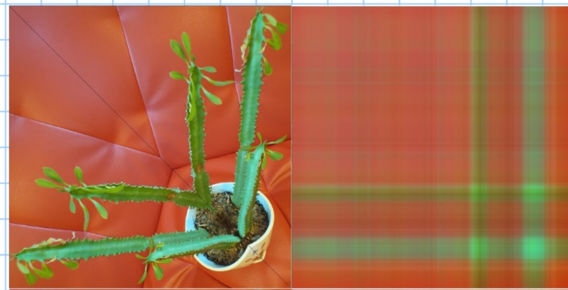
83강 특이값 분해 (Singular Value Decomposition)

* EVD → 대칭행렬



* $A = PDP^T$ (P는 직교행렬, D는 대각행렬)
  → 대칭행렬

  A → 대칭행렬 x, 정사각행렬 O

  $A = UDV^T$ (U, V는 직교행렬)

* SVD

  → 실수행렬

  $n \times n$ 행렬 A (꼭 대칭행렬일 필요는 없다.)

  $\mathrm{rank}(A) = k$ $\quad A = U\Sigma V^T$

  (단, $\Sigma = \begin{bmatrix} \sigma_1 & & & & & \\ & \sigma_2 & & & O & \\ & & \ddots & & & \\ & & & \sigma_k & & \\ & O & & & 0 & \\ & & & & & \ddots \; 0 \end{bmatrix}_{n \times n}$)

  (U, V는 직교행렬)

pf) A^TA : 대칭행렬 ($\because (A^TA)^T = A^TA$)

$\Rightarrow A^TA = VDV^T$ 직교대각화를 했다고 하자.

$D = \begin{bmatrix} \lambda_1 & & O \\ & \lambda_2 & \\ O & \ddots & \lambda_n \end{bmatrix}$ $\quad V = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$

기약행사다리꼴

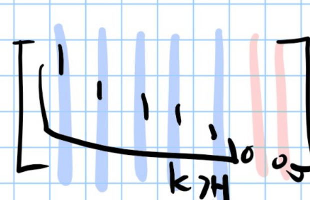
$\forall x \in \mathbb{R}^n$, $\|Ax\|^2 = Ax \cdot Ax = x \cdot A^TAx = x \cdot \lambda_i x = \lambda_i (x \cdot x)$

$= \lambda_i \|x\|^2 \Rightarrow \lambda_i \geq 0$

* $\mathrm{rank}(A) = \mathrm{rank}(A^TA)$ → 64강 $\mathrm{null}(A^TA) = \mathrm{null}(A)$

  $= \mathrm{rank}(D)$ → 76강

  $= k$



$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$

$\lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_n = 0$

$\{Av_1, Av_2, \cdots, Av_n\}$ $\quad v_1 \sim v_n \quad A^TA = VDV^T$

↳ 직교

$V = [v_1 \cdots v_n]$, $\lambda_1 \cdots \lambda_n$

$$Av_i \cdot Av_j = v_i \cdot \underbrace{A^TAv_j}$$
$$= v_i \cdot \lambda_j v_j = \lambda_j (v_i \cdot v_j) = 0$$

$$\|Av_i\|^2 = Av_i \cdot Av_i = v_i \cdot A^TAv_i = \underline{v_i} \cdot \lambda_i \underline{v_i} = \lambda_i (v_i \cdot v_i)$$
$$= \lambda_i \|v_i\|^2 = \lambda_i$$

$\|Av_i\| = \sqrt{\lambda_i} = 0 \qquad i \geq k+1$

$$\left\{\frac{1}{\sqrt{\lambda_1}}Av_1, \frac{1}{\sqrt{\lambda_2}}Av_2, \cdots, \frac{1}{\sqrt{\lambda_k}}Av_k\right\}$$

$\{u_1, u_2, \cdots, u_k\} \cdots \quad u_i = \dfrac{1}{\sqrt{\lambda_i}}Av_i$

$\mathbb{R}^n$의 정규직교기저

$\underbrace{\{u_1, \cdots, u_n\}}$ → 7강 → 정규직교기저

$U = [u_1 | \cdots | u_n]_{1\times n}$ $\qquad \Sigma = \begin{bmatrix} \sqrt{\lambda_1} & & & & & \\ & \sqrt{\lambda_2} & & & 0 & \\ & & \ddots & & & \\ & 0 & & \sqrt{\lambda_k} & & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix}_{n\times n}$ $\qquad \sqrt{\lambda} \Rightarrow$ 특이값

$\frac{1}{\sqrt{\lambda_1}}Av_1$

$$U\Sigma = [\sqrt{\lambda_1}u_1 \quad \sqrt{\lambda_2}u_2 \quad \cdots \quad \sqrt{\lambda_k}u_k \quad 0 \quad 0 \cdots 0]$$
$$= [Av_1 \quad Av_2 \quad \cdots \quad Av_k \quad \underline{Av_{k+1} \cdots Av_n}]$$
$$= AV \qquad V^{-1} = V^T$$

$$U\Sigma = AV \quad \Rightarrow \quad A = U\Sigma V^T$$

↳ 직교 ↳ 직교