제 47강 기저 변환

$E = \{e_1, e_2, \cdots, e_n\}$　　$B = \{v_1, v_2, \cdots, v_n\}$

$V \ni a = (a_1, a_2, \cdots, a_n)_E \longrightarrow (\qquad)_B$

$= a_1 e_1 + a_2 e_2 + \cdots + a_n e_n$

$e_1 = (1, 0, \cdots, 0)_E = (x_{11}, x_{12}, \cdots, x_{1n})_B$

$= x_{11} v_1 + x_{12} v_2 + \cdots + x_{1n} v_n$

$= \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} x_{11} \\ \vdots \\ x_{1n} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$　　$\begin{bmatrix} x_{11} \\ \vdots \\ x_{1n} \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$e_2 = (x_{21}, x_{22}, \cdots, x_{2n})_B$

$\begin{bmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{2n} \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$\vdots$

e_n

$e_2 \ B$

$$\begin{array}{cccc} x_1 & x_2 & & \\ \downarrow & \downarrow & & \end{array}$$

$$\boxed{\begin{bmatrix} x_{11} & x_{21} & \cdots & x_{n1} \\ x_{12} & x_{22} & & x_{n2} \\ \vdots & \vdots & & \vdots \\ x_{1n} & x_{2n} & & x_{nn} \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}^{-1}}$$

Thm. 벡터공간 V의 standard unit vector 들의 기저 $B = \{v_1, v_2, \cdots, v_n\}$에 대한 좌표를 각각 $x_1, x_2, \cdots, x_n$이라 할 때,

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}^{-1}$$ 이다

$V \ni \vec{a} = (a_1, a_2, \cdots, a_n)_E = (y_1, \cdots, y_n)_B$

$= a_1 e_1 + a_2 e_2 + \cdots + a_n e_n$

$= a_1 (x_{11}, x_{12}, \cdots, x_{1n})_B + a_2 (x_2)_B + \cdots + a_n (x_n)_B$

$= a_1 (x_{11} v_1 + x_{12} v_2 + \cdots + x_{1n} v_n)$

$+ a_2 (x_{21} v_1 + x_{22} v_2 + \cdots + x_{2n} v_n)$

$+ \cdots$

$+ a_n (x_{n1} v_1 + x_{n2} v_2 + \cdots + x_{nn} v_n)$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} a_1 x_{11} + a_2 x_{21} + \cdots + a_n x_{n1} \\ a_1 x_{12} + a_2 x_{22} + \cdots + a_n x_{n2} \\ \vdots \\ a_1 x_{1n} + a_2 x_{2n} + \cdots + a_n x_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} x_{11} & x_{21} & \cdots & x_{n1} \\ x_{12} & x_{22} & \cdots & x_{n2} \\ \vdots & & & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{nn} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}}$$

$= (a_1 x_{11} + a_2 x_{21} + \cdots + a_n x_{n1}) v_1$

$+ (a_1 x_{12} + a_2 x_{22} + \cdots + a_n x_{n2}) v_2$

$+ \cdots$

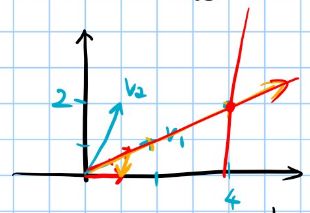
$+ (a_1 x_{1n} + a_2 x_{2n} + \cdots + a_n x_{nn}) v_n$

B의 축 $= \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}^{-1} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$

Thm 벡터공간 V의 임의의 벡터 $\vec{a} = (a_1, a_2, \cdots, a_n)_E$ 의 기저 $B = \{v_1, v_2, \cdots, v_n\}$ 에 대한 좌표를 $\vec{a} = (y_1, y_2, \cdots, y_n)_B$ 라 하면,

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \boxed{\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}^{-1} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}}$$ 을 만족한다.

예제) $\mathbb{R}^2$ $\quad E = \{e_1, e_2\}$



$v_1 = (2, 1)_E$

$v_2 = (1, 2)_E$

$B = \{v_1, v_2\}$

$$e_1 = (1, 0)_E = (a, b)_B = \left(\tfrac{2}{3}, -\tfrac{1}{3}\right)_B \qquad \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{-1} \begin{bmatrix} e_1 & e_2 \end{bmatrix}$$

$$e_2 = (0, 1)_E = (c, d)_B = \left(-\tfrac{1}{3}, \tfrac{2}{3}\right)_B$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$$

$$\boxed{\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}} = \frac{1}{4-1} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$(4, 2)_E = (x, y)_B$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \boxed{\begin{bmatrix} v_1 & v_2 \end{bmatrix}^{-1} \begin{bmatrix} e_1 & e_2 \end{bmatrix}} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{8}{3} - \frac{2}{3} \\ -\frac{4}{3} + \frac{4}{3} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}_B$$