91강 이차형식(2) (Quadratic Forms)

* 이차곡선

$$Ax^2+By^2+Cxy+Dx+Ey+F=0$$

↳ 회전 평행이동

* 이차형식의 정의 ⇒ 모든 항이 2차원인 식 → 변수가 2개 곱해져 있다.

x_1, x_2, x_3 → 2개를 뽑아서 만들수 있는 경우의 수

$$a_1x_1^2+a_2x_2^2+a_3x_3^2+2a_4x_1x_2+2a_5x_1x_3+2a_6x_2x_3 = ㉠$$

$x_1 \quad x_2$

$$a_1x_1^2+a_2x_2^2+2a_3x_1x_2 = [x_1 \quad x_2]\begin{bmatrix} a_1 & a_3 \\ a_3 & a_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

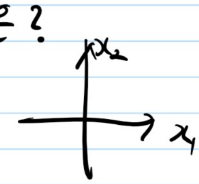
→ 대칭행렬

$$㉠ = [x_1 \quad x_2 \quad x_3]\begin{bmatrix} a_1 & a_4 & a_5 \\ a_4 & a_2 & a_6 \\ a_5 & a_6 & a_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$x=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ 대칭행렬 A에 대해서 x^TAx 꼴을 A에 대한 이차형식이라고 한다.

$$Q_A(x) = x^TAx = x^T(Ax) = x\cdot(Ax) = Ax\cdot x$$

* $a_1x_1^2+a_2x_2^2+2a_3x_1x_2 = k$ 의 그래프 모양은?



$A=\begin{bmatrix} a_1 & a_3 \\ a_3 & a_2 \end{bmatrix}$ $\quad x=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$Q_A(x)=k$

$x^TAx=k$

A : 대칭행렬 ⟺ 정규직교대각화가능 (80강, 81강) → 그람슈미트 $p^{-1}=p^T$

$A = pDp^T$ (D: 대각행렬, p: 직교행렬)

$y=\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ $\boxed{x=py}$ 치환 ⇒

축변환 → orthogonal

$D=p^TAp$

$Q_A(x)=k$

$Q_A(py)=k$

↳ $(py)^TA(py)=k$

$y^Tp^TApy=k$

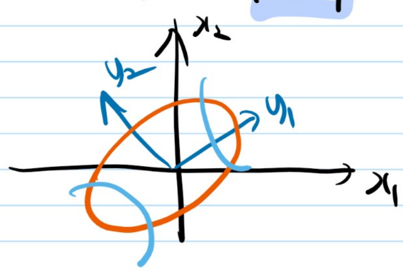
$y^TDy=k$

↳ 대각행렬 ⇒ 대칭

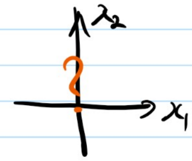
$Q_D(y)=k$

$Q_A(x)=Q_D(y)$ → 대각.

$\begin{cases} A=pDp^T \\ y=p^Tx \end{cases}$ change of variable

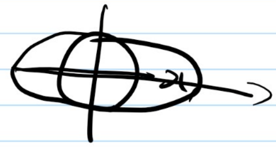


ex) $31x_1^2 + 21x_2^2 - 10\sqrt{3}x_1x_2 = 144$



↳ $a_1x_1^2 + a_2x_2^2 + 2a_3x_1x_2 = k$

$Q_A(x) = x^TAx = 144$ $k=144$



$A = \begin{bmatrix} 31 & -5\sqrt{3} \\ -5\sqrt{3} & 21 \end{bmatrix}$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$Q_A(x) = Q_D(y)$ 단, D는 대각행렬 ($A = PDP^T$)

정규직교 대각화. (축에 대한 확대 변환) $x = Py$

A의 고유값 (81강)

$\det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda-31 & 5\sqrt{3} \\ 5\sqrt{3} & \lambda-21 \end{bmatrix}\right) = (\lambda-31)(\lambda-21) - (5\sqrt{3})^2$

$= \lambda^2 - 52\lambda + 651 - 75$

$= \lambda^2 - 52\lambda + 576$

-36
-16

$= (\lambda-36)(\lambda-16) = 0$

$\lambda = 36, 16$

$A = \begin{bmatrix} 31 & -5\sqrt{3} \\ -5\sqrt{3} & 21 \end{bmatrix}$

i) $\lambda = 36$

$Ax = 36x$ $(A-36I)x = 0$

$\begin{bmatrix} 31-36 & -5\sqrt{3} \\ -5\sqrt{3} & 21-36 \end{bmatrix} x = 0$

$\Rightarrow \begin{bmatrix} 1 & \sqrt{3} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

고유값의 정의? $Ax = \lambda x$ 자명해 만을 가지지 않게 하는 λ 값.

$(A - \lambda I)x = 0$ 동차 연립 선형 방정식

↳ 비가역 행렬

$x_1 + \sqrt{3}x_2 = 0$

$x_1 = -\sqrt{3}x_2$ $v_1 = \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$ $\frac{3}{4} + \frac{1}{4} = 1$

$x_1^2 + x_2^2 = 1$

$3x_2^2 + x_2^2 = 1 \rightarrow x_2^2 = \frac{1}{4}$ $x_2 = \pm\frac{1}{2}$

ii) $\lambda = 16$ $A - 16I = \begin{bmatrix} 31-16 & -5\sqrt{3} \\ & \end{bmatrix} = \begin{bmatrix} 15 & -\frac{5\sqrt{3}}{15} \\ & \end{bmatrix}$

↳ $\begin{bmatrix} 1 & -\frac{\sqrt{3}}{3} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$3x_1 - \sqrt{3}x_2 = 0 \qquad V_2 = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$

$K = 144$

$A = PDP^T$

$P = [v_1 \ v_2] = \begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \qquad D = \begin{bmatrix} 36 & 0 \\ 0 & 16 \end{bmatrix}$

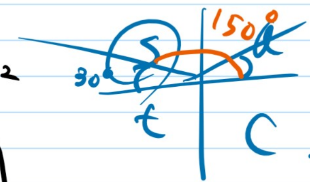
$x = Py \qquad Q_A(x) = Q_A(Py) = Q_D(y)$

$Px = y \qquad \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \qquad = (Py)^T A (Py) = y^T \underbrace{P^T A P}\, y \qquad \cos 30° = \frac{\sqrt{3}}{2}$

$\sin 30° =$

$= y^T D y = Q_D(y) \quad \frac{1}{2}$

$Q_D(y) = [y_1 \ y_2] \begin{bmatrix} 36 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$



$\Rightarrow 36y_1^2 + 16y_2^2 = 144$

$\Rightarrow \frac{y_1^2}{4} + \frac{y_2^2}{9} = 1$