Min And Max Height of a B-Tree

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1 Introduction

A B-Tree is a generalization of binary search tree in which a node can have more than two children. The maximum number of children the B-Tree can have is specified by its order. A B-Tree of order m can have m-1 keys in each node. So by using the definition of a B-Tree, we are interested in determining the minimum and maximum height given the order m and total number of keys n.

2 Minimum Height

The minimum height of B-Tree is obtained when all the nodes are completely filled. Now let's calculate the total number of keys.

Max number of keys

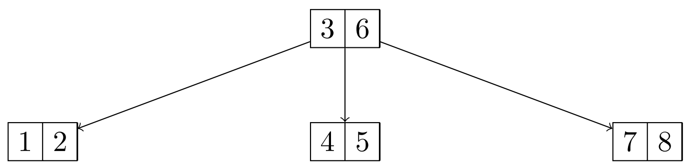
$$\begin{aligned}
&= (m-1) + m(m-1) + m^2(m-1) + ... + m^h(m-1) \\
&= (m-1)\{1 + m + m^2 + ... + m^h\} \\
&= (m-1).1\frac{m^{h+1}-1}{(m-1)} \\
n &= m^{h+1} - 1
\end{aligned}$$

To find the minimum height, we rearrange the equation as given below

$$\begin{aligned}
n + 1 &= m^{h+1} \\
h + 1 &= \lceil log_m(n+1) \rceil \\
h_{min} &= \lceil log_m(n+1) \rceil - 1
\end{aligned}$$

Given below is an example of a B-Tree of order 3 and 8 key elements. The tree has a height of 1 (considering root as 0) which is satisfied by the equation

$$\begin{aligned}
h_{min} &= \lceil log_3(8+1) \rceil - 1 \\
h_{min} &= 2 - 1 \\
h_{min} &= 1
\end{aligned}$$



3 Maximum Height

To obtain maximum height of the B-tree, we have to fill each node with the minimum number of keys and by maintaining the B-tree properties. That is, each node should contain atleast $\left\lceil \frac{m}{2} \right\rceil keys.$

Take d = $\left\lceil \frac{m}{2} \right\rceil$

Consider the table below

Height	Min no of nodes	Min no of keys
0	1	1
1	2	$2(\lceil \frac{m}{2} \rceil - 1)$ = 2(d-1)
2	2d	2d(d-1)
3	2d^2	2d$^2(d-1)$
..	..	..
h	2d^{h-1}	2d$^{h-1}(d-1)$

Min number of keys

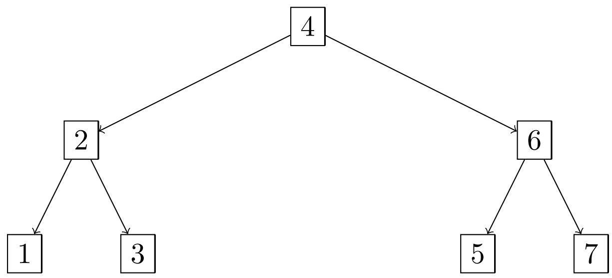
$$\begin{aligned}
&= 1 + 2(d-1) + 2d(d-1) + 2d^2(d-1) + ... + 2d^{h-1}(d-1) \\
&= 1 + 2(d-1)\{1 + d + d^2 + ... + d^{h-1}\} \\
&= 1 + 2(d-1)\{\frac{d^h - 1}{(d-1)}\} \\
&= 1 + 2(d^h - 1) \\
&= 1 + 2\left\lceil \frac{m}{2} \right\rceil^h - 2 \\
n &= 2\left\lceil \frac{m}{2} \right\rceil^h - 1
\end{aligned}$$

To find the maximum height h of the B-Tree we can simply rearrange the equation

$$\begin{aligned}
\left\lceil \frac{m}{2} \right\rceil^h &= \frac{n+1}{2} \\
h &= \lfloor log_{\lceil \frac{m}{2} \rceil} \frac{n+1}{2} \rfloor \\
h_{max} &= \lfloor log_d \frac{n+1}{2} \rfloor, where\ d = \lceil \frac{m}{2} \rceil
\end{aligned}$$

The B-Tree shown below has the same number of key elements and has order 3. The tree is drawn such that it has maximum possible height i.e 2 in this case.

$$\begin{aligned}
h_{max} &= \lfloor log_2 \frac{8+1}{2} \rfloor, where\ d = \lceil \frac{m}{2} \rceil = 2 \\
&= \lfloor 2.169 \rfloor \\
&= 2
\end{aligned}$$


4
2
6
1
3
5
7