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SA 00 (2025) 1–3

Wall functions and Reynolds stress models

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Abstract

The Reynolds stress models of Launder, Reece and Rodi and of Speziale, Sarkar and Gatski are analysed in the context of proportional wall functions for the individual Reynolds stresses. The analysis shows an inconsistency between the models and the wall functions. i.e. Both cannot be true.

1. Introduction

An equation for the transport of the Reynolds stresses may be derived from the Navier-Stokes equations as

$$\frac{\partial \overline{u_i u_j}}{\partial t} + \frac{\partial U_k \overline{u_i u_j}}{\partial x_k} = -\frac{\partial \overline{u_i u_j u_k}}{\partial x_k} - \overline{u_j u_k}\frac{\partial U_i}{\partial x_k} - \overline{u_i u_k}\frac{\partial U_j}{\partial x_k} - \frac{1}{\rho}\overline{u_j \frac{\partial p}{\partial x_i}} - \frac{1}{\rho}\overline{u_i \frac{\partial p}{\partial x_j}} + \nu \frac{\partial^2 \overline{u_i u_j}}{\partial x_k x_k} - 2\nu \overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}} \tag{1}$$

which, for convenience, is rewritten as

$$\frac{\partial \overline{u_i u_j}}{\partial t} + \frac{\partial U_k \overline{u_i u_j}}{\partial x_k} = P_{ij} + \frac{\partial}{\partial x_k}\left(\nu \frac{\partial \overline{u_i u_j}}{\partial x_k} + D_{ijk}\right) - E_{ij} \tag{2}$$

where the pressure terms have been absorbed into the terms P_{ij} (production from shear stress), D_{ijk} (turbulent diffussion) and E_{ij} (dissipation).

The model of Launder, Reece and Rodi [1] uses the model of Rotta [2] with the constant $C_1 = 3.6$

$$\frac{E_{ij}}{\epsilon} = \frac{2}{3}\delta_{ij} + C_1 b_{ij} \tag{3}$$

where the anisotropy tensor is defined in terms of the Reynolds stresses and their contraction the turbulent kinetic energy $k = \overline{u_i u_i}/2$

$$2kb_{ij} = \overline{u_i u_j} - \frac{1}{3}2k\delta_{ij} \tag{4}$$

Apparently, this is insufficient to model the decay of anisotropic turbulence and was extended by Speziale, Sarkar and Gatski [3] to

$$\frac{E_{ij}}{\epsilon} = \frac{2}{3}\delta_{ij} + C_1 b_{ij} + C_2\left(b_{ik}b_{kj} - \frac{1}{3}b_{lk}b_{kl}\delta_{ij}\right) \tag{5}$$

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where $C_1 = 3.4$ and $C_2 = -4.2$. Note that the term that C_2 multiplies is orthogonal to the isotropy tensor but not the anisotropy. If the three terms were orthogonal then they would form a basis for the Reynolds stresses.

In the log layer it is assumed by some pieces of software that the Reynolds stresses are in constant ratios to each other. In particular for the flow $U = (U_1(x_2), 0, 0)$ the wall functions are given by

$$\overline{u_i u_j} = \begin{pmatrix} 1.098 & -0.255 & 0 \\ -0.255 & 0.247 & 0 \\ 0 & 0 & 0.655 \end{pmatrix} k \tag{6}$$

2. Calculations

Assuming that the Reynolds stresses aren't constant, which is true for channel flow, then the viscous diffusion term can be replaced by

$$\nu \frac{\partial^2 \overline{u_i u_j}}{\partial x_k x_k} = \nu \frac{\overline{u_i u_j}}{k} \frac{\partial^2 k}{\partial x_k x_k} \tag{7}$$

This allows the equation for the Reynolds stresses to be contracted by members of basis to form multiples of the equation for the turbulent kinetic energy. To facilitate this the dissipation terms have been expanded in terms of the basis. i.e. The second order term is square of the anisotropy tensor made orthogonal to both the isotropy and anisotropy tensors.

$$\frac{E_{ij}}{\epsilon} = \frac{2}{3}\delta_{ij} + C_1 b_{ij} + C_2 B_{ij} \tag{8}$$

Hence contracting with the isotropy tensor

$$\nu \delta_{ij} \frac{\overline{u_i u_j}}{k} \frac{\partial^2 k}{\partial x_k x_k} = 2\nu \frac{\partial^2 k}{\partial x_k x_k},\ E_{ij}\delta_{ij} = 2\epsilon \tag{9}$$

Contracting with the anisotropy tensor

$$b_{ij} b_{ij} = 0.123,\ b_{ij} \frac{\overline{u_i u_j}}{k} = 0.246 \Rightarrow C_1 = 2 \tag{10}$$

Finally contracting with B_{ij} results in $C_2 = 0$. i.e. The model of Rotta but with the constant corresponding to the first order balance at the wall.

$$\frac{E_{ij}}{\epsilon} = \frac{2}{3}\delta_{ij} + 2b_{ij} = \frac{\overline{u_i u_j}}{k} \tag{11}$$

3. Discussion

There are a number of possibilities for the inconsistency between the calculated values of C_1 and C_2 and the LRR and SSG models.

Firstly, that the calculations give true model constants and thus the LRR and SSG models are wrong. However, this gives the model of Rotta which is known not to reproduce the decay of anisotropic turbulence.

Secondly, that C_1 and C_2 are functions of the invariants of the anisotropy tensor that take the calculated values in the log layer. i.e. They are not constants as assumed by the LRR and SSG models.

Thirdly, that the assumption of proportionality is wrong and thus the calculations are invalid. i.e. The wall functions are not proportional. Note that other wall functions are available. In fact, although the log layer is due to a matched asymptotic expansion, if the resulting formulae gave a perfect balance to the equations for the velocity, Reynolds stresses and dissipation then there would be the question of how the solution would diverge at the ends to meet the wall and main flow conditions.

References

[1] B. E. Launder, G. J. Reece, W. Rodi, Progress in the development of a reynolds-stress turbulence closure, Journal of Fluid Mechanics 68 (3) (1975) 537–566. doi:10.1017/S0022112075001814.

[2] J. Rotta, Statistische theorie nichthomogener turbulenz, Zeitschrift für Physik 129 (1951) 547–572. doi:https://doi.org/10.1007/BF01330059.

[3] C. G. Speziale, S. Sarkar, T. B. Gatski, Modelling the pressure–strain correlation of turbulence: an invariant dynamical systems approach, Journal of Fluid Mechanics 227 (1991) 245–272. doi:10.1017/S0022112091000101.