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EEM/EEE314 Automatic Control Systems

Exam-style questions with solutions

Part 4: Laplace transform and transfer functions

Abbreviations:

ODE: ordinary differential equation

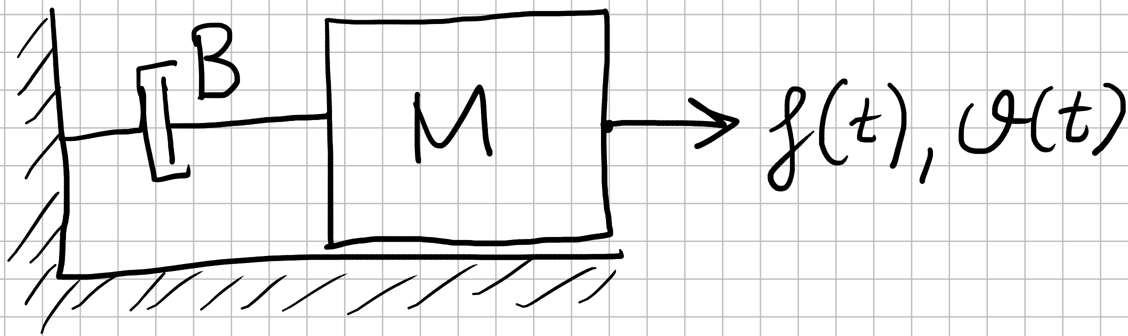
N2L: Newton's second law of motion

LHS: left-hand side (of the equation)

RHS: right-hand side (of the equation)

opamp: operational amplifier

Question 1: Consider the schematic of a translational mechanical system depicted below, consisting of a mass and a damper
(with parameters M and B). Speed of the mass is denoted as v(t).
An external force is being applied to the system,
denoted as f(t). There are no other forces acting on the system.
Assuming that the external force is in the form of a step function f(t) = f0*1(t)
(with 1(t) denoting the unit step function; and f0 a constant), and that all initial conditions are zero, find the speed of the mass at time equal to t = 5.



Solution: We first need to find the ODE model of the system, and then solve it for $f(t) = f_0 \cdot 1(t)$ to find the solution for $v(5)$.

Writing N2L:

$$\underbrace{f - B \cdot v = M \cdot \dot{v}}$$

this is the ODE model of the system

rearranging to have v terms on LHS:

$$M \dot{v} + B v = f$$

To solve the ODE (using Laplace transform method), we proceed as follows:

Take Laplace transform of both sides:

$$\mathcal{L}\{M \cdot \dot{v} + B v\} = \mathcal{L}\{f(t)\}$$

Laplace transform is linear, thus:

$$\mathcal{L}\{M \dot{v}\} + \mathcal{L}\{B v\} = \mathcal{L}\{f(t)\}$$

$$M \cdot \mathcal{L}\{\dot{v}\} + B \cdot \mathcal{L}\{v\} = \mathcal{L}\{f(t)\}$$

We define the following symbols:
(following standard notational convention of using uppercase letters for Laplace transforms of general signals)

$$\mathcal{L}\{v(t)\} = V(s)$$

$$\mathcal{L}\{f(t)\} = F(s)$$

Furthermore, from the Laplace transforms table, we know that:

$$\mathcal{L}\{\dot{v}\} = s V(s) - \cancel{v(0)}^{\,\rightarrow 0}$$

(all initial conditions are zero!)

$$\mathcal{L}\{\dot{v}\} = s V(s)$$

Thus, we have:

$$M \cdot s V(s) + B \cdot V(s) = F(s)$$

$$(M \cdot s + B) \cdot V(s) = F(s)$$

$$V(s) = \frac{1}{Ms + B} \cdot F(s)$$

We found the relationship between $f(t)$ and $v(t)$ in the Laplace domain. Now we need to substitute for the specific $f(t)$ given in the question and then take the inverse Laplace transform to find the solution to the ODE.

In the question it says:

$$f(t) = f_0 \cdot 1(t)$$

Taking the Laplace transform:

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{f_0 \cdot 1(t)\}$$
$$= f_0 \cdot \mathcal{L}\{1(t)\}$$

From the Laplace transform table, we know that: $\mathcal{L}\{1(t)\} = \frac{1}{s}$

Thus, for $f(t)$, we have:

$$\mathcal{L}\{f(t)\} = F(s) = f_0 \cdot \frac{1}{s}$$

Substituting this in the following

$$V(s) = \frac{1}{Ms + B} \cdot F(s)$$

$$V(s) = \frac{1}{Ms + B} \cdot f_0 \cdot \frac{1}{s}$$

Rewriting to have 1 as the coefficient of the s term in the denominator:

$$V(s) = \frac{f_0}{M} \cdot \frac{1}{s \cdot (s + B/M)}$$

Now we need to take the inverse Laplace transform of $V(s)$ to obtain the ODE's solution $v(t)$.

From the Laplace transform table, we know that:

$$\mathcal{L}\left\{\frac{1}{a}\cdot\left(1-e^{-at}\right)\right\} = \frac{1}{s\cdot(s+a)} \quad \left(\begin{array}{c} a\in\mathbb{R}, \\ \text{constant}\end{array}\right)$$

To see this, use $\mathcal{L}\left\{\frac{1}{a}\cdot 1(t)\right\} = \frac{1}{a}\cdot\frac{1}{s}$ and $\mathcal{L}\left\{e^{-at}\right\} = \frac{1}{s+a}$ together, as follows:

$$\frac{1}{a}\left(1-e^{-at}\right) = \frac{1}{a} - \frac{1}{a}e^{-at}$$

Taking Laplace transform of RHS

$$\mathcal{L}\left\{\frac{1}{a}\right\} = \frac{1}{a}\cdot\frac{1}{s} \quad \left(\text{since } \mathcal{L}\{1\} = \frac{1}{s}\right)$$

$$\mathcal{L}\left\{\frac{1}{a}\cdot e^{-at}\right\} = \frac{1}{a}\cdot\frac{1}{s+a} \quad \left(\text{since } \mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}\right)$$

combining the two we have:

$$\frac{1}{a}\cdot\frac{1}{s} - \frac{1}{a}\cdot\frac{1}{s+a} = \frac{1}{a}\cdot\left(\frac{1}{s}-\frac{1}{s+a}\right) = \frac{1}{s\cdot(s+a)}$$

Since $\mathcal{L}\left\{\frac{1}{a}(1-e^{-at})\right\} = \frac{1}{s(s+a)}$,

the inverse Laplace transform of

the RHS of $V(s) = \frac{f_0}{M} \cdot \frac{1}{s \cdot (s + B/M)}$

is as follows:

$$v(t) = \frac{f_0}{\cancel{M}} \cdot \frac{\cancel{M}}{B} \cdot \left(1 - e^{-B \cdot t/M}\right)$$

$$v(t) = \frac{f_0}{B}\left(1 - e^{-Bt/M}\right)$$

which is the solution of the ODE.

Evaluating this at time $t = 5$,

we find:

$$\boxed{v(5) = \frac{f_0}{B}\left(1 - e^{5B/M}\right)}$$

Question 2: Consider the electrical circuit from question 1, Part 1: Electrical systems. Assume that all initial conditions are zero. Find the transfer function model of the system from v(t) to q(t).

Solution: The ODE model of the system is (see solution of Q1, Part 1):

$$v = R\dot{q} + L\cdot\ddot{q} + \frac{1}{C}q$$

Taking Laplace transforms (considering all initial conditions zero) of all terms, we have:

$$V(s) = L\cdot s^2\cdot Q(s) + R\cdot s\cdot Q(s) + \frac{1}{C}Q(s)$$

$$V(s) = Q(s)\cdot\left(Ls^2 + Rs + \frac{1}{C}\right)$$

The transfer function model for this system is the ratio of the Laplace transforms of q(t) (the output) and v(t) (the input). Thus:

$$\frac{Q(s)}{V(s)} = \boxed{\frac{1}{Ls^2 + Rs + 1/C}}$$

Question 3: Consider the mechanical system from question 1, Part 2: Mechanical systems. Assume that all initial conditions are zero. Find the transfer function model of the system from f(t) to q(t).

Solution: The ODE model of the system is (see solution of Q1, Part 2):

$$M \cdot \ddot{q} + D \cdot \dot{q} + Kq = f$$

Taking Laplace transforms (considering all initial conditions zero) of all terms, we have:

$$Ms^2 Q(s) + Ds\,Q(s) + K\,Q(s) = F(s)$$

$$Q(s) \cdot (Ms^2 + Ds + K) = F(s)$$

$$\frac{Q(s)}{F(s)} = \boxed{\frac{1}{Ms^2 + Ds + K}}$$

Question 4: Consider the electromechanical system from question 1, Part 3: Electromechanical systems. Assume that: 1) L and B parameters are equal to zero. 2) External load torque d(t) is equal to zero.
3) All initial conditions are zero.
Find the transfer function model of the system from v(t) to ω(t).

Solution: The ODE model of the system is (see solution of Q1, Part 3)

$$\frac{LJ}{K}\ddot{\omega} + \frac{RJ+LB}{K}\dot{\omega} + \left(\frac{RB}{K}+K\right)\omega = \ldots$$

$$\ldots v - \frac{L}{K}\dot{d} - \frac{R}{K}d$$

Using the assumptions $L=0$, $B=0$, and $d(t)=0$, the ODE model simplifies to:

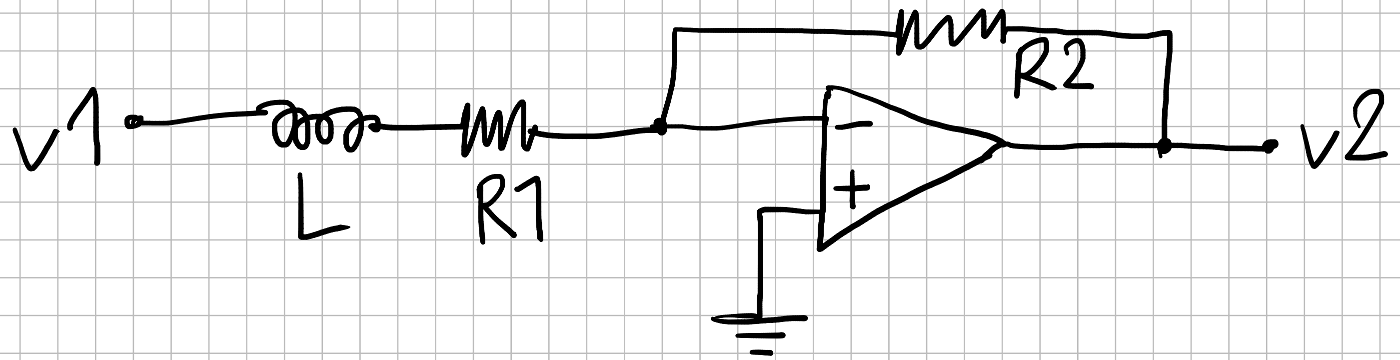
$$\frac{RJ}{K}\dot{\omega} + K\omega = v$$

Taking Laplace transforms (considering all initial conditions zero) of all terms, we have:

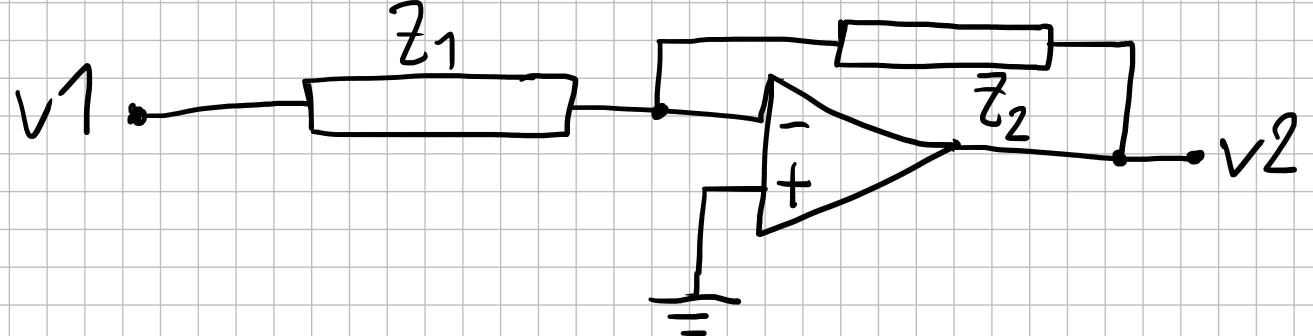
$$\frac{RJ}{K}s\Omega(s) + K\cdot\Omega(s) = V(s)$$

$$\frac{\Omega(s)}{V(s)} = \boxed{\frac{1}{\frac{RJ}{K}\cdot s + K}}$$

Question 5: Consider the schematic of an electronic circuit depicted below, consisting of an ideal opamp, together with two resistors and an inductor (with parameters R1, R2, and L, respectively).
The voltages at the two ends of the circuit are denoted as v1(t) and v2(t).
Find the transfer function model of the system from v1(t) to v2(t).



Solution: From the general form of the inverting opamp circuit, we know:



$$\frac{V2}{V1} = -\frac{Z_2}{Z_1} \quad \text{(transfer function)}$$

Specifically for the circuit in the question, we have: $Z_1 = Ls + R1$, $Z_2 = R2$

Thus:

$$\frac{V2}{V1} = \boxed{\frac{-R2}{Ls + R1}}$$