

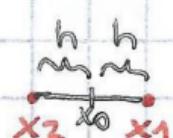
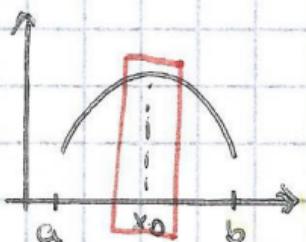
# Finite Difference Method

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We want to solve this problem

$$\begin{cases} u'' = 1 \\ u(a) = \alpha \\ u(b) = \beta \end{cases} \quad \begin{array}{l} x \in (a, b) \\ a, \beta \in \mathbb{R} \end{array}$$

If we want to find  $u'(x_0)$  with  $x_0 \in (a, b)$ , we make  $[a, b]$  discrete in  $n$  points  
 $a = x_0 < x_1 < \dots < x_i < \dots < x_n = b$



$$h = \frac{b-a}{n-1}$$

$$\begin{aligned} x_1 &= x_0 + h \\ x_2 &= x_0 - h \end{aligned}$$

$$u'(x_0) \approx f(u(x_0), u(x_2), u(x_1))$$

Consider Taylor's expansion:

$$v(x_0) = v(x_0)$$

$$v(x_0+h) = v(x_0) + v'(x_0)h + \frac{v''(x_0)h^2}{2} + \frac{v'''(x_0)h^3}{6} + O(h^4)$$

$$v(x_0-h) = v(x_0) - v'(x_0)h + \frac{v''(x_0)h^2}{2} - \frac{v'''(x_0)h^3}{6} + O(h^4)$$

$$D^+ := \frac{v(x_0+h) - v(x_0)}{h} = v'(x_0) + \frac{v''(x_0)h}{2} + \frac{v'''(x_0)h^2}{6} + O(h^3) = v'(x_0) + O(h)$$

$$D^- := \frac{v(x_0) - v(x_0-h)}{h} = v'(x_0) - \frac{v''(x_0)h}{2} + \frac{v'''(x_0)h^2}{6} + O(h^3) = v'(x_0) + O(h)$$

$$D^c := \frac{v(x_0+h) - v(x_0-h)}{2h} = \frac{D^+ + D^-}{2} = v'(x_0) + O(h)$$

$$\begin{aligned} D_c^2(v(x_0)) &= D^+(D^- - v(x_0)) = D^+ \left( \frac{v(x_0) - v(x_0-h)}{h} \right) = \\ &= \frac{1}{h} [D^+(v(x_0)) - D^+(v(x_0-h))] = \frac{1}{h} \left[ \frac{v(x_0+h) - v(x_0)}{h} - \frac{v(x_0) - v(x_0-h)}{h} \right] = \\ &= \frac{1}{h^2} [v(x_0+h) - 2v(x_0) + v(x_0-h)] \stackrel{\text{use Taylor expansion of } v(x_0+h)}{\approx} v''(x_0) + O(h^2) \\ &\Rightarrow J^k(x_0) = \sum_{i=0}^k v(x_i) x_i \end{aligned}$$

## Unknown coefficients method

We want to find the coefficients  $\alpha$  and  $\beta$  of  $v'(x_0) \approx \alpha v(x_0) + \beta v(x_0+h)$

$$\alpha v(x_0) = \alpha v(x_0)$$

$$\beta v(x_0+h) = \beta v(x_0) + \beta v'(x_0)h + \beta \frac{v''(x_0)h^2}{2} + \beta \frac{v'''(x_0)h^3}{6} + O(h^4)$$

$$\alpha v(x_0) + \beta v(x_0+h) = (\alpha + \beta)v(x_0) + \beta v'(x_0)h + \beta \frac{v''(x_0)h^2}{2} + \beta \frac{v'''(x_0)h^3}{6} + O(h^4)$$

$$\text{We want } \begin{cases} \alpha + \beta = 0 \\ \beta h = 1 \end{cases} \Rightarrow \begin{cases} \alpha = -1/h \\ \beta = 1/h \end{cases}$$

$$\Rightarrow v'(x_0) \approx \alpha v(x_0) + \beta v(x_0+h) = \frac{v(x_0+h) - v(x_0)}{h}$$

Analogously for  $v''(x_0) \approx \alpha v(x_0) + \beta v(x_0+h) + \gamma v(x_0-h)$

$$\alpha v(x_0) = \alpha v(x_0)$$

$$\beta v(x_0+h) = \beta v(x_0) + \beta v'(x_0)h + \beta \frac{v''(x_0)h^2}{2} + O(h^3)$$

$$\gamma v(x_0-h) = \gamma v(x_0) - \gamma v'(x_0)h + \gamma \frac{v''(x_0)h^2}{2} + O(h^3)$$

$$\text{We want } \begin{cases} \alpha + \beta + \gamma = 0 \\ \beta h - \gamma h = 0 \\ \frac{\beta h^2}{2} + \frac{\gamma h^2}{2} = 1 \end{cases} \Rightarrow \begin{cases} \beta = \gamma \\ \beta h^2 = 1 \\ \alpha + \beta + \gamma = 0 \end{cases} \Rightarrow \begin{cases} \beta = \gamma \\ \beta = 1/h^2 \\ \alpha = -2/h^2 \end{cases}$$

$$\Rightarrow v''(x_0) \approx \alpha v(x_0) + \beta v(x_0+h) + \gamma v(x_0-h) = \frac{v(x_0+h) - 2v(x_0) + v(x_0-h)}{h^2}$$

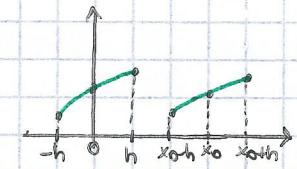
Another method for obtaining them. Let's interpolate the 3 points

$$(x_0 - h, u(x_0 - h)), (x_0, u(x_0)), (x_0 + h, u(x_0 + h))$$

$$y = ax^2 + bx + c \text{ such that } \begin{cases} y(x_0 - h) = u(x_0 - h) \\ y(x_0) = u(x_0) \\ y(x_0 + h) = u(x_0 + h) \end{cases}$$

Considering the translation  $x^* = x - x_0$ :

$$\begin{aligned} x_0 &\rightarrow 0 \\ x_0 + h &\rightarrow h \\ x_0 - h &\rightarrow -h \end{aligned}$$



$$y(0) = c = u(0)$$

$$\begin{aligned} * \quad y(h) &= ah^2 + bh + u(0) = u(h) \\ ** \quad y(-h) &= ah^2 - bh + u(0) = u(-h) \end{aligned}$$

$$*: a = \frac{u(h) - u(0) - bh}{h^2} \Rightarrow **: \frac{u(h) - u(0) - bh}{h^2} = \frac{h^2 - bh + u(0)}{h^2} = u(-h)$$

$$\Rightarrow b = \frac{u(-h) - u(h)}{-2h} = \frac{u(h) - u(-h)}{2h}$$

$$\Rightarrow a = \frac{u(h) - u(0) - \frac{u(h)}{2} + \frac{u(-h)}{2}}{h^2} = \frac{\frac{u(h)}{2} - u(0) + \frac{u(-h)}{2}}{h^2} = \frac{u(h) - 2u(0) + u(-h)}{2h^2}$$

$$\Rightarrow y = \frac{u(h) - 2u(0) + u(-h)}{2h^2} x^2 + \frac{u(h) - u(-h)}{2h} x + u(0)$$

$$\Rightarrow y' = \frac{u(h) - 2u(0) + u(-h)}{h^2} x + \frac{u(h) - u(-h)}{2h} = y'(0)$$

$$y'' = \frac{u(h) - 2u(0) + u(-h)}{h^2} \Leftrightarrow D_C^2(u(x_0))$$

## Hermite Interpolation

Considering  $u(x_0), u(x_0 - h), u(x_0 + h)$  and their derivatives  $u'(x_0 - h), u'(x_0 + h)$  we have 5 info so we can try to interpolate with a polynomial of order 4.

We have to consider both  $y$  and  $y'$ :

$$\begin{aligned} y &= ax^4 + bx^3 + cx^2 + dx + e \\ y' &= 4ax^3 + 3bx^2 + 2cx + d \end{aligned}$$

$$y(0) = e = u(0)$$

$$y(h) = ah^4 + bh^3 + ch^2 + dh + u(0) = u(h)$$

$$y(-h) = ah^4 - bh^3 + ch^2 - dh + u(0) = u(-h)$$

$$y'(h) = 4ah^3 + 3bh^2 + 2ch + d = y'(h)$$

$$y'(-h) = -4ah^3 + 3bh^2 - 2ch + d = y'(-h)$$

If you don't know  $u'$ :

$$\hat{y}(x_0) = \frac{y(x_0 + h) - y(x_0)}{h} \approx y'(x_0)$$

The process of calculations is not so important.

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$$y(0) = d = -\frac{1}{4} [y'(h) + y'(-h)] + \frac{3}{4} [y(h) - y(-h)] \approx y'(0)$$

$$\text{Translating it's } \dot{y}'(h) = -\frac{1}{4} [y'(2h) + y'(0)] + \frac{3}{4} [y(2h) - y(0)]$$

It is possible to assemble them in terms of a linear system with A, B matrices:

$$A \dot{y}' = B y \Rightarrow \dot{y}' = A^{-1} B y$$

Pros: with 3 points and their derivative, we have a order of accuracy circa higher i.e. 4 instead of 2

Cons: we have to compute  $A^{-1}$  which could be critical (if for example A is not well conditioned)

## Compact Finite Difference Schemes

$$\begin{cases} -u'' = f & x \in (a, b) \\ u(a) = \alpha & \alpha, \beta \in \mathbb{R} \\ u(b) = \beta \end{cases}$$

$a = x_0 < x_1 < \dots < x_N = b$  subdivision of  $[a, b]$   
with  $h = (x_{i+1} - x_i) \neq i$

We write the differential equation above for a generic  $x_i$ :

$$u'' \approx D^2(u(x_i)) = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \quad \text{with } u(x_i) = u_i$$

So the analytical formula  $-u'' = f$  becomes numerical in  $x_i$ :

$$-\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = f_i$$

Doing it  $\forall i$ , we can write it with a matrix:

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ -1/h^2 & 3/h^2 & -1/h^2 & 0 & \dots & 0 \\ 0 & -1/h^2 & 3/h^2 & -1/h^2 & 0 & \dots & 0 \\ \vdots & & & & & & \vdots \\ 0 & \dots & \dots & \dots & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_i \\ u_{i-1} \\ \vdots \\ u_N \end{bmatrix} = \begin{bmatrix} \alpha \\ f_1 \\ \vdots \\ f_i \\ \vdots \\ \beta \end{bmatrix}$$

$$x_1 \Rightarrow -\frac{u_2 - 2u_1 + \alpha}{h^2} = f_1 \Rightarrow -\frac{u_2 - 2u_1}{h^2} = f_1 + \frac{\alpha}{h^2}$$

$$x_{N-1} \Rightarrow -\frac{\beta - 2u_{N-1} + u_{N-2}}{h^2} = f_{N-1} \Rightarrow -\frac{u_{N-2} - 2u_{N-1}}{h^2} = f_{N-1} + \frac{\beta}{h^2}$$

$$\Rightarrow \frac{1}{h^2} \begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & \dots & 0 \\ \vdots & & & \ddots & 0 \\ 0 & \dots & 0 & -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix} = \begin{bmatrix} f_1 + \frac{\alpha}{h^2} \\ f_2 \\ \vdots \\ f_{N-1} + \frac{\beta}{h^2} \end{bmatrix}$$

Now we try to consider the Neumann Boundary Condition on the left extreme of the interval.

Using the forward approximation  $u'(a) \approx \frac{u_1 - u_0}{h}$  ( $D^+$ ) (1<sup>st</sup> order accuracy)

$$\begin{bmatrix} -1/h & 1/h & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_N \end{bmatrix} = \begin{bmatrix} \alpha \\ f_1 \\ \vdots \\ \beta \end{bmatrix}$$

If instead we use  $u'(a) \approx \alpha u_0 + \beta u_1 + \gamma u_2$  (2<sup>nd</sup> order accuracy)

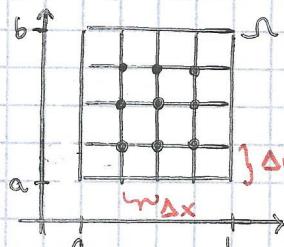
Also in  $x_0$  we want to consider 3 points (before and after  $x_0$ ):  $x_1 < a = x_0$

$$\Rightarrow v'(a) \approx \frac{v_1 - v_{-1}}{2h} \Rightarrow \text{(2nd order accuracy)}$$

$$-\frac{v_1 - 2v_0 + v_{-1}}{h^2} = f_0 \quad \text{EXTRA POINT TECHNIQUE}$$

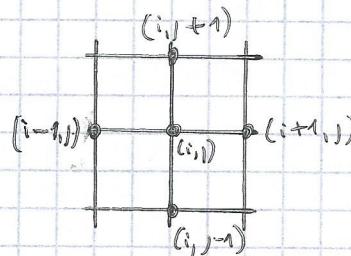
Consider a domain in  $\mathbb{R}^2$ , so a square.  $\Omega = [a, b]^2$

$$\begin{cases} -\nabla^2 v = f & \text{in } \Omega \subseteq \mathbb{R}^2 \\ v = v_0 \in \mathbb{R} & \text{on } \partial\Omega \end{cases}$$



generic point  $(i, j)$

$$\Delta x = \Delta y = h$$



$$\text{The Laplacian is } -\nabla^2 v = f \Rightarrow -\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} = f$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{v(i+1,j) + v(i-1,j) - 2v(i,j)}{h^2}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{v(i,j+1) + v(i,j-1) - 2v(i,j)}{h^2}$$

$$\text{In the generic point } (i,j) : -\frac{v(i+1,j) + v(i-1,j) - 2v(i,j)}{h^2} - \frac{v(i,j+1) + v(i,j-1) - 2v(i,j)}{h^2} = f(i,j)$$

$$\Rightarrow -\frac{v(i+1,j) + v(i-1,j) + v(i,j+1) + v(i,j-1) - 4v(i,j)}{h^2} = f(i,j)$$

Example:

$$\begin{cases} av'' + bv' + cv = f & \text{in } (0,1) \\ v(0) = \alpha \\ v(1) + v'(1) = \beta \end{cases}$$

1) Consider a backward FD for  $v'$ , a central FD for  $v''$  such that the order of accuracy is  $O(h^2)$

2) Try to make an extension to a 2D framework

## Finite Elements Methods

$$\begin{cases} -v'' = 1 & \text{in } (0,1) \\ v(0) = v(1) = 0 \end{cases}$$

Test functions  $v \in V$ ,  $v \in C^0([0,1])$

$$-v'' = 1 \Rightarrow -v''v = 1 \cdot v$$

$$\int_0^1 -v''v = \int_0^1 1 \cdot v$$

$$\int_0^1 v'v - [v'v]_0^1 = \int_0^1 1 \cdot v \Rightarrow \int_0^1 v'v = \int_0^1 v$$

$$v(0) = v(1) = 0$$

$$V = \{v \in H^1(0,1) : v(0)=v(1)=0\} = H_0^1(0,1)$$

$$a(u,v) = \int_0^1 u'v' \quad ? \Rightarrow a(u,v) = b(v)$$

$$b(v) = \int_0^1 v$$

bilinear  
form

linear  
form