Time Series Analysis

Basics

Sample Autocorrelations

$$\gamma_0 = \frac{1}{N}\sum_{t=1}^{N}(x_t - \bar{x})$$
$$\gamma_k = \frac{1}{N}\sum_{t=1}^{N-k}(x_t - \bar{x})(x_{t+k} - \bar{x})$$
$$\rho_0 = 1$$
$$\rho_k = \frac{\gamma_k}{\gamma_0}$$

Autoregressive Models

AR(1) Models

$$X_t - \phi X_{t-1} = a_t$$
$$(1 - \phi z) = 0$$

AR(1) Properties

- Positive ϕ
  - Realizations appear to be wandering (aperiodic)
  - Autocorrelations are damped exponentials.
  - Spectral densities have peaks at zero
- Negative ϕ
  - Realizations appear to be oscillating
  - Autocorrelations are damped oscillating exponentials
  - Spectral densities have peaks at $f = 0.5$

AR(2) Models

$$X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} = a_t$$
$$(1 - \phi_1 z - \phi_2 z^2) = 0$$

AR(2) Properties

- Two Real Roots - Both Pos
  - The realization will appear to be wandering
  - The autocorrelations will be exponentially damped
  - There will be a peak at 0
- Two Real Roots - Both Neg
  - The realization will appear to be oscillating
  - The autocorrelations will be damped oscillating exponentials
  - There will be a peak at 0.5
- Two Real Roots - One Each
  - The realization will appear to be wandering and an oscillation will run on the realization
  - The autocorrelations will be exponentially damped with a hint of oscillation
  - There will be peaks at 0 and 0.5 in the spectal density
- One Complex
  - The realization will appear to have a pseudo-cyclic behavior with a cycle length of $\frac{1}{f_0}$
  - The autocorrelations will be damped exponentials oscillating in a sinusoid envelope with a frequency of f_0
  - There will be a peak at f_0 (between 0 and 0.5)

$$f_0 = \frac{1}{2\pi} cos^{-1}\left(\frac{\phi_1}{2\sqrt{-\phi_2}}\right)$$

AR(p) Models

$$X_t - \beta + \phi_1 X_{t-1} + \phi_2 X_{t-2} + ... + \phi_2 X_{t-p} = a_t$$
$$x_t - \phi_1 B X_t - \phi_2 B^2 X_t - ... - \phi_p B^p X_t = a_t$$

Key Concepts

- An AR(p) model is stationary if and only if all the roots of the characteristic equation are outside the unit circle.
- Any AR(p) characteristic equation can be numerically factored into 1st and 2nd order elements.
- These factors are interpreted as contributing AR(1) and AR(2) behaviors to the total behavior of the AR(p) model.

Factor Contributions

AR(p) models reflect a contribution of AR(1) and AR(2) contributions. Roots that are close to the unit circle will be the dominate behavior.

- First order factors $(1 - \phi_1 B)$
  - Associated with real roots
  - Contribute AR(1)-type behavior to the AR(p) model
  - Associated with a system frequency of 0 if ϕ_1 is positive or 0.5 if ϕ_1 is negative
- Second order factors $(1 - \phi_1 B - \phi_2 B^2)$
  - Associated with complex roots
  - Contribute cyclic AR(2) behavior to the AR(p) model
  - Associated with a system frequency of f_0

Moving Average Models

MA(1) Models

$$X_t = a_t - \theta a_{t-1}$$
$$(1 - \theta_1 z) = 0$$
$$\rho_0 = 0$$
$$\rho_1 = \frac{-\theta_1}{1 + \theta_1^2}$$
$$\rho_k = 0|_{k>1}$$

MA(2) Models

$$X_t = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}$$
$$(1 - \theta_1 z - \theta_2 z^2) = 0$$
$$\rho_0 = 0$$
$$\rho_1 = \frac{-\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2}$$
$$\rho_k = 0|_{k>2}$$

MA(q) Models

$$X_t = a_t - \theta_1 a_{t-1} - ... - \theta_2 a_{t-q}$$
$$x_t = a_t - \theta_1 B a_t - ... - \theta_q B^q X_t$$

Key Concepts

- MA models are a finite GLP
- MA models are always stationary
- MA models are invertable iff all the roots are outside of the unit circle.

ARMA(p,q) Models

$$X_t = \beta + \phi_1 X_{t-1} + ... + \phi_2 X_{t-p} = a_t - \theta_1 a_{t-1} - ... - \theta_2 a_{t-q}$$
$$x_t - \phi_1 B X_t - ... - \phi_p B^p X_t = a_t - \theta_1 B a_t - ... - \theta_q B^q X_t$$

Key Concepts

- Valid when the model is stationary and invertable
  - Stationary: roots of $\phi(z)$ are outside the unit circle
  - Invertable: roots of $\theta(z)$ are outside the unit circle
- $\phi(z)$ and $\theta(z)$ have no common factors (check)