by the formula $\theta''^\sharp = \theta^\sharp \circ \psi^*(\theta'^\sharp)$ (**0**, 3.5.5). From this we conclude that preschemes form a *category*; using the usual notation, we will write $\mathrm{Hom}(X, Y)$ to mean the set of morphisms from a prescheme X to a prescheme Y.

Example (2.2.3). — If U is an open subset of X then the canonical injection (**0**, 4.1.2) of the induced prescheme $(U, \mathcal{O}_X|U)$ into $(X, \mathcal{O}_X)$ is a morphism of preschemes; it is further a *monomorphism* of ringed spaces (and *a fortiori* a monomorphism of preschemes), which quickly follows from (**0**, 4.1.1).

Proposition (2.2.4). — *Let* $(X, \mathcal{O}_X)$ *be a prescheme, and* $(S, \mathcal{O}_S)$ *an affine scheme associated to a ring* A*. Then there exists a canonical bijective correspondence between morphisms of preschemes from* $(X, \mathcal{O}_X)$ *to* $(S, \mathcal{O}_S)$ *and ring homomorphisms from* A *to* $\Gamma(X, \mathcal{O}_X)$*.*

Note first that, if $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are two arbitrary ringed spaces, a morphism (ψ, θ) from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ canonically defines a ring homomorphism $\Gamma(\theta) : \Gamma(Y, \mathcal{O}_Y) \to \Gamma(Y, \psi_*(\mathcal{O}_X)) = \Gamma(X, \mathcal{O}_X)$. In the case that we consider, everything boils down to showing that any homomorphism $\phi : A \to \Gamma(X, \mathcal{O}_X)$ is of the form $\Gamma(\theta)$ for one and only one θ. Now, by hypothesis there is a covering (V_α) of X by affine opens; by composing of ϕ with the restriction homomorphism $\Gamma(X, \mathcal{O}_X) \to \Gamma(V_\alpha, \mathcal{O}_X|V_\alpha)$ we obtain a homomorphism $\phi_\alpha : A \to \Gamma(V_\alpha, \mathcal{O}_X|V_\alpha)$ that corresponds to a unique morphism $(\psi_\alpha, \theta_\alpha)$ from the prescheme $(V_\alpha, \mathcal{O}_X|V_\alpha)$ to $(S, \mathcal{O}_S)$, according to Theorem (1.7.3). Furthermore, for each pair of indices (α, β), each point of $V_\alpha \cap V_\beta$ admits an open affine neighbourhood W contained inside $V_\alpha \cap V_\beta$ (2.1.3); it is clear that by composing ϕ_α and ϕ_β with the restriction homomorphisms to W, we obtain the same homomorphism $\Gamma(S, \mathcal{O}_S) \to \Gamma(W, \mathcal{O}_X|W)$, so, thanks to the relations $(\theta_\alpha^\sharp)_x = (\phi_\alpha)_x$ for all $x \in V_\alpha$ and all α (1.6.1), the restriction to W of the morphisms $(\psi_\alpha, \theta_\alpha)$ and $(\psi_\beta, \theta_\beta)$ coincide. From this we conclude that there is a morphism $(\psi, \theta) : (X, \mathcal{O}_X) \to (S, \mathcal{O}_S)$ of ringed spaces, and only one such that its restriction to each V_α is $(\psi_\alpha, \theta_\alpha)$, and it is clear that this morphism is a morphism of preschemes and such that $\Gamma(\theta) = \phi$.

Let $u : \mathrm{A} \to \Gamma(\mathrm{X}, \mathscr{O}_\mathrm{X})$ be a ring homomorphism, and $v = (\psi, \theta)$ the corresponding morphism $(\mathrm{X}, \mathscr{O}_\mathrm{X}) \to (\mathrm{S}, \mathscr{O}_\mathrm{S})$. For each $f \in \mathrm{A}$ we have that

$$\psi^{-1}(\mathrm{D}(f)) = \mathrm{X}_{u(f)} \tag{2.2.4.1}$$

with the notation of (**0**, 5.5.2) relative to the locally free sheaf $\mathscr{O}_\mathrm{X}$. In fact, it suffices to verify this formula when X itself is affine, and then this is nothing but (1.2.2.2).

Proposition (2.2.5). — *Under the hypotheses of Proposition (2.2.4), let* $\phi : \mathrm{A} \to \Gamma(\mathrm{X}, \mathscr{O}_\mathrm{X})$ *be a ring homomorphism,* $f : (\mathrm{X}, \mathscr{O}_\mathrm{X}) \to (\mathrm{S}, \mathscr{O}_\mathrm{S})$ *the corresponding morphism of preschemes,* $\mathscr{G}$ *(resp.* $\mathscr{F}$*) an* $\mathscr{O}_\mathrm{X}$*-module (resp.* $\mathscr{O}_\mathrm{S}$*-module), and* $\mathrm{M} = \Gamma(\mathrm{S}, \mathscr{F})$. *Then there exists a canonical bijective correspondence between* f*-morphisms* $\mathscr{F} \to \mathscr{G}$ *(**0**, 4.4.1) and* A*-homomorphisms* $\mathrm{M} \to$ 100
$(\Gamma(\mathrm{X}, \mathscr{G}))_{[\phi]}$.

Proof. Reasoning as in Proposition (2.2.4), we reduce to the case where X is affine, and the proposition then follows from Proposition (1.6.3) and from Corollary (1.3.8). □

(2.2.6). We say that a morphism of preschemes $(\psi, \theta) : (\mathrm{X}, \mathscr{O}_\mathrm{X}) \to (\mathrm{Y}, \mathscr{O}_\mathrm{Y})$ is *open* (resp. *closed*) if, for all open subsets U of X (resp. all closed subsets F of X), $\psi(\mathrm{U})$ is open (resp. $\psi(\mathrm{F})$ is closed) in Y. We say that (ψ, θ) is *dominant* if $\psi(\mathrm{X})$ is dense in Y, and *surjective* if ψ is surjective. We will point out that these conditions rely only on the continuous map ψ.

Proposition (2.2.7). — *Let*

$$f = (\psi, \theta) : (\mathrm{X}, \mathscr{O}_\mathrm{X}) \to (\mathrm{Y}, \mathscr{O}_\mathrm{Y});$$
$$g = (\psi', \theta') : (\mathrm{Y}, \mathscr{O}_\mathrm{Y}) \to (\mathrm{Z}, \mathscr{O}_\mathrm{Z})$$

be two morphisms of preschemes.

(i) *If f and g are both open (resp. closed, dominant, surjective), then so is $g \circ f$.*
(ii) *If f is surjective and $g \circ f$ closed, then g is closed.*
(iii) *If $g \circ f$ is surjective, then g is surjective.*

Proof. Claims (i) and (iii) are evident. Write $g \circ f = (\psi'', \theta'')$. If F is closed in Y then $\psi^{-1}(\mathrm{F})$ is closed in X, so $\psi''(\psi^{-1}(\mathrm{F}))$ is closed in Z; but since ψ is surjective, $\psi(\psi^{-1}(\mathrm{F})) = \mathrm{F}$, so $\psi''(\psi^{-1}(\mathrm{F})) = \psi'(\mathrm{F})$, which proves (ii). □

Proposition (2.2.8). — *Let $f = (\psi, \theta)$ be a morphism $(\mathrm{X}, \mathscr{O}_\mathrm{X}) \to (\mathrm{Y}, \mathscr{O}_\mathrm{Y})$, and (U_α) an open cover of* Y. *For f to be open (resp. closed, surjective, dominant), it is necessary and sufficient that its restriction to each induced prescheme $(\psi^{-1}(\mathrm{U}_\alpha), \mathscr{O}_\mathrm{X}|\psi^{-1}(\mathrm{U}_\alpha))$, considered as a morphism of preschemes from this induced prescheme to the induced prescheme $(\mathrm{U}_\alpha, \mathscr{O}_\mathrm{Y}|\mathrm{U}_\alpha)$ is open (resp. closed, surjective, dominant).*

Proof. The proposition follows immediately from the definitions, taking into account the fact that a subset F of Y is closed (resp. open, dense) in Y if and only if each of the $\mathrm{F} \cap \mathrm{U}_\alpha$ are closed (resp. open, dense) in U_α. □

(2.2.9). Let $(\mathrm{X}, \mathscr{O}_\mathrm{X})$ and $(\mathrm{Y}, \mathscr{O}_\mathrm{Y})$ be two preschemes; suppose that X (resp. Y) has a finite number of irreducible components X_i (resp. Y_i) $(1 \leqslant i \leqslant n)$; let ξ_i (resp. η_i) be the generic point of X_i (resp. Y_i) (2.1.5). We say that a morphism

$$f = (\psi, \theta) : (\mathrm{X}, \mathscr{O}_\mathrm{X}) \to (\mathrm{Y}, \mathscr{O}_\mathrm{Y})$$

is *birational* if, for all i, $\psi^{-1}(\eta_i) = \{\xi_i\}$ and $\theta^\sharp_{\xi_i} : \mathscr{O}_{\eta_i} \to \mathscr{O}_{\xi_i}$ is an *isomorphism*. It is clear that a birational morphism is dominant (**0**, 2.1.8), and thus it is surjective if it is also closed.

Notational conventions (2.2.10). — In all that follows, when there is no risk of confusion, we *suppress* the structure sheaf (resp. the morphism of structure sheaves) from the notation of a prescheme (resp. morphism of preschemes). If U is an open subset of the underlying space X of a prescheme, then whenever we speak of U as a prescheme we always mean the induced prescheme on U.

2.3. Gluing preschemes.

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(2.3.1). It follows from definition (2.1.2) that every ringed space obtained by *gluing* preschemes (**0**, 4.1.6) is again a prescheme. In particular, although every prescheme admits, by definition, a cover by affine open sets, we see that every prescheme can actually be obtained by *gluing affine schemes*.

(Example). 2.3.2 Let K be a field, and $\mathrm{B} = \mathrm{K}[s]$, $\mathrm{C} = \mathrm{K}[t]$ be two polynomial rings in one indeterminate over K, and define $\mathrm{X}_1 = \mathrm{Spec}(\mathrm{B})$, $\mathrm{X}_2 = \mathrm{Spec}(\mathrm{C})$, which are two isomorphic affine schemes. In X_1 (resp. X_2), let U_{12} (resp. U_{21}) be the affine open $\mathrm{D}(s)$ (resp. $\mathrm{D}(t)$) where the ring B_s (resp. C_t) is formed of rational fractions of the form $f(s)/s^m$ (resp. $g(t)/t^n$) with $f \in \mathrm{B}$ (resp. $g \in \mathrm{C}$). Let u_{12} be the isomorphism of preschemes $\mathrm{U}_{21} \to \mathrm{U}_{12}$ corresponding (2.2.4) to the isomorphism from B to C that, to $f(s)/s^m$, associates the rational fraction $f(1/t)/(1/t^m)$. We can glue X_1 and X_2 along U_{12} and U_{21} by using u_{12}, because there is clearly no gluing condition. We later show that the prescheme X obtained in this manner is a particular case of a general method of construction (**II**, 2.4.3). Here we only show that X *is not an affine scheme*; this will follow from the fact that the ring $\Gamma(X, \mathcal{O}_X)$ is *isomorphic* to K, and so its spectrum reduces to a point. Indeed, a section of $\mathcal{O}_X$ over X has a restriction over X_1 (resp. X_2), identified to an affine open of X, that is a polynomial $f(s)$ (resp. $g(t)$), and it follows from the definitions that we should have $g(t) = f(1/t)$, which is not possible if $f = g \in K$.

2.4. **Local schemes.**

(2.4.1). We say that an affine scheme is a *local scheme* if it is the affine scheme associated to a local ring A; then there exists in $X = \operatorname{Spec}(A)$ a single *closed point* $a \in X$, and for all other $b \in X$ we have that $a \in \overline{\{b\}}$ (1.1.7).

For all preschemes Y and points $y \in Y$, the local scheme $\operatorname{Spec}(\mathcal{O}_y)$ is called the *local scheme of* Y *at the point* y. Let V be an affine open of Y containing y, and B the ring of the affine scheme V; then $\mathcal{O}_y$ is canonically identified with B_y (1.3.4), and the canonical homomorphism $B \to B_y$ thus corresponds (1.6.1) to a morphism of preschemes $\operatorname{Spec}(\mathcal{O}_y) \to V$. If we compose this morphism with the canonical injection $V \to Y$, then we obtain a morphism $\operatorname{Spec}(\mathcal{O}_y) \to Y$, which is *independent* of the affine open V (containing y) that we chose: indeed, if V' is some other affine open containing y, then there exists a third affine open W containing y and such that $W \subset V \cap V'$ (2.1.3); we can thus assume that $V \subset V'$, and then if B' is the ring of V', everything comes down to remarking that the diagram

$$\begin{tikzcd} B' \arrow[rr] \arrow[dr] & & B \arrow[dl] \\ & \mathcal{O}_y & \end{tikzcd}$$

is commutative (**0**, 1.5.1). The morphism

$$\operatorname{Spec}(\mathcal{O}_y) \longrightarrow Y$$

thus defined is said to be *canonical*.

Proposition (2.4.2). — *Let* $(Y, \mathcal{O}_Y)$ *be a prescheme; for all* $y \in Y$*, let* (ψ, θ) *be the canonical morphism* $(\operatorname{Spec}(\mathcal{O}_y), \widetilde{\mathcal{O}_y}) \to (Y, \mathcal{O}_Y)$*. Then* ψ *is a homeomorphism from* $\operatorname{Spec}(\mathcal{O}_y)$ *to the subspace* S_y *of* Y *given by the* z *such that* $y \in \overline{\{z\}}$ *(or, equivalently, the generalizations of* y *(**0**, 2.1.2); furthermore, if* $z = \psi(\mathfrak{p})$*, then* $\theta_z^\sharp : \mathcal{O}_z \to (\mathcal{O}_y)_\mathfrak{p}$ *is an isomorphism;* (ψ, θ) *is thus a monomorphism of ringed spaces.*

As the unique closed point a of $\operatorname{Spec}(\mathcal{O}_y)$ is contained in the closure of any point of this space, and since $\psi(a) = y$, the image of $\operatorname{Spec}(\mathcal{O}_y)$ under the continuous map ψ is contained in S_y. Since S_y is contained in every affine open containing y, one can consider just the case where Y is an affine scheme; but then this proposition follows from (1.6.2).

We see (2.1.5) *that there is a bijective correspondence between* $\operatorname{Spec}(\mathcal{O}_y)$ *and the set of closed irreducible subsets of* Y *containing* y.

Corollary (2.4.3). — *For* $y \in Y$ *to be the generic point of an irreducible component of* Y*, it is necessary and sufficient that the only prime ideal of the local ring* $\mathcal{O}_y$ *is its maximal ideal (in other words, that* $\mathcal{O}_y$ *is of dimension zero).*

Proposition (2.4.4). — *Let* $(X, \mathcal{O}_X)$ *be a local scheme of a ring* A*,* a *its unique closed point, and* $(Y, \mathcal{O}_Y)$ *a prescheme. Every morphism* $u = (\psi, \theta) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *then factorizes uniquely as* $X \to \operatorname{Spec}(\mathcal{O}_{\psi(a)}) \to Y$*, where the second arrow denotes the canonical morphism, and the first corresponds to a local homomorphism* $\mathcal{O}_{\psi(a)} \to A$*. This establishes a canonical bijective correspondence between the set of morphisms* $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ *and the set of local homomorphisms* $\mathcal{O}_y \to A$ *for* $(y \in Y)$*.*

Indeed, for all $x \in X$, we have that $a \in \overline{\{x\}}$, so $\psi(a) \in \overline{\{\psi(x)\}}$, which shows that $\psi(X)$ is contained in every affine open containing $\psi(a)$. So it suffices to consider the case where $(Y, \mathcal{O}_Y)$ is an affine scheme of ring B, and we then have that $u = ({}^a\phi, \tilde{\phi})$, where $\phi \in \operatorname{Hom}(B, A)$ (1.7.3). Further, we have that $\phi^{-1}(\mathfrak{j}_a) = \mathfrak{j}_{\psi(a)}$, and hence that the image under ϕ of any element of $B - \mathfrak{j}_{\psi(a)}$ is invertible in the local ring A; the factorization in the result follows from the universal property of the ring of fractions (**0**, 1.2.4). Conversely, to each local homomorphism $\mathcal{O}_y \to A$ there exists a unique corresponding morphism $(\psi, \theta) : X \to \operatorname{Spec}(\mathcal{O}_y)$ such that $\psi(a) = y$ (1.7.3), and, by composing with the canonical morphism $\operatorname{Spec}(\mathcal{O}_y) \to Y$, we obtain a morphism $X \to Y$, which proves the proposition.

(2.4.5). The affine schemes whose ring is a field K have an underlying space that is just a point. If A is a local ring with maximal ideal $\mathfrak{m}$, then each local homomorphism $A \to K$ has kernel equal to $\mathfrak{m}$, and so factorizes as $A \to A/\mathfrak{m} \to K$, where the second arrow is a monomorphism. The morphisms $\operatorname{Spec}(K) \to \operatorname{Spec}(A)$ thus correspond bijectively to monomorphisms of fields $A/\mathfrak{m} \to K$.

Let $(Y, \mathcal{O}_Y)$ be a prescheme; for each $y \in Y$ and each ideal $\mathfrak{a}_y$ of $\mathcal{O}_y$, the canonical homomorphism $\mathcal{O}_y \to \mathcal{O}_y/\mathfrak{a}_y$ defines a morphism $\operatorname{Spec}(\mathcal{O}_y/\mathfrak{a}_y) \to \operatorname{Spec}(\mathcal{O}_y)$; if we compose this with the canonical morphism $\operatorname{Spec}(\mathcal{O}_y) \to Y$, then we obtain a morphism $\operatorname{Spec}(\mathcal{O}_y/\mathfrak{a}_y) \to Y$, again said to be *canonical*. For $\mathfrak{a}_y = \mathfrak{m}_y$, this says that $\mathcal{O}_y/\mathfrak{a}_y = k(y)$, and so Proposition (2.4.4) says that:

Corollary (2.4.6). — *Let* $(X, \mathscr{O}_X)$ *be a local scheme whose ring* K *is a field,* ξ *be the unique point of* X*, and* $(Y, \mathscr{O}_Y)$ *a prescheme. Then each morphism* $u : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ *factorizes uniquely as* $X \to \operatorname{Spec}(k(\psi(\xi))) \to Y$*, where the second arrow denotes the canonical morphism, and the first corresponds to a monomorphism* $k(\psi(\xi)) \to K$*. This establishes a canonical bijective correspondance between the set of morphisms* $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ *and the set of monomorphisms* $k(y) \to K$ *(for* $y \in Y$*).*

Corollary (2.4.7). — *For all* $y \in Y$*, each canonical morphism* $\operatorname{Spec}(\mathscr{O}_y/\mathfrak{a}_y) \to Y$ *is a monomorphism of ringed spaces.*

Proof. We have already seen this when $\mathfrak{a}_y = 0$ (2.4.2), and it suffices to apply Corollary (1.7.5). □

Remark. — 2.4.8 Let X be a local scheme, and a its unique closed point. Since every affine open containing a is necessarily in the whole of X, every *invertible* $\mathscr{O}_X$-module (**0**, 5.4.1) is necessarily *isomorphic to* $\mathscr{O}_X$ (or, as we say, again, *trivial*). This property does not hold in general for an arbitrary affine scheme $\operatorname{Spec}(A)$; we will see in Chapter V that if A is a normal ring then this is true when A is *factorial* (?).

2.5. **Preschemes over a prescheme.**

Definition (2.5.1). — Given a prescheme S, we say that the data of a prescheme X and a morphism of preschemes $\phi : X \to S$ defines a prescheme X *over the prescheme* S, or an S-*prescheme*; we say that S is the *base prescheme* of the S-prescheme X. The morphism ϕ is called the *structure morphism* of the S-prescheme X. When S is an affine scheme of ring A, we also say that X endowed with ϕ is a prescheme *over the ring* A (or an A-*prescheme*).

It follows from (2.2.4) that the data of a prescheme over a ring A is equivalent to the data of a prescheme $(X, \mathscr{O}_X)$ whose structure sheaf $\mathscr{O}_X$ is a sheaf of A-*algebras*. *An arbitrary prescheme can always be considered as a* **Z**-*prescheme in a unique way.*

If $\phi : X \to S$ is the structure morphism of an S-prescheme X, we say that a point $x \in X$ is *over a point* $s \in S$ if $\phi(x) = s$. We say that X *dominates* S if ϕ is a dominant morphism (2.2.6).

(2.5.2). Let X and Y be two S-preschemes; we say that a morphism of preschemes $u : X \to Y$ is a *morphism of preschemes over* S (or an S-*morphism*) if the diagram

$$\begin{tikzcd} X \arrow[rr, "u"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

(where the diagonal arrows are the structure morphisms) is commutative: this ensures that, for all $s \in S$ and $x \in X$ over s, $u(x)$ is also above s.

From this definition it follows immediately that the composition of two S-morphisms is an S-morphism; S-preschemes thus form a *category*.

We denote by $\operatorname{Hom}_S(X, Y)$ the set of S-morphisms from an S-prescheme X to an S-prescheme Y; the identity morphism of an S-prescheme is denoted by 1_X.

When S is an affine scheme of ring A, we will also say A-*morphism* instead of S-morphism. 104

(2.5.3). If X is an S-prescheme, and $v : X' \to X$ a morphism of preschemes, then the composition $X' \to X \to S$ endows X' with the structure of an S-prescheme; in particular, every prescheme induced by an open set U of X can be considered as an S-prescheme by the canonical injection.

If $u : X \to Y$ is an S-morphism of S-preschemes, then the restriction of u to any prescheme induced by an open subset U of X is also an S-morphism $U \to Y$. Conversely, let (U_α) be an open cover of X, and for each α let $u_\alpha : U_\alpha \to Y$ be an S-morphism; if, for all pairs of indices (α, β), the restrictions of u_α and u_β to $U_\alpha \cap U_\beta$ agree, then there exists an S-morphism $X \to Y$, and only one such that the restriction to each U_α is u_α.

If $u : X \to Y$ is an S-morphism such that $u(X) \subset V$, where V is an open subset of Y, then u, considered as a morphism from X to V, is also an S-morphism.

(2.5.4). Let $S' \to S$ be a morphism of preschemes; for all S'-preschemes, the composition $X \to S' \to S$ endows X with the structure of an S-prescheme. Conversely, suppose that S' is the induced prescheme of an open subset of S; let X be an S-prescheme and suppose that the structure morphism $f : X \to S$ is such that $f(X) \subset S'$; then we can consider X as an S'-prescheme. In this latter case, if Y is another S-prescheme whose structure morphism sends the underlying space to S', then every S-morphism from X to Y is also an S'-morphism.

(2.5.5). If X is an S-prescheme, with structure morphism $\phi : X \to S$, we define an S-*section of* X to be an S-morphism from S to X, that is to say a morphism of preschemes $\psi : S \to X$ such that $\phi \circ \psi$ is the identity on S. We denote by $\Gamma(X/S)$ the set of S-sections of X.

3. Products of preschemes

3.1. **Sums of preschemes.** Let (X_α) be any family of preschemes; let X be a topological space which is the *sum* of the underlying spaces X_α; X is then the union of the pairwise disjoint open subspaces X'_α, and for each α there is a homomorphism ϕ_α from X_α to X'_α. If we equip each of the X'_α with the sheaf $(\phi_\alpha)_*(\mathcal{O}_{X_\alpha})$, it is clear that X becomes a prescheme, which we call the *sum* of the family of preschemes (X_α) and which we denote $\amalg_\alpha X_\alpha$. If Y is a prescheme, the map $f \mapsto (f \circ \phi_\alpha)$ is a *bijection* from the set $\mathrm{Hom}(X, Y)$ to the product set $\Pi_\alpha \mathrm{Hom}(X_\alpha, Y)$. In particular, if the X_α are S-preschemes, with structure morphisms ψ_α, X is an S-prescheme by the unique morphism $\psi : X \to S$ such that $\psi \circ \phi_\alpha = \psi_\alpha$ for each α. The sum of two preschemes X, Y is denoted by $X \amalg Y$. It is immediate that if $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$, $X \amalg Y$ canonically identifies with $\mathrm{Spec}(A \times B)$.

3.2. **Products of preschemes.**

Definition (3.2.1). — Given two S-preschemes X, Y, we say that a triple (Z, p_1, p_2) consisting of an S-prescheme Z and of two S-morphisms $p_1 : Z \to X$, $p_2 : Z \to Y$, is a product of the S-preschemes X and Y, if, for each S-prescheme T, the map $f \mapsto (p_1 \circ f, p_2 \circ f)$ is a bijection from the set of S-morphisms from T to Z, to the set of pairs consisting of an S-morphism $T \to X$ and an S-morphism $T \to Y$ (in other words, a bijection

$$\mathrm{Hom}_S(T, Z) \xrightarrow{\sim} \mathrm{Hom}_S(T, X) \times \mathrm{Hom}_S(T, Y)).$$

There is therefore a general notion of a *product* of two objects in a category, applied to the category of S-preschemes (T, I, 1.1); in particular, a product of two S-preschemes is *unique* up to a unique S-isomorphism. Because of this uniqueness, most of the time we will denote a product of two S-preschemes X, Y by the notation $X \times_S Y$ (or simply $X \times Y$ when there is no chance of confusion), the morphisms p_1, p_2 (called the *canonical projections* of $X \times_S$ to X and Y, respectively) are suppressed in the notation. If $g : T \to X$, $h : T \to Y$ are two S-morphisms, we denote by $(g, h)_S$, or simply (g, h), the S-morphism $f : T \to X \times_S Y$ such that $p_1 \circ f = g$, $p_2 \circ f = h$. If X', Y' are two S-preschemes, p'_1, p'_2 the canonical projections of $X' \times_S Y'$ (assumed to exist), $u : X' \to X$, $v : Y' \to Y$ two S-morphisms, then we write $u \times_S v$ (or simply $u \times v$) for the S-morphism $(u \circ p'_1, v \circ p'_2)_S$ from $X' \times_S Y'$ to $X \times_S Y$.

When S is an affine scheme of ring A, we often replace S by A is the above notations.

Proposition (3.2.2). — *Let* X, Y, S *be three affine schemes,* B, C, A *their respective rings. Let* $Z = \mathrm{Spec}(B \otimes_A C)$, p_1, p_2 *the* S-*morphisms corresponding* (2.2.4) *to the canonical* A-*homomorphisms* $u : b \mapsto b \otimes 1$ *and* $v : c \mapsto 1 \otimes c$ *from* B *and* C *to* $B \otimes_A C$*; then* (Z, p_1, p_2) *is a product of* X *and* Y.

Proof. According to (2.2.4), it suffices to check that if, to each A-homomorphism $f : B \otimes_A C \to L$ (where L is an A-algebra), we associate the pair $(f \circ u, f \circ v)$, then we define a bijection $\mathrm{Hom}_A(B \otimes_A C, L) \xrightarrow{\sim} \mathrm{Hom}_A(B, L) \times \mathrm{Hom}_A(C, L)$,[3] which follows immediately from the definitions and the relation $b \otimes c = (b \otimes 1)(1 \otimes c)$. ☐

Corollary (3.2.3). — *Let* T *be an affine scheme of ring* D, $\alpha = ({}^a\rho, \widetilde{\rho})$ *(resp.* $\beta = ({}^a\sigma, \widetilde{\sigma})$*) an* S-*morphism* $T \to X$ *(resp.* $T \to Y$*), where* ρ *(resp.* σ*) is an* A-*homomorphism from* B *(resp.* C*) to* D*; then* $(\alpha, \beta)_S = ({}^a\tau, \widetilde{\tau})$*, where* τ *is the homomorphism* $B \otimes_A C \to D$ *such that* $\tau(b \otimes c) = \rho(b)\sigma(c)$.

Proposition (3.2.4). — *Let* $f : S' \to S$ *be a monomorphism of preschemes* (T, I, 1.1)*,* X, Y *two* S′-*preschemes, which are also considered as* S-*preschemes by means of* f. *Each product of* S-*preschemes* X, Y *is then a product of* S′-*preschemes* X, Y*, and conversely.*

Proof. Let $\phi : X \to S'$, $\psi : Y \to S'$ be the structure morphisms. If T is an S-prescheme, $u : T \to X$, $v : T \to Y$ two S-morphisms, we have by definition $f \circ \phi \circ u = f \circ \psi \circ v = \theta$, the structure morphism of T; the hypotheses on f imply that $\phi \circ u = \psi \circ v = \theta'$, and we can consider T as an S′-prescheme with structure morphism θ', u and v as S′-morphisms. The conclusion of the proposition follows immediately, taking into account (3.2.1). ☐

Corollary (3.2.5). — *Let* X, Y *be two* S-*preschemes,* $\phi : X \to S$, $\psi : Y \to S$ *their structure morphisms,* S′ *an open subset of* S *wuch that* $\phi(X) \subset S'$, $\psi(Y) \subset S'$. *Each product of* S-*preschemes* X, Y *is then also a product of* S′-*preschemes* X, Y*, and conversely.*

It suffices to apply (3.2.4) to the canonical injection $S' \to S$.

Theorem (3.2.6). — *Given two* S-*preschemes* X, Y*, there exists a product* $X \times_S Y$.

Proof. We proceed in several steps.

Lemma (3.2.6.1). — *Let* (Z, p, q) *be a product of* X *and* Y*,* U, V *two open subsets of* X, Y*, respectively. If we put* $W = p^{-1}(U) \cap q^{-1}(V)$*, then the triple consisting of* W *and the restrictions of* p *and* q *to* W *(considered as the morphisms* $W \to U$, $W \to V$*, respectively) is a product of* U *and* V.

[3] The notation Hom_A denotes here the set of homomorphisms of A-*algebras*.

Indeed, if T is an S-prescheme, we can identify the S-morphisms $T \to W$ and the S-morphisms $T \to Z$ mapping T to W. If then $g : T \to U$, $h : T \to V$ are any two S-morphisms, we can consider them as S-morphisms from T to X and Y respectively, and by hypothesis there is then a unique S-morphism $f : T \to Z$ such that $g = p \circ f$, $h = q \circ f$. As $p(f(Y)) \subset U$, $q(f(T)) \subset V$, we have

$$f(T) \subset p^{-1}(U) \cap q^{-1}(V) = W,$$

hence our assertion.

Lemma (3.2.6.2). — *Let* Z *be an* S*-prescheme,* $p : Z \to X$, $q : Z \to Y$ *two* S*-morphisms,* (U_α) *an open cover of* X, (V_λ) *an open cover of* Y*. Suppose that for each pair* (α, λ)*, the* S*-prescheme* $W_{\alpha\lambda} = p^{-1}(U_\alpha) \cap q^{-1}(V_\lambda)$ *and the restrictions of* p *and* q *to* $W_{\alpha\lambda}$ *form a product of* U_α *and* V_λ*. Then* (Z, p, q) *is a product of* X *and* Y*.*

We first show that, if f_1, f_2 are two S-morphisms $T \to Z$, then the relations $p \circ f_1 = p \circ f_2$ and $q \circ f_1 = q \circ f_2$ imply $f_1 = f_2$. Indeed, Z is the union of the $W_{\alpha\lambda}$, so the $f_1^{-1}(W_{\alpha\lambda})$ form an open cover of T, and similarly for $f_2^{-1}(W_{\alpha\lambda})$. In addition, we have

$$f_1^{-1}(W_{\alpha\lambda}) = f_1^{-1}(p^{-1}(U_\alpha)) \cap f_1^{-1}(q^{-1}(V_\lambda)) = f_2^{-1}(p^{-1}(U_\alpha)) \cap f_2^{-1}(q^{-1}(V_\lambda)) = f_2^{-1}(W_{\alpha\lambda})$$

by hypothesis, and it reduces to seeing that the the restrictions of f_1 and f_2 to $f_1^{-1}(W_{\alpha\lambda}) = f_2^{-1}(W_{\alpha\lambda})$ are identical for each pair of indices. But as these restrictions can be considered as S-morphisms from $f_1^{-1}(W_{\alpha\lambda})$ to $W_{\alpha\lambda}$, our assertion follows from the hypotheses and Definition (3.2.1).

Suppose now that we are given two S-morphisms $g : T \to X$, $h : T \to Y$. Put $T_{\alpha\lambda} = g^{-1}(U_\alpha) \cap h^{-1}(V_\lambda)$; the $T_{\alpha\lambda}$ form an open cover of T. By hypothesis, there exists an S-morphism $f_{\alpha\lambda}$ such that $p \circ f_{\alpha\lambda}$ and $q \circ f_{\alpha\lambda}$ are the respective restrictions of g and h to $T_{\alpha\lambda}$. In addition, we show that the restrictions of $f_{\alpha\lambda}$ and $f_{\beta\mu}$ to $T_{\alpha\lambda} \cap T_{\beta\mu}$ coincide, which would finish the proof of Lemma (3.2.6.2). The images of $T_{\alpha\lambda} \cap T_{\beta\mu}$ under $f_{\alpha\lambda}$ and $f_{\beta\mu}$ are contained in $W_{\alpha\lambda} \cap W_{\beta\mu}$ by definition. As

$$W_{\alpha\lambda} \cap W_{\beta\mu} = p^{-1}(U_\alpha \cap U_\beta) \cap q^{-1}(V_\lambda \cap V_\mu),$$

it follows from Lemma (3.2.6.1) that $W_{\alpha\lambda} \cap W_{\beta\mu}$ and the restrictions to this prescheme of p and q form a *product* of $U_\alpha \cap U_\beta$ and $V_\lambda \cap V_\mu$. As $p \circ f_{\alpha\lambda}$ and $p \circ f_{\beta\mu}$ coincide on $T_{\alpha\lambda} \cap T_{\beta\mu}$ and similarly for $q \circ f_{\alpha\lambda}$ and $q \circ f_{\beta\mu}$, we see that $f_{\alpha\lambda}$ and $f_{\beta\mu}$ coincide on $T_{\alpha\lambda} \cap T_{\beta\mu}$, q.e.d.

Lemma (3.2.6.3). — *Let* (U_α) *be an open cover of* X, (V_λ) *an open cover of* Y*, and suppose that for each pair* (α, λ)*, there exists a product of* U_α *and* V_λ*; then there exists a product of* X *and* Y*.* 107

Applying Lemma (3.2.6.1) to the open sets $U_\alpha \cap U_\beta$ and $V_\lambda \cap V_\mu$, we see that there exists a product of S-preschemes induced respectively by X and Y on these open sets; in addition, the uniqueness of the product shows that, if we set $i = (\alpha, \lambda)$, $j = (\beta, \mu)$, then there is a canonical isomorphism h_{ij} (resp. h_{ji}) from this product to an S-prescheme W_{ij} (resp. W_{ji}) induced by $U_\alpha \times_S V_\lambda$ (resp. $U_\beta \times_S V_\mu$) on an open set; $f_{ij} = h_{ij} \circ h_{ji}^{-1}$ is then an isomorphism from W_{ji} to W_{ij}. In addition, for a third pair $k = (\gamma, \nu)$, we have $f_{ik} = f_{ij} \circ f_{jk}$ on $W_{ki} \cap W_{kj}$, as it follows from applying Lemma (3.2.6.1) to the open sets $U_\alpha \cap U_\beta \cap U_\gamma$ and $V_\lambda \cap V_\mu \cap V_\nu$ in U_β and V_μ, respectively. It follows that we have a prescheme Z, an open cover (Z_i) of the underlying space of Z, and for each i and isomorphism g_i from the induced prescheme Z_i to the prescheme $U_\alpha \times_S V_\lambda$, so that for each pair (i, j), we have $f_{ij} = g_i \circ g_j^{-1}$ (2.3.1); in addition, we have $g_i(Z_i \cap Z_j) = W_{ij}$. If p_i, q_i, θ_i are the projections and the structure morphism of the S-prescheme $U_\alpha \times_S V_\lambda$, we immediately note that $p_i \circ g_i = p_j \circ g_j$ on $Z_i \cap Z_j$, and similarly for the two other morphisms. We can thus define the morphisms of preschemes $p : Z \to X$ (resp. $q : Z \to Y$, $\theta : Z \to S$) by the condition that p (resp. q, θ) coincide with $p_i \circ g_i$ (resp. $q_i \circ g_i$, $\theta_i \circ g_i$) on each of the Z_i; Z, equipped with θ, is then an S-prescheme. We now show that $Z'_i = p^{-1}(U_\alpha) \cap q^{-1}(V_\lambda)$ is equal to Z_i. For each index $j = (\beta, \mu)$, we have $Z_j \cap Z'_i = g_j^{-1}(p_j^{-1}(U_\alpha) \cap q_j^{-1}(V_\lambda))$. We have

$$p_j^{-1}(U_\alpha) \cap q_j^{-1}(V_\lambda) = p_j^{-1}(U_\alpha \cap U_\beta) \cap q_j^{-1}(V_\lambda \cap V_\mu);$$

according to Lemma (3.2.6.1), the restrictions of p_j and q_j to $p_j^{-1}(U_\alpha) \cap q_j^{-1}(V_\lambda)$ define on this S-prescheme the structure of a product of $U_\alpha \cap U_\beta$ and $V_\lambda \cap V_\mu$; but the uniqueness of the product then implies that $p_j^{-1}(U_\alpha) \cap q_j^{-1}(V_\lambda) = W_{ji}$. As a result we have $Z_j \cap Z'_i = Z_j \cap Z_i$ for each j, hence $Z'_i = Z_i$. We then deduce from Lemma (3.2.6.2) that (Z, p, q) is a product of X and Y.

Lemma (3.2.6.4). — *Let* $\phi : X \to S$, $\psi : Y \to S$ *be the structure morphisms of* X *and* Y*,* (S_i) *and open cover of* S*, and set* $X_i = \phi^{-1}(S_i)$*,* $Y_i = \psi^{-1}(S_i)$*. If each of the products* $X_i \times_S Y_i$ *exists, then* $X \times_S Y$ *exists.*

According to Lemma (3.2.6.3), everything comes down to proving that the products $X_i \times_S Y_i$ exists for any i and j. Set $X_{ij} = X_i \cap X_j = \phi^{-1}(S_i \cap S_j)$, $Y_{ij} = Y_i \cap Y_j = \psi^{-1}(S_i \cap S_j)$; according to Lemma (3.2.6.1), the product $Z_{ij} = X_{ij} \times_S Y_{ij}$ exists. We now note that if T is an S-prescheme and if $g : T \to X_i$, $h : T \to Y_j$ are S-morphisms, then we necessarily have that $\phi(g(T)) = \psi(h(T)) \subset S_i \cap S_j$ according to the definition of an S-morphisms, thus $g(T) \subset X_{ij}$ and $h(T) \subset Y_{ij}$; it is then immediate that Z_{ij} is the product of X_i and Y_j.

(3.2.6.5). We can now complete the proof of Theorem (3.2.6). If S is an *affine scheme*, there are covers (U_α), (V_λ) of X and Y respectively, consisting of affine opens; as $U_\alpha \times_S V_\lambda$ exists according to (3.2.2), there similarly exists $X \times_S Y$ by Lemma (3.2.6.3). If S is any prescheme, there is a cover (S_i) of S consisting of affine opens. If $\phi : X \to S$, $\psi : Y \to S$ are the structure morphisms, and if we set $X_i = \phi^{-1}(S_i)$, $Y_i = \psi^{-1}(S_i)$, the products $X_i \times_{S_i} Y_i$ exist according to the above; but then the products $X_i \times_S Y_i$ also exist (3.2.5), therefore $X \times_S Y$ similarly exists by Lemma (3.2.6.4).

□

Corollary (3.2.7). — *Let $Z = X \times_S Y$ be the product of two S-preschemes, p, q the projections from Z to X and Y, ϕ (resp. ψ) the structure morphism of X (resp. Y). Let S' be an open subset of S, U (resp. V) an open subset of X (resp. Y) contained in $\phi^{-1}(S')$ (resp. $\psi^{-1}(S')$). Then the product $U \times_{S'} V$ canonically identifies with the prescheme induced on Z by $p^{-1}(U) \cap q^{-1}(V)$ (considered as a S'-prescheme). In addition, if $f : T \to X$, $g : T \to Y$ are S-morphisms such that $f(T) \subset U$, $g(T) \subset V$, the S'-morphism $(f,g)_{S'}$ identifies with the restriction of $(f,g)_S$ to $p^{-1}(U) \cap q^{-1}(V)$.*

Proof. This follows from Corollary (3.2.5) and Lemma (3.2.6.1). □

(3.2.8). Let (X_α), (Y_λ) be two familes of S-preschemes, X (resp. Y) the sum of the family (X_α) (resp. (Y_λ)) (3.1). Then $X \times_S Y$ identifies with the *sum* of the family $(X_\alpha \times_S Y_\lambda)$; this follows immediately from Lemma (3.2.6.3).

(3.2.9). [4] It follows from (1.8.1) that we can state (3.2.2) in the following manner: $Z = \mathrm{Spec}(B \otimes_A C)$ is not only a product of $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(C)$ in the category of S-*preschemes*, but also in the category of *locally ringed spaces over* S (with a definition of S-morphisms modeled on that of (2.5.2)). The proof of (3.2.6) also proves that for any two S-preschemes X, Y, the prescheme $X \times_S Y$ is not only the product of X and Y in the category of S-preschemes, but also in the category of locally ringed spaces over the prescheme S.

3.3. Formal properties of the product; change of the base prescheme.

(3.3.1). The reader will notice that all the properties stated in this section, except (3.3.13) and (3.3.15), are true without modification in any category, whenever the products involved in the statements exist (since it is clear that the notions of an S-object and of an S-morphism can be defined exactly as in (2.5) for any object S of the category).

(3.3.2). First, $X \times_S Y$ is a *covariant bifunctor* in X and Y on the category of S-preschemes: it suffices in fact to note that the diagram

$$\begin{tikzcd} X \times Y \arrow[r, "f \times 1"] \arrow[d] & X' \times Y \arrow[r, "f' \times 1"] \arrow[d] & X'' \times Y \arrow[d] \\ X \arrow[r, "f"] & X' \arrow[r, "f'"] & X'' \end{tikzcd}$$

is commutative.

Proposition (3.3.3). — *For each S-prescheme X, the first (resp. second) projection from $X \times_S S$ (resp. $S \times_S X$) is a functorial isomorphism from $X \times_S S$ (resp. $S \times_S X$) to X, whose inverse isomorphism is $(1_X, \phi)_S$ (resp. $(\phi, 1_X)_S$), where we denote by ϕ the structure morphism $X \to S$; therefore we can write, up to a canonical isomorphism,*

$$X \times_S S = S \times_S X = X.$$

Proof. It suffices to prove that the triple $(X, 1_X, \phi)$ is a product of X and S. If T is an S-prescheme, the only S-morphism from T to S is necessarily the structure morphism $\psi : T \to S$. If f is an S-morphism from T to X, we necessarily have $\psi = \phi \circ f$, hence our assertion. □

Corollary (3.3.4). — *Let X, Y be two S-preschemes, $\phi : X \to S$, $\psi : Y \to S$ their structure morphisms. If we canonically identify X with $X \times_S S$ and Y with $S \times_S Y$, the projections $X \times_S Y \to X$ and $X \times_S Y \to Y$ identify respectively with $1_X \times \psi$ and $\phi \times 1_Y$.*

The proof is immediate and is left to the reader.

(3.3.5). We can define in a manner similar to (3.2) the product of a finite number n of S-preschemes, the existence of these products following from (3.2.6) by induction on n, and noting that $(X_1 \times_S X_2 \times_S \cdots \times_S X_{n-1}) \times_S X_n$ satisfies the definition of a product. The uniqueness of the product implies, as in any category, its *commutativity* and *associativity* properties. If, for example, p_1, p_2, p_3 denote the projections from $X_1 \times_S X_2 \times_S X_3$, and if we identify this prescheme with $(X_1 \times_S X_2) \times_S X_3$, then the projection to $X_1 \times_S X_2$ is identified with $(p_1, p_2)_S$.

[4] [Trans] (3.2.9) is from the errata of EGA II, on page 221.

(3.3.6). Let S, S' be two preschemes, $\phi : S' \to S$ a morphism, which makes S' an S-prescheme. For each S-prescheme X, consider the product $X \times_S S'$, and let p and π' be the projections to X and S' respectively. Equipped with π', this product is an S'-prescheme; when we consider it as such, we denote it by $X_{(S')}$ or $X_{(\phi)}$, and we say that this is the prescheme obtained by *base change* from S to S', by means of the morphism ϕ, or the *inverse image* of X by ϕ. We note that if π is the structure morphism of X, θ the structure morphism of $X \times_S S'$, considered as an S-prescheme, then the diagram

$$\begin{tikzcd} X \arrow[d, "\pi"'] & X_{(S')} \arrow[l, "p"'] \arrow[d, "\pi'"] \arrow[dl, "\theta"'] \\ S & S' \arrow[l, "\phi"] \end{tikzcd}$$

is commutative.

(3.3.7). With the notation of (3.3.6), for each S-morphism $f : X \to Y$, we denote by $f_{(S')}$ the S'-morphism $f \times_S 1 : X_{(S')} \to Y_{(S')}$, and we say that $f_{(S')}$ is the *base change* (or *inverse image*) of f by ϕ. Therefore, $X_{(S')}$ is a *covariant functor* in X, from the category of S-preschemes to that of S'-preschemes.

(3.3.8). The prescheme $X_{(S')}$ can be considered as a solution to a *universal mapping problem*: each S'-prescheme T is also an S-prescheme via ϕ; each S-morphism $g : T \to X$ is then uniquely written as $g = p \circ f$, where f is an S'-morphism $T \to X_{(S')}$, as it follows from the definition of the product applied to the S-morphisms f and $\psi : T \to S'$ (the structure morphism of T).

Proposition (3.3.9). — *("Transitivity of base change"). Let S'' be a prescheme, $\phi' : S'' \to S$ a morphism. For each S-prescheme X, there exists an canonical functorial isomorphism from the S''-prescheme $(X_{(\phi)})_{(\phi')}$ to the S''-prescheme $X_{(\phi\circ\phi')}$.*

Proof. Let T be a S''-prescheme, ψ its structure morphism, and g an S-morphism from T to X (T being considered as an S-prescheme with structure morphism $\phi\circ\phi'\circ\psi$). As T is also a S'-prescheme with structure morphism $\phi'\circ\psi$, we can write $g = p \circ g'$, where g' is an S'-morphism $T \to X_{(\phi)}$, and then $g' = p' \circ g''$, where g'' is an S''-morphism $T \to (X_{(\phi)})_{(\phi')}$:

$$\begin{tikzcd} X \arrow[d, "\pi"'] & X_{(\phi)} \arrow[l, "p"'] \arrow[d, "\pi'"'] & (X_{(\phi)})_{(\phi')} \arrow[l, "p'"'] \arrow[d, "\pi''"] \\ S & S \arrow[l, "\phi"] & S''. \arrow[l, "\phi'"] \end{tikzcd}$$

Hence the result follows by the uniqueness of the solution to a universal mapping problem. □ 110

This result can be written as the equality (up to a canonical isomorphism) $(X_{(S')})_{(S'')} = X_{(S'')}$, if there is no chance of confusion, or also

$$(X \times_S S') \times_{S'} S'' = X \times_S S''; \tag{3.3.9.1}$$

the functorial nature of the isomorphism defined in (3.3.9) can similarly be expressed by the transitivity formula for base change morphisms

$$(f_{(S')})_{(S'')} = f_{(S'')} \tag{3.3.9.2}$$

for each S-morphism $f : X \to Y$.

Corollary (3.3.10). — *If X and Y are two S-preschemes, then there exists a canonical functorial isomorphism from the S'-prescheme $X_{(S')} \times_{S'} Y_{(S')}$ to the S'-prescheme $(X \times_S Y)_{(S')}$.*

Proof. We have, up to canonical isomorphism,

$$(X \times_S S') \times_{S'} (Y \times_S S') = X \times_S (Y \times_S S') = (X \times_S Y) \times_S S'$$

according to (3.3.9.1) and the associativity of products of S-preschemes. □

The functorial nature of the isomorphism defined in Corollary (3.3.10) can be expressed by the formula

$$(u_{(S')}, v_{(S')})_{S'} = ((u, v)_S)_{(S')} \tag{3.3.10.1}$$

for each pair of S-morphisms $u : T \to X$, $v : T \to Y$.

In other words, the base change functor $X_{(S')}$ *commutes with products*; it also commutes with sums (3.2.8).

Corollary (3.3.11). — *Let Y be an S-prescheme, $f : X \to Y$ a morphism which makes X a Y-prescheme (and as a result also an S-prescheme). The prescheme $X_{(S')}$ then identifies with the product $X \times_Y Y_{(S')}$, the projection $X \times_Y Y_{(S')} \to Y_{(S')}$ identifying with $f_{(S')}$.*

Proof. Let $\psi : Y \to S$ be the structure morphism of Y; we have the commutative diagram

$$\begin{tikzcd} S' \arrow[d] & Y_{(S')} \arrow[l] \arrow[d] & X_{(S')} \arrow[l, "f_{(S')}"'] \arrow[d] \\ S & Y \arrow[l, "\psi"] & X. \arrow[l, "f"] \end{tikzcd}$$

We have that $Y_{(S')}$ identifies with $S'_{(\psi)}$ and $X_{(S')}$ with $S'_{(\psi\circ f)}$; taking into account (3.3.9) and (3.3.4), we deduce the corollary. □

(3.3.12). Let $f : X \to X'$, $g : Y \to Y'$ be two S-morphisms which are *monomorphisms* of preschemes (T, I, 1.1); then $f \times_S g$ is a *monomorphism*. Indeed, if p and q are the projections of $X \times_S Y$, p', q' those of $X' \times_S Y'$, and u, v two S-morphisms $T \to X \times_S Y$, then the relation $(f \times_S g) \circ u = (f \times_S g) \circ v$ implies that $p' \circ (f \times_S g) \circ u = p' \circ (f \times_S g) \circ v$, in other words, $f \circ p \circ u = f \circ p \circ v$, and as f is a monomorphism, $p \circ u = p \circ v$; using the fact that g is a monomorphism, we similarly obtain $q \circ u = q \circ v$, hence $u = v$.

It follows that for each base change $S' \to S$,

$$f_{(S')} : X_{(S')} \longrightarrow Y_{(S')}$$

is a monomorphism.

(3.3.13). Let S, S′ be two affine schemes of rings A, A′ respectively; a morphism $S' \to S$ then corresponds to a ring homomorphism $A \to A'$. If X is an S-prescheme, we denote by $X_{(A')}$ or $X \otimes_A A'$ the S′-prescheme $X_{(S')}$; when X is also affine of ring B, $X_{(A')}$ is affine of ring $B_{(A')} = B \otimes_A A'$ obtained by extension by scalars from the A-algebra B to A′.

(3.3.14). With the notation of (3.3.6), for each S-*morphism* $f : S' \to X$, $f' = (f, 1_{S'})_S$ is an S′-morphism $S' \to X' = X_{(S')}$ such that $p \circ f' = f$, $\pi' \circ f' = 1_{S'}$, in other words an S′-*section of of* X′; conversely, if f' is such an S′-section, $f = p \circ f'$ is an S-morphism $S' \to X$. We thus define a canonical *bijective correspondence*

$$\mathrm{Hom}_S(S', X) \xrightarrow{\sim} \mathrm{Hom}_{S'}(S', X').$$

We say that f' is the *graph morphism* of f, and we denote it by Γ_f.

(3.3.15). Given a prescheme X, which we can always consider it as a **Z**-prescheme, it follows in particular from (3.3.14) that the X-*sections* of $X \otimes_{\mathbf{Z}} \mathbf{Z}[T]$ (where T is an indeterminate) bijectively correspond to *morphisms* $\mathbf{Z}[T] \to X$. Let us show that these X-sections also bijectively correspond to *sections of the structure sheaf* $\mathscr{O}_X$ *over* X. Indeed, let (U_α) be a cover of X by the affine opens; let $u : X \to X \otimes_{\mathbf{Z}} \mathbf{Z}[T]$ be an X-morphism and let u_α be its restriction to U_α; if A_α is the ring of the affine scheme U_α, then $U_\alpha \otimes_{\mathbf{Z}} \mathbf{Z}[T]$ is an affine scheme of ring $A_\alpha[T]$ (3.2.2), and u_α canonically corresponds to an A_α-homomorphism $A_\alpha[T] \to A_\alpha$ (1.7.3). Now, as such a homomorphism is completely determined by the data of the image of T in A_α, let $s_\alpha \in A_\alpha = \Gamma(U_\alpha, \mathscr{O}_X)$, and if we suppose that the restrictions of u_α and u_β to an open affine $V \subset U_\alpha \cap U_\beta$ coincide, then we see immediately that s_α and s_β coincide on V; thus the family (s_α) consists of the restrictions to U_α of a section s of $\mathscr{O}_X$ over X; convsersely, it is clear that such a section defines a family (u_α) of morphisms which are the restrictions to U_α of an X-morphism $X \to X \otimes_{\mathbf{Z}} \mathbf{Z}[T]$. This result is generalized in (**II**, 1.7.12).

3.4. Points of a prescheme with values in a prescheme; geometric points.

(3.4.1). Let X be a prescheme; for each prescheme T, we then denote by X(T) the set Hom(T, X) of morphism $T \to X$, and the elements of this set are called *the points of* X *with values in* T. If we associate to each morphism $f : T \to T'$ the map $u' \mapsto u' \circ f$ from X(T′) to X(T), we see that, for X fixed, X(T) is a *contravariant functor in* T, from the category of preschemes to that of sets. In addition, each morphism of preschemes $g : X \to Y$ defines a functorial homomorphism $X(T) \to Y(T)$, which sends $v \in X(T)$ to $g \circ v$.

(3.4.2). Given three sets P, Q, R and two maps $\phi : P \to R$, $\psi : Q \to R$, we define the *fibre product of* P *and* Q *over* R (relative to ϕ and ψ) as the subset of the product set $P \times Q$ consisting of the pairs (p, q) such that $\phi(p) = \psi(q)$; we denote it by $P \times_R Q$. Definition (3.2.1) of the product of S-preschemes can be interpreted, with the notation of (3.4.1), via the formula

(3.4.2.1) $$(X \times_S Y)(T) = X(T) \times_{S(T)} Y(T).$$

the maps $X(T) \to S(T)$ and $Y(T) \to S(T)$ corresponding to the structure morphisms $X \to S$ and $Y \to S$.

(3.4.3). If we are given a prescheme S and we consider only the S-preschemes and S-morphisms, then we will denote by $X(T)_S$ the set $\mathrm{Hom}_S(T, X)$ of S-morphisms $T \to X$, and suppress the subscript S when there is no chance of confusion; we say that the elements of $X(T)_S$ are the *points* (or S-*points* when there is a possibility of confusion) *of the* S-*prescheme* X *with values in the* S-*prescheme* T. In particular, an S-*section* of X is none other than a *point of* X *with values in* S. The formuka (3.4.2.1) can then be written as

(3.4.3.1) $$(X \times_S Y)(T)_S = X(T)_S \times Y(T)_S;$$

more generally, if Z is an S-prescheme, and X, Y, T are Z-preschemes (thus *ipso facto* S-preschemes), then we have

(3.4.3.2) $$(\mathrm{X} \times_{\mathrm{Z}} \mathrm{Y})(\mathrm{T})_{\mathrm{S}} = \mathrm{X}(\mathrm{T})_{\mathrm{S}} \times_{\mathrm{Z}(\mathrm{T})_{\mathrm{S}}} \mathrm{Y}(\mathrm{T})_{\mathrm{S}}.$$

We note that to show that a triple (W, r, s) consisting of an S-prescheme W and two S-morphisms $r : \mathrm{W} \to \mathrm{X}$, $s : \mathrm{W} \to \mathrm{Y}$ is a product of X and Y (over Z), it suffices by definition to check that for *each* S-*prescheme* T, the diagram

$$\begin{tikzcd} \mathrm{W}(\mathrm{T})_{\mathrm{S}} \arrow[r, "r'"] \arrow[d, "s'"'] & \mathrm{X}(\mathrm{T})_{\mathrm{S}} \arrow[d, "\phi'"] \\ \mathrm{Y}(\mathrm{T})_{\mathrm{S}} \arrow[r, "\psi'"] & \mathrm{Z}(\mathrm{T})_{\mathrm{S}} \end{tikzcd}$$

makes $\mathrm{W}(\mathrm{T})_{\mathrm{S}}$ the fibre product of $\mathrm{X}(\mathrm{T})_{\mathrm{S}}$ and $\mathrm{Y}(\mathrm{T})_{\mathrm{S}}$ over $\mathrm{Z}(\mathrm{T})_{\mathrm{S}}$, where r' and s' correspond to r and s, ϕ' and ψ' to the structure morphisms $\phi : \mathrm{X} \to \mathrm{Z}$, $\psi : \mathrm{Y} \to \mathrm{Z}$.

(3.4.4). When T (resp. S) in the above is an affine scheme of ring B (resp. A), we replace T (resp. S) by B (resp. A) in the above notations, and we then call the elements of X(B) the *points of* X *with values in the ring* B, and the elements of $\mathrm{X}(\mathrm{B})_{\mathrm{A}}$ the *points of the* A-*prescheme* X *with values in the* A-*algebra* B. We note that X(B) and $\mathrm{X}(\mathrm{B})_{\mathrm{A}}$ are *covariant* functors in B. We similarly write $\mathrm{X}(\mathrm{T})_{\mathrm{A}}$ for the set of points of the A-prescheme X with values in the A-prescheme T.

(3.4.5). Consider in particular that case where T is of the form Spec(A), where A is a *local* ring; the elements of X(A) then bijectively correspond to *local* homomorphisms $\mathscr{O}_x \to \mathrm{A}$ for $x \in \mathrm{X}$ (2.2.4); we say that the point x of the underlying space of X is the *location*[5] of the point of X with values in A to which it corresponds.

More particularly, we call the *geometric points* of a prescheme S the *points of* X *with values in a field* K: the data of such a point is equivalent to the data of its location x in the underlying subspace of X, and of an *extension* K of $k(x)$; K will be 113
called the *field of values* of the corresponding geometric point, and we say that this geometric point is *located at* x. We also define a map $\mathrm{X}(\mathrm{K}) \to \mathrm{X}$, sending a geometric point with values in K to its location.

If $\mathrm{S}' = \mathrm{Spec}(\mathrm{K})$ is an S-prescheme (in other words, if K is considered as an extension of the residue field $k(s)$, where $s \in \mathrm{S}$) and if X is an S-prescheme, then an element of $\mathrm{X}(\mathrm{K})_{\mathrm{S}}$, or as we say, a *geometric point of* X *lying over* s *with values in* K, consists of the data of a $k(s)$-monomorphism from the residue field $k(x)$ to K, where x is a point of X *lying over* s (therefore $k(x)$ is an extension of $k(s)$).

In particular, if $\mathrm{S} = \mathrm{Spec}(\mathrm{K}) = \{\xi\}$, *then the geometric points of* X *with values in* K *identify with the points* $x \in \mathrm{X}$ *such that* $k(x) = \mathrm{K}$; we say that these latter points are the K-*rational points of the* K-*prescheme* X; if K′ is an extension of K, then the geometric points of X with values in K′ bijectively correspond to the K′-rational points of $\mathrm{X}' = \mathrm{X}_{(\mathrm{K}')}$ (3.3.14).

Lemma (3.4.6). — *Let* X_i *($1 \leqslant i \leqslant n$) be* S*-preschemes,* s *a point of* S*,* x_i *($1 \leqslant i \leqslant n$) a point of* X_i *lying over* s*. Then there exists an extension* K *of* $k(s)$ *and a geometric point of the product* $\mathrm{Y} = \mathrm{X}_1 \times_{\mathrm{S}} \mathrm{X}_2 \times_{\mathrm{S}} \cdots \times_{\mathrm{S}} \mathrm{X}_n$*, with values in* K*, whose projections to the* X_i *are located at* x_i*.*

4. Subpreschemes and immersion morphisms

4.1. Subpreschemes.

(4.1.1). As the notion of a quasi-coherent sheaf (**0**, 5.1.3) is local, a quasi-coherent $\mathscr{O}_{\mathrm{X}}$-module $\mathscr{F}$ over a prescheme X can be defined by the condition that, for each affine open V of X, $\mathscr{F}|\mathrm{V}$ is isomorphic to the sheaf associated to a $\Gamma(\mathrm{V}, \mathscr{O}_{\mathrm{X}})$-module (1.4.1). It is clear that over a prescheme X, the structure sheaf $\mathscr{O}_{\mathrm{X}}$ is quasi-coherent and that the kernels, cokernels, and images of homomorphisms of quasi-coherent $\mathscr{O}_{\mathrm{X}}$-modules, as well as inductive limits and direct sums of quasi-coherent $\mathscr{O}_{\mathrm{X}}$-modules, are also quasi-coherent (Theorem (1.3.7) and Corollary (1.3.9)).

Proposition (4.1.2). — *Let* X *be a prescheme,* $\mathscr{J}$ *a quasi-coherent sheaf of ideals of* $\mathscr{O}_{\mathrm{X}}$*. The support* Y *of the sheaf* $\mathscr{O}_{\mathrm{X}}/\mathscr{J}$ *is then closed, and if we denote by* $\mathscr{O}_{\mathrm{Y}}$ *the restriction of* $\mathscr{O}_{\mathrm{X}}/\mathscr{J}$ *to* Y*, then* $(\mathrm{Y}, \mathscr{O}_{\mathrm{Y}})$ *is a prescheme.*

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Proof. It evidently suffices (2.1.3) to consider the case where X is an affine scheme, and to show that in this case Y is closed in X and is an *affine scheme*. Indeeed, if $\mathrm{X} = \mathrm{Spec}(\mathrm{A})$, then we have $\mathscr{O}_{\mathrm{X}} = \widetilde{\mathrm{A}}$ and $\mathscr{J} = \widetilde{\mathfrak{J}}$, where $\mathfrak{J}$ is an ideal of A (1.4.1); Y is then equal to the closed subset $\mathrm{V}(\mathfrak{J})$ of X and identifies with the prime spectrum of the ring $\mathrm{B} = \mathrm{A}/\mathfrak{J}$ (1.1.1.11); in addition, if ϕ is the canonical homomorphism $\mathrm{A} \to \mathrm{B} = \mathrm{A}/\mathfrak{J}$, then the direct image ${}^a\phi_*(\widetilde{\mathrm{B}})$ canonically identifies with the sheaf $\widetilde{\mathrm{A}}/\widetilde{\mathfrak{J}} = \mathscr{O}_{\mathrm{X}}/\mathscr{J}$ (Proposition (1.6.3) and Corollary (1.3.9)), which finishes the proof. □

We say that $(\mathrm{Y}, \mathscr{O}_{\mathrm{Y}})$ is the *subprescheme* of $(\mathrm{X}, \mathscr{O}_{\mathrm{X}})$ *defined by the sheaf of ideals* $\mathscr{J}$; this is a particular case of the more general notion of *subprescheme*:

Definition (4.1.3). — We say that a ringed space $(\mathrm{Y}, \mathscr{O}_{\mathrm{Y}})$ is a subprescheme of a prescheme $(\mathrm{X}, \mathscr{O}_{\mathrm{X}})$ if:

[5] [Trans] We say that the geometric point lies over this x.

1st Y is a localy closed subspace of X;
2nd if U denotes the largest open subset of X containing Y such that Y is closed in U (*equivalently*, the complement in X of the boundary of Y with respect to $\overline{Y}$), then $(Y, \mathscr{O}_Y)$ is a subprescheme of $(U, \mathscr{O}_X|U)$ defined by a quasi-coherent sheaf of ideals of $\mathscr{O}_X|U$.

We say that the subprescheme $(Y, \mathscr{O}_Y)$ of $(X, \mathscr{O}_X)$ is closed if Y is closed in X (in which case $U = X$).

It follows immediately from this definition and Proposition (4.1.2) that the closed subpreschemes of X are in canonical *bijective correspondence* with the quasi-coherent sheaf of ideals $\mathscr{J}$ of $\mathscr{O}_X$, since if two such sheaves $\mathscr{J}$, $\mathscr{J}'$ have the same (closed) support Y and if the restrictions of $\mathscr{O}_X/\mathscr{J}$ and $\mathscr{O}_X/\mathscr{J}'$ to Y are identical, then we have $\mathscr{J}' = \mathscr{J}$.

(4.1.4). Let $(Y, \mathscr{O}_Y)$ be a subprescheme of X, U the largest open subset of X containing Y and in which Y is closed, V an open subset of X contained in U; then $V \cap Y$ is closed in V. In addition, if Y is defined by the quasi-coherent sheaf of ideals $\mathscr{J}$ of $\mathscr{O}_X|U$, then $\mathscr{J}|V$ is a quasi-coherent sheaf of ideals of $\mathscr{O}_X|V$, and it is immediate that the prescheme induced by Y on $Y \cap V$ is the closed subprescheme of V defined by the sheaf of ideals $\mathscr{J}|V$. Conversely:

Proposition (4.1.5). — *Let* $(Y, \mathscr{O}_Y)$ *be a ringed space such that* Y *is a subspace of* X *and there exists a cover* (V_α) *of* Y *by open subsets of* X *such that for each* α, $Y \cap V_\alpha$ *is closed in* V_α *and the ringed space* $(Y \cap V_\alpha, \mathscr{O}_Y|(Y \cap V_\alpha))$ *is a closed subprescheme of the prescheme induced on* V_α *by* X*. Then* $(Y, \mathscr{O}_Y)$ *is a subprescheme of* X.

Proof. The hypotheses imply that Y is locally closed in X and that the largest open U containing Y in which is closed contains all the V_α; we can thus reduce to the case where $U = X$ and Y is closed in X. We then define a quasi-coherent sheaf of ideals $\mathscr{J}$ of $\mathscr{O}_X$ by taking $\mathscr{J}|V_\alpha$ to be the sheaf of ideals of $\mathscr{O}_X|V_\alpha$ which define the closed subprescheme $(Y \cap V_\alpha, \mathscr{O}_Y|(Y \cap V_\alpha))$, and for each open subset W of X not intersecting Y, $\mathscr{J}|W = \mathscr{O}_X|W$. We check immediately according to Definition (4.1.3) and (4.1.4) that there exists a unique sheaf of ideals $\mathscr{J}$ satisfying these conditions and that define the closed subprescheme $(Y, \mathscr{O}_Y)$. □

In particular, the *induced* prescheme by X on an *open subset* of X is a *subprescheme* of X.

Proposition (4.1.6). — *A subprescheme (resp. a closed subprescheme) of a subprescheme (resp. closed subprescheme) of* X *canonically identifies with a subprescheme (resp. closed subprescheme) of* X.

Proof. Since a locally closed subset of a locally closed subspace of X is a locally closed subspace of X, it is clear (4.1.5) that the question is local and that we can thus suppose that X is affine; the proposition then follows from the canonical identification of $A/\mathfrak{J}'$ and $(A/\mathfrak{J})/(\mathfrak{J}'/\mathfrak{J})$ when $\mathfrak{J}$, $\mathfrak{J}'$ are two ideals of a ring A such that $\mathfrak{J} \subset \mathfrak{J}'$. □

We will always make the previous identification.

(4.1.7). Let Y be a subprescheme of a prescheme X, and denote by ψ the canonical injection $Y \to X$ of the *underlying subspaces*; we know that the inverse image $\psi^*(\mathscr{O}_X)$ is the restriction $\mathscr{O}_X|Y$ (**0**, 3.1.7). If, for each $y \in Y$, we denote by ω_y the canonical homomorphism $(\mathscr{O}_X)_y \to (\mathscr{O}_Y)_y$, then these homomorphisms are the restrictions to stalks of a *surjective* homomorphism ω of sheaves of rings $\mathscr{O}_X|Y \to \mathscr{O}_Y$: indeed, is suffices to check locally on Y, that is to say, we can suppose that X is affine and that the subprescheme Y is closed; if in this case $\mathscr{J}$ is the sheaf of ideals in $\mathscr{O}_X$ which defines Y, then the ω_y are none other than the restriction to stalks of the homomorphism $\mathscr{O}_X|Y \to (\mathscr{O}_X/\mathscr{J})|Y$. We have thus defined a *monomorphism of ringed spaces* (**0**, 4.1.1) $j = (\psi, \omega^\flat)$ which is evidently a morphism $Y \to X$ of preschemes (2.2.1), and we call this the *canonical injection morphism*.

If $f : X \to Z$ is a morphism, we then say that the composite morphism $Y \xrightarrow{j} X \xrightarrow{f} Z$ is the *restriction* of f to the subprescheme Y.

5. Reduced preschemes; separation conditions

6. Finiteness conditions

7. Rational maps

8. Chevalley schemes

8.1. **Allied local rings.** For each local ring A, we denote by $\mathfrak{m}(A)$ the maximal ideal of A.

Lemma (8.1.1). — *Let* A *and* B *be two local rings such that* $A \subset B$*; then the following conditions are equivalent: (i)* $\mathfrak{m}(B) \cap A = \mathfrak{m}(A)$*; (ii)* $\mathfrak{m}(A) \subset \mathfrak{m}(B)$*; (iii)* 1 *is not an element of the ideal of* B *generated by* $\mathfrak{m}(A)$.

Proof. It is evident that (i) implies (ii), and (ii) implies (iii); lastly, if (iii) is true, then $\mathfrak{m}(B) \cap A$ contains $\mathfrak{m}(A)$ and does not contain 1, and is thus equal to $\mathfrak{m}(A)$.

When the equivalent conditions of (8.1.1) are satisfied, we say that B *dominates* A; this is equivalent to saying that the injection $A \to B$ is a *local* homomorphism. It is clear that, in the set of local subrings of a ring R, the relation given by domination is an order (?). $\square$

(8.1.2). Now consider a *field* R. For all subrings A of R, we denote by $L(A)$ the set of local rings $A_{\mathfrak{p}}$, where $\mathfrak{p}$ runs over the prime spectrum of A; they are identified with the subrings of R containing A. Since $\mathfrak{p} = (\mathfrak{p}A_{\mathfrak{p}}) \cap A$, the map $\mathfrak{p} \mapsto A_{\mathfrak{p}}$ from $\operatorname{Spec}(A)$ to $L(A)$ is bijective.

Lemma (8.1.3). — *Let* R *be a field, and* A *a subring of* R*. For a local subring* M *of* R *to dominate a ring* $A_{\mathfrak{p}} \in L(A)$ *it is necessary and sufficient that* $A \subset M$*; the local ring* $A_{\mathfrak{p}}$ *dominated by* M *is then unique, and corresponds to* $\mathfrak{p} = \mathfrak{m}(M) \cap A$.

Proof. If M dominates $A_{\mathfrak{p}}$, then $\mathfrak{m}(M) \cap A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$, by (8.1.1), whence the uniqueness of $\mathfrak{p}$; on the other hand, if $A \subset M$, then $\mathfrak{m}M \cap A = \mathfrak{p}$ is prime in A, and since $A - \mathfrak{p} \subset M$, we have that $A_{\mathfrak{p}} \subset M$ and $\mathfrak{p}A_{\mathfrak{p}} \subset \mathfrak{m}(M)$, so M dominates $A_{\mathfrak{p}}$. $\square$ 165

Lemma (8.1.4). — *Let* R *be a field,* M *and* N *two local subrings of* R*, and* P *the subring of* R *generated by* $M \cup N$*. Then the following conditions are equivalent:*

(i) *There exists a prime ideal* $\mathfrak{p}$ *of* P *such that* $\mathfrak{m}(M) = \mathfrak{p} \cap M$ *and* $\mathfrak{m}(N) = \mathfrak{p} \cap N$.
(ii) *The ideal* $\mathfrak{a}$ *generated in* P *by* $\mathfrak{m}(M) \cup \mathfrak{m}(N)$ *is distinct from* P.
(iii) *There exists a local subring* Q *of* R *simultaneously dominating both* M *and* N.

Proof. It is clear that (i) implies (ii); conversely, if $\mathfrak{a} \neq P$, then $\mathfrak{a}$ is contained in a maximal ideal $\mathfrak{n}$ of P, and since $1 \notin \mathfrak{n}$, $\mathfrak{n} \cap M$ contains $\mathfrak{m}(M)$ and is distinct from M, so $\mathfrak{n} \cap M = \mathfrak{m}(M)$, and similarly $\mathfrak{n} \cap N = \mathfrak{m}(N)$. It is clear that, if Q dominates both M and N, then $P \subset Q$ and $\mathfrak{m}(M) = \mathfrak{m}(Q) \cap M = (\mathfrak{m}(Q) \cap P) \cap M$, and $\mathfrak{m}(N) = (\mathfrak{m}(Q) \cap P) \cap N$, so (iii) implies (i); the converse is evident when we take $Q = P_{\mathfrak{p}}$. $\square$

When the conditions of (8.1.4) are satisfied, we say, with C. Chevalley, that the local rings M and N are *allied*.

Proposition (8.1.5). — *Let* A *and* B *be two subrings of a field* R*, and* C *the subring of* R *generated by* $A \cup B$*. Then the following conditions are equivalent:*

(i) *For every local ring* Q *containing* A *and* B*, we have that* $A_{\mathfrak{p}} = B_{\mathfrak{q}}$*, where* $\mathfrak{p} = \mathfrak{m}(Q) \cap A$ *and* $\mathfrak{q} = \mathfrak{m}(Q) \cap B$.
(ii) *For all prime ideals* $\mathfrak{r}$ *of* C*, we have that* $A_{\mathfrak{p}} = B_{\mathfrak{q}}$*, where* $\mathfrak{p} = \mathfrak{r} \cap A$ *and* $\mathfrak{q} = \mathfrak{r} \cap B$.
(iii) *If* $M \in L(A)$ *and* $N \in L(B)$ *are allied, then they are identical.*
(iv) $L(A) \cap L(B) = L(C)$.

Proof. Lemmas (8.1.3) and (8.1.4) prove that (i) and (iii) are equivalent; it is clear that (i) implies (ii) by taking $Q = C_{\mathfrak{r}}$; conversely, (ii) implies (i), because if Q contains $A \cup B$ then it contains C, and if $\mathfrak{r} = \mathfrak{m}(Q) \cap C$ then $\mathfrak{p} = \mathfrak{r} \cap A$ and $\mathfrak{q} = \mathfrak{r} \cap B$, from (8.1.3). It is immediate that (iv) implies (i), because if Q contains $A \cup B$ then it dominates a local ring $C_{\mathfrak{r}} \in L(C)$ by (8.1.3); by hypothesis we have that $C_{\mathfrak{r}} \in L(A) \cap L(B)$, and (8.1.1) and (8.1.3) prove that $C_{\mathfrak{r}} = A_{\mathfrak{p}} = B_{\mathfrak{q}}$. We prove finally that (iii) implies (iv). Let $Q \in L(C)$; Q dominates some $M \in L(A)$ and some $N \in L(B)$ (8.1.3), so M and N, being allied, are identical by hypothesis. As we then have that $C \subset M$, we know that M dominates some $Q' \in L(C)$ (8.1.3), so Q dominates Q', whence necessarily (8.1.3) $Q = Q' = M$, so $Q \in L(A) \cap L(B)$. Conversely, if $Q \in L(A) \cap L(B)$, then $C \subset Q$, so (8.1.3) Q dominates some $Q'' \in L(C) \subset L(A) \cap L(B)$; Q and Q'', being allied, are identical, so $Q'' = Q \in L(C)$, which completes the proof. $\square$

8.2. Local rings of an integral scheme.

(8.2.1). Let X be an *integral* prescheme, and R its field of rational functions, identical to the local ring of the generic point a of X; for all $x \in X$, we know that $\mathcal{O}_x$ can be canonically identified with a subring of R (7.1.5), and for every rational function $f \in R$, the domain of definition $\delta(f)$ of f is the open set of $x \in X$ such that $f \in \mathcal{O}_x$. It thus follows from (7.2.6) that, for every open $U \subset X$, we have

(8.2.1.1)
$$\Gamma(U, \mathcal{O}_X) = \bigcap_{x \in U} \mathcal{O}_x.$$

Proposition (8.2.2). — *Let* X *be an integral prescheme, and* R *its field of rational fractions. For* X *to be a scheme, it is necessary and sufficient that the relation "*$\mathcal{O}_x$ *and* $\mathcal{O}_y$ *are allied" (8.1.4), for points* x, y *of* X*, implies that* $x = y$. 166

Proof. We suppose that this condition is verified, and aim to show that X is separated. Let U and V be two distinct affine opens of X, with rings A and B, identified with subrings of R; U (resp. V) is thus identified (8.1.2) with $L(A)$ (resp. $L(B)$), and the hypotheses tell us (8.1.5) that C is the subring of R generated by $A \cup B$, and $W = U \cap V$ is identified with $L(A) \cap L(B) = L(C)$. Furthermore, we know ([CC], p. 5-03 (?), 4 *bis*) that every subring E of R is equal to the intersection of the local rings belonging to $L(E)$; C is thus identified with the intersection of the rings $\mathcal{O}_z$ for $z \in W$, or, equivalently (8.2.1.1) with $\Gamma(W, \mathcal{O}_X)$. So consider the subprescheme induced by X on W; to the identity (?) morphism $\phi : C \to \Gamma(W, \mathcal{O}_X)$ there corresponds (2.2.4) a morphism $\Phi = (\psi, \theta) : W \to \operatorname{Spec}(C)$; we will see that Φ is an *isomorphism* of preschemes, whence W is an *affine* open. The identification of W with $L(C) = \mathrm{Spec}(C)$ shows that ψ is *bijective*.On the other hand, for all $x \in W$, $\theta_x^\sharp$ is the injection $C_{\mathfrak{r}} \to \mathcal{O}_x$, where $\mathfrak{r} = \mathfrak{m}_x \cap C$, and by definition $C_{\mathfrak{r}}$ is identified with $\mathcal{O}_x$, so $\theta_x^\sharp$ is bijective. It thus remains to show that ψ is a *homeomorphism*, i.e. that for every closed subset $F \subset W$, $\psi(F)$ is closed in $\mathrm{Spec}(C)$. But F is the trace over (?) W of closed subspace of U, of the form $V(\mathfrak{a})$, where $\mathfrak{a}$ is an ideal of A; we show that $\psi(F) = V(\mathfrak{a}C)$, which proves our claim. In fact, the prime ideals of C containing $\mathfrak{a}C$ are the prime ideals of C containing $\mathfrak{a}$, and so are the ideals of the form $\psi(x) = \mathfrak{m}_x \cap C$, where $\mathfrak{a} \subset \mathfrak{m}_x$ and $x \in W$; since $\mathfrak{a} \subset \mathfrak{m}_x$ is equivalent to $x \in V(\mathfrak{a}) = W \cap F$ for $x \in U$, we do indeed have that $\psi(F) = V(\mathfrak{a}C)$.

It follows that X is separated, because $U \cap V$ is affine and its ring C is generated by the union $A \cup B$ of the rings of U and V (5.5.6).

Conversely, suppose that X is separated, and let x, y be two points of X such that $\mathcal{O}_x$ and $\mathcal{O}_y$ are allied. Let U (resp. V) be an affine open containing x (resp. y), of ring A (resp. B); we then know that $U \cap V$ is affine and that its ring C is generated by $A \cup B$ (5.5.6). If $\mathfrak{p} = \mathfrak{m}_x \cap A$ and $\mathfrak{q} = \mathfrak{m}_y \cap B$, then $A_{\mathfrak{p}} = \mathcal{O}_x$ and $B_{\mathfrak{q}} = \mathcal{O}_y$, and since $A_{\mathfrak{p}}$ and $B_{\mathfrak{q}}$ are allied, there exists a prime ideal $\mathfrak{r}$ of C such that $\mathfrak{p} = \mathfrak{r} \cap A$ and $\mathfrak{q} = \mathfrak{r} \cap B$ (8.1.4). But then there exists a point $z \in U \cap V$ such that $\mathfrak{r} = \mathfrak{m}_z \cap C$, since $U \cap V$ is affine, and so evidently $x = z$ and $y = z$, whence $x = y$. □

Corollary (8.2.3). — *Let* X *be an integral scheme, and* x, y *two points of* X.*In order that* $x \in \overline{\{y\}}$, *it is necessary and sufficient that* $\mathcal{O}_x \subset \mathcal{O}_y$, *or, equivalently, that every rational function defined at* x *is also defined at* y.

Proof. The condition is evidently necessary because the domain of definition $\delta(f)$ of a rational function $f \in R$ is open; we now show that it is sufficient.If $\mathcal{O}_x \subset \mathcal{O}_y$, then there exists a prime ideal $\mathfrak{p}$ of $\mathcal{O}_x$ such that $\mathcal{O}_y$ dominates $(\mathcal{O}_x)_{\mathfrak{p}}$ (8.1.3); but (2.4.2) there exists $z \in X$ such that $x \in \overline{\{z\}}$ and $\mathcal{O}_z = (\mathcal{O}_x)_{\mathfrak{p}}$; since $\mathcal{O}_z$ and $\mathcal{O}_y$ are allied, we have that $z = y$ by (8.2.2), whence the corollary. □

Corollary (8.2.4). — *If* X *is an integral scheme then the map* $x \to \mathcal{O}_x$ *is injective; equivalently, if* x *and* y *are two distinct points of* X, *then there exists a rational function defined at one of these points but not the other.*

Proof. This follows from (8.2.3) and the axiom (T_0) (2.1.4). □

Corollary (8.2.5). — *Let* X *be an integral scheme whose underlying space is Noetherian; letting* f *run over the field* R *of rational functions on* X, *the sets* $\delta(f)$ *generate the topology of* X.

In fact, every closed subset of X is thus a finite union of irreducible closed subsets, i.e., of the form $\overline{\{y\}}$ (2.1.5). But, if $x \notin \overline{\{y\}}$, then there exists a rational function f defined at x but not at y (8.2.3), or, equivalently, we have that $x \in \delta(f)$ and $\delta(f)$ is not contained in $\overline{\{y\}}$. The complement of $\overline{\{y\}}$ is thus a union of sets of the form $\delta(f)$, and by virtue of the first remark, every open subset of X is the union of finite intersections of open sets of the form $\delta(f)$.

(8.2.6). Corollary (8.2.5) shows that the topology of X is entirely characterised by the data of the local rings $(\mathcal{O}_x)_{x\in X}$ that have R as their field of fractions. It amounts to the same to say that the closed subsets of X are defined in the following manner: given a finite subset $\{x_1, \ldots, x_n\}$ of X, consider the set of $y \in X$ such that $\mathcal{O}_y \subset \mathcal{O}_{x_i}$ for at least one index i, and these sets (over all choices of $\{x_1, \ldots, x_n\}$) are the closed subsets of X. Further, once the topology on X is known, the structure sheaf $\mathcal{O}_X$ is also determined by the family of the $\mathcal{O}_x$, since $\Gamma(U, \mathcal{O}_X) = \bigcap_{x\in U} \mathcal{O}_x$ by (8.2.1.1). The family $(\mathcal{O}_X)_{x\in X}$ thus completely determines the prescheme X when X is an integral scheme whose underlying space is Noetherian.

Proposition (8.2.7). — *Let* X, Y *be two integral schemes,* $f : X \to Y$ *a dominant morphism (2.2.6), and* K *(resp.*L*) the field of rational functions on* X *(resp.*Y*). Then* L *can be identified with a subfield of* K, *and for all* $x \in X$, $\mathcal{O}_{f(x)}$ *is the unique local ring of* Y *dominated by* $\mathcal{O}_x$.

Proof. If $f = (\psi, \theta)$ and a is the generic point of X, then $\psi(a)$ is the generic point of Y (**0**, 2.1.5); $\theta_a^\sharp$ is then a monomorphism of fields, from $L = \mathcal{O}_{\psi(a)}$ to $K = \mathcal{O}_a$. Since every non-empty affine open U of Y contains $\psi(a)$, it follows from (2.2.4) that the homomorphism $\Gamma(U, \mathcal{O}_Y) \to \Gamma(\psi^{-1}(U), \mathcal{O}_X)$ corresponding to f is the restriction of $\theta_a^\sharp$ to $\Gamma(U, \mathcal{O}_Y)$. So, for every $x \in X$, $\theta_x^\sharp$ is the restriction to $\mathcal{O}_{\psi(a)}$ of $\theta_a^\sharp$, and is thus a monomorphism. We also know that $\theta_x^\sharp$ is a local homomorphism, so, if we identify L with a subfield of K by $\theta_a^\sharp$, $\mathcal{O}_{\psi(x)}$ is dominated by $\mathcal{O}_x$ (8.1.1); it is also the only local ring of Y dominated by $\mathcal{O}_x$, since two local rings of Y that are allied are identical (8.2.2). □

Proposition (8.2.8). — *Let* X *be an irreducible prescheme; and* $f : X \to Y$ *a local immersion (resp. a local isomorphism); and suppose further that* f *is separated. Then* f *is an immersion (resp. an open immersion).*

Proof. Let $f = (\psi, \theta)$; it suffices, in both cases, to prove that ψ is a *homeomorphism* from X to $\psi(X)$ (4.5.3). Replacing f by f_{red} ((5.1.6) and (5.5.1), (vi)), we can assume that X and Y are *reduced*. If Y' is the closed reduced subprescheme of Y having $\overline{\psi(X)}$ as its underlying space, then f factorizes as $X \xrightarrow{f'} Y' \xrightarrow{j} Y$, where j is the canonical injection (5.2.2). It follows from (5.5.1), (v) that f' is again a separated morphism; further, f' is again a local immersion (resp. a local isomorphism), because, since the condition is local on X and Y, we can reduce ourselves to the case where f is a closed immersion (resp. open immersion), and then our claim follows immediately from (4.2.2).

We can thus suppose that f is a *dominant* morphism, which leads to the fact that Y is, itself, irreducible (**0**, 2.1.5), and so X and Y are both *integral*. Further, the condition being local on Y, we can suppose that Y is an affine scheme; since f is separated, X is a scheme (5.5.1), (ii), and we are finally at the hypotheses of Proposition (8.2.7). Then, for all $x \in \mathrm{X}$, $\theta_x^\sharp$ is injective; but the hypothesis that f is a local immersion implies that $\theta_x^\sharp$ is surjective (4.2.2), so $\theta_x^\sharp$ is bijective, or, equivalently (with the identification of Proposition (8.2.7)) we have that $\mathscr{O}_{\psi(x)} = \mathscr{O}_x$. This implies, by Corollary (8.2.4), that ψ is an *injective* map, which already proves the proposition when f is a local isomorphism (4.5.3). When we suppose that f is only a local immersion, for all $x \in \mathrm{X}$ there exists an open neighbourhood U of x in X and an open neighbourhood V of $\psi(x)$ in Y such that the restriction of ψ to U is a homeomorphism from U to a *closed* subset of V. But U is dense in X, so $\psi(\mathrm{U})$ is dense in Y and *a fortiori* in V, which proves that $\psi(\mathrm{U}) = \mathrm{V}$; since ψ is injective, $\psi^{-1}(\mathrm{V}) = \mathrm{U}$ and this proves that ψ is a homeomorphism from X to $\psi(\mathrm{X})$. □

8.3. **Chevalley schemes.**

(8.3.1). Let X be a *Noetherian* integral scheme, and R its field of rational functions; we denote by X′ the set of local subrings $\mathscr{O}_x \subset \mathrm{R}$, where x runs over all points of X. The set X′ satisfies the three following conditions:

(Sch. 1) For all $\mathrm{M} \in \mathrm{X}'$, R is the field of fractions of M.

(Sch. 2) There exists a finite set of Noetherian subrings A_i of R such that $\mathrm{X}' = \bigcup_i \mathrm{L}(\mathrm{A}_i)$, and, for all pairs of indices i, j, the subring A_{ij} of R generated by $\mathrm{A}_i \cup \mathrm{A}_j$ is an algebra of finite type over A_i.

(Sch. 3) Two elements M and N of X′ that are allied are identical.

We have seen in (8.2.1) that (Sch. 1) is satisfied, and (Sch. 3) follows from (8.2.2). To show (Sch. 2), it suffices to cover X by a finite number of affine opens U_i, whose rings are Noetherian, and to take $\mathrm{A}_i = \Gamma(\mathrm{U}_i, \mathscr{O}_\mathrm{X})$; the hypothesis that X is a scheme implies that $\mathrm{U}_i \cap \mathrm{U}_j$ is affine, and also that $\Gamma(\mathrm{U}_i \cap \mathrm{U}_j, \mathscr{O}_\mathrm{X}) = \mathrm{A}_{ij}$ (5.5.6); further, since the space U_i is Noetherian, the immersion $\mathrm{U}_i \cap \mathrm{U}_j \to \mathrm{U}_i$ is of finite type (6.3.5), so A_{ij} is an A_i-algebra of finite type (6.3.3).

(8.3.2). The structures whose axioms are (Sch. 1), (Sch. 2), and (Sch. 3), generalise "schemes" in the sense of C. Chevalley, who supposes furthermore that R is an extension of finite type of a field K, and that the A_i are K-algebras of finite type (which renders a part of (Sch. 2) useless) [CC]. Conversely, if we have such a structure on a set X′, then we can associate to it an integral scheme X by using the remarks from (8.2.6): the underlying space of X is equal to X′ endowed with the topology defined in (8.2.6), and with the sheaf $\mathscr{O}_\mathrm{X}$ such that $\Gamma(\mathrm{U}, \mathscr{O}_\mathrm{X}) = \bigcap_{x \in \mathrm{U}} \mathscr{O}_x$ for all open $\mathrm{U} \subset \mathrm{X}$, with the evident definition of restriction homomorphisms. We leave to the reader the task of verifying that we obtain thusly an integral scheme, whose local rings are the elements of X′; we will not use this result in what follows.

9. Supplement on quasi-coherent sheaves

9.1. **Tensor product of quasi-coherent sheaves.** 169

Proposition (9.1.1). — *Let X be a prescheme (resp. a locally Noetherian prescheme). Let $\mathscr{F}$ and $\mathscr{G}$ be two quasi-coherent (resp. coherent) $\mathscr{O}_\mathrm{X}$-modules; then $\mathscr{F} \otimes_{\mathscr{O}_\mathrm{X}} \mathscr{G}$ is quasi-coherent (resp. coherent) and of finite type if $\mathscr{F}$ and $\mathscr{G}$ are of finite type. If $\mathscr{F}$ admits a finite presentation and if $\mathscr{G}$ is quasi-coherent (resp. coherent), then $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ is quasi-coherent (resp. coherent).*

Proof. Being a local property, we can suppose that X is affine (resp. Noetherian affine); further, if $\mathscr{F}$ is coherent, then we can assume that it is the cokernel of a homomorphism $\mathscr{O}_\mathrm{X}^m \to \mathscr{O}_\mathrm{X}^n$. The claims pertaining to quasi-coherent sheaves then follow from the Corollaries (1.3.12) and (1.3.9); the claims pertaining to coherent sheaves follow from Theorem (1.5.1) and from the fact that, if M and N are modules of finite type over a Noetherian ring A, $\mathrm{M} \otimes_\mathrm{A} \mathrm{N}$ and $\mathrm{Hom}_\mathrm{A}(\mathrm{M}, \mathrm{N})$ are A-modules of finite type. □

Definition (9.1.2). — Let X and Y be two S-preschemes, p and q the projections of $\mathrm{X} \times_\mathrm{S} \mathrm{Y}$, and $\mathscr{F}$ (resp. $\mathscr{G}$) a quasi-coherent $\mathscr{O}_\mathrm{X}$-module (resp. quasi-coherent $\mathscr{O}_\mathrm{Y}$-module). We define the tensor product of $\mathscr{F}$ and $\mathscr{G}$ over $\mathscr{O}_\mathrm{S}$ (*or* over S), denoted by $\mathscr{F} \otimes_{\mathscr{O}_\mathrm{S}} \mathscr{G}$ (*or* $\mathscr{F} \otimes_\mathrm{S} \mathscr{G}$) to be the tensor product $p^*(\mathscr{F}) \otimes_{\mathscr{O}_{\mathrm{X}\times_\mathrm{S}\mathrm{Y}}} q^*(\mathscr{G})$ over the prescheme $\mathrm{X} \times_\mathrm{S} \mathrm{Y}$.

If X_i $(1 \leqslant i \leqslant n)$ are S-preschemes, and $\mathscr{F}_i$ are quasi-coherent $\mathscr{O}_{\mathrm{X}_i}$-modules $(1 \leqslant i \leqslant n)$, then we define similarly the tensor product $\mathscr{F}_1 \otimes_\mathrm{S} \mathscr{F}_2 \otimes_\mathrm{S} \cdots \otimes_\mathrm{S} \mathscr{F}_n$ over the prescheme $\mathrm{Z} = \mathrm{X}_1 \times_\mathrm{S} \mathrm{X}_2 \times_\mathrm{S} \cdots \times_\mathrm{S} \mathrm{X}_n$; it is a *quasi-coherent* $\mathscr{O}_\mathrm{Z}$-module by virtue of (9.1.1) and (**0**, 5.1.4); it is *coherent* if the $\mathscr{F}_i$ are coherent and Z is *locally Noetherian*, by virtue of (9.1.1), (**0**, 5.3.11), and (6.1.1).

Note that if we take $\mathrm{X} = \mathrm{Y} = \mathrm{S}$ then definition (9.1.2) gives us back the tensor product of $\mathscr{O}_\mathrm{S}$-modules. Furthermore, as $q^*(\mathscr{O}_\mathrm{Y}) = \mathscr{O}_{\mathrm{X}\times_\mathrm{S}\mathrm{Y}}$ (**0**, 4.3.4), the product $\mathscr{F} \otimes_\mathrm{S} \mathscr{O}_\mathrm{Y}$ is canonically identified with $p^*(\mathscr{F})$, and, in the same way, $\mathscr{O}_\mathrm{X} \otimes_\mathrm{S} \mathscr{G}$ is canonically identified with $q^*(\mathscr{G})$. In particular, if we take $\mathrm{Y} = \mathrm{S}$ and denote by f the structure morphism $\mathrm{X} \to \mathrm{Y}$, we have that $\mathscr{O}_X \otimes_Y \mathscr{G} = f^*(\mathscr{G})$: the ordinary tensor product and the inverse image thus appear as particular cases of the general tensor product.

Definition (9.1.2) leads immediately to the fact that, for fixed X and Y, $\mathscr{F} \otimes_S \mathscr{G}$ is an *additive covariant bifunctor that is right exact* in $\mathscr{F}$ and $\mathscr{G}$.

Proposition (9.1.3). — *Let* S, X, Y *be three affine schemes of rings* A, B, C *(respectively), with* B *and* C *being* A*-algebras. Let* M *(resp.* N*) be a* B*-module (resp.* C*-module), and* $\mathscr{F} = \widetilde{M}$ *(resp.* $\mathscr{G} = \widetilde{N}$*) the associated quasi-coherent sheaf; then* $\mathscr{F} \otimes_S \mathscr{G}$ *is canonically isomorphic to the sheaf associated to the* $(B \otimes_A C)$*-module* $M \otimes_A N$.

Proof. According to Proposition (1.6.5), $\mathscr{F} \otimes_S \mathscr{G}$ is canonically isomorphic to the sheaf associated to the $(B \otimes_A C)$-module

$$\big(M \otimes_B (B \otimes_A C)\big) \otimes_{B \otimes_A C} \big((B \otimes_A C) \otimes_C N\big)$$

and by the canonical isomorphisms between tensor products, this latter module is isomorphic to

$$M \otimes_B (B \otimes_A C) \otimes_C N = (M \otimes_B B) \otimes_A (C \otimes_C N) = M \otimes_A N.$$

□

Proposition (9.1.4). — *Let* $f : T \to X$*, and* $g : T \to Y$ *be two* S*-morphisms, and* $\mathscr{F}$ *(resp.* $\mathscr{G}$*) a quasi-coherent* $\mathscr{O}_X$*-module (resp. quasi-coherent* $\mathscr{O}_Y$*-module). Then*

$$(f, g)_S^*(\mathscr{F} \otimes_S \mathscr{G}) = f^*(\mathscr{F}) \otimes_{\mathscr{O}_T} g^*(\mathscr{G}).$$

Proof. If p, q are the projections of $X \times_S Y$, then the formula in fact follows from the relations $(f, g)_S^* \circ p^* = f^*$ and $(f, g)_S^* \circ q^* = g^*$ (**0**, 3.5.5), and the fact that the inverse image of a tensor product of algebraic sheaves is the tensor product of their inverse images (**0**, 4.3.3). □

Corollary (9.1.5). — *Let* $f : X \to X'$ *and* $g : Y \to Y'$ *be* S*-morphisms, and* $\mathscr{F}'$ *(resp.* $\mathscr{G}'$*) a quasi-coherent* $\mathscr{O}_{X'}$*-module (resp. quasi-coherent* $\mathscr{O}_{Y'}$*-module). Then*

$$(f, g)_S^*(\mathscr{F}' \otimes_S \mathscr{G}') = f^*(\mathscr{F}') \otimes_S g^*(\mathscr{G}')$$

Proof. This follows from (9.1.4) and the fact that $f \times_S g = (f \circ p, g \circ q)_S$, where p, q are the projections of $X \times_S Y$. □

Corollary (9.1.6). — *Let* X, Y, Z *be three* S*-preschemes, and* $\mathscr{F}$ *(resp.* $\mathscr{G}$*,* $\mathscr{H}$*) a quasi-coherent* $\mathscr{O}_X$*-module (resp. quasi-coherent* $\mathscr{O}_Y$*-module, quasi-coherent* $\mathscr{O}_Z$*-module); then the sheaf* $\mathscr{F} \otimes_S \mathscr{G} \otimes_S \mathscr{H}$ *is the inverse image of* $(\mathscr{F} \otimes_S \mathscr{G}) \otimes_S \mathscr{H}$ *by the canonical isomorphism from* $X \times_S Y \times_S Z$ *to* $(X \times_S Y) \times_S Z$.

Proof. This isomorphism is given by $(p_1, p_2)_S \times_S p_3$, where p_1, p_2, p_3 are the projections of $X \times_S Y \times_S Z$.

Similarly, the inverse image of $\mathscr{G} \otimes_S \mathscr{F}$ under the canonical isomorphism from $X \times_S Y$ to $Y \times_S X$ is $\mathscr{F} \otimes_S \mathscr{G}$. □

Corollary (9.1.7). — *If* X *is an* S*-prescheme, then every quasi-coherent* $\mathscr{O}_X$*-module* $\mathscr{F}$ *is the inverse image of* $\mathscr{F} \otimes_S \mathscr{O}_S$ *by the canonical isomorphism from* X *to* $X \times_S S$ *(3.3.3).*

Proof. This isomorphism is $(1_X, \phi)_S$, where ϕ is the structure morphism $X \to S$, and the corollary follows from (9.1.4) and the fact that $\phi^*(\mathscr{O}_S) = \mathscr{O}_X$. □

(9.1.8). Let X be an S-prescheme, $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module, and $\phi : S' \to S$ a morphism; we denote by $\mathscr{F}_{(\phi)}$ or $\mathscr{F}_{(S')}$ the quasi-coherent sheaf $\mathscr{F} \otimes_S \mathscr{O}_{S'}$ over $X \times_S S' = X_{(\phi)} = X_{(S')}$; so $\mathscr{F}_{(S')} = p^*(\mathscr{F})$, where p is the projection $X_{(S')} \to X$.

Proposition (9.1.9). — *Let* $\phi'' : S'' \to S'$ *be a morphism. For every quasi-coherent* $\mathscr{O}_X$*-module* $\mathscr{F}$ *on the* S*-prescheme* X*,* $(\mathscr{F}_{(\phi)})_{(\phi')}$ *is the inverse image of* $\mathscr{F}_{(\phi \circ \phi')}$ *by the canonical isomorphism* $(X_{(\phi)})_{(\phi')} \xrightarrow{\sim} X_{(\phi \circ \phi')}$ *(3.3.9).*

Proof. This follows immediately from the definitions and from (3.3.9), and is written

$$(\mathscr{F} \otimes_S \mathscr{O}_{S'}) \otimes_{S'} \mathscr{O}_{S''} = \mathscr{F} \otimes_S \mathscr{O}_{S''}. \tag{9.1.9.1}$$

□

Proposition (9.1.10). — *Let* Y *be an* S*-prescheme, and* $f : X \to Y$ *an* S*-morphism. For every quasi-coherent* $\mathscr{O}_Y$*-module and every morphism* $S' \to S$*, we have that* $(f_{(S')})^*(\mathscr{G}_{(S')}) = (f^*(\mathscr{G}))_{(S')}$.

Proof. This follows immediately from the commutativity of the diagram

$$\begin{tikzcd} X_{(S')} \arrow[r, "f_{(S')}"] \arrow[d] & Y_{(S')} \arrow[d] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

□

Corollary (9.1.11). — *Let* X *and* Y *be* S*-preschemes, and* $\mathscr{F}$ *(resp.* $\mathscr{G}$*) a quasi-coherent* $\mathscr{O}_X$*-module (resp. quasi-coherent* $\mathscr{O}_Y$*-module). Then the inverse image of the sheaf* $(\mathscr{F}_{(S')})\otimes_{(S')}(\mathscr{G}_{(S')})$ *by the canonical isomorphism* $(X\times_S Y)_{(S')}\xrightarrow{\sim}(X_{(S')})\times_{S'}(Y_{(S')})$ *(3.3.10) is equal to* $(\mathscr{F}\otimes_S\mathscr{G})_{(S')}$.

Proof. If p,q are the projections of $X\times_S Y$, then the isomorphism in question is nothing but $(p_{(S')},q_{(S')})_{S'}$; the corollary then follows from the Propositions (9.1.4) and (9.1.10). □

Proposition (9.1.12). — *With the notation from Definition (9.1.2), let* z *be a point of* $X\times_S Y$, $x=p(z)$*, and* $y=q(z)$*; the stalk* $(\mathscr{F}\otimes_S\mathscr{G})_z$ *is isomorphic to* $(\mathscr{F}_x\otimes_{\mathscr{O}_x}\mathscr{O}_z)\otimes_{\mathscr{O}_z}(\mathscr{G}_y\otimes_{\mathscr{O}_y}\mathscr{O}_z)=\mathscr{F}_x\otimes_{\mathscr{O}_x}\mathscr{O}_z\otimes_{\mathscr{O}_y}\otimes\mathscr{G}_y$.

Proof. As we can reduce to the affine case, the proposition follows from equation (1.6.5.1). □

Corollary (9.1.13). — *If* $\mathscr{F}$ *and* $\mathscr{G}$ *are of finite type, then we have that*

$$\mathrm{Supp}(\mathscr{F}\otimes_S\mathscr{G})=p^{-1}(\mathrm{Supp}(\mathscr{F}))\cap q^{-1}(\mathrm{Supp}(\mathscr{G})).$$

Proof. Since $p^*(\mathscr{F})$ and $q^*(\mathscr{G})$ are both of finite type over $\mathscr{O}_{X\times_S Y}$, we reduce, by Proposition (9.1.12) and by (**0**, 1.7.5), to the case where $\mathscr{G}=\mathscr{O}_Y$, that is, it remains to prove the following equation:

$$\mathrm{Supp}(p^{-1}(\mathscr{F}))=p^{-1}(\mathrm{Supp}(\mathscr{F})). \tag{9.1.13.1}$$

The same reasoning as in (**0**, 1.7.5) leads us to prove that, for all $z\in X\times_S Y$, we have $\mathscr{O}_z/\mathfrak{m}_x\mathscr{O}_z\neq 0$ (with $x=p(z)$), which follows from the fact that the homomorphism $\mathscr{O}_x\to\mathscr{O}_z$ is *local*, by hypothesis. □

We leave it to the reader to extend the results in this section to the more general case of arbitrarily (but finitely) many factors, instead of just two.

9.2. **Direct image of a quasi-coherent sheaf.**

Proposition (9.2.1). — *Let* $f:X\to Y$ *be a morphism of preschemes. We suppose that there exists a cover* (Y_α) *of* Y *by affine opens having the following property: every* $f^{-1}(Y_\alpha)$ *admits a finite cover* $(X_{\alpha i})$ *by affine opens contained in* $f^{-1}(Y_\alpha)$ *such that every intersection* $X_{\alpha i}\cap X_{\alpha j}$ *is itself a finite union of affine opens. With these hypotheses, for every quasi-coherent* $\mathscr{O}_X$*-module* $\mathscr{F}$, $f_*(\mathscr{F})$ *is a quasi-coherent* $\mathscr{O}_Y$*-module.*

Proof. Since this is a local condition on Y, we can assume that Y is equal to one of the Y_α, and thus omit the indices α.

(a) First, suppose that the $X_i\cap X_j$ are themselves *affine* opens. We set $\mathscr{F}_i=\mathscr{F}|X_i$ and $\mathscr{F}_{ij}=\mathscr{F}|(X_i\cap X_j)$, and let $\mathscr{F}'_i$ and $\mathscr{F}'_{ij}$ be the images of $\mathscr{F}_i$ and $\mathscr{F}_{ij}$ (respectively) by the restriction of f to X_i and $X_i\cap X_j$ (respectively); we know that the $\mathscr{F}'_i$ and $\mathscr{F}'_{ij}$ are quasi-coherent (1.6.3). Set $\mathscr{G}=\bigoplus_i\mathscr{F}'_i$ and $\mathscr{H}=\bigoplus_{i,j}\mathscr{F}'_{ij}$; $\mathscr{G}$ and $\mathscr{H}$ are quasi-coherent $\mathscr{O}_Y$-modules; we will define a homomorphism $u:\mathscr{G}\to\mathscr{H}$ such that $f_*(\mathscr{F})$ is the *kernel* of u; it will follow from this that $f_*(\mathscr{F})$ is quasi-coherent (1.3.9). It suffices to define u as a homomorphism of presheaves; taking into account the definitions of $\mathscr{G}$ and $\mathscr{H}$, it thus suffices, for every open subset $W\subset Y$, to define a homomorphism 172

$$u_W:\bigoplus_i\Gamma(f^{-1}(W)\cap X_i,\mathscr{F})\longrightarrow\bigoplus_{i,j}\Gamma(f^{-1}(W)\cap X_i\cap X_j,\mathscr{F})$$

in such a way that it satisfies the usual compatibility conditions when W varies. If, for every section $s_i\in\Gamma(f^{-1}(W)\cap X_i,\mathscr{F})$, we denote by $s_{i|j}$ the restriction to $f^{-1}(W)\cap X_i\cap X_j$, then we set

$$u_W\big((s_i)\big)=(s_{i|j}-s_{j|i})$$

and the compatibility conditions are clearly satisfied. To prove that the kernel $\mathscr{R}$ of u is $f_*(\mathscr{F})$, we define a homomorphism from $f_*(\mathscr{F})$ to $\mathscr{R}$ by sending each section $s\in\Gamma(f^{-1}(W),\mathscr{F})$ to the family (s_i), where s_i is the restriction of s to $f^{-1}(W)\cap X_i$; the axioms (F1) and (F2) of sheaves (G, II, 1.1) tell us that this homomorphism is *bijective*, which finishes the proof in this case.

(b) In the general case, the same reasoning applies once we have established that the $\mathscr{F}_{ij}$ are quasi-coherent. But, by hypothesis, $X_i\cap X_j$ is a finite union of affine opens X_{ijk}; and since the X_{ijk} are affine opens *in a scheme*, the intersection of any two of them is again an affine open (5.5.6). We are thus led to the first case, and so we have proved (9.2.1).

□

Corollary (9.2.2). — *The conclusion of (9.2.1) holds true in each of the following cases:*

(a) f *is separated and quasi-compact.*
(b) f *is separated and of finite type.*
(c) f *is quasi-compact and the underlying space of* X *is locally Noetherian.*

Proof. In the case *(a)*, the $X_{\alpha i} \cap X_{\alpha j}$ are affine (5.5.6). Case *(b)* is a particular case of *(a)* (6.6.3). Finally, in case *(c)*, we can reduce to the case where Y is affine and the underlying space of X is Noetherian; then X admits a finite cover of affine opens (X_i), and the $X_i \cap X_j$, being quasi-compact, are finite unions of affine opens (2.1.3). □

9.3. **Extension of sections of quasi-coherent sheaves.**

Theorem (9.3.1). — *Let X be a prescheme whose underlying space is Noetherian, or a scheme whose underlying space is quasi-compact. Let $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module (**0**, 5.4.1), f a section of $\mathscr{L}$ over X, X_f the open set of $x \in X$ such that $f(x) \neq 0$ (**0**, 5.5.1), and $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module.*

(i) *If $s \in \Gamma(X, \mathscr{F})$ is such that $s|X_f = 0$, then there exists an integer $n > 0$ such that $s \otimes f^{\otimes n} = 0$.*
(ii) *For every section $s \in \Gamma(X_f, \mathscr{F})$, there exists an integer $n > 0$ such that $s \otimes f^{\otimes n}$ extends to a section of $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$ over X.*

Proof.

(i) Since the underlying space of X is quasi-compact, and thus the union of finitely-many affine opens U_i with $\mathscr{L}|U_i$ is isomorphic to $\mathscr{O}_X|U_i$, we can reduce to the case where X is affine and $\mathscr{L} = \mathscr{O}_X$. In this case, f is identified with an element of A(X), and we have that $X_f = D(f)$; s is identified with an element of an A(X)-module M, and $s|X_f$ to the corresponding element of M_f, and the result is then trivial, recalling the definition of a module of fractions.
(ii) Again, X is a finite union of affine opens U_i $(1 \leqslant i \leqslant r)$ such that $\mathscr{L}|U_i \cong \mathscr{O}_X|U_i$, and for every i, $(s \otimes f^{\otimes n})|(U_i \cap X_f)$ is identified (by the aforementioned isomorphism) with $(f|(U_i \cap X_f))^n(s|(U_i \cap X_f))$. We then know (1.4.1) that there exists an integer $n > 0$ such that, for all i, $(s \otimes f^{\otimes n})|(U_i \cap X_f)$ extends to a section s_i of $\mathscr{F} \otimes \mathscr{L}^{\otimes n}$ over U_i. Let $s_{i|j}$ be the restriction of s_i to $U_i \cap U_j$; by definition we have that $s_{i|j} - s_{j|i} = 0$ on $X_f \cap U_i \cap U_j$. But, if X is a Noetherian space, then $U_i \cap U_j$ is quasi-compact; if X is a scheme, then $U_i \cap U_j$ is an affine open (5.5.6), and so again quasi-compact. By virtue of (i), there thus exists an integer m (independent of i and j) such that $(s_{i|j} - s_{j|i}) \otimes f^{\otimes m} = 0$. It immediately follows that there exists a section s' of $\mathscr{F} \otimes \mathscr{L}^{\otimes (n+m)}$ over X, restricting to $s_i \otimes f^{\otimes m}$ over each U_i, and restricting to $s \otimes f^{\otimes (n+m)}$ over X_f.

□

The following corollaries give an interpretation of Theorem (9.3.1) in a more algebraic language:

Corollary (9.3.2). — *With the hypotheses of (9.3.1), consider the graded ring $A_* = \Gamma_*(\mathscr{L})$ and the graded A_*-module $M_* = \Gamma_*(\mathscr{L}, \mathscr{F})$ (**0**, 5.4.6). If $f \in A_n$, where $n \in \mathbf{Z}$, then there is a canonical isomorphism $\Gamma(X_f, \mathscr{F}) \xrightarrow{\sim} ((M_*)_f)_0$ (the subgroup of the module of fractions $(M_*)_f$ consisting of elements of degree 0).*

Corollary (9.3.3). — *Suppose that the hypotheses of (9.3.1) are satisfied, and suppose further that $\mathscr{L} = \mathscr{O}_X$. Then, setting $A = \Gamma(X, \mathscr{O}_X)$ and $M = \Gamma(X, \mathscr{F})$, the A_f-module $\Gamma(X_f, \mathscr{F})$ is canonically isomorphic to M_f.*

Proposition (9.3.4). — *Let X be a Noetherian prescheme, $\mathscr{F}$ a coherent $\mathscr{O}_X$-module, and $\mathscr{J}$ a coherent sheaf of ideals in $\mathscr{O}_X$, such that the support of $\mathscr{F}$ is contained in that of $\mathscr{O}_X|\mathscr{J}$. Then there exists a whole number $n > 0$ such that $\mathscr{J}^n\mathscr{F} = 0$.*

Proof. Since X is a union of finitely-many affine opens whose rings are Noetherian, we can suppose that X is affine of Noetherian ring A; then $\mathscr{F} = \widetilde{M}$, where $M = \Gamma(X, \mathscr{F})$ is an A-module of finite type, and $\mathscr{J} = \widetilde{\mathfrak{J}}$, where $\mathfrak{J} = \Gamma(X, \mathscr{J})$ is an ideal of A ((1.4.1) and (1.5.1)). Since A is Noetherian, $\mathfrak{J}$ admits a finite system of generators f_i $(1 \leqslant i \leqslant m)$. By hypothesis, every section of $\mathscr{F}$ over X is zero on each of the $D(f_i)$; if s_j $(1 \leqslant j \leqslant q)$ are sections of $\mathscr{F}$ generating M, then there exists a whole number h, independent of i and j, such that $f_i^h s_j = 0$ (1.4.1), whence $f_i^h s = 0$ for all $s \in M$. We thus conclude that if $n = mh$ then $\mathfrak{J}^n M = 0$, and so the corresponding $\mathscr{O}_X$-module $\mathscr{J}^n\mathscr{F} = \widetilde{\mathfrak{J}^n M}$ (1.3.13) is zero. □

Corollary (9.3.5). — *With the hypotheses of (9.3.4), there exists a closed subprescheme Y of X, whose underlying space is the support of $\mathscr{O}_X/\mathscr{J}$, such that, if $j : Y \to X$ is the canonical injection, then $\mathscr{F} = j_*(j^*(\mathscr{F}))$.*

Proof. First, note that the supports of $\mathscr{O}_X/\mathscr{J}$ and $\mathscr{O}_X/\mathscr{J}^n$ are the same, since, if $\mathscr{J}_x = \mathscr{O}_x$, then $\mathscr{J}_x^n = \mathscr{O}_x$, and we also have that $\mathscr{J}_x^n \subset \mathscr{J}_x$ for all $x \in X$. We can, thanks to (9.3.4), thus suppose that $\mathscr{J}\mathscr{F} = 0$; we can then take Y to be the closed subprescheme of X defined by $\mathscr{J}$, and since $\mathscr{F}$ is then an $(\mathscr{O}_X/\mathscr{J})$-module, the conclusion follows immediately. □

9.4. **Extension of quasi-coherent sheaves.**

(9.4.1). Let X be a topological space, $\mathscr{F}$ a sheaf of sets (resp. of groups, of rings) on X, U an open subset of X, $\psi : U \to X$ the canonical injection, and $\mathscr{G}$ a subsheaf of $\mathscr{F}|U = \psi^*(\mathscr{F})$. Since ψ_* is left exact, $\psi_*(\mathscr{G})$ is a subsheaf of $\psi_*(\psi^*(\mathscr{F}))$; if we denote by ρ the canonical homomorphism $\mathscr{F} \to \psi_*(\psi^*(\mathscr{F}))$ (**0**, 3.5.3), then we denote by $\overline{\mathscr{G}}$ the subsheaf $\rho^{-1}(\psi_*(\mathscr{G}))$ of $\mathscr{F}$. It follows immediately from the definitions that, for every open subset V of X, $\Gamma(V, \overline{\mathscr{G}})$ consists of sections $s \in \Gamma(V, \mathscr{F})$ whose restriction to $V \cap U$ is a section of $\mathscr{G}$ over $V \cap U$. We thus have that $\overline{\mathscr{G}}|U = \psi^*(\overline{\mathscr{G}}) = \mathscr{G}$, and that $\overline{\mathscr{G}}$ is the *biggest* subsheaf of $\mathscr{F}$ that restricts to $\mathscr{G}$ over U; we say that $\overline{\mathscr{G}}$ is the *canonical extension* of the subsheaf $\mathscr{G}$ of $\mathscr{F}|\mathrm{U}$ to a subsheaf of $\mathscr{F}$.

Proposition (9.4.2). — *Let* X *be a prescheme,* U *an open subset of* X *such that the canonical injection* $j : \mathrm{U} \to \mathrm{X}$ *is a quasi-compact morphism (which will be the case for all* U *if the underlying space of* X *is locally Noetherian (6.6.4), (i)). Then:*

(i) *For every quasi-coherent* $(\mathscr{O}_{\mathrm{X}}|\mathrm{U})$*-module* $\mathscr{G}$*,* $j_*(\mathscr{G})$ *is a quasi-coherent* $\mathscr{O}_{\mathrm{X}}$*-module, and* $j_*(\mathscr{G})|\mathrm{U} = j^*(j_*(\mathscr{G})) = \mathscr{G}$*.*

(ii) *For every quasi-coherent* $\mathscr{O}_{\mathrm{X}}$*-module* $\mathscr{F}$ *and every quasi-coherent* $(\mathscr{O}_{\mathrm{X}}|\mathrm{U})$*-submodule* $\mathscr{G}$*, the canonical extension* $\overline{\mathscr{G}}$ *of* $\mathscr{G}$ *(9.4.1) is a quasi-coherent* $\mathscr{O}_{\mathrm{X}}$*-submodule of* $\mathscr{F}$*.*

Proof. If $j = (\psi, \theta)$ (ψ being the injection $\mathrm{U} \to \mathrm{X}$ of underlying spaces), then by definition we have that $j_*(\mathscr{G}) = \psi_*(\mathscr{G})$ for every $(\mathscr{O}_{\mathrm{X}}|\mathrm{U})$-module $\mathscr{G}$, and, further, that $j^*(\mathscr{H}) = \psi^*(\mathscr{H}) = \mathscr{H}|\mathrm{U}$ for every $\mathscr{O}_{\mathrm{X}}$-module $\mathscr{H}$, by definition of the prescheme induced over an open subset. So (i) is thus a particular case of ((9.2.2), (a)); for the same reason, $j_*(j^*(\mathscr{F}))$ is quasi-coherent, and since $\overline{\mathscr{G}}$ is the inverse image of $j_*(\mathscr{G})$ by the homomorphism $\rho : \mathscr{F} \to j_*(j^*(\mathscr{F}))$, (ii) follows from (4.1.1). □

Note that the hypothesis that the morphism $j : \mathrm{U} \to \mathrm{X}$ is quasi-compact holds whenever the open subset U is *quasi-compact* and X is a *scheme*: indeed, U is then a union of finitely-many affine opens U_i, and for every affine open V of X, $\mathrm{V} \cap \mathrm{U}_i$ is an affine open (5.5.6), and thus quasi-compact.

Corollary (9.4.3). — *Let* X *be a prescheme,* U *a quasi-compact open subset of* X *such that the injection morphism* $j : \mathrm{U} \to \mathrm{X}$ *is quasi-compact. Suppose as well that every quasi-coherent* $\mathscr{O}_{\mathrm{X}}$*-module is the inductive limit of its quasi-coherent* $\mathscr{O}_{\mathrm{X}}$*-submodules of finite type (which will be the case if* X *is an affine scheme). Then let* $\mathscr{F}$ *be a quasi-coherent* $\mathscr{O}_{\mathrm{X}}$*-module, and* $\mathscr{G}$ *a quasi-coherent* $(\mathscr{O}_{\mathrm{X}}|\mathrm{U})$*-submodule of finite type of* $\mathscr{F}|\mathrm{U}$*. Then there exists a quasi-coherent* $\mathscr{O}_{\mathrm{X}}$*-submodule* $\mathscr{G}'$ *of* $\mathscr{F}$ *of finite type such that* $\mathscr{G}'|\mathrm{U} = \mathscr{G}$*.*

Proof. We have $\mathscr{G} = \overline{\mathscr{G}}|\mathrm{U}$, and $\overline{\mathscr{G}}$ is quasi-coherent, from (9.4.2), and so the inductive limit of its quasi-coherent $\mathscr{O}_{\mathrm{X}}$-submodules $\mathscr{H}_\lambda$ of finite type. It follows that $\mathscr{G}$ is the inductive limit of the $\mathscr{H}_\lambda|\mathrm{U}$, and thus equal to one of the $\mathscr{H}_\lambda|\mathrm{U}$ since it is of finite type (**0**, 5.2.3). □

Remark (9.4.4). — Suppose that for *every* affine open $\mathrm{U} \subset \mathrm{X}$, the injection morphism $\mathrm{U} \to \mathrm{X}$ is quasi-compact. Then, if the conclusion of (9.4.3) holds for every affine open U and every quasi-coherent $(\mathscr{O}_{\mathrm{X}}|\mathrm{U})$-submodule $\mathscr{G}$ of $\mathscr{F}|\mathrm{U}$ of finite type, it follows that $\mathscr{F}$ is the inductive limit of its quasi-coherent $\mathscr{O}_{\mathrm{X}}$-submodules of finite type. Indeed, for every affine open 175
$\mathrm{U} \subset \mathrm{X}$, we have that $\mathscr{F}|\mathrm{U} = \widetilde{\mathrm{M}}$, where M is an A(U)-module, and since the latter is the inductive limit of its quasi-coherent submodules of finite type, $\mathscr{F}|\mathrm{U}$ is the inductive limit of its $(\mathscr{O}_{\mathrm{X}}|\mathrm{U})$-submodules of finite type (1.3.9). But, by hypothesis, each of these submodules is induced on U by a quasi-coherent $\mathscr{O}_{\mathrm{X}}$-submodule $\mathscr{G}_{\lambda,\mathrm{U}}$ of $\mathscr{F}$ of finite type. The finite sums of the $\mathscr{G}_{\lambda,\mathrm{U}}$ are again quasi-coherent $\mathscr{O}_{\mathrm{X}}$-modules of finite type, because the property is local, and the case where X is affine was covered in (1.3.10); it is clear then that $\mathscr{F}$ is the inductive limit of these finite sums, whence our claim.

Corollary (9.4.5). — *Under the hypotheses of Corollary (9.4.3), for every quasi-coherent* $(\mathscr{O}_{\mathrm{X}}|\mathrm{U})$*-module* $\mathscr{G}$ *of finite type, there exists a quasi-coherent* $\mathscr{O}_{\mathrm{X}}$*-module* $\mathscr{G}'$ *of finite type such that* $\mathscr{G}'|\mathrm{U} = \mathscr{G}$*.*

Proof. Since $\mathscr{F} = j_*(\mathscr{G})$ is quasi-coherent (9.4.2) and $\mathscr{F}|\mathrm{U} = \mathscr{G}$, it suffices to apply Corollary (9.4.3) to $\mathscr{F}$. □

Lemma (9.4.6). — *Let* X *be a prescheme,* L *a well-ordered set,* $(\mathrm{V}_\lambda)_{\lambda \in \mathrm{L}}$ *a cover of* X *by affine opens, and* U *an open subset of* X*; for all* $\lambda \in \mathrm{L}$*, we set* $\mathrm{W}_\lambda = \bigcup_{\mu<\lambda} \mathrm{V}_\mu$*. Suppose that: (1) for every* $\lambda \in \mathrm{L}$*,* $\mathrm{V}_\lambda \cap \mathrm{W}_\lambda$ *is quasi-compact; (2) the immersion morphism* $\mathrm{U} \to \mathrm{X}$ *is quasi-compact. Then, for every quasi-coherent* $\mathscr{O}_{\mathrm{X}}$*-module* $\mathscr{F}$ *and every quasi-coherent* $(\mathscr{O}_{\mathrm{X}}|\mathrm{U})$*-submodule* $\mathscr{G}$ *of* $\mathscr{F}|\mathrm{U}$ *of finite type, there exists a quasi-coherent* $\mathscr{O}_{\mathrm{X}}$*-submodule* $\mathscr{G}'$ *of* $\mathscr{F}$ *of finite type such that* $\mathscr{G}'|\mathrm{U} = \mathscr{G}$*.*

Proof. Let $\mathrm{U}_\lambda = \mathrm{U} \cup \mathrm{W}_\lambda$; we will define a family $(\mathscr{G}'_\lambda)$ by induction, where $\mathscr{G}'_\lambda$ is a quasi-coherent $(\mathscr{O}_{\mathrm{X}}|\mathrm{U}_\lambda)$-submodule of $\mathscr{F}|\mathrm{U}_\lambda$ of finite type, such that $\mathscr{G}'_\lambda|\mathrm{U}_\mu = \mathscr{G}'_\mu$ for $\mu < \lambda$ and $\mathscr{G}'_\lambda|\mathrm{U} = \mathscr{G}$. The unique $\mathscr{O}_{\mathrm{X}}$-submodule $\mathscr{G}'$ of $\mathscr{F}$ such that $\mathscr{G}'|\mathrm{U}_\lambda = \mathscr{G}'$ for all $\lambda \in \mathrm{L}$ (**0**, 3.3.1) gives us what we want. So suppose that the $\mathscr{G}'_\mu$ are defined and have the preceding properties for $\mu < \lambda$; if λ does not have a predecessor then we take for $\mathscr{G}'_\lambda$ the unique $(\mathscr{O}_{\mathrm{X}}|\mathrm{U}_\lambda)$-submodule of $\mathscr{F}|\mathrm{U}_\lambda$ such that $\mathscr{G}'_\lambda|\mathrm{U}_\mu = \mathscr{G}'_\mu$ for all $\mu < \lambda$, which is allowed since the U_μ with $\mu < \lambda$ then form a cover of U_λ. If, conversely, $\lambda = \mu + 1$, then $\mathrm{U}_\lambda = \mathrm{U}_\mu \cup \mathrm{V}_\mu$, and it suffices to define a quasi-coherent $(\mathscr{O}_{\mathrm{X}}|\mathrm{V}_\mu)$-submodule $\mathscr{G}''_\mu$ of $\mathscr{F}|\mathrm{V}_\mu$ of finite type such that

$$\mathscr{G}''_\mu|(\mathrm{U}_\mu \cap \mathrm{V}_\mu) = \mathscr{G}'_\mu|(\mathrm{U}_\mu \cap \mathrm{V}_\mu);$$

and then to take for $\mathscr{G}'_\lambda$ the $(\mathscr{O}_{\mathrm{X}}|\mathrm{U}_\lambda)$-submodule of $\mathscr{F}|\mathrm{U}_\lambda$ such that $\mathscr{G}'_\lambda|\mathrm{U}_\mu = \mathscr{G}'_\mu$ and $\mathscr{G}'_\lambda|\mathrm{V}_\mu = \mathscr{G}''_\mu$ (**0**, 3.3.1). But, since V_μ is affine, the existence of $\mathscr{G}''_\mu$ is guaranteed by (9.4.3) as soon as we show that $\mathrm{U}_\mu \cap \mathrm{V}_\mu$ is quasi-compact; but $\mathrm{U}_\mu \cap \mathrm{V}_\mu$ is the union of $\mathrm{U} \cap \mathrm{V}_\mu$ and $\mathrm{W}_\mu \cap \mathrm{V}_\mu$, which are both quasi-compact by virtue of the hypothesis. □

Theorem (9.4.7). — *Let* X *be a prescheme, and* U *an open set of* X*. Suppose that one of the following conditions is verified:*

(a) *the underlying space of* X *is locally Noetherian;*
(b) X *is a quasi-compact scheme and* U *is a quasi-compact open.*

Then, for every quasi-coherent $\mathscr{O}_X$*-module* $\mathscr{F}$ *and every quasi-coherent* $(\mathscr{O}_X|U)$*-submodule* $\mathscr{G}$ *of* $\mathscr{F}|U$ *of finite type, there exists a quasi-coherent* $\mathscr{O}_X$*-submodule* $\mathscr{G}'$ *of* $\mathscr{F}$ *of finite type such that* $\mathscr{G}'|U = \mathscr{G}$*.*

Proof. Let $(V_\lambda)_{\lambda\in L}$ be a cover of X by affine opens, with L assumed to be finite in case (b); since L is equipped with the structure of a well-ordered set, it suffices to check that the conditions of (9.4.6) are satisfied. It is clear in the case of (a), as the spaces V_λ are Noetherian. For case (b), the $V_\lambda \cap \lambda_\mu$ are affine (5.5.6), and thus quasi-compact, and since L is finite, $V_\lambda \cap W_\lambda$ is quasi-compact. Whence the theorem. □

Corollary (9.4.8). — *Under the hypotheses of (9.4.7), for every quasi-coherent* $(\mathscr{O}_X|U)$*-module* $\mathscr{G}$ *of finite type, there exists a quasi-coherent* $\mathscr{O}_X$*-module* $\mathscr{G}'$ *of finite type such that* $\mathscr{G}'|U = \mathscr{G}$*.*

Proof. It suffices to apply (9.4.7) to $\mathscr{F} = j_*(\mathscr{G})$, which is quasi-coherent (9.4.2) and such that $\mathscr{F}|U = \mathscr{G}$. □

Corollary (9.4.9). — *Let* X *be a prescheme whose underlying space is locally Noetherian, or a quasi-compact scheme. Then every quasi-coherent* $\mathscr{O}_X$*-module is the inductive limit of its quasi-coherent* $\mathscr{O}_X$*-submodules of finite type.*

Proof. This follows from Theorem (9.4.7) and Remark (9.4.4). □

Corollary (9.4.10). — *Under the hypotheses of (9.4.9), if a quasi-coherent* $\mathscr{O}_X$*-module* $\mathscr{F}$ *is such that every quasi-coherent* $\mathscr{O}_X$*-submodule of finite type of* $\mathscr{F}$ *is generated by its sections over* X*, then* $\mathscr{F}$ *is generated by its sections over* X*.*

Proof. Let U be an affine open neighbourhood of a point $x \in X$, and let s be a section of $\mathscr{F}$ over U; the $\mathscr{O}_X$-submodule $\mathscr{G}$ of $\mathscr{F}|U$ generated by s is quasi-coherent and of finite type, so there exists a quasi-coherent $\mathscr{O}_X$-submodule $\mathscr{G}'$ of $\mathscr{F}$ of finite type such that $\mathscr{G}'|U = \mathscr{G}$ (9.4.7). By hypothesis, there is thus a finite number of sections t_i of $\mathscr{G}'$ over X and of sections a_i of $\mathscr{O}_X$ over a neighbourhood $V \subset U$ of x such that $s|V = \sum_i a_i(t_i|V)$, which proves the corollary. □

9.5. **Closed image of a prescheme; closure of a subprescheme.**

Proposition (9.5.1). — *Let* $f : X \to Y$ *be a morphism of preschemes such that* $f_*(\mathscr{O}_X)$ *is a quasi-coherent* $\mathscr{O}_Y$*-module (which will be the case if* f *is quasi-compact and if in addition* f *is either separated or* X *is locally Noetherian (9.2.2)). Then there exists a smaller subprescheme* Y' *of* Y *such that* f *factors through the canonical injection* $j : Y' \to Y$ *(or, equivalently (4.4.1), such that the subprescheme* $f^{-1}(Y')$ *of* X *is identical to* X*).*

More precisely:

Corollary (9.5.2). — *Under the conditions of (9.5.1), let* $f = (\psi, \theta)$*, and let* $\mathscr{J}$ *be the (quasi-coherent) kernel of the homomorphism* $\theta : \mathscr{O}_Y \to f_*(\mathscr{O}_X)$*. Then the closed subprescheme* Y' *of* Y *defined by* $\mathscr{J}$ *satisfies the conditions of (9.5.1).*

Proof. Since the functor ψ^* is exact, the canonical factorization $\theta : \mathscr{O}_Y \to \mathscr{O}_Y/\mathscr{J} \xrightarrow{\theta'} \psi_*(\mathscr{O}_X)$ gives (**0**, 3.5.4.3) a factorization $\theta^\sharp : \psi^*(\mathscr{O}_Y) \to \psi^*(\mathscr{O}_Y)/\psi^*(\mathscr{J}) \xrightarrow{\theta'^\sharp} \mathscr{O}_X$; since $\theta^\sharp_x$ is a local homomorphism for every $x \in X$, the same is true of ${\theta'}_x^\sharp$; if we denote by ψ_0 the continuous map ψ considered as a map from X to X', and by θ_0 the restriction $\theta'|X' : (\mathscr{O}_Y/\mathscr{J})|X' \to \psi_*(\mathscr{O}_X)|X' = (\psi_0)_*(\mathscr{O}_X)$, we see that $f_0 = (\psi_0, \theta_0)$ is a morphism of preschemes $X \to X'$ (2.2.1) such that $f = j \circ f_0$. Now, if X'' is a second closed subprescheme of Y, defined by a quasi-coherent sheaf of ideals $\mathscr{J}'$ of $\mathscr{O}_Y$, such that f factors through the injection $j' : X'' \to Y$, then we should immediately have that $\psi(X) \subset X''$, and so $X' \subset X''$, since X'' is closed. Furthermore, for all $y \in X''$, θ should factorize as $\mathscr{O}_y \to \mathscr{O}_y/\mathscr{J}'_y \to (\psi_*(\mathscr{O}_X))_y$, which by definition leads to $\mathscr{J}'_y \subset \mathscr{J}_y$, and thus X' is a closed subprescheme of X'' (4.1.10). □

Definition (9.5.3). — Whenever there exists a smaller subprescheme Y' of Y such that f factors through the canonical injection $j : Y' \to Y$, we say that Y' is the *closed image* prescheme of X under the morphism f.

Proposition (9.5.4). — *If* $f_*(\mathscr{O}_X)$ *is a quasi-coherent* $\mathscr{O}_Y$*-module, then the underlying space of the closed image of* X *under* f *is the closure* $\overline{f(X)}$ *in* Y*.*

Proof. As the support of $f_*(\mathscr{O}_X)$ is contained in $\overline{f(X)}$, we have (with the notation of (9.5.2)) $\mathscr{J}_y = \mathscr{O}_y$ for $y \notin \overline{f(X)}$, thus the support of $\mathscr{O}_Y/\mathscr{J}$ is contained in $\overline{f(X)}$. In addition, this support is closed and contains $f(X)$: indeed, if $y \in f(X)$, the unit element of the ring $(\psi_*(\mathscr{O}_X))_y$ is not zero, being the germ at y of the section

$$1 \in \Gamma(X, \mathscr{O}_X) = \Gamma(Y, \psi_*(\mathscr{O}_X));$$

as it is the image under θ of the unit element of $\mathscr{O}_y$, the latter does not belong to $\mathscr{J}_y$, hence $\mathscr{O}_y/\mathscr{J}_y \neq 0$; this finishes the proof. □

Proposition (9.5.5). — *(Transitivity of closed images). Let $f: \mathrm{X} \to \mathrm{Y}$ and $g: \mathrm{Y} \to \mathrm{Z}$ be two morphisms of preschemes; we suppose that the closed image Y' of X under f exists, and that if g' is the restriction of g to Y', then the closed image Z' of Y' under g' exists. Then the closed image of X under $g \circ f$ exists and is equal to Z'.*

Proof. It suffices (9.5.1) to show that Z' is the smallest closed subprescheme Z_1 of Z such that the closed subprescheme $(g \circ f)^{-1}(\mathrm{Z}_1)$ of X (equal to $f^{-1}(g^{-1}(\mathrm{Z}_1))$ by Corollary (4.4.2)) is equal to X; it is equivalent to say that Z' is the smallest closed subprescheme of Z such that f factors (?) through the injection $g^{-1}(\mathrm{Z}_1) \to \mathrm{Y}$ (4.4.1). By virtue of the existence of the closed image Y', every Z_1 with this property is such that $g^{-1}(\mathrm{Z}_1)$ factors (?) through Y', which is equivalent to saying that $j^{-1}(g^{-1}(\mathrm{Z}_1)) = g'^{-1}(\mathrm{Z}_1) = \mathrm{Y}'$, denoting by j the injection $\mathrm{Y}' \to \mathrm{Y}$. By the definition of Z', we indeed conclude that Z' is the smallest closed subprescheme of Z satisfying the preceding condition. □

Corollary (9.5.6). — *Let $f: \mathrm{X} \to \mathrm{Y}$ be an S-morphism such that Y is the closed image of X under f. Let Z be an S-scheme; if two S-morphisms g_1, g_2 from Y to Z are such that $g_1 \circ f = g_2 \circ f$ then $g_1 = g_2$.*

Proof. Let $h = (g_1, g_2)_{\mathrm{S}}: \mathrm{Y} \to \mathrm{Z} \times_{\mathrm{S}} \mathrm{Z}$; since the diagonal $\mathrm{T} = \Delta_{\mathrm{Z}}(\mathrm{Z})$ is a closed subprescheme of $\mathrm{Z} \times_{\mathrm{S}} \mathrm{Z}$, $\mathrm{Y}' = h^{-1}(\mathrm{T})$ is a closed subprescheme of Y (4.4.1). Let $u = g_1 \circ f = g_2 \circ f$; we then have, by definition of the product, $h' = h \circ f = (u, u)_{\mathrm{S}}$, so $h \circ f = \Delta_{\mathrm{Z}} \circ u$; since $\Delta_{\mathrm{Z}}^{-1}(\mathrm{T}) = \mathrm{Z}$, we have $h'^{-1}(\mathrm{T}) = u^{-1}(\mathrm{Z}) = \mathrm{X}$, so $f^{-1}(\mathrm{Y}') = \mathrm{X}$. From this, we conclude (4.4.1) that the canonical injection $\mathrm{Y}' \to \mathrm{Y}$ factors (?) through f, so $\mathrm{Y}' = \mathrm{Y}$ by hypothesis; it then follows (4.4.1) that h factorises as $\Delta_{\mathrm{Z}} \circ v$, where v is a morphism $\mathrm{Y} \to \mathrm{Z}$, which implies that $g_1 = g_2 = v$. □

Remark (9.5.7). — If X and Y are S-schemes, proposition (9.5.6) implies that, when Y if the closed image of X under f, 178
f is an *epimorphism* in the category of S-*schemes* (T, 1.1). We will show in chapter V that, conversely, if the closed image Y' of X under f exists and if f is an epimorphism of S-schemes, then we necessarily have $\mathrm{Y}' = \mathrm{Y}$.

Proposition (9.5.8). — *Suppose that the hypotheses of (9.5.1) are satisfied, and let Y' be the closed image of X under f. For every open V of Y, let $f_{\mathrm{V}}: f^{-1}(\mathrm{V}) \to \mathrm{V}$ be the restriction of f; then the closed image of $f^{-1}(\mathrm{V})$ under f_{V} in V exists and is equal to the prescheme induced by Y' on the open $\mathrm{V} \cap \mathrm{Y}'$ of Y'* (said otherwise, to the subprescheme $\inf(\mathrm{V}, \mathrm{Y})$) of Y (4.4.3).

Proof. Let $\mathrm{X}' = f^{-1}(\mathrm{V})$; since the direct image of $\mathscr{O}_{\mathrm{X}'}$ by f_{V} is exactly the restriction of $f_*(\mathscr{O}_{\mathrm{X}})$ to V, it is clear that the kernel $\mathscr{J}'$ of the homomorphism $\mathscr{O}_{\mathrm{V}} \to (f_{\mathrm{V}})_*(\mathscr{O}_{\mathrm{X}'})$ is the restriction of $\mathscr{J}$ to V, from where the proposition quickly follows. □

We will see that this result can be understood as saying that taking the closed image commutes with an extension $\mathrm{Y}_1 \to \mathrm{Y}$ of the base prescheme, which is an *open immersion*. We will see in chapter IV that it is the same for an extension $\mathrm{Y}_1 \to \mathrm{Y}$ which is a *flat* morphism, provided that f is separated and quasi-compact.

Proposition (9.5.9). — *Let $f: \mathrm{X} \to \mathrm{Y}$ be a morphism such that the closed image Y' of X under f exists.*

(1) *If X is reduced, then so too is Y'.*
(2) *If the hypotheses of (9.5.1) are satisfied and X is irreducible* (resp. *integral (?)), then so too is Y'.*

Proof. By hypothesis, the morphism f factorises as $\mathrm{X} \xrightarrow{g} \mathrm{Y}' \xrightarrow{j} \mathrm{Y}$, where j is the canonical injection. As X is reduced, g factorises as $\mathrm{X} \xrightarrow{h} \mathrm{Y}'_{\mathrm{red}} \xrightarrow{j'} \mathrm{Y}'$, where j' is the canonical injection (5.2.2), and it then follows from the definition of Y' that $\mathrm{Y}'_{\mathrm{red}} = \mathrm{Y}'$. If, further, the conditions of (9.5.1) are satisfied, then it follows from (9.5.4) that $f(\mathrm{X})$ is dense in Y'; if X is irreducible, then so too is Y' (**0**, 2.1.5). The claim about integral preschemes follows from the conjunction of the two others. □

Proposition (9.5.10). — *Let Y be a subprescheme of a prescheme X, such that the canonical injection $i: \mathrm{Y} \to \mathrm{X}$ is a quasi-compact morphism. Then there exists a smaller closed subprescheme $\overline{\mathrm{Y}}$ of X containing (?) Y; its underlying space is the closure of that of Y; the latter is open in its closure, and the prescheme Y is induced on this open by $\overline{\mathrm{Y}}$.*

Proof. It suffices to apply (9.5.1) to the injection j, which is separated (5.5.1) and quasi-compact by hypothesis; (9.5.1) thus proves the existence of $\overline{\mathrm{Y}}$ and (9.5.4) shows that its underlying space is the closure of Y in X; since Y is locally closed in X, it is open in $\overline{\mathrm{Y}}$, and the last claim comes from (9.5.8) applied to an open V of X such that Y is closed in V. □

With the above notation, if the injection $\mathrm{V} \to \mathrm{X}$ is quasi-compact, and if $\mathscr{J}$ is the quasi-coherent sheaf of ideals of $\mathscr{O}_{\mathrm{X}}|\mathrm{V}$ defining the closed subprescheme Y of V, it follows from (9.5.1) that the quasi-coherent sheaf of ideals of $\mathscr{O}_{\mathrm{X}}$ defining $\overline{\mathrm{Y}}$ is the canonical extension (9.4.1) $\overline{\mathscr{J}}$ of $\mathscr{J}$, because it is evidently the biggest quasi-coherent subsheaf of ideals of $\mathscr{O}_{\mathrm{X}}$ inducing $\mathscr{J}$ on V.

Corollary (9.5.11). — *Under the hypotheses of (9.5.10), every section of $\mathscr{O}_{\overline{\mathrm{Y}}}$ over an open V of $\overline{\mathrm{Y}}$ that is null on $\mathrm{V} \cap \mathrm{Y}$ is null.* 179

Proof. By (9.5.8), we can reduce to the case where $\mathrm{V} = \overline{\mathrm{Y}}$. If we take into account that the sections of $\mathscr{O}_{\overline{\mathrm{Y}}}$ over $\overline{\mathrm{Y}}$ canonically correspond to the $\overline{\mathrm{Y}}$-sections of $\overline{\mathrm{Y}} \otimes_{\mathrm{Z}} \mathrm{Z}[\mathrm{T}]$ (3.3.15) and that the latter is separated over $\overline{\mathrm{Y}}$, the corollary appears as a specific case of (9.5.6). □

When there exists a smaller closed subprescheme Y of X containing (?) a subprescheme Y of X, we say that Y′ is the *closure* of Y in X, when it doesn't lead to any confusion.

9.6. **Quasi-coherent sheaves of algebras; change of structure sheaf.**

Proposition (9.6.1). — *Let* X *be a prescheme,* $\mathscr{B}$ *a quasi-coherent* $\mathscr{O}_{\mathrm{X}}$*-algebra (**0**, 5.1.3). For a* $\mathscr{B}$*-module* $\mathscr{F}$ *to be quasi-coherent (on the ringed space* $(\mathrm{X},\mathscr{B})$*) it is necessary and sufficient that* $\mathscr{F}$ *be a quasi-coherent* $\mathscr{O}_{\mathrm{X}}$*-module.*

Proof. Since the question is a local one, we can assume X to be affine, given by the ring A, and thus $\mathscr{B}=\widetilde{\mathrm{B}}$, where B is an A-algebra (1.4.3). If $\mathscr{F}$ is quasi-coherent on the ringed space $(\mathrm{X},\mathscr{B})$ then we can also assume that $\mathscr{F}$ is the cokernel of a $\mathscr{B}$-homomorphism $\mathscr{B}^{(\mathrm{I})}\to\mathscr{B}^{(\mathrm{J})}$; since this homomorphism is also an $\mathscr{O}_{\mathrm{X}}$-homomorphism of $\mathscr{O}_{\mathrm{X}}$-modules, and $\mathscr{B}^{(\mathrm{I})}$ and $\mathscr{B}^{(\mathrm{J})}$ are quasi-coherent $\mathscr{O}_{\mathrm{X}}$-modules (1.3.9, (ii)), $\mathscr{F}$ is also a quasi-coherent $\mathscr{O}_{\mathrm{X}}$-module (1.3.9, (i)).

Conversely, if $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_{\mathrm{X}}$-module, then $\mathscr{F}=\widetilde{\mathrm{M}}$, where M is a B-module (1.4.3); M is isomorphic to the cokernel of a B-homomorphism $\mathrm{B}^{(\mathrm{I})}\to\mathrm{B}^{(\mathrm{J})}$, so $\mathscr{F}$ is a $\mathscr{B}$-module isomorphic to the cokernel of the corresponding homomorphism $\mathscr{B}^{(\mathrm{I})}\to\mathscr{B}^{(\mathrm{J})}$ (1.3.13), which finishes the proof. □

In particular, if $\mathscr{F}$ and $\mathscr{G}$ are two quasi-coherent $\mathscr{B}$-modules, $\mathscr{F}\otimes_{\mathscr{B}}\mathscr{G}$ is a quasi-coherent $\mathscr{B}$-module; similarly for $\mathscr{H}om(\mathscr{F},\mathscr{G})$ whenever we further suppose that $\mathscr{F}$ admits a finite presentation (1.3.13).

(9.6.2). Given a prescheme X, we say that a quasi-coherent $\mathscr{O}_{\mathrm{X}}$-algebra $\mathscr{B}$ is of *finite type* if, for all $x\in\mathrm{X}$, there exists an open *affine* neighbourhood U of x such that $\Gamma(\mathrm{U},\mathscr{B})=\mathrm{B}$ is an algebra of finite type over $\Gamma(\mathrm{U},\mathscr{O}_{\mathrm{X}})=\mathrm{A}$. We then have that $\mathscr{B}|\mathrm{U}=\widetilde{\mathrm{B}}$ and, for all $f\in\mathrm{A}$, the induced $(\mathscr{O}_{\mathrm{X}}|\mathrm{D}(f))$-algebra $\mathscr{B}|\mathrm{D}(f)$ is of finite type, because it is isomorphic to $(\mathrm{B}_f)^\sim$, and $\mathrm{B}_f=\mathrm{B}\otimes_{\mathrm{A}}\mathrm{A}_f$ is clearly an algebra of finite type over A_f. Since the $\mathrm{D}(f)$ form a basis of the topology of U, we thus conclude that if $\mathscr{B}$ is a quasi-coherent $\mathscr{O}_{\mathrm{X}}$-algebra of finite type then, for every open V of X, $\mathscr{B}|\mathrm{V}$ is a quasi-coherent $(\mathscr{O}_{\mathrm{X}}|\mathrm{V})$-algebra of finite type.

Proposition (9.6.3). — *Let* X *be a locally Noetherian prescheme. Then every quasi-coherent* $\mathscr{O}_{\mathrm{X}}$*-algebra* $\mathscr{B}$ *of finite type is a coherent sheaf of rings (**0**, 5.3.7).*

Proof. We can once again restrict to the case where X is an affine scheme given by a Noetherian ring A, and where $\mathscr{B}=\widetilde{\mathrm{B}}$, B being an A-algebra of finite type; B is then a Noetherian ring. With this, it remains to prove that the kernel $\mathscr{N}$ of a $\mathscr{B}$-homomorphism $\mathscr{B}^m\to\mathscr{B}$ is a $\mathscr{B}$-module of finite type; but it is isomorphic (as a $\mathscr{B}$-module) to $\widetilde{\mathrm{N}}$, where N is the kernel of the corresponding homomorphism of B-modules $\mathrm{B}^m\to\mathrm{B}$ (1.3.13). Since B is Noetherian, the submodule N of B^m is a B-module of finite type, so there exists a homomorphism $\mathrm{B}^p\to\mathrm{B}^m$ with image N; since the sequence $\mathrm{B}^p\to\mathrm{B}^m\to\mathrm{B}$ is exact, so too is the corresponding sequence $\mathscr{B}^p\to\mathscr{B}^m\to\mathscr{B}$ (1.3.5) and since $\mathscr{N}$ is the image of $\mathscr{B}^p\to\mathscr{B}^m$ (1.3.9, (i)), the proposition is proved. □

Corollary (9.6.4). — *Under the hypotheses of (9.6.3), for a* $\mathscr{B}$*-module* $\mathscr{F}$ *to be coherent, it is necessary and sufficient that it be a quasi-coherent* $\mathscr{O}_{\mathrm{X}}$*-module and a* $\mathscr{B}$*-module of finite type. If this is the case, and if* $\mathscr{G}$ *is a sub-*$\mathscr{B}$*-module or a quotient module of* $\mathscr{F}$*, in order for* $\mathscr{G}$ *to be a coherent* $\mathscr{B}$*-module, it is necessary and sufficient that is be a quasi-coherent* $\mathscr{O}_{\mathrm{X}}$*-module.*

Proof. Taking into account (9.6.1), the conditions on $\mathscr{F}$ are clearly necessary; we will show that they are sufficient. We can restrict to the case where X is affine given by a Noetherian ring A, $\mathscr{B}=\widetilde{\mathrm{B}}$, where B is an A-algebra of finite type, $\mathscr{F}=\widetilde{\mathrm{M}}$, where M is a B-module, and where there exists a surjective $\mathscr{B}$-homomorphism $\mathscr{B}^m\to\mathscr{F}\to 0$. We then have the corresponding exact sequence $\mathrm{B}^m\to\mathrm{M}\to 0$, so M is a B-module of finite type; further, the kernel P of the homomorphism $\mathrm{B}^m\to\mathrm{M}$ is then a B-module of finite type, since B is Noetherian. We thus conclude (1.3.13) that $\mathscr{F}$ is the cokernel of a $\mathscr{B}$-homomorphism $\mathscr{B}^m\to\mathscr{B}^n$, and is thus coherent, since $\mathscr{B}$ is a coherent sheaf of rings (**0**, 5.3.4). The same reasoning shows that a quasi-coherent sub-$\mathscr{B}$-module (resp. a quotient $\mathscr{B}$-module) of $\mathscr{F}$ is of finite type, from whence the second part of the corollary. □

Proposition (9.6.5). — *Let* X *be a quasi-compact scheme or a prescheme whose underlying space is Noetherian. For all quasi-compact* $\mathscr{O}_{\mathrm{X}}$*-algebras* $\mathscr{B}$ *of finite type, there exists a quasi-coherent sub-*$\mathscr{O}_{\mathrm{X}}$*-module* $\mathscr{E}$ *of* $\mathscr{B}$ *of finite type such that* $\mathscr{E}$ *generates (**0**, 4.1.4) the* $\mathscr{O}_{\mathrm{X}}$*-algebra* $\mathscr{B}$*.*

Proof. In fact, by the hypothesis there exists a finite cover (U_i) of X consisting of open affines such that $\Gamma(\mathrm{U}_i,\mathscr{B})=\mathrm{B}_i$ is an algebra of finite type over $\Gamma(\mathrm{U}_i,\mathscr{O}_{\mathrm{X}})=\mathrm{A}_i$; let E_i be a subA_i-module of B_i of finite type that generates the A_i-algebra B_i; thanks to (9.4.7), there exists a sub-$\mathscr{O}_{\mathrm{X}}$-module $\mathscr{E}_i$ of $\mathscr{B}$, quasi-coherent and of finite type, such that $\mathscr{E}_i|\mathrm{U}_i=\widetilde{\mathrm{E}}_i$. It is clear that the sum $\mathscr{E}$ of the $\mathscr{E}_i$ is the desired object. □

Proposition (9.6.6). — *Let* X *be a prescheme whose underlying space is locally Noetherian, or a quasi-compact scheme. Then every quasi-coherent* $\mathscr{O}_{\mathrm{X}}$*-algebra* $\mathscr{B}$ *is the inductive limit of its quasi-coherent sub-*$\mathscr{O}_{\mathrm{X}}$*-algebras of finite type.*

Proof. In fact, it follows from (9.4.9) that $\mathscr{B}$ is the inductive limit (as an $\mathscr{O}_{\mathrm{X}}$-module) of its quasi-coherent sub-$\mathscr{O}_{\mathrm{X}}$-modules of finite type; the latter generating quasi-coherent sub-$\mathscr{O}_{\mathrm{X}}$-*algebras* of $\mathscr{B}$ of finite type (1.3.14), and so $\mathscr{B}$ is a fortiori their inductive limit. □

10. Formal schemes

10.1. **Formal affine schemes.**

(10.1.1). Let A be an *admissible* topological ring (**0**, 7.1.2); for each ideal of definition $\mathfrak{J}$ of A, $\mathrm{Spec}(A/\mathfrak{J})$ identifies with the closed subspace $V(\mathfrak{J})$ of Spec(A) (1.1.11), the set of *open* prime ideals of A; this topological space does not depend on the ideal of definition $\mathfrak{J}$ considered; we denote this topological space by $\mathfrak{X}$. Let $(\mathfrak{J}_\lambda)$ be a fundamental system of neighbourhoods of 0 in A, consisting of ideals of definition, and for each λ, let $\mathscr{O}_\lambda$ be the structure sheaf of $\mathrm{Spec}(A/\mathfrak{J}_\lambda)$; this sheaf is induced on $\mathfrak{X}$ by $\widetilde{A/\mathfrak{J}_\lambda}$ (which is zero outside of $\mathfrak{X}$). For $\mathfrak{J}_\mu \subset \mathfrak{J}_\lambda$, the canonical homomorphism $A/\mathfrak{J}_\mu \to A/\mathfrak{J}_\lambda$ thus defines a homomorphism $u_{\lambda\mu} : \mathscr{O}_\mu \to \mathscr{O}_\lambda$ of sheaves of rings (1.6.1), and $(\mathscr{O}_\lambda)$ is a *projective system of sheaves of rings* for these homomorphisms. As the topology of $\mathfrak{X}$ admits a basis consisting of quasi-compact open subsets, we can associate to each $\mathscr{O}_\lambda$ a *sheaf of psuedo-discrete topological rings* (**0**, 3.8.1) which ... (?) 181

We denote by $\mathscr{O}_{\mathfrak{X}}$ the *sheaf of topological rings* on $\mathfrak{X}$, the projective limit of the system $(\mathscr{O}_\lambda)$; for each *quasi-compact* open subset U of $\mathfrak{X}$, then $\Gamma(U, \mathscr{O}_{\mathfrak{X}})$ is a topological ring, the projective limit of the system of *discrete* rings $\Gamma(U, \mathscr{O}_\lambda)$ (3.2.6).

Definition (10.1.2). — Given an admissible topological ring A, we define the formal spectrum of A, and denote it by Spf(A), to be the closed subspace $\mathfrak{X}$ of Spec(A) consisting of the open prime ideals of A. We say that a topologically ringed space is a formal affine scheme if it is isomorphic to a formal spectrum $\mathrm{Spf}(A) = \mathfrak{X}$ equipped with a sheaf of topological rings $\mathscr{O}_{\mathfrak{X}}$ which is the projective limit of sheaves of psuedo-discrete topological rings $(\widetilde{A/\mathfrak{J}_\lambda})|\mathfrak{X}$, where $\mathfrak{J}_\lambda$ varies over the filtered set of ideals of definition for A.

When we speak of a *formal spectrum* $\mathfrak{X} = \mathrm{Spf}(A)$ as a formal affine scheme,

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