Proving that a hash has no funnels

Standard Model

This paper will assume that hash functions are constructed using a **standard model**. Hash functions have some internal state, and a permutation **mix** is used to mix this internal state. Another function, **combine**, is used to combine input blocks with the current internal state.

Standard model for hash functions

```
initialize the internal state;
for (each block of the input)
{
  combine (the internal state, the current input block);
  mix( the internal state);
}
value = postprocess( the internal state );
return (value);
```

Although most hashes fit this model, some (for example MD4 [MD4] and Rogaway's bucket hash [Rogaway]) do not.

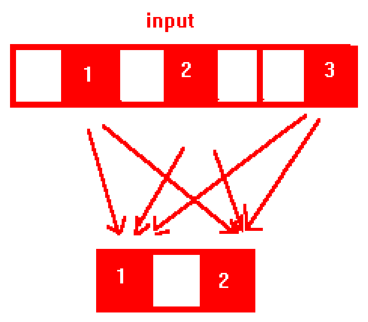
Funneling

Consider the hash function

```
char hashfun( char *key, int len)
{
  char hash;
  int  i;
  for (hash=0, i=0; i<len; ++i) hash=hash^key[i];
  return (hash %= 101);           /* 101 is prime */
}
```

(This function requires 5*m*+3 instructions to hash *m* bytes. Compare that to 6*m*+35 and 5*m*+41 for the two new hash functions.) The keys "ab" and "ba" hash to the same value. The only difference between the two keys is the first bit of each byte. What is the problem here? (Ignore commutativity for the moment.)

A hash function is bad if it causes collisions when keys differ in only a few bits. This always happens when the effects of a number of input bits are funneled into a smaller number of bits in the internal state. Funneling formalizes this concept.



A funnel of 3 input bits into 2 output bits

Let K (keys) be the set of all *k*-bit values and *V* (hash values) be the set of all *v*-bit values. Let a hash function *h : K -> V* be given. Bits *i* in 1..*k* **affects** bit *j* in 1..*v* if two keys differing only at bit *i* will differ at bit *j* about half the time, that is between *epsilon* and 1-*epsilon* for some *epsilon* near zero.

Define *h* to be a **funneling hash** if there is some subset of *t* of the input bits which can only affect bits *u* in the internal state, and $|t| > |u|$ and $v > |u|$. *h* has a **funnel** of those *t* input bits into those *u* bits of the internal state. The set of keys differing only in the input bits of the funnel can produce no more than half that number of hash values. (More precisely, if *epsilon* is zero, those $2^{|t|}$ keys can produce no more than $2^{|u|}$ out of 2^{v} hash values.) Keys differing in only a few bits is a common pattern in human and computer keys, so a funneling hash is seriously flawed.

Funneling: there is an efficient test for funnels.

Lemma One-to-one:

> A mapping of a key block to the internal state cannot cause funnels within that block if the mapping is one-to-one (sends every distinct input to a distinct output).

Proof: A set of 2^t keys differing in only t bits in a key block will map to 2^t distinct values in the internal state after the mapping, and 2^t states cannot be represented in less than t bits, so those t original bits will affect t or more bits in the internal state.

Q.E.D. one-to-one

Lemma Mix:

> A mixing step cannot cause funnels in previous input blocks if the mixing step is a permutation, unless the mixing step compresses already-mixed data.

Proof: If there are no funnels yet, the t input bits from previous blocks have affected at least t bits of the internal state. The set of 2^t internal states differing in only those t bits will map to 2^t distinct internal states after a permutation, and 2^t states cannot be represented in less than t bits, so those t original bits will affect t or more bits of the internal state after the permutation.

It may be that those t input bits could cause only $(1/(1\text{-}epsilon))^t$ variations of those t internal state bits, in which case some compression is possible.

Q.E.D. mix

Corollary Combine:

> A combining step cannot cause funnels in previous input blocks if it is a permutation of the internal state, unless the combining step compresses already-mixed data. (This does not rule out funnels with input bits spread between the previous input blocks and current or future input blocks.)

The proof is the same as for the mixing step.

Q.E.D. combine

When the mixing step is nonlinear, it is very rare that any single function can cancel out the effects of an already-mixed block, and even rarer that the combining step is that function. Let f be the mixing step and + be the combining step. Let k_1 and k_2 be text blocks, and z the internal state of the hash function. Every (k_2, z) fully defines a function that maps $f(k_2 + f(k_1 + z))$ to $f(k_2 + f(0 + z))$. When f is linear this function is always $ff(-k_1) + x$. But when f is nonlinear, the mapping can be different for every k_2 and z. This is how nonlinearity can make it is impossible to cancel the effects of already-mixed data.

Avalanche

The property of **avalanche** is that every one of the k input bits changes every one of the v output bits about half the time [Schneier]. Whether a function achieves avalanche can be tested by applying the function about $k\log(2v)$ times.

Lemma Postprocess:

> If the postprocessing step simply selects a subset of internal state bits to be the result, then the final mixing and postprocessing step introduce no funnels if the final mixing step and postprocessing step together achieve avalanche.

Proof: Assume there were no funnels before the final mixing step. This implies that every input bit has affected at least one bit of the internal state.

If the mixing and postprocessing step together achieve avalanche, then every bit of the internal state before the mix affected every one of the v bits of the result. No input bit affected less than v bits, so no input bit is in a funnel smaller than v bits. By the definition of a funnel, that means there are no funnels at all.

Users do not always use their entire result. If the user only selects q of those v bits, all q of those bits are still affected by every bit of the internal state from before the final mixing step, thus by every input bit.

Q.E.D. postprocess

Corollary Everybit

If the combining step is to XOR an input block the size of the internal state with the internal state, then lemma postprocess is if and only if.

Users do not always use their entire result. Suppose that for every bit of the internal state there is one input bit from the final block that affected only that bit. If there is a result bit that is not affected by some bit of the internal state, then the user might select only that result bit, and that would be a funnel of 1 bit into 0.

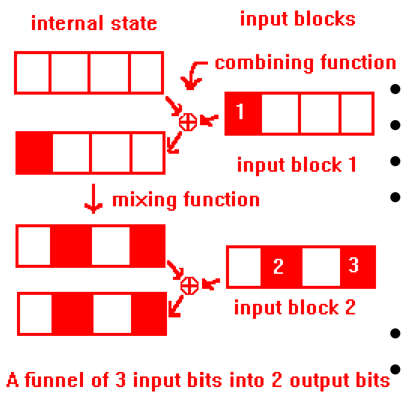
Q.E.D. everybit

Hashes with No Funnels

Theorem Nofunnel:

Unless the mixing step compresses already-mixed data, a hash matching our standard model has no funnels if these conditions all hold:

1. The combining step is a permutation of the internal state and a one-to-one mapping of an input block the size of the internal state,
2. The mixing function is a permutation of the internal state,
3. The postprocessing step just selects v bits of the internal state to be the result,
4. The mixing step causes avalanche in the result, and
5. The mixing step, when run either forwards or backwards, causes every bit of the internal state to affect at least v bits of the internal state.
6. For every intermediate point in the mixing step, consider running the mixing step forward to that point from the previous combine, and backward to that point from the next combine. For every set Y of bits in the internal state in the middle, there some set X of bits in the previous input block and Z in the next input block that affect those Y bits. For every Y with less than v bits, |X|+|Z| must be less than or equal to |Y|.



There are a number of cases to consider:

- All the input bits come from one block.
- All the input bits come from two adjacent blocks.
- All the input bits come from two blocks that are not adjacent.
- The input bits are in more than 3 blocks.

Also, and independent of that,

- The funnel may occur during the combining step.
- The funnel may occur during the mixing step.
- The funnel may occur during the postprocessing step.

Suppose all the input bits are in one block. Point 1 and lemma Combine shows the funnel does not occur during the combining step, point 2 and lemma Mix during the mixing step, and points 3 and 4 and lemma Postprocess during the postprocessing step. So input bits in one block do not cause funnels.

The combining step will cause every bit in input block k_1 to affect at least one bit of the internal state. If the mixing step caused some internal bit to affect only u bits of the internal state, then the next input block k_2 will also cause every one of those u bits to be affected by perhaps just one new input bit. This is a total of u+1 input bits affecting u bits of the internal state, which is a funnel if and only if $u < v$.

Since the mixing step is reversible, it makes sense to ask the same questions in reverse. Affecting v bits forward does not imply that v bits will be affected in reverse. In the mix-combine scheme, if the mixing step does not achieve avalanche in reverse, there will be a bit in k_2 that affects only u bits of the internal state (at the time k_1 is combined). This is again a funnel of u+1 into u if and only if $u < v$.

Any two bits affect at least as many bits forward and backwards as any one of those bits (by our definitions of affect and funnel), so the problems with one bit forward and one bit backward are all that need to be considered. If every bit affects at least v bits forward and backward, then there are no funnels with bits spread between adjacent blocks.

There is another case. The problem is with a bit affecting at least v bits by the time the next block is combined, but at some later point in time affecting fewer bits. This requires the mixing function to compress already-mixed data.

Q.E.D. nofunnel