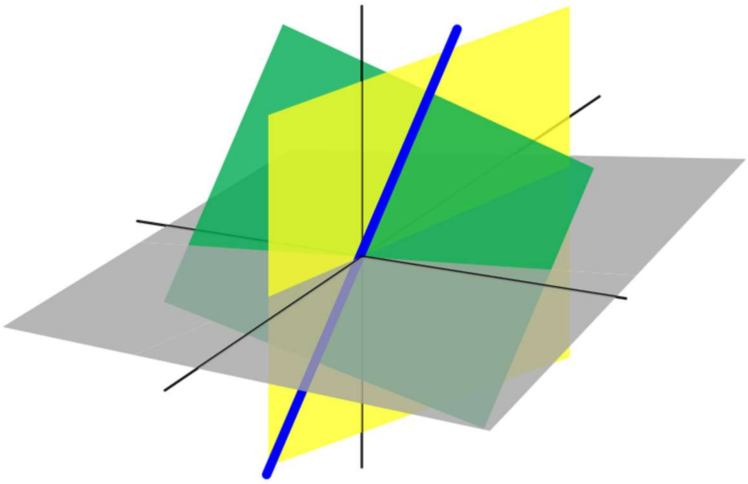
Atma Ram Sanatan Dharma College
University of Delhi



Linear Algebra

Assignment for Paper Code 32355202

Submitted By

Sudipto Ghosh

College Roll No. 19/78003

BSc (Hons) Computer Science

Submitted To

Mr Anant Tiwari

Department of Mathematics

Sudipto Ghosh (19/78003)

ASSIGNMENT

Q1. consider the linear transformation $L: \mathbb{R}^3 \to \mathbb{R}^4$ given by $L(v) = Av$ where

$$A = \begin{bmatrix} 4 & -2 & 8 \\ 7 & 1 & 5 \\ -2 & -1 & 0 \\ 3 & -2 & 7 \end{bmatrix}$$

(i) find Ker(L) & range (L)
(ii) verify dimension theorem.

Ans Consider B = reduced row echelon form of A, ie row reducing A,

$$\Rightarrow \begin{bmatrix} 4 & -2 & 8 \\ 7 & 1 & 5 \\ -2 & -1 & 0 \\ 3 & -2 & 7 \end{bmatrix} \quad R_1 \to R_1/4$$

$$\Rightarrow \begin{bmatrix} 1 & -1/2 & 2 \\ 7 & 1 & 5 \\ -2 & -1 & 0 \\ 3 & -2 & 7 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 7R_1 \\ R_3 \to R_3 + 2R_1 \\ R_4 \to R_4 - 3R_1 \end{array}$$

$$\Rightarrow \begin{bmatrix} 1 & -1/2 & 2 \\ 0 & 9/2 & -9 \\ 0 & -2 & 4 \\ 0 & -1/2 & 1 \end{bmatrix} \quad R_2 \to \frac{2}{9} R_2$$

$$\Rightarrow \begin{bmatrix} 1 & -1/2 & 2 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \\ 0 & -1/2 & 1 \end{bmatrix} \quad \begin{array}{l} R_3 \to R_3 + 2R_2 \\ R_4 \to R_4 + \frac{1}{2} R_2 \end{array}$$

$$\Rightarrow \begin{bmatrix} 1 & -1/2 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_1 \to R_1 + \frac{1}{2} R_2$$

$$\Rightarrow \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = B$$

From B, we see that x_3 is independent, let $x_3 = c \in \mathbb{R}$ we get the following for $BX = 0$,

$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_1 + x_3 = 0 \\ x_2 - 2x_3 = 0 \end{cases}$$

Now as $x_3 = c$, $c \in \mathbb{R}$

$x_1 = -c$

$x_2 = +2c$

i.e. $X = c \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ for some $c \in \mathbb{R}$

$$\therefore \ker(L) = \left\{ c \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\} (c \in \mathbb{R})$$

$$\Rightarrow \ker(L) = \operatorname{span} \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\} \quad -(1)$$

Now, range(L) is spanned by the columns of A having non zero pivot entries in B and the first 2 columns of B qualify,

$$\therefore \text{range}(L) = \operatorname{span} \left\{ \begin{bmatrix} 4 \\ 7 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \\ -2 \end{bmatrix} \right\} \quad -(2)$$

Now according to the dimension theorem,

$\dim(\ker(L)) + \dim(\text{range}(L)) = \dim(V)$
for a linear transformation $L : V \to W$

∘ RHS Here V is $\mathbb{R}^3$. $(\because \dim(\mathbb{R}^n) = n)$

$\Rightarrow \dim(\mathbb{R}^3) = \underline{\underline{3}}$

∘ LHS $\dim(\ker(L)) = 1$

$+ \dim(\text{range}(L)) = 2$

$\dim(\ker(L)) + \dim(\text{range}(L)) = \underline{\underline{3}}$

As LHS = RHS, dimension theorem is verified! $-(3)$

Q2 consider $L: \mathbb{P}_2 \to \mathbb{R}^2$ given by $L(p) = [p(1), p'(1)]$ for $p \in \mathbb{P}_2$ verify dimension theorem.

Ans. for $L: \mathbb{P}_2 \to \mathbb{R}^2$, consider some p in $\mathbb{P}_2$ as

$$p = ax^2 + bx + c$$

Now, $L(p) = \begin{bmatrix} a+b+c \\ 2a+b \end{bmatrix}$.

consider the standard bases for $\mathbb{P}_2$ and $\mathbb{R}^2$ as B and C.

where

$$B = (x^2, x, 1) \;;\; C = (e_1, e_2)$$

Observing effect of L on standard basis B.

$$\to L(x^2) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \to L(x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \to L(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Therefore, transformation matrix A_{BC} is given by

$$A_{BC} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

Rowreducing A_{BC} and let $\beta = \text{rref}(A_{BC})$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \quad R_2 \to R_2 - 2R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix} \quad R_2 \to -R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \quad R_1 \to R_1 - R_2$$

$$\Rightarrow \begin{array}{c} \begin{matrix} x^2 & x & 1 \end{matrix} \\ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \end{array} = \beta$$

In β, the third column is independent consider constant $\alpha \in \mathbb{R}$

For system $\beta X = 0$,

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x^2 \\ x \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We get, $\begin{cases} x^2 - 1 = 0 \\ x + 2 = 0 \end{cases}$

Now as constant $= \alpha$, $\alpha \in \mathbb{R}$

$$x^2 = \alpha$$
$$x = -2\alpha$$

i.e. $X = \left\{ \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \middle| \ \alpha \in \mathbb{R} \right\}$

$\therefore$ $\ker(L) = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\} \Rightarrow \underline{\dim(\ker(L)) = 1}$

From β, x^2 and x columns have pivot entries,

$\therefore$ $\text{range}(L) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \Rightarrow \underline{\dim(\text{range}(L)) = 2}$

Now, according to dimension theorem,

$\dim(\ker(L)) + \dim(\text{range}(L)) = \dim(V)$
for linear transformation $L : V \to W$

- RHS Here V is $\mathbb{P}_2$
  $\Rightarrow \dim(\mathbb{P}_2) = \underline{\underline{3}}$ $\quad (\because \dim(\mathbb{P}_n) = n+1)$

- LHS $\dim(\ker(L)) = 1$
  $+ \ \dim(\text{range}(L)) = 2$
  $\dim(\ker(L)) + \dim(\text{range}(L)) = \underline{\underline{3}}$

As LHS = RHS, the dimension theorem is verified! — 3

Q3. Consider $L : M_{22} \to M_{32}$ given by $L\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = \begin{bmatrix} 0 & -a_{12} \\ -a_{21} & 0 \\ 0 & 0 \end{bmatrix}$.
Find $\ker(L)$ and $\text{range}(L)$.

Ans. Consider the standard basis of $M_{22} = B$, and for $M_{32} = C$

$$B = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

Observing effect of L on elements of B,

$\rightarrow L\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad \rightarrow L\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\rightarrow L\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad \rightarrow L\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Therefore, transition matrix A_{BC} is given by,

$$A_{BC} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Row reducing A_{BC} and let $\beta = \text{rref}(A_{BC})$,

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_2 \to -R_2 \\ R_3 \to -R_3 \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \beta$$

In β, first and fourth columns are independent, let $\underbrace{a_{11}}_{=a}, \underbrace{a_{22}}_{=b} \in \mathbb{R}$ for $\beta X = 0$,

$$\beta \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

we get, $a_{12} = 0$, $a_{21} = 0$

ie. $X = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$

$\therefore \ker(L) = \left\{ a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \,\middle|\, a, b \in \mathbb{R} \right\}$

$\Rightarrow \ker(L) = \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ — (1)

from β, second and third columns have pivot entries,

$\therefore \text{range}(L) = \text{span}\left\{ \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$ — (2)

Q4. Verify that $\mathbb{P}_3 \cong \mathbb{R}^4$ as $L : \mathbb{P}_3 \to \mathbb{R}^4$ given by $L(p) = [p(-1), p(0), p(1), p(2)]$ for $p \in \mathbb{P}_3$

Ans. In order to show $\mathbb{P}_3 \cong \mathbb{R}^4$, we must show that L is an isomorphism from $\mathbb{P}_3$ to $\mathbb{R}^4$.

To show L is an isomorphism, we must show that:

→ L is a linear transformation

→ L is one-one.

→ L is onto.

(I) <u>To show: L is a linear transformation</u>

Suppose L is a linear transformation, then the following properties must hold:

(1) $L(p_1 + p_2) = L(p_1) + L(p_2)$ for $p_1, p_2 \in \mathbb{P}_3$

(2) $L(\alpha p) = \alpha L(p)$ for some $\alpha \in \mathbb{R}$, $p \in \mathbb{P}_3$

(II) <u>To show: L is one-one</u>

As L is one-one iff $L(p_1) = L(p_2)$ implies $p_1 = p_2$ for $p_1, p_2 \in \mathbb{P}_3$ we must show the same.

(III) <u>To show: L is onto.</u>

L is onto iff $\dim(\text{range}(L)) = \dim(W)$ for $L : V \to W$, Consider $L: \mathbb{P}_3 \to \mathbb{R}^4$ given by $L(p) = [p(-1), p(0), p(1), p(2)]$ for $p \in \mathbb{P}_3$ & consider some p in $\mathbb{P}_3$ where $p = ax^3 + bx^2 + c + d$,

then $L(p) = \begin{bmatrix} -a+b-c+d \\ d \\ a+b+c+d \\ 8a+4b+2c+d \end{bmatrix}$

Let $B = (x^3, x^2, x, 1)$ and $C = (e_1, e_2, e_3, e_4)$ be the standard bases for $\mathbb{P}_3$ and $\mathbb{R}^4$ respectively,

Now observing effect of L on each element of B,

$\Rightarrow L(x^3) = [-1, 0, 1, 8]$ $\quad \to L(x^2) = [1, 0, 1, 4]$

$\Rightarrow L(x) = [-1, 0, 1, 2]$ $\quad \to L(1) = [1, 1, 1, 1]$

Therefore, transformation matrix A_{BC} is given by,

$$A_{BC} = \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \end{bmatrix}$$

Also $|A_{BC}| = 12 \neq 0$,
$\Rightarrow A_{BC}$ is invertible.
$\Rightarrow A_{BC}^{-1}$ exists.
$\Rightarrow L^{-1}$ is possible.

Now, (I) Proving L is a linear transformation.

If L is a linear transformation,

(1) $L(p_1 + p_2) = L(p_1) + L(p_2)$

Consider $\left.\begin{matrix} p_1 = a_1x^3 + b_1x^2 + c_1x + d_1 \\ p_2 = a_2x^3 + b_2x^2 + c_2x + d_2 \end{matrix}\right\} \in \mathbb{P}_3$.

LHS $L(p_1 + p_2)$

$\Rightarrow L((a_1+a_2)x^3 + (b_1+b_2)x^2 + (c_1+c_2)x + (d_1+d_2))$

$$\Rightarrow \begin{bmatrix} -(a_1+a_2) + (b_1+b_2) - (c_1+c_2) + (d_1+d_2) \\ (d_1+d_2) \\ (a_1+a_2) + (b_1+b_2) + (c_1+c_2) + (d_1+d_2) \\ 8(a_1+a_2) + 4(b_1+b_2) + 2(c_1+c_2) + (d_1+d_2) \end{bmatrix}$$

RHS $L(p_1) + L(p_2)$

$\Rightarrow L(a_1x^3 + b_1x^2 + c_1x + d_1) + L(a_2x^3 + b_2x^2 + c_2x + d_2)$

$$\Rightarrow \begin{bmatrix} -a_1+b_1-c_1+d_1 \\ d_1 \\ a_1+b_1+c_1+d_1 \\ 8a_1+4b_1+2c_1+d_1 \end{bmatrix} + \begin{bmatrix} -a_2+b_2-c_2+d_2 \\ d_2 \\ a_2+b_2+c_2+d_2 \\ 8a_1+4b_2+2c_2+d_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -(a_1+a_2)+(b_1+b_2)-(c_1+c_2)+(d_1+d_2) \\ (d_1+d_2) \\ (a_1+a_2)+(b_1+b_2)+(c_1+c_2)+(d_1+d_2) \\ 8(a_1+a_2)+4(b_1+b_2)+2(c_1+c_2)+(d_1+d_2) \end{bmatrix}$$

As LHS = RHS,

$\Rightarrow L(p_1+p_2) = L(p_1)+L(p_2)$

(II) $\underline{L(\alpha p) = \alpha L(p)}$

consider $\alpha \in \mathbb{R}$, $p = ax^3+bx^2+cx+d \in P_3$

$\underline{\text{LHS}}$ $L(\alpha p) = L(\alpha a x^3 + \alpha b x^2 + \alpha c x + \alpha d)$

$$\Rightarrow \begin{bmatrix} -(\alpha a)+(\alpha b)-(\alpha c)+(\alpha d) \\ (\alpha d) \\ (\alpha a)+(\alpha b)+(\alpha c)+(\alpha d) \\ 8(\alpha a)+4(\alpha b)+2(\alpha c)+(\alpha d) \end{bmatrix}$$

$\underline{\text{RHS}}$ $\alpha(L(p)) = \alpha \cdot L(ax^3+bx^2+cx+d)$

$$\Rightarrow \alpha \cdot \begin{bmatrix} -a+b-c+d \\ d \\ a+b+c+d \\ 8a+4b+2c+d \end{bmatrix} \Rightarrow \begin{bmatrix} -(\alpha a)+(\alpha b)-(\alpha c)+(\alpha d) \\ (\alpha d) \\ (\alpha a)+(\alpha b)+(\alpha c)+(\alpha d) \\ 8(\alpha a)+4(\alpha b)+2(\alpha c)+(\alpha d) \end{bmatrix}$$

As LHS = RHS,

$\Rightarrow L(\alpha \cdot p) = \alpha \cdot L(p)$

$\therefore$ $\underline{L : P_3 \to \mathbb{R}^4 \text{ is a linear transformation!}}$

Now, (II) $\underline{\text{Proving } L \text{ is one one}}$

Now, $[L(p)]_c = A_{BC}[P]_B$ as A_{BC} exists.

Also, L is one-one iff $L(p_1) = L(p_2)$ implies $p_1 = p_2$ for $p_1, p_2 \in P_3$

Suppose L is one-one,

then $L(p_1) = L(p_2)$

$\Rightarrow A_{BC} \cdot p_1 = A_{BC} \cdot p_2$

$\Rightarrow A_{BC}^{-1} A_{BC} \cdot p_1 = A_{BC}^{-1} A_{BC}\, p_2$ $(\because |A_{BC}| \neq 0)$

$\Rightarrow p_1 = p_2$

As $L(p_1) = L(p_2) \longrightarrow p_1 = p_2$,

$\therefore$ $\underline{L \text{ is one-one!}}$

Now, (III) $\underline{\text{Proving } L \text{ is onto}}$

L is onto iff $\dim(\text{range}(L)) = \dim(\mathbb{R}^4)$

finding range(L),

Sudipto Ghosh (19/78003)

let B be the reduced row echelon form of ABC,

Row reducing ABC,

$$\Rightarrow \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \end{bmatrix} \quad R_1 \to -R_1$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 8R_1 \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 12 & -6 & 9 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 12 & -6 & 9 \end{bmatrix} \quad R_2 \to \frac{1}{2} R_2$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 12 & -6 & 9 \end{bmatrix} \quad R_4 \to R_4 - 12R_2$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -6 & -3 \end{bmatrix} \quad R_3 \leftrightarrow R_4$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -6 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_3 \to \frac{-R_3}{6}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - \frac{R_4}{2} \\ R_2 \to R_2 - R_4 \\ R_1 \to R_1 + R_4 \end{matrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_1 \to R_1 - R_3$$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad R_1 \to R_1 + R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \beta$$

In $\beta = \text{rref}(A)$, all four columns have pivot entries,

$$\Rightarrow \text{range}(L) = \text{span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Now, $\dim(\mathbb{R}^4) = 4$ $\quad (\because \dim(\mathbb{R}^n) = n)$

and $\dim(\text{range}(L)) = 4$

$\Rightarrow \dim(\text{range}(L)) = \dim(\mathbb{R}^4)$

$\therefore$ L is onto !

Hence, as L is a linear transformation, is one-one and onto, it is an isomorphism from $\mathbb{P}_3$ to $\mathbb{R}^4$.

$\therefore \mathbb{P}_3 \cong \mathbb{R}^4$ proved.

Q5. Consider the linear transformation $L: \mathbb{P}_3 \to \mathbb{R}^3$ given by $L(dx^3 + cx^2 + bx + a) = [a+b, 2c, d-a]$ where $\mathcal{B}$ and $\mathcal{C}$ are the standard bases for $\mathbb{P}_3$ and $\mathbb{R}^3$ respectively.

Find A_{BC} and A_{DE} where D and E are the ordered bases $D = (x^3+x^2, x^2+x, x+1, 1) \in \mathbb{P}_3$ and $E = \left(\begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix} \right) \in \mathbb{R}^3$.

Ans. Standard basis for P_3 and $\mathbb{R}^3$ are,

$B = (x^3, x^2, x, 1)$

$C = (e_1, e_2, e_3)$

Observing effect of L on P_3,

$\rightarrow L(x^3) = [0, 0, 1]$ $\rightarrow L(x^2) = [0, 2, 0]$

$\rightarrow L(x) = [1, 0, 0]$ $\rightarrow L(1) = [1, 0, -1]$

Therefore, transformation matrices A_{BC} is given by,

$$\therefore A_{BC} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

Now. $A_{DE} = Q_{E\leftarrow C} A_{BC} P^{-1}_{D\leftarrow B}$ where $Q_{E\leftarrow C}$ and $P_{D\leftarrow B}$ are the transition matrices from C to E and B to D.

(1) Finding $Q_{E\leftarrow C}$

Using change of basis and row reducing $[E \mid C]$,

$$\Rightarrow \left[\begin{array}{ccc|ccc} -2 & 1 & 3 & 1 & 0 & 0 \\ 1 & -3 & -6 & 0 & 1 & 0 \\ -3 & 0 & 2 & 0 & 0 & 1 \end{array}\right] R_1 \to -R_1/2$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1/2 & -3/2 & -1/2 & 0 & 0 \\ 1 & -3 & -6 & 0 & 1 & 0 \\ -3 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 + 3R_1 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1/2 & -3/2 & -1/2 & 0 & 0 \\ 0 & -5/2 & -9/2 & 1/2 & 1 & 0 \\ 0 & -3/2 & -5/2 & -3/2 & 0 & 1 \end{array}\right] R_2 \to -\frac{2}{5} R_2$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1/2 & -3/2 & -1/2 & 0 & 0 \\ 0 & 1 & 9/5 & -1/5 & -2/5 & 0 \\ 0 & -3/2 & -5/2 & -3/2 & 0 & 1 \end{array}\right] R_3 \to R_3 + \frac{3}{2} R_2$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1/2 & -3/2 & -1/2 & 0 & 0 \\ 0 & 1 & 9/5 & -1/5 & -2/5 & 0 \\ 0 & 0 & 1/5 & -9/5 & -3/5 & 1 \end{array}\right] R_3 \to R_3(5)$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1/2 & -3/2 & -1/2 & 0 & 0 \\ 0 & 1 & 9/5 & -1/5 & -2/5 & 0 \\ 0 & 0 & 1 & -9 & -3 & 5 \end{array}\right] \begin{array}{l} R_2 \to R_2 - \frac{9}{5} R_3 \\ R_1 \to R_1 + \frac{3}{2} R_3 \end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & -1/2 & 0 & -14 & -9/2 & 15/2 \\ 0 & 1 & 0 & 16 & 5 & -9 \\ 0 & 0 & 1 & -9 & -3 & 5 \end{array}\right] \quad R_1 \to R_1 + \frac{R_2}{2}$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -6 & -2 & 3 \\ 0 & 1 & 0 & 16 & 5 & -9 \\ 0 & 0 & 1 & -9 & -3 & 5 \end{array}\right] \quad (\text{in the form } [I_3 \mid \wp_{E\leftarrow C}]$$

$$\Rightarrow \wp_{E\leftarrow C} = \begin{bmatrix} -6 & -2 & 3 \\ 16 & 5 & -9 \\ -9 & -3 & 5 \end{bmatrix}$$

(2) <u>Finding</u> $\underline{P^{-1}_{D\leftarrow B} = P_{B\leftarrow D}}$,

Row reducing $[B \mid D]$,

$$\Rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right] \quad (\text{in the form } [I_4 \mid P_{B\leftarrow D}]$$

(3) <u>Computing</u> $\underline{A_{DE}}$

As $A_{DE} = \wp_{E\leftarrow C} A_{BC} P^{-1}_{D\leftarrow B}$

$$\Rightarrow A_{DE} = \begin{bmatrix} -6 & -2 & 3 \\ 16 & 5 & -9 \\ -9 & -3 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow A_{DE} = \begin{bmatrix} 3 & -4 & 6 & -9 \\ -9 & 10 & 16 & 25 \\ 5 & -6 & -9 & -14 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow A_{DE} = \begin{bmatrix} -1 & -10 & -15 & -9 \\ 1 & 26 & 41 & 25 \\ -1 & -15 & -23 & -14 \end{bmatrix}$$

Sudipto Ghosh (19/78003)

Q6. In $\mathbb{R}^4$, $W = \text{span}\left(\left\{\begin{bmatrix}2\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\2\\2\end{bmatrix}\right\}\right)$ and $v = \begin{bmatrix}-1\\3\\3\\2\end{bmatrix}$

(i) find $\text{proj}_W v = ?$

(ii) find $w_1 \in W$ and $w_2 \in W^\perp$ (as $v = w_1 + w_2$)

(iii) also find $W^\perp = ?$ (as orthogonal basis)

Ans (i) finding $\text{proj}_W v$

we need an orthonormal basis to compute $\text{proj}_W v$, we can normalize the orthogonal basis to get the orthonormal basis.

Checking if $W \subset \mathbb{R}^4$ ~~is spanned by the~~ has orthogonal basis,

$$\begin{bmatrix}2\\-1\\1\\0\end{bmatrix} \cdot \begin{bmatrix}1\\-1\\2\\2\end{bmatrix} = 5 \neq 0$$

$\Rightarrow$ given basis is not orthogonal basis.

finding orthogonal basis T

using Gram-Schmidt method,

$\rightarrow v_1 = w_1 = [2, -1, 1, 0]$

$\rightarrow v_2 = w_2 - \text{proj}_{v_1} w_2 = w_2 - \left(\frac{w_2 \cdot v_1}{\|v_1\|^2}\right) \cdot v_1$

$v_2 = [1, -1, 2, 2] - \left(\frac{5}{6}\right)[2, -1, 1, 0]$

$v_2 = \left[\frac{-4}{6}, \frac{-1}{6}, \frac{7}{6}, \frac{12}{6}\right]$

$v_2 = [-4, -1, 7, 12]$ (ignoring fractions)

Now, $T = \left\{\begin{bmatrix}2\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}-4\\-1\\7\\12\end{bmatrix}\right\}$ is an orthogonal basis for W and $W = \text{span}(T)$.

finding orthonormal basis U,

normalizing vectors in T,

Now, $U = \left\{\left[\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, 0\right], \left[\frac{-4}{\sqrt{210}}, \frac{-1}{\sqrt{210}}, \frac{7}{\sqrt{210}}, \frac{12}{\sqrt{210}}\right]\right\}$

is an orthonormal basis for W and $W = \text{span}(U)$

Now, given the orthonormal basis $U = \left\{ \left[\frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, 0\right], \left[\frac{-4}{\sqrt{210}}, \frac{-1}{\sqrt{210}}, \frac{7}{\sqrt{210}}, \frac{12}{\sqrt{210}}\right] \right\}$.

$\left\{ \text{proj}_W v = (v \cdot u_1) u_1 + (v \cdot u_2) u_2 \right\}$ and $v = [1, 3, 3, 2]$

$$\Rightarrow \text{proj}_W v = \left(\begin{bmatrix} -1 \\ 3 \\ 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ 1/\sqrt{6} \\ 0 \end{bmatrix} \right) \begin{bmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ 1/\sqrt{6} \\ 0 \end{bmatrix} + \left(\begin{bmatrix} -1 \\ 3 \\ 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -4/\sqrt{210} \\ -1/\sqrt{210} \\ 7/\sqrt{210} \\ 12/\sqrt{210} \end{bmatrix} \right) \begin{bmatrix} -4/\sqrt{210} \\ -1/\sqrt{210} \\ 7/\sqrt{210} \\ 12/\sqrt{210} \end{bmatrix}$$

$$= \left(\frac{-\sqrt{6}}{3} \right) \begin{bmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ 1/\sqrt{6} \\ 0 \end{bmatrix} + \left(\frac{46}{\sqrt{210}} \right) \begin{bmatrix} -4/\sqrt{210} \\ -1/\sqrt{210} \\ 7/\sqrt{210} \\ 12/\sqrt{210} \end{bmatrix}$$

$$= \begin{bmatrix} -2/3 \\ 1/3 \\ -1/3 \\ 0 \end{bmatrix} + \begin{bmatrix} -92/105 \\ -23/105 \\ 23/15 \\ 92/35 \end{bmatrix}$$

$$\Rightarrow \text{proj}_W v = \left[\frac{-54}{35}, \frac{4}{35}, \frac{42}{35}, \frac{92}{35} \right]$$

(ii) We know that if $W \subset \mathbb{R}^n$ and $v \in \mathbb{R}^n$, then v can be expressed as $v = w_1 + w_2$ where $w_1 = \text{proj}_W v \in W$ and $w_2 = v - \text{proj}_W v \in W^\perp$.

Therefore, $w_1 = \text{proj}_W v$

$$\Rightarrow w_1 = \left[\frac{-54}{35}, \frac{4}{35}, \frac{42}{35}, \frac{92}{35} \right] \in W \quad — (a)$$

and $w_2 = v - \text{proj}_W v$

$$\Rightarrow w_2 = [-1, 3, 3, 2] - \left[\frac{-54}{35}, \frac{4}{35}, \frac{42}{35}, \frac{92}{35} \right]$$

$$\Rightarrow w_2 = \left[\frac{19}{35}, \frac{101}{35}, \frac{63}{35}, \frac{-22}{35} \right] \in W^\perp \quad — (b)$$

Verifying $w_1 + w_2 = v$ with (a) and (b),

$$\underline{\text{LHS}} \Rightarrow w_1 + w_2 = \left[\frac{-54+19}{35}, \frac{4+101}{35}, \frac{42+63}{35}, \frac{92-22}{35} \right]$$

$$= [-1, 3, 3, 2] = v = \underline{\text{RHS}}$$

$\therefore$ $\underline{v = w_1 + w_2}$ for some $w_1 \in W$, $w_2 \in W^\perp$

(iii) finding $W^\perp$.

Given $w_1 = [2, -1, 1, 0]$
$w_2 = [1, -1, 2, 2]$

Let $v_i \in W^\perp$ and using gram-schmidt method,

then, $\vec{v_1} = w_1 = [2, -1, 1, 0]$

$$\rightarrow v_2 = w_2 - \frac{(w_2 \cdot v_1)}{\|v_1\|^2} v_1 = [1, -1, 2, 2] - \frac{5}{6}[2, -1, 1, 0]$$

$$v_2 = [-4, -1, 7, 12]$$

Therefore, orthogonal basis $T = \{[2, -1, 1, 0], [-4, -1, 7, 12]\}$

Enlarging $[T \mid I]$, (to $\mathbb{R}^4$) and row reducing,

$$\begin{array}{cccc} (v_1) & (v_2) & (w_3) & (w_4) \end{array}$$

$$= \left[\begin{array}{cc|cccc} 2 & -4 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 7 & 0 & 0 & 1 & 0 \\ 0 & 12 & 0 & 0 & 0 & 1 \end{array}\right] R_1 \rightarrow R_1/2$$

$$= \begin{bmatrix} 1 & -2 & 1/2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 7 & 0 & 0 & 1 & 0 \\ 0 & 12 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_2 \rightarrow R_2 + R_1 \\ R_3 \rightarrow R_3 - R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & -2 & 1/2 & 0 & 0 & 0 \\ 0 & -3 & 1/2 & 1 & 0 & 0 \\ 0 & 9 & -1/2 & 0 & 1 & 0 \\ 0 & 12 & 0 & 0 & 0 & 1 \end{bmatrix} R_2 \rightarrow -\frac{R_2}{3}$$

$$= \begin{bmatrix} 1 & -2 & 1/2 & 0 & 0 & 0 \\ 0 & 1 & -1/6 & -1/3 & 0 & 0 \\ 0 & 9 & -1/2 & 0 & 1 & 0 \\ 0 & 12 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_3 \rightarrow R_3 - 9R_2 \\ R_4 \rightarrow R_4 - 12R_2 \end{matrix}$$

$$= \begin{bmatrix} 1 & -2 & 1/2 & 0 & 0 & 0 \\ 0 & 1 & -1/6 & -1/3 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 2 & 4 & 0 & 1 \end{bmatrix} R_4 \rightarrow R_4 - 2R_3$$

$$= \begin{bmatrix} 1 & -2 & 1/2 & 0 & 0 & 0 \\ 0 & 1 & -1/6 & -1/3 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & -2 & -2 & 1 \end{bmatrix} \quad R_4 \to \frac{-R_4}{2}$$

$$= \begin{bmatrix} 1 & -2 & 1/2 & 0 & 0 & 0 \\ 0 & 1 & -1/6 & -1/3 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1/2 \end{bmatrix} \quad \begin{array}{l} R_3 \to R_3 - 3R_4 \\ R_2 \to R_2 + \frac{R_4}{3} \end{array}$$

$$= \begin{bmatrix} 1 & -2 & 1/2 & 0 & 0 & 0 \\ 0 & 1 & -1/6 & 0 & 1/3 & -1/6 \\ 0 & 0 & 1 & 0 & -2 & 3/2 \\ 0 & 0 & 0 & 1 & 1 & -1/2 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 + \frac{R_3}{6} \\ R_1 \to R_1 - \frac{R_3}{2} \end{array}$$

$$= \begin{bmatrix} 1 & -2 & 0 & 0 & 1 & -3/4 \\ 0 & 1 & 0 & 0 & 0 & 1/12 \\ 0 & 0 & 1 & 0 & -2 & 3/2 \\ 0 & 0 & 0 & 1 & 1 & -1/2 \end{bmatrix} \quad R_1 \to R_1 + 2R_2$$

$$= \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 1 & -7/12 \\ 0 & 1 & 0 & 0 & 0 & 1/12 \\ 0 & 0 & 1 & 0 & -2 & 3/2 \\ 0 & 0 & 0 & 1 & 1 & -1/2 \end{array}\right]$$

$\underbrace{\qquad\qquad}_{(L.I.)}$

Now, $w_3 = [1, 0, 0, 0]$, $w_4 = [0, 1, 0, 0]$

$\Rightarrow$ New orthogonal basis,

$$T^* = \{[2, -1, 1, 0], [-4, -1, 7, 12], [1, 0, 0, 0], [0, 1, 0, 0]\}$$

using gram schmidt for the orthogonal basis,

$$\to v_3 = w_3 - \left(\frac{w_3 \cdot v_1}{\|v_1\|^2}\right) v_1 - \left(\frac{w_3 \cdot v_2}{\|v_2\|^2}\right) v_2 .$$

$$v_3 = [1, 0, 0, 0] - \frac{2}{6}[2, -1, 1, 0] + \frac{4}{210}[-4, -1, 7, 12]$$

$$v_3 = \left[\frac{9}{35}, \frac{11}{35}, \frac{-7}{35}, \frac{8}{35}\right] \quad -(a)$$

$$\rightarrow v_4 = w_4 - \left(\frac{w_4 \cdot v_1}{\|v_1\|^2}\right) v_1 - \left(\frac{w_4 \cdot v_2}{\|v_2\|^2}\right) v_2 - \left(\frac{w_4 \cdot v_3}{\|v_3\|^2}\right) v_3$$

$$v_4 = [0,1,0,0] + \frac{1}{6}[2,-1,1,0] + \frac{1}{210}[-4,-1,7,12]$$

$$- \frac{11}{9}\left[\frac{9}{35}, \frac{11}{35}, \frac{-7}{35}, \frac{8}{35}\right]$$

$$v_4 = \left[\frac{11}{35}, \frac{29}{35}, \frac{7}{35}, \frac{2}{35}\right] - \frac{11}{9}\left[\frac{9}{35}, \frac{11}{35}, \frac{-7}{35}, \frac{8}{35}\right]$$

$$v_4 = \left[0, \frac{4}{9}, \frac{4}{9}, \frac{-2}{9}\right] \quad - (6)$$

As $v_3, v_4 \in W^\perp$,

$$\therefore \quad W^\perp = \operatorname{span}\left\{\begin{bmatrix} 9/35 \\ 11/35 \\ -7/35 \\ 8/35 \end{bmatrix}, \begin{bmatrix} 0 \\ 4/9 \\ 4/9 \\ -2/9 \end{bmatrix}\right\}$$

where orthogonal basis of $W^\perp$ is $\left(\begin{bmatrix} 9/35 \\ 11/35 \\ -7/35 \\ 8/35 \end{bmatrix}, \begin{bmatrix} 0 \\ 4/9 \\ 4/9 \\ -2/9 \end{bmatrix}\right)$.