HOMEWORK ASSIGNMENT 2
WAVEFIELD IMAGING EE4595

This homework assignment consists of **2** problems.
Submit your solutions as a pdf file.
Deadline: 09-06-2020, 23:59

Problem 1. To find a minimum of a function $f : \mathbb{R}^n \to \mathbb{R}$, we apply the method of steepest descent. Specifically, we start with an initial guess $\mathsf{x}^{[0]}$ and construct a sequence of vectors according to the update equation

$$\mathsf{x}^{[k+1]} = \mathsf{x}^{[k]} - \tau^{[k]} \nabla f(\mathsf{x}^{[k]}),$$

where $\nabla f(\mathsf{x}^{[k]})$ is the gradient of f at $\mathsf{x}^{[k]}$ and

$$\tau^{[k]} = \min_{\tau > 0} f[\mathsf{x}^{[k]} - \tau \nabla f(\mathsf{x}^{[k]})]$$

and we hope, of course, that $\mathsf{x}^{[k]}$ converges to a minimum of f.

The iterates $\mathsf{x}^{[k]}$ of the steepest descent algorithm generally exhibit a "zigzag" pattern, that is, the iterates satisfy the orthogonality relation

$$\left(\mathsf{x}^{[k+2]} - \mathsf{x}^{[k+1]}\right)^T \left(\mathsf{x}^{[k+1]} - \mathsf{x}^{[k]}\right) = 0 \qquad \text{for } k = 0, 1, \ldots . \quad (*)$$

a) Plot four vectors $\mathsf{x}^{[0]}$, $\mathsf{x}^{[1]}$,$\mathsf{x}^{[2]}$, and $\mathsf{x}^{[3]}$ belonging to $\mathbb{R}^2$ that satisfy Eq. $(*)$ thereby illustrating the zigzag pattern.

b) Prove that the iterates $\mathsf{x}^{[k]}$ of the steepest descent method satisfy Eq. $(*)$.

Consider now the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{a}{2}x_2^2, \qquad a > 0.$$

We apply the steepest descent method to find the minimum of this function starting with the initial guess

$$\mathsf{x}^{[0]} = \begin{pmatrix} a \\ 1 \end{pmatrix}.$$

c) Show that the steepest descent method generates the sequence

$$\mathsf{x}^{[k]} = \left(\frac{a-1}{a+1}\right)^k \begin{pmatrix} a \\ (-1)^k \end{pmatrix} \quad \text{for } k = 0, 1, 2, \ldots .$$

d) Take $a = 16$ and use Matlab to plot some level curves of f. Include the steepest descent points in this figure.

We introduce the change of variables

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

where α is a nonzero constant.

e) For what value(s) of α does the steepest descent algorithm converge in a single iteration?

Now consider the function

$$f(\mathsf{x}) = \|\mathsf{d}^{\mathrm{sc}} - \mathsf{G}_{\mathrm{SB}}\mathsf{x}\|_2^2,$$

where d^{sc} is an m-by-1 vector containing the scattered field measurements, G_{SB} is a discretized version of the Born-data operator $\mathcal{G}_{\mathrm{SB}}$ (see lecture notes and slides), and x is an n-by-1 vector representing the contrast of the object. We start the steepest descent method with the initial guess $\mathsf{x}^{[0]} = \mathsf{0}$ and to limit the computational costs, we perform a single iteration only.

f) Show that

$$\mathsf{x}^{[1]} = \frac{\|\mathsf{G}_{\mathrm{SB}}^H \mathsf{d}^{\mathrm{sc}}\|_2^2}{\|\mathsf{G}_{\mathrm{SB}}\mathsf{G}_{\mathrm{SB}}^H \mathsf{d}^{\mathrm{sc}}\|_2^2} \mathsf{G}_{\mathrm{SB}}^H \mathsf{d}^{\mathrm{sc}}$$

and give an interpretation of this reconstruction.

Suppose we are only interested in qualitative information about the object, that is, we are only interested in an image of the contrast profile and particular contrast values are of no interest to us.

g) What do you propose to further reduce the computational costs?

Problem 2. We are given a one-dimensional configuration that varies in the z-direction only. The background material in this configuration is characterized by a constant wave speed c_{b}. A delta source is located at $z = -a$ with $a > 0$ and the field $\hat{u}^{\mathrm{inc}}$ that is generated satisfies the equation

$$\left(\partial_z^2 - \hat{\gamma}_{\mathrm{b}}^2\right) \hat{u}^{\mathrm{inc}} = -2\hat{\gamma}_{\mathrm{b}}\delta(z + a),$$

with $\hat{\gamma}_{\mathrm{b}} = s/c_{\mathrm{b}}$ and $\mathrm{Re}(s) > 0$.

a) Show that the background field $\hat{u}^{\mathrm{inc}}$ is given by

$$\hat{u}^{\mathrm{inc}}(z, s) = \exp(-\hat{\gamma}_{\mathrm{b}}|z + a|).$$

We now include a slab in our configuration. The slab occupies the domain $0 < z < d$ with $d > 0$ and is characterized by a wave speed $c(z)$ for $0 < z < d$. To take the presence of this slab into account, we set up a scattering formalism and introduce the scattered field as

$$\hat{u}^{\mathrm{sc}}(z, s) = \hat{u}(z, s) - \hat{u}^{\mathrm{inc}}(z, s), \quad z \in \mathbb{R},$$

where $\hat{u}$ is the total field.

b) Show that the scattered field is given by

$$\hat{u}^{\mathrm{sc}}(z,s) = \frac{\hat{\gamma}_{\mathrm{b}}}{2} \int_{z'=0}^{d} \exp(-\hat{\gamma}_{\mathrm{b}}|z-z'|)\chi(z')\,\hat{u}(z',s)\,\mathrm{d}z', \qquad z \in \mathbb{R},$$

with a contrast function

$$\chi(z) = 1 - \left[\frac{c_{\mathrm{b}}}{c(z)}\right]^2.$$

c) Use the definition of the scattered field and show that the total field $\hat{u}$ inside the slab satisfies the integral equation

$$\hat{u}(z,s) - \frac{\hat{\gamma}_{\mathrm{b}}}{2} \int_{z'=0}^{d} \exp(-\hat{\gamma}_{\mathrm{b}}|z-z'|)\chi(z')\,\hat{u}(z',s)\,\mathrm{d}z' = \hat{u}^{\mathrm{inc}}(z,s) \quad \text{for } 0 < z < d.$$

We now take $s = \mathrm{j}\omega$ and a slab width $d = \alpha\lambda_{\mathrm{b}}$ with $\alpha > 0$ and λ_{b} is the wavelength in the background medium. Furthermore, we introduce the grid nodes

$$z_k = (k - 1/2)\Delta z \qquad \text{for } k = 1, 2, ..., n$$

with $n\Delta z = d$. Write the integral in the above equation as

$$\int_{z'=0}^{d} ...\,\mathrm{d}z' = \int_{z'=z_{1/2}}^{z_{3/2}} ...\,\mathrm{d}z' + \int_{z'=z_{3/2}}^{z_{5/2}} ...\,\mathrm{d}z' + ... + \int_{z'=z_{n-1/2}}^{z_{n+1/2}} ...\,\mathrm{d}z'$$

and apply the midpoint rule to each subintegral.

d) Show that this discretization procedure leads to the equation

$$\hat{u}(z,\mathrm{j}\omega) - \mathrm{j}\pi\frac{\Delta z}{\lambda_{\mathrm{b}}} \sum_{k=1}^{n} \exp(-\mathrm{j}2\pi\frac{|z-z_k|}{\lambda_{\mathrm{b}}})\chi(z_k)\hat{u}(z_k,\mathrm{j}\omega) = \hat{u}^{\mathrm{inc}}(z,\mathrm{j}\omega), \qquad 0 < z < d.$$

Finally, to arrive at a system of equations for the total field, we take $z = z_j$, $j = 1, 2, ..., n$, in the above equation. Arranging all field unknowns in the n-by-1 vector

$$\mathsf{u} = [\hat{u}(z_1,\mathrm{j}\omega), \hat{u}(z_2,\mathrm{j}\omega), ..., \hat{u}(z_n,\mathrm{j}\omega)]^T$$

and the background field values in the n-by-1 vector

$$\mathsf{u}^{\mathrm{inc}} = [\hat{u}^{\mathrm{inc}}(z_1,\mathrm{j}\omega), \hat{u}^{\mathrm{inc}}(z_2,\mathrm{j}\omega), ..., \hat{u}^{\mathrm{inc}}(z_n,\mathrm{j}\omega)]^T$$

we arrive at the discretized system

$$(\mathsf{I} - \mathsf{GC})\,\mathsf{u} = \mathsf{u}^{\mathrm{inc}} \qquad (*)$$

where I is the identity matrix of order n and C is the contrast matrix containing the contrast values at the sample points.

e) Give matrix C explicitly.

f) Give an explicit expression for the elements of matrix G.

g) Show that matrix G is symmetric and Toeplitz.

We place the delta source at $z = -a = -\lambda_{\mathrm{b}}$ and first take $\alpha = 1$, which means that the width of the slab is exactly equal to the background wavelength.

h) Take $n = 256$ and use Matlab to solve equation $(*)$ for $c(z) = 0.95c_{\mathrm{b}}$, for $0 < z < d$. Plot the magnitude of the solution and the magnitude of the incident field on the domain $0 < z < d$.

i) Take $n = 256$ and use Matlab to solve equation $(*)$ for $c(z) = \frac{1}{8}c_{\mathrm{b}}$ for $0 < z < d$. Plot the magnitude of the solution and the magnitude of the incident field on the domain $0 < z < d$.

Subsequently, take $\alpha = 5$, which means that the width of the slab is five times the background wavelength.

j) Take $n = 256$ and use Matlab to solve equation $(*)$ for $c(z) = 0.95c_{\mathrm{b}}$, for $0 < z < d$. Plot the magnitude of the solution and the magnitude of the incident field on the domain $0 < z < d$.

k) Take $n = 256$ and use Matlab to solve equation $(*)$ for $c(z) = \frac{1}{8}c_{\mathrm{b}}$ for $0 < z < d$. Plot the magnitude of the solution and the magnitude of the incident field on the domain $0 < z < d$.

l) Why do we compare the solution with the incident field in the above cases?