74강 대수적 중복도와 기하적 중복도 (2)

(Algebraic multiplicity and geometric multiplicity (2))

Thm A : $m \times m$ 행렬.
고유값 λ의 대수적 중복도를 a
기하적 중복도를 g 라고 하면
항상 $a \geq g$ 이다.

pf) $g = \dim(S)$ $\quad S = \{ x \in \mathbb{R}^m \mid Ax = \lambda x \}$ $\quad (A-\lambda I)x = 0$

$B = \{ x_1, x_2, \cdots, x_g \}$ $\quad g \leq m$

$S \subseteq \mathbb{R}^m$
↳ $E = \{ x_1, x_2, \cdots, x_m \}$

$T = [\, x_1\ x_2 \cdots x_g \mid y_1 \cdots y_{m-g} \,] = [\, X \,]$

$AT = A[\, x_1\ x_2 \cdots x_g \mid y_1 \cdots y_{m-g} \,]_{1\times m}$ $\quad AT : (m\times m)\cdot(m\times m) = (m\times m)$

$= [\, Ax_1\ Ax_2 \cdots Ax_g \mid Ay_1\ Ay_2 \cdots Ay_{m-g} \,]$

$= [\, \lambda x_1\ \lambda x_2 \cdots \lambda x_m \mid Ay_1\ Ay_2 \cdots Ay_{m-g} \,]$

$= [\, \lambda x_m \,]$ ①×② $= [\, X \,]_{m\times g} [\, \lambda I \mid Q \,]_{1\times 2}$ $\quad$ ↳ $m\times m$

$\begin{array}{c|c} g\times g & g\times(m-g) \\ \hline (m-g)\times g & (m-g)\times(m-g) \end{array}$ $\quad m\times m$

$[\, x_1 \cdots x_g \,] = X$, $[\, y_1 \cdots y_{m-g} \,] = Y$

$XP + YR = \lambda X$, $XQ + YS = AY$

$[\, x_1\ x_2 \cdots x_g \,][\, ? \,] + [\, y_1\ y_2 \cdots y_{m-g} \,] O$

↳ λI $\quad X \cdot \lambda I = \lambda X$

$AT = T \cdot \lambda I$
$= \lambda X$ $\quad A = T \cdot \lambda I \cdot T^{-1}$

$[\, x_1 x_2 \cdots x_g \,] Q + [\, y_1 \cdots y_{m-g} \,] \cdot S = A [\, y_1 \cdots y_{m-g} \,]$

↳ $[\, Q_1 Q_2 \cdots Q_{m-g} \,]$ $\quad$ ↳ $[\, S_1 S_2 \cdots S_{m-g} \,]$

$f(x) = Ax$ $\quad m\times m$
$\mathbb{R}^m \to \mathbb{R}^m$

$XQ_1 + YS_1 = Ay_1 = f(y_1)$
$XQ_2 + YS_2 = Ay_2 = f(y_2)$
$\vdots$
$XQ_{m-g} + YS_{m-g} = Ay_{m-g} = f(y_{m-g})$

$$AT = T \begin{bmatrix} \lambda I & Q \\ O & S \end{bmatrix}$$

$$= [\, X\ Y \,] \begin{bmatrix} \lambda I & Q \\ O & S \end{bmatrix}$$

$$A = T \begin{bmatrix} \lambda I & Q \\ O & S \end{bmatrix} T^{-1}$$

① det
② tr
③ 고유값 → 특성방정식

$$\det(cI - A) = \det\left(cI - \begin{bmatrix} \lambda I & Q \\ O & S \end{bmatrix} \right) = \det(cI - \lambda I)$$

cI : $m\times m$, $\begin{bmatrix} \lambda I & Q \\ O & S \end{bmatrix}$: $g\times g$

$$= \det\left(\begin{bmatrix} (c-\lambda)I & -Q \\ O & cI - S \end{bmatrix} \right) = (c-\lambda)^m$$

↳ $g\times g$ $\quad$ ↳ $(m-g)\times(m-g)$

$= \det((c-\lambda)I)\det(cI - S)$

$= (c-\lambda)^g \det(cI - S)$

$a \geq g$.