Notes relatives à la méthode asymptotique de Lyapunov–Schmidt–Koiter

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These are my notes on the LSK method for the analysis of the stability and bifurcation(s) of a conservative system. These notes are based on several references: Koiter's initial PhD thesis (Koiter 1945) as well as some graphical illustrations from his lecture notes (Koiter and Heijden 2009). I enjoyed the concise presentation of Nguyen (2000) as well as the lecture notes of Triantafyllidis (2017). Finally, the chapter by Potier-Ferry (1987) helped me clear some issues.

I hope the reader will find these notes useful, even though there are still a few points which I do not fully understand (they are clearly indicated in the text).

1. Notations

L'espace des champs cinématiquement admissibles est noté U. On suppose qu'il a la structure d'espace vectoriel. L'énergie du système est notée $\mathscr{E}(u, \lambda)$, où λ désigne un paramètre de chargement. Soit $u^*(\lambda)$ la branche fondamentale. Par définition

$$\mathscr{E}_{,u}[u^*(\lambda), \lambda; \hat{u}] = 0 \quad \text{pour tout} \quad \hat{u} \in U, \tag{1}$$

and, deriving twice w.r.t. λ, we find successively, for all $\hat{u} \in U$

$$\mathscr{E}_{,uu}[u^*(\lambda), \lambda; \dot{u}^*(\lambda), \hat{u}] + \mathscr{E}_{,u\lambda}[u^*(\lambda), \lambda; \hat{u}] = 0 \tag{2}$$

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and

$$\mathscr{E}_{,uuu}[u^*(\lambda),\lambda;\dot{u}^*(\lambda),\dot{u}^*(\lambda),\hat{u}] + 2\mathscr{E}_{,uu\lambda}[u^*(\lambda),\lambda;\dot{u}^*(\lambda),\hat{u}] + \mathscr{E}_{,u\lambda\lambda}[u^*(\lambda),\lambda;\hat{u}] + \mathscr{E}_{,uu}[u^*(\lambda),\lambda;\ddot{u}^*(\lambda),\hat{u}] = 0 \quad (3)$$

Il sera commode d'introduire les notations suivantes

$$\mathscr{E}_2(\lambda) = \mathscr{E}_{,uu}[u^*(\lambda),\lambda], \quad \mathscr{E}_3(\lambda) = \mathscr{E}_{,uuu}[u^*(\lambda),\lambda], \quad \mathscr{E}_4(\lambda) = \mathscr{E}_{,uuuu}[u^*(\lambda),\lambda]. \quad (4)$$

Noter que $\mathscr{E}_2$, $\mathscr{E}_3$ et $\mathscr{E}_4$ sont des formes bi-, tri- et quadri-linéaires, respectivement. L'application de ces formes à des éléments de U sera notée $\mathscr{E}_2(\lambda;u,v)$, $\mathscr{E}_3(\lambda;u,v,w)$, etc. La dérivée de ces formes par rapport à λ sera notée à l'aide d'un point supérieur ($\dot{\mathscr{E}}_2, \dot{\mathscr{E}}_3, \ldots$).

On suppose que l'équilibre est stable pour des valeurs suffisamment petites de λ. Plus précisément, on suppose que $\mathscr{E}_2(\lambda)$ est définie positive pour tout $\lambda < \lambda_0$. Pour $\lambda = \lambda_0$, la forme quadratique $\mathscr{E}_2(\lambda_0)$ n'est plus que positive. On note $u_0 = u^*(\lambda_0)$, $\dot{u}_0 = \dot{u}^*(\lambda_0)$ et $\ddot{u}_0 = \ddot{u}^*(\lambda_0)$ de sorte que les Éqs. (2) et (3) s'écrivent, en $\lambda = \lambda_0$

$$\mathscr{E}_{,uu}(u_0,\lambda_0;\dot{u}_0,\bullet) + \mathscr{E}_{,u\lambda}(u_0,\lambda_0;\bullet) = 0 \quad (5)$$

$$\mathscr{E}_{,uuu}(u_0,\lambda_0;\dot{u}_0,\dot{u}_0,\bullet) + 2\mathscr{E}_{,uu\lambda}(u_0,\lambda_0;\dot{u}_0,\bullet) + \mathscr{E}_{,u\lambda\lambda}(u_0,\lambda_0;\bullet) + \mathscr{E}_{,uu}(u_0,\lambda_0;\ddot{u}_0,\bullet) = 0 \quad (6)$$

On s'intéresse dans ce qui suit à toutes les courbes d'équilibre qui passent par le point (u_0,λ_0).

Noter que dans ce qui suit, on convient que les formes $\mathscr{E}_2$, $\mathscr{E}_3$ et $\mathscr{E}_4$ sont implicitement évaluées en λ_0 lorsque λ n'est pas rappelé : ainsi, on notera $\mathscr{E}_2(\bullet,\bullet)$ plutôt que $\mathscr{E}_2(\lambda_0;\bullet,\bullet)$.

Par hypothèse, $\mathscr{E}_2(\lambda_0)$ est positive, sans être définie positive ; soit V son noyau, qui forme un sous-espace vectoriel de U. On suppose que V est de dimension finie $m = \dim V$. Soit $(v_1,\ldots,v_m)$ une base orthonormée de ce noyau pour le produit scalaire $\langle\bullet,\bullet\rangle$(qui n'est pas précisé pour le moment). On introduit le sous-espace supplémentaire orthogonal W de V dans U

$$U = V \overset{\perp}{\otimes} W. \quad (7)$$

Remark 1. The bilinear form $\mathscr{E}_2$ being elliptic over W, variational problems of the type: find $w \in W$ such that

$$\mathscr{E}_2(w,\hat{w}) + \ell(\hat{w}) = 0 \quad \text{for all} \quad \hat{w} \in W \quad (8)$$

are well-posed for any linear form ℓ over W. In particular, for $\ell = 0$, the unique solution to the variational problem

$$\mathscr{E}_2(w,\hat{w}) = 0 \quad \text{for all} \quad \hat{w} \in W \quad (9)$$

is $w = 0$.

For $1 \le i, j \le m$, we introduce the solutions $w_i, w_{ij} \in W$ to the following variational problems

$$\mathscr{E}_2(\lambda_0; w_i, \hat{w}) + 2\dot{\mathscr{E}}_2(\lambda_0; v_i, \hat{w}) = 0, \tag{10}$$
$$\mathscr{E}_2(\lambda_0; w_{ij}, \hat{w}) + \mathscr{E}_3(\lambda_0; v_i, v_j, \hat{w}) = 0, \tag{11}$$

for all $\hat{w} \in W$. Since w_i and w_{ij} belong to W, we have $\langle w_i, v\rangle = \langle w_{ij}, v\rangle = 0$ for all $v \in V$. Since $\mathscr{E}_2(\lambda_0; \bullet, \bullet)$ is symmetric, it can be verified that $w_{ij} = w_{ji}$. We also introduce the following tensors, defined in V

$$E_{ijk} = \mathscr{E}_3(\lambda_0; v_i, v_j, v_k) + \mathscr{E}_2(\lambda_0; v_i, w_{jk}) + \mathscr{E}_2(\lambda_0; v_j, w_{ki}) + \mathscr{E}_2(\lambda_0; v_k, w_{ij}), \tag{12}$$
$$E_{ijkl} = \mathscr{E}_4(\lambda_0; v_i, v_j, v_k, v_l) + \mathscr{E}_3(\lambda_0; v_i, v_j, w_{kl}) + \mathscr{E}_3(\lambda_0; v_i, v_k, w_{lj}) + \mathscr{E}_3(\lambda_0; v_i, v_l, w_{jk}), \tag{13}$$
$$F_{ij} = \dot{\mathscr{E}}_2(\lambda_0; v_i, v_j) + \frac{1}{2}\Big[\mathscr{E}_2(\lambda_0; v_i, w_j) + \mathscr{E}_2(\lambda_0; v_j, w_i)\Big], \tag{14}$$

as well as the derivatives

$$\mathring{E}_{ijk} = \dot{\mathscr{E}}_3(\lambda_0; v_i, v_j, v_k) + \dot{\mathscr{E}}_2(\lambda_0; v_i, w_{jk}) + \dot{\mathscr{E}}_2(\lambda_0; v_j, w_{ki}) + \dot{\mathscr{E}}_2(\lambda_0; v_k, w_{ij}), \tag{15}$$
$$\mathring{F}_{ij} = \ddot{\mathscr{E}}_2(\lambda_0; v_i, v_j) + \frac{1}{2}\Big[\dot{\mathscr{E}}_2(\lambda_0; v_i, w_j) + \dot{\mathscr{E}}_2(\lambda_0; v_j, w_i)\Big]. \tag{16}$$

Note that, since $\mathscr{E}_2(\lambda_0; v_i, \bullet) = 0$, the above expressions simplify as follows

$$E_{ijk} = \mathscr{E}_3(\lambda_0; v_i, v_j, v_k), \tag{17}$$
$$E_{ijkl} = \mathscr{E}_4(\lambda_0; v_i, v_j, v_k, v_l) + \mathscr{E}_3(\lambda_0; v_i, v_j, w_{kl}) + \mathscr{E}_3(\lambda_0; v_i, v_k, w_{jl}) + \mathscr{E}_3(\lambda_0; v_i, v_l, w_{jk}), \tag{18}$$
$$F_{ij} = \dot{\mathscr{E}}_2(\lambda_0; v_i, v_j). \tag{19}$$

The tensors E_{ijk}, F_{ij}, $\mathring{E}_{ijk}$ and $\mathring{F}_{ij}$ are fully symmetric. Furthermore, the following expression of E_{ijkl} result from Eq. (11)

$$E_{ijkl} = \mathscr{E}_4(\lambda_0; v_i, v_j, v_k, v_l) - \mathscr{E}_2(\lambda_0; w_{ij}, w_{kl}) - \mathscr{E}_2(\lambda_0; w_{ik}, w_{jl}) - \mathscr{E}_2(\lambda_0; w_{il}, w_{jk}), \tag{20}$$

which shows that E_{ijkl} is also fully symmetric. To close this section, the following useful identities are derived from applications of Eqs. (10) and (11)

$$\begin{aligned}
\mathring{F}_{ij} &= \ddot{\mathscr{E}}_2(\lambda_0; v_i, v_j) + \frac{1}{2}\Big[\dot{\mathscr{E}}_2(\lambda_0; v_i, w_j) + \dot{\mathscr{E}}_2(\lambda_0; w_i, v_j)\Big] = \ddot{\mathscr{E}}_2(\lambda_0; v_i, v_j) + \dot{\mathscr{E}}_2(\lambda_0; v_j, w_i) \\
&= \ddot{\mathscr{E}}_2(\lambda_0; v_i, v_j) - \frac{1}{2}\mathscr{E}_2(\lambda_0; w_i, w_j) = \ddot{\mathscr{E}}_2(\lambda_0; v_i, v_j) + \dot{\mathscr{E}}_2(\lambda_0; v_i, w_j),
\end{aligned} \tag{21}$$

$$\begin{aligned}
\mathring{E}_{ijk} &= \dot{\mathscr{E}}_3(\lambda_0; v_i, v_j, v_k) + \dot{\mathscr{E}}_2(\lambda_0; v_i, w_{jk}) + \dot{\mathscr{E}}_2(\lambda_0; v_j, w_{ik}) + \dot{\mathscr{E}}_2(\lambda_0; v_k, w_{ij}) \\
&= \dot{\mathscr{E}}_3(\lambda_0; v_i, v_j, v_k) - \frac{1}{2}\Big[\mathscr{E}_2(\lambda_0; w_i, w_{jk}) + \mathscr{E}_2(\lambda_0; w_j, w_{ik}) + \mathscr{E}_2(\lambda_0; w_k, w_{ij})\Big].
\end{aligned} \tag{22}$$

2. Analysis of the critical point

In this section, we discuss the stability of the critical point (u_0, λ_0). To this end, we evaluate the potential energy in a neighboring state $u_0 + u$, where $u \in U$ is "small".

We have, to the fourth order

$$\mathscr{E}(u_0 + u, \lambda_0) - \mathscr{E}(u_0, \lambda_0) = \frac{1}{2}\mathscr{E}_2(\lambda_0; u, u) + \frac{1}{6}\mathscr{E}_3(\lambda_0; u, u, u) + \frac{1}{24}\mathscr{E}_4(\lambda_0; u, u, u, u) + o(\langle u, u\rangle^2), \tag{23}$$

where the linear term has been omitted, u_0 being a critical point of the energy. Since $v \in V$, we have $\mathscr{E}_2(\lambda_0; v, \bullet) = 0$. We now expand u as $u = \xi v + \eta w$, with $\xi, \eta \in \mathbb{R}$ and $v \in V$ and $w \in W$ are fixed, orthogonal directions. Owing to the multi-linearity and symmetry of the successive differential of $\mathscr{E}$, the above expression expands as follows

$$\begin{aligned}\mathscr{E}(u_0 + u, \lambda_0) - \mathscr{E}(u_0, \lambda_0) = {}& \frac{1}{2}\eta^2\mathscr{E}_2(\lambda_0; w, w) + \frac{1}{6}\xi^3\mathscr{E}_3(\lambda_0; v, v, v)\\ &+ \frac{1}{2}\xi^2\eta\mathscr{E}_3(\lambda_0; v, v, w) + \frac{1}{2}\xi\eta^2\mathscr{E}_3(\lambda_0; v, w, w)\\ &+ \frac{1}{6}\eta^3\mathscr{E}_3(\lambda_0; w, w, w) + \frac{1}{24}\xi^4\mathscr{E}_4(\lambda_0; v, v, v, v)\\ &+ \frac{1}{6}\xi^3\eta\mathscr{E}_4(\lambda_0; v, v, v, w) + \frac{1}{4}\xi^2\eta^2\mathscr{E}_4(\lambda_0; v, v, w, w)\\ &+ \frac{1}{6}\xi\eta^3\mathscr{E}_4(\lambda_0; v, w, w, w) + \frac{1}{24}\eta^4\mathscr{E}_4(\lambda_0; w, w, w, w)\\ &+ o\Big[\big(\xi^2 + \eta^2\big)^2\Big].\end{aligned} \tag{24}$$

For the equilibrium to be stable, the above expression must be ≥ 0 for all ξ et η small enough. Taking first $\eta = 0$, we get the following necessary conditions

$$\mathscr{E}_3(\lambda_0; v, v, v) = 0 \quad \text{and} \quad \mathscr{E}_4(\lambda_0; v, v, v, v) \geq 0 \quad \text{for all} \quad v \in V. \tag{25}$$

Remark 2. Note that, from Theorem 3, the first of these two conditions is equivalent to $E_{ijk} = 0$, for all $i, j, k = 1, \dots m$.

In other words, if there exists $v \in V$ such that $\mathscr{E}_3(\lambda_0; v, v, v) \neq 0$ or $\mathscr{E}_4(v, v, v, v) < 0$, then the equilibrium is unstable at the critical point. The above conditions are not sufficient. Indeed, assuming conditions (25) to hold, we now take $\eta = \xi^2$

$$\mathscr{E}(u_0 + u, \lambda_0) - \mathscr{E}(u_0, \lambda_0) = \frac{1}{2}\xi^4\Big[\mathscr{E}_2(\lambda_0; w, w) + \mathscr{E}_3(\lambda_0; v, v, w) + \frac{1}{12}\mathscr{E}_4(\lambda_0; v, v, v, v)\Big] + o(\xi^4) \tag{26}$$

and we get the further necessary condition

$$\mathscr{E}_2(w, w) + \mathscr{E}_3(v, v, w) + \frac{1}{12}\mathscr{E}_4(v, v, v, v) \geq 0 \quad \text{for all} \quad v \in V \quad \text{and} \quad w \in W. \tag{27}$$

The direction $v \in V$ being fixed, the above expression is minimal when w satisfies the following variational problem

$$2\mathscr{E}_2(w, \hat{w}) + \mathscr{E}_3(v, v, \hat{w}) = 0 \quad \text{for all} \quad \hat{w} \in W. \tag{28}$$

Expanding $v \in V$ in the (v_i) basis, it is observed that the solution to the above variational problem is $w = \frac{1}{2}\xi^i \xi^j w_{ij}$, where w_{ij} is the solution to the elementary variational problem (11). For this value of w, condition (27) reads

$$\left[\mathscr{E}_4(v_i, v_j, v_k, v_l) - 3\mathscr{E}_2(w_{ij}, w_{kl})\right]\xi^i \xi^j \xi^k \xi^l \geq 0 \quad \text{for all} \quad \xi^1, \dots, \xi^m \in \mathbb{R}, \tag{29}$$

which, in view of definition (20) of E_{ijkl}, is equivalent to

$$E_{ijkl}\xi^i \xi^j \xi^k \xi^l \geq 0 \quad \text{for all} \quad \xi^1, \dots, \xi^m \in \mathbb{R}. \tag{30}$$

Note that Eq. (28) implies $\mathscr{E}_4(\lambda_0; v, v, v, v) \geq 0$, which becomes a redundant necessary condition. Indeed, plugging $w = \xi^i \xi^j w_{ij}$ into Eq. (28) cancels the first two terms. To sum up, we have the following necessary conditions for stability

$$E_{ijk}\xi^i \xi^j \xi^k = 0 \quad \text{and} \quad E_{ijkl}\xi^i \xi^j \xi^k \xi^l \geq 0 \quad \text{for all} \quad \xi^1, \dots, \xi^m \in \mathbb{R}. \tag{31}$$

Conversely, the following condition is sufficient to ensure stability of the critical point

$$E_{ijk}\xi^i \xi^j \xi^k = 0 \quad \text{and} \quad E_{ijkl}\xi^i \xi^j \xi^k \xi^l > 0 \quad \text{for all} \quad \xi^1, \dots, \xi^m \in \mathbb{R}. \tag{32}$$

3. Analysis of bifurcated branches

In this section, we show that, besides the fundamental branch $u^*(\lambda)$, other (bifurcated) equilibrium branches may pass through the critical point (u_0, λ_0). The starting point is the characterization of an equilibrium by the stationarity of the energy, which defines all equilibrium branches as implicit functions, which can be expanded with respect to $\xi_1, \dots, \xi_m$ and $(\lambda - \lambda_0)$.

The first approach (see Sec. 3.1) is the (historical) Lyapunov–Schmidt decomposition of the equilibrium branch over V and W. However, this approach leads to tedious derivations. This approach has historical and pedagogical value: in particular, it provides a meaning to w_i and w_{ij} defined by Eqs. (10) and (11). An alternative, more systematic approach is developped in Sec. 3.2; I discovered this alternative approach through the paper by Chakrabarti, Mora, Richard, Phou, Fromental, Pomeau, and Audoly (2018, Appendix A).

3.1. The Lyapunov–Schmidt decomposition

The following decomposition of the equilibrium state u at the load-level $\lambda$is postulated

$$u = u^*(\lambda) + \xi_i v_i + w, \quad \text{with} \quad w \in W. \tag{33}$$

It follows from the orthogonality of V and W that $\langle v_i, w\rangle = 0$ for all $i = 1, \dots, m$. Stationarity of the energy is expressed as follows

$$\mathscr{E}_{,u}[u^*(\lambda) + \xi_i v_i + w, \lambda; \hat{u}] = 0, \quad \text{for all} \quad \hat{u} \in U \tag{34}$$

or, equivalently

$$\mathscr{E}_{,u}[u^*(\lambda) + \xi_i v_i + w, \lambda; \hat{v}] = 0, \quad \text{for all} \quad \hat{v} \in V \tag{35}$$

and

$$\mathscr{E}_{,u}[u^*(\lambda) + \xi_i v_i + w, \lambda; \hat{w}] = 0, \quad \text{for all} \quad \hat{w} \in W. \tag{36}$$

The method proceeds in three steps. In Step 1, Eq. (36) is used to define w as an implicit function of ξ_1, ..., ξ_m and λ. Then, in Step 2, Eq. (36) is used to define λ as an implicit function of ξ_1, ..., ξ_m. Finally, a parametrization η of ξ_1, ...ξ_m is introduced in Step 3 and the Taylor expansion of u and λ with respect to η is derived. These steps are presented below.

Step 1: w as a function of ξ_i and λ In this paragraph, $\hat{w}$ denotes an arbitrary test function in W. From the implicit function theorem, Eq. (36) defines a function $(\xi_1, \dots, \xi_m, \lambda) \mapsto w(\xi_1, \dots, \xi_m, \lambda)$ in the neighborhood of $(\xi_1, \dots, \xi_m, \lambda) = (0, \dots, 0, \lambda_0)$. Why the theorem applies will be clarified below. Eq. (36) is first differentiated w.r.t. ξ_i

$$\mathscr{E}_{,uu}(u^* + \xi_k v_k + w, \lambda; v_i + w_{,i}, \hat{w}) = 0. \tag{37}$$

Substituting $\xi_1 = \cdots = \xi_m = 0, \lambda = \lambda_0$ in the above equations and observing that $\mathscr{E}_2(\lambda_0; v_i, W) = 0$ since $v_i \in V$, we get

$$\mathscr{E}_2(\lambda_0; v_i + w_{,i}, \hat{w}) = \mathscr{E}_2(\lambda_0; w_{,i}, \hat{w}) = 0. \tag{38}$$

Since $w \in W$ for all ξ^i and λ, we have $w_{,i} \in W$ and, Remark 1 leads to $w_{,i} = 0$ at the point $\xi_1 = 0, \dots, \xi_m = 0$ and $\lambda = \lambda_0$. Eq. (36) is then differentiated w.r.t. λ

$$\mathscr{E}_{,uu}(u^* + \xi_i v_i + w, \lambda; \dot{u}^* + w_{,\lambda}, \hat{w}) + \mathscr{E}_{,u\lambda}(u^* + \xi_i v_i + w, \lambda; \hat{w}) = 0 \tag{39}$$

and, at $\xi_1 = \dots = \xi_m = 0$

$$\mathscr{E}_{,uu}(u^*, \lambda; w_{,\lambda}, \hat{w}) + \underbrace{\mathscr{E}_{,uu}(u^*, \lambda; \dot{u}^*, \hat{w}) + \mathscr{E}_{,u\lambda}(u^*, \lambda; \hat{w})}_{=0 \quad \text{see Eq. (2)}} = \mathscr{E}_2(\lambda; w_{,\lambda}, \hat{w}) = 0, \tag{40}$$

which proves similarly that the derivative of w with respect to λ vanishes at the critical point. We have found so far that

$$\left.\frac{\partial w}{\partial \xi_1}\right|_{\xi_1=\cdots=\xi_m=0,\lambda=\lambda_0} = \dots = \left.\frac{\partial w}{\partial \xi_m}\right|_{\xi_1=\cdots=\xi_m=0,\lambda=\lambda_0} = \left.\frac{\partial w}{\partial \lambda}\right|_{\xi_1=\cdots=\xi_m=0,\lambda=\lambda_0} = 0. \tag{41}$$

To express the second-order derivatives of w, Eq. (37) is differentiated first with respect to ξ_j, then with respect to λ. This delivers

$$\mathscr{E}_{,uuu}(u^* + \xi_k v_k + w, \lambda; v_i + w_{,i}, v_j + w_{,j}, \hat{w}) + \mathscr{E}_{,uu}(u^* + \xi_k v_k + w, \lambda; w_{,ij}, \hat{w}) = 0 \tag{42}$$

and

$$\begin{aligned}
&\mathscr{E}_{,uuu}(u^* + \xi_k v_k + w, \lambda; v_i + w_{,i}, \dot{u}^* + w_{,\lambda}, \hat{w}) \\
&+\mathscr{E}_{,uu\lambda}(u^* + \xi_k v_k + w, \lambda; v_i + w_{,i}, \hat{w}) + \mathscr{E}_{,uu}(u^* + \xi_k v_k + w, \lambda; w_{,i\lambda}, \hat{w}) = 0
\end{aligned} \tag{43}$$

and, at $\xi_1 = \cdots = \xi_m = 0, \lambda = \lambda_0$ (recalling that, at this point, $w_{,1} = \cdots = w_{,m} = w_{,\lambda} = 0$)

$$\mathscr{E}_3(\lambda_0; v_i, v_j, \hat{w}) + \mathscr{E}_2(\lambda_0; w_{,ij}, \hat{w}) = 0 \quad \text{and} \quad \dot{\mathscr{E}}_2(\lambda_0; v_i, \hat{w}) + \mathscr{E}_2(\lambda_0; w_{,i\lambda}, \hat{w}) = 0. \tag{44}$$

The variational problems (10) and (11) are recognized, leading to

$$\left.\frac{\partial^2 w}{\partial \xi_i \partial \xi_j}\right|_{\xi_1=\cdots=\xi_m=0,\lambda=\lambda_0} = w_{ij} \quad \text{and} \quad \left.\frac{\partial^2 w}{\partial \lambda \partial \xi_i}\right|_{\xi_1=\cdots=\xi_m=0,\lambda=\lambda_0} = w_i. \tag{45}$$

Finally, differentiating Eq. (40) w.r.t. λ leads to

$$\dot{\mathscr{E}}_2(\lambda; w_{,\lambda}, \hat{w}) + \mathscr{E}_2(\lambda; w_{,\lambda\lambda}, \hat{w}) = 0 \tag{46}$$

and, at $\lambda = \lambda_0$

$$\left.\frac{\partial^2 w}{\partial \lambda^2}\right|_{\xi_1=\cdots=\xi_m=0,\lambda=\lambda_0} = 0. \tag{47}$$

We have obtained the following Taylor expansion of the component w of the LSK expansion of u

$$w(\xi_1, \dots, \xi_m, \lambda) = \tfrac{1}{2}\xi_i \xi_j w_{ij} + \big(\lambda - \lambda_0\big)\xi_i w_i + o\Big(\xi_1^2 + \cdots + \xi_m^2 + \big(\lambda - \lambda_0\big)^2\Big). \tag{48}$$

Step 2: λ as a function of ξ_i We now turn to Eq. (35). Since w is a function of λ and ξ_k ($k = 1, \dots, m$) this equation implicitly defines λ as a function of ξ_k, the derivatives of which can be evaluated at $\xi_1 = \cdots = \xi_m = 0$. In this paragraph, $\hat{v}$ denotes an arbitrary element of V. Besides, unless otherwise mentioned, the differentials of the energy $\mathscr{E}_{,uu}, \mathscr{E}_{,u\lambda}, \mathscr{E}_{,\lambda\lambda}, \mathscr{E}_{,uuu}$ ... are evaluated at $u = u^*(\lambda) + \xi_k v_k + w(\xi_k, \lambda)$. Differentiating first Eq. (35) with respect to ξ_i

$$\mathscr{E}_{,uu}[v_i + w_{,i} + \lambda_{,i}(\dot{u}^* + w_{,\lambda}), \hat{v}] + \lambda_{,i}\mathscr{E}_{,u\lambda}(\hat{v}) = 0, \tag{49}$$

then with respect to ξ_j

$$\begin{aligned}
&\mathscr{E}_{,uuu}[v_i + w_{,i} + \lambda_{,i}(\dot{u}^* + w_{,\lambda}), v_j + w_{,j} + \lambda_{,j}(\dot{u}^* + w_{,\lambda}), \hat{v}] \\
&+\lambda_{,j}\mathscr{E}_{,uu\lambda}[v_i + w_{,i} + \lambda_{,i}(\dot{u}^* + w_{,\lambda}), \hat{v}] \\
&+\mathscr{E}_{,uu}[w_{,ij} + \lambda_{,ij}(\dot{u}^* + w_{,\lambda}) + \lambda_{,i}\lambda_{,j}(\ddot{u}^* + w_{,\lambda\lambda}), \hat{v}] \\
&+\lambda_{,ij}\mathscr{E}_{,u\lambda}(\hat{v}) + \lambda_{,i}\mathscr{E}_{,uu\lambda}[v_j + w_{,j} + \lambda_{,j}(\dot{u}^* + w_{,\lambda}), \hat{v}] + \lambda_{,i}\lambda_{,j}\mathscr{E}_{,u\lambda\lambda}(\hat{v}) = 0,
\end{aligned} \tag{50}$$

Eqs. (49) and (50) are then evaluated at $\xi_1 = \cdots = \xi_m = 0$, delivering

$$\underbrace{\mathscr{E}_{,uu}(u_0, \lambda_0; v_i, \hat{v})}_{=0 \text{ since } \hat{v}\in V} + \lambda_{,i}\underbrace{\Big[\mathscr{E}_{,uu}(u_0, \lambda_0; \dot{u}_0, \hat{v}) + \mathscr{E}_{,u\lambda}(u_0, \lambda_0; \hat{v})\Big]}_{=0 \text{ from Eq. (2)}} = 0, \tag{51}$$

and

$$
\begin{aligned}
&\mathscr{E}_{,uuu}(u_0,\lambda_0;v_i,v_j,\hat{v}) + \underbrace{\mathscr{E}_{,uu}(u_0,\lambda_0;w_{ij},\hat{v})}_{=0 \text{ since } \hat{v}\in V} \\
&+\lambda_{,i}\Big[\mathscr{E}_{,uuu}(u_0,\lambda_0;v_j,\dot{u}_0,\hat{v}) + \mathscr{E}_{,uu\lambda}[u_0,\lambda_0;v_j,\hat{v}]\Big] \\
&+\lambda_{,j}\Big[\mathscr{E}_{,uuu}(u_0,\lambda_0;v_i,\dot{u}_0,\hat{v}) + \mathscr{E}_{,uu\lambda}(u_0,\lambda_0;v_i,\hat{v})\Big] \\
&+\lambda_{,ij}\underbrace{\Big[\mathscr{E}_{,uu}(u_0,\lambda_0;\dot{u}_0,\hat{v}) + \mathscr{E}_{,u\lambda}(u_0,\lambda_0;\hat{v})\Big]}_{=0 \text{ from Eq. (2)}} \\
&+\lambda_{,i}\lambda_{,j}\underbrace{\Big[\mathscr{E}_{,uuu}(u_0,\lambda_0;\dot{u}_0,\dot{u}_0,\hat{v}) + 2\mathscr{E}_{,uu\lambda}(u_0,\lambda_0;\dot{u}_0,\hat{v}) + \mathscr{E}_{,u\lambda\lambda}(u_0,\lambda_0;\hat{v}) + \mathscr{E}_{,uu}(u_0,\lambda_0;\ddot{u}_0,\hat{v})\Big]}_{=0 \text{ from Eq. (3)}} = 0
\end{aligned}
\tag{52}
$$

Eq. (51) is non-informative (identically satisfied), while Eq. (52) simplifies as follows

$$
\begin{aligned}
&\mathscr{E}_{,uuu}(u_0,\lambda_0;v_i,v_j,\hat{v}) + \lambda_{,i}\underbrace{\Big[\mathscr{E}_{,uuu}(u_0,\lambda_0;v_j,\dot{u}_0,\hat{v}) + \mathscr{E}_{,uu\lambda}(u_0,\lambda_0;v_j,\hat{v})\Big]}_{=\dot{\mathscr{E}}_2(\lambda_0;v_j,\hat{v})} \\
&+\lambda_{,j}\underbrace{\Big[\mathscr{E}_{,uuu}(u_0,\lambda_0;v_i,\dot{u}_0,\hat{v}) + \mathscr{E}_{,uu\lambda}(u_0,\lambda_0;v_i,\hat{v})\Big]}_{\lambda_{,j}\dot{\mathscr{E}}_2(\lambda_0;v_i,\hat{v})} = 0
\end{aligned}
\tag{53}
$$

and, recognizing derivatives of $\mathscr{E}_2$ with respect to λ, we finally get

$$
\mathscr{E}_3(\lambda_0;v_i,v_j,\hat{v}) + \lambda_{,i}\dot{\mathscr{E}}_2(\lambda_0;v_j,\hat{v}) + \lambda_{,j}\dot{\mathscr{E}}_2(\lambda_0;v_i,\hat{v}) = 0. \tag{54}
$$

Testing with $v_k \in V$, the above equation reads

$$
\mathscr{E}_3(\lambda_0;v_i,v_j,v_k) + \lambda_{,i}\dot{\mathscr{E}}_2(\lambda_0;v_j,v_k) + \lambda_{,j}\dot{\mathscr{E}}_2(\lambda_0;v_i,v_k) = 0, \tag{55}
$$

or, with Eqs. (17) and (19)

$$
E_{ijk} + F_{jk}\left.\frac{\partial\lambda}{\partial\xi_i}\right|_{\xi_1=\cdots=\xi_m=0} + F_{ik}\left.\frac{\partial\lambda}{\partial\xi_j}\right|_{\xi_1=\cdots=\xi_m=0} = 0. \tag{56}
$$

In order to evaluate the second order partial derivatives of λ, Eq. (50) should be further differentiated with respect to ξ^k. This leads to extremely tedious derivations, and we will adopt an alternative approach in Sec. 3.2.

Step 3: parametrization of the bifurcated branch The bifurcated branch is a curve $(u,\lambda) \in \mathbb{R}^{m+1}$, which is parametrized by η: $[u(\eta),\lambda(\eta)]$, with $u(0) = u_0$ and $\lambda(0) = \lambda_0$; primed quantities denoting derivatives with respect to η, we introduce

$$
\xi_i^I = \xi_i'(0), \quad \xi_i^{II} = \xi_i''(0), \quad \dots, \quad \lambda^I = \lambda'(0), \quad \dots \tag{57}
$$

and first observe that

$$\lambda^I = \xi_i^I \frac{\partial \lambda}{\partial \xi_i}\bigg|_{\xi_1=\cdots=\xi_m=0} \tag{58}$$

Multiplying both sides of Eq. (56) by $\xi_i^I \xi_j^I$ therefore results in the following identity

$$\begin{aligned} 0 &= E_{ijk}\xi_i^I\xi_j^I + F_{jk}\xi_i^I\xi_j^I \frac{\partial \lambda}{\partial \xi_i}\bigg|_{\xi_1=\cdots=\xi_m=0} + F_{ik}\xi_i^I\xi_j^I \frac{\partial \lambda}{\partial \xi_j}\bigg|_{\xi_1=\cdots=\xi_m=0} \\ &= E_{ijk}\xi_i^I\xi_j^I + F_{jk}\lambda^I\xi_j^I + F_{ik}\xi_i^I\lambda^I \end{aligned} \tag{59}$$

and, rearranging

$$E_{ijk}\xi_j^I\xi_k^I + 2\lambda^I F_{ij}\xi_j^I = 0, \tag{60}$$

to be compared with Eq. (71). We now turn to w

$$w'(\eta) = w_{,i}\xi_i' + w_{,\lambda}\lambda' \quad \text{and} \quad w''(\eta) = w_{,ij}\xi_i'\xi_j' + 2w_{,i\lambda}\xi_i'\lambda' + w_{,i}\xi_i'' + w_{,\lambda\lambda}\lambda'^2 + w_{,\lambda}\lambda'' \tag{61}$$

and, at $\eta = 0$

$$w'(0) = 0 \quad \text{and} \quad w''(0) = \xi_i^I\xi_j^{II} w_{ij} + 2\lambda^I\xi_i^I w_i \tag{62}$$

and we get the Taylor expansion of the bifurcated branch as $\eta \to 0$

$$u(\eta) = u^*[\lambda(\eta)] + \xi_i^I v_i + \frac{1}{2}\left(\xi_i^{II} v_i + \xi_i^I\xi_j^{II} w_{ij} + 2\lambda^I\xi_i^I w_i\right) + o(\eta^2), \tag{63}$$

to be compared with Eq. (75).

3.2. Alternative route to the asymptotic expansions

Following the Appendix A of Ref. (Chakrabarti, Mora, Richard, Phou, Fromental, Pomeau, and Audoly 2018), we introduce the following parametrization of the bifurcated branch

$$\lambda = \lambda_0 + \eta\lambda_1 + \frac{1}{2}\eta^2\lambda_2 + \frac{1}{6}\eta^3\lambda_3 + \cdots, \tag{64}$$

$$u = u^*(\lambda) + \eta u_1 + \frac{1}{2}\eta^2 u_2 + \frac{1}{6}\eta^3 u_3 + \cdots, \tag{65}$$

where the parameter η is not specified, but for the fact that $\eta = 0$ corresponds to the critical point (u_0, λ_0). Note that, similarly to the Lyapunov–Schmidt decomposition (33), the bifurcated branch $u(\lambda)$ is compared to the fundamental branch $u^*(\lambda)$, rather than with the critical point u_0.

Les coefficients λ_k et u_k des développements (64) et (65) sont identifiés en écrivant que l'énergie est stationnaire le long de la courbe d'équilibre, c'est-à-dire que le résidu $\mathscr{E}_{,u}[u(\eta), \lambda(\eta)]$ est nul. Le développement limité du résidu est établi au voisinage de $\eta = 0$ dans l'annexe A.2 [voir Éq. (118)]. En écrivant que tous ses termes s'annulent, on trouve successivement, pour tout $\hat{u} \in U$

$$\mathscr{E}_2(\lambda_0; u_1, \hat{u}) = 0, \tag{66}$$

$$\mathscr{E}_3(\lambda_0; u_1, u_1, \hat{u}) + 2\lambda_1 \dot{\mathscr{E}}_2(\lambda_0; u_1, \hat{u}) + \mathscr{E}_2(\lambda_0; u_2, \hat{u}) = 0, \tag{67}$$

$$\begin{aligned}\mathscr{E}_4(\lambda_0; u_1, u_1, u_1, \hat{u}) + 3\mathscr{E}_3(\lambda_0; u_1, u_2, \hat{u}) + \mathscr{E}_2(\lambda_0; u_3, \hat{u})\\ +3\lambda_1 \dot{\mathscr{E}}_3(\lambda_0; u_1, u_1, \hat{u}) + 3\lambda_1 \dot{\mathscr{E}}_2(\lambda_0; u_2, \hat{u})\\ +3\lambda_1^2 \ddot{\mathscr{E}}_2(\lambda_0; u_1, \hat{u}) + 3\lambda_2 \dot{\mathscr{E}}_2(\lambda_0; u_1, \hat{u}) = 0.\end{aligned} \tag{68}$$

On déduit de l'équation (66) que $u_1 \in V$. En prenant la fonction test également dans V, on déduit de l'équation (67) que u_1 est solution du problème suivant : trouver $u_1 \in V$ tel que

$$\frac{1}{2}\mathscr{E}_3(\lambda_0; u_1, u_1, \hat{v}) + \lambda_1 \dot{\mathscr{E}}_2(\lambda_0; u_1, \hat{v}) = 0, \tag{69}$$

pour tout $\hat{v} \in V$. The above problem can be transformed into a system of scalar equations. Indeed, expanding the $u_1 \in V$ in the basis $(v_i)_{1 \le i \le m}$ as follows

$$u_1 = \xi_1^i v_i \tag{70}$$

and plugging the definitions (17) and (19) of E_{ijk} and F_{ij} into Eq. (69)

$$\frac{1}{2}E_{ijk}\xi_1^j \xi_1^k + \lambda_1 F_{ij}\xi_1^j = 0. \tag{71}$$

On obtient ainsi un système de m équations quadratiques à $(m+1)$ inconnues, qui permet en général de déterminer les valeurs de λ_1 et u_1(voir discussion ci-après).

Afin de déterminer les termes suivants du développement asymptotique de la branche bifurquée, soit λ_2 et u_2, on introduit la décomposition

$$u_2 = \xi_2^i v_i + \tilde{u}_2, \tag{72}$$

où $\tilde{u}_2 \in W$ est la projection orthogonale de u_2 sur W. On a alors $\mathscr{E}_2(u_2, \hat{u}) = \mathscr{E}_2(\tilde{u}_2, \hat{u})$ et l'équation (67) s'écrit

$$\mathscr{E}_3(\lambda_0; u_1, u_1, \hat{u}) + 2\lambda_1 \dot{\mathscr{E}}_2(\lambda_0; u_1, \hat{u}) + \mathscr{E}_2(\lambda_0; \tilde{u}_2, \hat{u}) = 0, \tag{73}$$

pour tout $\hat{u} \in U$. En prenant cette fois-ci la fonction test dans l'espace W, on obtient le problème variationnel suivant : trouver $\tilde{u}_2 \in W$ tel que

$$\mathscr{E}_2(\lambda_0; \tilde{u}_2, \hat{w}) + \xi_1^i \xi_1^j \mathscr{E}_3(\lambda_0; v_i, v_j, \hat{w}) + 2\lambda_1 \xi_1^i \dot{\mathscr{E}}_2(\lambda_0; v_i, \hat{w}) = 0, \tag{74}$$

pour tout $\hat{w} \in W$. The solution to the variational problem (74) is expressed as a linear combination of the w_i and w_{ij} [defined by the variational problems (10) and (11)]: $\tilde{u}_2 = \xi_1^i \xi_1^j w_{ij} + \lambda_1 \xi_1^i w_i$ and

$$u_2 = \xi_2^i v_i + \xi_1^i \xi_1^j w_{ij} + \lambda_1 \xi_1^i w_i. \tag{75}$$

Plugging expressions (70) and (75) into Eq. (68) and taking further $\hat{u} = v_i$ [remember that $\mathscr{E}_2(\lambda_0; v_i, \bullet) = 0$], we then get

$$\begin{aligned}&\Big[\mathscr{E}_4(\lambda_0; v_i, v_j, v_k, v_l) + 3\mathscr{E}_3(\lambda_0; v_i, v_j, w_{kl})\Big]\xi_1^j \xi_1^k \xi_1^l\\ &\quad + 3\lambda_1\Big[\mathscr{E}_3(\lambda_0; v_i, v_j, w_k) + \dot{\mathscr{E}}_3(\lambda_0; v_i, v_j, v_k) + \dot{\mathscr{E}}_2(\lambda_0; v_i, w_{jk})\Big]\xi_1^j \xi_1^k\\ &\quad + 3\Big[\lambda_1^2 \dot{\mathscr{E}}_2(\lambda_0; v_i, w_j) + \lambda_1^2 \ddot{\mathscr{E}}_2(\lambda_0; v_i, v_j) + \lambda_2 \dot{\mathscr{E}}_2(\lambda_0; v_i, v_j)\Big]\xi_1^j\\ &\quad + 3\Big[\mathscr{E}_3(\lambda_0; v_i, v_j, v_k)\xi_1^k + \lambda_1 \dot{\mathscr{E}}_2(\lambda_0; v_i, v_j)\Big]\xi_2^j = 0\end{aligned}$$

It results from the variational problems (10) and (11) that

$$\mathscr{E}_3(\lambda_0; v_i, v_j, w_k) = -\mathscr{E}_2(\lambda_0; w_{ij}, w_k) = 2\dot{\mathscr{E}}_2(\lambda_0; v_k, w_{ij}),$$

therefore

$$\begin{aligned}\mathscr{E}_3(\lambda_0; v_i, v_j, w_k)\xi_1^j\xi_1^k &= \tfrac{1}{2}\Big[\mathscr{E}_3(\lambda_0; v_i, v_j, w_k) + \mathscr{E}_3(\lambda_0; v_i, v_k, w_j)\Big]\xi_1^j\xi_1^k \\ &= \Big[\dot{\mathscr{E}}_2(\lambda_0; v_k, w_{ij}) + \dot{\mathscr{E}}_2(\lambda_0; v_j, w_{ik})\Big]\xi_1^j\xi_1^k.\end{aligned}$$

Similarly,

$$\begin{aligned}\dot{\mathscr{E}}_2(\lambda_0; v_i, w_j) &= -\tfrac{1}{2}\mathscr{E}_2(\lambda_0; w_i, w_j) = -\tfrac{1}{2}\mathscr{E}_2(\lambda_0; w_j, w_i) = \dot{\mathscr{E}}_2(\lambda_0; v_j, w_i) \\ &= \tfrac{1}{2}\Big[\dot{\mathscr{E}}_2(\lambda_0; v_i, w_j) + \dot{\mathscr{E}}_2(\lambda_0; v_j, w_i)\Big].\end{aligned}$$

Finally, the definitions (15), (16), (17), (18) and (19) of E_{ijk}, E_{ijkl}, F_{ij}, $\mathring{E}_{ijk}$ and $\mathring{F}_{ij}$ lead to the following compact bifurcation equation

$$\tfrac{1}{3}E_{ijkl}\xi_1^j\xi_1^k\xi_1^l + \lambda_1\Big(\mathring{E}_{ijk}\xi_1^k + \lambda_1\mathring{F}_{ij}\Big)\xi_1^j + \Big(E_{ijk}\xi_1^k + \lambda_1 F_{ij}\Big)\xi_2^j + \lambda_2 F_{ij}\xi_1^j = 0. \tag{76}$$

In order to analyse the stability of the bifurcated branches thus found, one must look at the Hessian of the energy. It is first observed that, on the fundamental branch

$$\mathscr{E}_2(\lambda; \hat{u}, \hat{v}) = \mathscr{E}_2(\lambda_0; \hat{u}, \hat{v}) + \big(\lambda - \lambda_0\big)\dot{\mathscr{E}}_2(\lambda_0; \hat{u}, \hat{v}) + o(\lambda - \lambda_0). \tag{77}$$

In what follows, it will be assumed that $\dot{\mathscr{E}}_2(\lambda_0) \neq 0$ and that $\mathscr{E}_2(\lambda)$ (which is positive definite over V for $\lambda < \lambda_0$ and null for $\lambda = \lambda_0$) is negative definite for $\lambda > \lambda_0$ sufficiently small (the fundamental branch is strictly unstable beyond the critical load). From the above expansion, it results that $\dot{\mathscr{E}}_2(\lambda_0)$ is negative definite over V. In other words, $-F_{ij}$ is a positive definite tensor. The asymptotic expansion of the Hessian of the energy along the bifurcated branch is derived in appendix A.4. For all $\hat{u}, \hat{v} \in U$

$$\begin{aligned}\mathscr{E}_{,uu}[u(\eta), \lambda(\eta); \hat{u}, \hat{v}] &= \mathscr{E}_2(\lambda_0; \hat{u}, \hat{v}) + \eta\Big[\mathscr{E}_3(\lambda_0; u_1, \hat{u}, \hat{v}) + \lambda_1\dot{\mathscr{E}}_2(\lambda_0; \hat{u}, \hat{v})\Big] \\ &+ \tfrac{1}{2}\eta^2\Big[\mathscr{E}_4(\lambda_0; u_1, u_1, \hat{u}, \hat{v}) + \mathscr{E}_3(\lambda_0; u_2, \hat{u}, \hat{v}) + 2\lambda_1\dot{\mathscr{E}}_3(\lambda_0; u_1, \hat{u}, \hat{v}) \\ &+ \lambda_1^2\ddot{\mathscr{E}}_2(\lambda_0; \hat{u}, \hat{v}) + \lambda_2\dot{\mathscr{E}}_2(\lambda_0; \hat{u}, \hat{v})\Big] + o(\eta^2).\end{aligned} \tag{78}$$

Stability analysis is performed by means of the eigenvalues $\alpha \in \mathbb{R}$ and eigenvectors $x \in U$ of the Hessian

$$\mathscr{E}_{,uu}[u(\eta), \lambda(\eta); x, \hat{u}] = \alpha\langle x, \hat{u}\rangle \quad \text{for all} \quad \hat{u} \in V, \tag{79}$$

where α and x are expanded to second order in η

$$\alpha = \alpha_0 + \eta\alpha_1 + \tfrac{1}{2}\eta^2\alpha_2 + o(\eta^2) \quad \text{and} \quad x = x_0 + \eta x_1 + \tfrac{1}{2}\eta^2 x_2 + o(\eta^2). \tag{80}$$

The following results are proved in Appendix A.5: first, (α_0, x_0) is necessarily an eigenpair of $\mathscr{E}_2(\lambda_0)$. Since $\mathscr{E}_2(\lambda_0)$ is positive, $\alpha_0 \geq 0$. If $\alpha_0 > 0$, then $\alpha > 0$ in the neighborhood of λ_0. Potentially unstable modes are therefore such that $\alpha_0 = 0$. In other words, $x_0 \in V$; furthermore, (α_1, χ_0^i) is an eigenpair of the symmetric tensor $(E_{ijk}\xi_1^k + \lambda_1 F_{ij})$

$$x_0 = \chi_0^i v_i \quad \text{and} \quad \left(E_{ijk}\xi_1^k + \lambda_1 F_{ij}\right)\chi_0^j = \alpha_1 \chi_0^i. \tag{81}$$

As for the higher order terms, it is also found that

$$x_1 = \chi_1^i v_i + \chi_0^i \xi_1^j w_{ij} + \frac{1}{2}\lambda_1 \chi_0^i w_i \tag{82}$$

and

$$\begin{aligned}\Big[E_{ijkl}\xi_1^k\xi_1^l + \lambda_1\big(2\mathring{E}_{ijk}\xi_1^k + \lambda_1\mathring{F}_{ij}\big) + E_{ijk}\xi_2^k + \lambda_2 F_{ij}\Big]\chi_0^j \\ + 2\big(E_{ijk}\xi_1^k + \lambda_1 F_{ij}\big)\chi_1^j = 2\alpha_1\chi_1^i + \alpha_2\chi_0^i.\end{aligned} \tag{83}$$

Finally, to close this analysis of the bifurcated branches, the following asymptotic expansion of the energy is derived in Appendix A.3

$$\begin{aligned}\mathscr{E}[u(\eta), \lambda(\eta)] = \mathscr{E}\{u^*[\lambda(\eta)], \lambda(\eta)\} + \frac{1}{6}\lambda_1\eta^3 F_{ij}\xi_1^i\xi_1^j \\ + \frac{1}{24}\eta^4\Big[E_{ijkl}\xi_1^i\xi_1^j\xi_1^k\xi_1^l + 4\lambda_1\mathring{E}_{ijk}\xi_1^i\xi_1^j\xi_1^k + 6\big(\lambda_1^2\mathring{F}_{ij} + \lambda_2 F_{ij}\big)\xi_1^i\xi_1^j\Big] + o(\eta^4).\end{aligned} \tag{84}$$

4. Discussion

In this section, we discuss the two main cases of bifurcations, namely asymmetric and symmetric. In each case, we analyse the stability of the bifurcated branch.

Remark 3. The boundary case is unclear to me. I think that whether a bifurcation is symmetric or asymmetric should depend on the value of λ_1 only. If $\lambda_1 \neq 0$, the bifurcated branch is asymmetric. Conversely, if $\lambda_1 = 0$ and $\lambda_2 \neq 0$, then the bifurcated branch is symmetric.

In the literature, the discussion is placed on E_{ijk}. If $\lambda_1 \neq 0$, surely one of the E_{ijk} is non-zero also. However, I believe it is not a sufficient condition: one of the bifurcated branches could be symmetric ($\lambda_1 = 0$), even if all E_{ijk} are not null. It is true however that all bifurcated branches are symmetric if, and only if, $E_{ijk} = 0$ for all $i, j, k = 1, \ldots, m$. Therefore, the two cases that will be discussed below are: (1) one of the bifurcated branches is asymmetric and (2) all bifurcated branches are symmetric. The mixed case "one of the bifurcated branches is symmetric" will not be discussed.

4.1. Asymmetric bifurcated branch ($\lambda_1 \neq 0$)

We first consider the situation where $\lambda_1 \neq 0$ on the bifurcated branch. The bifurcation equation (71) shows that necessarily, E_{ijk} is not identically nul. This equation has at most $(2^m - 1)$ pairs of real solutions (λ_1, u_1) et $(-\lambda_1, -u_1)$; furthermore, multiplication by ξ_1^i shows that

$$\lambda_1 = -\frac{E_{ijk}\xi_1^i \xi_1^j \xi_1^k}{2F_{ij}\xi_1^i \xi_1^j}. \tag{85}$$

Remark 4. I can't prove that the bifurcation equation (71) has at most $(2^m - 1)$ pairs of real solutions.

Along the bifurcated branch, we have $\lambda = \lambda_0 + \eta\lambda_1 + o(\eta)$, and η can be eliminated. In other words, $\eta = \lambda$ ($\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = \cdots = 0$) can be selected as a parameter. It is therefore possible to express the bifurcated branch as a function of λ, $u(\lambda)$. For example, combining Eqs. (69) and (78), we find that

$$\begin{aligned}\mathscr{E}_{,uu}[u(\eta), \lambda(\eta); u_1, u_1] &= \eta\Big[\mathscr{E}_3(\lambda_0; u_1, u_1, u_1) + \lambda_1\dot{\mathscr{E}}_2(\lambda_0; u_1, u_1)\Big] + o(\eta) \\ &= -\eta\lambda_1\dot{\mathscr{E}}_2(\lambda_0; u_1, u_1) + o(\eta),\end{aligned} \tag{86}$$

or

$$\mathscr{E}_{,uu}[u(\lambda), \lambda; u_1, u_1] = -\big(\lambda - \lambda_0\big)\dot{\mathscr{E}}_2(\lambda_0; u_1, u_1) + o(\lambda - \lambda_0). \tag{87}$$

For $\lambda < \lambda_0$, the above quantity is negative (since $\dot{\mathscr{E}}_2$ is negative definite). In other words

For asymmetric bifurcations, below the critical load, the bifurcated branch is unstable

To investigate the stability above the critical load, we need to analyse the sign of the eigenvalues α of the Hessian. At first order, $\alpha = \eta\alpha_1 + o(\eta)$, where α_1 is an eigenvalue of $(E_{ijk}\xi_1^k + \lambda_1 F_{ij})$. Let $\alpha_{\min}$ and $\alpha_{\max}$ be the minimum and maximum eigenvalues of this second-order tensor. Three cases must be discussed

1. If $\alpha_{\min}\alpha_{\max} > 0$, then $(E_{ijk}\xi_1^k + \lambda_1 F_{ij})$ is positive or negative definite: all eigenvalues have the same sign, $\epsilon \in \{-1, +1\}$. Then the sign of the eigenvalues α of the Hessian is $\epsilon\eta$ and there is a stability switch at the critical load. Since the bifurcated branch is unstable below the critical load, this means that it is stable above the critical load.

2. If $\alpha_{\min}\alpha_{\max} < 0$, then the extremal eigenvalues of the Hessian are $\eta\alpha_{\min}$ and $\eta\alpha_{\max}$, the product of which is $\eta^2\alpha_{\min}\alpha_{\max} < 0$. The bifurcated branch is unstable for all values of λ.

3. If $\alpha_{\min}\alpha_{\max} = 0$, the analysis is inconclusive.

To close this section, it is observed that the dominant term of the expansion (84) of the potential energy along the bifurcated branch is of the third order in η

$$\mathscr{E}[u(\eta), \lambda(\eta)] = \mathscr{E}\{u^*[\lambda(\eta)], \lambda(\eta)\} + \tfrac{1}{6}\lambda_1\eta^3 F_{ij}\xi_1^i\xi_1^j + o(\eta^3). \tag{88}$$

Eliminating λ and plugging expression (85) of λ_1 delivers the expression of the potential energy, where λ is the parameter

$$
\begin{aligned}
\mathscr{E}[u(\lambda), \lambda] &= \mathscr{E}[u^*(\lambda), \lambda] + \frac{\left(\lambda - \lambda_0\right)^3}{6\lambda_1^2} F_{ij}\xi_1^i\xi_1^j + o(\lambda^3) \\
&= \mathscr{E}[u^*(\lambda), \lambda] + \frac{2\left(F_{ij}\xi_1^i\xi_1^j\right)^3}{3\left(E_{ijk}\xi_1^i\xi_1^j\xi_1^k\right)^2}\left(\lambda - \lambda_0\right)^3 + o(\lambda^3).
\end{aligned}
\tag{89}
$$

Recalling that $F_{ij}\xi_1^i\xi_1^j < 0$, it is found that, above the critical load, the potential energy is smaller along the bifurcated branch than along the fundamental branch.

Remark 5. As expected, the above expression does not depend on the scaling of u_1 (of the ξ_1^i).

Remark 6. It has been shown in Sec. 2 that, when E_{ijk} is not identically null, the bifurcation point is unstable.

4.2. A particular case of symmetric bifurcation

We now consider the case $E_{ijk} = 0$ for all $i, j, k = 1, \dots, m$. Then [see Eq. (71)] $\lambda_1 = 0$ on all bifurcated branches. It is assumed that, on the bifurcated branch under consideration, the next term of the expansion of λ is non-zero: $\lambda_2 \neq 0$. The bifurcation is symmetric, and the bifurcation equation (76) reduces to

$$
\tfrac{1}{3} E_{ijkl}\xi_1^j\xi_1^k\xi_1^l + \lambda_2 F_{ij}\xi_1^j = 0,
\tag{90}
$$

which has at most $(3^m - 1)/2$ pairs of real solutions (λ_2, u_1) and $(-\lambda_2, -u_1)$. Upon multiplication by ξ_1^i, the above equation delivers the following expression of λ_2

$$
\lambda_2 = -\frac{E_{ijkl}\xi_1^i\xi_1^j\xi_1^k\xi_1^l}{3F_{ij}\xi_1^i\xi_1^j}.
\tag{91}
$$

Since $F_{ij}\xi_1^i\xi_1^j < 0$, λ_2 has the same sign as $E_{ijkl}\xi_1^i\xi_1^j\xi_1^k\xi_1^l$. In other words, if $E_{ijkl}\xi_1^i\xi_1^j\xi_1^k\xi_1^l > 0$, (resp. < 0) then the bifurcated branch exists above (resp. below) the critical load λ_0 only.

Remark 7. I can't prove that the bifurcation equation (90) has at most $(3^m - 1)/2$ pairs of real solutions.

Turning now to the eigenpairs of the Hessian of the energy along the bifurcated branch, Eq. (81) shows that $\alpha_1 = 0$. Then $\alpha = \alpha_2\eta^2/2 + o(\eta^2)$ and, from Eq. (83)

$$
\left(E_{ijkl}\xi_1^k\xi_1^l + \lambda_2 F_{ij}\right)\chi_0^j = \alpha_2\chi_0^i.
\tag{92}
$$

If $(E_{ijkl}\xi_1^k\xi_1^l + \lambda_2 F_{ij})$ is positive definite, then the bifurcated branch is stable (note that, in that case, the bifurcated branch exists above the critical load only). If one of the eigenvalues of this tensor is < 0, then the bifurcated branch is unstable. The stability is undecided when all eigenvalues are ≥ 0.

Remark 8. Note that, from Eq. (90),

$$E_{ijkl}\xi_1^i\xi_1^j\xi_1^k\xi_1^l + \lambda_2 F_{ij}\xi_1^i\xi_1^j = \tfrac{2}{3}E_{ijkl}\xi_1^i\xi_1^j\xi_1^k\xi_1^l \tag{93}$$

To conclude this section, it is observed that, when $\lambda_1 = 0$, the dominant term of the potential energy along the bifurcated branch is of the fourth order [see Eq. (84)]. Combining with Eq. (91),

$$\begin{aligned}\mathscr{E}[u(\eta), \lambda(\eta)] &= \mathscr{E}\{u^*[\lambda(\eta)], \lambda(\eta)\} + \tfrac{1}{24}\eta^4\Big(E_{ijkl}\xi_1^i\xi_1^j\xi_1^k\xi_1^l + 6\lambda_2 F_{ij}\xi_1^i\xi_1^j\Big) + o(\eta^4)\\ &= \mathscr{E}\{u^*[\lambda(\eta)], \lambda(\eta)\} - \tfrac{1}{24}\eta^4 E_{ijkl}\xi_1^i\xi_1^j\xi_1^k\xi_1^l + o(\eta^4).\end{aligned} \tag{94}$$

The expansion $\lambda = \lambda_0 + \lambda_2\eta^2/2 + o(\eta^2)$ can be inverted as follows

$$\eta^4 = \frac{4\big(\lambda - \lambda_0\big)^2}{\lambda_2^2} + o(\lambda^2) = \frac{36\big(F_{ij}\xi_1^i\xi_1^j\big)^2}{\big(E_{ijkl}\xi_1^i\xi_1^j\xi_1^k\xi_1^l\big)^2}\big(\lambda - \lambda_0\big)^2 \tag{95}$$

and expression (94) reads

$$\mathscr{E}[u(\eta), \lambda(\eta)] = \mathscr{E}\{u^*[\lambda(\eta)], \lambda(\eta)\} - \frac{3\big(F_{ij}\xi_1^i\xi_1^j\big)^2}{2E_{ijkl}\xi_1^i\xi_1^j\xi_1^k\xi_1^l}\big(\lambda - \lambda_0\big)^2 + o(\lambda^2). \tag{96}$$

Again, the above expression does not depend on the scaling of u_1 (of the ξ_1^i). Note that, if $E_{ijkl}\xi_1^i\xi_1^j\xi_1^k\xi_1^l > 0$, then only loads that are greater than the critical load can be reached on the bifurcated branch, where the energy is lower than the fundamental branch.

The above discussion simplifies considerably when there is only one buckling mode ($m = 1$). This is addressed in the next section.

5. The case of a single mode

In this section, we discuss the case $m = 1$; all tensors considered above (F_{ij}, E_{ijk}, E_{ijkl}) then reduce to simple scalars. To avoid ambiguity, indices are kept: F_{11}, E_{11}, E_{11}. Since $\dot{\mathscr{E}}_2(\lambda_0)$ is negative definite over V, we have $F_{11} < 0$.

It is first observed that the following conditions are necessary to ensure stability of the critical point

$$E_{111} = 0 \quad \text{and} \quad E_{1111} \geq 0, \tag{97}$$

which shows that asymmetric bifurcation points are always unstable.

5.1. Asymmetric bifurcations

We first consider the case $E_{111} \neq 0$. Owing to the discussion above, the bifurcation point is unstable. Setting $\lambda_1 = 1$, Eq. (71) delivers

$$E_{111}\xi_1^1 + 2F_{11} = 0 \quad \text{and} \quad u(\lambda) = u^*(\lambda) - \frac{2F_{11}}{E_{111}}\big(\lambda - \lambda_0\big)v_1 + o(\lambda - \lambda_0). \tag{98}$$

Furthermore, the hessian of the energy along the bifurcated branch is retrieved from Eq. (87)

$$\begin{aligned}\mathscr{E}_{,uu}[u(\eta), \lambda(\eta), v_1, v_1] &= \eta\Big(E_{111}\xi_1^1 + \lambda_1 F_{11}\Big) + o(\eta) = -2\eta F_{11} + o(\eta) \\ &= -2F_{11}\big(\lambda - \lambda_0\big) + o(\lambda - \lambda_0).\end{aligned} \tag{99}$$

Asymmetric bifurcations branches are unstable for $\lambda \leq \lambda_0$ and stable for $\lambda > \lambda_0$ (stability switch).

5.2. Symmetric bifurcations

We now consider the case $E_{111} = 0$. From the general discussion of Sec. 2, the bifurcation point is stable if $E_{1111} > 0$ and unstable if $E_{1111} < 0$. The bifurcation equation (90) reduces to

$$E_{1111}\big(\xi_1^1\big)^2 + 3\lambda_2 F_{11} = 0, \tag{100}$$

which in particular shows that λ_2 has the same sign as E_{1111}. Since the expansion of λ reads: $\lambda = \lambda_0 + \lambda_2\eta^2$, the bifurcation branch exists only for loads above the critical load ($\lambda \geq \lambda_0$) if $E_{1111} > 0$ and only for loads below the critical load ($\lambda \leq \lambda_0$) if $E_{1111} < 0$.

From Eq. (78), the hessian of the energy along the bifurcated branch reads

$$\mathscr{E}_{,uu}[u(\eta), \lambda(\eta); v_1, v_1] = \tfrac{1}{2}\eta^2\Big[E_{1111}\big(\xi_1^1\big)^2 + \lambda_2 F_{11}\Big] + o(\eta^2) = -\eta^2\lambda_2 F_{11} + o(\eta^2),$$

which has the sign of λ_2. Therefore the Hessian is positive (resp. negative) definite if $E_{1111} > 0$ (resp < 0).

To sum up, if $E_{1111} > 0$, then the bifurcation branch (including the critical point) is stable and exists only for loads greater than the critical load. Conversely, if $E_{1111} < 0$, then the bifurcation branch (including the critical point) is unstable and exists only for loads lower than the critical load.

6. Propriétés des formes bilinéaires symétriques, positives

Dans ce qui suit, $\mathscr{B}$ désigne une forme bilinéaire symétrique et positive sur l'espace vectoriel U. On définit son noyau $\ker \mathscr{B}$ de la façon suivante

$$\ker \mathscr{B} = \Big\{u \in U, \mathscr{B}(u, u) = 0\Big\}. \tag{101}$$

Theorem 1. Le noyau d'une forme bilinéaire, symétrique et positive est un sous-espace vectoriel.

Proof. Soient $u, v \in \ker \mathscr{B}$, $\alpha \in \mathbb{R}$ et $w = u + \alpha v$. Montrons que $w \in \ker \mathscr{B}$. Il suffit d'évaluer $\mathscr{B}(w, w)$

$$\mathscr{B}(w, w) = \mathscr{B}(u + \alpha v, u + \alpha v) = \mathscr{B}(u, u) + 2\alpha \mathscr{B}(u, v) + \alpha^2 \mathscr{B}(v, v), \tag{102}$$

où l'on a tenu compte de la symétrie de $\mathscr{B}$ pour écrire que $\mathscr{B}(u, v) = \mathscr{B}(v, u)$. Comme $u, v \in \ker \mathscr{B}$, le premier et le dernier terme sont nuls, soit $\mathscr{B}(w, w) = 2\alpha \mathscr{B}(u, v)$. La forme bilinéaire étant positive, cette grandeur est positive, quelle que soit la valeur de $\alpha \in \mathbb{R}$. On en déduit donc que $\mathscr{B}(u, v) = 0$, puis que $\mathscr{B}(w, w) = 0$ et donc que $w \in \ker \mathscr{B}$.

Theorem 2. Soit $u \in V$. Alors

$$u \in \ker \mathscr{B} \quad \text{ssi} \quad \text{pour tout } v \in V, \mathscr{B}(u, v) = 0. \tag{103}$$

Proof. Soient $u \in \ker \mathscr{B}$, $v \in V$ et $\alpha \in \mathbb{R}$. Comme précédemment, on écrit que $\mathscr{B}(w, w) \geq 0$, avec $w = \alpha u + v$

$$\mathscr{B}(w, w) = 2\alpha \mathscr{B}(u, v) + \mathscr{B}(v, v) \geq 0, \tag{104}$$

où l'on a tenu compte de ce que $\mathscr{B}(u, u) = 0$. L'expression précédente, affine en α, a un signe constant. Le terme linéaire en α est donc nul, soit $\mathscr{B}(u, v) = 0$. Réciproquement, si $\mathscr{B}(u, v) = 0$ pour tout $v \in V$, alors $\mathscr{B}(u, u) = 0$(en prenant $v = u$).

Theorem 3. Let $\mathscr{T}$ be a trilinear, symmetric form, such that

$$\mathscr{T}(u, u, u) = 0 \quad \text{for all} \quad u \in U. \tag{105}$$

Then

$$\mathscr{T}(u, v, w) = 0 \quad \text{for all} \quad u, v, w \in U. \tag{106}$$

Proof. The form $\mathscr{T}$ being trilinear and symmetric, we have, for all $u, v, w \in U$ and $\alpha, \beta \in \mathbb{R}$

$$\begin{aligned}\mathscr{T}(u + \alpha v + \beta w, u + \alpha v + \beta w, u + \alpha v + \beta w) &= \mathscr{T}(u, u, u) + 3\alpha \mathscr{T}(u, u, v) \\ &+ 3\beta \mathscr{T}(u, u, w) + 3\alpha^2 \mathscr{T}(u, u, v) + 6\alpha\beta \mathscr{T}(u, v, w) + 3\beta^2 \mathscr{T}(u, u, w) \\ &+ \alpha^3 \mathscr{T}(v, v, v) + 3\alpha^2\beta \mathscr{T}(v, v, w) + 3\alpha\beta^2 \mathscr{T}(v, w, w) + \beta^3 \mathscr{T}(w, w, w)\end{aligned} \tag{107}$$

and, upon simplification using Eq. (105)

$$\begin{aligned}3\alpha \mathscr{T}(u, u, v) &+ 3\beta \mathscr{T}(u, u, w) + 3\alpha^2 \mathscr{T}(u, v, v) + 6\alpha\beta \mathscr{T}(u, v, w) \\ &+ 3\beta^2 \mathscr{T}(u, w, w) + 3\alpha^2\beta \mathscr{T}(v, v, w) + 3\alpha\beta^2 \mathscr{T}(v, w, w) = 0.\end{aligned} \tag{108}$$

In particular taking successively $\alpha = \pm 1$, $\beta = 0$ and $w = 0$ delivers

$$\pm 3\mathscr{T}(u, u, v) + 3\mathscr{T}(u, u, v) = 0 \quad \text{for all} \quad u, v \in U, \tag{109}$$

from which it results that

$$\mathscr{T}(u, u, v) = 0 \quad \text{for all} \quad u, v \in U. \tag{110}$$

Plugging into Eq. (108) with $\alpha = \beta = 1$ results in: $\mathscr{T}(u, v, w) = 0$ for all $u, v, w \in U$.

A. Développements limités le long d'une branche bifurquée du diagramme d'équilibre

A.1. Principe du calcul

On pose dans ce qui suit

$$\Lambda(\eta) = \lambda(\eta) - \lambda_0 = \eta\lambda_1 + \frac{1}{2}\eta^2\lambda_2 + \frac{1}{6}\eta^3\lambda_3 + \cdots, \tag{111}$$

$$U(\eta) = u(\eta) - u^*[\lambda(\eta)] = \eta u_1 + \frac{1}{2}\eta^2 u_2 + \frac{1}{6}\eta^3 u_3 + \cdots. \tag{112}$$

On considère une fonctionnelle $\mathscr{F}$ de u et λ : $\mathscr{F}(u, \lambda)$. Cette fonctionnelle est évaluée le long de la branche bifurquée. En d'autres termes, on considère

$$f(\eta) = F\{u^*[\lambda_0 + \Lambda(\eta)] + U(\eta), \lambda_0 + \Lambda(\eta)\}.$$

On souhaite établir un développement limité de f au voisinage de $\eta = 0$, ce qui conduit à calculer les dérivées successives de f en $\eta = 0$, puisque

$$f(\eta) = f(0) + \eta f'(0) + \frac{1}{2}\eta^2 f''(0) + \cdots.$$

Pour calculer ces dérivées, il sera commode d'introduire la fonction auxiliaire F

$$F(\eta, \lambda) = \mathscr{F}[u^*(\lambda) + U(\eta), \lambda],$$

dans laquelle les variables λ et η sont provisoirement considérées comme indépendantes. On a $f(\eta) = F[\eta, \lambda_0 + \Lambda(\eta)]$, d'où l'on déduit successivement que

$$\begin{aligned}
f'(\eta) &= \partial_\eta F + \Lambda' \partial_\lambda F,\\
f''(\eta) &= \partial^2_{\eta\eta} F + 2\Lambda' \partial^2_{\eta\lambda} F + \Lambda'^2 \partial^2_{\lambda\lambda} F + \Lambda'' \partial_\lambda F,\\
f'''(\eta) &= \partial^3_{\eta\eta\eta} F + 3\Lambda' \partial^3_{\eta\eta\lambda} F + 3\Lambda'^2 \partial^3_{\eta\lambda\lambda} F + \lambda'^3 \partial^3_{\lambda\lambda\lambda} F\\
&\quad + 3\Lambda'' \partial^2_{\eta\lambda} F + 3\Lambda'\Lambda'' \partial^2_{\lambda\lambda} F + \Lambda''' \partial_\lambda F,\\
f''''(\eta) &= \partial^4_{\eta\eta\eta\eta} F + 4\Lambda' \partial^4_{\eta\eta\eta\lambda} F + 6\Lambda'^2 \partial^4_{\eta\eta\lambda\lambda} F + 4\Lambda'^3 \partial^4_{\eta\lambda\lambda\lambda} F + \Lambda'^4 \partial^4_{\lambda\lambda\lambda\lambda} F\\
&\quad + 6\Lambda'' \partial^3_{\eta\eta\lambda} F + 12\Lambda'\Lambda'' \partial^3_{\eta\lambda\lambda} F + 6\Lambda'^2\Lambda'' \partial^3_{\lambda\lambda\lambda} F\\
&\quad + 4\Lambda''' \partial^2_{\eta\lambda} F + \left(3\Lambda''^2 + 4\Lambda'\Lambda'''\right) \partial^2_{\lambda\lambda} F + \lambda'''' \partial_\lambda F,
\end{aligned}$$

où Λ et ses dérivées sont évaluées en η, tandis que F et ses dérivées partielles sont évaluées en $[\eta, \lambda_0 + \Lambda(\eta)]$. En $\eta = 0$, les relations précédentes s'écrivent

$$f'(0) = \partial_\eta F + \lambda_1 \partial_\lambda F, \tag{113}$$

$$f''(0) = \partial^2_{\eta\eta} F + 2\lambda_1 \partial^2_{\eta\lambda} F + \lambda_1^2 \partial^2_{\lambda\lambda} F + \lambda_2 \partial_\lambda F, \tag{114}$$

$$\begin{aligned} f'''(0) = {} & \partial^3_{\eta\eta\eta} F + 3\lambda_1 \partial^3_{\eta\eta\lambda} F + 3\lambda_1^2 \partial^3_{\eta\lambda\lambda} F + \lambda_1^3 \partial^3_{\lambda\lambda\lambda} F \\ & + 3\lambda_2 \partial^2_{\eta\lambda} F + 3\lambda_1 \lambda_2 \partial^2_{\lambda\lambda} F + \lambda_3 \partial_\lambda F, \end{aligned} \tag{115}$$

$$\begin{aligned} f''''(0) = {} & \partial^4_{\eta\eta\eta\eta} F + 4\lambda_1 \partial^4_{\eta\eta\eta\lambda} F + 6\lambda_1^2 \partial^4_{\eta\eta\lambda\lambda} F + 4\lambda_1^3 \partial^4_{\eta\lambda\lambda\lambda} F + \lambda_1^4 \partial^4_{\lambda\lambda\lambda\lambda} F \\ & + 6\lambda_2 \partial^3_{\eta\eta\lambda} F + 12\lambda_1 \lambda_2 \partial^3_{\eta\lambda\lambda} F + 6\lambda_1^2 \lambda_2 \partial^3_{\lambda\lambda\lambda} F \\ & + 4\lambda_3 \partial^2_{\eta\lambda} F + \left(3\lambda_2^2 + 4\lambda_1 \lambda_3\right) \partial^2_{\lambda\lambda} F + \lambda_4 \partial_\lambda F, \end{aligned} \tag{116}$$

où F et ses dérivées sont maintenant évaluées en $(\eta = 0, \lambda = \lambda_0)$.

A.2. Développement limité du résidu

On cherche un développement limité du résidu (c'est-à-dire de la première variation de l'énergie). La fonction test $\hat{u} \in U$ étant fixée, la méthode précédente est donc appliquée avec

$$f(\eta) = \mathscr{E}_{,u}[u(\eta), \lambda(\eta); \hat{u}] \quad \text{et} \quad F(\eta, \lambda) = \mathscr{E}_{,u}[u^*(\lambda) + U(\eta), \lambda; \hat{u}]. \tag{117}$$

On remarque tout d'abord que $F(0, \lambda) = \mathscr{E}_{,u}[u^*(\lambda), \lambda; \hat{u}] = 0$, puisque $u^*(\lambda)$ est un point d'équilibre. En dérivant par rapport à λ, on obtient

$$\frac{\partial^k F}{\partial \lambda^k}(0, \lambda) = 0 \quad \text{pour tout} \quad k \geq 0.$$

En dérivant par rapport à η l'expression (117) de F, on obtient successivement

$$\partial_\eta F(\eta, \lambda) = \mathscr{E}_{,uu}[u^*(\lambda) + U(\eta), \lambda; U'(\eta), \hat{u}],$$

$$\begin{aligned} \partial^2_{\eta\eta} F(\eta, \lambda) = {} & \mathscr{E}_{,uuu}[u^*(\lambda) + U(\eta), \lambda; U'(\eta), U'(\eta), \hat{u}] \\ & + \mathscr{E}_{,uu}[u^*(\lambda) + U(\eta), \lambda; U''(\eta), \hat{u}], \end{aligned}$$

$$\begin{aligned} \partial^3_{\eta\eta\eta} F(\eta, \lambda) = {} & \mathscr{E}_{,uuuu}[u^*(\lambda) + U(\eta), \lambda; U'(\eta), U'(\eta), U'(\eta), \hat{u}] \\ & + 3\mathscr{E}_{,uuu}[u^*(\lambda) + U(\eta), \lambda; U'(\eta), U''(\eta), \hat{u}] \\ & + \mathscr{E}_{,uu}[u^*(\lambda) + U(\eta), \lambda; U'''(\eta), \hat{u}], \end{aligned}$$

soit, en $\eta = 0$

$$\partial_\eta F(0, \lambda) = \mathscr{E}_2(\lambda; u_1, \hat{u}),$$

$$\partial^2_{\eta\eta} F(0, \lambda) = \mathscr{E}_3(\lambda; u_1, u_1, \hat{u}) + \mathscr{E}_2(\lambda; u_2, \hat{u}),$$

$$\partial^3_{\eta\eta\eta} F(0, \lambda) = \mathscr{E}_4(\lambda; u_1, u_1, u_1, \hat{u}) + 3\mathscr{E}_3(\lambda; u_1, u_2, \hat{u}) + \mathscr{E}_2(\lambda; u_3, \hat{u}).$$

Les dérivées croisées de F en $(0, \lambda)$ s'obtiennent par simple dérivation des relations précédentes par rapport à λ

$$\partial^2_{\eta\lambda} F(0,\lambda) = \dot{\mathscr{E}}_2(\lambda; u_1, \hat{u}), \quad \partial^3_{\eta\lambda\lambda} F(0,\lambda) = \ddot{\mathscr{E}}_2(\lambda; u_1, \hat{u}),$$

$$\partial^3_{\eta\eta\lambda} F(0,\lambda) = \dot{\mathscr{E}}_3(\lambda; u_1, u_1, \hat{u}) + \dot{\mathscr{E}}_2(\lambda; u_2, \hat{u}).$$

En insérant les résultats précédents dans les relations générales (113)–(116), on trouve alors les expressions suivantes des dérivées successives de f en $\eta = 0$

$$\begin{aligned}
f'(0) &= \mathscr{E}_2(\lambda_0; u_1, \hat{u}),\\
f''(0) &= \mathscr{E}_3(\lambda_0; u_1, u_1, \hat{u}) + \mathscr{E}_2(\lambda_0; u_2, \hat{u}) + 2\lambda_1 \dot{\mathscr{E}}_2(\lambda_0; u_1, \hat{u}),\\
f'''(0) &= \mathscr{E}_4(\lambda_0; u_1, u_1, u_1, \hat{u}) + 3\mathscr{E}_3(\lambda_0; u_1, u_2, \hat{u}) + \mathscr{E}_2(\lambda_0; u_3, \hat{u})\\
&\quad + 3\lambda_1 \dot{\mathscr{E}}_3(\lambda_0; u_1, u_1, \hat{u}) + 3\lambda_1 \dot{\mathscr{E}}_2(\lambda_0; u_2, \hat{u})\\
&\quad + 3\lambda_1^2 \ddot{\mathscr{E}}_2(\lambda_0; u_1, \hat{u}) + 3\lambda_2 \dot{\mathscr{E}}_2(\lambda_0; u_1, \hat{u}).
\end{aligned}$$

On en déduit finalement le développement limité à l'ordre 3 en η du résidu

$$\begin{aligned}
\mathscr{E}_{,u}[u(\eta), \lambda(\eta)] &= \eta \mathscr{E}_2(\lambda_0; u_1, \hat{u}) + \tfrac{1}{2}\eta^2 \Big[\mathscr{E}_3(\lambda_0; u_1, u_1, \hat{u}) + \mathscr{E}_2(\lambda_0; u_2, \hat{u})\\
&\quad + 2\lambda_1 \dot{\mathscr{E}}_2(\lambda_0; u_1, \hat{u}) \Big] + \tfrac{1}{6}\eta^3 \Big[\mathscr{E}_4(\lambda_0; u_1, u_1, u_1, \hat{u}) + 3\mathscr{E}_3(\lambda_0; u_1, u_2, \hat{u})\\
&\quad + \mathscr{E}_2(\lambda_0; u_3, \hat{u}) + 3\lambda_1 \dot{\mathscr{E}}_3(\lambda_0; u_1, u_1, \hat{u}) + 3\lambda_1 \dot{\mathscr{E}}_2(\lambda_0; u_2, \hat{u})\\
&\quad + 3\lambda_1^2 \ddot{\mathscr{E}}_2(\lambda_0; u_1, \hat{u}) + 3\lambda_2 \dot{\mathscr{E}}_2(\lambda_0; u_1, \hat{u}) \Big] + o(\eta^3).
\end{aligned} \tag{118}$$

A.3. Développement limité de l'énergie

On s'intéresse ici à l'écart d'énergie, pour un chargement λ donné, entre la branche bifurquée et la branche fondamentale, soit

$$F(\eta, \lambda) = \mathscr{E}[u^*(\lambda) + U(\eta), \lambda] - \mathscr{E}[u^*(\lambda), \lambda] \quad \text{et} \quad f(\eta) = F[\eta, \lambda_0 + \Lambda(\eta)]. \tag{119}$$

On observe tout d'abord que $F(0, \lambda) = 0$ pour tout λ, donc

$$\frac{\partial^k F}{\partial \lambda^k}(0, \lambda) = 0 \quad \text{pour tout} \quad k \geq 0,$$

tandis que les dérivées de F par rapport à η s'écrivent

$$\begin{aligned}
\partial_\eta F(\eta, \lambda) &= \mathscr{E}_{,u}(U'),\\
\partial^2_{\eta\eta} F(\eta, \lambda) &= \mathscr{E}_{,uu}(U', U') + \mathscr{E}_{,u}(U''),\\
\partial^3_{\eta\eta\eta} F(\eta, \lambda) &= \mathscr{E}_{,uuu}(U', U', U') + 3\mathscr{E}_{,uu}(U', U'') + \mathscr{E}_{,u}(U'''),\\
\partial^4_{\eta\eta\eta\eta} F &= \mathscr{E}_{,uuuu}(U', U', U', U') + 6\mathscr{E}_{,uuu}(U', U', U'')\\
&\quad + 3\mathscr{E}_{,uu}(U'', U'') + 4\mathscr{E}_{,uu}(U', U''') + \mathscr{E}_{,u}(U''''),
\end{aligned}$$

où les différentielles successives de $\mathscr{E}$ sont évaluées en $[u^*(\lambda)+U(\eta),\lambda]$, tandis que les dérivées successives de U sont évaluées en η. Les relations précédentes s'écrivent, en $\eta=0$, en observant que $\mathscr{E}_{,u}[u^*(\lambda),\lambda]=0$

$$\begin{aligned}
\partial_\eta F(0,\lambda) &= 0,\\
\partial^2_{\eta\eta}F(0,\lambda) &= \mathscr{E}_2(\lambda;u_1,u_1),\\
\partial^3_{\eta\eta\eta}F(0,\lambda) &= \mathscr{E}_3(\lambda;u_1,u_1,u_1)+3\mathscr{E}_2(\lambda;u_1,u_2),\\
\partial^4_{\eta\eta\eta\eta}F(\eta,\lambda) &= \mathscr{E}_4(\lambda;u_1,u_1,u_1,u_1)+6\mathscr{E}_3(\lambda;u_1,u_1,u_2)+3\mathscr{E}_2(\lambda;u_2,u_2)+4\mathscr{E}_2(\lambda;u_1,u_3).
\end{aligned}$$

En dérivant alors par rapport à λ, on en déduit que

$$\begin{aligned}
\partial^2_{\eta\lambda}F(0,\lambda) &= 0, & \partial^4_{\eta\eta\eta\lambda}F(0,\lambda) &= \dot{\mathscr{E}}_3(\lambda;u_1,u_1,u_1)+3\dot{\mathscr{E}}_2(\lambda;u_1,u_2),\\
\partial^3_{\eta\eta\lambda}F(0,\lambda) &= \dot{\mathscr{E}}_2(\lambda;u_1,u_1), & \partial^4_{\eta\eta\lambda\lambda}F(0,\lambda) &= \ddot{\mathscr{E}}_2(\lambda;u_1,u_1),\\
\partial^3_{\eta\lambda\lambda}F(0,\lambda) &= 0, & \partial^4_{\eta\lambda\lambda\lambda}F(0,\lambda) &= 0
\end{aligned}$$

et finalement

$$\begin{aligned}
f'(0) = 0,\qquad & f''(0) = \mathscr{E}_2(\lambda_0;u_1,u_1),\\
f'''(0) &= \mathscr{E}_3(\lambda_0;u_1,u_1,u_1)+3\mathscr{E}_2(\lambda_0;u_1,u_2)+3\lambda_1\dot{\mathscr{E}}_2(\lambda_0;u_1,u_1),\\
f''''(0) &= \mathscr{E}_4(\lambda_0;u_1,u_1,u_1,u_1)+6\mathscr{E}_3(\lambda_0;u_1,u_1,u_2)+3\mathscr{E}_2(\lambda_0;u_2,u_2)+4\mathscr{E}_2(\lambda_0;u_1,u_3)\\
&\quad +4\lambda_1\dot{\mathscr{E}}_3(\lambda_0;u_1,u_1,u_1)+12\lambda_1\dot{\mathscr{E}}_2(\lambda_0;u_1,u_2)+6\lambda_1^2\ddot{\mathscr{E}}_2(\lambda_0;u_1,u_1)+6\lambda_2\dot{\mathscr{E}}_2(\lambda_0;u_1,u_1).
\end{aligned}$$

Les relations précédentes se simplifient notamment en tenant compte de ce que $u_1\in V:\mathscr{E}_2(\lambda_0;u_1,u_i)=0$ pour $i=1,2,3$. On trouve ainsi $f''(0)=0$ et

$$f'''(0) = -\lambda_1 G_{ij}\xi_1^i\xi_1^j, \tag{120}$$

en utilisant l'équation de bifurcation (69). En introduisant les décompositions (70) et (75) de u_1 et u_2, on trouve tout d'abord, pour $\mathscr{E}_3(\lambda_0;u_1,u_1,u_2)$

$$\begin{aligned}
\mathscr{E}_3(\lambda_0;u_1,u_1,u_2) &= \mathscr{E}_3(v_i,v_j,v_k)\xi_1^i\xi_1^j\xi_2^k+\mathscr{E}_3(v_i,v_j,w_{kl})\xi_1^i\xi_1^j\xi_1^k\xi_1^l+\lambda_1\mathscr{E}_3(v_i,v_j,w_k)\xi_1^i\xi_1^j\xi_1^k\\
&= \mathscr{E}_3(v_i,v_j,v_k)\xi_1^i\xi_1^j\xi_2^k+\mathscr{E}_3(v_i,v_j,w_{kl})\xi_1^i\xi_1^j\xi_1^k\xi_1^l-\lambda_1\mathscr{E}_2(w_{ij},w_k)\xi_1^i\xi_1^j\xi_1^k,
\end{aligned}$$

en tenant compte de la définition (11) des w_{ij}. Dans le dernier terme de l'expression précédente, les indices i,j et k sont muets, donc

$$\begin{aligned}
\mathscr{E}_3(\lambda_0;u_1,u_1,u_2) &= \mathscr{E}_3(v_i,v_j,v_k)\xi_1^i\xi_1^j\xi_2^k+\mathscr{E}_3(v_i,v_j,w_{kl})\xi_1^i\xi_1^j\xi_1^k\xi_1^l-\lambda_1\mathscr{E}_2(w_i,w_{jk})\xi_1^i\xi_1^j\xi_1^k\\
&= \mathscr{E}_3(v_i,v_j,v_k)\xi_1^i\xi_1^j\xi_2^k+\mathscr{E}_3(v_i,v_j,w_{kl})\xi_1^i\xi_1^j\xi_1^k\xi_1^l+2\lambda_1\dot{\mathscr{E}}_2(v_i,w_{jk})\xi_1^i\xi_1^j\xi_1^k,
\end{aligned}$$

en introduisant cette fois-ci la définition (10) de w_i. On procède de même pour le terme suivant, soit $\mathscr{E}_2(u_2, u_2)$

$$
\begin{aligned}
\mathscr{E}_2(u_2, u_2) &= \mathscr{E}_2(\xi_2^i v_i + \xi_1^i \xi_1^j w_{ij} + \lambda_1 \xi_1^i w_i, \xi_2^k v_k + \xi_1^k \xi_1^l w_{kl} + \lambda_1 \xi_1^k w_k) \\
&= \mathscr{E}_2(\xi_1^i \xi_1^j w_{ij} + \lambda_1 \xi_1^i w_i, \xi_1^k \xi_1^l w_{kl} + \lambda_1 \xi_1^k w_k) \\
&= \mathscr{E}_2(w_{ij}, w_{kl}) \xi_1^i \xi_1^j \xi_1^k \xi_1^l + 2\lambda_1 \mathscr{E}_2(w_i, w_{jk}) \xi_1^i \xi_1^j \xi_1^k + \lambda_1^2 \mathscr{E}_2(w_i, w_j) \xi_1^i \xi_1^j \\
&= \mathscr{E}_2(w_{ij}, w_{kl}) \xi_1^i \xi_1^j \xi_1^k \xi_1^l + 2\lambda_1 \mathscr{E}_2(w_i, w_{jk}) \xi_1^i \xi_1^j \xi_1^k + \tfrac{1}{2}\lambda_1^2 \Big[\mathscr{E}_2(w_i, w_j) + \mathscr{E}_2(w_j, w_i)\Big] \xi_1^i \xi_1^j \\
&= -\mathscr{E}_3(v_i, v_j, w_{kl}) \xi_1^i \xi_1^j \xi_1^k \xi_1^l - 4\lambda_1 \dot{\mathscr{E}}_2(v_i, w_{jk}) \xi_1^i \xi_1^j \xi_1^k - \lambda_1^2 \Big[\dot{\mathscr{E}}_2(v_i, w_j) + \dot{\mathscr{E}}_2(v_j, w_i)\Big] \xi_1^i \xi_1^j
\end{aligned}
$$

et enfin

$$
\begin{aligned}
\dot{\mathscr{E}}_2(u_1, u_2) &= \dot{\mathscr{E}}_2(v_i, v_j) \xi_1^i \xi_2^j + \dot{\mathscr{E}}_2(v_i, w_{jk}) \xi_1^i \xi_1^j \xi_1^k + \lambda_1 \dot{\mathscr{E}}_2(v_i, w_j) \xi_1^i \xi_1^j \\
&= \dot{\mathscr{E}}_2(v_i, v_j) \xi_1^i \xi_2^j + \dot{\mathscr{E}}_2(v_i, w_{jk}) \xi_1^i \xi_1^j \xi_1^k + \tfrac{1}{2}\lambda_1 [\dot{\mathscr{E}}_2(v_i, w_j) + \dot{\mathscr{E}}_2(v_j, w_i)] \xi_1^i \xi_1^j.
\end{aligned}
$$

En rassemblant les résultats précédents, on trouve pour $f''''(0)$

$$
\begin{aligned}
f''''(0) = &\Big[\mathscr{E}_4(v_i, v_j, v_k, v_l) + 3\mathscr{E}_3(v_i, v_j, w_{kl})\Big] \xi_1^i \xi_1^j \xi_1^k \xi_1^l + 4\lambda_1 \Big[\dot{\mathscr{E}}_3(v_i, v_j, v_k) + 3\dot{\mathscr{E}}_2(v_i, w_{jk})\Big] \xi_1^i \xi_1^j \xi_1^k \\
&+ \Big\{3\lambda_1^2 \Big[2\ddot{\mathscr{E}}_2(v_i, v_j) + \dot{\mathscr{E}}_2(v_i, w_j) + \dot{\mathscr{E}}_2(v_j, w_i)\Big] + 6\lambda_2 \dot{\mathscr{E}}_2(v_i, v_j)\Big\} \xi_1^i \xi_1^j \\
&+ 6\Big[\mathscr{E}_3(v_i, v_j, v_k) \xi_1^k + 2\lambda_1 \dot{\mathscr{E}}_2(v_i, v_j)\Big] \xi_1^i \xi_2^j,
\end{aligned}
$$

et on observe que le dernier terme(en $\xi_1^i \xi_2^j$) est nul, du fait de l'équation de bifurcation (71). On obtient donc

$$
f''''(0) = E_{ijkl} \xi_1^i \xi_1^j \xi_1^k \xi_1^l + 4\lambda_1 \mathring{E}_{ijk} \xi_1^i \xi_1^j \xi_1^k - 6\Big(\lambda_1^2 \mathring{G}_{ij} + \lambda_2 G_{ij}\Big) \xi_1^i \xi_1^j. \tag{121}
$$

Le développement limité (84) est alors obtenu en rassemblant les expressions précédentes de $f'(0)$, $f''(0)$, $f'''(0)$ et $f''''(0)$.

A.4. Développement limité de la hessienne

On cherche maintenant un développement limité de la hessienne de l'énergie. Les fonctions test $\hat{u}, \hat{v} \in U$ étant fixées, on applique la méthode du §A.1 à la fonction $f(\eta) = F[\eta, \lambda_0 + \Lambda(\eta)]$, avec

$$
F(\eta, \lambda) = \mathscr{E}_{,uu}[u^*(\lambda) + U(\eta), \lambda; \hat{u}, \hat{v}].
$$

On observe tout d'abord que $F(0, \lambda) = \mathscr{E}_2(\lambda; \hat{u}, \hat{v})$, soit, en dérivant par rapport à λ

$$
\partial_\lambda F(0, \lambda) = \dot{\mathscr{E}}_2(\lambda; \hat{u}, \hat{v}) \quad \text{et} \quad \partial^2_{\lambda\lambda} F(0, \lambda) = \ddot{\mathscr{E}}_2(\lambda; \hat{u}, \hat{v}).
$$

On trouve de même successivement

$$
\begin{aligned}
\partial_\eta F(\eta, \lambda) &= \mathscr{E}_{,uuu}(U', \hat{u}, \hat{v}), \\
\partial^2_{\eta\eta} F(\eta, \lambda) &= \mathscr{E}_{,uuuu}(U', U', \hat{u}, \hat{v}) + \mathscr{E}_{,uuu}(U'', \hat{u}, \hat{v}),
\end{aligned}
$$

où les différentielles successives de $\mathscr{E}$ sont évaluées en $[u^*(\lambda)+U(\eta),\lambda]$, tandis que les dérivées successives de U sont évaluées en η. Les relations précédentes s'écrivent en $\eta=0$

$$\begin{aligned}\partial_\eta F(0,\lambda) &= \mathscr{E}_3(\lambda;u_1,\hat{u},\hat{v}),\\ \partial^2_{\eta\eta}F(0,\lambda) &= \mathscr{E}_4(\lambda;u_1,u_1,\hat{u},\hat{v})+\mathscr{E}_3(\lambda;u_2,\hat{u},\hat{v}),\end{aligned}$$

et en dérivant cette fois par rapport à λ

$$\partial^2_{\eta\lambda}F(0,\lambda)=\dot{\mathscr{E}}_3(\lambda;u_1,\hat{u},\hat{v}).$$

En insérant les résultats précédents dans les expressions (113) et (114), on trouve

$$\begin{aligned}f'(0) &= \mathscr{E}_3(u_1,\hat{u},\hat{v})+\lambda_1\dot{\mathscr{E}}_2(\hat{u},\hat{v}),\\ f''(0) &= \mathscr{E}_4(u_1,u_1,\hat{u},\hat{v})+\mathscr{E}_3(u_2,\hat{u},\hat{v})+2\lambda_1\dot{\mathscr{E}}_3(u_1,\hat{u},\hat{v})+\lambda_1^2\ddot{\mathscr{E}}_2(\hat{u},\hat{v})+\lambda_2\dot{\mathscr{E}}_2(\hat{u},\hat{v}).\end{aligned}$$

qui conduisent finalement au développement limité (78).

A.5. Asymptotic expansions of the eigenvalues and eigenvectors of the Hessian

In this appendix, Eqs. (81), (82) and (83) are derived. The postulated expansions (80) are plugged into the asymptotic expansion (78) of the Hessian on the one hand

$$\begin{aligned}\mathscr{E}_{,uu}[u(\eta),\lambda(\eta);x,\hat{u}] &= \mathscr{E}_2(x_0,\hat{u})+\eta\Big[\mathscr{E}_2(x_1,\hat{u})+\mathscr{E}_3(u_1,x_0,\hat{u})+\lambda_1\dot{\mathscr{E}}_2(x_0,\hat{u})\Big]\\ &+\tfrac{1}{2}\eta^2\Big[\mathscr{E}_2(x_2,\hat{u})+2\mathscr{E}_3(u_1,x_1,\hat{u})+2\lambda_1\dot{\mathscr{E}}_2(x_1,\hat{u})+\mathscr{E}_4(u_1,u_1,x_0,\hat{u})\\ &\quad+\mathscr{E}_3(u_2,x_0,\hat{u})+2\lambda_1\dot{\mathscr{E}}_3(u_1,x_0,\hat{u})+\lambda_1^2\ddot{\mathscr{E}}_2(x_0,\hat{u})+\lambda_2\dot{\mathscr{E}}_2(x_0,\hat{u})\Big]+o(\eta^2)\end{aligned}$$

(where the $\mathscr{E}_k$ and $\dot{\mathscr{E}}_k$ are all evaluated at $\lambda=\lambda_0$) and into the scalar product $\alpha\langle x,\hat{u}\rangle$ on the other hand

$$\begin{aligned}\alpha\langle x,\hat{u}\rangle &= \alpha_0\langle x_0,\hat{u}\rangle+\eta\Big(\alpha_1\langle x_0,\hat{u}\rangle+\alpha_0\langle x_1,\hat{u}\rangle\Big)\\ &\qquad+\tfrac{1}{2}\eta^2\Big(\alpha_0\langle x_2,\hat{u}\rangle+2\alpha_1\langle x_1,\hat{u}\rangle+\alpha_2\langle x_0,\hat{u}\rangle\Big)+o(\eta^2).\end{aligned}$$

Equating both expressions for all $\hat{u}\in U$ [see Eq. (79)] leads to three variational problems (for the η^0, η^1 and η^2 terms) that are discussed below.

Variational problem of order 0 Find $x_0\in U$ and $\alpha_0\in\mathbb{R}$ such that, for all $\hat{u}\in U$

$$\mathscr{E}_2(x_0,\hat{u})=\alpha_0\langle x_0,\hat{u}\rangle.$$

The above equation shows that (α_0,x_0) is an eigenpair of $\mathscr{E}_2(\lambda_0)$. As discussed in Sec. 3, only the case $\alpha_0=0$ is relevant. Then $x_0\in V$, which is expressed by the expansion (81) of x_0.

Variational problem of order 1 Find $x_1 \in U$ and $\alpha_1 \in \mathbb{R}$ such that, for all $\hat{u} \in U$

$$\mathscr{E}_2(x_1, \hat{u}) + \mathscr{E}_3(u_1, x_0, \hat{u}) + \lambda_1 \dot{\mathscr{E}}_2(x_0, \hat{u}) = \alpha_1 \langle x_0, \hat{u} \rangle, \tag{122}$$

or, equivalently, plugging the expansions (70) and (81) of u_1 and x_0 in the v_i basis

$$\mathscr{E}_2(x_1, \hat{u}) + \mathscr{E}_3(v_j, v_k, \hat{u}) \chi_0^j \xi_1^k + \lambda_1 \dot{\mathscr{E}}_2(v_j, \hat{u}) \chi_0^j = \alpha_1 \chi_0^j \langle v_j, \hat{u} \rangle. \tag{123}$$

For $\hat{u} = v_i$, observing that $\langle v_i, v_j \rangle = \delta_{ij}$ since (v_i) is orthonormal, the above equation reads

$$\left[\mathscr{E}_3(\lambda_0; v_i, v_j, v_k) \xi_1^k + \lambda_1 \dot{\mathscr{E}}_2(\lambda_0; v_i, v_j) \right] \chi_0^j = \alpha_1 \chi_0^i, \tag{124}$$

which reduces to Eq. (81).

The test function is now picked in $W = V^{\perp}$, and x_1 is decomposed as the sum of its projections onto V and W: $x_1 = \chi_1^i v_i + y_1$, where $y_1 \in W$. Eq. (123) then delivers the following variational problem: find $y_1 \in W$ such that, for all $\hat{w} \in W$,

$$\mathscr{E}_2(y_1, \hat{w}) + \mathscr{E}_3(v_i, v_j, \hat{w}) \chi_0^i \xi_1^j + \lambda_1 \dot{\mathscr{E}}_2(v_i, \hat{w}) \chi_0^i = 0, \tag{125}$$

(observe that $\langle v_j, \hat{w} \rangle$ since V and W are orthogonal subspaces). The solution to the above problem is expressed as a linear combination of the w_i and w_{ij} defined by the variational problems (10) and (11), respectively: $y_1 = \chi_0^i \xi_1^j w_{ij} + \frac{1}{2} \lambda_1 \chi_0^i w_i$, and the decomposition (82) is retrieved.

Variational problem of order 2 For all $\hat{u} \in U$,

$$\begin{aligned} &\mathscr{E}_2(x_2, \hat{u}) + 2\mathscr{E}_3(u_1, x_1, \hat{u}) + 2\lambda_1 \dot{\mathscr{E}}_2(x_1, \hat{u}) + \mathscr{E}_4(u_1, u_1, x_0, \hat{u}) + \mathscr{E}_3(u_2, x_0, \hat{u}) \\ &\qquad + 2\lambda_1 \dot{\mathscr{E}}_3(u_1, x_0, \hat{u}) + \lambda_1^2 \ddot{\mathscr{E}}_2(x_0, \hat{u}) + \lambda_2 \dot{\mathscr{E}}_2(x_0, \hat{u}) = 2\alpha_1 \langle x_1, \hat{u} \rangle + \alpha_2 \langle x_0, \hat{u} \rangle. \end{aligned}$$

For $\hat{u} = \hat{v}_i$, plugging the decompositions (70), (75), (81) and (82) of u_1, u_2, x_0 et x_1 delivers

$$\begin{aligned} &\left[\mathscr{E}_4(v_i, v_j, v_k, v_l) + 2\mathscr{E}_3(v_i, w_{jk}, v_l) + \mathscr{E}_3(v_i, v_j, w_{kl}) \right] \chi_0^j \xi_1^k \xi_1^l \\ &\quad + \lambda_1 \left[\mathscr{E}_3(v_i, w_j, v_k) + 2\dot{\mathscr{E}}_2(v_i, w_{jk}) + \mathscr{E}_3(v_i, v_j, w_k) + 2\dot{\mathscr{E}}_3(v_i, v_j, v_k) \right] \chi_0^j \xi_1^k \\ &\quad + \lambda_1^2 \left[\dot{\mathscr{E}}_2(v_i, w_j) + \ddot{\mathscr{E}}_2(v_i, v_j) \right] \chi_0^j + \left[\mathscr{E}_3(v_i, v_j, v_k) \xi_2^k + \lambda_2 \dot{\mathscr{E}}_2(v_i, v_j) \right] \chi_0^j \\ &\quad + 2\left[\mathscr{E}_3(v_i, v_j, v_k) \xi_1^k + \lambda_1 \dot{\mathscr{E}}_2(v_i, v_j) \right] \chi_1^j = 2\alpha_1 \chi_1^i + \alpha_2 \chi_0^i. \end{aligned}$$

The $\chi_0^j \xi_1^k$ term is transformed with Eqs. (10) and (11)

$$\begin{aligned} &\left[\mathscr{E}_4(v_i, v_j, v_k, v_l) + \mathscr{E}_3(v_i, w_{jk}, v_l) + \mathscr{E}_3(v_i, w_{jl}, v_k) + \mathscr{E}_3(v_i, v_j, w_{kl}) \right] \chi_0^j \xi_1^k \xi_1^l \\ &\quad + \lambda_1 \left[-\mathscr{E}_2(w_{ik}, w_j) - \mathscr{E}_2(w_i, w_{jk}) - \mathscr{E}_2(w_{ij}, w_k) + 2\dot{\mathscr{E}}_3(v_i, v_j, v_k) \right] \chi_0^j \xi_1^k \\ &\quad + \lambda_1^2 \left[\dot{\mathscr{E}}_2(v_i, w_j) + \ddot{\mathscr{E}}_2(v_i, v_j) \right] \chi_0^j + \left[\mathscr{E}_3(v_i, v_j, v_k) \xi_2^k + \lambda_2 \dot{\mathscr{E}}_2(v_i, v_j) \right] \chi_0^j \\ &\quad + 2\left[\mathscr{E}_3(v_i, v_j, v_k) \xi_1^k + \lambda_1 \dot{\mathscr{E}}_2(v_i, v_j) \right] \chi_1^j = 2\alpha_1 \chi_1^i + \alpha_2 \chi_0^i, \end{aligned}$$

and Eq. (83) results from the application of Eqs. (21) and (22).