1) Let G be a group of order pq, where p and q are distinct primes. Prove that G is abelian.

Proof: The product of subgroups P and Q, denoted PQ, is a subgroup of G with order

$$|PQ| = \frac{|P||Q|}{|P \cap Q|} = \frac{PQ}{1} = PQ.$$

Thus, $G = PQ$. Since Q is a normal subgroup, G is the internal direct product of P and Q, which means $G \cong P \times Q$.

Since ~~Q is~~ P and Q are both cyclic groups of prime order, they are abelian. The direct product of abelian group is abelian. So, G is an abelian group

2) Prove that if G is a group of order p^2, where p is prime, then G is abelian if and only if it has $p+1$ subgroup of order p.

Proof: Every group of order p^2 is abelian. There are exactly two types: C_{p^2} (has 1 subgroup of order p) and $C_p \times C_p$ (has $p+1$ subgroups of order p).

The correct equivalence is : "$G \cong C_p \times C_p \Leftrightarrow G$ has $p+1$ subgroup of order p". Both groups are abelian.

4) Let G be a group and N be a normal sub-group of G. If G/N is cyclic and N is cyclic, prove that G is abelian.

Proof: The statement as written is false.

counter example: Dihedral group D_{2n} (symmetries of an n-gon). Let R be the cyclic rotation subgroup of order n (so R is cyclic and normal), and $D_{2n}/R \cong C_2$ is cyclic. But D_{2n} is non-abelian for $n \geq 3$

7) Let G be a group and $a, b \in G$. Prove that if $a^4 = b^2$ and $ab = ba$, then $(ab)^6 = e$.

Answer: The statement is false.

For commutativity $(ab)^6 = a^6 b^6$

using $b^2 = a^4$ give $b^6 = (b^2)^3 = (a^4)^3 = a^{12}$

Thus $(ab)^6 = a^6 a^{12} = a^{18}$

Nothing in the hypotheses forces $a^{18} = e$. So the claim is not generally true.