74강 대수적 중복도와 기하적 중복도 (2)

(Algebraic multiplicity and geometric multiplicity (2))

Thm A : $m\times m$ 행렬.

고유값 λ의 대수적 중복도를 a

기하적 중복도를 g 라고 하면

항상 $a \ge g$ 이다.

pf) $g = \dim(S)$ $\quad S = \{x \in \mathbb{R}^m \mid Ax = \lambda x\}$ $\quad (A-\lambda I)x = 0$

$B = \{x_1, x_2, \cdots, x_g\}$

$g \le m$

$S \subseteq \mathbb{R}^m$

$\hookrightarrow E = \{x_1, x_2, \cdots, x_m\}$

$T = [x_1\ x_2 \cdots x_g \mid y_1 \cdots y_{m-g}] = [X\ ]$

$AT = \underset{1\times1}{\boxed{A}}\underset{1\times m}{[x_1 | x_2 | \cdots | x_g | y_1 | \cdots | y_{m-g}]}$ $\quad AT : (m\times m)\cdot(m\times m) = (m\times m)$

$= [Ax_1\ Ax_2 \cdots Ax_g \mid Ay_1\ Ay_2 \cdots Ay_{m-g}]$

$= [\lambda x_1\ \lambda x_2 \cdots \lambda x_m \mid Ay_1\ Ay_2 \cdots Ay_{m-g}]$

$= [\ \lambda X_m\ ]_{①\times②} = \underset{m\times g}{[X}\ \underset{1\times 2}{\cancel{X}]}\begin{bmatrix} \boxed{\lambda I} & \cancel{Q} \\ \cancel{O} & \cancel{S} \end{bmatrix}_{m\times m}$

$g\times g$	$g\times(m-g)$
$(m-g)\times g$	$(m-g)\times(m-g)$

$m\times m$

$[x_1 \cdots x_g] = X$, $[y_1 \cdots y_{m-g}] = Y$

$XP + YR = \lambda X$, $\quad XQ + YS = AY$ $\quad AT = T\cdot\lambda I$

$[x_1\ x_2 \cdots x_g][\ ?\ ] + \cancel{[y_1\ y_2 \cdots y_{m-g}]\cdot O}$ $\quad = \lambda X$ $\quad A = T\cdot\lambda I\cdot T^{-1}$

$\hookrightarrow \lambda I$ $\qquad X\cdot\lambda I = \lambda X$

$[x_1 x_2 \cdots x_g]Q + [y_1 \cdots y_{m-g}]\cdot S = A[y_1 \cdots y_{m-g}]$

$\hookrightarrow [Q_1 Q_2 \cdots Q_{m-g}]$ $\qquad \hookrightarrow [S_1 S_2 \cdots S_{m-g}]$

$f(x) = \underset{m\times m}{A}x$ $\quad \mathbb{R}^m \to \mathbb{R}^m$

$XQ_1 + YS_1 = Ay_1 = f(y_1)$

$XQ_2 + YS_2 = Ay_2 = f(y_2)$

$\vdots$

$XQ_{m-g} + YS_{m-g} = Ay_{m-g} = f(y_{m-g})$

$$AT = T\begin{bmatrix} \lambda I & Q \\ O & S \end{bmatrix}$$

$$= [X\ Y]\begin{bmatrix} \lambda I & Q \\ O & S \end{bmatrix}$$

$$A = T\begin{bmatrix} \lambda I & Q \\ O & S \end{bmatrix}T^{-1}$$

① det
② tr
③ 고유값 → 특성방정식

$\det(cI - A) = \det(c\underset{m\times m}{I} - \cancel{\begin{bmatrix} \overset{g\times g}{\lambda I} & Q \\ O & S \end{bmatrix}}) = \det(cI - \lambda I) = (c-\lambda)^m$

$= \det\left(\begin{bmatrix} (c-\lambda)I & -Q \\ O & cI - S \end{bmatrix}\right)$ $\quad (c-\lambda)I \hookrightarrow g\times g$, $\quad cI-S \hookrightarrow (m-g)\times(m-g)$

$= \det((c-\lambda)I)\det(cI - S)$

$= (c-\lambda)^g \det(cI - S)$ $\qquad a \ge g.$