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A discrete stochastic model of turbulence

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Abstract

Direct Numerical Simulation of the decay of isotropic turbulence was performed using a discrete stochastic model adapted from Molecular Dynamics. Although resulting in the expected power law the resulting exponent was extreme.

1. Introduction

Fluid flow is usually modelled by the Navier-Stokes equations which, for an incompressible fluid, are given by the equation for velocity u_i and pressure p over position x_i and time t as

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j x_j} \tag{1}$$

with an equation for continuity of

$$\frac{\partial u_j}{\partial x_j} = 0 \tag{2}$$

For the decay of turbulence equation 1 can be multiplied by the velocity and averaged and the spatial derivatives of the averaged terms discarded to produce an equation for the Reynolds stresses $\overline{u_i u_j}$ of

$$\frac{\partial \overline{u_i u_j}}{\partial t} = -\frac{1}{\rho}\overline{u_j \frac{\partial p}{\partial x_i}} - \frac{1}{\rho}\overline{u_i \frac{\partial p}{\partial x_j}} - \epsilon_{ij},\ \epsilon_{ij} = 2\nu \overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_j}{\partial x_k}} \tag{3}$$

Defining twice the turbulent kinetic energy as $2k = \overline{u_i u_i}$ and contracting equation 3 results in the pressure terms evaluating to zero by use of equation 2

$$\frac{\partial k}{\partial t} = -\epsilon,\ \epsilon = \nu \overline{\frac{\partial u_i}{\partial x_k}\frac{\partial u_i}{\partial x_k}} \tag{4}$$

To solve the equations then requires the set of equations to be closed. It is assumed that the dissipation can be modelled as

$$\frac{\partial \epsilon}{\partial t} = -C_\epsilon \frac{\epsilon^2}{k} \tag{5}$$

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Making the usual assumption that the decay follows a power law then

$$k = At^{-n} \Rightarrow \epsilon = nAt^{-n-1} \Rightarrow C_\epsilon = \frac{n+1}{n} \tag{6}$$

For the standard model C_ϵ has been tuned to $1.92\,(\Rightarrow n = 1.09)$. Baumert [1] derived a value of $n = 1$ leading to $C_\epsilon = 2$. Other values exist due to experiment, theory and DNS. For Saffman turbulence the Loitsiansky integral is conserved and

$$n = \frac{6}{5} \Rightarrow C_\epsilon = \frac{11}{6} \approx 1.83 \tag{7}$$

For Batchelor turbulence the Birkhoff integral is conserved and

$$n = \frac{10}{7} \Rightarrow C_\epsilon = \frac{17}{10} = 1.7 \tag{8}$$

2. Discrete stochastic simulation

In Molecular Dynamics gases are modelled by tracking their position and velocity which requires significant computing power. Here, instead, only the velocity is tracked and a stochastic model is used for the collisions.

An oblique inelastic collision between particles of equal mass is modelled as transferring an impulse

$$\vec{i} = ((1-\alpha)\,(\vec{v}_B - \vec{v}_A)\cdot\vec{n})\,\vec{n} \tag{9}$$

between the velocity vectors for particles A, $\vec{v}_A$, and B, $\vec{v}_B$. These particles collide such that the unit vector between their centres is $\vec{n}$ with the inelasticity controlled by the parameter α.

$$\vec{v}_A \to \vec{v}_A + \vec{i},\ \vec{v}_B \to \vec{v}_B - \vec{i} \tag{10}$$

Hence momentum is conserved

$$\vec{v}_A + \vec{v}_B \to \vec{v}_A + \vec{v}_B \tag{11}$$

but kinetic energy isn't

$$\vec{v}_A\cdot\vec{v}_A + \vec{v}_B\cdot\vec{v}_B \to \vec{v}_A\cdot\vec{v}_A + \vec{v}_B\cdot\vec{v}_B - 2\alpha\,(1-\alpha)\left((\vec{v}_B - \vec{v}_A)\cdot\vec{n}\right)^2 \tag{12}$$

For this particular model the unit vector, $\vec{n}$, is random and uniformly distributed. To determine which particles collide a list of their indices is formed which is then shuffled and traversed pairwise. The relative speed of a pair is then calculated and, if less than some newly calculated random number, the pair are deemed not to collide.

A number of simulations were undertaken with $\alpha \in [0.05, 0.5)$. The population was of size 6000 whose velocities were initially of magnitude one and axis aligned such that the average was zero. The random speed of collision was uniformly distributed in $[0, 0.1]$.

After an initial exponential decay, a power law decay occurs as shown in figure 1. The resulting slope has $n = 2.1$ leading to $C_\epsilon = 1.48$, the most extreme values yet mentioned.

3. Discussion

Given the enormous computing power required to perform Direct Numerical Simulation of turbulence at high Reynolds numbers using the Navier-Stokes equations an attempt was made based on the similarly expensive Molecular Dynamics but simplified by a discrete stochastic collision model. Although the power law decay of isotropic turbulence was reproduced the exponent was seen to be an extreme value. Perhaps with more work the method could become successful however it may be the case that the power law was merely spurious.

References

[1] H. Z. Baumert, Universal equations and constants of turbulent motion, Physica Scripta 2013 (T155) (2013) 014001. doi:10.1088/0031-8949/2013/T155/014001.
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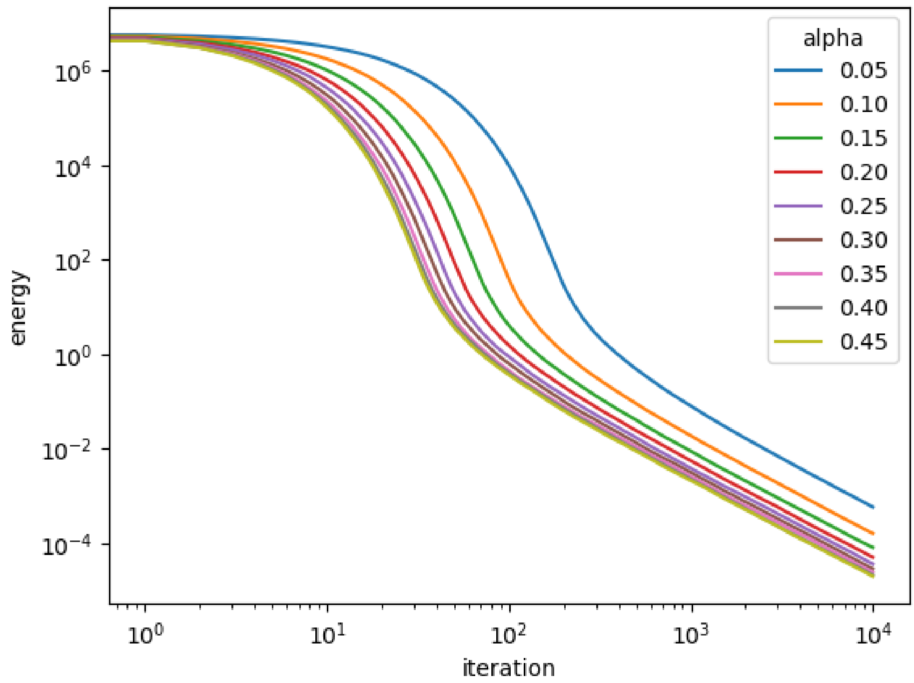

Figure 1. Decay of isotropic turbulence