Balanced Allocations, Cuckoo Hash Tables, and Such

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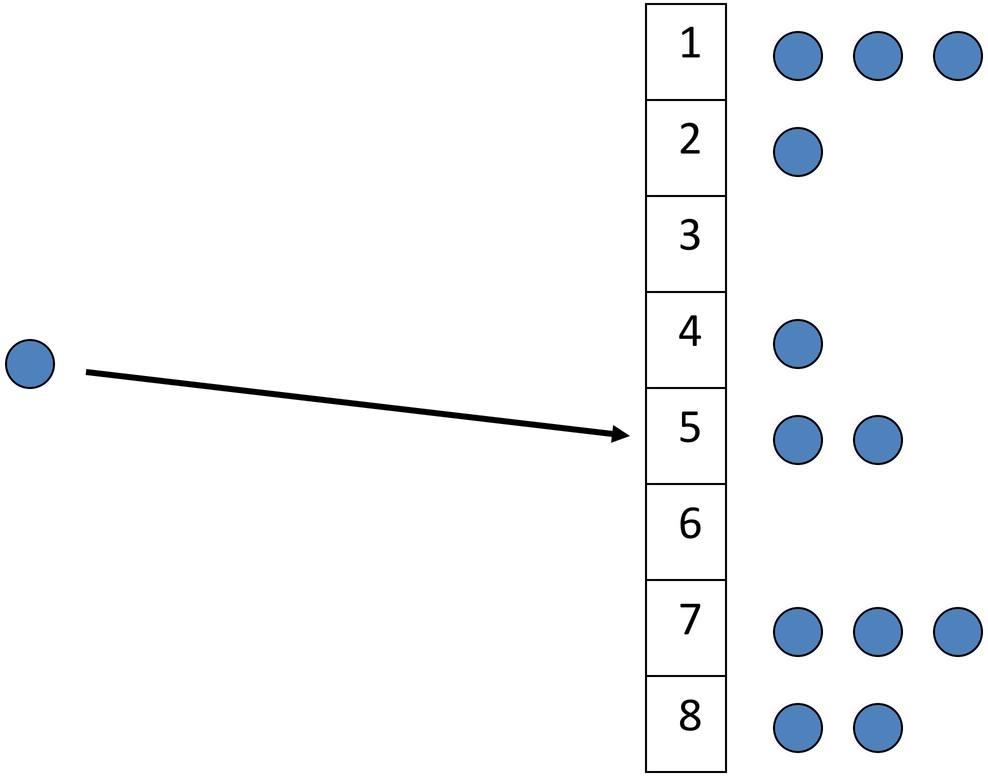
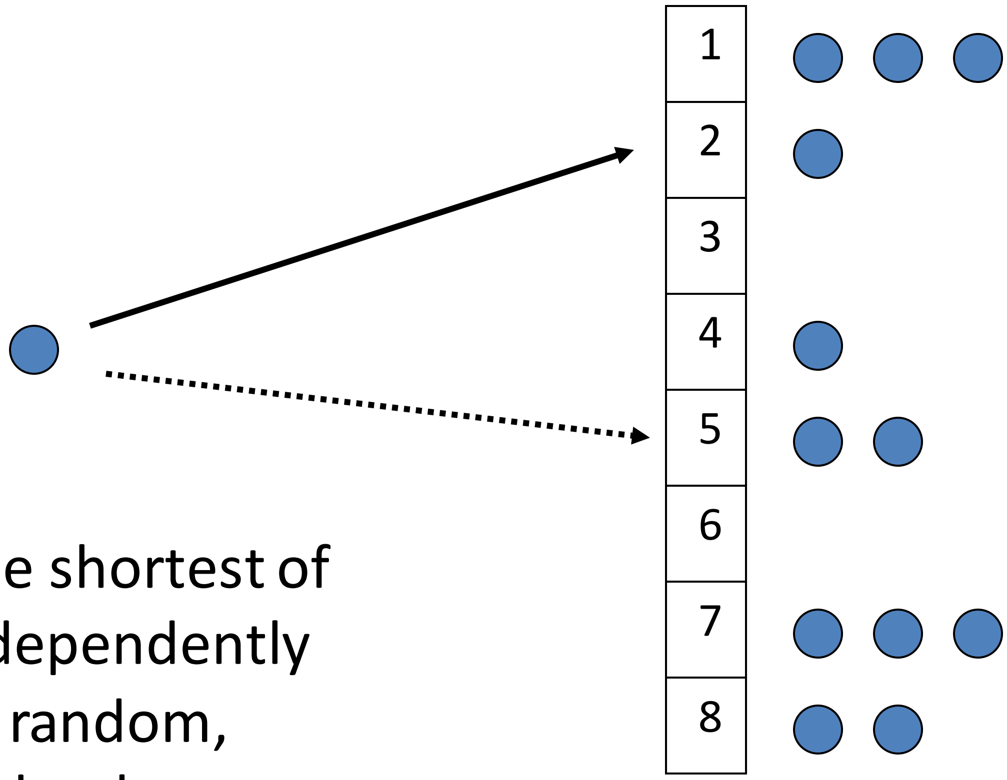
Hashing Model



Hashing Model: *d*-Random



Item placed in the shortest of *d* bins chosen independently and uniformly at random, breaking ties randomly.

Azar, Broder, Karlin, Upfal (STOC 94)

Throw n balls into n bins randomly.
Maximum load is $\log n / \log \log n$.

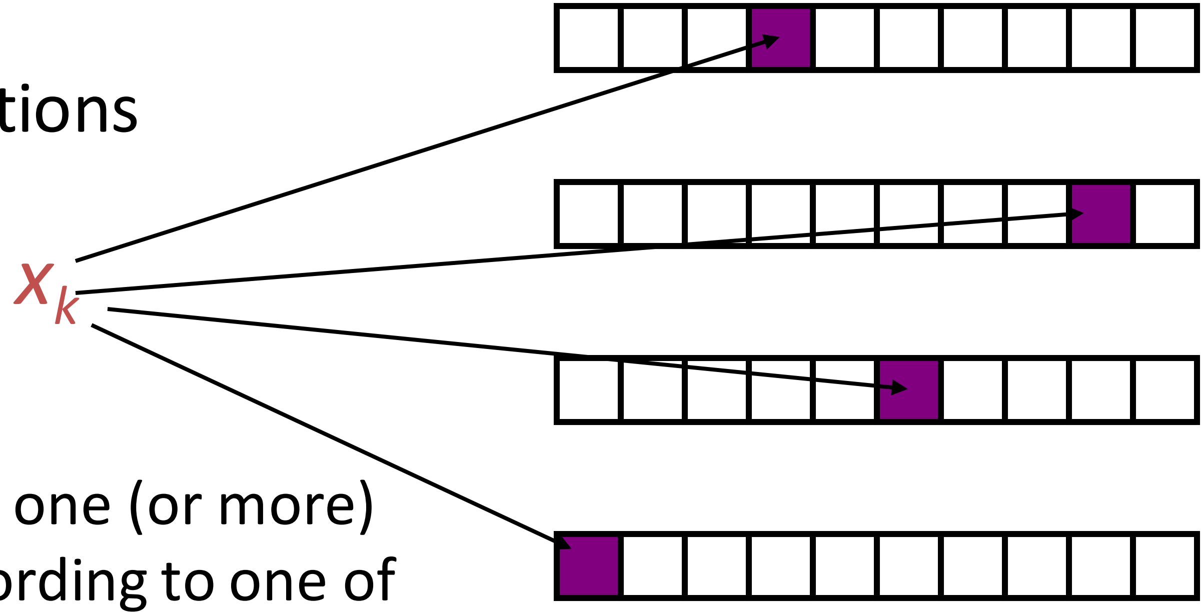
Place n balls into n bins sequentially, each ball going into the least loaded of d random locations.

Maximum load is $\log \log n / \log d$.
Two choices: $\log \log n / \log 2$.

Multiple-Choice Hashing

$S = \{x_1, x_2, \ldots, x_n\}$ of items

d hash functions



Item placed in one (or more) locations, according to one of the hash functions.

d subtables

Multiple-Choice Hashing

- Can significantly reduce load.
  - Choice allows much more even spread of items over buckets.
- But need to check *d* locations.
  - Parallelizable, but still a cost -- e.g. pin count in router design.

Multiple-Choice Hashing

- Can significantly reduce load.
  - Choice allows much more even spread of items over buckets.
- But need to check *d* locations.
  - Parallelizable, but still a cost -- e.g. pin count in router design.
  - Can sometimes use Bloom filters (or variants) to track which location(s) to check for an item.

Example: Cached Hash Tables

- May want hash table to live in cache: each bucket corresponds to a hash line.
  - Example: hash tables in routers.
  - Consider 4-6 items per hash line.
- With one choice, large and highly variable maximum load.
  - Lots of empty, lightly loaded bins.
- Using two choices, most bins mostly full: more efficient memory usage.
- Application: IP routing.

Analysis Methods

- Layered induction
- Witness trees
- Fluid limits

How Many Empty Bins?

- [Hajek 88]

Let $x(t)$ be fraction of non-empty bins;
throw all n balls in 1 second.

$$E[\Delta x(t)] = (1 - x(t)^d)/n\,;\, \Delta t = 1/n$$

$$dx/dt = 1 - x^d$$

Solve at $t = 1$, given $x(0) = 0$. Obtain $x(1) = 0.7616..$ for $d = 2$, matches simulations.

Generalize to Loads

Let $s_k(t)$ be fraction of bins with load at least k.

$$E[\Delta s_k(t)] = (s_{k-1}(t)^d - s_k(t)^d)/n\,;\ \Delta t = 1/n$$

All choices must have load at least k-1,
but not all can have load at least k.

$$ds_k/dt = s_{k-1}^d - s_k^d$$

Successively solve equations at $t = 1$.

Double Exponential Decrease

$$ds_k / dt = s_{k-1}^d - s_k^d$$

$$ds_k / dt \leq s_{k-1}^d(t) \leq s_{k-1}^d(1)$$

$$s_k(1) \leq s_{k-1}^d(1) \leq s_{k-2}^{d^2}(1) \ldots \leq s_1^{d^{k-1}}(1)$$

This is crux of the ABKU proof; implies maximum load of log log n / log d.

Kurtz's Theorem

- Over fixed time intervals and for fixed finite dimensional processes, the deviation of the random process from the solution to the differential equations obeys Chernoff-like bounds.

$$\Pr[\sup_{0\le t\le T,1\le i\le k} |s_i(t)-\hat{s}_i(t)| > \varepsilon] \le c_1 k \exp(-c_2 n \varepsilon^2)$$

Power of Two Generalizations

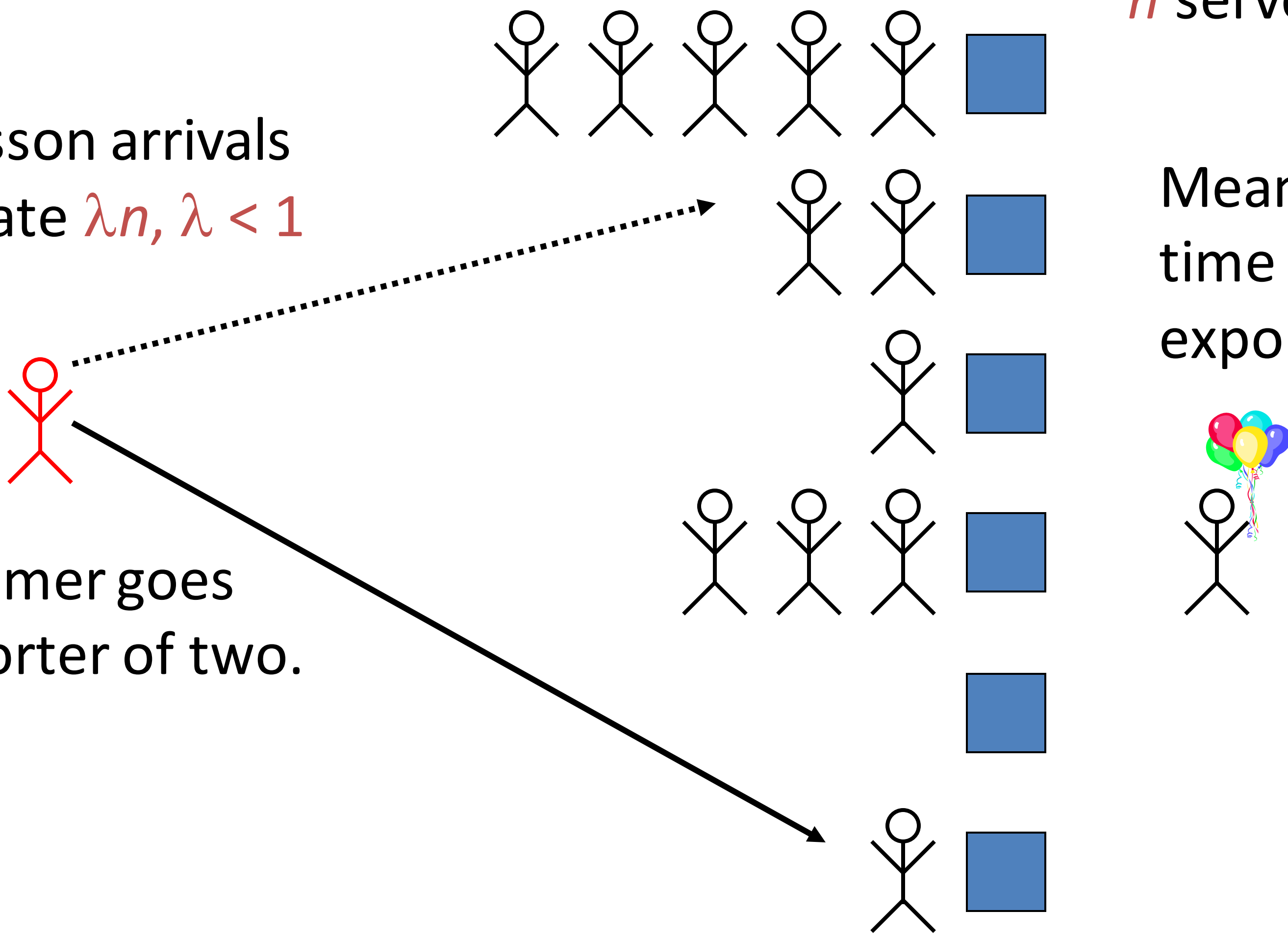
- Many more results…
  - Asymmetric load balancing
  - More balls than bins
  - Weighted balls
  - Unbalanced initial conditions
  - Insertions and deletion
  - Non-uniform chocies

Supermarket Model

n servers

Poisson arrivals at rate λn, $\lambda < 1$

Mean service time is 1, exponential



Customer goes to shorter of two.

Mathematical Description

- Let s_k be fraction of queues with at least k customers.
- System state: $(s_0(t), s_1(t), s_2(t), \ldots)$
- Fraction of queues with k customers is

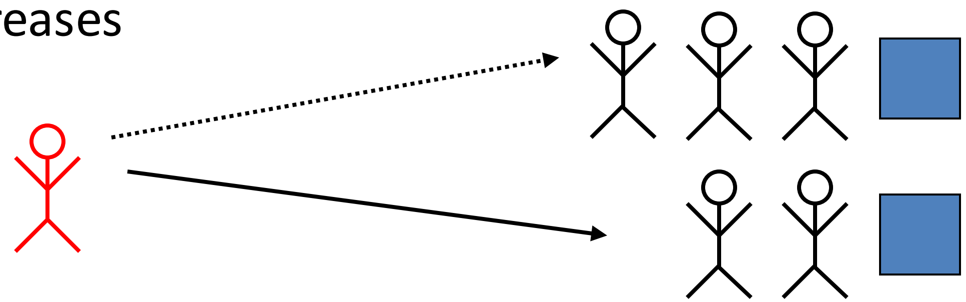
$$s_k - s_{k+1}$$

- Smallest of d random choices has k-1 customers with probability

$$s_{k-1}^d - s_k^d$$

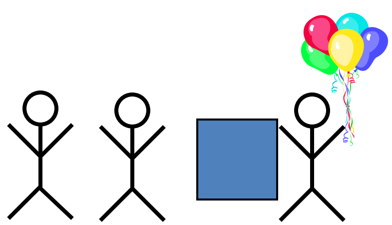
Setting Differential Equations

rate s_k increases



$$(\lambda n\, dt)\left(s_{k-1}^2 - s_k^2\right)/n$$

rate s_k decreases



$$n\left(s_k - s_{k+1}\right)(dt)/n$$

System behavior

- Expected behavior of process as differential equations.

$$ds_k / dt = \lambda\left(s_{k-1}^2 - s_k^2\right) - \left(s_k - s_{k+1}\right)$$

- Converges to fixed point

$$\pi = (\pi_0(t),\ \pi_1(t),\ \pi_2(t), \ldots) \quad \text{where}$$

$$ds_k / dt = 0$$

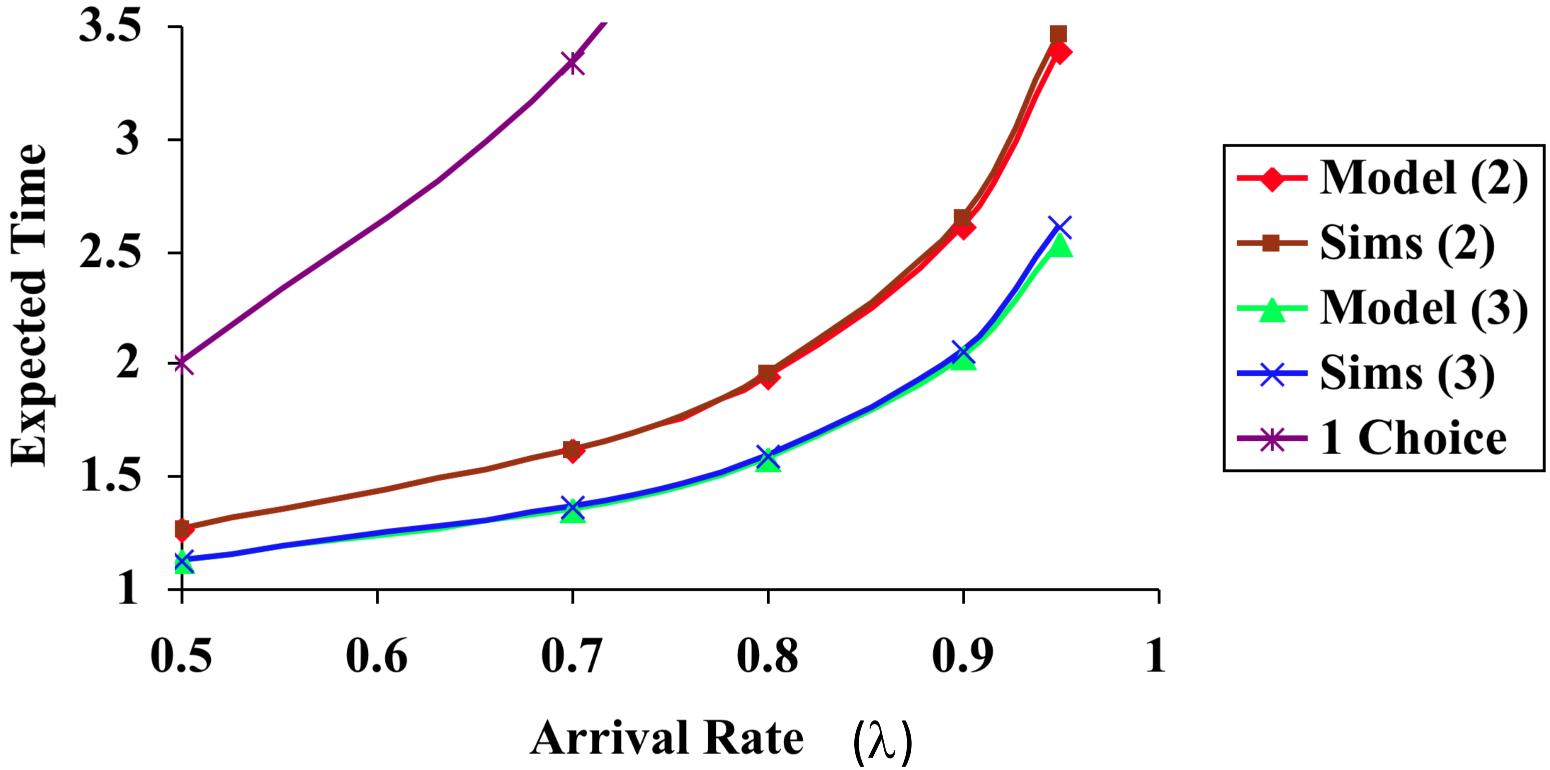
- At fixed point tails decrease doubly exponentially:

$$\pi_k = \lambda^{2^k - 1}$$

Relation to the Real World

- One choice yields exponential tails
  - $\pi_k = \lambda^k$ vs. $\pi_k = \lambda^{2^k - 1}$
- Two choices yields exponential gains in
  - Maximum queue size.
  - Average time in the system.
- Gains observable even for "small" systems.
  - 128 servers, average time in system within 2% of limiting model for $\lambda < 0.9$.

Simulations vs. Predictions



Balanced Allocations with Double Hashing

- Double hashing yields “same performance” for balanced allocations as fully random hashing.
  - Method 1: Double hashing starts with 2 random choices. Need to show adding choices can only help load distribution. So double hashing better than 2 random choices.
  - Method 2: Witness trees. Gets double hashing has same $\log \log n / \log d$ high order term.
  - Method 3: Differential equations. Same differential equations govern behavior of double hashing. Hence same empirical performance.

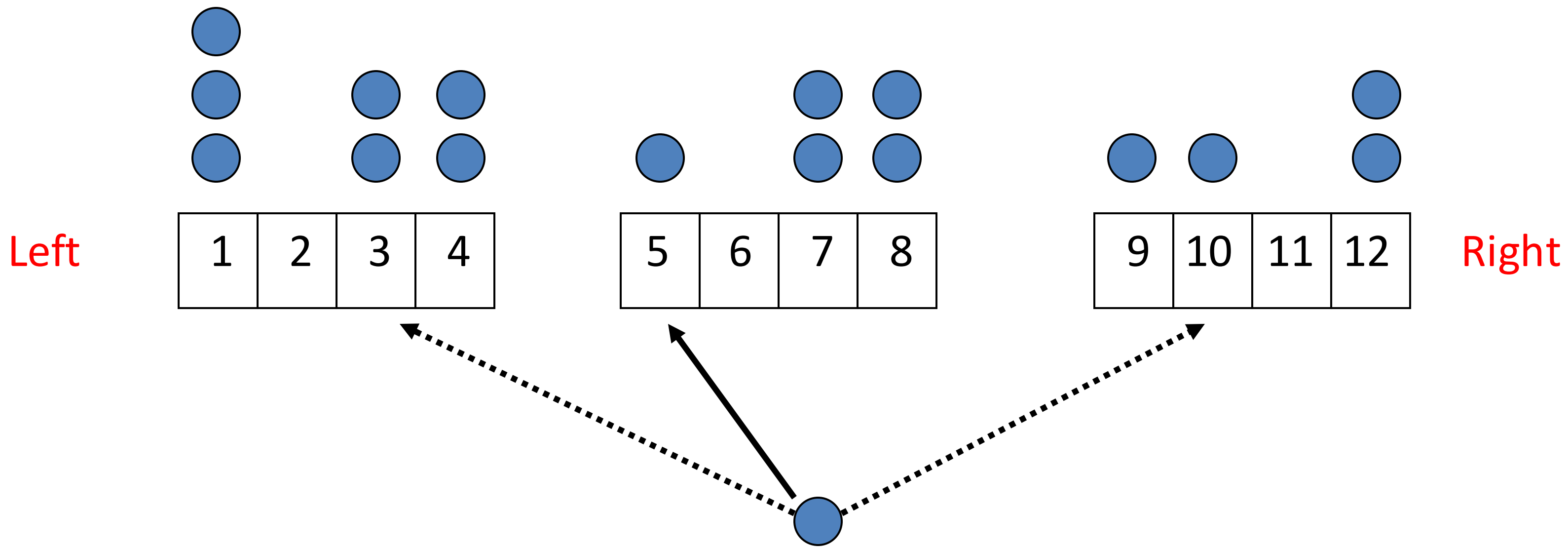
Variation: Double Hashing

- Let h_1 and h_2 be hash functions.
- For $i = 0, 1, 2, \ldots, k - 1$ and some f, let

$$g_i(x) = h_1(x) + i h_2(x) \bmod m$$

  - So 2 hash functions can mimic k hash functions.

Hashing Model: *d*-Left



Item placed in the shortest of *d* bins,
one chosen independently and uniformly
from each of *d* disjoint groups (of equal size).
Ties broken by placing toward the left.
Lookups can be parallelized!

d-Left Scheme

- *d*-Left is better than *d*-Random [V 99]
  - *d*-Random: max. load with *n* balls and *n* bins is log log *n* / log *d* with *d* choices.
  - *d*-Left: log log *n* / *d* with *d* choices.
- Extensions [MV 99]:
  - Gives simple differential equations analysis for *d*-Left schemes.
  - Provides analysis, numerical results.
  - Suggests further improvements.

Comparison

- Considering n balls, n bins.
- For d-Random, fraction of bins with load at least k falls like 2^{-d^k}
  - Max. load $\log \log n / \log d$
- For d-Left, fraction of bins with load at least k falls like $2^{-\phi(d)^{dk}}$
  - $\phi(d)$ is rate of growth of generalized Fib. numbers
  - Max. load $\log \log n / d \log \phi(d)$

$$\phi_2 = 1.618\ldots, \phi_3 = 1.839\ldots;\ 2^{(d-1)/d} < \phi_d < 2.$$

Example of *d*-left hashing

- Consider 4-left performance with average load of 6, using differential equations.

Insertions only

Load $\geq$ 1	1.0000
Load $\geq$ 2	1.0000
Load $\geq$ 3	1.0000
Load $\geq$ 4	0.9999
Load $\geq$ 5	0.9971
Load $\geq$ 6	0.8747
Load $\geq$ 7	0.1283
Load $\geq$ 8	1.273e-10
Load $\geq$ 9	2.460e-138

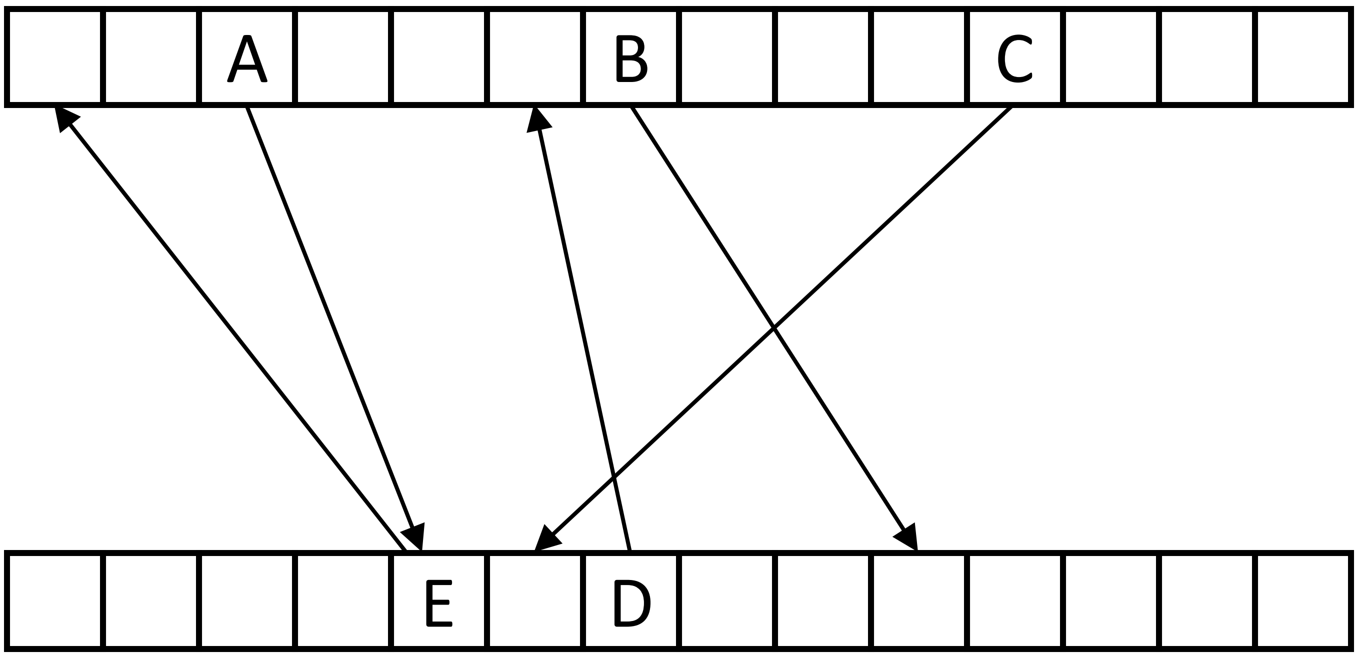
Alternating insertions/deletions
Steady state

Load $\geq$ 1	1.0000
Load $\geq$ 2	0.9999
Load $\geq$ 3	0.9990
Load $\geq$ 4	0.9920
Load $\geq$ 5	0.9505
Load $\geq$ 6	0.7669
Load $\geq$ 7	0.2894
Load $\geq$ 8	0.0023
Load $\geq$ 9	1.681e-27

Cuckoo Hashing

- Basic scheme: each element gets two possible locations (uniformly at random).
- To insert *x*, check both locations for *x*. If one is empty, insert.
- If both are full, *x* kicks out an old element *y*. Then *y* moves to its other location.
- If that location is full, *y* kicks out *z*, and so on, until an empty slot is found.

Cuckoo Hashing Examples



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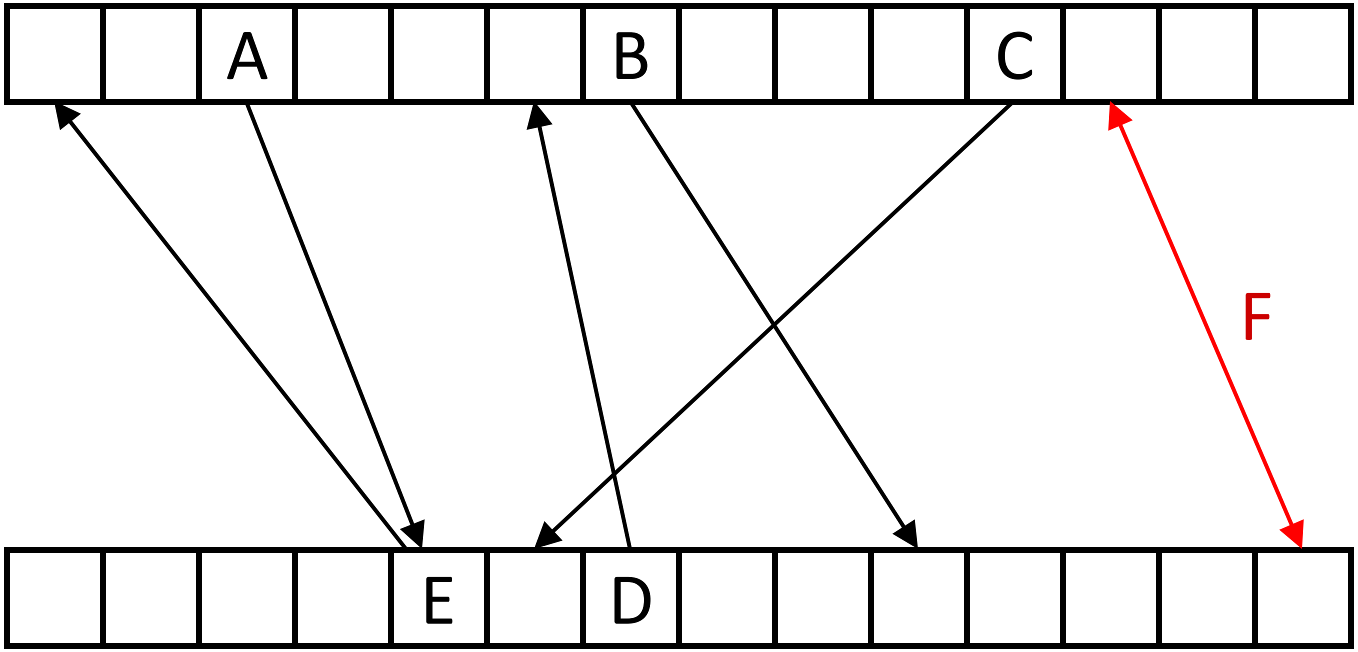

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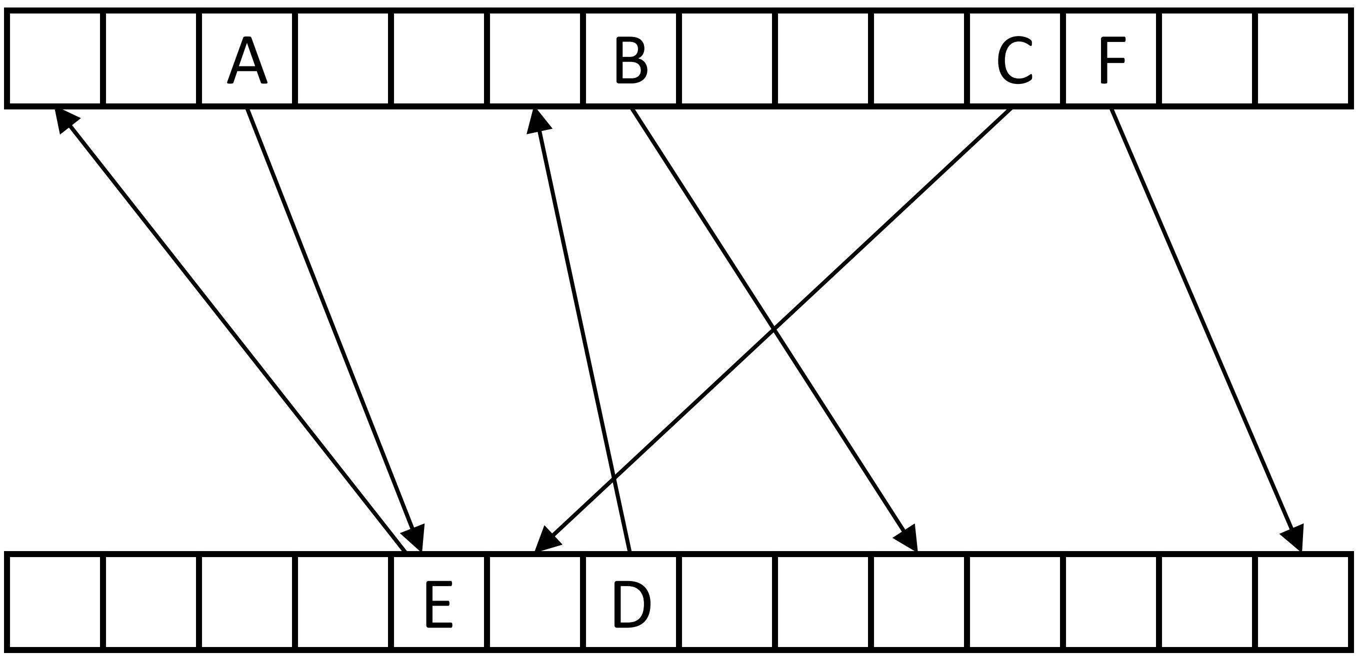

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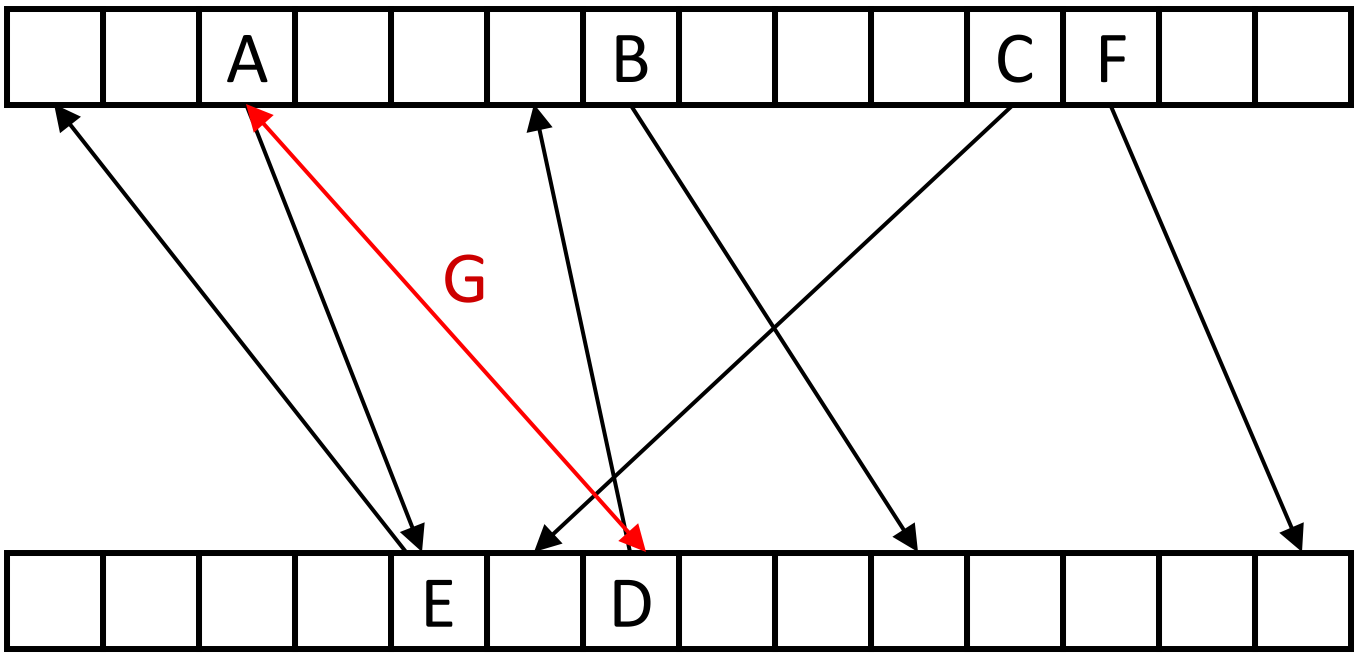

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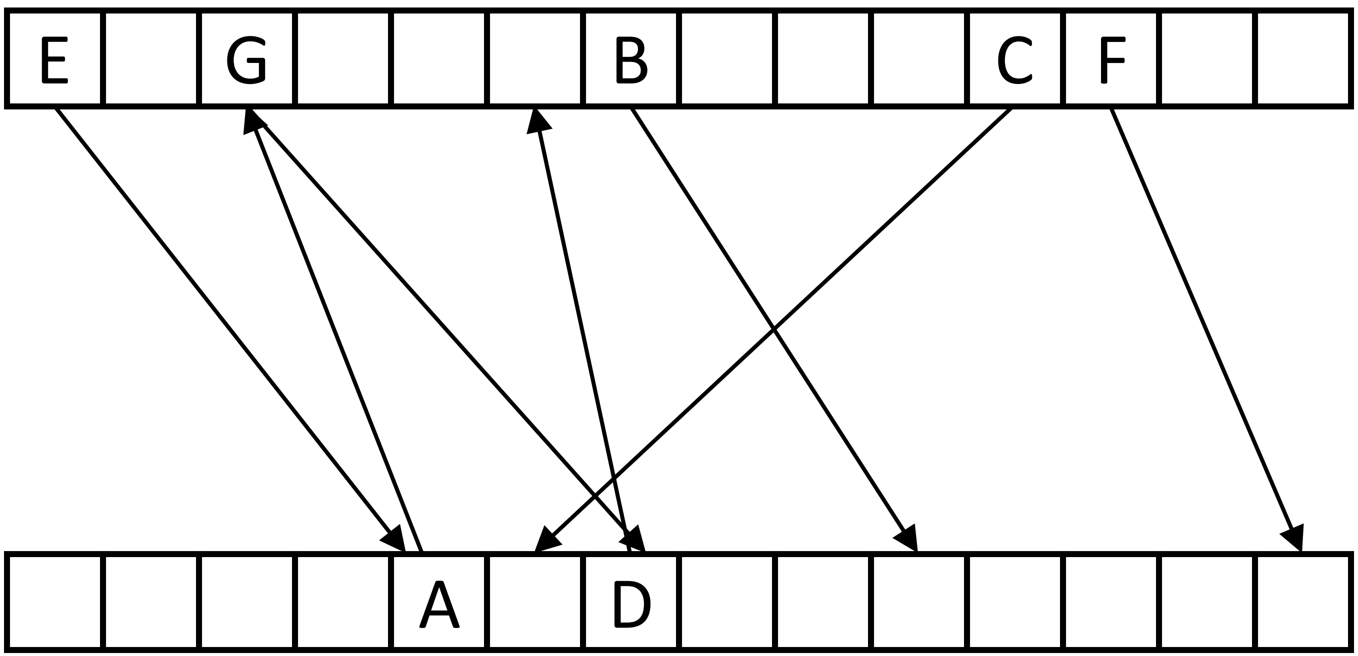

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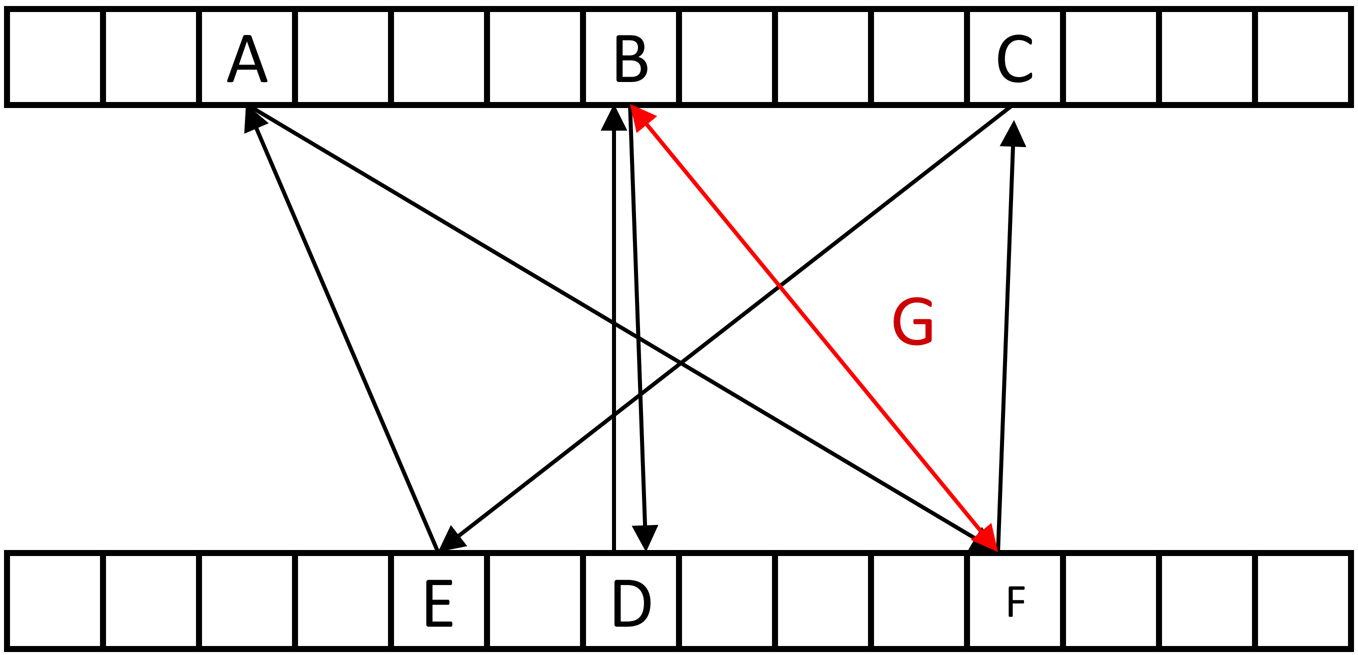


Multiple Choice vs. Cuckoo Hashing?

- Multiple-choice hashing yields tables with
  - High memory utilization.
  - Constant time look-ups.
  - Simplicity – easily coded, parallelized.
- Cuckoo hashing expands on this, combining multiple choices with ability to move elements.
  - Is moving elements worth the cost?
- Practical potential, and theoretically interesting!

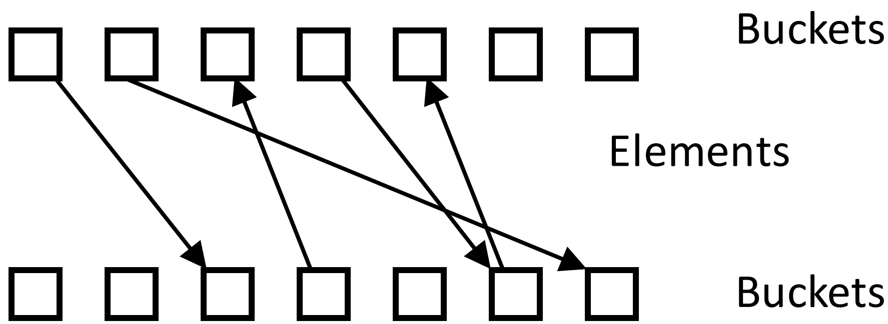
Good Properties of Cuckoo Hashing

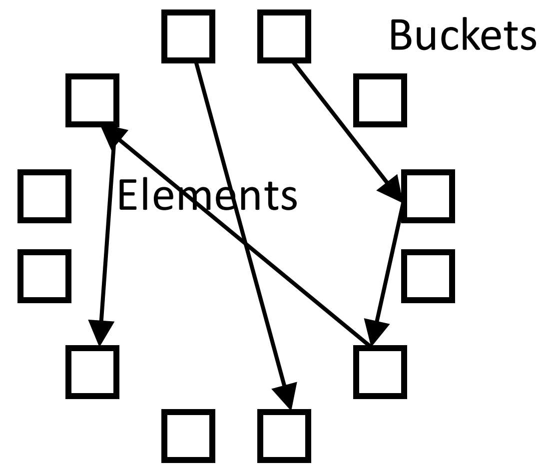
- *Worst case constant lookup time.*
- High memory utilizations possible.
- Simple to build, design.

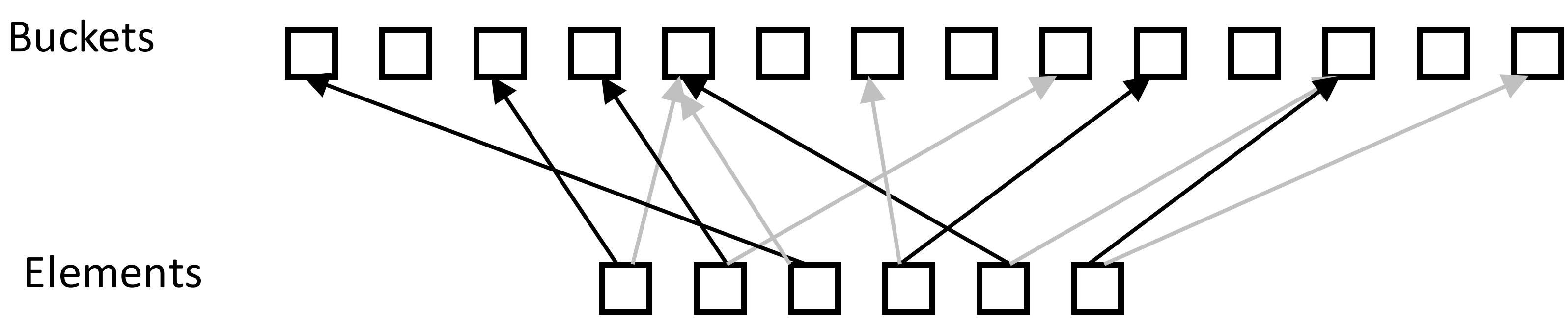
Cuckoo Hashing Failures

- Bad case 1: inserted element runs into cycles.
- Bad case 2: inserted element has very long path before insertion completes.
  - Could be on a long cycle.
- Bad cases occur with very small probability when load is sufficiently low.
- Theoretical solution: re-hash everything if a failure occurs.

Various Representations







Basic Performance

- For 2 choices, load less than 50%, n elements gives failure rate of $\Theta(1/n)$; maximum insert time $O(\log n)$.
- Related to random graph representation.
  - Each element is an edge, buckets are vertices.
  - Edge corresponds to two random choices of an element.
  - Small load implies small acyclic or unicyclic components, of size at most $O(\log n)$.

Natural Extensions

- More than 2 choices per element.
  - Very different : hypergraphs instead of graphs.
  - D. Fotakis, R. Pagh, P. Sanders, and P. Spirakis.
  - Space efficient hash tables with worst case constant access time.
- More than 1 element per bucket.
  - M. Dietzfelbinger and C. Weidling.
  - Balanced allocation and dictionaries with tightly packed constant size bins.

Thresholds

Bucket Size 1

Choices	1	2	3	4	5	6
Load	0.5	0.918	0.976	0.992	0.997	0.999

2 Choices

Bucket size	1	2	3	4	5	8	10
Load	0.5	0.897	0.959	0.980	0.989	0.997	0.999

Proofs for Thresholds

- Most insight comes from viewing the process a branching tree from a node.
  - Cuckoo process as a hypergraph.
    - Each “edge” corresponds to a key, vertices are buckets.
  - Random neighborhood.
    - Distribution known/understood.
  - Locally a tree (with high probability).
- Then one fixes up the tree argument.
  - Challenging details.

Related to *l*-core

- The *l*-core is the maximal subgraph where each vertex has degree at least *l*.
- Can be found by peeling.
  - Take any vertex of degree at most *l*-1, remove it and corresponding edges.
- For cuckoo hashing with buckets of size *l*-1
  - If a bucket has at most *l*-1 keys that could be assigned to it, assign the keys to that bucket.
  - Remove bucket and keys from consideration.
- Cuckoo hashing succeeds when *l*-core is empty.
- Harder: cuckoo hashing succeeds when *l*-core has x remaining vertices, but less than (*l*-1)*x* edges.
  - The remaining core can be "matched".

Recursion Argument

- Let b be a node.
- Let q_h = probability node b is peeled after h rounds.
- Let p_j = probability node at distance h-j from b is peeled after j rounds. (Note $p_0 = 0$.) $p = \lim p_j$.

$$p_1 = \Pr\left[\mathrm{Bin}\left(\binom{m-1}{k-1}, \ k! \cdot \frac{c}{m^{k-1}}\right) \leq \ell - 2\right]$$

$$= \Pr[\mathrm{Po}(kc) \leq \ell - 2] \pm o(1),$$

$$p_{j+1} = \Pr\left[\mathrm{Bin}\left(\binom{m-1}{k-1}, \ k! \cdot \frac{c}{m^{k-1}} \cdot (1-p_j)^{k-1}\right) \leq \ell - 2\right]$$

$$= \Pr[\mathrm{Po}(kc(1-p_j)^{k-1}) \leq \ell - 2] \pm o(1), \text{ for } j = 1, \ldots, h-2.$$

$$p = \Pr[\mathrm{Po}(kc(1-p)^{k-1}) \leq \ell - 2].$$

$$q_h = \Pr[\mathrm{Po}(kc(1-p_{h-1})^{k-1}) \leq \ell - 1] \pm o(1).$$

Stashes

- A failure in cuckoo hashing occurs whenever one element can't be placed.
- Is that really necessary?
- What if we could keep one element unplaced? Or eight? Or $O(\log n)$? Or εn?
- Goal : Reduce the failure probability.

Motivation : CAMs

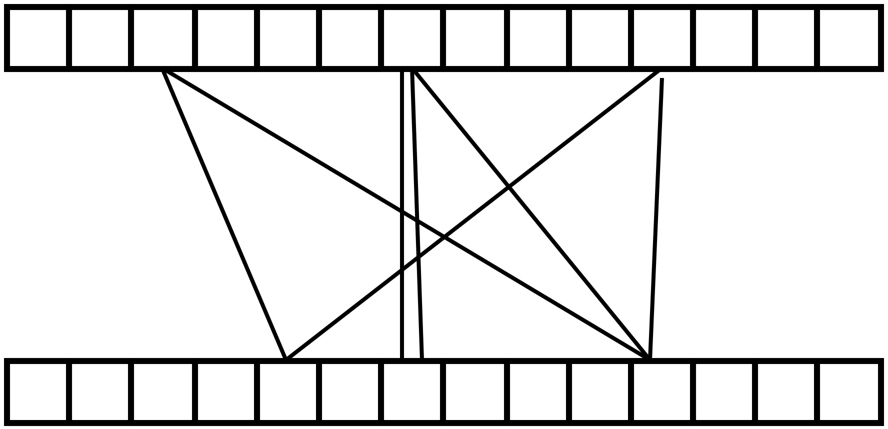
- CAM = content addressable memory
  - Fully associative lookup.
  - Usually expensive, so must be kept small.
  - Hardware solution, or a dedicated cache line in software.
- Not usually considered in theoretical work, but very useful in practice.
- Can we bridge this gap?
  - What can CAMs do for us?

A CAM-Stash

- Use a CAM to stash away elements that would cause failure.
- Intuition: if failures were independent, probability that *s* elements cause failures goes to $\Theta(1/n^s)$.
  - Failures not independent, but nearly so.
  - A stash holding a *constant* number of elements greatly reduces failure probability.
  - Implemented as hardware CAM or cache line.
- Lookup requires also looking at stash.
  - But generally empty.

Analysis Method

- Treat cells as vertices, elements as edges in bipartite graph.
- Count components that have excess edges to be placed in stash.
- Random graph analysis to bound excess edges.



6 vertices, 7 edges:
1 edge must go into stash.

A Simple Experiment

- 10,000 elements, table of size 24,000, 2 choices per element, 10^7 trials.

Stash Size	Needed Trials
0	9989861
1	10040
2	97
3	2
4	0

Random Walk Cuckoo Hashing

- When it is time to kick something out, choose one randomly.
- Small state, effective.
- Intuition : if fraction *p* of the buckets are empty, random walk "should" have fraction *p* of finding empty bucket at each step.
  - Clearly wrong, but nice intuition.
  - Suggests logarithmic time to find an empty slot.

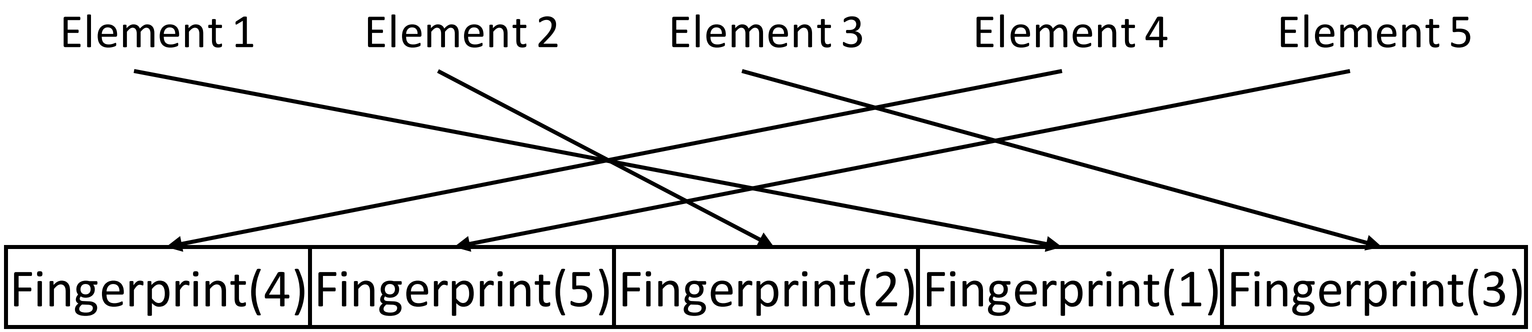
Some Progress

- Polylogarithmic bounds on insertion time.
- Open question: better bounds on performance of random walk cuckoo hashing?

Bloom Filters via Hash Tables

- Recall one could obtain an optimal static Bloom filter using perfect hashing
- Can we use multiple-choice hashing/cuckoo hashing to get a "near-perfect" hash table for a Bloom filter type object?

Perfect Hashing Approach



Near-Perfect Hash Functions via *d*-left Hashing

- Maximum load equals 1
  - Requires significant space to avoid all collisions, or some small fraction of spillovers.
- Maximum load greater than 1
  - Multiple buckets must be checked, and multiple cells in a bucket must be checked.
  - Not perfect in space usage.
    - In practice, 75% space usage is very easy.
    - In theory, can do even better.
- False positives increase with bucket size.

Modern Update : Cuckoo Filters

- Use a cuckoo hash table to obtain a near-perfect hash table
- Store a fingerprint in the hash table
- Can support insertion and deletion of keys
- Very space efficient
  - From cuckoo hash table construction, with buckets that hold multiple keys.

Cuckoo Filters : Issues

- Consider cuckoo hash table, 2 choice per key, 4 fingerprints of keys per bucket.
- Buckets fill, an item has to be moved.
- How do we know where to move it?
  - We don’t have the key any more.
  - Just the fingerprint.

Partial-key Cuckoo Hashing

- Can't use the key when moving a key.
- So we have to use the fingerprint instead.

$$h_1(x) = \text{hash}(x)$$

$$h_2(x) = h_1(x) \oplus \text{hash}(x\text{'s fingerprint})$$

- Note fingerprint is the same in both locations.
- Can compute h_1 from h_2 and vice versa with the stored fingerprint.

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- Note fingerprint is the same in both locations.
- Can compute h_1 from h_2 and vice versa with the stored fingerprint.
- But now the two choices are limited, not completely random. Will this still work?

Partial-key Cuckoo Hashing

- Does it work?
- In practice, yes.
  - Essentially no discernible change in the threshold under reasonable settings.
- In theory, no.
  - You “need” logarithmic sized fingerprints…
  - But with a small constant factor.
  - So in practice it ends up OK.
- Open problem – better provable bounds on performance of partial-key cuckoo hashing.

Bit-Saving Tricks

- Every bit counts for space purposes.
- Bucket size of 4.
- Sort the fingerprints.
- Take the first 4 most significant bits.
- After sorting there are 3876 possible outcomes.
  - Less that 2^{12}.
  - So use only 12 bits to represent these 16.
  - Saves 1 bit per item.

Cuckoo Filter Performance

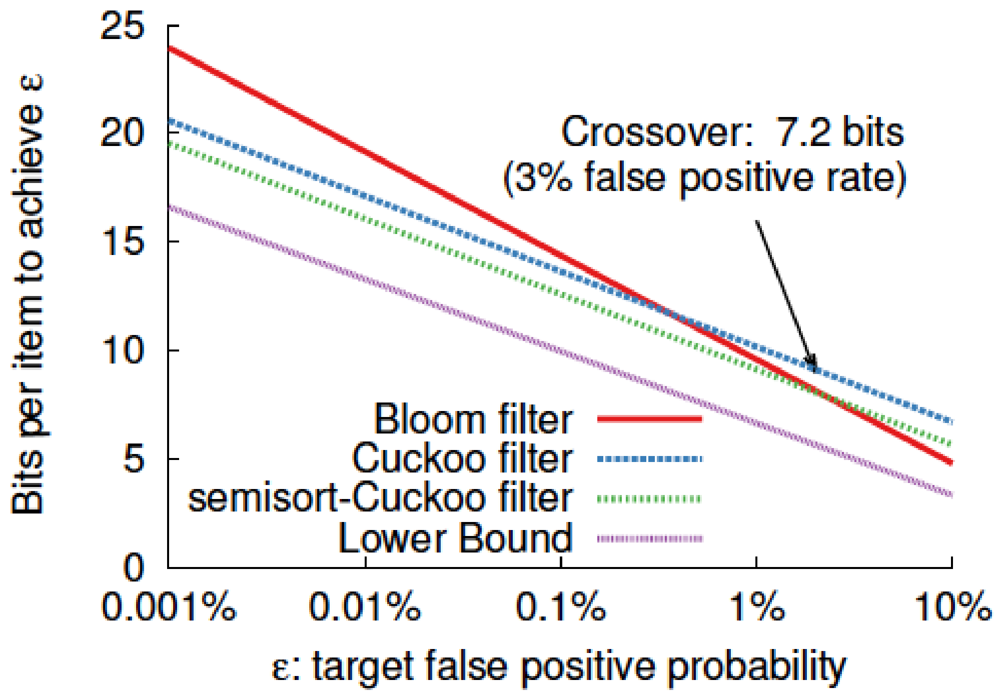

Figure 4: False positive rate vs. space cost per element. For low false positive rates ($< 3\%$), cuckoo filters require fewer bits per element than the space-optimized Bloom filters. The load factors to calculate space cost of cuckoo filters are obtained empirically.

Conclusion

- Power of two (or more) choices
  - A little choice usually goes a long way
- More and more uses for multiple-choice and cuckoo hashing
- Still lots of theoretical questions on cuckoo hashing to solve.

Exercise (Hard)

- Derive a family of differential equations that describe the d-left scheme of Vöcking.
- Assuming the differential equations are accurate, show that they yield a "Fibonacci exponential" decrease in the fraction of bins with load j as j increases for the case of n balls being placed into n bins.

Exercise

- Partial-key cuckoo hashing, with 2 choices per key, bucket size b, fingerprint size f bits, n items, table size $m = cn$ buckets for constant c.
- A failure happens if $2b+1$ keys map to the same pair of buckets.
- What is the expected number of sets of $2b+1$ keys that map to the same pair of buckets?
- What value of f is needed so this is o(1)?

Open Problems

- Better analysis of random walk cuckoo hashing.
- Better analysis of partial-key cuckoo hashing.
- Analyzing double hashing+cuckoo hashing. Can one prove the same thresholds apply?
- Analyzing double hashing and peeling on random hypergraphs. Can one prove the same thresholds apply?