autoregressive_models

March 3, 2020

1 Autoregressive (AR) Models

Autoregressive models are expressed as dependence on lagged values. These types of models are generally good are modeling stationary processes that move forward in time.

First order AR models AR(1) and second order AR models AR(2) are discussed first. Then the details from these based models are used to discuss the generalizations to pth order AR models AR(p).

1.1 1st Order: AR(1)

The first order autoregressive model AR(1) is expressed as

$$X_t = \beta + \phi X_{t-1} + a_t$$

where $\beta = (1 - \phi)\mu$

This model is sometimes written in zero-mean form as

$$X_t = \phi X_{t-1} + a_t$$

The interpretation of AR(1) is "the value of the process at time t depends on the value of the process at $t - 1$ plus a random noise component."

1.1.1 Properties of AR(1)

There are several properties that depend on the value of ϕ and the sign of ϕ.

General

- An AR(1) is **stationary if and only if** $|\phi| < 1$

  - The expexted value of an AR(1) model is μ
  - The variance is finite if and only if $|\phi| < 1$
  - The autocorrelation depends only on k: $\rho_k = \phi^k, k > 0$

Positive ϕ

- Realizations apprear to be wandering (aperiodic).
- Autocorrelations are damped exponentials.
- Spectral densities have peaks at sero.

Negative ϕ

- Realizations seem to be oscillating.
- Autocorrelations are damped oscillating exponentials
- Spectral densities have peaks at $f = 0.5$

1.1.2 Non-Stationary "AR(1) Types"

Realizations with $|\phi| = 1$, look similar to realizations that are encountered in practice. These are a special case of ARIMA realizations. Realizations with $|\phi| > 1$, do not looks like real realizations in practice.

1.1.3 The Characteristic Equation

The characteristic equation of an AR(1) model can be found by using backshift notation to factor the model. Shown here with the zero mean form AR(1).

$$X_t - \phi X_{t-1} = a_t$$

Noting that $BX_t = X_{t-1}$ and setting a_t to 0

$$(1 - \phi B)X_t = 0$$

The characteristic equation is

$$(1 - \phi B) = 0$$

and the root is

$$root = \frac{1}{\phi}$$

Since AR(1) is stationary if and only if $|\phi| < 1$, then an AR(1) model will be stationary if the root of the characteristic equation is $\left|\frac{1}{\phi}\right| > 1$.

1.2 2nd Order: AR(2)

The first order autoregressive model AR(1) is expressed as

$$X_t = \beta + \phi_1 X_{t-1} + \phi_2 X_{t-2} + a_t$$

where $\beta = (1 - \phi_1 - \phi_2) + \mu$
The zero-mean for of the model is written as

2

$$x_t - \phi_1 B X_t - \phi_2 B^2 X_t = a_t$$

Which is typically written in one of the following forms:

- $(1 - \phi_1 B - \phi_2 B^2) X_t = a_t$
- $\phi(B) X_t = a_t$ where $\phi(B) = (1 - \phi_1 B - \phi_2 B^2)$

The interpretation of AR(2) is "the value at time t is a linear combination of two values at the two previous time periods plus a random noise component.

1.2.1 Properties of AR(2)

- An AR(2) model is **stationary if and only if the roots of the characteristic equation are outside of the unit circle**.

  - The expected value of an AR(2) model is μ
  - The variance is defined as

$$\sigma_X^2 = \frac{\sigma_a^2}{1 - \phi_1 \rho_1 - \phi_2 \rho_2}$$

1.2.2 The Characteristic Equation

Starting from the zero-mean model, we have

$$x_t - \phi_1 B X_t - \phi_2 B^2 X_t = a_t$$

Then setting a_t to 0, we have the characteristic equation

$$1 - \phi_1 B - \phi_2 B^2 = 0$$

There will either be two real roots or the roots will be a complex conjugate pair.

1.2.3 Behavior Based on the Roots

The beahvior of the AR(2) model is based on the roots of the characteristic equation.

Two Real Roots - Positive and Negative Roots

- The realization will appear to be wandering and an oscillation will run on the realization
- The autocorrelations will be exponentially damped with a hint of oscillation
- There will be peaks at 0 and 0.5 in the spectal density

Two Real Roots - Both Positive

- The realization will appear to be wandering
- The autocorrelations will be exponentially damped
- There will be a peak at 0

Two Real Roots - Both Negative

- The realization will appear to be oscillating
- The autocorrelations will be damped oscillating exponentials
- There will be a peak at 0.5

Two Complex Roots

- The realization will appear to have a pseudo-cyclic behavior with a cycle length of $\frac{1}{f_0}$
- The autocorrelations will be damped exponentials oscillating in a sinusoid envelope with a frequency of f_0
- There will be a peak at f_0 (between 0 and 0.5)

Where f_0 is given by

$$f_0 = \frac{1}{2\pi} cos^{-1} \left(\frac{\phi_1}{2\sqrt{-\phi_2}} \right)$$

1.3   Generalization to AR(p)

___

An AR(p) model is expressed as

$$X_t = \beta + \phi_1 X_{t-1} + \phi_2 X_{t-2} + ... + \phi_2 X_{t-p} + a_t$$

where $\beta = (1 - \phi_1 - \phi_2 - ... - \phi_p) + \mu$
The zero-mean for of the model is written as

$$x_t - \phi_1 B X_t - \phi_2 B^2 X_t - \phi_p B^p X_t = a_t$$

1.3.1   Key Concepts

- An AR(p) model is stationary if and only if all the roots of the characterisitc equation are outside the unit circle.
- Any AR(p) characteristic equation can be numerically factored into 1st and 2nd order elements.
- These factors are interpreted as contributing AR(1) and AR(2) behaviors to the total behavior of the AR(p) model.

1.3.2   Factor Contributions

AR(p) models reflect a contribution of AR(1) and AR(2) contributions.

- First order factors $(1 - \phi_1 B)$

    - Associated with real roots
    - Contribute AR(1)-type behavior to the AR(p) model
    - Associated with a system frequency of 0 if ϕ_1 is positive or 0.5 if ϕ_1 is negative

- Second order factors $(1 - \phi_1 B - \phi_2 B^2)$

– Associated with complex roots
  – Contribute cyclic AR(2) behavior to the AR(p) model
  – Associated with a system frequency of f_0

Roots that are close to the unit circle will be the dominate behavior.

1.3.3 Factor Table

High order AR models are summerized in a factor table to help present the behaviors that are present in the model.
  Example
  The model is

$$X_t - 1.95X_{t-1} - 1.85X_{t-2} - 0.855X_{t-3} = a_t$$

factors to

$$(1 - 0.95B)(1 - B + 0.9B^2)$$

Then the table is constructed from the factors

Factor	Root(s)	Abs Reciprocal	System Frequency
1- 0.95 B	1.053	0.95	0
1 - B + 0.95 B^2	$0.556 \pm 0.896i$	0.95	0.16

Notes from the table: * The model is stationary because all roots are outside the unit circle * There is wandering behavior from the AR(1) component * There is psuedo-cyclic behavior from the AR(2) component * There are peaks in the spectral density at 0.16 and 0

1.4 References

- [1] W. Woodward and B. Salder, "Autoregressive Models and Filtering", SMU, 2019
- [2] W. Woodward and B. Salder, "AR(2) and AR(p)", SMU, 2019