10/03

What is NUMERICAL ANALYSIS?
- algorithm, convergence
- approximation of math problems
- estimate of error

So there are 3 properties:
- approximation (algorithms)
- implementation (computer code)
- error analysis

Elements of Functional Analysis

VECTOR SPACE: $(V, +, \cdot), \mathbb{R}$ such that

- $v, w \in V \Rightarrow z = v + w \in V$
- $\alpha v \in V$
- $\alpha v + \beta w \in V$
- $0 \in V$
- $1 \in \mathbb{R}$

$f \in C^0([0,1])$ in $\mathbb{R}^n$

For $n=2$ $\quad x = (x_1, x_2),\ y = (y_1, y_2) \in \mathbb{R}^2 \Rightarrow z = (x_1 + y_1, x_2 + y_2) \quad \alpha x = (\alpha x_1, \alpha x_2)$

NORM: $\|\cdot\| : V \to \mathbb{R}_0^+$ with
- positivity: $\|v\| \geq 0 \quad (= 0 \iff v = 0)$
- $\|\alpha v\| = |\alpha|\,\|v\|$
- $\|v + w\| \leq \|v\| + \|w\|$

x_p-norms $= \left(\sum_{i=1}^{2} |x_i|^p \right)^{1/p}$ for $p \in \mathbb{N}$

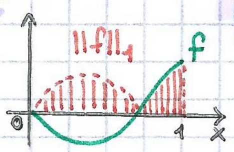
$\lim_{p \to +\infty} \|v\|_p = \max |v_i|$ because given $v = (1, -2)$

$\|v\|_2 = (|1|^2 + |-2|^2)^{1/2} = \sqrt{5}$

$\vdots$

$\|v\|_\infty = \max |v_i| = \max\{|1|, |-2|\} = |2|$

$$\|f\|_p = \left(\int |f|^p \, dx \right)^{1/p}$$



$$\|f\|_\infty = \max_{x \in [0,1]} |f|$$

$$\|f\|_1 = \int_0^1 |f| \, dx$$

METRIC: $d : V \times V \to \mathbb{R}_0^+$ such that
- $d(v,w) \geq 0 \quad (= 0 \iff v = w)$
- $d(v,w) = d(w,v)$
- $d(v,w) \leq d(v,z) + d(z,w)$

proof $d(x,y) = \|x - y\|$ is a metric:

The first two are trivial, for the third: $\|x - z\| \leq \|x - y\| + \|y - z\|$

We can write $\|x - z\| = \|x - y + y - z\|$ where $A = x - y$, $B = y - z$
so it becomes $\|A + B\| \leq \|A\| + \|B\|$ $\square$

$A = \{(0,1),(1,0)\}$ $\quad \mathbb{R}^2 = \text{span}(A)$

$v = \sum_{i=1}^{2} \alpha_i v_i \qquad \alpha_i \in A$

P^n space of polynomials of order n

For $n=2$: $v = \alpha_0 + \alpha_1 x + \alpha_2 x^2 \in P^2 \Rightarrow P^2 = \text{span}\{1, x, x^2\}$