Modes of internal symmetry - taking a closer look at Messiaen's modes of limited transposition

Stefaan.Himpe@gmail.com

06/March/2014

1 Introduction

After reading Olivier Messiaen's book "The technique of my Musical Language" I started wondering how exactly his "modes of limited transposition" create the sound they create. Major questions for me became:

- What is so special about the property of *limited transposition* that makes the music derived from it work so well?
- Is music derived from a random subset of a chromatic scale on some sense "crippled" compared to music written based on Messiaen's modes? Can we find some kind of intuitive explanation why Messiaen's modes work well, arguably better than other non-diatonic modes? Can we use this insight to propose other modes that perhaps do not have limited transposition, but still might result in an interesting, fresh, sound?

I set out to do a series of experiments and came up with some insights that I intend to explain here. I apologize in advance if what I'm about to describe is already well-known and obvious to more informed readers. Please accept that I'm writing this down only to further my own understanding of the matter. I do not claim to have found something new.

2 Symmetry? What symmetry?

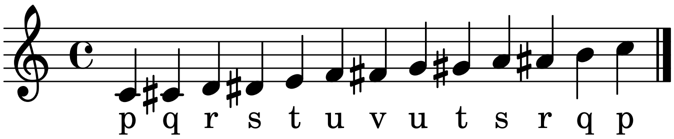
Messiaen himself explains how his modes can be thought of as consisting of *symmetrical* groups of notes. The exact nature of this symmetry was not entirely clear to me. In fact, at first sight it seemed more like repeating patterns than symmetry, and this is what I wanted to clarify.

As a starting point I set out to systematically enumerate all modes derived from a chromatic scale that have intervals symmetrically distributed around the f# (the middle note of the chromatic scale starting on c). Note that during the experiments I don't directly take into account any property of *limited transposition* but I will find back many of Messiaen's modes by considering only symmetry arguments anyway.

2.1 Systematic enumeration

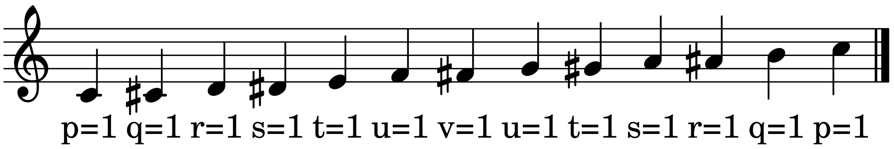
The first thing to explain is how the modes with intervals symmetrically distributed around f# can be systematically enumerated.

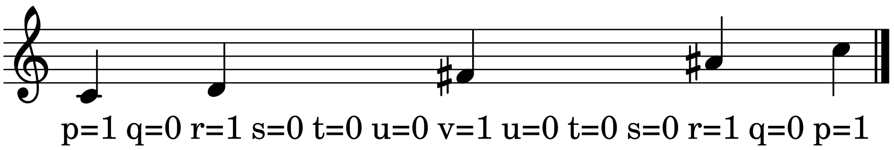
I started from a chromatic scale. On that scale I added symbols under each note. Note how the symbols that occur left of f# return later on to the right of f#. This is important to keep symmetry of the intervals around f#, as will hopefully become clearer in the next step. For now, remember that the symbols under the complete chromatic scale form a *palindrome*, i.e. if you read them left-to-right you get exactly the same sequence of symbols as when you read them right-to-left.



In what follows we will now construct modes by assigning values 0 or 1 to each of the symbols p,q,r,s,t,u,v. A value of 0 indicates that the note to which the symbol is attached is not to be selected from the chromatic scale while constructing a mode. A value of 1 indicates that the notes to which the symbol is attached are to be selected from the chromatic scale.

E.g. if we set p=1, q=1, r=1, s=1, t=1, u=1, v=1 we retain all the notes from the chromatic scale and end up with the chromatic scale itself. We can say that the chromatic scale is characterized by a "key" (p,q,r,s,t,u,v) = (1,1,1,1,1,1,1). If we set p=1, q=0, r=1, s=0, t=0, u=0, v=1, we end up with a mode that contains the notes c, d, f#, a#, c'. This mode is characterized by a key (p,q,r,s,t,u,v) = (1,0,1,0,0,0,1).





This method for constructing modes has some properties:

- Any combination of values 0,1 assigned to all of p, q, r, s, t, u, v results in a mode derived from the chromatic scale with intervals distributed symmetrically around f#.
- A different combination of values 0,1 assigned to p, q, r, s, t, u, v results in a different mode. In other words, there are no two different keys (p,q,r,s,t,u,v) that result in the same set of notes selected from the chromatic scale.
- If we look at all possible ways that we can assign 0, 1 to p, q, r, s, t, u, v, we construct all possible subsets of the chromatic scale with intervals distributed symmetrically around f#. In other words, we don't skip any modes symmetrical around f# by using this method.

- All in all, this means that any (p,q,r,s,t,u,v) key uniquely defines one mode with intervals symmetrically distributed around f#.
- There are $2^7 = 127$ different ways to assign the values 0, 1 to the variables p, q, r, s, t, u, v. This means that there are 127 unique modes with symmetrical distribution of intervals around f#.

2.2 Intermezzo: binary numbers

In computer science a key like (0,1,1,0,0,1,0) can be interpreted as a binary number. Each number 1 or 0 is called a "bit". For every binary number, there's an equivalent decimal number and vice versa. In the systematic enumeration of modes in the appendices of this explanation, I use decimal equivalents of binary numbers, because they take much less typesetting space. If you want to make sense of the explanations that follow, it's useful to understand how binary numbers relate to decimal numbers, as explained now:

2.2.1 Conversion from binary number to decimal number

In order to convert from a binary number to a decimal number, one writes powers of two underneath the binary number and then multiplies and adds the results. As an example, consider conversion of number (0,1,1,0,0,1,0) to decimal:

0	1	1	0	0	1	0	$\leftarrow$ write binary digits	
64	32	16	8	4	2	1	$\leftarrow$ write powers of 2	
$*\downarrow$	$*\downarrow$	$*\downarrow$	$*\downarrow$	$*\downarrow$	$*\downarrow$	$*\downarrow$	$\downarrow$ multiply one by one	
0	32	16	0	0	2	0	$\rightarrow$ sum to get:	$32 + 16 + 2 = 50$

Note that if I add extra 0's to the left of a binary number, the decimal number doesn't change. The same is true for a decimal number: if you write 6 or you write 06 you really have the same number.

2.2.2 Conversion from a decimal number to a binary number

In order to convert a decimal number back to a binary number one keeps on dividing the number by 2, and notes down the rest after division. As an example, consider converting 50 back to binary representation:

- We start from 50
- divide 50 by 2 to get 25, with rest after division=0
- divide 25 by 2 to get 12, with rest after division=1
- divide 12 by 2 to get 06, with rest after division=0
- divide 06 by 2 to get 03, with rest after division=0
- divide 03 by 2 to get 01, with rest after division=1
- divide 01 by 2 to get 00, with rest after division=1

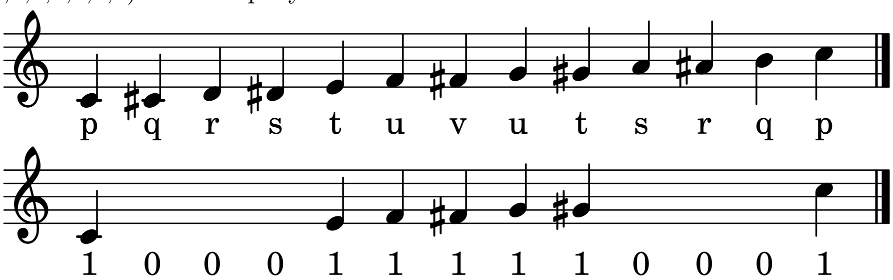
If you now look at the rests after division from bottom to top, you get (1,1,0,0,1,0). Remember from section 2.2.1 that one can add zeros to the left of any number without changing its value. Since we prefer to work with binary numbers (keys) of length 7 (i.e. the number of symbols p,q,...,v) we turn the binary number into key (0, 1, 1, 0, 0, 1, 0).

2.2.3 Tip about binary/decimal conversion

Practically all operating systems nowadays have some built-in calculator application that knows how to convert between binary and decimal should you ever need to do so.

3 What's the point? How do these binary numbers help us analyze music?

Using this enumeration method, I listed all the 127 modes that are symmetrical around f# as can be seen in section 7. I've sorted the modes by length. Each mode is annotated with a few numbers, e.g. 74:7-71. The number 74 means it's the 74th mode in the list of modes sorted by length. The number 7 means that the mode consists of 7 notes, distributed symmetrically around f#. The number 71 is the decimal number that corresponds to the binary key (p,q,r,s,t,u,v) = (1,0,0,0,1,1,1) that uniquely defines the mode:

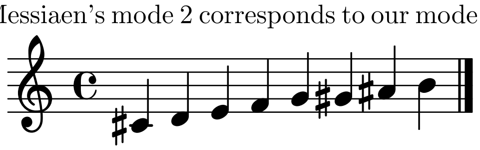


3.1 The symmetry of Messiaen's modes around f#

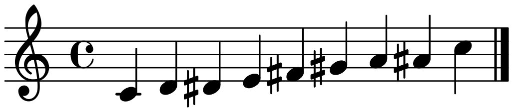
A first thing that struck me as interesting is that all of Messiaen's modes can be found back in the list of 127 modes enumerated in section 7 (albeit not always in first transposition, which is caused by our restriction to only look at modes symmetrical around f#, and not to look at modes symmetrical around – say – g.):

- Messiaen's mode 1 corresponds to our mode 85 (p,q,r,s,t,u,v) = (1,0,1,0,1,0,1)

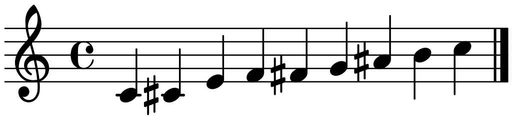
- Messiaen's mode 2 corresponds to our mode 54 (p,q,r,s,t,u,v) = (0,1,1,0,1,1,0)



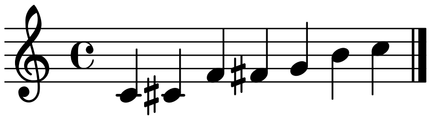
- Messiaen's mode 3 corresponds to our mode 93 (p,q,r,s,t,u,v) = (1,0,1,1,1,0,1)



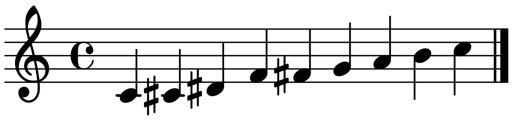
- Messiaen's mode 4 corresponds to our mode 103 (p,q,r,s,t,u,v) = (1,1,0,0,1,1,1)



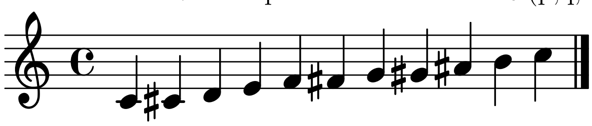
- Messiaen's mode 5 corresponds to our mode 99 (p,q,r,s,t,u,v) = (1,1,0,0,0,1,1)



- Messiaen's mode 6 corresponds to our mode 107 (p,q,r,s,t,u,v) = (1,1,0,1,0,1,1)



- Messiaen's mode 7 corresponds to our mode 119 (p,q,r,s,t,u,v) = (1,1,1,0,1,1,1)



3.2 More symmetry in Messiaen's modes

Now take a close look at the binary keys for Messiaen's modes. *In all but one of his modes, the binary keys are themselves palindromes*, indicating that not only are the modes symmetrical around f#, but they also have extra internal symmetry around d# (left-hand side of f#) and around a (right-hand side of f#). Having a palindromic key thus implies that there's a second level of symmetrical interval distribution inside the upper and lower halves of the modes.

It would not be correct to say that only modes with palindromic binary keys sound good. Look e.g. at mode of internal symmetry number 90 (1,0,1,1,0,1,0), which is better known as "c dorian". The binary key is not palindromic but it does show other extra symmetries in its structure (observe how its binary key becomes a palindrome if you leave out the last 0). Similarly, Messiaen's fourth mode of limited transposition also becomes a palindrome if you leave out the last 1 in its binary key, indicating that it has significant internal symmetries beyond the always present symmetry around f#.

I'd like to speculate here that additional internal symmetries play an important role in making modes (and their "modes") sound good. The human brain is optimized for pattern matching. It is sensitive to symmetries and the listener probably subconsciously picks up the patterns and hears the symmetries present in the intervals that make up the mode. (If that were true, we might even nominate the "Dorian" mode as the most natural of modes based on a diatonic scale.) Modes with extra internal symmetries look like good candidates for harmony and melody experiments.

3.3 Are there any modes with similar internal symmetries that are not Messiaen modes?

3.3.1 Perfect internal symmetry

In section 8 all modes are listed that have palindromic binary keys. Note that almost all of Messiaen's modes appear here in one form or another, and that some modes appear that are not part of Messiaen's musical language. This is because we used *internal symmetry* instead of *limited transposition* as criterion. From these results, it's quite clear that there's a close connection between the two criteria. It's also interesting that a number of (shorter) modes appear which, to the best of my knowledge, were not used directly by Messiaen, but which may be interesting for further harmonic and melodic experiments.

In what follows, remember that the notation "mode x-y" means "a mode with x notes and key y, where y should be converted to binary to see which notes are present in the mode. Compare the descriptions given here to the modes as listed in section 8.

- Mode 02-008 (0,0,0,1,0,0,0) is a ditonic mode. Two notes may be a bit limited for a composition. Interesting though that it is (enharmonically equivalent to) a tritone, which is one of the most important building blocks for Messiaen's musical language.
- Mode 04-020 (0,0,1,0,1,0,0) is a tetratonic mode.
- Mode 06-028 (0,0,1,1,1,0,0) is a hexatonic mode.
- Mode 04-034 (0,1,0,0,0,1,0) is a tetratonic mode. Interesting about this mode is that there's even more symmetry present in the lower and upper half of the binary key. This is a third level of symmetry in the mode.
- Mode 06-042 (0,1,0,1,0,1,0) is a hexatonic mode. Interesting about this mode is that there's even more symmetry present in the lower and upper half of the binary key. This is a third level of symmetry in the mode.
- Mode 08-054 (0,1,1,0,1,1,0) is an octotonic mode. It's also known as Messiaen's second mode of limited transposition.
- Mode 10-062 (0,1,1,1,1,1,0) is a decatonic mode. This mode is not listed by Messiaen. However, if we extend this mode with a c# at the right, we get a "mode" of Messiaen's seventh mode of limited transposition built on note d.
- Mode 03-065 (1,0,0,0,0,0,1) is a tritonic mode. It consists of 2 tritones.
- Mode 05-073 (1,0,0,1,0,0,1) is a pentatonic mode.
- Mode 07-085 (1,0,1,0,1,0,1) is a heptatonic mode. It's also known as Messiaen's first mode of limited transposition, or as the whole-tone scale.

Interesting about this mode is that there's even more symmetry present in the lower and upper half of the binary key. This is a third level of symmetry in the mode.

- Mode 09-093 (1,0,1,1,1,0,1) is a nonatonic mode. This is Messiaen's third mode of limited transposition. Interesting about this mode is that there's even more symmetry present in the lower and upper half of the binary key. This is a third level of symmetry in the mode.
- Mode 07-099 (1,1,0,0,0,1,1) is a heptatonic mode. This is Messiaen's fifth mode of limited transposition.
- Mode 09-107 (1,1,0,1,0,1,1) is a nonatonic mode. This is a "mode" of Messiaen's sixth mode of limited transposition built on note c#.
- Mode 11-119 (1,1,1,0,1,1,1) is an undecatonic mode. This is a "mode" of Messiaen's seventh mode of limited transposition built on note b. Interesting about this mode is that there's even more symmetry present in the lower and upper half of the binary key. This is a third level of symmetry in the mode.
- Mode 13-127 (1,1,1,1,1,1,1) is the chromatic scale itself. This mode also has a third and even fourth level of symmetry. Interestingly though, this mode is so extremely unconstrained that it no longer offers guarantees to elicit a certain mood/sound/color/character.

3.3.2 Partial internal symmetry

Now follows a list of modes that are partially palindromic as follows: the binary keys of the modes listed here become palindromic if you leave out either the first or last bit. They are listed in musical form in section 9. These form another subset of modes (slightly less "perfect" than the modes in the previous section). Probably some of these modes are better known under other names, but my knowledge of existing scales is not large enough to recognize all of them.

By reducing the constraints on symmetry also other interesting modes can be selected (e.g. modes that are a palindromic if you leave out 2 outer bits), but listing those is left as an exercise to the interested reader.

- 001 (0,0,0,0,0,0,1) is a mode consisting of a single note f#. This is a bit limited to compose with :)
- 012 (0,0,0,1,1,0,0) tetratonic
- 018 (0,0,1,0,0,1,0) tetratonic
- 024 (0,0,1,1,0,0,0) tetratonic
- 025 (0,0,1,1,0,0,1) tetratonic. Sounds quite exotic (arabic?).
- 030 (0,0,1,1,1,1,0) octatonic
- 033 (0,1,0,0,0,0,1) tritonic
- 036 (0,1,0,0,1,0,0) tetratonic

- 037 (0,1,0,0,1,0,1) pentatonic
- 045 (0,1,0,1,1,0,1) heptatonic. This is c# aeolian mode (natural minor diatonic scale).
- 051 (0,1,1,0,0,1,1) heptatonic. Sounds quite exotic (arabic?)
- 060 (0,1,1,1,1,0,0) octatonic
- 061 (0,1,1,1,1,0,1) nonatonic
- 063 (0,1,1,1,1,1,1) undecatonic. Like a chromatic scale but without note c.
- 064 (1,0,0,0,0,0,0) ditonic. Consists of only notes c.
- 066 (1,0,0,0,0,1,0) tetratonic. Consists of the notes of a sus4 chord built on c.
- 067 (1,0,0,0,0,1,1) pentatonic.
- 076 (1,0,0,1,1,0,0) hexatonic.
- 082 (1,0,1,0,0,1,0) hexatonic.
- 090 (1,0,1,1,0,1,0) octatonic. This is really just the dorian mode of c.
- 091 (1,0,1,1,0,1,1) nonatonic. Like the dorian mode of c but with f# added.
- 094 (1,0,1,1,1,1,0) decatonic.
- 097 (1,1,0,0,0,0,1) pentatonic.
- 102 (1,1,0,0,1,1,0) octatonic. Sounds quite exotic (arabic?).
- 103 (1,1,0,0,1,1,1) is the fourth mode of limited transposition of Messiaen.
- 109 (1,1,0,1,1,0,1) nonatonic. Left halve sounds darker than right halve.
- 115 (1,1,1,0,0,1,1) nonatonic.
- 126 (1,1,1,1,1,1,0) dodecatonic. Like a chromatic scale, but with f# left out.
- 127 (1,1,1,1,1,1,1) is the chromatic scale. This is the only key that also appears in the fully palindromic modes.

4 Deriving harmonies from modes of internal symmetry

Another question I was struggling with was why Messiaen chose to build chords based on the interval of a fourth. During my investigation I think I saw a possible explanation as follows:

- Sections 10, 11, 12, 13 systematically list the chords built on the modes of internal symmetry by taking notes from consecutive mode (scale) degrees, every second mode degree, every third mode degree, every fourth mode degree respectively.
- As is clearly visible, the fewer notes are in a mode that is symmetrical around f#, the more widely spaced those notes are (on average).
- In wider spaced modes (many 0's in the binary key), it makes sense to build chords from mode degrees that are close enough to each other.
- In more dense modes (many 1's in the binary key), it makes sense to build chords from mode degrees that are spaced further apart. If we use notes too close together, every chord sounds very dissonant.
- Since Messiaen's modes are relatively dense compared to - say - a diatonic scale (Messiaen's modes all have 8 or more notes, whereas a diatonic scale has 7 notes), my guess is that the harmonies as built using every second mode degree sound a bit too harsh, and Messiaen therefore decided to to use every third mode degree (i.e. using fourths).

5 Conclusions

To summarize:

- First we systematically listed all modes with internal symmetry around f# derived from a chromatic scale. We found that there are 127 such modes.
- Then, we saw how all of Messiaen's modes are part of this list of 127 modes, meaning that they at least have some symmetry in interval distribution.
- After that, we noticed how all but one of the Messiaen modes contain extra symmetries in the lower half and upper half of the mode (palindromic keys). We listed all modes that have both symmetry around f# and the extra symmetries in the lower half and upper half of the mode, and discovered some potentially interesting modes not directly used by Messiaen (with fewer notes). Even though we didn't care about the property of *limited transposition*, by just considering symmetry arguments, we arrived at a very similar set of modes.
- While doing so, we also formulated a possible explanation for why Messiaen may have chosen to build his chords using fourths, and we speculated about how internal symmetry may make Messiaen's modes sound superior to other, randomly chosen modes.

6 Some resources

While investigating I made extensive use of Jackson Hardaker's Messiaen mode visualizer: http://messiaen.jacksonhardaker.com/

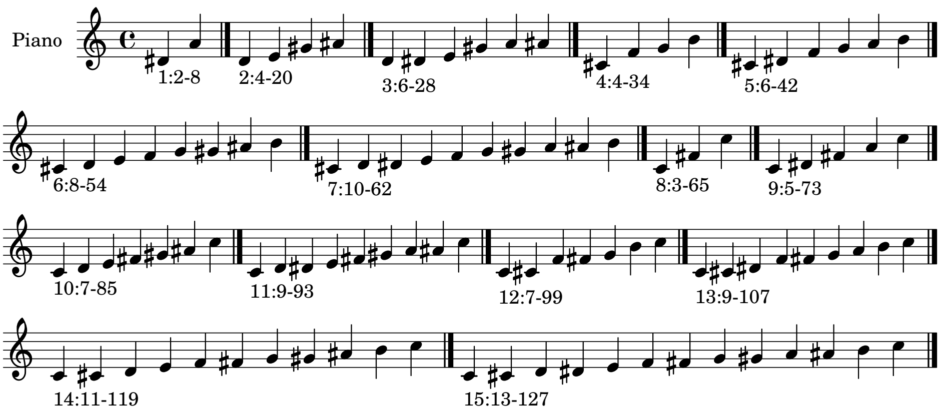
The complete code required to reproduce the experiments and the text in this document (together with midi files for the listed modes and chords in the following sections) can be found online at https://github.com/shimpe/mints. To recreate this document you need at the very least the following free software python 2.x, lilypond, LaTeX and a .pdf viewer (I used okular, but any viewer should do). The provided build script is written in bash. For windows or other systems, you may need to translate to an appropriate format.

Feel free to send your questions and remarks with respect to this document to stefaan.himpe@gmail.com.

7 All modes with internal symmetry around f#, ordered by length


80:7-99
81:7-101
82:7-105
83:7-113
84:8-30
85:8-46
86:8-54
87:8-58
88:8-60
89:8-78
90:8-86
91:8-90
92:8-92
93:8-102
94:8-106
95:8-108
96:8-114
97:8-116
98:8-120
99:9-31
100:9-47
101:9-55
102:9-59
103:9-61
104:9-79
105:9-87
106:9-91
107:9-93
108:9-103
109:9-107
110:9-109
111:9-115
112:9-117
113:9-121
114:10-62
115:10-94
116:10-110
117:10-118
118:10-122
119:10-124
120:11-63
121:11-95
122:11-111
123:11-119
124:11-123
125:11-125
126:12-126
127:13-127

8 All modes with palindromic keys



9 All modes with partially palindromic keys

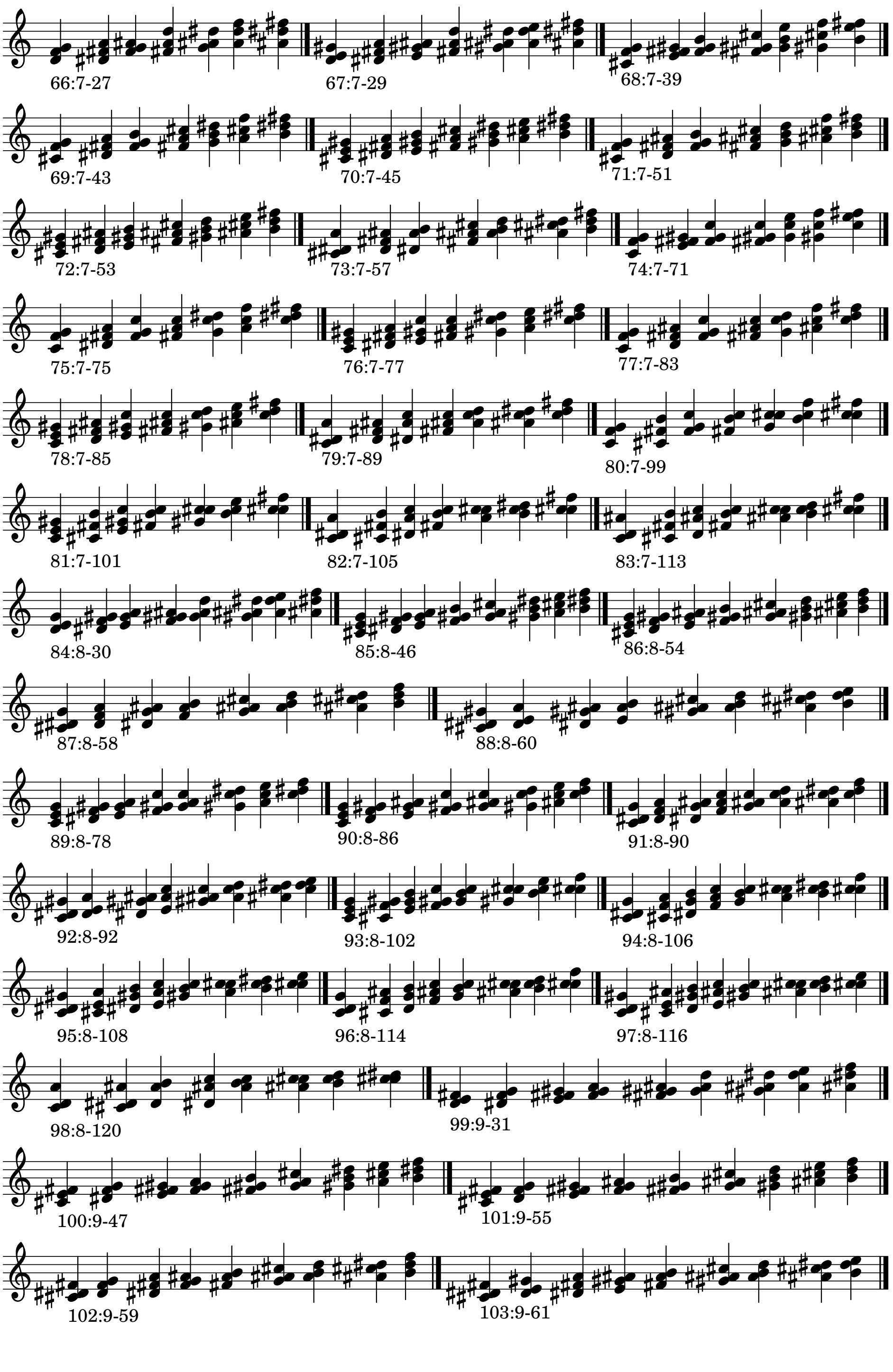
10 All chords built by stacking consecutive notes from the modes of internal symmetry


58:6-84
59:6-88
60:6-98
61:6-100
62:6-104
63:6-112
64:7-15
65:7-23
66:7-27
67:7-29
68:7-39
69:7-43
70:7-45
71:7-51
72:7-53
73:7-57
74:7-71
75:7-75
76:7-77
77:7-83
78:7-85
79:7-89
80:7-99
81:7-101
82:7-105
83:7-113
84:8-30
85:8-46
86:8-54
87:8-58
88:8-60
89:8-78
90:8-86
91:8-90
92:8-92
93:8-102


94:8-106
95:8-108
96:8-114
97:8-116
98:8-120
99:9-31
100:9-47
101:9-55
102:9-59
103:9-61
104:9-79
105:9-87
106:9-91
107:9-93
108:9-103
109:9-107
110:9-109
111:9-115
112:9-117
113:9-121
114:10-62
115:10-94
116:10-110
117:10-118
118:10-122
119:10-124
120:11-63
121:11-95
122:11-111

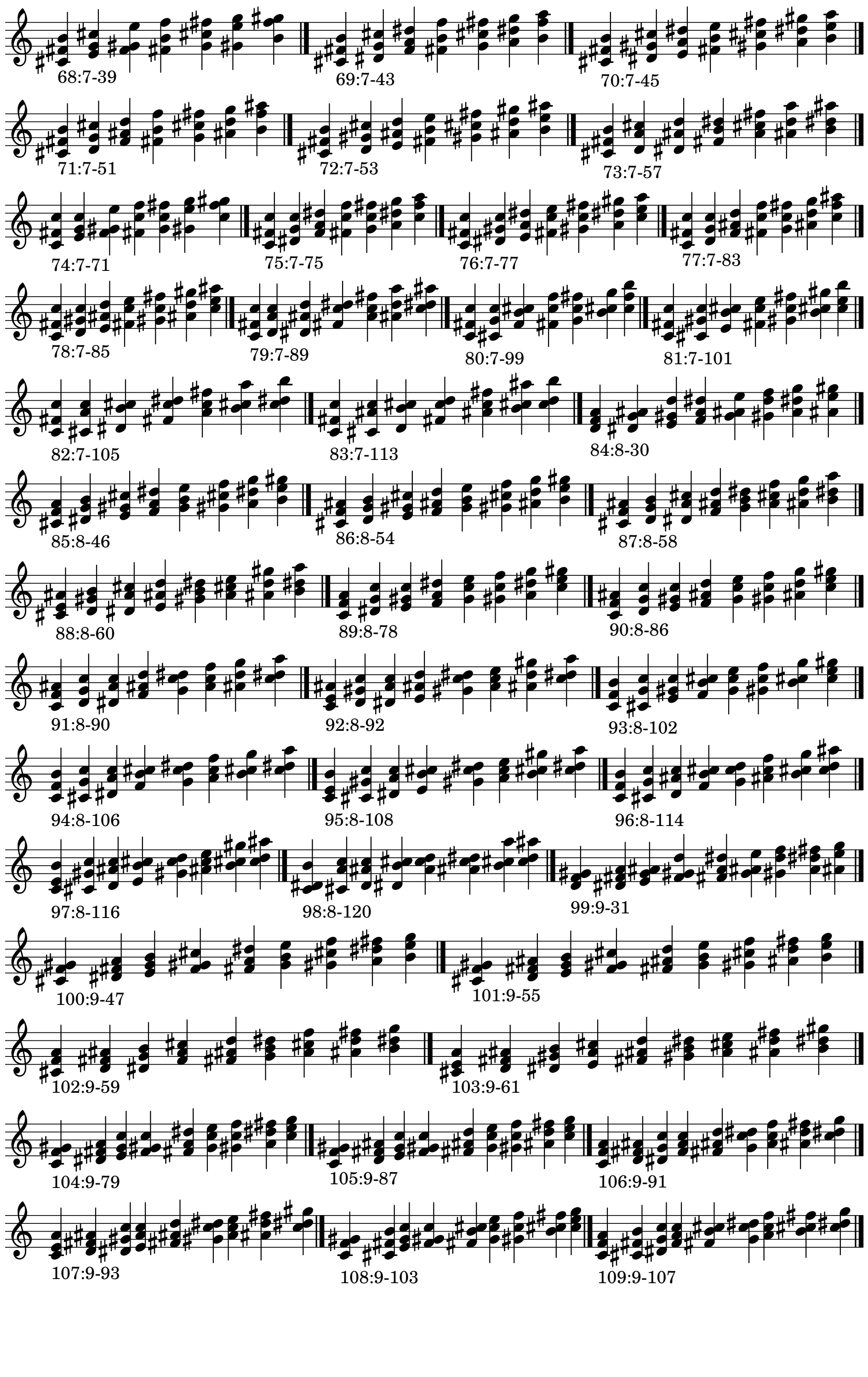

123:11-119
124:11-123
125:11-125
126:12-126
127:13-127

11 All chords built by stacking every second note from the modes of internal symmetry


66:7-27
67:7-29
68:7-39
69:7-43
70:7-45
71:7-51
72:7-53
73:7-57
74:7-71
75:7-75
76:7-77
77:7-83
78:7-85
79:7-89
80:7-99
81:7-101
82:7-105
83:7-113
84:8-30
85:8-46
86:8-54
87:8-58
88:8-60
89:8-78
90:8-86
91:8-90
92:8-92
93:8-102
94:8-106
95:8-108
96:8-114
97:8-116
98:8-120
99:9-31
100:9-47
101:9-55
102:9-59
103:9-61

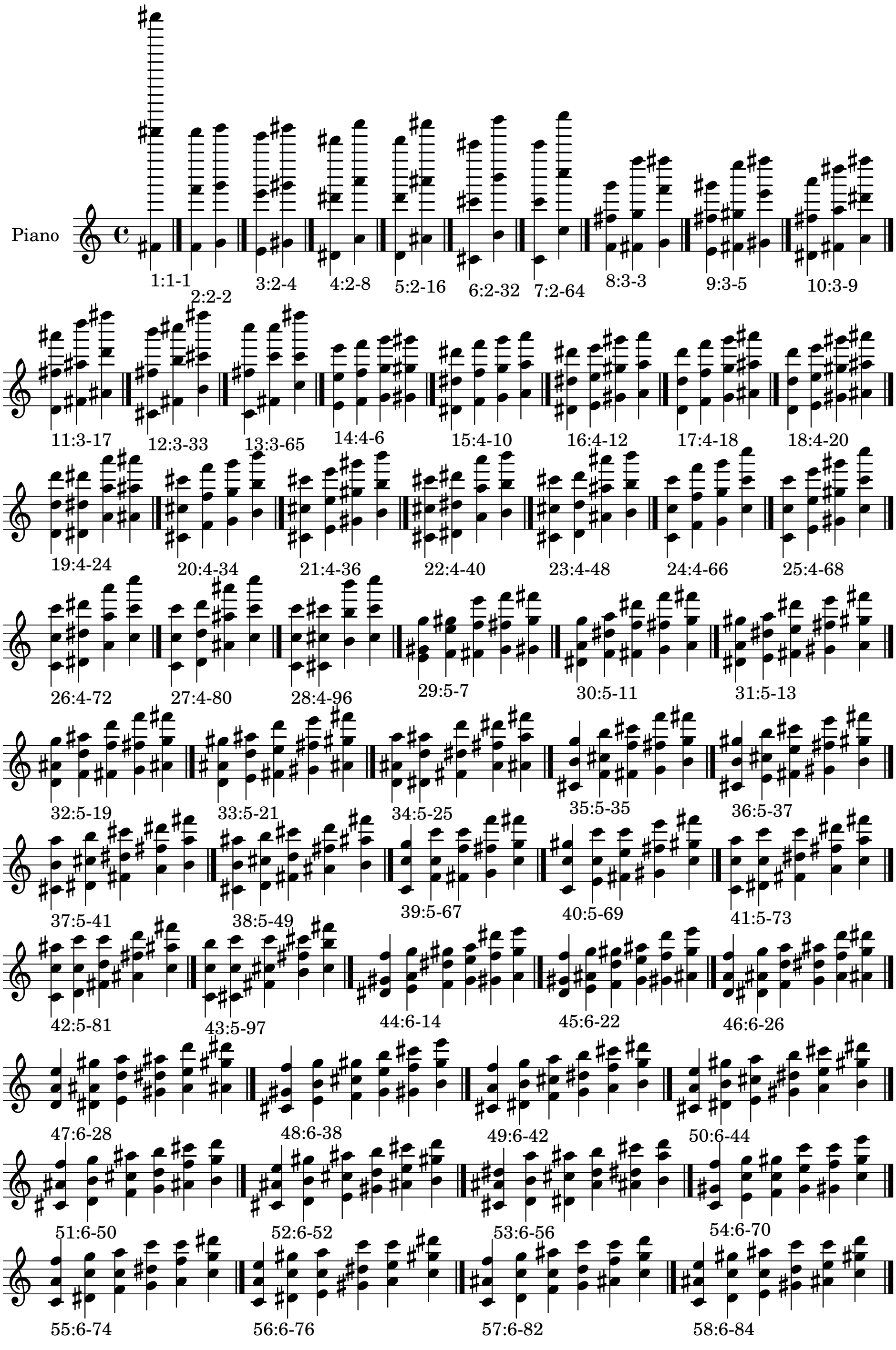

104:9-79
105:9-87
106:9-91
107:9-93
108:9-103
109:9-107
110:9-109
111:9-115
112:9-117
113:9-121
114:10-62
115:10-94
116:10-110
117:10-118
118:10-122
119:10-124
120:11-63
121:11-95
122:11-111
123:11-119
124:11-123
125:11-125
126:12-126
127:13-127

12 All chords built by stacking every third note from the modes of internal symmetry

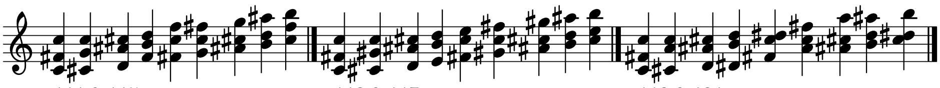

68:7-39
69:7-43
70:7-45
71:7-51
72:7-53
73:7-57
74:7-71
75:7-75
76:7-77
77:7-83
78:7-85
79:7-89
80:7-99
81:7-101
82:7-105
83:7-113
84:8-30
85:8-46
86:8-54
87:8-58
88:8-60
89:8-78
90:8-86
91:8-90
92:8-92
93:8-102
94:8-106
95:8-108
96:8-114
97:8-116
98:8-120
99:9-31
100:9-47
101:9-55
102:9-59
103:9-61
104:9-79
105:9-87
106:9-91
107:9-93
108:9-103
109:9-107


110:9-109
111:9-115
112:9-117
113:9-121
114:10-62
115:10-94
116:10-110
117:10-118
118:10-122
119:10-124
120:11-63
121:11-95
122:11-111
123:11-119
124:11-123
125:11-125
126:12-126
127:13-127

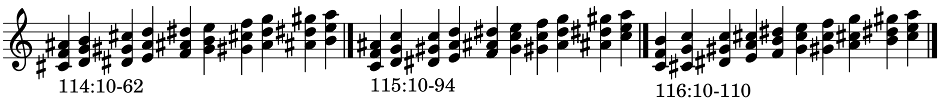
13 All chords built by stacking every fourth note from the modes of internal symmetry



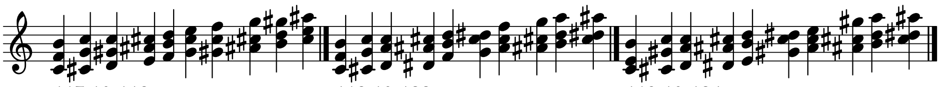

59:6-88
60:6-98
61:6-100
62:6-104
63:6-112
64:7-15
65:7-23
66:7-27
67:7-29
68:7-39
69:7-43
70:7-45
71:7-51
72:7-53
73:7-57
74:7-71
75:7-75
76:7-77
77:7-83
78:7-85
79:7-89
80:7-99
81:7-101
82:7-105
83:7-113
84:8-30
85:8-46
86:8-54
87:8-58
88:8-60
89:8-78
90:8-86
91:8-90
92:8-92
93:8-102
94:8-106
95:8-108
96:8-114
97:8-116
98:8-120
99:9-31
100:9-47
101:9-55
102:9-59
103:9-61
104:9-79
105:9-87
106:9-91
107:9-93
108:9-103
109:9-107
110:9-109



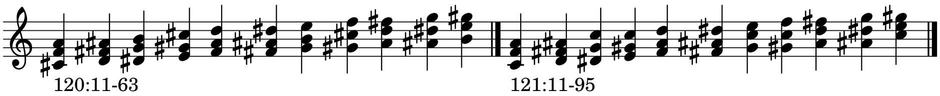
111:9-115 112:9-117 113:9-121



114:10-62 115:10-94 116:10-110



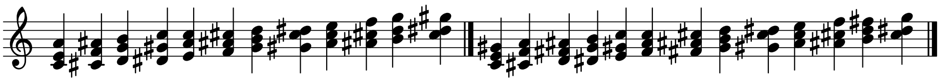
117:10-118 118:10-122 119:10-124



120:11-63 121:11-95

122:11-111 123:11-119

124:11-123 125:11-125



126:12-126 127:13-127