Interpolation

We're going to approximate the elements of a space of infinite dimension by using a finite dimensional subspace. This subspace is generated by a set of linear independent vectors.

We want to approximate continuous functions. For example we consider $P^n \subseteq V = C^0([0,1])$ where P^n is the set of polynomials of degree n.

Recall that each P^n is finite dimension because it's possible to obtain:

$$\forall p \in P^n \quad \exists \{w_i\}_{i=0}^n \;|\; p = \sum_{i=1}^n w_i \tilde{v}_i$$

$$P^n = \text{span}\{\tilde{v}_i\}_{i=0}^n \qquad P^n = \text{span}\{1, x, x^2, \dots, x^n\}$$

The points $\{x_i\}_{i=1}^n$ are called **INTERPOLATION POINTS** and $\{f(x_i)\}_{i=1}^n$ are the values assumed by a function f on interpolation points.

$\forall i \quad p(x_i) = f(x_i)$ for p polynomial

So our problem can be rephrased in terms of a linear system.

If we have 3 points, we look for a 2 degree polynomial.

$$p_2(x) = ax^2 + bx + c \quad \text{where} \begin{cases} p_2(x_0) = f_0 \\ p_2(x_1) = f_1 \\ p_2(x_2) = f_2 \end{cases} \Rightarrow \begin{pmatrix} x_0^2 & x_0 & 1 \\ x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ f_2 \end{pmatrix}$$

VANDERMOND MATRIX: $\underline{V}$ | this becomes $\underline{V} \cdot \underline{a} = \underline{b}$

Ao solves our problem: $\underline{a} = \underline{V}^{-1} \underline{b}$

Remark: the Vandermond matrix is bad conditioned when we increase n. Indeed increasing n, the conditioning number is larger and larger. So we can't solve this problem using the monomial basis.

CONDITIONING NUMBER OF THE POLYNOMIAL INTERPOLATION:

$f: C^0([0,1]) \to \mathbb{R}^n$ it's $\mathbb{R}^n$: we want the weight of the polynomials

or

$f: C^0([0,1]) \to C^0([0,1])$ it's $C^0([0,1])$: polynomials are continuous functions and we want to know where the info of our functions is important

$I^n: \; u \in C^0([0,1])$

1^{st} perspective: $p \in \mathbb{R}^n$, $w = \sum p_i v_i$ $\qquad$ 2^{nd} perspective: $w \in C^0([0,1])$

We want to see if it's well defined in the 1^{st} perspective:

$$\frac{\|f(x+\delta x) - f(x)\|_y}{\|\delta x\|_x} = \frac{\|I^n(u+\delta u) - I^n(u)\|_{\ell\infty}}{\|\delta u\|_{L\infty}} = \frac{\|I^n(u) + I^n(\delta u) - I^n(u)\|_{\ell\infty}}{\|\delta u\|_{L\infty}} =$$

this is a function

$$= \frac{\|I^n(\delta u)\|}{\|\delta u\|_{\ell\infty}} = \frac{\|v^{-1}\delta u\|}{\|\delta w\|} \leq \frac{\|v^{-1}\|_{\ell\infty}\|\delta u\|_{L\infty}}{\|\delta u\|_{L\infty}} = \|v^{-1}\|_{\ell\infty}$$

$\underline{v} \cdot \underline{p} = \underline{u} \Rightarrow \underline{p} = \underline{v}^{-1} u \Rightarrow \delta \underline{p} = \underline{v}^{-1}\delta u$ $\quad \rightarrow K_{abs} = \|v^{-1}\|_{\ell\infty}$ well posed ☑

If we consider $\underline{V} = \underline{I}$, we can MINIMIZE THE CONDITIONING NUMBER for this problem.
In this case we are not considering the monomial basis anymore, but we are changing the basis $u = \sum_{i=1}^{n} p_i v_i$

The coefficients p_i are exactly the value of the function at the interpolation points.
So our basis is given by the polynomials $v_i = \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{x - a_j}{a_j - a_i}$ with a_i interpolation points.

On the point we're considering $v_i = 1$, on all the other points $v_i = 0$.

This basis is optimal in the sense that we obtain the minimum value $\|v^{-1}\|_{\ell\infty}$ (the conditioning number)

Considering the 2^{nd} perspective:

$$\frac{\|f(x+\delta x) - f(x)\|}{\|\delta x\|} = \frac{\|I^n(u+\delta u) - I^n(u)\|_{L\infty}}{\|\delta u\|_{L\infty}} = \frac{\|I^n(\delta u)\|_{L\infty}}{\|\delta u\|_{L\infty}} =$$

$$= \frac{\|\sum_{i=1}^{n} \delta u_i e_i\|_{L\infty}}{\|\delta u\|_{L\infty}} =$$

$I^n(u) = \sum_{i=1}^{n} p_i v_i = \sum_{i=1}^{n} u_i e_i$
monomial basis $\qquad$ Lagrangian basis

$$= \frac{\|\sum_{i=1}^{n} |e_i|\|_{L\infty} \|\delta u\|_{L\infty}}{\|\delta u\|_{L\infty}} = \|\sum_{i=1}^{n} |e_i|\|_{L\infty}$$

LEBESGUE FUNCTION: $\Lambda_i = \sum_{i=1}^{n} |e_i|$

$\Rightarrow K_{abs} = \|\Lambda\|_{L\infty}$ $\quad$ we want to minimize it

Study the behaviour of Λ:

it goes to $+\infty$ as log

1) $\forall A^n \in \mathbb{R}^n \quad \exists c > 0 \mid \|\Lambda^n\|_{L\infty} \geq \frac{2}{\pi}\log(n-1) - c$

2) $\forall A^n \in \mathbb{R}^n \quad \exists f \mid \lim_{n\to+\infty} \|I^n(f) - f\| = \infty$

$\forall n \; \exists$ function $\mid$ the interpolation of f is not good

$p = I^n(f)$ best approximation $\Longleftrightarrow \forall q \in P^n : \|p - f\|_{L\infty} \leq \|q - f\|_{L\infty}$

polynomials of degree n

$I^n(p)=p$ for p polynomial

$$\|I^n(f)-f\|_{L^\infty} = \|I^n(f)-p+p-f\|_{L^\infty} \leq \|I^n(f)-I^n(p)\| + \|p-f\| \leq$$

$$\leq \|I^n(f-p)\|_{L^\infty} + \|p-f\|_{L^\infty} \leq \|I^n\| \, \|p-f\|_{L^\infty} + \|p-f\|_{L^\infty} = (1+\|I^n\|)\,\|p-f\|_{L^\infty}$$

$$\Rightarrow \|q-f\|_{L^\infty} \leq (1+\|I^n\|)\,\|p-f\|_{L^\infty}$$

$$\|I^n\| = \sup_{\|u\|\neq 0} \frac{\|I^n u\|}{\|u\|} = \|\Lambda\|_{L^\infty}$$

Theorem:

$f \in C^{n+1}([0,1])$, $(a_i)_{i=1}^{n}$ interpolation points in $(0,1)$

$$\forall x \in (0,1)\ \exists \beta \mid (f - \underset{\overset{\shortparallel}{p}}{I(f)})\{\beta\} = \frac{f^{n+1}(\beta)}{(n+1)!}\,\omega(x)$$

ω is called CHARACTERISTIC POLYNOMIAL $\omega(x) = \prod_{i=0}^{n} (x-a_i)$

proof:

$\forall x \in (0,1)$ we define $G(t) = (f(t)-p(t))\,\omega(x) - (f(x)-p(x))\,\omega(t)$

and recall that $f(a_i) = p(a_i)$ implies $G(a_i)=0$ $\forall i = 0,\dots,n$

So $G(t)$ has at least $n+1$ roots and $t=x$ is a root too

$\Rightarrow$ $n+2$ roots of the equation $G(t)=0$

$\Rightarrow$ $\underset{\overset{\shortparallel}{A(t)}}{G^n(t)}$ has 2 zeroes

$\exists\, s_1, s_2 : A(s_1)=A(s_2)=0$ $\quad A'(t) = G^{n+1}(t)=0$

$$G^{n+1}(t) = f^{n+1}(t)\,\omega(x) - (f(x)-p(x))\,(n+1)! = 0 \quad \text{for } t=\beta$$

$$\Rightarrow f-p = \frac{f^{n+1}\omega}{(n+1)!}$$

$$\|f-p\|_{L^\infty} \leq \frac{1}{(n+1)!} \underbrace{\|\omega\|_{L^\infty}}_{=(b-a)^n} \|f^{n+1}\|_{L^\infty}$$

$(b-a)^n$ if we consider an interval $[a,b]$ ▨