Time Series Analysis

Basics
Sample Autocorrelations

$$\gamma_0 = \frac{1}{N} \sum_{t=1}^{N} (x_t - \bar{x})$$

$$\gamma_k = \frac{1}{N} \sum_{t=1}^{N-k} (x_t - \bar{x})(x_{t+k} - \bar{x})$$

$$\rho_0 = 1$$

$$\rho_k = \frac{\gamma_k}{\gamma_0}$$

Autoregressive Models
AR(1)

$$X_t - \phi X_{t-1} = a_t$$
$$(1 - \phi z) = 0$$

AR(1) Properties
- Positive ϕ

  - Realizations appear to be wandering (aperiodic)
  - Autocorrelations are damped exponentials.
  - Spectral densities have peaks at zero

- Negative ϕ

  - Realizations appear to be oscillating
  - Autocorrelations are damped oscillating exponentials
  - Spectral densities have peaks at $f = 0.5$

AR(2) Models

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = a_t$$
$$(1 - \phi_1 z - \phi_2 z^2) = 0$$

AR(2) Properties
- Two Real Roots - Both Pos

  - The realization will appear to be wandering
  - The autocorrelations will be exponentially damped
  - There will be a peak at 0

- Two Real Roots - Both Neg

  - The realization will appear to be oscillating
  - The autocorrelations will be damped oscillating exponentials
  - There will be a peak at 0.5

- Two Real Roots - One Each

  - The realization will appear to be wandering and an oscillation will run on the realization
  - The autocorrelations will be exponentially damped with a hint of oscillation
  - There will be peaks at 0 and 0.5 in the spectal density

- One Complex

  - The realization will appear to have a pseudo-cyclic behavior with a cycle length of $\frac{1}{f_0}$
  - The autocorrelations will be damped exponentials oscillating in a sinusoid envelope with a frequency of f_0
  - There will be a peak at f_0 (between 0 and 0.5)

$$f_0 = \frac{1}{2\pi} cos^{-1} \left(\frac{\phi_1}{2\sqrt{-\phi_2}} \right)$$

AR(p)

$$X_t = \beta + \phi_1 X_{t-1} + \phi_2 X_{t-2} + ... + \phi_2 X_{t-p} + a_t$$
$$x_t - \phi_1 B X_t - \phi_2 B^2 X_t - \phi_p B^p X_t = a_t$$

Key Concepts
- An AR(p) model is stationary if and only if all the roots of the characteristic equation are outside the unit circle.

- Any AR(p) characteristic equation can be numerically factored into 1st and 2nd order elements.

- These factors are interpreted as contributing AR(1) and AR(2) behaviors to the total behavior of the AR(p) model.

Factor Contributions

AR(p) models reflect a contribution of AR(1) and AR(2) contributions. Roots that are close to the unit circle will be the dominate behavior.

- First order factors $(1 - \phi_1 B)$

  - Associated with real roots

  - Contribute AR(1)-type behavior to the AR(p) model

  - Associated with a system frequency of 0 if ϕ_1 is positive or 0.5 if ϕ_1 is negative

- Second order factors $(1 - \phi_1 B - \phi_2 B^2)$

  - Associated with complex roots

  - Contribute cyclic AR(2) behavior to the AR(p) model

  - Associated with a system frequency of f_0