

91강 이차형식(2) (Quadratic Forms)

* 이차곡선

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$$

회전 평행이동

→ 변수가 2개 곱해져 있다.

* 이차형식의 정의 ⇒ 모든 항이 2차원인 식

$x_1, x_2, x_3 \rightarrow$ 2개를 뽑아서 만들 수 있는 경우의 수

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + 2a_4x_1x_2 + 2a_5x_1x_3 + 2a_6x_2x_3 = 0$$

x_1, x_2

$$a_1x_1^2 + a_2x_2^2 + 2a_3x_1x_2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_1 & a_3 \\ a_3 & a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

대칭행렬

$$0 = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a_1 & a_4 & a_5 \\ a_4 & a_2 & a_6 \\ a_5 & a_6 & a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

A에 대한

$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ 대칭행렬 A에 대해서 $x^T A x$ 꼴을 이차형식이라고 한다.

$$Q_A(x) = x^T A x = \underline{x^T} (\underline{Ax}) = x \cdot (Ax) = Ax \cdot x$$

* $a_1x_1^2 + a_2x_2^2 + 2a_3x_1x_2 = k$ 의 그래프 모양은?

$$A = \begin{bmatrix} a_1 & a_3 \\ a_3 & a_2 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



$$Q_A(x) = k$$

$$x^T A x = k$$

A : 대칭행렬 \iff 정규 직교대각화 가능 (80강, 81강) $\rightarrow P^{-1} = P^T$

$$A = P D P^T \quad (D: \text{대각행렬}, P: \text{직교행렬})$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$x = P y$$

$$\text{좌변} \Rightarrow Q_A(x) = k$$

$$Q_A(x) = Q_D(y)$$

$$Q_A(Py) = k$$

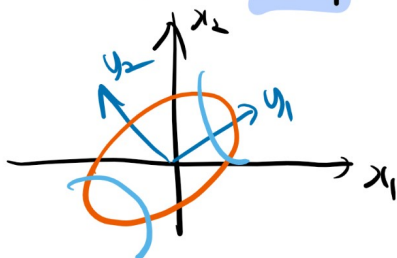
$$(Py)^T A (Py) = k$$

$$y^T P^T A P y = k$$

$$y^T D y = k$$

대각행렬 \Rightarrow 대칭

$$Q_D(y) = k$$



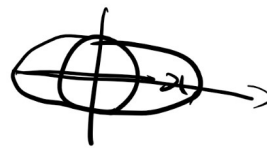
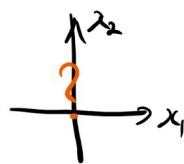
change of variable

$$\text{ex) } 3x_1^2 + 2x_2^2 - 10\sqrt{3}x_1x_2 = 144$$

$$\hookrightarrow a_1x_1^2 + a_2x_2^2 + 2a_3x_1x_2 = k$$

$$Q_A(x) = x^T A x = 144 \quad k=144$$

$$A = \begin{bmatrix} 3 & -5\sqrt{3} \\ -5\sqrt{3} & 2 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



$$Q_A(x) = Q_D(y) \quad \text{단, } D \text{는 대각행렬 } (A = P D P^T)$$

정규직교대각화. (축에 대한 확대변환)

$$x = P y$$

A의 고유값 (81강)

$$\begin{aligned} \det(\lambda I - A) &= \det \left(\begin{bmatrix} \lambda - 3 & 5\sqrt{3} \\ 5\sqrt{3} & \lambda - 2 \end{bmatrix} \right) = (\lambda - 3)(\lambda - 2) - (5\sqrt{3})^2 \\ &= \lambda^2 - 5\lambda + 6 - 75 \\ &= \lambda^2 - 5\lambda - 69 \\ &= (\lambda - 36)(\lambda - 16) = 0 \end{aligned}$$

$$\lambda = 36, 16$$

$$A = \begin{bmatrix} 3 & -5\sqrt{3} \\ -5\sqrt{3} & 2 \end{bmatrix}$$

$$\text{i) } \lambda = 36$$

$$Ax = 36x$$

$$(A - 36I)x = 0$$

$$\begin{bmatrix} 3-36 & -5\sqrt{3} \\ -5\sqrt{3} & 2-36 \end{bmatrix} x = 0$$

$$\Rightarrow \begin{bmatrix} 1 & \sqrt{3} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

고유값의 정의? $Ax \leq \lambda x$ 라면 해 만족하지 않게 하는 λ 값.

$$(A - \lambda I)x = 0 \quad \text{동차 연립 선형 방정식}$$

$$x_1 + \sqrt{3}x_2 = 0$$

$$x_1 = -\sqrt{3}x_2$$

$$x_1^2 + x_2^2 = 1$$

$$3x_2^2 + x_2^2 = 1 \rightarrow x_2^2 = \frac{1}{4} \quad x_2 = \pm \frac{1}{2}$$

$$v_1 = \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix} \quad \frac{3}{4} + \frac{1}{4} = 1$$

$$\text{ii) } \lambda = 16$$

$$A - 16I = \begin{bmatrix} 3-16 & -5\sqrt{3} \\ -5\sqrt{3} & 2-16 \end{bmatrix} = \begin{bmatrix} -13 & -5\sqrt{3} \\ -5\sqrt{3} & -14 \end{bmatrix}$$

$$\hookrightarrow \begin{bmatrix} 1 & \frac{\sqrt{3}}{3} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3x_1 - \sqrt{3}x_2 = 0$$

$$v_2 = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$K=144$$

$$A = PDP^T$$

$$P = [v_1 \ v_2] = \begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$D = \begin{bmatrix} 36 & 0 \\ 0 & 16 \end{bmatrix}$$

$$x = Py$$

$$Q_A(x) = Q_A(Py) = Q_D(y)$$

$$Px = y$$

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

$$= (py)^T A (py) = y^T P^T A P y$$

$$= y^T D y = Q_D(y)$$

$$\cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\sin 30^\circ = \frac{1}{2}$$

$$Q_D(y) = [y_1 \ y_2] \begin{bmatrix} 36 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\Rightarrow 36y_1^2 + 16y_2^2 = 144$$

$$\Rightarrow \frac{y_1^2}{4} + \frac{y_2^2}{9} = 1$$

