Notes on the Lyapunov–Schmidt–Koiter asymptotic method

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1 Welcome!

These are my notes on the LSK method for the analysis of the stability and bifurcation(s) of a conservative system. These notes are based on several references: the initial PhD thesis of Warner Tjardus Koiter (1945) as well as some graphical illustrations from his lecture notes (W. T. Koiter and Heijden 2009). I enjoyed the concise presentation of Nguyen (2000) as well as the lecture notes of Triantafyllidis (2017). Finally, the chapter by Potier-Ferry (1987) helped me clear some issues.

I hope the reader will find these notes useful, even though there are still a few points which I do not fully understand (they are clearly indicated in the text).

2 Setting-up the mathematical stage

In this chapter, we define the problem mathematically. Symbols and notation are introduced in 2.1, as well as the fundamental assumptions that we make. Then, the *kernel* of the hessian of the energy is introduced in 2.2 as a subspace of the space of admissible states.

2.1 Mathematical setting

The space of admissible states of the system under consideration is denoted U. It has the structure of a real vector space. The energy of the system is $\mathscr{E}(u, \lambda)$, where λ denotes a loading parameter. It is assumed that the fundamental branch of the equilibrium diagram, $u^\star(\lambda)$ is known. Then the energy is stationary with respect to the state u along the whole branch. In other words, for all $\hat{u} \in U$

$$\mathscr{E}_{,u}[u^\star(\lambda), \lambda; \hat{u}] = 0, \tag{2.1}$$

where $\mathscr{E}_{,u}(u, \lambda; \hat{u})$ denotes the (real) value of the differential of the energy $\mathscr{E}$ with respect to the state u, evaluated at (u, λ), for the test function $\hat{u}$. Similarly, evaluation of the second-, third-, etc., order differential of the energy will be denoted $\mathscr{E}_{,uu}(u, \lambda; \hat{u}, \hat{v})$, $\mathscr{E}_{,uuu}(u, \lambda; \hat{u}, \hat{v}, \hat{w})$, etc. It is assumed that a finite value $\lambda_0 > 0$ of λ can be found (critical load), such that

1. $\mathscr{E}_{,uu}[u(\lambda), \lambda] > 0$ for all $0 < \lambda < \lambda_0$,
2. $\mathscr{E}_{,uu}(u_0, \lambda_0) \geq 0$ but $\mathscr{E}_{,uu}(u_0, \lambda_0) \not> 0$,
3. $\mathscr{E}_{,uu}[u(\lambda), \lambda] < 0$ for $\lambda > \lambda_0$, close enough to λ_0.

The load λ_0 will be referred to as the *critical load*; similarly, the state u_0 of the system at the critical load will be referred to as the *critical state*; finally, the pair (u_0, λ_0) is the *critical point* of the system. Assumption 1 implies that equilibria along the fundamental branch are *stable below the critical load*. Conversely, it results from assumption 3 that equilibrium points on the fundamental branch are *unstable above the critical load*. Stability at the critical load is yet undetermined.

The goal of these notes is to analyze *all* equilibrium paths that pass through the critical point (u_0, λ_0).

We introduce the following notations

$$u_0 = u^\star(\lambda_0), \quad \dot{u}_0 = \left.\frac{\mathrm{d}u^\star}{\mathrm{d}\lambda}\right|_{\lambda=\lambda_0}, \quad \ddot{u}_0 = \left.\frac{\mathrm{d}^2u^\star}{\mathrm{d}\lambda^2}\right|_{\lambda=\lambda_0}, \quad \dddot{u}_0 = \ldots, \quad \ddddot{u}_0 = \ldots$$

and

$$\mathscr{E}_2 = \mathscr{E}_{,uu}(u_0, \lambda_0), \quad \mathscr{E}_3 = \mathscr{E}_{,uuu}(u_0, \lambda_0), \quad \mathscr{E}_4 = \mathscr{E}_{,uuuu}(u_0, \lambda_0).$$

Note that $\mathscr{E}_2$, $\mathscr{E}_3$ and $\mathscr{E}_4$ thus defined are bi-, tri- and quadrilinear forms, respectively. The following derivatives are also introduced

$$\dot{\mathscr{E}}_2(\hat{u},\hat{v}) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\mathscr{E}_{,uu}[u^\star(\lambda),\lambda;\hat{u},\hat{v}]\Big|_{\lambda=\lambda_0} = \mathscr{E}_{,uuu}(u_0,\lambda_0;\dot{u}_0,\hat{u},\hat{v}) + \mathscr{E}_{,uu\lambda}(u_0,\lambda_0;\hat{u},\hat{v}) \tag{2.2}$$

$$\begin{aligned}\ddot{\mathscr{E}}_2(\hat{u},\hat{v}) = \frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\mathscr{E}_{,uu}[u^\star(\lambda),\lambda;\hat{u},\hat{v}]\Big|_{\lambda=\lambda_0} &= \mathscr{E}_{,uuuu}(u_0,\lambda_0;\dot{u}_0,\dot{u}_0,\hat{u},\hat{v}) + 2\mathscr{E}_{,uuu\lambda}(u_0,\lambda_0;\dot{u}_0,\hat{u},\hat{v})\\ &\quad + \mathscr{E}_{,uu\lambda\lambda}(u_0,\lambda_0;\hat{u},\hat{v}) + \mathscr{E}_{,uuu}(u_0,\lambda_0,\ddot{u}_0),\end{aligned}$$

and, similarly, $\dot{\mathscr{E}}_3$, $\ddot{\mathscr{E}}_3$, etc.

2.2 Kernel of the hessian of the energy

The *kernel* of the hessian of the energy, $\mathscr{E}_2$, is defined as follows

$$V = \left\{u \in U, \mathscr{E}_2(u,u) = 0\right\}.$$

Since $\mathscr{E}_2$ is a bilinear, symmetric and positive (but not poitive definite!) form, V is a vector subspace of U.

To prove this result, we must show that, for all $u, v \in V$ and $\alpha \in \mathbb{R}$, $w = u + \alpha\, v \in V$. From the bilinearity and symmetry of $\mathscr{E}_2$

$$\mathscr{E}_2(w,w) = \mathscr{E}_2(u+\alpha\,v, u+\alpha\,v) = \mathscr{E}_2(u,u) + 2\alpha\,\mathscr{E}_2(u,v) + \alpha^2\,\mathscr{E}_2(v,v),$$

Since $u, v \in \ker\mathscr{E}_2$, the first and the last term vanish, and the above identity reduces to

$$\mathscr{E}_2(w,w) = 2\alpha\,\mathscr{E}_2(u,v)$$

The bilinear form $\mathscr{E}_2$ is positive, therefore the left-hand side is positive, *for all values of* $\alpha \in \mathbb{R}$. The quantity $\mathscr{E}_2(u,v) = 0$ is necessarily null, and $\mathscr{E}_2(w,w) = 0$, which proves that $w \in V$ and that V is a vector subspace of U. The following characterization of V holds

$$v \in V \quad \Longleftrightarrow \quad \mathscr{E}_2(v,\hat{u}) = 0 \quad \text{for all} \quad \hat{u} \in U. \tag{2.3}$$

Indeed, if for all $\hat{u} \in U$, $\mathscr{E}_2(v,\hat{u}) = 0$, then in particular $\mathscr{E}_2(v,v) = 0$ and $v \in V$. Conversely, let $v \in V$, $\hat{u} \in U$ and $\alpha \in \mathbb{R}$. Similarly to the previous proof, we write that $\mathscr{E}_2(w,w) \geq 0$, with $w = \hat{u} + \alpha\, v$

$$\mathscr{E}_2(w,w) = \mathscr{E}_2(\hat{u},\hat{u}) + 2\alpha\,\mathscr{E}_2(\hat{u},v) + \alpha^2\,\mathscr{E}_2(v,v) = 2\alpha\,\mathscr{E}_2(\hat{u},v) + \mathscr{E}_2(\hat{u},\hat{u}) \geq 0,$$

($\mathscr{E}_2(v,v) = 0$ since $v \in V$). The above expression is of degree 1 in α, with a constant sign. Therefore the linear term in α must vanish: $\mathscr{E}_2(\hat{u},v) = 0$, which proves the characterization (2.3) of V.

It will be assumed in the remainder of these notes the dimension of V is finite: $m = \dim V < +\infty$; m is the *multiplicity* of the critical point. A (finite) basis $(v_1, \dots, v_m)$ of V can therefore be introduced, that is orthonormal in the sense of $\langle \bullet, \bullet \rangle$

$$\langle v_i, v_j \rangle = \delta_{ij} \quad \text{for all} \quad i, j = 1, \dots, m.$$

To close this section, we define the complementary subspace W, orthogonal to V for the scalar product $\langle \bullet, \bullet \rangle$

$$U = V \overset{\perp}{\otimes} W \quad \text{and} \quad \langle \hat{v}, \hat{w} \rangle = 0 \quad \text{for all} \quad \hat{v} \in V \quad \text{and} \quad \hat{w} \in W.$$

2.3 Two canonical variational problems

The bilinear form $\mathscr{E}_2$ is elliptic over W. Therefore, variational problems of the type: find $w \in W$ such that

$$\mathscr{E}_2(w, \hat{w}) + \ell(\hat{w}) = 0 \quad \text{for all} \quad \hat{w} \in W$$

are well-posed for any linear form ℓ over W. In particular, for $\ell = 0$, the unique solution to the variational problem

$$\mathscr{E}_2(w, \hat{w}) = 0 \quad \text{for all} \quad \hat{w} \in W$$

is $w = 0$. For $\ell(\bullet) = \dot{\mathscr{E}}_2(v_i, \bullet)$ and $\ell(\bullet) = \mathscr{E}_3(v_i, v_j, \bullet)$, $w_i \in W$ and $w_{ij} \in W$ are defined as the unique solutions in W of the following variational problems

$$\mathscr{E}_2(w_i, \hat{w}) + \dot{\mathscr{E}}_2(v_i, \hat{w}) = 0 \tag{2.4}$$

and

$$\mathscr{E}_2(w_{ij}, \hat{w}) + \mathscr{E}_3(v_i, v_j, \hat{w}) = 0, \tag{2.5}$$

for all $\hat{w} \in W$. These variational problems (and their solutions) will pop-up recurrently in what follows.

2.4 Additional symbols

We will make use of the following symbols

$$\dot{E}_{ij} = \dot{\mathscr{E}}_2(v_i, v_j), \tag{2.6}$$

$$\ddot{E}_{ij} = \ddot{\mathscr{E}}_2(v_i, v_j) + \dot{\mathscr{E}}_2(v_i, w_j) + \dot{\mathscr{E}}_2(v_j, w_i), \tag{2.7}$$

$$E_{ijk} = \mathscr{E}_3(v_i, v_j, v_k), \tag{2.8}$$

$$\dot{E}_{ijk} = \dot{\mathscr{E}}_3(v_i, v_j, v_k) + \dot{\mathscr{E}}_2(v_i, w_{jk}) + \dot{\mathscr{E}}_2(v_j, w_{ik}) + \dot{\mathscr{E}}_2(v_k, w_{ij}), \tag{2.9}$$

$$\begin{aligned} E_{ijkl} = {} & \mathscr{E}_4(v_i, v_j, v_k, v_l) + \mathscr{E}_3(v_i, v_j, w_{kl}) \\ & + \mathscr{E}_3(v_i, v_k, w_{jl}) + \mathscr{E}_3(v_i, v_l, w_{jk}). \end{aligned} \tag{2.10}$$

These symbols define second, third and fourth-order, fully symmetric, tensors over V.

Consistency of the above definitions

Since $\mathscr{E}_2(v, \bullet) = 0$ for all $v \in V$, definitions (2.6) and (2.8) of $\dot{E}_{ij}$ and E_{ijk} can also be written

$$\dot{E}_{ij} = \dot{\mathscr{E}}_2(v_i, v_j) + \mathscr{E}_2(v_i, w_j) + \mathscr{E}_2(v_j, w_i)$$

and

$$E_{ijk} = \mathscr{E}_3(v_i, v_j, v_k) + \mathscr{E}_2(v_i, w_{jk}) + \mathscr{E}_2(v_j, w_{ik}) + \mathscr{E}_2(v_k, w_{ij}),$$

wich shows the consistency with definitions (2.7) and (2.9) of $\ddot{E}_{ij}$ and $\dot{E}_{ijk}$.

3 Setting-up the computational stage

This chapter lays the ground for all symbolic calculations that are to follow. The SymPy library is imported and initialized in 3.1. Then, the energy of the system is rewritten with a minimum set of parameters.

3.1 Importing all necessary modules

We will use the Python library SymPy for symbolic mathematics and rely also on the IPython library (in particular, IPython.display.Math) for pretty LaTeX output. Some useful functions are defined in the `lsk.display` module, which will be systematically imported. Also, all relevant symbols are defined in the `lsk.symbols` module (see Chapter 9).

Without loss of generality, it will be assumed in all symbolic calculations that $\lambda_0 = 0$ and $u_0 = 0$. The general case $\lambda_0 \neq 0$ and $u_0 \neq 0$ is readily recovered through the substitution $\lambda \leftrightarrow \lambda - \lambda_0$ and $u \leftrightarrow u - u_0$.

The following developments involve the energy $(u, \lambda) \mapsto \mathscr{E}(u, \lambda)$ and its differentials at the critical point (u_0, λ_0), as well as the fundamental path $\lambda \mapsto u^*(\lambda)$ and its derivatives at $\lambda = \lambda_0$. It will therefore be convenient to express $\mathscr{E}$ and $u^\star$ as Taylor expansions with respect to u and λ.

```
from sympy import *
from lsk.display import *
from lsk.symbols import *
```

We start with the Taylor expansion of the energy $\mathscr{E}$. We define its differentials at the critical point. These differentials are stored in a dictionary. Values are indexed with the order of the differentials with respect to u and λ.

```
d = {
    r"\E_{,\lambda}(u_0, \lambda_0)": E_λ,
    r"\E_{,uu}(u_0, \lambda_0)": E2,
    r"\E_{,u\lambda}(u_0, \lambda_0)": E_uλ,
    r"\E_{,\lambda\lambda}(u_0, \lambda_0)": E_λλ,
    r"\E_{,uuu}(u_0, \lambda_0)": E3,
    r"\E_{,uu\lambda}(u_0, \lambda_0)": E_uuλ,
    r"\E_{,u\lambda\lambda}(u_0, \lambda_0)": E_uλλ,
    r"\E_{,\lambda\lambda\lambda}(u_0, \lambda_0)": E_λλλ,
    r"\E_{,uuuu}(u_0, \lambda_0)": E4,
```

```python
    r"\E_{,uuu\lambda}(u_0, \lambda_0)": E_uuuλ,
    r"\E_{,uu\lambda\lambda}(u_0, \lambda_0)": E_uuλλ,
    r"\E_{,u\lambda\lambda\lambda}(u_0, \lambda_0)": E_uλλλ,
    r"\E_{,\lambda\lambda\lambda\lambda}(u_0, \lambda_0)": E_λλλλ,
}

display_latex_dict(d, num_cols=3)
```

$$\begin{array}{rrr}
\mathscr{E}_{,\lambda}(u_0,\lambda_0) = \mathscr{E}_{\lambda} & \mathscr{E}_{,uu}(u_0,\lambda_0) = \mathscr{E}_{2} & \mathscr{E}_{,u\lambda}(u_0,\lambda_0) = \mathscr{E}_{u\lambda} \\
\mathscr{E}_{,\lambda\lambda}(u_0,\lambda_0) = \mathscr{E}_{\lambda\lambda} & \mathscr{E}_{,uuu}(u_0,\lambda_0) = \mathscr{E}_{3} & \mathscr{E}_{,uu\lambda}(u_0,\lambda_0) = \mathscr{E}_{uu\lambda} \\
\mathscr{E}_{,u\lambda\lambda}(u_0,\lambda_0) = \mathscr{E}_{u\lambda\lambda} & \mathscr{E}_{,\lambda\lambda\lambda}(u_0,\lambda_0) = \mathscr{E}_{\lambda\lambda\lambda} & \mathscr{E}_{,uuuu}(u_0,\lambda_0) = \mathscr{E}_{4} \\
\mathscr{E}_{,uuu\lambda}(u_0,\lambda_0) = \mathscr{E}_{uuu\lambda} & \mathscr{E}_{,uu\lambda\lambda}(u_0,\lambda_0) = \mathscr{E}_{uu\lambda\lambda} & \mathscr{E}_{,u\lambda\lambda\lambda}(u_0,\lambda_0) = \mathscr{E}_{u\lambda\lambda\lambda} \\
\mathscr{E}_{,\lambda\lambda\lambda\lambda}(u_0,\lambda_0) = \mathscr{E}_{\lambda\lambda\lambda\lambda} & &
\end{array}$$

where

- $\mathscr{E}_{\lambda}$, $\mathscr{E}_{\lambda\lambda}$, $\mathscr{E}_{\lambda\lambda\lambda}$ and $\mathscr{E}_{\lambda\lambda\lambda\lambda}$ are *scalar quantities*,
- $\mathscr{E}_{u\lambda}$, $\mathscr{E}_{u\lambda\lambda}$ and $\mathscr{E}_{u\lambda\lambda\lambda}$ are *linear forms*,
- $\mathscr{E}_{2}$, $\mathscr{E}_{uu\lambda}$ and $\mathscr{E}_{uu\lambda\lambda}$ are *bilinear forms*,
- $\mathscr{E}_{3}$ and $\mathscr{E}_{uuu\lambda}$ are *trilinear forms*,
- $\mathscr{E}_{4}$ is a *quadrilinear form*.

! Important

Note that all these differentials are defined as SymPy *scalars*. A definition as a SymPy *function* (e.g. `E2 = Function(r"\E_2")`) would be more appropriate. However, SymPy would fail to account for multilinearity or symmetry of these forms. Therefore, we use the following trick: all multi-linear forms are defined as scalars, and the standard multiplication operator `*` means function application. In other words, `E2 * (α * u + β * v) * w` (resp. `E2 * u * v - E2 * v * u`) should be understood as `E2(α * u + β * v, w)` (resp. `E2(u, v) - E2(v, u)`). In both cases, the expressions would are correctly simplified.
Whether the symbols in an expression are true scalars or vectors (elements of U) should be clear from the context. For example, in the expression: `λ * E2 * u * v`, the first `*` is a true multiplication, while the other `*` refer to function application.

The energy $\mathscr{E}(u,\lambda)$ is now expressed as a Taylor expansion about the critical point. We use the function `create_E` that is defined in the `lsk` module. We will get back to the optional parameter `simplify_mixed_derivatives` later.

```python
u, λ = symbols(r"u \lambda")
E = (λ * E_λ + (E2 * u**2 + 2 * λ * E_uλ * u + λ**2 * E_λλ) / 2
     + (E3 * u**3 + 3 * λ * E_uuλ * u**2 + 3 * λ**2 * E_uλλ * u + λ**3 * E_λλλ) / 6
     + (E4 * u**4 + 4 * λ * E_uuuλ * u**3 + 6 * λ**2 * E_uuλλ * u**2
        + 4 * λ**3 * E_uλλλ * u + λ**4 * E_λλλλ) / 24).expand()
```

```
display_latex_long_equation(r"\E(u, \lambda)", E, terms_per_line=7)
```

$$\mathscr{E}(u,\lambda) = \frac{\mathscr{E}_2 u^2}{2} + \frac{\mathscr{E}_3 u^3}{6} + \frac{\mathscr{E}_4 u^4}{24} + \frac{\mathscr{E}_{\lambda\lambda\lambda\lambda}\lambda^4}{24} + \frac{\mathscr{E}_{\lambda\lambda\lambda}\lambda^3}{6} + \frac{\mathscr{E}_{\lambda\lambda}\lambda^2}{2} + \mathscr{E}_\lambda \lambda$$
$$+ \frac{\mathscr{E}_{u\lambda\lambda\lambda}\lambda^3 u}{6} + \frac{\mathscr{E}_{u\lambda\lambda}\lambda^2 u}{2} + \mathscr{E}_{u\lambda}\lambda u + \frac{\mathscr{E}_{uu\lambda\lambda}\lambda^2 u^2}{4} + \frac{\mathscr{E}_{uu\lambda}\lambda u^2}{2} + \frac{\mathscr{E}_{uuu\lambda}\lambda u^3}{6}$$

The fundamental path $\lambda \mapsto u^\star(\lambda)$ is also defined through its Taylor expansion.

```
u_star = (λ * u0_dot
          + λ**2 * u0_ddot / 2
          + λ**3 * u0_dddot / 6
          + λ**4 * u0_ddddot / 24)
```

```
display_latex_equation(r"u^\star(\lambda)", u_star)
```

$$u^\star(\lambda) = \frac{\ddddot{u}_0 \lambda^4}{24} + \frac{\dddot{u}_0 \lambda^3}{6} + \frac{\ddot{u}_0 \lambda^2}{2} + \dot{u}_0 \lambda$$

Where $\dot{u}_0$, $\ddot{u}_0$, etc denote the derivatives of $u^\star$ with respect to λ, at $\lambda = \lambda_0$.

```
d = {f"\\frac{{\\D^{k}u^\\star}}{{\\D \\lambda^{k}}}"
     "\\biggr \\rvert_{{\\lambda=\\lambda_0}}" : x
     for k, x in enumerate([u0_dot, u0_ddot, u0_dddot, u0_ddddot], start=1)}
display_latex_dict(d, num_cols=4)
```

$$\frac{\mathrm{d}^1 u^\star}{\mathrm{d}\lambda^1}\biggr\rvert_{\lambda=\lambda_0} = \dot{u}_0 \quad \frac{\mathrm{d}^2 u^\star}{\mathrm{d}\lambda^2}\biggr\rvert_{\lambda=\lambda_0} = \ddot{u}_0 \quad \frac{\mathrm{d}^3 u^\star}{\mathrm{d}\lambda^3}\biggr\rvert_{\lambda=\lambda_0} = \dddot{u}_0 \quad \frac{\mathrm{d}^4 u^\star}{\mathrm{d}\lambda^4}\biggr\rvert_{\lambda=\lambda_0} = \ddddot{u}_0$$

3.2 Elimination of the derivatives of the jacobian w.r.t. λ

Since the fundamental path $\lambda \mapsto u^\star(\lambda)$ is an equilibrium path, the various differentials of the energy at the critical point are not linearly independent. To express the relationships between these forms, we define $\mathscr{R}^\star(\lambda; \bullet)$ as the jacobian of the energy along the fundamental path $u^\star(\lambda)$

$$\mathscr{R}^\star(\lambda;\bullet) = \mathscr{E}_{,u}[u^\star(\lambda), \lambda; \bullet].$$

Combining the expansions of $\lambda \mapsto u^\star(\lambda)$ and $(u,\lambda) \mapsto \mathscr{E}(u,\lambda)$ delivers and expansion of $\mathscr{R}^\star$ with respect to the powers of λ, up to the fourth order.

```
R_star = (E.diff(u) * u_hat).subs(u, u_star).series(λ, 0, 5).removeO().expand()
```

```
display_latex_long_equation(r"\mathcal{R}^\ast(\lambda;"
                            + sympy.latex(u_hat) + ")",
```

```
                       R_star, terms_per_line=4)
```

$$\begin{aligned}\mathscr{R}^*(\lambda;\hat{u}) = \mathscr{E}_2\dot{u}_0\hat{u}\lambda + \frac{\mathscr{E}_{u\lambda\lambda\lambda}\hat{u}\lambda^3}{6} + \frac{\mathscr{E}_{u\lambda\lambda}\hat{u}\lambda^2}{2} + \mathscr{E}_{u\lambda}\hat{u}\lambda \\ + \frac{\mathscr{E}_2\ddot{u}_0\hat{u}\lambda^2}{2} + \frac{\mathscr{E}_3\dot{u}_0^2\hat{u}\lambda^2}{2} + \frac{\mathscr{E}_{uu\lambda\lambda}\dot{u}_0\hat{u}\lambda^3}{2} + \mathscr{E}_{uu\lambda}\dot{u}_0\hat{u}\lambda^2 \\ + \frac{\mathscr{E}_2\dddot{u}_0\hat{u}\lambda^3}{6} + \frac{\mathscr{E}_{uu\lambda\lambda}\ddot{u}_0\hat{u}\lambda^4}{4} + \frac{\mathscr{E}_{uu\lambda}\ddot{u}_0\hat{u}\lambda^3}{2} + \frac{\mathscr{E}_{uuu\lambda}\dot{u}_0^2\hat{u}\lambda^3}{2} \\ + \frac{\mathscr{E}_2\ddddot{u}_0\hat{u}\lambda^4}{24} + \frac{\mathscr{E}_3\ddot{u}_0^2\hat{u}\lambda^4}{8} + \frac{\mathscr{E}_4\dot{u}_0^3\hat{u}\lambda^3}{6} + \frac{\mathscr{E}_{uu\lambda}\dddot{u}_0\hat{u}\lambda^4}{6} \\ + \frac{\mathscr{E}_3\dddot{u}_0\dot{u}_0\hat{u}\lambda^4}{6} + \frac{\mathscr{E}_3\ddot{u}_0\dot{u}_0\hat{u}\lambda^3}{2} + \frac{\mathscr{E}_4\ddot{u}_0\dot{u}_0^2\hat{u}\lambda^4}{4} + \frac{\mathscr{E}_{uuu\lambda}\ddot{u}_0\dot{u}_0\hat{u}\lambda^4}{2}\end{aligned}$$

Of course, since $\lambda \mapsto u^*(\lambda)$ is an equilibrium path, we have $\mathscr{R}^*(\lambda;\bullet) = 0$ for all λ. Therefore, all coefficients of the above polynomial in λ are null, which delivers expressions of $\mathscr{E}_{u\lambda}$, $\mathscr{E}_{u\lambda\lambda}$ and $\mathscr{E}_{u\lambda\lambda\lambda}$. Each term is analyzed in term below. Expressions of the mixed derivatives are to be stored in the `mixed1` dictionary.

```
mixed1 = dict()
```

3.2.1 The term of order 0

This term is uniformly null and therefore delivers no informations.

```
assert R_star.coeff(λ, 0) == 0
```

3.2.2 The term of order 1

This term delivers the following equation

```
eq = Eq(R_star.coeff(λ, 1), 0)
```

```
display(eq)
```

$\mathscr{E}_2\dot{u}_0\hat{u} + \mathscr{E}_{u\lambda}\hat{u} = 0$

for all $\hat{u} \in U$. This equation delivers the following expression of $\mathscr{E}_{,u\lambda}(u_0,\lambda_0)$

```
sol = solve(eq, E_uλ)
mixed1[E_uλ] = sol[0]
```

```
display_latex_equation(E_uλ, mixed1[E_uλ])
```

$\mathscr{E}_{u\lambda} = -\mathscr{E}_2 \dot{u}_0$

3.2.3 The term of order 2

```
eq = Eq(R_star.coeff(λ, 2).subs(mixed1), 0)
```

```
display(eq)
```

$$\frac{\mathscr{E}_2 \ddot{u}_0 \hat{u}}{2} + \frac{\mathscr{E}_3 \dot{u}_0^2 \hat{u}}{2} + \frac{\mathscr{E}_{u\lambda\lambda} \hat{u}}{2} + \mathscr{E}_{uu\lambda} \dot{u}_0 \hat{u} = 0$$

for all $\hat{u} \in U$. This equation delivers the following expression of $\mathscr{E}_{,u\lambda\lambda}(u_0, \lambda_0)$

```
sol = solve(eq, E_uλλ)
mixed1[E_uλλ] = sol[0]
```

```
display_latex_equation(E_uλλ, mixed1[E_uλλ])
```

$\mathscr{E}_{u\lambda\lambda} = -\mathscr{E}_2 \ddot{u}_0 - \mathscr{E}_3 \dot{u}_0^2 - 2\mathscr{E}_{uu\lambda} \dot{u}_0$

3.2.4 The term of order 3

```
eq = Eq(R_star.coeff(λ, 3).subs(mixed1), 0).expand()
```

```
display(eq)
```

$$\frac{\mathscr{E}_2 \dddot{u}_0 \hat{u}}{6} + \frac{\mathscr{E}_3 \ddot{u}_0 \dot{u}_0 \hat{u}}{2} + \frac{\mathscr{E}_4 \dot{u}_0^3 \hat{u}}{6} + \frac{\mathscr{E}_{u\lambda\lambda\lambda} \hat{u}}{6} + \frac{\mathscr{E}_{uu\lambda\lambda} \dot{u}_0 \hat{u}}{2} + \frac{\mathscr{E}_{uu\lambda} \ddot{u}_0 \hat{u}}{2} + \frac{\mathscr{E}_{uuu\lambda} \dot{u}_0^2 \hat{u}}{2} = 0$$

for all $\hat{u} \in U$. This equation delivers the following expression of $\mathscr{E}_{,u\lambda\lambda\lambda}(u_0, \lambda_0)$

```
sol = solve(eq, E_uλλλ)
mixed1[E_uλλλ] = sol[0]
```

```
display_latex_equation(E_uλλ, mixed1[E_uλλ])
```

$\mathscr{E}_{u\lambda\lambda} = -\mathscr{E}_2 \ddot{u}_0 - \mathscr{E}_3 \dot{u}_0^2 - 2\mathscr{E}_{uu\lambda} \dot{u}_0$

3.3 Elimination of the remaining mixed derivatives

So far, we have found the following expressions

```
display_latex_dict(mixed1, num_cols=1)
```

$$\mathscr{E}_{u\lambda} = -\mathscr{E}_2\dot{u}_0$$
$$\mathscr{E}_{u\lambda\lambda} = -\mathscr{E}_2\ddot{u}_0 - \mathscr{E}_3\dot{u}_0^2 - 2\mathscr{E}_{uu\lambda}\dot{u}_0$$
$$\mathscr{E}_{u\lambda\lambda\lambda} = -\mathscr{E}_2\dddot{u}_0 - 3\mathscr{E}_3\ddot{u}_0\dot{u}_0 - \mathscr{E}_4\dot{u}_0^3 - 3\mathscr{E}_{uu\lambda\lambda}\dot{u}_0 - 3\mathscr{E}_{uu\lambda}\ddot{u}_0 - 3\mathscr{E}_{uuu\lambda}\dot{u}_0^2$$

We want to get rid of the remaining mixed derivatives, namely: $\mathscr{E}_{uu\lambda}$, $\mathscr{E}_{uuu\lambda}$ and $\mathscr{E}_{uu\lambda\lambda}$. To do so, we introduce the derivatives $\dot{\mathscr{E}}_2$, $\ddot{\mathscr{E}}_2$ and $\dot{\mathscr{E}}_3$ defined in Chapter 2.

```
E_uu_star = E.diff(u, 2).subs(u, u_star).expand()
E_uuu_star = E.diff(u, 3).subs(u, u_star).expand()
mixed2 = dict()
```

The mixed derivative $\mathscr{E}_{uu\lambda}$ can first be expressed as a function of $\dot{\mathscr{E}}_2$.

```
x = E_uuλ
lhs = E2_dot
rhs = E_uu_star.coeff(λ, 1)
```

```
display_latex_equation(lhs, rhs)
```

$$\dot{\mathscr{E}}_2 = \mathscr{E}_3\dot{u}_0 + \mathscr{E}_{uu\lambda}$$

```
sol = solve(Eq(lhs, rhs), x)
mixed2[x] = sol[0]
```

```
display_latex_equation(x, mixed2[x])
```

$$\mathscr{E}_{uu\lambda} = -\mathscr{E}_3\dot{u}_0 + \dot{\mathscr{E}}_2$$

Then, the expression of $\dot{E}_3$ delivers an expression of the mixed derivative $\mathscr{E}_{uuu\lambda}$.

```
x = E_uuuλ
lhs = E3_dot
rhs = E_uuu_star.coeff(λ, 1)
```

```
display_latex_equation(lhs, rhs)
```

$$\dot{\mathscr{E}}_3 = \mathscr{E}_4\dot{u}_0 + \mathscr{E}_{uuu\lambda}$$

```
sol = solve(Eq(lhs, rhs), x)
mixed2[x] = sol[0]
```

```
display_latex_equation(x, mixed2[x])
```

$\mathscr{E}_{uuu\lambda} = -\mathscr{E}_4\dot{u}_0 + \dot{\mathscr{E}}_3$

Finally, $\ddot{\mathscr{E}}_2$ delivers an expression of the mixed derivative $\mathscr{E}_{uu\lambda\lambda}$.

```
x = E_uuλλ
lhs = E2_ddot
rhs = 2 * E_uu_star.coeff(λ, 2).subs(mixed2).expand()
```

```
display_latex_equation(lhs, rhs)
```

$\ddot{\mathscr{E}}_2 = \mathscr{E}_3\ddot{u}_0 - \mathscr{E}_4\dot{u}_0^2 + \mathscr{E}_{uu\lambda\lambda} + 2\dot{\mathscr{E}}_3\dot{u}_0$

```
sol = solve(Eq(lhs, rhs), x)
mixed2[x] = sol[0]
```

```
display_latex_equation(x, mixed2[x])
```

$\mathscr{E}_{uu\lambda\lambda} = -\mathscr{E}_3\ddot{u}_0 + \mathscr{E}_4\dot{u}_0^2 + \ddot{\mathscr{E}}_2 - 2\dot{\mathscr{E}}_3\dot{u}_0$

3.4 Summary: final expression of the energy

The following expressions were derived in 3.2

```
display_latex_dict(mixed1, num_cols=1)
```

$$\begin{aligned}
\mathscr{E}_{u\lambda} &= -\mathscr{E}_2\dot{u}_0\\
\mathscr{E}_{u\lambda\lambda} &= -\mathscr{E}_2\ddot{u}_0 - \mathscr{E}_3\dot{u}_0^2 - 2\mathscr{E}_{uu\lambda}\dot{u}_0\\
\mathscr{E}_{u\lambda\lambda\lambda} &= -\mathscr{E}_2\dddot{u}_0 - 3\mathscr{E}_3\ddot{u}_0\dot{u}_0 - \mathscr{E}_4\dot{u}_0^3 - 3\mathscr{E}_{uu\lambda\lambda}\dot{u}_0 - 3\mathscr{E}_{uu\lambda}\ddot{u}_0 - 3\mathscr{E}_{uuu\lambda}\dot{u}_0^2
\end{aligned}$$

and in 3.3

```
display_latex_dict(mixed2, num_cols=1)
```

$$\begin{aligned}
\mathscr{E}_{uu\lambda} &= -\mathscr{E}_3\dot{u}_0 + \dot{\mathscr{E}}_2\\
\mathscr{E}_{uuu\lambda} &= -\mathscr{E}_4\dot{u}_0 + \dot{\mathscr{E}}_3\\
\mathscr{E}_{uu\lambda\lambda} &= -\mathscr{E}_3\ddot{u}_0 + \mathscr{E}_4\dot{u}_0^2 + \ddot{\mathscr{E}}_2 - 2\dot{\mathscr{E}}_3\dot{u}_0
\end{aligned}$$

Combining the above results allows to fully eliminate the mixed derivatives

```
mixed = {k: v.subs(mixed2).expand() for k, v in mixed1.items()}
mixed.update(mixed2)

display_latex_dict(mixed, num_cols=1)
```

$$
\begin{aligned}
\mathscr{E}_{u\lambda} &= -\mathscr{E}_2\dot{u}_0 \\
\mathscr{E}_{u\lambda\lambda} &= -\mathscr{E}_2\ddot{u}_0 + \mathscr{E}_3\dot{u}_0^2 - 2\dot{\mathscr{E}}_2\dot{u}_0 \\
\mathscr{E}_{u\lambda\lambda\lambda} &= -\mathscr{E}_2\dddot{u}_0 + 3\mathscr{E}_3\ddot{u}_0\dot{u}_0 - \mathscr{E}_4\dot{u}_0^3 - 3\ddot{\mathscr{E}}_2\dot{u}_0 - 3\ddot{u}_0\dot{\mathscr{E}}_2 + 3\dot{\mathscr{E}}_3\dot{u}_0^2 \\
\mathscr{E}_{uu\lambda} &= -\mathscr{E}_3\dot{u}_0 + \dot{\mathscr{E}}_2 \\
\mathscr{E}_{uuu\lambda} &= -\mathscr{E}_4\dot{u}_0 + \dot{\mathscr{E}}_3 \\
\mathscr{E}_{uu\lambda\lambda} &= -\mathscr{E}_3\ddot{u}_0 + \mathscr{E}_4\dot{u}_0^2 + \ddot{\mathscr{E}}_2 - 2\dot{\mathscr{E}}_3\dot{u}_0
\end{aligned}
$$

These expressions can be plugged into the expansion of the energy.

```
E = E.subs(mixed).expand()
```

```
display_latex_long_equation(r"\E(u, \lambda)", E, terms_per_line=5)
```

$$
\begin{aligned}
\mathscr{E}(u,\lambda) = {} & \frac{\mathscr{E}_2u^2}{2} + \frac{\mathscr{E}_3u^3}{6} + \frac{\mathscr{E}_{\lambda\lambda\lambda}\lambda^3}{6} + \frac{\mathscr{E}_{\lambda\lambda}\lambda^2}{2} + \mathscr{E}_\lambda\lambda \\
& + \frac{\mathscr{E}_4u^4}{24} + \frac{\mathscr{E}_{\lambda\lambda\lambda\lambda}\lambda^4}{24} + \frac{\ddot{\mathscr{E}}_2\lambda^2u^2}{4} + \frac{\dot{\mathscr{E}}_2\lambda u^2}{2} + \frac{\dot{\mathscr{E}}_3\lambda u^3}{6} \\
& + -\frac{\mathscr{E}_2\ddot{u}_0\lambda^2u}{2} - \mathscr{E}_2\dot{u}_0\lambda u + \frac{\mathscr{E}_3\dot{u}_0^2\lambda^2u}{2} - \dot{\mathscr{E}}_2\dot{u}_0\lambda^2u + \frac{\dot{\mathscr{E}}_3\dot{u}_0^2\lambda^3u}{2} \\
& + -\frac{\mathscr{E}_3\ddot{u}_0\lambda^2u^2}{4} - \frac{\mathscr{E}_3\dot{u}_0\lambda u^2}{2} - \frac{\ddot{\mathscr{E}}_2\dot{u}_0\lambda^3u}{2} - \frac{\ddot{u}_0\dot{\mathscr{E}}_2\lambda^3u}{2} - \frac{\dot{\mathscr{E}}_3\dot{u}_0\lambda^2u^2}{2} \\
& + -\frac{\mathscr{E}_2\dddot{u}_0\lambda^3u}{6} + \frac{\mathscr{E}_3\ddot{u}_0\dot{u}_0\lambda^3u}{2} - \frac{\mathscr{E}_4\dot{u}_0^3\lambda^3u}{6} + \frac{\mathscr{E}_4\dot{u}_0^2\lambda^2u^2}{4} - \frac{\mathscr{E}_4\dot{u}_0\lambda u^3}{6}
\end{aligned}
$$

From which we deduce the expression of the residual $\mathscr{E}_{,u}$

```
E_u = E.diff(u)
display_latex_long_equation(r"\E_{,u}(u, \lambda)", E_u, terms_per_line=5)
```

$$
\begin{aligned}
\mathscr{E}_{,u}(u,\lambda) = {} & \mathscr{E}_2u + \frac{\mathscr{E}_3\dot{u}_0^2\lambda^2}{2} + \frac{\mathscr{E}_3u^2}{2} + \frac{\mathscr{E}_4u^3}{6} + \dot{\mathscr{E}}_2\lambda u \\
& + -\mathscr{E}_2\dot{u}_0\lambda + \frac{\ddot{\mathscr{E}}_2\lambda^2u}{2} - \dot{\mathscr{E}}_2\dot{u}_0\lambda^2 + \frac{\dot{\mathscr{E}}_3\dot{u}_0^2\lambda^3}{2} + \frac{\dot{\mathscr{E}}_3\lambda u^2}{2} \\
& + -\frac{\mathscr{E}_2\dddot{u}_0\lambda^3}{6} - \frac{\mathscr{E}_2\ddot{u}_0\lambda^2}{2} - \frac{\mathscr{E}_4\dot{u}_0^3\lambda^3}{6} - \frac{\ddot{\mathscr{E}}_2\dot{u}_0\lambda^3}{2} - \frac{\ddot{u}_0\dot{\mathscr{E}}_2\lambda^3}{2} \\
& + \frac{\mathscr{E}_3\ddot{u}_0\dot{u}_0\lambda^3}{2} - \frac{\mathscr{E}_3\ddot{u}_0\lambda^2u}{2} - \mathscr{E}_3\dot{u}_0\lambda u + \frac{\mathscr{E}_4\dot{u}_0^2\lambda^2u}{2} - \dot{\mathscr{E}}_3\dot{u}_0\lambda^2u \\
& + -\frac{\mathscr{E}_4\dot{u}_0\lambda u^2}{2}
\end{aligned}
$$

```
E_uu = E.diff(u, 2)
display_latex_long_equation(r"\E_{,uu}(u, \lambda)", E_uu, terms_per_line=7)
```

$$\mathscr{E}_{,uu}(u,\lambda) = \mathscr{E}_2 + \mathscr{E}_3 u + \frac{\mathscr{E}_4 \dot{u}_0^2 \lambda^2}{2} + \frac{\mathscr{E}_4 u^2}{2} + \frac{\ddot{\mathscr{E}}_2 \lambda^2}{2} + \dot{\mathscr{E}}_2 \lambda + \dot{\mathscr{E}}_3 \lambda u$$
$$+ -\frac{\mathscr{E}_3 \ddot{u}_0 \lambda^2}{2} - \mathscr{E}_3 \dot{u}_0 \lambda - \mathscr{E}_4 \dot{u}_0 \lambda u - \dot{\mathscr{E}}_3 \dot{u}_0 \lambda^2$$

3.5 Implementation in the `lsk.energy` module

This module exposes three functions

- `create_E(u, λ)` : asymptotic expansion of the energy $\mathscr{E}(u,\lambda)$,
- `create_E_u(u, λ)` : asymptotic expansion of the jacobian $\mathscr{E}_{,u}(u,\lambda)$,
- `create_E_u(u, λ)` : asymptotic expansion of the hessian $\mathscr{E}_{,uu}(u,\lambda)$,
- `create_u_star(λ)` : asymptotic expansion of the fundamental branch $u^\star(\lambda)$,

where u and λ must be `SymPy` expressions.

```
import lsk.energy

%psource lsk.energy
```

```
from lsk.symbols import *

__mixed_derivatives = {
    E_uλ : -E2 * u0_dot,
    E_uλλ : -E2 * u0_ddot - 2 * E2_dot * u0_dot + E3 * u0_dot**2,
    E_uλλλ : (
        -E2 * u0_dddot
        - 3 * E2_dot * u0_ddot
        - 3 * E2_ddot * u0_dot
        + 3 * E3 * u0_dot * u0_ddot
        + 3 * E3_dot * u0_dot**2
        - E4 * u0_dot**3
    ),
    E_uuλ : E2_dot - E3 * u0_dot,
    E_uuλλ : E4 * u0_dot**2 - 2 * E3_dot * u0_dot - E3 * u0_ddot + E2_ddot,
    E_uuuλ : E3_dot - E4 * u0_dot,
}


def create_E(u, λ):
    out = (λ * E_λ + (E2 * u**2 + 2 * λ * E_uλ * u + λ**2 * E_λλ) / 2
           + (E3 * u**3 + 3 * λ * E_uuλ * u**2 + 3 * λ**2 * E_uλλ * u
              + λ**3 * E_λλλ) / 6
           + (E4 * u**4 + 4 * λ * E_uuuλ * u**3 + 6 * λ**2 * E_uuλλ * u**2
              + 4 * λ**3 * E_uλλλ * u + λ**4 * E_λλλλ) / 24)
```

```
    return out.subs(__mixed_derivatives).expand()


def create_E_u(u, λ):
    out = (E2 * u + λ * E_uλ
           + (E3 * u**2 + 2 * λ * E_uuλ * u + λ**2 * E_uλλ) / 2
           + (E4 * u**3 + 3 * λ * E_uuuλ * u**2 + 3 * λ**2 * E_uuλλ * u
              + λ**3 * E_uλλλ) / 6)
    return out.subs(__mixed_derivatives).expand()


def create_E_uu(u, λ):
    out =  (E2 + E3 * u + λ * E_uuλ
            + (E4 * u**2 + 2 * λ * E_uuuλ * u + λ**2 * E_uuλλ) / 2)
    return out.subs(__mixed_derivatives).expand()


def create_u_star(λ):
    return (λ * u0_dot + λ**2 * u0_ddot / 2 + λ**3 * u0_dddot / 6
            + λ**4 * u0_ddddot / 24).expand()
```

And these functions can be tested against the expressions found above.

```
assert E == lsk.energy.create_E(u, λ)
assert E_u == lsk.energy.create_E_u(u, λ)
assert E_uu == lsk.energy.create_E_uu(u, λ)
```

4 Bifurcation equations

4.1 Introduction

In this chapter, the bifurcation analysis of the perfect system is performed symbolically. It is assumed that through the critical point passes a second equilibrium curve $\lambda \mapsto u(\lambda)$, besides the fundamental branch $\lambda \mapsto u^\star(\lambda)$. We seek an asymptotic expansion of $u(\lambda)$ for $\lambda \to \lambda_0$ (critical load). It will be convenient to introduce an auxiliary parametrization η such that the bifurcated branch is defined as the set of points $(u(\eta), \lambda(\eta))$. The functions $\eta \mapsto \lambda(\eta)$ and $\eta \mapsto u(\eta)$ are expanded as follows

$$\lambda(\eta) = \lambda^{(0)} + \eta\,\lambda^{(1)} + \tfrac{1}{2}\eta^2\,\lambda^{(2)} + \ldots \tag{4.1}$$

and

$$u(\eta) = u^*[\lambda(\eta)] + \eta\,u^{(1)} + \tfrac{1}{2}\eta^2\,u^{(2)} + \ldots \tag{4.2}$$

It will be shown below that

$$u^{(1)} = \xi_i^{(1)}\,v_i \quad \text{and} \quad u^{(2)} = \xi_i^{(2)}\,v_i + \xi_i^{(1)}\,\xi_j^{(1)}\,w_{ij} + 2\lambda^{(1)}\,\xi_i^{(1)}\,w_{i\lambda}, \tag{4.3}$$

where the coefficients $\lambda^{(1)}$, $\lambda^{(2)}$, $\xi_i^{(1)}$ and $\xi_i^{(2)}$ solve the **first bifurcation equation**

$$\tfrac{1}{2}E_{ijk}\,\xi_j^{(1)}\,\xi_k^{(1)} + \lambda^{(1)}\,\dot{E}_{ij}\,\xi_j^{(1)} = 0 \tag{4.4}$$

and the **second bifurcation equation**

$$\begin{aligned}\tfrac{1}{3}E_{ijkl}\,\xi_j^{(1)}\,\xi_k^{(1)}\,\xi_l^{(1)} &+ \lambda^{(1)}\left(\dot{E}_{ijk}\,\xi_k^{(1)} + \lambda^{(1)}\,\ddot{E}_{ij}\right)\xi_j^{(1)} \\ &+\left(E_{ijk}\,\xi_k^{(1)} + \lambda^{(1)}\,\dot{E}_{ij}\right)\xi_j^{(2)} + \lambda^{(2)}\,\dot{E}_{ij}\,\xi_j^{(1)} = 0.\end{aligned} \tag{4.5}$$

The tensors $\dot{E}_{ij}$, $\ddot{E}_{ij}$, E_{ijk}, $\dot{E}_{ijk}$ and E_{ijkl} have been defined in Chapter 2 (see 2.4).

The starting point is the symbolic expression of the residual $(u, \lambda) \mapsto \mathscr{E}_{,u}(u, \lambda)$ that was derived in Chapter 3. We plug the postulated expansions (4.1) and (4.2) into the equilibrium equation

$$\mathscr{E}_{,u}[u(\eta), \lambda(\eta); \hat{u}] = 0 \quad \text{for all} \quad \hat{u} \in U.$$

The coefficients of η^0, η^1, etc deliver a series of variational problems from which $u^{(k)}$ and $\lambda^{(k)}$ are identified.

```
from sympy import *
from lsk.display import *
from lsk.energy import *
from lsk.symbols import *
```

The asymptotic expansion of λ according to Eq. (4.1) is first postulated and plugged into the expression of $u^\star$. The resulting symbolic expressions are combined to define the asymptotic expansion of u according to Eq. (4.2).

```
λ = η * λ1 + η**2 / 2 * λ2 + η**3 / 6 * λ3 + η**4 / 24 * λ4 + O(η**5)
u_star = create_u_star(λ)
u = u_star + η * u1 + η**2 / 2 * u2 + η**3 / 6 * u3 + η**4 / 24 * u4
```

```
display_latex_equation(r"\lambda(\eta)", λ)
display_latex_long_equation(r"u^\star(\eta)", u_star, terms_per_line=6)
display_latex_long_equation(r"u(\eta)", u, terms_per_line=6)
```

$$\lambda(\eta) = \eta\lambda^{(1)} + \frac{\eta^2\lambda^{(2)}}{2} + \frac{\eta^3\lambda^{(3)}}{6} + \frac{\eta^4\lambda^{(4)}}{24} + O\left(\eta^5\right)$$

$$\begin{aligned} u^\star(\eta) = {} & \frac{\dddot{u}_0\eta^3{\lambda^{(1)}}^3}{6} + \frac{\ddot{u}_0\eta^4{\lambda^{(2)}}^2}{8} + \frac{\ddot{u}_0\eta^2{\lambda^{(1)}}^2}{2} + \frac{\dot{u}_0\eta^3\lambda^{(3)}}{6} + \frac{\dot{u}_0\eta^2\lambda^{(2)}}{2} + \dot{u}_0\eta\lambda^{(1)} \\ & + \frac{\dot{u}_0\eta^4\lambda^{(4)}}{24} + \frac{\ddot{u}_0\eta^3\lambda^{(1)}\lambda^{(2)}}{2} + \frac{\ddot{u}_0\eta^4\lambda^{(1)}\lambda^{(3)}}{6} + \frac{\dddot{u}_0\eta^4{\lambda^{(1)}}^2\lambda^{(2)}}{4} + \frac{\ddddot{u}_0\eta^4{\lambda^{(1)}}^4}{24} + O\left(\eta^5\right) \end{aligned}$$

$$\begin{aligned} u(\eta) = {} & \frac{\ddot{u}_0\eta^2{\lambda^{(1)}}^2}{2} + \dot{u}_0\eta\lambda^{(1)} + \frac{\eta^4u^{(4)}}{24} + \frac{\eta^3u^{(3)}}{6} + \frac{\eta^2u^{(2)}}{2} + \eta u^{(1)} \\ & + \frac{\ddddot{u}_0\eta^4{\lambda^{(1)}}^4}{24} + \frac{\dddot{u}_0\eta^3{\lambda^{(1)}}^3}{6} + \frac{\ddot{u}_0\eta^4{\lambda^{(2)}}^2}{8} + \frac{\dot{u}_0\eta^4\lambda^{(4)}}{24} + \frac{\dot{u}_0\eta^3\lambda^{(3)}}{6} + \frac{\dot{u}_0\eta^2\lambda^{(2)}}{2} \\ & + \frac{\ddot{u}_0\eta^3\lambda^{(1)}\lambda^{(2)}}{2} + \frac{\ddot{u}_0\eta^4\lambda^{(1)}\lambda^{(3)}}{6} + \frac{\dddot{u}_0\eta^4{\lambda^{(1)}}^2\lambda^{(2)}}{4} + O\left(\eta^5\right) \end{aligned}$$

These expressions are then used to compute an asymptotic expansion of the residual $\mathscr{E}_{,u}$ along the bifurcated branch.

```
E_u = (create_E_u(u, λ) * u_hat).expand()
```

The general form of this expansion is

$$\begin{aligned} \mathscr{E}_{,u}[u(\eta), \lambda(\eta); \hat{u}] = {} & \mathscr{E}_{,u}^{(0)}(\hat{u}) + \eta\,\mathscr{E}_{,u}^{(1)}(u^{(1)}; \hat{u}) + \tfrac{1}{2}\eta^2\,\mathscr{E}_{,u}^{(2)}(u^{(1)}, u^{(2)}, \lambda^{(1)}; \hat{u}) \\ & + \tfrac{1}{6}\eta^3\,\mathscr{E}_{,u}^{(3)}(u^{(1)}, u^{(2)}, u^{(3)}, \lambda^{(1)}, \lambda^{(2)}; \hat{u}) + \dots, \end{aligned}$$

which delivers the following variational problems

$$\mathscr{E}^{(k)}_{,u}(u^{(1)}, \dots, u^{(k)}, \lambda^{(1)}, \dots, \lambda^{(k-1)}; \hat{u}) \quad \text{for all} \quad \hat{u} \in U, \quad k = 0, 1, 2, \dots$$

These problems are studied successively in the following sections. Note that the variational problem of order 0 is in fact uninformative, since $\mathscr{E}^{(0)}_{,u} = 0$.

```
assert E_u.coeff(η, 0) == 0
```

4.2 The variational problem of order 1

This problem reads

$$\mathscr{E}_2(u^{(1)}, \hat{u}) = 0 \quad \text{for all} \quad \hat{u} \in U.$$

```
assert E_u.coeff(η, 1) == E2 * u1 * u_hat
```

Therefore $u^{(1)} \in V$ and we introduce the following decomposition

$$u^{(1)} = \xi^{(1)}_i v_i, \tag{4.6}$$

where $\xi^{(1)}_1, \dots, \xi^{(1)}_m$ are yet unknown scalars.

4.3 The variational problem of order 2

```
E_u2 = E_u.coeff(η, 2)
```

The term in η^2 of the residual delivers the following variational problem

```
display_latex_equation(E_u2, 0)
```

$$\frac{\mathscr{E}_2 \hat{u} u^{(2)}}{2} + \frac{\mathscr{E}_3 \hat{u} {u^{(1)}}^2}{2} + \dot{\mathscr{E}}_2 \hat{u} \lambda^{(1)} u^{(1)} = 0$$

for all $\hat{u} \in U$. Testing with $\hat{v} \in V$, the term in $\mathscr{E}_2(u^{(2)}, \hat{v})$ vanishes and we get the following variational problem

```
lhs1 = E_u2.subs(u_hat, v_hat).subs(E2 *  v_hat, 0)
```

```
display_latex_equation(lhs1, 0)
```

$$\frac{\mathscr{E}_3 \hat{v} {u^{(1)}}^2}{2} + \dot{\mathscr{E}}_2 \hat{v} \lambda^{(1)} u^{(1)} = 0$$

to be understood as

$$\tfrac{1}{2}\mathscr{E}_3(u^{(1)}, u^{(1)}, \hat{v}) + \lambda^{(1)}\,\dot{\mathscr{E}}_2(u^{(1)}, \hat{v}) = 0, \tag{4.7}$$

for all $\hat{v} \in V$. The above equation fully defines $u^{(1)}$. Indeed, plugging the decomposition (4.6) delivers the equivalent equations

$$\tfrac{1}{2}E_{ijk}\,\xi_j^{(1)}\,\xi_k^{(1)} + \lambda^{(1)}\,\dot{E}_{ij}\,\xi_j^{(1)} = 0,$$

where we have introduced $\dot{E}_{ij}$ and E_{ijk}, defined by Eqs. (2.6) and (2.8), respectively. We finally retrieve the first bifurcation equation (4.4).

We now test the same equation with $\hat{w} \in W$, plugging the expansion (4.6) of $u^{(1)}$

$$\tfrac{1}{2}\mathscr{E}_2(u^{(2)}, \hat{w}) + \tfrac{1}{2}\xi_i^{(1)}\,\xi_j^{(1)}\,\mathscr{E}_3(v_i, v_j, \hat{w}) + \lambda^{(1)}\,\xi_i^{(1)}\,\dot{E}_2(v_i, \hat{w}) = 0.$$

The second-order term $u^{(2)}$ is projected onto V and W

$$u^{(2)} = u_V^{(2)} + u_W^{(2)}, \quad \text{where} \quad u_V^{(2)} = \xi_i^{(2)}\,v_i \in V \quad \text{and} \quad u_W^{(2)} \in W.$$

Plugging this decomposition delivers

$$\tfrac{1}{2}\mathscr{E}_2(u_W^{(2)}, \hat{w}) + \tfrac{1}{2}\xi_i^{(1)}\,\xi_j^{(1)}\,\mathscr{E}_3(v_i, v_j, \hat{w}) + \lambda^{(1)}\,\xi_i^{(1)}\,\dot{E}_2(v_i, \hat{w}) = 0.$$

The unique solution to this variational problem can be expressed as a function of $w_{i\lambda}$ and w_{ij}, (see 2.3)

$$u_W^{(2)} = \xi_i^{(1)}\,\xi_j^{(1)}\,w_{ij} + 2\lambda^{(1)}\,\xi_i^{(1)}\,w_{i\lambda},$$

which proves the decomposition (4.3) of $u^{(2)}$.

4.4 The variational problem of order 3

We now turn to the third-order term of the residual $\mathscr{E}_{,u}$. It involves $u^{(3)}$, that gets eliminated when testing with $v_i \in V$.

```
lhs2 = E_u.coeff(η, 3).subs(u_hat, v[i]).subs(E2 * v[i], 0)
```

We get the following equations for $i = 1, \dots, m$

```
display_latex_equation(lhs2, 0)
```

$$\frac{\mathscr{E}_3 u^{(1)} u^{(2)} v_i}{2} + \frac{\mathscr{E}_4 {u^{(1)}}^3 v_i}{6} + \frac{\ddot{\mathscr{E}}_2 {\lambda^{(1)}}^2 u^{(1)} v_i}{2} + \frac{\dot{\mathscr{E}}_2 \lambda^{(1)} u^{(2)} v_i}{2} + \frac{\dot{\mathscr{E}}_2 u^{(1)} \lambda^{(2)} v_i}{2} + \frac{\dot{\mathscr{E}}_3 \lambda^{(1)} {u^{(1)}}^2 v_i}{2} = 0$$

For the terms that involve only v_i and $u^{(1)}$, we use Eqs. (2.6), (2.7) and (2.9) together with the decomposition $u^{(1)} = \xi_i^{(1)} v_i$.

```
d = {
    E2_dot * u1 * v[i]: ξ1[j] * E_dot[i, j],
    E2_ddot * u1 * v[i]: (ξ1[j] * (E_ddot[i, j]
                                   - E2_dot * v[i] * w[j])
                          - E2_dot * u1 * w[i]),
    E3_dot * u1 * u1 * v[i]: (ξ1[j] * ξ1[k] * (E_dot[i, j, k]
                                               - E2_dot * v[i] * w[j, k])
                              - 2 * ξ1[j] * E2_dot * u1 * w[i, j]),
}

lhs2 = lhs2.subs(d).expand()
```

```
display_latex_equation(lhs2, 0)
```

$$\frac{\mathscr{E}_3 u^{(1)} u^{(2)} v_i}{2} + \frac{\mathscr{E}_4 {u^{(1)}}^3 v_i}{6} - \frac{\dot{\mathscr{E}}_2 {\lambda^{(1)}}^2 u^{(1)} w_i}{2} - \frac{\dot{\mathscr{E}}_2 {\lambda^{(1)}}^2 v_i w_j {\xi^{(1)}}_j}{2} - \dot{\mathscr{E}}_2 \lambda^{(1)} u^{(1)} w_{i,j} {\xi^{(1)}}_j + \frac{\dot{\mathscr{E}}_2 \lambda^{(1)} u^{(2)} v_i}{2} - \frac{\dot{\mathscr{E}}_2 \lambda^{(1)} v_i w_{j,k} {\xi^{(1)}}_j {\xi^{(1)}}_k}{2} + \frac{{\lambda^{(1)}}^2 \ddot{E}_{i,j} {\xi^{(1)}}_j}{2} + \frac{\lambda^{(1)} \dot{E}_{i,j,k} {\xi^{(1)}}_j {\xi^{(1)}}_k}{2} + \frac{\lambda^{(2)} \dot{E}_{i,j} {\xi^{(1)}}_j}{2} = 0$$

In the remainder of this section, we simplify this equation. To do so, we rely heavily on Eqs. (2.5) and (2.4) as well as the definitions of 2.4. We first apply the following simplification

$$\dot{\mathscr{E}}_2(u^{(2)}, v_i) = \dot{E}_{ij}\, \xi_j^{(2)} + \xi_j^{(1)}\, \xi_k^{(1)}\, \dot{\mathscr{E}}_2(w_{jk}, v_i) + 2\lambda^{(1)}\, \dot{\mathscr{E}}_2(w_i, u^{(1)}).$$

Proof

$$\begin{aligned}
\mathscr{E}_2(u^{(2)}, v_i) &= \dot{E}_{ij}\,\xi_j^{(2)} + \xi_j^{(1)}\,\xi_k^{(1)}\,\dot{\mathscr{E}}_2(w_{jk}, v_i) + 2\lambda^{(1)}\,\xi_j^{(1)}\,\dot{\mathscr{E}}_2(w_j, v_i)\\
&= \dot{E}_{ij}\,\xi_j^{(2)} + \xi_j^{(1)}\,\xi_k^{(1)}\,\dot{\mathscr{E}}_2(w_{jk}, v_i) - 2\lambda^{(1)}\,\xi_j^{(1)}\,\dot{\mathscr{E}}_2(v_i, w_j)\\
&= \dot{E}_{ij}\,\xi_j^{(2)} + \xi_j^{(1)}\,\xi_k^{(1)}\,\dot{\mathscr{E}}_2(w_{jk}, v_i) + 2\lambda^{(1)}\,\xi_j^{(1)}\,\dot{\mathscr{E}}_2(w_i, v_j)\\
&= \dot{E}_{ij}\,\xi_j^{(2)} + \xi_j^{(1)}\,\xi_k^{(1)}\,\dot{\mathscr{E}}_2(w_{jk}, v_i) + 2\lambda^{(1)}\,\dot{\mathscr{E}}_2(w_i, u^{(1)}).
\end{aligned}$$

```
expr = (E_dot[i, j] * ξ2[j] + ξ1[j] * ξ1[k] * E2_dot * v[i] * w[j, k]
        + 2 * λ1 * E2_dot * u1 * w[i])

lhs2 = lhs2.subs(E2_dot * u2 * v[i], expr).expand()
```

```
display_latex_equation(lhs2, 0)
```

$$\frac{\mathscr{E}_3 u^{(1)} u^{(2)} v_i}{2} + \frac{\mathscr{E}_4 {u^{(1)}}^3 v_i}{6} + \frac{\dot{\mathscr{E}}_2 {\lambda^{(1)}}^2 u^{(1)} w_i}{2} - \frac{\dot{\mathscr{E}}_2 {\lambda^{(1)}}^2 v_i w_j {\xi^{(1)}}_j}{2} - \dot{\mathscr{E}}_2 \lambda^{(1)} u^{(1)} w_{i,j} {\xi^{(1)}}_j + \frac{{\lambda^{(1)}}^2 \ddot{E}_{i,j} {\xi^{(1)}}_j}{2} + \frac{\lambda^{(1)} \dot{E}_{i,j,k} {\xi^{(1)}}_j {\xi^{(1)}}_k}{2} + \frac{\lambda^{(1)} \dot{E}_{i,j} {\xi^{(2)}}_j}{2} + \frac{\lambda^{(2)} \dot{E}_{i,j} {\xi^{(1)}}_j}{2} = 0$$

Then,

$$\begin{aligned}
\mathscr{E}_3(u^{(1)}, u^{(2)}, v_i) = {}& E_{ijk}\,\xi_j^{(2)}\,\xi_k^{(1)} + \tfrac{1}{3}\xi_j^{(1)}\,\xi_k^{(1)}\,\xi_l^{(1)}\,E_{ijkl}\\
&- \tfrac{1}{3}\mathscr{E}_4(u^{(1)}, u^{(1)}, u^{(1)}, v_i) + 2\lambda^{(1)}\,\xi_j^{(1)}\,\dot{\mathscr{E}}_2(u^{(1)}, w_{ij}).
\end{aligned}$$

Proof

$$\begin{aligned}
\mathscr{E}_3(u^{(1)}, u^{(2)}, v_i) &= \xi_j^{(2)}\,\xi_k^{(1)}\,\mathscr{E}_3(v_i, v_j, v_k) + \xi_j^{(1)}\,\xi_k^{(1)}\,\xi_l^{(1)}\,\mathscr{E}_3(v_i, v_j, w_{kl})\\
&\quad + 2\lambda^{(1)}\,\xi_j^{(2)}\,\xi_k^{(1)}\,\mathscr{E}_3(v_i, v_j, w_k)\\
&= E_{ijk}\,\xi_j^{(2)}\,\xi_k^{(1)}\\
&\quad + \tfrac{1}{3}\xi_j^{(1)}\,\xi_k^{(1)}\,\xi_l^{(1)}\left[\mathscr{E}_3(v_i, v_j, w_{kl}) + \mathscr{E}_3(v_i, v_k, w_{jl}) + \mathscr{E}_3(v_i, v_l, w_{jk})\right]\\
&\quad - 2\lambda^{(1)}\,\xi_j^{(1)}\,\xi_k^{(1)}\,\dot{\mathscr{E}}_2(w_{ij}, w_k)\\
&= E_{ijk}\,\xi_j^{(2)}\,\xi_k^{(1)} + \tfrac{1}{3}\xi_j^{(1)}\,\xi_k^{(1)}\,\xi_l^{(1)}\left[E_{ijkl} - \mathscr{E}_4(v_i, v_j, v_k, v_l)\right]\\
&\quad + 2\lambda^{(1)}\,\xi_j^{(1)}\,\xi_k^{(1)}\,\dot{\mathscr{E}}_2(w_{ij}, v_k)
\end{aligned}$$

and the above identity is retrieved.

```python
expr = (E[i, j, k] * ξ2[j] * ξ1[k]
        + E[i, j, k, l] * ξ1[j] * ξ1[k] * ξ1[l] / 3
        - E4 * v[i] * u1**3 / 3
        + 2 * λ1 * ξ1[j] * E2_dot * u1 * w[i, j])

lhs2 = lhs2.subs(E3 * u1 * u2 * v[i], expr).expand()
```

```python
display_latex_equation(lhs2, 0)
```

$$\frac{\dot{\mathscr{E}}_2 {\lambda^{(1)}}^2 u^{(1)} w_i}{2} - \frac{\dot{\mathscr{E}}_2 {\lambda^{(1)}}^2 v_i w_j {\xi^{(1)}}_j}{2} + \frac{{\lambda^{(1)}}^2 \ddot{E}_{i,j} {\xi^{(1)}}_j}{2} + \frac{\lambda^{(1)} \dot{E}_{i,j,k} {\xi^{(1)}}_j {\xi^{(1)}}_k}{2} + \frac{\lambda^{(1)} \dot{E}_{i,j} {\xi^{(2)}}_j}{2} + \frac{\lambda^{(2)} \dot{E}_{i,j} {\xi^{(1)}}_j}{2} + \frac{E_{i,j,k,l} {\xi^{(1)}}_j {\xi^{(1)}}_k {\xi^{(1)}}_l}{6} + \frac{E_{i,j,k} {\xi^{(1)}}_k {\xi^{(2)}}_j}{2} = 0$$

Finally,

$$\xi_j^{(1)}\, \dot{\mathscr{E}}_2(w_j, v_i) = -\xi_j^{(1)}\, \dot{\mathscr{E}}_2(w_i, w_j) = \xi_j^{(1)}\, \dot{\mathscr{E}}_2(v_j, w_i) = \dot{\mathscr{E}}_2(u^{(1)}, w_i).$$

```python
lhs2 = lhs2.subs(ξ1[j] * E2_dot * v[i] * w[j], E2_dot * u1 * w[i])
```

```python
display_latex_equation(lhs2, 0)
```

$$\frac{{\lambda^{(1)}}^2 \ddot{E}_{i,j} {\xi^{(1)}}_j}{2} + \frac{\lambda^{(1)} \dot{E}_{i,j,k} {\xi^{(1)}}_j {\xi^{(1)}}_k}{2} + \frac{\lambda^{(1)} \dot{E}_{i,j} {\xi^{(2)}}_j}{2} + \frac{\lambda^{(2)} \dot{E}_{i,j} {\xi^{(1)}}_j}{2} + \frac{E_{i,j,k,l} {\xi^{(1)}}_j {\xi^{(1)}}_k {\xi^{(1)}}_l}{6} + \frac{E_{i,j,k} {\xi^{(1)}}_k {\xi^{(2)}}_j}{2} = 0$$

And the second bifurcation equation (4.5) is retrieved.

```python
expected = (E[i, j, k, l] * ξ1[j] * ξ1[k] * ξ1[l] / 3
            + λ1 * (E_dot[i, j, k] * ξ1[k] + λ1 * E_ddot[i, j]) * ξ1[j]
            + (E[i, j, k] * ξ1[k] + λ1 * E_dot[i, j]) * ξ2[j]
            + λ2 * E_dot[i, j] * ξ1[j])

assert expand(2 * lhs2 - expected) == 0
```

5 Asymptotic expansion of the energy and its hessian

The present chapter is organized as follows. In 5.1, we derive the following asymptotic expansion of the energy along the bifurcated branch

$$
\begin{aligned}
\mathscr{E}[u(\eta), \lambda(\eta)] = \mathscr{E}[u^\star \circ \lambda(\eta), \lambda(\eta)] &+ \tfrac{1}{6}\eta^3\, \lambda^{(1)}\, \xi_i^{(1)}\, \xi_j^{(1)}\, \dot{E}_{ij} \\
&+ \tfrac{1}{24}\eta^4 \Big\{E_{ijkl}\, \xi_i^{(1)}\, \xi_j^{(1)}\, \xi_k^{(1)}\, \xi_l^{(1)} + 4\lambda^{(1)}\, \dot{E}_{ijk}\, \xi_i^{(1)}\, \xi_j^{(1)}\, \xi_k^{(1)} \\
&+ 6\Big[\big(\lambda^{(1)}\big)^2\, \ddot{E}_{ij} + \lambda^{(2)}\, \dot{E}_{ij}\Big]\, \xi_i^{(1)}\, \xi_j^{(1)}\Big\} + o(\eta^4).
\end{aligned} \tag{5.1}
$$

Then, in 5.2, we derive the asymptotic expansion of the hessian of the energy along the bifurcated branch

$$
\begin{aligned}
\mathscr{E}_{,uu}[u(\eta), \lambda(\eta); \bullet, \bullet] = \mathscr{E}_2(\bullet, \bullet) &+ \eta\left[\mathscr{E}_3(u^{(1)}, \bullet, \bullet) + \lambda^{(1)}\, \dot{\mathscr{E}}_2(\bullet, \bullet)\right] \\
&+ \tfrac{1}{2}\eta^2\Big[\mathscr{E}_4(u^{(1)}, u^{(1)}, \bullet, \bullet) + \mathscr{E}_3(u^{(2)}, \bullet, \bullet) \\
&+ 2\lambda^{(1)}\, \dot{\mathscr{E}}_3(u^{(1)}, \bullet, \bullet) + (\lambda^{(1)})^2\, \ddot{\mathscr{E}}_2(\bullet, \bullet) + \lambda^{(2)}\, \dot{\mathscr{E}}_2(\bullet, \bullet)\Big].
\end{aligned} \tag{5.2}
$$

```
from sympy import *
from lsk.display import *
from lsk.energy import *
from lsk.symbols import *
```

As in Chapter 4, λ and $u^\star$ are defined as asymptotic expansions of the powers of η.

```
λ = η * λ1 + η**2 / 2 * λ2 + η**3 / 6 * λ3 + η**4 / 24 * λ4 + O(η**5)
u_star = create_u_star(λ)
```

```
display_latex_equation(r"\lambda(\eta)", λ)
display_latex_long_equation(r"u^\star(\eta)", u_star, terms_per_line=5)
```

$$\lambda(\eta) = \eta\lambda^{(1)} + \frac{\eta^2\lambda^{(2)}}{2} + \frac{\eta^3\lambda^{(3)}}{6} + \frac{\eta^4\lambda^{(4)}}{24} + O\left(\eta^5\right)$$

$$
\begin{aligned}
u^{\star}(\eta) = &\frac{\dddot{u}_0\eta^3{\lambda^{(1)}}^3}{6} + \frac{\ddot{u}_0\eta^2{\lambda^{(1)}}^2}{2} + \frac{\dot{u}_0\eta^3\lambda^{(3)}}{6} + \frac{\ddot{u}_0\eta^2\lambda^{(2)}}{2} + \dot{u}_0\eta\lambda^{(1)} \\
&+ \frac{\ddddot{u}_0\eta^4{\lambda^{(1)}}^4}{24} + \frac{\dddot{u}_0\eta^4{\lambda^{(1)}}^2\lambda^{(2)}}{4} + \frac{\ddot{u}_0\eta^4{\lambda^{(2)}}^2}{8} + \frac{\ddot{u}_0\eta^3\lambda^{(1)}\lambda^{(2)}}{2} + \frac{\dot{u}_0\eta^4\lambda^{(4)}}{24} \\
&+ \frac{\ddot{u}_0\eta^4\lambda^{(1)}\lambda^{(3)}}{6} + O\left(\eta^5\right)
\end{aligned}
$$

The bifurcated branch $u(\eta)$ is also expanded. Moreover, the second-order term, $u^{(2)}$ is expressed as the orthogonal decomposition

$$
u^{(2)} = u_V^{(2)} + u_W^{(2)}, \quad \text{with} \quad u_V^{(2)} \in V \quad \text{and} \quad u_W^{(2)} \in W.
$$

It was in fact shown in Chapter 4 that

$$
u_W^{(2)} = \xi_i^{(1)}\,\xi_j^{(1)}\,w_{ij} + 2\lambda^{(1)}\,\xi_i^{(1)}\,w_i,
$$

while $u_V^{(2)}$ is expanded as follows:

$$
u_V^{(2)} = \xi_i^{(2)}\,v_i.
$$

```
u = expand(u_star + η * u1 + η**2 / 2 * (u2_V + u2_W)
           + η**3 / 6 * u3 + η**4 / 24 * u4)
```

```
display_latex_long_equation(r"u(\eta)", u, terms_per_line=5)
```

$$
\begin{aligned}
u(\eta) = &\frac{\eta^4u^{(4)}}{24} + \frac{\eta^3u^{(3)}}{6} + \frac{\eta^2u_V^{(2)}}{2} + \frac{\eta^2u_W^{(2)}}{2} + \eta u^{(1)} \\
&+ \frac{\dddot{u}_0\eta^3{\lambda^{(1)}}^3}{6} + \frac{\ddot{u}_0\eta^2{\lambda^{(1)}}^2}{2} + \frac{\dot{u}_0\eta^3\lambda^{(3)}}{6} + \frac{\ddot{u}_0\eta^2\lambda^{(2)}}{2} + \dot{u}_0\eta\lambda^{(1)} \\
&+ \frac{\ddddot{u}_0\eta^4{\lambda^{(1)}}^4}{24} + \frac{\dddot{u}_0\eta^4{\lambda^{(1)}}^2\lambda^{(2)}}{4} + \frac{\ddot{u}_0\eta^4{\lambda^{(2)}}^2}{8} + \frac{\ddot{u}_0\eta^3\lambda^{(1)}\lambda^{(2)}}{2} + \frac{\dot{u}_0\eta^4\lambda^{(4)}}{24} \\
&+ \frac{\ddot{u}_0\eta^4\lambda^{(1)}\lambda^{(3)}}{6} + O\left(\eta^5\right)
\end{aligned}
$$

5.1 Asymptotic expansion of the energy

We expand the following quantity at the critical point

$$
\Delta\mathscr{E}(\eta) = \mathscr{E}[u(\eta), \lambda(\eta)] - \mathscr{E}[u^* \circ \lambda(\eta), \lambda(\eta)],
$$

where η is the parametrization of the bifurcated branch introduced in Chapter 4 ($\eta = 0$ at the critical point).

The asymptotic expansion of $\Delta\mathscr{E}$ results from plugging the expansions of $\lambda(\eta)$ and $u(\eta)$ defined above into the general expression of the energy that was derived in Chapter 3.

We then use these expansions to evaluate the energy along the **fundamental** and **bifurcated** branches, as well as the difference $\Delta\mathscr{E}$ of these two quantities. The resulting expressions are too long to be displayed. We first apply some elementary simplifications.

```
ΔE = (create_E(u, λ) - create_E(u_star, λ)).expand().subs({
    E2 * u1 : 0,
    E2 * u2_V : 0
})
```

```
assert ΔE.coeff(η, 0) == 0
assert ΔE.coeff(η, 1) == 0
assert ΔE.coeff(η, 2) == 0
```

Simplification of the third- and fourth- order terms is performed below, first observing that, for all $v \in V$

$$\mathscr{E}_3(u^{(1)}, u^{(1)}, v) = -2\lambda^{(1)}\,\dot{\mathscr{E}}_2(u^{(1)}, v). \tag{5.3}$$

Proof

Let $v = \xi_i\, v_i \in V$

$$\begin{aligned}\mathscr{E}_3(u^{(1)}, u^{(1)}, v) &= \xi_i\,\xi_j^{(1)}\,\xi_k^{(1)}\,\mathscr{E}_3(v_i, v_j, v_k) = E_{ijk}\,\xi_i\,\xi_j^{(1)}\,\xi_k^{(1)}\\ &= -2\lambda^{(1)}\,\dot{E}_{ij}\,\xi_i\,\xi_j^{(1)} = -2\lambda^{(1)}\,\dot{\mathscr{E}}_2(u^{(1)}, v),\end{aligned}$$

where we have used the definitions (2.6) and (2.8) of $\dot{E}_{ij}$ and E_{ijk} and the first bifurcation equation (4.4).

Note

Note that the first bifurcation equation was used to prove Eq. (5.3).

```
d = {
    E3 * u1 * u1 * u1 : -2 * λ1 * E2_dot * u1 * u1,
    E3 * u1 * u1 * u2_V : -2 * λ1 * E2_dot * u1 * u2_V,
}

ΔE = ΔE.subs(d).expand()
```

The subsequent simplifications rely heavily on definitions (2.5) and (2.4) of w_{ij} and w_i, definitions (2.6), (2.7), (2.8), (2.9) and (2.10) of $\dot{E}_{ij}$, $\ddot{E}_{ij}$, E_{ijk}, $\dot{E}_{ijk}$ and E_{ijkl}. Finally, expressions (4.3) of $u^{(1)}$ and $u^{(2)}$ are used.

$$\begin{aligned}
\mathscr{E}_2(u_W^{(2)}, u_W^{(2)}) &= \xi_i^{(1)}\, \xi_j^{(1)}\, \xi_k^{(1)}\, \xi_l^{(1)}\, \mathscr{E}_2(w_{ij}, w_{kl}) + 4\,\lambda^{(1)}\, \xi_i^{(1)}\, \xi_j^{(1)}\, \xi_k^{(1)}\, \mathscr{E}_2(w_{ij}, w_k) \\
&\quad + 4\left(\lambda^{(1)}\right)^2 \xi_i^{(1)}\, \xi_j^{(1)}\, \mathscr{E}_2(w_i, w_j) \\
&= \tfrac{1}{3}\left[\mathscr{E}_4(u^{(1)}, u^{(1)}, u^{(1)}, u^{(1)}) - E_{ijkl}\, \xi_i^{(1)}\, \xi_j^{(1)}\, \xi_k^{(1)}\, \xi_l^{(1)}\right] \\
&\quad + \tfrac{4}{3}\lambda^{(1)}\left[\dot{\mathscr{E}}_3(u^{(1)}, u^{(1)}, u^{(1)}) - \dot{E}_{ijk}\, \xi_i^{(1)}\, \xi_j^{(1)}\, \xi_k^{(1)}\right] \\
&\quad + 2\left(\lambda^{(1)}\right)^2\left[\ddot{\mathscr{E}}_2(u^{(1)}, u^{(1)}) - \ddot{E}_{ij}\, \xi_i^{(1)}\, \xi_j^{(1)}\right]
\end{aligned} \tag{5.4}$$

```
expr = ((E4 * u1 * u1 * u1 * u1
         - E[i, j, k, l] * ξ1[i] * ξ1[j] * ξ1[k] * ξ1[l]) / 3
        + 4 * λ1 / 3 * (E3_dot * u1 * u1 * u1
                        - E_dot[i, j, k] * ξ1[i] * ξ1[j] * ξ1[k])
        + 2 * λ1**2 * (E2_ddot * u1 * u1
                       - E_ddot[i, j] * ξ1[i] * ξ1[j]))

ΔE = ΔE.subs(E2 * u2_W * u2_W, expr).expand()
```

$$\begin{aligned}
\dot{\mathscr{E}}_2(u^{(1)}, u_W^{(2)}) &= \tfrac{1}{3}\left[\dot{E}_{ijk}\, \xi_i^{(1)}\, \xi_j^{(1)}\, \xi_k^{(1)} - \dot{\mathscr{E}}_3(u^{(1)}, u^{(1)}, u^{(1)})\right] \\
&\quad + \lambda^{(1)}\left[\ddot{E}_{ij}\, \xi_i^{(1)}\, \xi_j^{(1)} - \ddot{\mathscr{E}}_2(u^{(1)}, u^{(1)})\right]
\end{aligned}$$

Proof

$$\begin{aligned}
\dot{\mathscr{E}}_2(u^{(1)}, u_W^{(2)}) &= \xi_i^{(1)}\, \xi_j^{(1)}\, \xi_k^{(1)}\, \dot{\mathscr{E}}_2(v_i, w_{jk}) + 2\lambda^{(1)}\, \xi_i^{(1)}\, \xi_j^{(1)}\, \dot{\mathscr{E}}_2(v_i, w_{j\lambda}) \\
&= \tfrac{1}{3}\xi_i^{(1)}\, \xi_j^{(1)}\, \xi_k^{(1)}\left[\dot{\mathscr{E}}_2(v_i, w_{jk}) + \dot{\mathscr{E}}_2(v_j, w_{ik}) + \dot{\mathscr{E}}_2(v_k, w_{ij})\right] \\
&\quad + \lambda^{(1)}\, \xi_i^{(1)}\, \xi_j^{(1)}\left[\dot{\mathscr{E}}_2(v_i, w_{j\lambda}) + \dot{\mathscr{E}}_2(v_j, w_{i\lambda})\right] \\
&= \tfrac{1}{3}\xi_i^{(1)}\, \xi_j^{(1)}\, \xi_k^{(1)}\left[\dot{E}_{ijk} - \dot{\mathscr{E}}_3(v_i, v_j, v_k)\right] \\
&\quad + \lambda^{(1)}\, \xi_i^{(1)}\, \xi_j^{(1)}\left[\ddot{E}_{ij} - \ddot{\mathscr{E}}_2(v_i, v_j)\right]
\end{aligned}$$

```
expr = ((E_dot[i, j, k] * ξ1[i] * ξ1[j] * ξ1[k]
         - E3_dot * u1 * u1 * u1) / 3
        + λ1 * (E_ddot[i, j] * ξ1[i] * ξ1[j]
                - E2_ddot * u1 * u1))

ΔE = ΔE.subs(E2_dot * u1 * u2_W, expr).expand()
```

$$
\begin{aligned}
\mathscr{E}_3(u^{(1)}, u^{(1)}, u_W^{(2)}) &= \xi_i^{(1)}\,\xi_j^{(1)}\,\xi_k^{(1)}\,\xi_l^{(1)}\,\mathscr{E}_3(v_i, v_j, w_{kl}) + 2\lambda^{(1)}\,\xi_i^{(1)}\,\xi_j^{(1)}\,\xi_k^{(1)}\,\mathscr{E}_3(v_i, v_j, w_{k\lambda}) \\
&= \frac{1}{3}\Big[E_{ijkl}\,\xi_i^{(1)}\,\xi_j^{(1)}\,\xi_k^{(1)}\,\xi_l^{(1)} - \mathscr{E}_4(u^{(1)}, u^{(1)}, u^{(1)}, u^{(1)})\Big] \\
&\quad + \frac{2}{3}\lambda^{(1)}\Big[\dot{E}_{ijk}\,\xi_i^{(1)}\,\xi_j^{(1)}\,\xi_k^{(1)} - \dot{\mathscr{E}}_3(u^{(1)}, u^{(1)}, u^{(1)})\Big]
\end{aligned}
$$

```
expr = ((E[i, j, k, l] * ξ1[i] * ξ1[j] * ξ1[k] * ξ1[l]
         - E4 * u1 * u1 * u1 * u1) / 3
        + 2 * λ1 / 3 * (E_dot[i, j, k] * ξ1[i] * ξ1[j] * ξ1[k]
                        - E3_dot * u1 * u1 * u1))

ΔE = ΔE.subs(E3 * u1 * u1 * u2_W, expr).expand()
```

The energy difference $\Delta\mathscr{E}$ is finally reordered as follows

$$
\Delta\mathscr{E} = \frac{1}{6}\eta^3\,\Delta\mathscr{E}^{(3)} + \frac{1}{24}\eta^4\,\Delta\mathscr{E}^{(4)},
$$

with

$$
\Delta\mathscr{E}^{(3)} = \lambda^{(1)}\,\dot{\mathscr{E}}_2(u^{(1)}, u^{(1)}) = \lambda^{(1)}\,\dot{E}_{ij}\,\xi_i^{(1)}\,\xi_j^{(1)}
$$

and

$$
\begin{aligned}
\Delta\mathscr{E}^{(4)} &= E_{ijkl}\,\xi_i^{(1)}\,\xi_j^{(1)}\,\xi_k^{(1)}\,\xi_l^{(1)} + 4\lambda^{(1)}\,\dot{E}_{ijk}\,\xi_i^{(1)}\,\xi_j^{(1)}\,\xi_k^{(1)} \\
&\quad + 6\Big[\big(\lambda^{(1)}\big)^2\,\ddot{E}_{ij} + \lambda^{(2)}\,\dot{E}_{ij}\Big]\xi_i^{(1)}\,\xi_j^{(1)}.
\end{aligned} \tag{5.5}
$$

and the asymptotic expansion (5.1) is retrieved.

```
ΔE3 = expand(6 * ΔE.coeff(η, 3))
ΔE4 = expand(24 * ΔE.coeff(η, 4))
```

```
display_latex_equation("\order[3]{\Delta\E}", ΔE3)
display_latex_equation("\order[4]{\Delta\E}", ΔE4)

assert ΔE3 == λ1 * E2_dot * u1 * u1

expected = (E[i, j, k, l] * ξ1[i] * ξ1[j] * ξ1[k] * ξ1[l]
            + 4 * λ1 * E_dot[i, j, k] * ξ1[i] * ξ1[j] * ξ1[k]
            + 6 * (λ1**2 * E_ddot[i, j] * ξ1[i] * ξ1[j]
                   + λ2 * E2_dot * u1 * u1)).expand()

assert ΔE4 == expected
```

$\Delta\mathscr{E}^{(3)} = \dot{\mathscr{E}}_2 \lambda^{(1)} u^{(1)^2}$

$\Delta\mathscr{E}^{(4)} = 6\dot{\mathscr{E}}_2 u^{(1)^2} \lambda^{(2)} + 6\lambda^{(1)^2} \ddot{E}_{i,j} \xi^{(1)}{}_i \xi^{(1)}{}_j + 4\lambda^{(1)} \dot{E}_{i,j,k} \xi^{(1)}{}_i \xi^{(1)}{}_j \xi^{(1)}{}_k + E_{i,j,k,l} \xi^{(1)}{}_i \xi^{(1)}{}_j \xi^{(1)}{}_k \xi^{(1)}{}_l$

5.2 Asymptotic expansion of the hessian of the energy

We now turn to the hessian of the energy, which is expanded to second order in η

$$\mathscr{E}_{,uu}[u(\eta), \lambda(\eta)] = \mathscr{H}^{(0)} + \eta\, \mathscr{H}^{(1)} + \frac{1}{2}\eta^2\, \mathscr{H}^{(2)}$$

```
E_uu = create_E_uu(u, λ).expand() + O(η**3)
```

```
H0 = E_uu.coeff(η, 0)
H1 = E_uu.coeff(η, 1)
H2 = expand(2 * E_uu.coeff(η, 2))
```

```
d = {
    r"\order[0]{\mathcal{H}}": H0,
    r"\order[1]{\mathcal{H}}": H1,
    r"\order[2]{\mathcal{H}}": H2,
    }

display_latex_dict(d, num_cols=1)
```

$\mathscr{H}^{(0)} = \mathscr{E}_2$

$\mathscr{H}^{(1)} = \mathscr{E}_3 u^{(1)} + \dot{\mathscr{E}}_2 \lambda^{(1)}$

$\mathscr{H}^{(2)} = \mathscr{E}_3 u_V^{(2)} + \mathscr{E}_3 u_W^{(2)} + \mathscr{E}_4 u^{(1)^2} + \ddot{\mathscr{E}}_2 \lambda^{(1)^2} + \dot{\mathscr{E}}_2 \lambda^{(2)} + 2\dot{\mathscr{E}}_3 \lambda^{(1)} u^{(1)}$

```
assert H0 == E2
assert H1 == E3 * u1 + λ1 * E2_dot
expected = (E4 * u1 * u1
            + E3 * u2_V
            + E3 * u2_W
            + 2 * λ1 * E3_dot * u1
            + λ1**2 * E2_ddot
            + λ2 * E2_dot)
assert H2 == expected
```

Using Eq. (4.3), the above expressions can be expanded as follows

$$\mathscr{H}^{(0)} = \mathscr{E}_2, \quad \mathscr{H}^{(1)} = \xi_i^{(1)}\, \mathscr{E}_3(v_i, \bullet, \bullet) + \lambda^{(1)}\, \dot{\mathscr{E}}_2,$$

and

$$\begin{aligned}
\mathscr{H}^{(2)} = {} & \xi_i^{(1)}\, \xi_j^{(1)}\, \mathscr{E}_4(v_i, v_j, \bullet, \bullet) + \xi_i^{(2)}\, \mathscr{E}_3(v_i, \bullet, \bullet) \\
& + \xi_i^{(1)}\, \xi_j^{(1)}\, \mathscr{E}_3(w_{ij}, \bullet, \bullet) + 2\lambda^{(1)}\, \xi_i^{(1)}\, \mathscr{E}_3(w_i, \bullet, \bullet) \\
& + 2\lambda^{(1)}\, \xi_i^{(1)}\, \dot{\mathscr{E}}_3(v_i, \bullet, \bullet) + (\lambda^{(1)})^2\, \ddot{\mathscr{E}}_2(\bullet, \bullet) + \lambda^{(2)}\, \dot{\mathscr{E}}_2(\bullet, \bullet),
\end{aligned}$$

and Eq. (5.2) is retrieved.

6 Eigenmodes of the hessian of the energy

In view of stability analysis, the eigenvalues α and eigenvectors x of the hessian of the energy are expanded in this chapter to second order in η

```
from sympy import *
from lsk.display import *
from lsk.symbols import *
from lsk.energy import *

x0_V, x0_W = symbols(r"{\order[0]{x}_V} {\order[0]{x}_W}")
x1_V, x1_W = symbols(r"{\order[1]{x}_V} {\order[1]{x}_W}")
x2_V, x2_W = symbols(r"{\order[2]{x}_V} {\order[2]{x}_W}")
α0, α1, α2 = symbols(r"{\order[0]{\alpha}} {\order[1]{\alpha}} {\order[2]{\alpha}}")

χ0 = IndexedBase(r"\order[0]{\chi}")
χ1 = IndexedBase(r"\order[1]{\chi}")

x = (x0_V + x0_W) + η * (x1_V + x1_W) + η**2 / 2 * (x2_V + x2_W) + O(η**3)
α = α0 + η * α1 + η**2 / 2 * α2 + O(η**3)

display_latex_dict({"x": x, r"\alpha": α}, num_cols=1)
```

$$x = x_W^{(0)} + x_V^{(0)} + \eta\left(x_V^{(1)} + x_W^{(1)}\right) + \frac{\eta^2\left(x_V^{(2)} + x_W^{(2)}\right)}{2} + O\left(\eta^3\right)$$

$$\alpha = \alpha^{(0)} + \eta\alpha^{(1)} + \frac{\eta^2\alpha^{(2)}}{2} + O\left(\eta^3\right)$$

Note that the eigenvector x is projected onto V and W : $x_V^{(k)} \in V$ and $x_W^{(k)} \in W$. It will be convenient to expand the V-component in the $(v_1, \dots, v_m)$ basis

$$x_V^{(k)} = \chi_i^{(k)} v_i.$$

We immediately have the following simplification rules

```
rules = {
    E2 * v[i]: 0,
    E2 * x0_V: 0,
    E2 * x1_V: 0,
    E2 * x2_V: 0
```

```
}
```

We focus in this chapter on potentially unstable eigenmodes, for which the eigenvalue α might be negative in the vicinity of $\eta = 0$. It will be shown that these eigenmodes are necessarily such that $\alpha^{(0)} = 0$. In that case, $x_W^{(0)} = 0$ and

$$x_W^{(1)} = \lambda^{(1)}\,\chi_j^{(0)}\,w_j + \chi_j^{(0)}\,\xi_k^{(1)}\,w_{jk}. \tag{6.1}$$

The coefficients $\alpha^{(1)}$, $\alpha^{(2)}$, $\chi_i^{(0)}$ and $\chi_i^{(1)}$ solve the following equations

$$\left(E_{ijk}\,\xi_k^{(1)} + \lambda^{(1)}\,\dot{E}_{ij}\right)\chi_j^{(0)} = \alpha^{(1)}\,\chi_i^{(0)}, \tag{6.2}$$

and

$$\begin{aligned}\Big[E_{ijkl}\,\xi_k^{(1)}\,\xi_l^{(1)} + \lambda^{(1)}\left(2\dot{E}_{ijk}\,\xi_k^{(1)} + \lambda^{(1)}\,\ddot{E}_{ij}\right) + E_{ijk}\,\xi_k^{(2)}\\ +\lambda^{(2)}\,\dot{E}_{ij}\Big]\chi_j^{(0)} + 2\Big(E_{ijk}\,\xi_k^{(1)} + \lambda^{(1)}\,\dot{E}_{ij}\Big)\,\chi_j^{(1)} = 2\alpha^{(1)}\,\chi_i^{(1)} + \alpha^{(2)}\,\chi_i^{(0)}.\end{aligned} \tag{6.3}$$

6.1 The eigenvalue problem

The eigenvalues $\alpha \in \mathbb{R}$ and eigenvectors $x \in U$ of the hessian are such that

$$\mathscr{E}_{,uu}[u(\eta), \lambda(\eta); x, \hat{u}] = \alpha\,\langle x, \hat{u}\rangle \quad \text{for all} \quad \hat{u} \in U, \tag{6.4}$$

where $\eta \mapsto \lambda(\eta)$ and $\eta \mapsto u(\eta)$ define the bifurcated branch.

In Chapter 4 and in Chapter 5, the following asymptotic expansions of $\lambda(\eta)$, $u(\eta)$ and $\mathscr{E}_{,uu}[u(\eta), \lambda(\eta); \bullet, \bullet]$ were derived

```
H0 = E2
H1 = E3 * u1 + λ1 * E2_dot
H2 = (E4 * u1 * u1  + E3 * (u2_V + u2_W)
      + 2 * λ1 * E3_dot * u1 + λ1**2 * E2_ddot + λ2 * E2_dot)

H = expand(H0 + η * H1 + η**2 / 2 * H2 + O(η**3))
```

```
display_latex_long_equation(r"\E_{uu}[u(\eta), λ(\eta)]", H, terms_per_line=6)
```

$$\begin{aligned}\mathscr{E}_{uu}[u(\eta), \lambda(\eta)] = \mathscr{E}_2 + \frac{\mathscr{E}_3\eta^2 u_V^{(2)}}{2} + \frac{\mathscr{E}_3\eta^2 u_W^{(2)}}{2} + \mathscr{E}_3\eta u^{(1)} + \frac{\mathscr{E}_4\eta^2 {u^{(1)}}^2}{2} + \dot{\mathscr{E}}_2\eta\lambda^{(1)}\\ + \dot{\mathscr{E}}_3\eta^2\lambda^{(1)}u^{(1)} + \frac{\dot{\mathscr{E}}_2\eta^2\lambda^{(2)}}{2} + \frac{\ddot{\mathscr{E}}_2\eta^2{\lambda^{(1)}}^2}{2} + O\left(\eta^3\right)\end{aligned}$$

We define the left-hand side and right-hand side of the variational problem that defines the eigenpairs (α, x) at all orders.

```
lhs = (H * x * u_hat).subs(rules).expand()
rhs = (α * x * u_hat).subs(rules).expand()
```

The terms of order 0, 1 and 2 in η are identified below.

6.2 The variational problem of order 0

```
lhs0 = lhs.coeff(η, 0).subs(rules)
rhs0 = rhs.coeff(η, 0)
```

The lowest-order problem reads: find $x_V^{(0)} \in V$, $x_W^{(0)} \in W$ and $\alpha^{(0)} \in \mathbb{R}$ such that, for all $\hat{u} \in U$

```
display_latex_equation(lhs0, rhs0)
```

$\mathscr{E}_2 \hat{u} x_W^{(0)} = \hat{u}\alpha^{(0)} x_V^{(0)} + \hat{u}\alpha^{(0)} x_W^{(0)}$

to be understood as

$$\mathscr{E}_2(x_W^{(0)}, \hat{u}) = \alpha^{(0)}\langle x_V^{(0)}, \hat{u}\rangle + \alpha^{(0)}\langle x_W^{(0)}, \hat{u}\rangle.$$

This variational equation is tested with an element $\hat{w} \in W$. Then, $\langle x_W^{(0)}, \hat{w}\rangle$ vanished and we are left with finding $x_W^{(0)} \in W$ such that, for all $\hat{w} \in W$,

$$\mathscr{E}_2(x_W^{(0)}, \hat{w}) = \alpha^{(0)}\langle x_W^{(0)}, \hat{w}\rangle.$$

If $x_W^{(0)} \neq 0$, then $\alpha^{(0)}$ is an eigenvalue of $\mathscr{E}_2$ over W. Therefore, $\alpha^{(0)} > 0$ and $\alpha > 0$ in the vicinity of the critical point $\lambda = \lambda_0$: the resulting eigenmode is *stable*. So, in order to find potentially unstable modes, we need to consider that $x_W^{(0)} = 0$ and $\alpha^{(0)} = 0$.

```
rules[α0] = 0
rules[x0_W] = 0
```

6.3 The variational problem of order 1

```
lhs1 = lhs.coeff(η, 1).subs(rules)
rhs1 = rhs.coeff(η, 1).subs(rules)
```

The problem reads: find $x_V^{(O)} \in V$, $x_W^{(1)} \in W$ and $\alpha^{(1)} \in \mathbb{R}$ such that, for all $\hat{u} \in U$

```
display_latex_equation(lhs1, rhs1)
```

$$\mathscr{E}_2 \hat{u} x_W^{(1)} + \mathscr{E}_3 \hat{u} x_V^{(0)} u^{(1)} + \dot{\mathscr{E}}_2 \hat{u} x_V^{(0)} \lambda^{(1)} = \hat{u} x_V^{(0)} \alpha^{(1)}$$

Testing first with $\hat{u} = v_i$ delivers the following variational problem: find $x_V^{(0)} \in V$ such that, for all $i = 1, \dots, m$

```
d = {
    u_hat: v[i],
    x0_V : χ0[j] * v[j],
    u1 : ξ1[k] * v[k]
}
lhs1a = lhs1.subs(d).subs(rules)
rhs1a = rhs1.subs(d).subs(rules).subs(χ0[j] * v[j] * v[i], χ0[i])

display_latex_equation(lhs1a, rhs1a)
```

$$\mathscr{E}_3 {\chi^{(0)}}_j v_i v_j v_k {\xi^{(1)}}_k + \dot{\mathscr{E}}_2 \lambda^{(1)} {\chi^{(0)}}_j v_i v_j = \alpha^{(1)} {\chi^{(0)}}_i$$

and Eq. (6.2) is retrieved.

The test function is now picked in $W = V^\perp$ and we get the following variational problem: find $x_W^{(1)} \in W$ such that, for all $\hat{w} \in W$,

```
d = {
    u_hat: w_hat,
    x0_V : χ0[i] * v[i],
    u1 : ξ1[j] * v[j]
}
lhs1b = lhs1.subs(d).subs(rules)
rhs1b = rhs1.subs(d).subs(rules).subs(v[i] * w_hat, 0)

display_latex_equation(lhs1b, rhs1b)
```

$$\mathscr{E}_2 \hat{w} x_W^{(1)} + \mathscr{E}_3 \hat{w} {\chi^{(0)}}_i v_i v_j {\xi^{(1)}}_j + \dot{\mathscr{E}}_2 \hat{w} \lambda^{(1)} {\chi^{(0)}}_i v_i = 0$$

(observe that, in the RHS, $\langle v_i, \hat{w} \rangle = 0$ since V and W are orthogonal subspaces). The solution to the above problem is expressed as a linear combination of the w_{ij} and w_i –defined by the variational problems (2.5) and (2.4), respectively–, delivering Eq. (6.1).

```
rules[x1_W] = χ0[k] * ξ1[l] * w[k, l] + λ1 * χ0[k] * w[k]
```

6.4 The variational problem of order 2

The terms of second order are tested against elements of V only.

```
lhs2 = expand(2 * lhs.coeff(η, 2).subs(u_hat, v[i]).subs(rules))
rhs2 = expand(2 * rhs.coeff(η, 2).subs(u_hat, v[i]).subs(rules))
```

Simplification of these expressions

Use some orthogonality conditions in the right-hand side.

```
rhs2 = rhs2.subs({
    x0_V * v[i]: χ0[i],
    x1_V * v[i]: χ1[i],
    v[i] * w[k]: 0,
    v[i] * w[k, l]: 0
})
```

Plug expansions

$$u^{(1)} = \xi_i^{(1)}\, v_i, \quad u_V^{(2)} = \xi_i^{(2)}\, v_i, \quad x_V^{(0)} = \chi_i^{(0)}\, v_i, \quad \text{and} \quad x_V^{(1)} = \chi_i^{(1)}\, v_i.$$

```
d = dict()

d[E2_dot * v[i] * x0_V] = χ0[j] * E2_dot * v[i] * v[j]
d[E2_dot * v[i] * x1_V] = χ1[j] * E2_dot * v[i] * v[j]
d[E2_ddot * x0_V * v[i]] = χ0[j] * E2_ddot * v[i] * v[j]
d[E3 * v[i] * x1_V * u1] = χ1[j] * ξ1[k] * E3 * v[i] * v[j] * v[k]
d[E3 * v[i] * x0_V * u2_V] = χ0[j] * ξ2[k] * E3 * v[i] * v[j] * v[k]
d[E3_dot * v[i] * x0_V * u1] = χ0[j] * ξ1[k] * E3_dot * v[i] * v[j] * v[k]
d[E4 * v[i] * x0_V * u1 * u1] = (χ0[j] * ξ1[k] * ξ1[l]
                                 * E4 * v[i] * v[j] * v[k] * v[l])

lhs2 = lhs2.subs(d)
```

Rename some indices.

```
lhs2 = lhs2.subs({
    χ0[k] * E2_dot * v[i] * w[k]: χ0[j] * E2_dot * v[i] * w[j],
    χ0[k] * ξ1[l] * w[k, l]: χ0[j] * ξ1[k] * w[j, k]
})
```

$$\begin{aligned}\mathscr{E}_3(x_V^{(0)}, u_W^{(2)}, v_i) &= \chi_j^{(0)}\, \xi_k^{(1)} \left[\xi_l^{(1)}\, \mathscr{E}_3(v_i, v_j, w_{kl}) + 2\lambda^{(1)}\, \mathscr{E}_3(v_i, v_j, w_k)\right] \\ &= \chi_j^{(0)}\, \xi_k^{(1)}\, \xi_l^{(1)}\, \mathscr{E}_3(v_i, v_j, w_{kl}) + 2\lambda^{(1)}\, \chi_j^{(0)}\, \xi_k^{(1)}\, \dot{\mathscr{E}}_2(v_k, w_{ij})\end{aligned}$$

```
lhs2 = lhs2.subs(E3 * v[i] * x0_V * u2_W,
                 (χ0[j] * ξ1[k] * ξ1[l]
                  * E3 * v[i] * v[j] * w[k,l]
                  + 2 * λ1 * χ0[j] * ξ1[k]
                  * E2_dot * v[k] * w[i, j])).expand()
```

$$\chi_k^{(0)}\, \mathscr{E}_3(v_i, u^{(1)}, w_k) = \chi_j^{(0)}\, \xi_k^{(1)}\, \mathscr{E}_3(v_i, v_k, w_j) = \chi_j^{(0)}\, \xi_k^{(1)}\, \dot{\mathscr{E}}_2(v_j, w_{ik})$$

```
lhs2 = lhs2.subs(χ0[k] * E3 * v[i] * u1 * w[k],
                 χ0[j] * ξ1[k] * E2_dot * v[j] * w[i, k]).expand()
```

$$\begin{aligned}\chi_j^{(0)}\, \xi_k^{(1)}\, \mathscr{E}_3(v_i, w_{j,k}, u^{(1)}) &= \chi_j^{(0)}\, \xi_k^{(1)}\, \xi_l^{(1)}\, \mathscr{E}_3(v_i, v_l, w_{j,k}) \\ &= \frac{1}{2}\chi_j^{(0)}\, \xi_k^{(1)}\, \xi_l^{(1)} \left[\mathscr{E}_3(v_i, v_k, w_{j,l}) + \mathscr{E}_3(v_i, v_l, w_{j,k})\right]\end{aligned}$$

```
lhs2 = lhs2.subs(χ0[j] * ξ1[k] * E3 * v[i] * w[j, k] * u1,
                 χ0[j] * ξ1[k] * ξ1[l] / 2
                 * (E3 * v[i] * v[k] * w[j, l]
                    + E3 * v[i] * v[l] * w[j, k])).expand()
```

```
lhs2 = lhs2.subs({
    E2_dot * v[i] * v[j]: E_dot[i, j],
    E2_ddot * v[i] * v[j]: (E_ddot[i, j] - 2 * E2_dot * v[i] * w[j]),
    E3 * v[i] * v[j] * v[k]: E[i, j, k],
    E3_dot * v[i] * v[j] * v[k]: (E_dot[i, j, k]
                                  - E2_dot * v[i] * w[j, k]
                                  - E2_dot * v[j] * w[i, k]
                                  - E2_dot * v[k] * w[i, j]),
    E4 *v[i] * v[j] * v[k] * v[l]: (E[i, j, k, l]
                                 - E3 * v[i] * (v[j] * w[k, l]
                                               + v[k] * w[j, l]
                                               + v[l] * w[j, k]))
}).expand()
```

And Eq. (6.3) is finally retrieved.

```
expected = ((E[i, j, k, l] * ξ1[k] * ξ1[l]
            + λ1 * (2 * E_dot[i, j, k] * ξ1[k] + λ1 * E_ddot[i, j])
            + E[i, j, k] * ξ2[k]
            + λ2 * E_dot[i, j]) * χ0[j]
            + 2 * (E[i, j, k] * ξ1[k] + λ1 * E_dot[i, j]) * χ1[j])
```

```
assert expand(lhs2 - expected) == 0

expected = 2 * α1 * χ1[i] + α2 * χ0[i]

assert expand(rhs2 - expected) == 0
```

7 Asymmetric bifurcation

In this chapter, we consider a bifurcated branch for which $\lambda^{(1)} \neq 0$. The bifurcation equation (4.4) shows that necessarily, E_{ijk} is not identically nul. This equation has at most $(2^m - 1)$ pairs of real solutions $(\lambda^{(1)}, u^{(1)})$ et $(-\lambda^{(1)}, -u^{(1)})$; furthermore, multiplication by $\xi_i^{(1)}$ shows that

$$\lambda^{(1)} = -\frac{E_{ijk}\,\xi_i^{(1)}\,\xi_j^{(1)}\,\xi_k^{(1)}}{2\dot{E}_{ij}\,\xi_i^{(1)}\,\xi_j^{(1)}}. \tag{7.1}$$

> **Note**
>
> I can't prove that the bifurcation equation (4.4) has at most $(2^m - 1)$ pairs of real solutions.

Along the bifurcated branch, we have $\lambda = \lambda_0 + \eta\,\lambda^{(1)} + o(\eta)$, and η can be eliminated. In other words, $\eta = \lambda$ ($\lambda^{(1)} = 1$ and $\lambda^{(2)} = \lambda^{(3)} = \cdots = 0$) can be selected as a parameter. It is therefore possible to express the bifurcated branch as a function of λ: $u(\lambda)$. For example, combining Eqs. (4.7) and (5.2), we find that

$$\begin{aligned}\mathscr{E}_{,uu}[u(\eta), \lambda(\eta); u^{(1)}, u^{(1)}] &= \eta\left[\mathscr{E}_3(u^{(1)}, u^{(1)}, u^{(1)}) + \lambda^{(1)}\,\dot{\mathscr{E}}_2(u^{(1)}, u^{(1)})\right] + o(\eta)\\ &= -\eta\,\lambda^{(1)}\,\dot{\mathscr{E}}_2(u^{(1)}, u^{(1)}) + o(\eta),\end{aligned}$$

or

$$\mathscr{E}_{,uu}[u(\lambda), \lambda; u^{(1)}, u^{(1)}] = -\big(\lambda - \lambda_0\big)\,\dot{\mathscr{E}}_2(u^{(1)}, u^{(1)}) + o(\lambda - \lambda_0). \tag{7.2}$$

For $\lambda < \lambda_0$, the above quantity is *negative* (since $\dot{\mathscr{E}}_2$ is negative definite). In other words: **for asymmetric bifurcations, below the critical load, the bifurcated branch is unstable.**

To investigate the stability above the critical load, we need to analyse the sign of the eigenvalues α of the Hessian. At first order, $\alpha = \eta\,\alpha^{(1)} + o(\eta)$, where $\alpha^{(1)}$ is an eigenvalue of $(E_{ijk}\,\xi_k^{(1)} + \lambda^{(1)}\,\dot{E}_{ij})$ (see Chapter 6). Let $\alpha_{\min}$ and $\alpha_{\max}$ be the minimum and maximum eigenvalues of this second-order tensor. Three cases must be discussed

1. If $\alpha_{\min}\,\alpha_{\max} > 0$, then $(E_{ijk}\,\xi_k^{(1)} + \lambda^{(1)}\,\dot{E}_{ij})$ is positive or negative definite: all eigenvalues have the same sign, $\epsilon \in \{-1, +1\}$. Then the sign of the eigenvalues α of the Hessian is $\epsilon\,\eta$ and there is a stability switch at the critical load. Since the bifurcated branch is unstable *below* the critical load, this means that it is *stable* above the critical load.

2. If $\alpha_{\min}\,\alpha_{\max} < 0$, then the extremal eigenvalues of the Hessian are $\eta\,\alpha_{\min}$ and $\eta\,\alpha_{\max}$, the product of which is $\eta^2\,\alpha_{\min}\,\alpha_{\max} < 0$. The bifurcated branch is *unstable* for all values of λ.
3. If $\alpha_{\min}\,\alpha_{\max} = 0$, the analysis is inconclusive.

To close this section, it is observed that the dominant term of the expansion (5.1) of the potential energy along the bifurcated branch is of the third order in η

$$\mathscr{E}[u(\eta), \lambda(\eta)] = \mathscr{E}[u^\star \circ \lambda(\eta), \lambda(\eta)\} + \tfrac{1}{6}\lambda^{(1)}\,\eta^3\,\dot{E}_{ij}\,\xi_i^{(1)}\,\xi_j^{(1)} + o(\eta^3).$$

Eliminating λ and plugging expression (7.1) of $\lambda^{(1)}$ delivers the expression of the potential energy, where λ is the parameter

$$\begin{aligned}\mathscr{E}[u(\lambda), \lambda] &= \mathscr{E}[u^\star(\lambda), \lambda] + \frac{\left(\lambda - \lambda_0\right)^3}{6\left(\lambda^{(1)}\right)^2}\,\dot{E}_{ij}\,\xi_i^{(1)}\,\xi_j^{(1)} + o(\lambda^3)\\ &= \mathscr{E}[u^\star(\lambda), \lambda] + \frac{2\left(\dot{E}_{ij}\,\xi_i^{(1)}\,\xi_j^{(1)}\right)^3}{3\left(E_{ijk}\,\xi_i^{(1)}\,\xi_j^{(1)}\,\xi_k^{(1)}\right)^2}\left(\lambda - \lambda_0\right)^3 + o(\lambda^3).\end{aligned}$$

Recalling that $\dot{E}_{ij}\,\xi_i^{(1)}\,\xi_j^{(1)} < 0$, it is found that, above the critical load, the potential energy is *smaller* along the bifurcated branch than along the fundamental branch.

> **ℹ Note**
>
> As expected, the above expression does not depend on the scaling of $u^{(1)}$ (of the $\xi_i^{(1)}$).

> **ℹ Note**
>
> It has been shown in **?@sec-xxx** that, when E_{ijk} is not identically null, the bifurcation point is *unstable*.

8 A particular case of symmetric bifurcation

In this chapter, we consider the case $E_{ijk} = 0$ for all $i, j, k = 1, \ldots, m$. Then, from Eq. (4.4), $\lambda^{(1)} = 0$ on *all* bifurcated branches. It is assumed that, on the bifurcated branch under consideration, the next term of the expansion of λ is non-zero: $\lambda^{(2)} \neq 0$. The bifurcation is *symmetric*, and the bifurcation equation (eq-20230124205642) reduces to

$$\tfrac{1}{3} E_{ijkl}\, \xi_j^{(1)}\, \xi_k^{(1)}\, \xi_l^{(1)} + \lambda^{(2)}\, \dot{E}_{ij}\, \xi_j^{(1)} = 0, \tag{8.1}$$

which has at most $(3^m - 1)/2$ pairs of real solutions $(\lambda^{(2)}, u^{(1)})$ and $(-\lambda^{(2)}, -u^{(1)})$.

> **Note**
>
> I can't prove that the bifurcation equation (8.1) has at most $(3^m - 1)/2$ pairs of real solutions.

Upon multiplication by $\xi_i^{(1)}$, the above equation delivers the following expression of $\lambda^{(2)}$

$$\lambda^{(2)} = -\frac{E_{ijkl}\, \xi_i^{(1)}\, \xi_j^{(1)}\, \xi_k^{(1)}\, \xi_l^{(1)}}{3 \dot{E}_{ij}\, \xi_i^{(1)}\, \xi_j^{(1)}}. \tag{8.2}$$

Since $\dot{E}_{ij}\, \xi_i^{(1)}\, \xi_j^{(1)} < 0$, $\lambda^{(2)}$ has the same sign as $E_{ijkl}\, \xi_i^{(1)}\, \xi_j^{(1)}\, \xi_k^{(1)}\, \xi_l^{(1)}$. In other words, if $E_{ijkl}\, \xi_i^{(1)}\, \xi_j^{(1)}\, \xi_k^{(1)}\, \xi_l^{(1)} > 0$, (resp. < 0) then the bifurcated branch exists *above* (resp. *below*) the critical load λ_0 only.

Turning now to the eigenpairs of the Hessian of the energy along the bifurcated branch, Eq. (6.2) shows that $\alpha^{(1)} = 0$. Then $\alpha = \alpha^{(2)} \eta^2/2 + o(\eta^2)$ and, from Eq. (6.3)

$$\left(E_{ijkl}\, \xi_k^{(1)}\, \xi_l^{(1)} + \lambda^{(2)}\, \dot{E}_{ij}\right) \chi_j^{(0)} = \alpha^{(2)}\, \chi_i^{(0)}.$$

If $(E_{ijkl}\, \xi_k^{(1)}\, \xi_l^{(1)} + \lambda^{(2)}\, \dot{E}_{ij})$ is positive definite, then the bifurcated branch is stable (note that, in that case, the bifurcated branch exists above the critical load only). If one of the eigenvalues of this tensor is < 0, then the bifurcated branch is unstable. The stability is undecided when all eigenvalues are ≥ 0.

> **Note**
>
> Note that, from Eq. (8.1),
>
> $$E_{ijkl}\, \xi_i^{(1)}\, \xi_j^{(1)}\, \xi_k^{(1)}\, \xi_l^{(1)} + \lambda^{(2)}\, \dot{E}_{ij}\, \xi_i^{(1)}\, \xi_j^{(1)} = \tfrac{2}{3} E_{ijkl}\, \xi_i^{(1)}\, \xi_j^{(1)}\, \xi_k^{(1)}\, \xi_l^{(1)}.$$

To conclude this section, it is observed that, when $\lambda^{(1)} = 0$, the dominant term of the potential energy along the bifurcated branch is of the fourth order, see Eq (5.1). Combining with Eq. (8.2),

$$\begin{aligned}\mathscr{E}[u(\eta), \lambda(\eta)] &= \mathscr{E}[u^\star \circ \lambda(\eta), \lambda(\eta)] + \tfrac{1}{24}\eta^4 \Big(E_{ijkl}\, \xi_i^{(1)}\, \xi_j^{(1)}\, \xi_k^{(1)}\, \xi_l^{(1)} \\ &\quad + 6\lambda^{(2)}\, \dot{E}_{ij}\, \xi_i^{(1)}\, \xi_j^{(1)}\Big) + o(\eta^4) \\ &= \mathscr{E}[u^\star \circ \lambda(\eta), \lambda(\eta)] - \tfrac{1}{24}\eta^4 E_{ijkl}\, \xi_i^{(1)}\, \xi_j^{(1)}\, \xi_k^{(1)}\, \xi_l^{(1)} + o(\eta^4).\end{aligned} \tag{8.3}$$

The expansion $\lambda = \lambda_0 + \lambda^{(2)}\,\eta^2/2 + o(\eta^2)$ can be inverted as follows

$$\eta^4 = \frac{4\big(\lambda - \lambda_0\big)^2}{\big(\lambda^{(2)}\big)^2} + o(\lambda^2) = \frac{36\big(\dot{E}_{ij}\, \xi_i^{(1)}\, \xi_j^{(1)}\big)^2}{\big(E_{ijkl}\, \xi_i^{(1)}\, \xi_j^{(1)}\, \xi_k^{(1)}\, \xi_l^{(1)}\big)^2}\big(\lambda - \lambda_0\big)^2$$

and expression (8.3) reads

$$\mathscr{E}[u(\eta), \lambda(\eta)] = \mathscr{E}[u^\star \circ \lambda(\eta), \lambda(\eta)] - \frac{3\big(\dot{E}_{ij}\, \xi_i^{(1)}\, \xi_j^{(1)}\big)^2}{2E_{ijkl}\, \xi_i^{(1)}\, \xi_j^{(1)}\, \xi_k^{(1)}\, \xi_l^{(1)}}\big(\lambda - \lambda_0\big)^2 + o(\lambda^2).$$

Again, the above expression does not depend on the scaling of $u^{(1)}$ (of the $\xi_i^{(1)}$). Note that, if $E_{ijkl}\, \xi_i^{(1)}\, \xi_j^{(1)}\, \xi_k^{(1)}\, \xi_l^{(1)} > 0$, then only loads that are greater than the critical load can be reached on the bifurcated branch, where the energy is lower than the fundamental branch.

9 The lsk.symbols module

This module defines all symbols required for the SymPy derivations (see below).

```
import lsk.symbols

%psource lsk.symbols
```

```
from sympy import Idx, IndexedBase, Symbol

E2 = Symbol(r"\E_2")
E3 = Symbol(r"\E_3")
E4 = Symbol(r"\E_4")
E2_dot = Symbol(r"\dot{\E}_2")
E3_dot = Symbol(r"\dot{\E}_3")
E2_ddot = Symbol(r"\ddot{\E}_2")
E_λ = Symbol(r"\E_{\lambda}")
E_λλ = Symbol(r"\E_{\lambda\lambda}")
E_λλλ = Symbol(r"\E_{\lambda\lambda\lambda}")
E_λλλλ = Symbol(r"\E_{\lambda\lambda\lambda\lambda}")

E_uλ = Symbol(r"\E_{u\lambda}")
E_uλλ = Symbol(r"\E_{u\lambda\lambda}")
E_uλλλ = Symbol(r"\E_{u\lambda\lambda\lambda}")
E_uuλ = Symbol(r"\E_{uu\lambda}")
E_uuλλ = Symbol(r"\E_{uu\lambda\lambda}")
E_uuuλ = Symbol(r"\E_{uuu\lambda}")

u0_dot = Symbol(r"\dot{u}_0")
u0_ddot = Symbol(r"\ddot{u}_0")
u0_dddot = Symbol(r"\dddot{u}_0")
u0_ddddot = Symbol(r"\ddddot{u}_0")

i = Idx("i")
j = Idx("j")
k = Idx("k")
l = Idx("l")

λ = IndexedBase(r"\lambda")
ξ = IndexedBase(r"\xi")
```

```
ξ1 = IndexedBase(r"{\order[1]{\xi}}")
ξ2 = IndexedBase(r"{\order[2]{\xi}}")
v = IndexedBase("v")
w = IndexedBase("w")
E = IndexedBase("E")
E_dot = IndexedBase("\dot{E}")
E_ddot = IndexedBase("\ddot{E}")

u1 = Symbol(r"{\order[1]{u}}")
u2 = Symbol(r"{\order[2]{u}}")
u2_V = Symbol(r"{\order[2]{u}_V}")
u2_W = Symbol(r"{\order[2]{u}_W}")
u3 = Symbol(r"{\order[3]{u}}")
u4 = Symbol(r"{\order[4]{u}}")

λ1 = Symbol(r"{\order[1]{\lambda}}")
λ2 = Symbol(r"{\order[2]{\lambda}}")
λ3 = Symbol(r"{\order[3]{\lambda}}")
λ4 = Symbol(r"{\order[4]{\lambda}}")

η = Symbol(r"\eta")

u_hat = Symbol(r"\hat{u}")
v_hat = Symbol(r"\hat{v}")
w_hat = Symbol(r"\hat{w}")
```

Note that the symbols: `λ`, `ξ`, `v`, `w`, `E`, `E_dot` and `E_ddot` are all instances of the class `IndexedBase`. In other words, they can be indexed with symbolic indices

- `v[i]` → v_i
- `E[i, j, k]` → E_{ijk}

etc.

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