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The affine growth of wealth

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Abstract

Economists fit both the log-normal and Pareto distributions to wealth depending on whether they are on the right tail or not. For Bernoulli trials it has been recently shown affine returns may be approximated by an appropriately scaled logit-normal distribution. A good fit for some of the wealth statistics is performed by this latter distribution whose parameters are derived using a mixture of Maximum Likelihood Estimation and a grid search.

1. Introduction

Economists believe that Plebs, such as the author, labour under Gibrat's Law whilst being looked down on by fat-tailed Patricians who presumably think it appropriate that they are instead ruled by the power law of Pareto. This class divide has seemingly not been challenged by Marxist economists who presumably prefer a uniform distribution except, of course, for themselves.[1] A more equal world would exist if the wealth of the populace was subject to a single distribution as it would indicate that there was a single stochastic process for all saving the logicians a headache.

By way of example, if there were some upper scale applicable then the logit-normal distribution may be appropriate. The logit-normal distribution is much overlooked but important, for example it moderates the growth of a well-mixed epidemic from exponential to logistic as shown by the author [1].

The author [2] also showed, for stock prices, that affine returns on coin tosses can be approximated by a stretched-out logit-normal distribution for the case when the growth converges to a finite support. The same shapes were also evident in the case where the support grows exponentially with t when $\beta > 1$ in equation 1.

$$\left(0, \delta \frac{1-\beta^t}{1-\beta}\right) \tag{1}$$

The process itself being described by the random variable S_t growing from $S_0 = 0$ with probability p

$$S_{t+1} = \beta S_t + \delta,\ 0 < \beta \neq 1,\ \delta > 0 \tag{2}$$

and shrinking with probability $q = 1 - p$

$$S_{t+1} = \alpha S_t,\ 0 < \alpha < 1 \tag{3}$$

When $\beta < 1$ the probability distribution converges. Applying the equations to the distribution leads to shrinks and shifts which, at the tails, means only one copy is applied. The resulting equations then show that both tails are power

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[1] Marx got one of his servants pregnant and refused to acknowledge the resulting progeny. Do as I say, not as I do.

laws. i.e. The logit-normal distribution may be appropriate for the right tail. In fact, it should be expected that only the one distribution should cover all the natural cases rather than the mish-mash of the existing approach. Whether the model is appropriate though hinges on whether it fits the data.

2. Parameter estimation

The Maximum Likelihood Estimators for the log-normal and logit-normal distributions are the same as for the normal distribution fitted to the values transformed by the appropriate function, so the fitting is done in a highly distorted space. The probability density function for the scaled logit-normal is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \frac{L}{x(L-x)} \exp\left(-\frac{(\log(x) - \log(L-x) - \mu)^2}{2\sigma^2}\right) \tag{4}$$

Taking logarithms

$$-\log(\sigma) - \frac{1}{2}\log(2\pi) + \log(L) - \log(x) - \log(L-x) - \frac{(\log(x) - \log(L-x) - \mu)^2}{2\sigma^2} \tag{5}$$

Differentiating with respect to μ, σ, L results in, respectively

$$\frac{\log(x) - \log(L-x) - \mu}{\sigma^2} \tag{6}$$

$$-\frac{1}{\sigma} + \frac{(\log(x) - \log(L-x) - \mu)^2}{\sigma^3} \tag{7}$$

$$\frac{1}{L} - \frac{1}{L-x} + \frac{\log(x) - \log(L-x) - \mu}{\sigma^2} \frac{1}{L-x} \tag{8}$$

Resulting in the usual, if L were equal to one, estimates of

$$\mu\Sigma_i 1 = \Sigma_i (\log(x_i) - \log(L - x_i)) \tag{9}$$

$$\sigma^2\Sigma_i 1 = \Sigma_i (\log(x_i) - \log(L - x_i) - \mu)^2 \tag{10}$$

The final derivative was not summed over the data and used to estimate L but was used to estimate the error. The mean square error between the formula and the histogram was also calculated. Finally, the result was checked to see whether it could be improved upon by searching through a grid of parameters.

3. Data analysis

3.1. Data for US household income

Mean household incomes for US neighbourhoods in 2017 were downloaded from Kaggle [3].

The Maximum Likelihood Estimation was performed with L from \$236,911 to \$50,000,000 with the mean square error decreasing to a minimum at \$5,406,911. Subsequently a matrix of tests was performed with a minimum at $L = \$4,810,000\,\mu = -4.34$, $\sigma^2 = 0.19$ is shown in figure 1.

4. Discussion

That the Maximum Likelihood Estimation wasn't sufficient to determine the parameters may be due to the support being semi-infinite. i.e. The length scale tends to infinity. A rough grid search, although ugly and a misuse of cheap computing power, seemed to be sufficient.

The fit for the household income was very good. Obviously more data needs to be fitted and comparison needs to be made with the other distributions. However, using a scaled logit-normal distribution does appear to be a promising option.

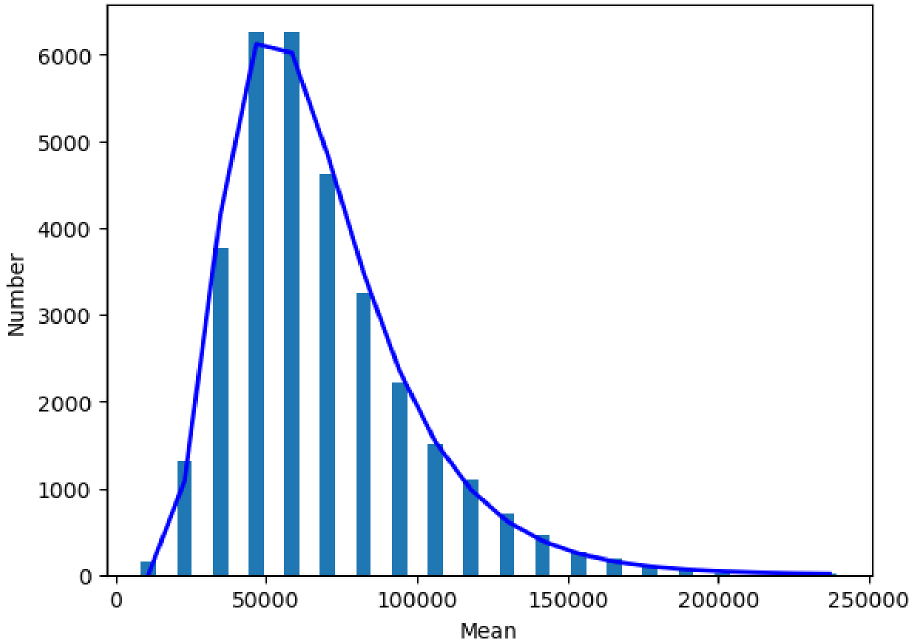

Figure 1. Fit of mean household incomes

References

[1] Z. Mason, Naturally extending the standard sir model to stochastic growth, TBD (2023). doi:10.31219/osf.io/y6ckv.
[2] Z. Mason, Affine returns on bernoulli trials in finance, TBD (2025).
[3] Golden Oak Research Group LLC, U.S. income database kaggle, `https://www.kaggle.com/datasets/goldenoakresearch/us-household-income-stat` (2017).