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- For few/no collisions over n keys, pick $m \geq n^3$. This is called *universe reduction*.
- For more complex hash functions H, like k-independent with larger k, we only have to deal with key universe polynomial in number of keys

$$U \xrightarrow{h} [n^3] \xrightarrow{H} R$$

Our focus in this lecture is H.

Fast and Powerful Hashing using Tabulation

Mikkel Thorup
University of Copenhagen

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Lecture covers results from

- Mihai Pătraşcu adn Mikkel Thorup: The power of simple tabulation hashing. J. ACM 59(3): 14 (2012). Announced at STOC 2011: 1-10
- Mihai Pătraşcu and Mikkel Thorup: Twisted Tabulation Hashing. SODA 2013: 209-228
- Mikkel Thorup: Simple Tabulation, Fast Expanders, Double Tabulation, and High Independence. FOCS 2013: 90-99.
- Søren Dahlgaard and Mikkel Thorup: Approximately Minwise Independence with Twisted Tabulation. SWAT 2014.
- And some recent stuff.

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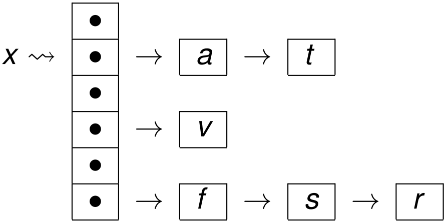
- Providing algorithmically important probabilisitic guarantees akin to those of truly random hashing, yet easy to implement.
- Bridging theory (assuming truly random hashing) with practice (needing something implementable).

Applications of Hashing $\rightsquigarrow$

Hash tables (n keys and $2n$ hashes: expect 1/2 keys per hash)

- chaining: follow pointers



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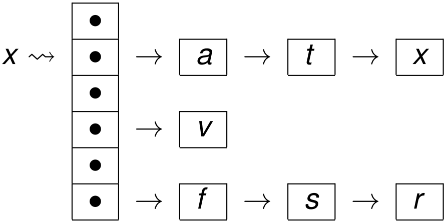

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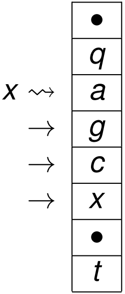

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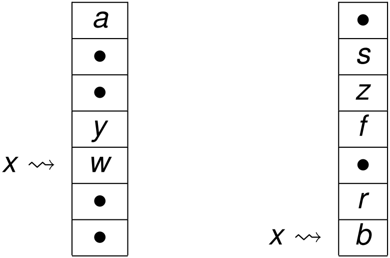

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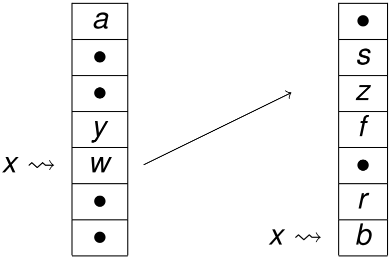

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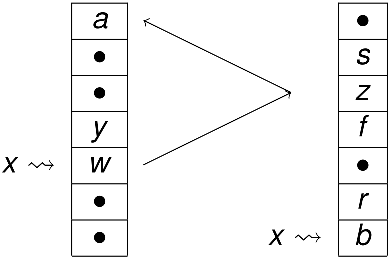

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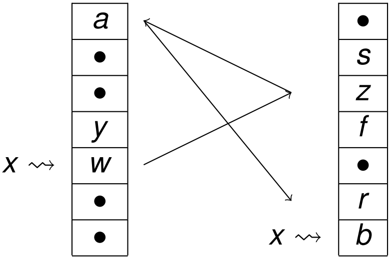

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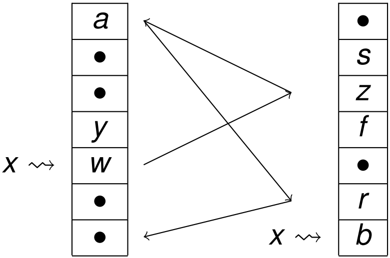

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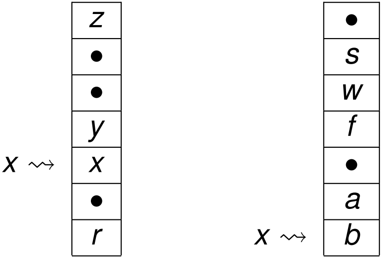

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Sketching, streaming, and sampling:

- second moment estimation: $F_2(\bar{x}) = \sum_i x_i^2$
- sketch A and B to later find $|A \cap B|/|A \cup B|$

$$|A \cap B|/|A \cup B| = \Pr_h[\min h(A) = \min h(B)]$$

We need h to be ε-minwise independent:

$$x \in S: \qquad \Pr[h(x) = \min h(S)] = \frac{1 \pm \varepsilon}{|S|}$$

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Important outside theory. These simple practical hash tables often bottlenecks in the processing of data—substantial fraction of worlds computational resources spent here.

Wegman & Carter [FOCS'77]

We do not have space for truly random hash functions, but

Family $\mathcal{H} = \{h : [u] \to [b]\}$ k-independent iff for random $h \in \mathcal{H}$:

- $(\forall) x \in [u]$, $h(x)$ is uniform in $[b]$;
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Many solutions for k-independent hashing proposed, but generally slow for $k > 3$ and too slow for $k > 5$.

How much independence needed?

Chaining $\mathbf{E}[t] = O(1)$ $\mathbf{E}[t^k] = O(1)$ $t = O\left(\frac{\lg n}{\lg \lg n}\right)$ w.h.p.	2 $2k+1$ $\Theta\left(\frac{\lg n}{\lg \lg n}\right)$	
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Independence has been the ruling measure for quality of hash functions for 30+ years, but is it right?

Simple Tabulation Hashing [Zobrist'70 chess]

- Key x divided into $c = O(1)$ characters $x_1, ..., x_c$, e.g., 32-bit key as 4×8-bit characters.

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- Not 4-independent: $h(a_1 a_2) \oplus h(a_1 b_2) \oplus h(b_1 a_2) \oplus h(b_1 b_2)$

$$\begin{aligned} = \quad & (R_1[a_1] \oplus R_2[a_2]) \oplus (R_1[a_1] \oplus R_2[b_2]) \oplus \\ & (R_1[b_1] \oplus R_2[a_2]) \oplus (R_1[b_1] \oplus R_2[b_2]) \; = \; 0. \end{aligned}$$

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New result: Despite its 4-dependence, simple tabulation suffices for all the above applications:

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We prove that dependence of simple tabulation is not harmful in any of the above applications.

Chaining/hashing into bins

Theorem Consider hashing n balls into $m \geq n^{1-1/(2c)}$ bins by simple tabulation. Let q be an additional *query ball*, and define X_q as the number of regular balls that hash into a bin chosen as a function of $h(q)$. Let $\mu = \mathbf{E}[X_q] = \frac{n}{m}$. The following probability bounds hold for any constant γ:

$$\Pr[X_q \geq (1+\delta)\mu] \leq \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{\Omega(\mu)} + m^{-\gamma}$$

$$\Pr[X_q \leq (1-\delta)\mu] \leq \left(\frac{e^{-\delta}}{(1-\delta)^{(1-\delta)}}\right)^{\Omega(\mu)} + m^{-\gamma}$$

With $m \leq n$ bins, every bin gets

$$n/m \pm O\left(\sqrt{n/m}\log^c n\right).$$

keys with probability $1 - n^{-\gamma}$.

Hashing into many bins

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Nothing like this lemma holds if we instead of simple tabulation assumed k-independent hashing with $k = O(1)$.

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Proof that for any positive constants ε, γ, if we hash n keys into m bins and $n \leq m^{1-\varepsilon}$, then all bins get less than $d = 2^{(1+\gamma)/\varepsilon}$ keys with probability $\geq 1 - m^{-\gamma}$.

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- Return $\{x\} \cup U'$ where U' independent subset of T'.

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Claim 2 The probability that there exists $u = (1+\gamma)/\varepsilon$ keys hashing independently to the same bin is $m^{-\gamma}$.

Hashing into many bins

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- Propability bound over all U is

$$n^u m^{u-1} \leq m^{(1-\varepsilon)u+1-u} = m^{1-\varepsilon u} = m^{-\gamma}.$$

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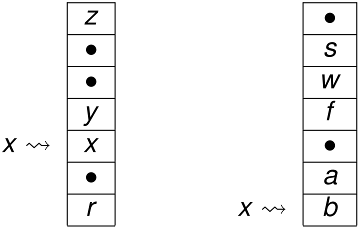
Fundamental point Simple tabulation is only 3-independent, but inside any set T of size $\omega(1)$, there is a subset $S \subseteq T$ of size $\omega(1)$ where all keys in S are hashed independently.

Cuckoo hashing

Each key placed in one of two hash locations.



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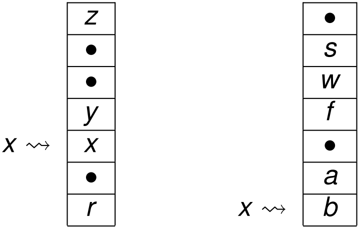

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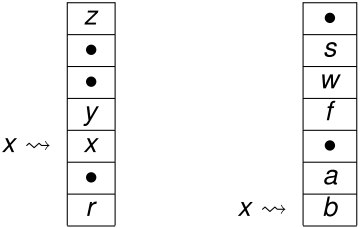

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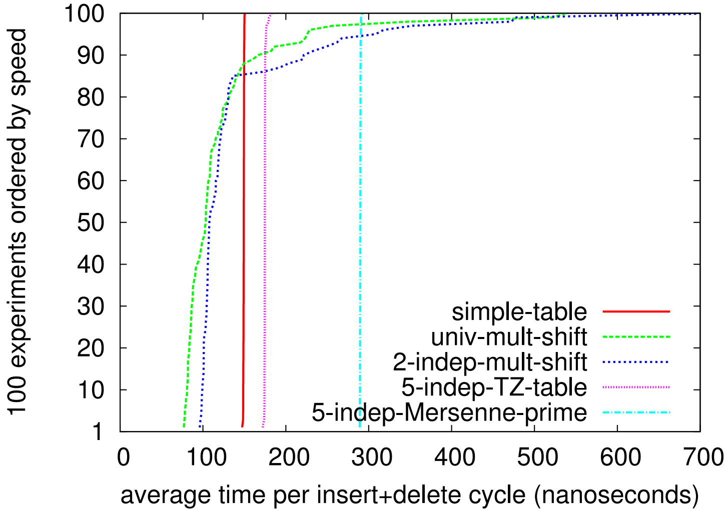
- Very delicate proof showing that obstruction can be used to code random tables R_i with few bits.

Speed

Hashing random keys		32-bit computer	64-bit computer
bits	hashing scheme	hashing time (ns)	
32	univ-mult-shift `(a*x)>>s`	1.87	2.33
32	2-indep-mult-shift	5.78	2.88
32	5-indep-Mersenne-prime	99.70	45.06
32	5-indep-TZ-table	10.12	12.66
32	simple-table	4.98	4.61
64	univ-mult-shift	7.05	3.14
64	2-indep-mult-shift	22.91	5.90
64	5-indep-Mersenne-prime	241.99	68.67
64	5-indep-TZ-table	75.81	59.84
64	simple-table	15.54	11.40

Experiments with help from Yin Zhang.

Robustness in linear probing for dense interval



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- Multiplicative hashing used in practice, but turns out to be very unreliable under typical denial-of-service (DoS) attacks based on consecutive IP addresses: systematic good performance 95% of the time, but systematic terrible performance 5% of the time [TZ'10].

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- Linear probing had gotten a reputation for being fastest in practice, but sometimes unreliable needing special protection against bad cases.
- Here we proved linear probing safe with good probabilistic performance for all input if we use simple tabulation.
- Simple tabulation also powerful for chaining, cuckoo hashing, and min-wise hashing:

  one simple and fast scheme for (almost) all your needs.

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C-code for twisted tabulation (32-bit key, 8-bit char)

```
INT32 TwistedTab32(INT32 x, INT64[4][256] H) {
  INT32 i; INT64 h=0; INT8 c;
  for (i=0;i<c;i++) {
    c=x;
    h^=H[i][c];
    x = x>> 8;
  }
  c=x^h;        //twisted character
  h^=H[i][c];
  h>>=8;        //dropping twister from hash
  return ((INT32) h);
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- With simple tabulation, additive term $(\max_i p_i)^{\gamma}$
  —in the hash tables we had $p \approx 1/n$.

Min-wise hashing with Søren Dahlgaard

- Min-wise hashing h of keys in U with bias ε:

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Speed of different schemes

Hashing scheme	time (ns)
2-indep: multiplication-shift [Dietzfelbinger'96]	1.27
3-indep: Polynomial with Mersenne prime $2^{61}-1$	7.72
5-indep: Polynomial with Mersenne prime $2^{61}-1$	14.09
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Pseudo-random number generator	time (ns)
random() from C-libraray	6.99
twisted-PRG()	1.18

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- Take any application using abstract truly random hash function, and prove that simple/twisted tabulation works.
- Could this be the first implementable hash function/RNG making classic quick sort work directly: using hash of i to generate index of ith pivot?
- Recently with Søren Dahlgaar, Mathias Bæk Tejs Knudsen, and Eva Rotenberg: simple tabulation yields good results for power of two choices, e.g., max load $O(\log \log n)$ w.h.p., placing n keys in $\Omega(n)$ bins (each key hash to two bins and goes for the lighter loaded).

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But if you really want high independence, then just apply simple tabulation twice...

k-independence [Wegman Carter FOCS'79]

Hashing universe U of keys into range R of hash values.

Random hash function $h : U \to R$ is k-independent iff:

- $(\forall) x \in U$, $h(x)$ is (almost) uniform in R;
- $(\forall) x_1, \ldots, x_k \in U$, $h(x_1), \ldots, h(x_k)$ are independent.

Prototypical example: Polynomial over prime field $\mathbb{Z}_p$:

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- but evaluating $h(x)$ takes $O(k)$ time.

Siegel on highly independence [FOCS'89]

Polynomials: k-independent hashing in k time using k space

Question Can we get k-independent hashing in $t < k$ time?

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Schemes are simple but analysis is not.

Simple tabulation

- Key $x = (x_0, ..., x_{c-1}) \in \Phi^c = U$, $c = O(1)$.
- For $i = 0, , ..., c - 1$, truly random character hash table:
  $h_i : \Phi \to R = b$-bit strings.
- Hash function $h : U \to R$ defined by

$$h(x_0, ..., x_{c-1}) = h_0[x_0] \oplus \cdots \oplus h_{c-1}[x_{c-1}]$$

- We saw not 4-independent, yet powerful enough for many algorithmic contexts otherwise requiring higher independence.

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We now claim that simple tabulation is expected to yield:

- Very fast unbalanced constant degree expanders.
- Applied twice yields highly independent hashing.

Simple tabulation as unbalanced expander

- Consider any function $f : U \to \Phi^d$.
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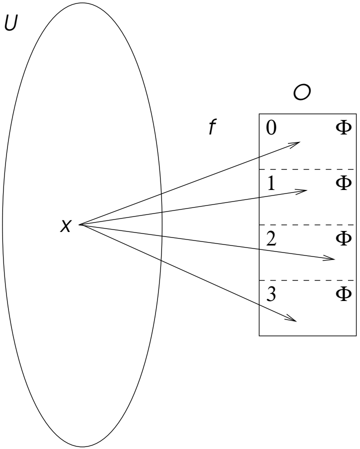

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Thm Let $h : \Phi^c \to \Phi^d$ random simple tabulation function with $d = 12c$. Then with probability $1 - o(1/|\Phi|)$,

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Note Best explicit unbalanced expanders [Guruswami et al. JACM'09] have logarithmic degrees, and would be orders of magnitude slower to compute.

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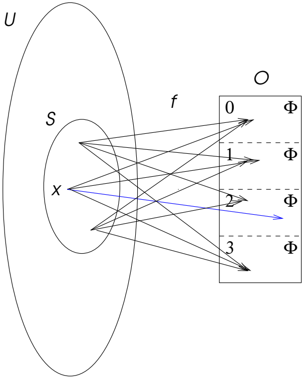

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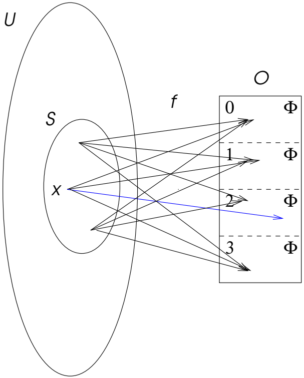

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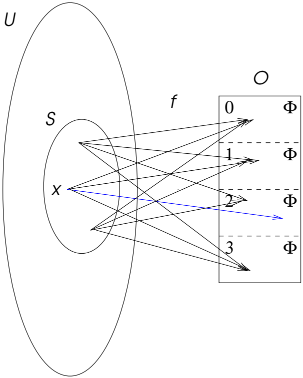

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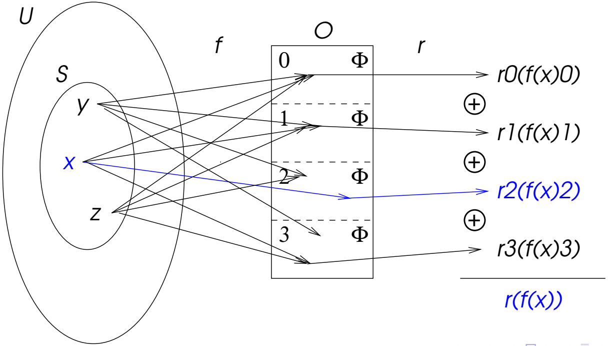

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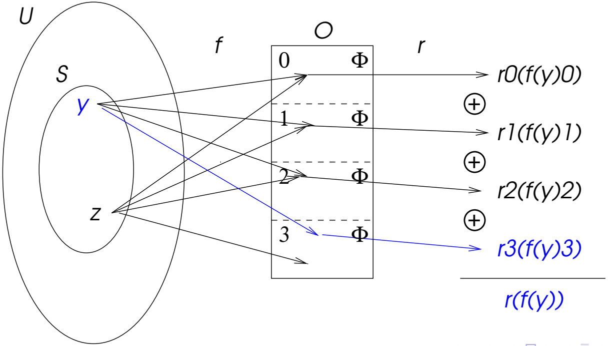

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Cor (Double Tabulation) If $d = 6c$ and $h : U = \Phi^c \to \Phi^d$ and $r : \Phi^d \to R$ are random simple tabulation functions, then $r \circ h : U \to R$ is $|\Phi|^{1/(5c)}$-independent with "universal" probability $1 - o(1/|\Phi|)$.

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Note A k-unique h can be used as a universal constant that only has to be guessed or constructed once.

With real constants

Constants in construction pretty good:

Thm For 32-bit keys divided in $c = 2$ characters from $\Phi = [2^{16}]$, if $h : \Phi^2 \to \Phi^{20}$ is random simple tabulation function, then h is 100-unique with probability $1 - 1.5 \times 10^{-42}$.

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Claiming that a USB flash drive with random bits represents a universal k-unique simple tabulation function, would be very safe hardware

Techniques

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- The number of possible derivations is small compared to the number of possible output characters, so the event is unlikely with random output position characters.

Concluding remarks

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- Simple, twisted, and double tabulation only needs pointers to randomly configured memory that does not need to be changed.
- In multi-core systems this leaves the cache clean with efficient concurrent processing.
- One could even imagine using special cheap and fast randomly configured read only memory.

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- Double tabulation hashing also gives exponentially better error bounds in min-wise hashing and Chernoff bounds than those obtained with other hashing schemes.

High independence

Negative (Siegel) $t < k$ time (cell probes) needs $|U|^{1/t}$ space.

Positive new results

1. $|U|^{\Omega(1/c^2)}$-independent in optimal $O(c)$ time using $|U|^{1/c}$ space (double tabluation).
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Recently With Søren Dahlgaar and Mathias Bæk Tejs Knudsen: for given set S of size $\leq |U|^{1/c}$, w.h.p., double tabulation truely random on S using $O(|U|^{1/c})$ space and $O(c)$ time.

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- Most people believe that deterministic constant time is not possible without randomization, but nobody can prove it.