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ABSTRACT PROBLEM: $\mathbb{X}$ input space, $\mathbb{Y}$ output space,
$x \in \mathbb{X}$, $y \in \mathbb{Y}$, $y = f(x)$:

1) $\forall x \in \mathbb{X} \quad \exists y \in \mathbb{Y} \mid y = f(x)$

2) $\exists! \, y$

3) $\forall$ perturbation $\delta_x \in \mathbb{X} \mid x + \delta_x \in \mathbb{X}$ — ABSOLUTE STABILITY

$\exists$ constant K_{abs} : $\|\delta_y\|_{\mathbb{Y}} \leq K_{abs} \|\delta_x\|_{\mathbb{X}}$

3') $\forall x \in \mathbb{X} \mid \|x\|_{\mathbb{X}} \neq 0, \ \|f(x)\|_{\mathbb{Y}} \neq 0$ — RELATIVE STABILITY

$\exists$ constant K_{rel} : $\dfrac{\|\delta_y\|_{\mathbb{Y}}}{\|y\|_{\mathbb{Y}}} \leq K_{rel} \dfrac{\|\delta_x\|_{\mathbb{X}}}{\|x\|_{\mathbb{Y}}}$

If it satisfies 1, 2, 3, it's a well posed problem in absolute terms

If it also satisfies 3', it's a well posed problem in relative terms

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* Well Posed Problems

Based on $y = f(x)$: $\|\delta_y\| = \|y + \delta_y - y\| = \|f(x+\delta_x) - f(x)\|$

For absolute stability: $\|f(x+\delta_x) - f(x)\|_{\mathbb{Y}} \leq K_{abs} \|\delta_x\|_{\mathbb{Y}}$

$\Rightarrow \dfrac{\|f(x+\delta_x) - f(x)\|}{\|\delta_x\|} = f'(x) \leq K_{abs}$ i.e. $K_{abs} = \sup\limits_{x \in \mathbb{X}} \sup\limits_{\delta_x} \dfrac{\|f(x+\delta_x) - f(x)\|}{\|\delta_x\|}$

Analogue for the relative stability but with normalization ($: \|y\|$ $: \|x\|$)

Example:

$x \in \mathbb{X} = \mathbb{R}^2 \qquad x = (x_1, x_2) \qquad y = x_1 + x_2 \in \mathbb{Y} = \mathbb{R}$

$\delta_x = (\delta_{x_1}, \delta_{x_2}) \in \mathbb{X} \qquad \delta_y = \delta_{x_1} + \delta_{x_2}$

$\|f(x+\delta_x) - f(x)\| = \|\delta_y\| = \|\delta_{x_1} + \delta_{x_2}\|$

$\|x\|_{\mathbb{X}} = \|x\|_{\ell_1} = |x_1| + |x_2|$

ℓ_p-norms $= \left(\sum\limits_{i=1}^{n} |x_i|^p\right)^{1/p} \qquad$ For $p = 1$: $\|y\|_{\mathbb{Y}} = |y|$

$$\frac{\| f(x+\delta x) - f(x) \|}{\|\delta x\|} = \frac{\|\delta y\|}{\|\delta x\|} = \frac{|x_1|+|x_2|}{|x_1+x_2|} \leq \frac{|\delta x_1|+|\delta x_2|}{|\delta x_1|+|\delta x_2|} = 1 \Rightarrow K_{abs}=1$$

$\frac{\|\delta y\|}{\|\delta x\|} = 1$ for before

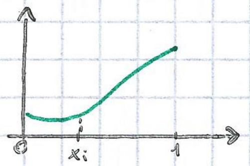
$$\frac{\frac{\|\delta y\|}{\|y\|}}{\frac{\|\delta x\|}{\|x\|}} = \frac{\|\delta y\|}{\|y\|}\,\frac{\|x\|}{\|\delta x\|} \leq 1\;\frac{|x_1|+|x_2|}{|x_1+x_2|} \quad \text{i.e. } K_{rel} = \sup_{x_1,x_2\in X} \frac{|x_1|+|x_2|}{|x_1+x_2|}$$

If $|x_1| \sim |x_2|$ and $x_1, x_2 \leq 0$, the problem is not well conditioned

Pretty common situation:

$f \in C^0([0,1]) \qquad x_i \in [0,1] \qquad h \in \mathbb{R}^+$

$$f'(x_i) \simeq \frac{f(x_i+h)-f(x_i)}{h}$$



From "10.10 lab2 with Girfoglio (03-04-06-MG-Well-posed-problems_and-interpolation).ipynb" line 4,
the last part of the plot where blue and red are very different is when h is big;
the first part of the plot where there is a plateau is when h is so small that the computer can't detect the difference between $f(x+h)$ and $f(x)$

Numerical approximation of an abstract problem

$y = f(x) \rightarrow$ x_n, y_n, f_n depend on n

We substitute the abstract problem with this sequence

We expect convergence to the analytical problem: $x_n \to x$, $y_n \to y$, $f_n \to f$

A numerical approximation is well-posed when each problem $y_n \to f_n(x_n)$ is well-posed $\forall n$

A numerical approximation is CONSISTENT when $\lim_{n\to+\infty} \|f_n(x_n) - f(x)\| = 0$

A numerical approximation is CONVERGENT when $\lim_{n\to+\infty} \|f_n(x) - f(x)\| = 0$

$$f: \; g \in C^0([0,1]) \to g'(\xi) = f(g(\xi)) \quad \text{for } \xi \in [0,1]$$

CENTRAL FORWARD DIFFERENCE FORMULA (CFD): $\frac{f(x+h)-f(x-h)}{2h}$

FORWARD DIFFERENCE FORMULA (FD): $\frac{f(x+h)-f(x)}{h}$ $\qquad f_n(g) \overset{h=1/n}{=} \frac{g(\xi+\frac{1}{n})-g(\xi)}{1/n}$

For construction FD is CONSISTENT: $\lim_{n\to+\infty} \|f_n(g_n) - f(g)\| = 0$

Lax theorem:
Given a numerical approximation which is consistent: convergence $\Leftrightarrow$ stability

Compute the velocity of consistency (i.e. of the error):

$$\|f_n(g)-f(g)\| = \left\| \frac{g(\xi+\frac{1}{n})-g(\xi)}{1/n} - g'(\xi)\right\| = \left\| \left[g\left(\xi+\frac{1}{n}\right)-g(\xi)\right]n - g'(\xi)\right\| \overset{\text{Taylor}}{=}$$

$$= \left\| \left[\cancel{g(\xi)} + g'(\xi)h + \frac{g''(\xi)}{2}h^2 + \sum_{k=3}^{\infty}\frac{g^k(\xi)}{k!}h^k - \cancel{g(\xi)}\right]n - g'(\xi)\right\| \overset{h=\frac{1}{n}}{=}$$

$$= \left\| \cancel{g'(\xi)} + \frac{g''(\xi)}{2}h + \sum_{k=3}^{\infty}\frac{g^k(\xi)}{k!}h^{k-1} - \cancel{g'(\xi)}\right\| =$$

$$= \left\| \sum_{k=2}^{\infty}\frac{g^k(\xi)}{k!}h^{k-1}\right\| = \left\| \underbrace{\frac{g''(\xi)}{2}\frac{1}{n}}_{\text{TRONCATION ERROR}} + O\left(\frac{1}{n^2}\right)\right\|$$

Order of accuracy: $\frac{1}{n}$ $\Rightarrow$ order 1

Analogously $\|f_n(g)-f(g)\| = \left\|\frac{g(\xi+1/n)-g(\xi-1/n)}{2\cdot 1/n} - g'(\xi)\right\|$. In this case the troncation error is better because it is of order $1/n^2$ $\Rightarrow$ order 2.

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OPERATIONAL NORM: $X, Y, f, y=f(x)$ $\qquad \|f\|_* = \sup_{\substack{x\in X \\ \|x\|\neq 0}} \frac{\|f(x)\|}{\|x\|}$

Consider $A: \mathbb{R}^n \to \mathbb{R}^m$ linear operator, it's a matrix and we can use the p-norms previously defined in this operational norm

In this case we have $\|A\|_* = \|A\|_p$ $\qquad \|Ax\|_p \leq \|A\|_p \|x\|_p$

$$\|A\|_p = \sup_{\substack{x\in X \\ \|x\|\neq 0}} \frac{\|Ax\|_p}{\|x\|_p}$$

Example:
$n=m$ $\quad X=Y=\mathbb{R}^n \quad y=Ax \quad A\in\mathbb{R}^{n\times n}$

We want to verify if the problem $y=Ax$ is well posed or not, that is if

$\exists\, K_{abs} \mid \forall x\in X, \forall \delta x \in X: x+\delta x \in X$

$$\Rightarrow \frac{\|f(x+\delta x)-f(x)\|_p}{\|\delta x\|_p} \leq K_{abs}$$

$f(x)=Ax \qquad f(x+\delta x) = A(x+\delta x) = Ax + A\delta x$

$$\Rightarrow \frac{\|Ax+A\delta x - Ax\|_p}{\|\delta x\|_p} = \frac{\|A\delta x\|_p}{\|\delta x\|_p} \leq \|A\|_p \frac{\|\delta x\|_p}{\|\delta x\|_p} = \|A\|_p = K_{abs}$$

$$\frac{\|\delta y\|}{\|y\|}\frac{\|x\|}{\|\delta x\|} \leq \|A\|_p \frac{\|x\|_p}{\|y\|_p}$$

In some conditions it's possible to define also the inverse function

$\begin{cases} y=f(x) \\ x=f^{-1}(y)\end{cases}$ but if $y=f(x)$ is well posed, it's not true in general that $x=f^{-1}(y)$ is again well posed

Now assume that in our case this operation is invertible, then

$$\frac{\|\delta y\|}{\|y\|}\,\frac{\|x\|}{\|\delta x\|} \leq \|A\|_p \frac{\|x\|_p}{\|y\|_p} \overset{\substack{\text{new thing}\\ !}}{\leq} \|A\|_p \|A^{-1}\|_p \frac{\|y\|_p}{\|y\|_p} = \|A\|_p \|A^{-1}\|_p = K_{rel}$$

$$\Rightarrow \begin{cases} K_{abs} = \|A\|_p \\ K_{rel} = \|A\|_p \|A^{-1}\|_p \end{cases}$$

So we need our matrix A to be invertible and the product small enough