Prove that the set of rational numbers $\mathbb{Q}$, equipped with two binary operations of addition and multiplication, forms a field

$\Rightarrow$ $\mathbb{Q} = \{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \}$ with ordinary addition and multiplication is a field.

Proof: We verify the field axioms.

1) Well-defineness of operations:
A rational number is an equivalence class of pairs (p, q) with $q \neq 0$ under $\frac{p}{q} = \frac{p'}{q'} \Leftrightarrow pq' = p'q$.
The usual formulas

$$\frac{p}{q} + \frac{r}{s} = \frac{ps + rq}{qs}, \quad \frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}$$

respect this equivalance, so addition and multiplication are well-defined on $\mathbb{Q}$.

2) Closure: If $\frac{p}{q}, \frac{r}{s} \in \mathbb{Q}$ (with $q, s \neq 0$), then

$$\frac{p}{q} + \frac{r}{s} = \frac{ps + rq}{qs} \in \mathbb{Q}, \quad \frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs} \in \mathbb{Q}$$

since integers are closed under $+$ and $\cdot$. $qs \neq 0$.

3) Associativity of $+$ and $\cdot$: Associativity follows from associativity in $\mathbb{Z}$ and the formulas for sum/product of fractions; e.g. for addition computer both $\left(\frac{p}{q} + \frac{r}{s}\right) + \frac{t}{u}$ and $\frac{p}{q} + \left(\frac{r}{s} + \frac{t}{u}\right)$

and simplify to the same fraction $\frac{psu + rqu + tqs}{qsu}$. Similar for multiplication.

4. Commutavity of + and $\cdot$: For commutativity in $\mathbb{Z}$:

$$\frac{p}{q} + \frac{r}{s} = \frac{ps + rq}{qs} = \frac{rq + ps}{sq} = \frac{r}{s} + \frac{p}{q},$$

and likewise $\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs} = \frac{rp}{sq} = \frac{r}{s} \cdot \frac{p}{q}$

5. Identities:

→ Additive identity: $0 = \frac{0}{1}$ for any $\frac{p}{q}$.

$$\frac{p}{q} + \frac{0}{1} = \frac{p \cdot 1 + 0 \cdot q}{q \cdot 1} = \frac{p}{q}$$

→ Multiplicative identity: $1 = \frac{1}{1}$ for any $\frac{p}{q}$

$$\frac{p}{q} \cdot \frac{1}{1} = \frac{p \cdot 1}{q \cdot 1} = \frac{p}{q}$$

6. Additive inverse:

For $\frac{p}{q} \in \mathbb{Q}$, the additive inverse is $-\frac{p}{q} = \frac{-p}{q}$ since

$$\frac{p}{q} + \frac{-p}{q} = \frac{pq + (-p)q}{q^2} = \frac{0}{q^2} = 0.$$

7. Multiplicative inverses (for non-zero elements):

If $\frac{p}{q} \in \mathbb{Q}$ and $\frac{p}{q} \neq 0$, then $p \neq 0$. The inverse is

$$\left(\frac{p}{q}\right)^{-1} = \frac{q}{p}$$

and indeed $\frac{p}{q} \cdot \frac{q}{p} = \frac{pq}{qp} = 1.$

8. Distributive law: For any $\frac{p}{q}, \frac{r}{s}, \frac{t}{u} \in Q$

$$\frac{p}{q}\left(\frac{r}{s} + \frac{t}{u}\right) = \frac{p}{q} \cdot \frac{ru + ts}{su} = \frac{p(ru+ts)}{qsu}$$

$$= \frac{pru + pts}{qsu}$$

$$= \frac{pr}{qs} + \frac{pt}{qu}$$

$$= \frac{p}{q} \cdot \frac{r}{s} + \frac{p}{q} \cdot \frac{t}{u}$$

so multiplication distributes over addition.

All field axioms hold, so $(Q, +, \cdot)$ is a field.

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