Finite Difference Method

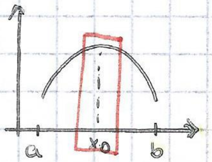
12/05

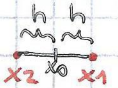
We want to solve this problem

$$\begin{cases} u'' = 1 & x \in (a,b) \\ u(a) = \alpha & \alpha, \beta \in \mathbb{R} \\ u(b) = \beta \end{cases}$$

If we want to find $u'(x_0)$ with $x_0 \in (a,b)$, we make $[a,b]$ discrete in n points

$$a = x_0 < x_1 < \ldots < x_i < \ldots < x_n = b$$





$$h = \frac{b-a}{n-1}$$

$$x_1 = x_0 + h$$
$$x_2 = x_0 - h$$

$$u'(x_0) \simeq f(u(x_0), u(x_2), u(x_1))$$

Consider Taylor's expansion:

$$\begin{cases} u(x_0) = u(x_0) \\ u(x_0+h) = u(x_0) + u'(x_0)h + \frac{u''(x_0)}{2}h^2 + \frac{u'''(x_0)}{6}h^3 + O(h^4) \\ u(x_0-h) = u(x_0) - u'(x_0)h + \frac{u''(x_0)}{2}h^2 - \frac{u'''(x_0)}{6}h^3 + O(h^4) \end{cases}$$

$$D^+ := \frac{u(x_0+h) - u(x_0)}{h} = u'(x_0) + \frac{u''(x_0)}{2}h + \frac{u'''(x_0)}{6}h^2 + O(h^3) = u'(x_0) + O(h)$$

$$D^- := \frac{u(x_0) - u(x_0-h)}{h} = u'(x_0) - \frac{u''(x_0)}{2}h + \frac{u'''(x_0)}{6}h^2 + O(h^3) = u'(x_0) + O(h)$$

$$D^c := \frac{u(x_0+h) - u(x_0-h)}{2h} = \frac{D^+ + D^-}{2} = u'(x_0) + O(h)$$

$$D_c^2(u(x_0)) = D^+(D^-u(x_0)) = D^+\left(\frac{u(x_0) - u(x_0-h)}{h}\right) =$$

$$= \frac{1}{h}\left[D^+(u(x_0)) - D^+(u(x_0-h))\right] = \frac{1}{h}\left[\frac{u(x_0+h)-u(x_0)}{h} - \frac{u(x_0)-u(x_0-h)}{h}\right] =$$

use Taylor expansion of $u(x_0 \pm h)$

$$= \frac{1}{h^2}\left[u(x_0+h) - 2u(x_0) + u(x_0-h)\right] \overset{!}{=} u''(x_0) + O(h^2)$$

$$\Rightarrow \quad u^k(x_0) = \sum_{i=0}^{2} u(x_i)\alpha_i$$

Unknown coefficients method

We want to find the coefficients α and β of $u'(x_0) \simeq \alpha u(x_0) + \beta u(x_0+h)$

$$\begin{cases} \alpha u(x_0) = \alpha u(x_0) \\ \beta u(x_0+h) = \beta u(x_0) + \beta u'(x_0)h + \beta\frac{u''(x_0)}{2}h^2 + \beta\frac{u'''(x_0)}{6}h^3 + O(h^4) \end{cases}$$

$$\alpha u(x_0) + \beta u(x_0+h) = (\alpha+\beta)u(x_0) + \beta u'(x_0)h + \beta\frac{u''(x_0)}{2}h^2 + \beta\frac{u'''(x_0)}{6}h^3 + O(h^4)$$

We want $\begin{cases} \alpha+\beta = 0 \\ \beta h = 1 \end{cases} \Rightarrow \begin{cases} \alpha = -1/h \\ \beta = 1/h \end{cases}$

$$\Rightarrow u'(x_0) \simeq \alpha u(x_0) + \beta u(x_0+h) = \frac{u(x_0+h) - u(x_0)}{h}$$

Analogously for $u''(x_0) \simeq \alpha u(x_0) + \beta u(x_0+h) + \gamma u(x_0-h)$

$$\begin{cases} \alpha u(x_0) = \alpha u(x_0) \\ \beta u(x_0+h) = \beta u(x_0) + \beta u'(x_0)h + \beta\frac{u''(x_0)}{2}h^2 + O(h^3) \\ \gamma u(x_0-h) = \gamma u(x_0) - \gamma u'(x_0)h + \gamma\frac{u''(x_0)}{2}h^2 + O(h^3) \end{cases}$$

We want $\begin{cases} \alpha+\beta+\gamma = 0 \\ \beta h - \gamma h = 0 \\ \frac{\beta h^2}{2} + \frac{\gamma h^2}{2} = 1 \end{cases} \Rightarrow \begin{cases} \beta = \gamma \\ \beta h^2 = 1 \\ \alpha+\beta+\gamma = 0 \end{cases} \Rightarrow \begin{cases} \beta = \gamma \\ \beta = 1/h^2 \\ \alpha = -2/h^2 \end{cases}$

$$\Rightarrow u''(x_0) \simeq \alpha u(x_0) + \beta u(x_0+h) + \gamma u(x_0-h) = \frac{u(x_0-h) - 2u(x_0) + u(x_0+h)}{h^2}$$

Another method for obtaining them. Let's interpolate the 3 points

$(x_0-h,\ u(x_0-h)),\quad (x_0,\ u(x_0)),\quad (x_0+h,\ u(x_0+h))$

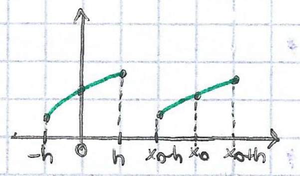
$y = ax^2+bx+c$ such that $\begin{cases} y(x_0-h) = u(x_0-h) \\ y(x_0) = u(x_0) \\ y(x_0+h) = u(x_0+h) \end{cases}$

Considering the translation $x^* = x - x_0$:

$x_0 \to 0$

$x_0+h \to h$

$x_0-h \to -h$



$$\begin{cases} y(0) = c = u(0) \\ * \quad y(h) = ah^2+bh+u(0) = u(h) \\ ** \quad y(-h) = ah^2-bh+u(0) = u(-h) \end{cases}$$

$*: \ a = \dfrac{u(h)-u(0)-bh}{h^2} \quad \Rightarrow ** : \ \dfrac{u(h)-u(0)-bh}{h^2} h^2 - bh + u(0) = u(-h)$

$$u(h) - 2bh = u(-h)$$

$$\Rightarrow b = \frac{u(-h)-u(h)}{-2h} = \frac{u(h)-u(-h)}{2h}$$

$$\Rightarrow a = \frac{u(h)-u(0)-\frac{u(h)}{2}+\frac{u(-h)}{2}}{h^2} = \frac{\frac{u(h)}{2}-u(0)+\frac{u(-h)}{2}}{h^2} = \frac{u(h)-2u(0)+u(-h)}{2h^2}$$

$$\Rightarrow y = \frac{u(h)-2u(0)+u(-h)}{2h^2}x^2 + \frac{u(h)-u(-h)}{2h}x + u(0)$$

$$\Rightarrow y' = \frac{u(h)-2u(0)+u(-h)}{h^2}x + \underbrace{\frac{u(h)-u(-h)}{2h}}_{=y'(0)}$$

$$y'' = \frac{u(h)-2u(0)+u(-h)}{h^2} \iff D^2_c(u(x_0))$$

Hermite Interpolation

Considering $u(x_0), u(x_0-h), u(x_0+h)$ and their derivatives $u'(x_0-h), u'(x_0+h)$ we have 5 info so we can try to interpolate with a polynomial of order 4.

We have to consider both y and y':

$y = ax^4+bx^3+cx^2+dx+e$

$y' = 4ax^3+3bx^2+2cx+d$

$$\begin{cases} y(0) = e = u(0) \\ y(h) = ah^4+bh^3+ch^2+dh+u(0) = u(h) \\ y(-h) = ah^4-bh^3+ch^2-dh+u(0) = u(-h) \\ y'(h) = 4ah^3+3bh^2+2ch+d = \hat{y}'(h) \\ y'(-h) = -4ah^3+3bh^2-2ch+d = \hat{y}'(-h) \end{cases}$$

$\Rightarrow \hat{y}(x_0) = \dfrac{y(x_0+h)-y(x_0)}{h} \simeq y'(x_0)$

The process of calculations is not so important. 12/07

$$y'(0) = d = -\frac{1}{4}\left[\hat{y}'(h)+\hat{y}'(-h)\right] + \frac{3}{4}\left[y(h)-y(-h)\right] \simeq \hat{y}'(0)$$

Translating it's $\tilde{y}'(h) = -\frac{1}{4}\left[\tilde{y}'(2h)+\tilde{y}'(0)\right] + \frac{3}{4}\left[y(2h)-y(0)\right]$

It is possible to assemble them in terms of a linear system with A, B matrices:

$$\underline{\underline{A}}\,\underline{y}' = \underline{\underline{B}}\,\underline{y} \quad \Rightarrow \quad \underline{y}' = \underline{\underline{A}}^{-1}\underline{\underline{B}}\,\underline{y}$$

Pro: with 3 points and their derivative, we have a order of accuracy circa higher i.e. 4 instead of 2

Con: we have to compute A^{-1} which could be critical (if for example A is not well conditioned)

Compact Finite Difference Schemes

$$\begin{cases} -u'' = f & x \in (a,b) \\ u(a) = \alpha & \alpha, \beta \in \mathbb{R} \\ u(b) = \beta \end{cases}$$

$a = x_0 < x_1 < \ldots < x_N = b$ subdivision of $[a,b]$ with $h = |x_{i+1} - x_i|$ $\forall i$

We write the differential equation above for a generic x_i:

$$u'' \simeq D^2(u(x_i)) = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \qquad \text{with } u(x_i) = u_i$$

So the analytical formula $-u'' = f$ becomes numerical in x_i:

$$-\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = f_i$$

Doing it $\forall i$, we can write it with a matrix:

$$\begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ -1/h^2 & 2/h^2 & -1/h^2 & 0 \ldots & 0 \\ 0 & -1/h^2 & 2/h^2 & -1/h^2 \; 0 \ldots & 0 \\ \vdots & & & & \vdots \\ 0 & \ldots & \ldots & 0 & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_i \\ \vdots \\ u_N \end{bmatrix} = \begin{bmatrix} \alpha \\ f_1 \\ \vdots \\ f_i \\ \vdots \\ \beta \end{bmatrix}$$

$$x_1 \Rightarrow -\frac{u_2 - 2u_1 + \alpha}{h^2} = f_1 \Rightarrow -\frac{u_2 - 2u_1}{h^2} = f_1 + \frac{\alpha}{h^2}$$

$$x_{N-1} \Rightarrow -\frac{\beta - 2u_{N-1} + u_{N-2}}{h^2} = f_{N-1} \Rightarrow -\frac{u_{N-2} - 2u_{N-1}}{h^2} = f_{N-1} + \frac{\beta}{h^2}$$

$$\Rightarrow \frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 & \ldots & \ldots & 0 \\ 0 & -1 & 2 & -1 & 0 \ldots & 0 \\ \vdots & & & & & \vdots \\ 0 & \ldots & 0 & -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_{N-1} \end{bmatrix} = \begin{bmatrix} f_1 + \frac{\alpha}{h^2} \\ f_2 \\ \vdots \\ f_{N-1} + \frac{\beta}{h^2} \end{bmatrix}$$

Now we try to consider the Neumann Boundary Condition on the left extreme of the interval.

Using the forward approximation $u'(a) \simeq \frac{u_1 - u_0}{h} = \alpha$ (D^+) (1st order accuracy)

$$\begin{bmatrix} -1/h & 1/h & 0 & \ldots & 0 \\ & & & & \\ & & & & \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_N \end{bmatrix} = \begin{bmatrix} \alpha \\ f_1 \\ \vdots \\ \beta \end{bmatrix}$$

If instead we use $u'(a) \simeq \alpha u_0 + \beta u_1 + \gamma u_2$ (2nd order accuracy)

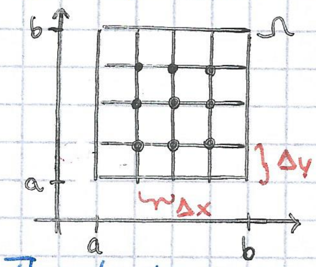
Also in x_0 we want to consider 3 points (before and after x_0): $x_{-1} < a = x_0$

$\Rightarrow u'(a) \simeq \frac{u_1 - u_{-1}}{2h} = \alpha$ (2nd order accuracy)

$-\frac{u_1 - 2u_0 + u_{-1}}{h^2} = f_0$ EXTRA POINT TECHNIQUE

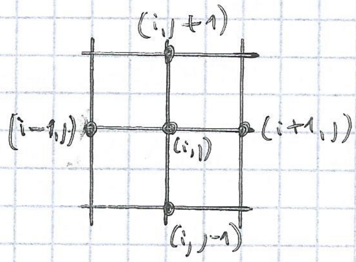
Consider a domain in $\mathbb{R}^2$, so a square $\Omega = [a,b]^2$

$$\begin{cases} -\nabla^2 u = f & \text{in } \Omega \subseteq \mathbb{R}^2 \\ u = u_b \in \mathbb{R} & \partial\Omega \end{cases}$$



generic point (i,j)

$\Delta x = \Delta y = h$



The laplacian is $-\nabla^2 u = f \Rightarrow -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u(i+1,j) + u(i-1,j) - 2u(i,j)}{h^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u(i,j+1) + u(i,j-1) - 2u(i,j)}{h^2}$$

In the generic point (i,j): $-\frac{u(i+1,j) + u(i-1,j) - 2u(i,j)}{h^2} - \frac{u(i,j+1) + u(i,j-1) - 2u(i,j)}{h^2} = f(i,j)$

$$\Rightarrow -\frac{u(i+1,j) + u(i-1,j) + u(i,j+1) + u(i,j-1) - 4u(i,j)}{h^2} = f(i,j)$$

Example:

$$\begin{cases} au'' + bu' + cu = f & \text{in } (0,1) \\ u(0) = \alpha \\ u(1) + u'(1) = \beta \end{cases}$$

1) Consider a backward FD for u', a central FD for u'' such that the order of accuracy is $O(h^2)$

2) Try to make an extension to a 2D framework

Finite Elements Methods

$$\begin{cases} -u'' = 1 & \text{in } (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

Test functions $v \in V$, $v \in C^\infty(0,1)$

$-u'' = 1 \Rightarrow -u''v = 1 \cdot v$

$$\int_0^1 -u''v = \int_0^1 1 \cdot v$$

$$\int_0^1 u'v' - \underbrace{\cancel{[u'v]_0^1}}_{v(0)=v(1)=0} = \int_0^1 1 \cdot v \quad \Rightarrow \int_0^1 u'v' = \int_0^1 v$$

$V = \{ v \in H^1(0,1) : v(0) = v(1) = 0 \} = H_0^1(0,1)$

$$\left.\begin{aligned} a(u,v) &= \int_0^1 u'v' \\ b(v) &= \int_0^1 v \end{aligned}\right\} \Longrightarrow a(u,v) = b(v)$$

bilinear form   linear form