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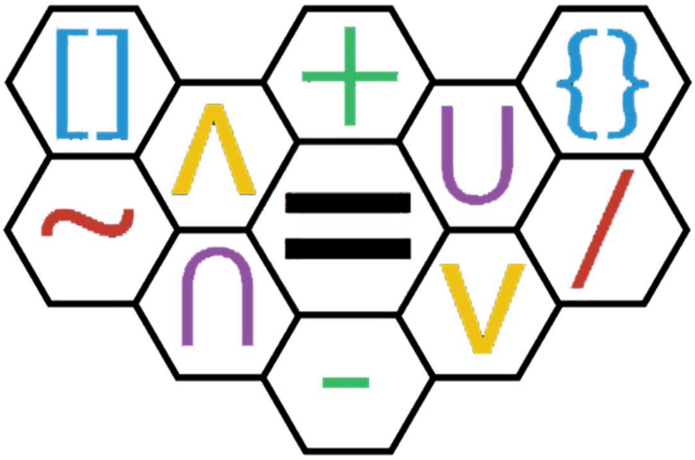

Discrete Structures

Assignment for Paper Code 32341202

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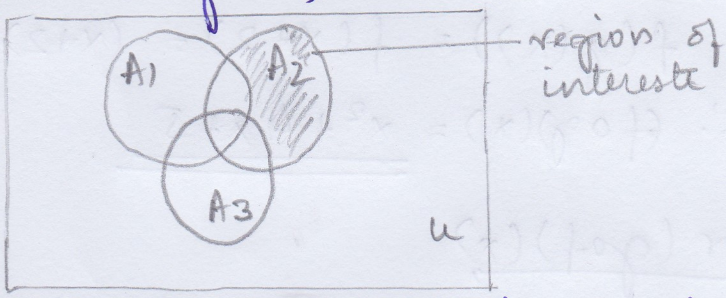
ASSIGNMENT

Section A

Q 1(a) Of a group of 20 students, 10 are interested in music 7 are interested in photography and 4 like swimming. Furthermore 4 are interested in both music & photography, 3 are interested in both music and swimming, 2 are interested in both photography and swimming and 1 is interested in music, photography and swimming. How many students are interested in photography but not in music and swimming?

Ans Let A_1 be the set of students interested in music.
Let A_2 be the set of students in photography interest.
Let A_3 be the set of students interested in swimming.
Using a venn diagram,



Now, from the diagram, the set of students interested in photography but not in music and swimming is given by $A_2 \cup A_1' \cup A_3'$

Also, $|A_2 \cup A_1' \cup A_3'| = |A_2| - |A_1 \cap A_2| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$

$\Rightarrow |A_2 \cup A_1' \cup A_3'| = 7 - 4 - 2 + 1 = 2$

$\therefore$ No. of students interested in photography but not in swimming and music is 2.

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Q 1(b) Prove that the given Boolean expression is a tautology using equivalence rules:

$$(\neg p \wedge q) \rightarrow (\neg(q \rightarrow p))$$

Ans $(\neg p \wedge q) \rightarrow (\neg(q \rightarrow p))$

$\Rightarrow \neg(\neg p \wedge q) \vee (\neg(\neg q \vee p))$ (conditional operator)

$\Rightarrow \neg(\neg p \wedge q) \vee (q \wedge \neg p)$ (demorgan's law)

$\Rightarrow \neg(\neg p \wedge q) \vee (\neg p \wedge q)$ $(A \cdot B = B \cdot A)$

$\Rightarrow T$ $(A + \neg A = T)$

Hence, proved!

Q1(c) Given $f(x) = x^2+1$ and $g(x) = x+2$. Find $(f o g)$ and $(g o f)$ where f and g are functions from R to R.

Ans For $(fog)(x)$

$$f(g(x)) = f(x+2) = (x+2)^2+1 = x^2+4x+5$$

$$\therefore (fog)(x) = \underline{x^2+4x+5}$$

For $(gof)(x)$

$$g(f(x)) = g(x^2+1) = (x^2+1)+2 = x^2+3$$

$$\therefore (gof)(x) = \underline{x^2+3}$$

Q1(d) Consider the following advertisement of a game:

(i) There are three statements in this advertisement

(ii) Two of them are not true.

(iii) The average IQ increase of people who learned this game is more than 20 points.

Prove that the statement (iii) is true using a truth table.

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Ans There are three statements (i), (ii), (iii).

(i)	(ii)	(iii)
T	T	T
T	T	F
$\boxed{T}$	$\boxed{F}$	$\boxed{T}$
T	F	F

(a) We know that there are three statements, it implies (i) is a tautology.

(b) If we assume (ii) is true, then it contradicts itself as then two statements, (i) and (ii) cannot be true. ∴ (ii) is false

(c) As (ii) is false, thus, two statements are true, ∴ (iii) must be true.

Hence proved!

(e) A dance pair means a woman and man dancing together. How many such dance pairs can be formed from a group of 6 women and 10 men?

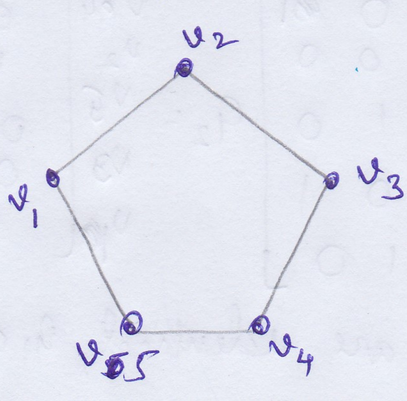
Ans As there are 6 women, we need to select 6 men from the pool of 10 men, that is, $^{10}C_6 = {}^{10}C_4$. Now, these pairs can be arranged in $6!$ ways.

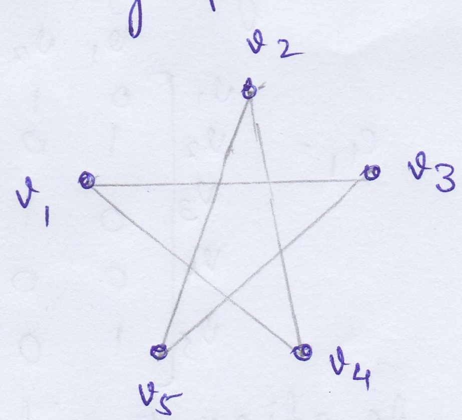
∴ No. of dance pairs that can be formed $= {}^{10}C_6 \times 6! = \frac{10!}{\cancel{6!}4!} \times \cancel{6!} = \frac{10!}{4!}$

$= \underline{151200}$.

However, the number of ways 1 pair can be formed is given by $^6C_1 \times {}^{10}C_1$ that is $\underline{60}$ ways.

(f). Determine whether the given graphs G_1 and G_2 are isomorphic or not.





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Ans Checking constraints,

for G_1, $|V_1| = 5$; $|E_1| = 5$

for G_2, $|V_2| = 5$; $|E_2| = 5$

Thus, $|V_1| = |V_2|$ and $|E_1| = |E_2|$.

Checking degrees of every node,

for G_1, $\deg(v_1) = 2$

$\deg(v_2) = 2$

$\deg(v_3) = 2$

$\deg(v_4) = 2$

$\deg(v_5) = 2$

for G_2, $\deg(v_1) = 2$

$\deg(v_2) = 2$

$\deg(v_3) = 2$

$\deg(v_4) = 2$

$\deg(v_5) = 2$

Checking mapping of nodes,

→ v_2 in $G_1 \longleftrightarrow v_2$ in G_2

→ v_1 in $G_1 \longleftrightarrow v_4$ in G_2

→ v_3 in $G_1 \longleftrightarrow v_5$ in G_2

→ v_4 in $G_1 \longleftrightarrow v_3$ in G_2

→ v_5 in $G_1 \longleftrightarrow v_1$ in G_2

Checking adjacency matrices,

$$G_1 = \begin{array}{c|ccccc} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \hline v_1 & 0 & 1 & 0 & 0 & 1 \\ v_2 & 1 & 0 & 1 & 0 & 0 \\ v_3 & 0 & 1 & 0 & 1 & 0 \\ v_4 & 0 & 0 & 1 & 0 & 1 \\ v_5 & 1 & 0 & 0 & 1 & 0 \end{array} \qquad G_2 = \begin{array}{c|ccccc} & v_4 & v_2 & v_5 & v_3 & v_1 \\ \hline v_4 & 0 & 1 & 0 & 0 & 1 \\ v_2 & 1 & 0 & 1 & 0 & 0 \\ v_5 & 0 & 1 & 0 & 1 & 0 \\ v_3 & 0 & 0 & 1 & 0 & 1 \\ v_1 & 1 & 0 & 0 & 1 & 0 \end{array}$$

As adjacent matrices are identical, G_1 and G_2 are isomorphic

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Q1(g) Does there exist a simple graph with seven vertices having the following degree sequence:

$$(1, 3, 3, 4, 5, 6, 6)$$

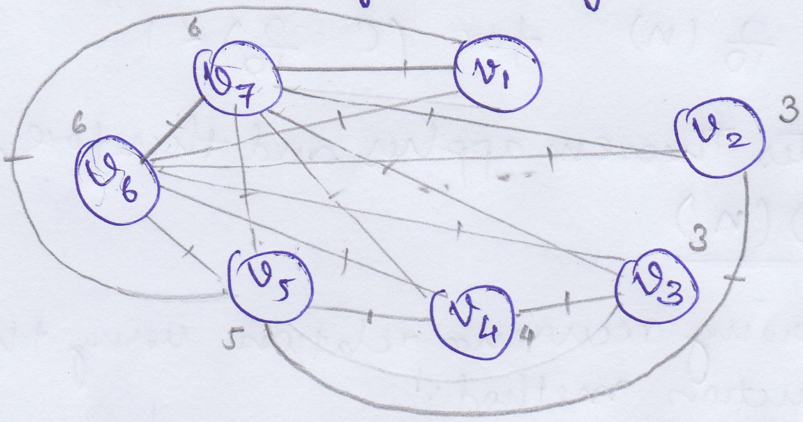
Ans For any graph, the sum of degrees is twice the number of edges.

$$\Rightarrow \sum_{i=01}^{n} deg(v_i) = 2|E|$$

$$\Rightarrow 1+3+3+4+5+6+6 = 2|E|$$

$$\Rightarrow |E| = \frac{28}{2} = 14.$$

Drawing all nodes and edges in descending order of their degrees, we get



we get a contradiction that one of the vertices cannot be a pendant vertex.

Now, maximum sum of degrees for a simple graph having n vertices is true for the complete graph K_n.

$\therefore$ Maximum sum of degrees $= \frac{n(n-1)}{2}$

for $n=7$,

maximum sum of degrees $= \frac{7(6)}{2} = 21$

For the given degree sequence,

$$\sum_{v \in V} deg(v) = 28$$

Therefore, no simple graph exists for the given degree sequence.

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1(h) use master method to find asymptotic bounds for the following reccurence relation.

$$T(n) = T\left(\frac{9n}{10}\right) + n$$

Ans In $T(n) = T\left(\frac{9n}{10}\right) + n$,

$a = 1$, $b = \frac{10}{9}$ and $f(n) = n$

Now, $\log_b a = \log_{\frac{10}{9}} 1 = 0$

As $f(n) = n = \Omega(n^{0+1})$ where $\varepsilon = 1$

and $f\left(\frac{9n}{10}\right) \leq c \cdot f(n)$

or, $\frac{9n}{10} \leq \frac{9}{10}(n)$ for $\left(c = \frac{9}{10}\right) < 1$

Case (3) of Master Theorem applies and therefore,

$T(n) = \Theta(n)$

1(i) Solve the following recurrence relation using the generating function method:

$$a_r = 3a_{r-1} + 2$$

where $r \geq 1$ and boundary condition $a_0 = 1$

Ans. We have, $a_r = 3a_{r-1} + 2$ for $r \geq 1$ and $a_0 = 1$

Multiplying both sides by x^r,

$\Rightarrow x^r a_r = 3x^r a_{r-1} + 2x^r$

Summing from $r = 1$ to $r = \infty$ $\forall r \geq 1$,

$$\Rightarrow \sum_{r=1}^{\infty} x^r a_r = 3\sum_{r=1}^{\infty} x^r a_{r-1} + 2\sum_{r=1}^{\infty} x^r$$

Considering LHS,

$$\sum_{r=1}^{\infty} x^r a_r = a_1 x^1 + a_2 x^2 + a_3 x^3 + \dots$$

$$= A(x) - a_0 x^0$$

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considering RHS,

$$3\sum_{r=1}^{\infty} x^r a_{r-1} + 2\sum_{r=1}^{\infty} x^r = 3(a_0x^1 + a_1x^2 + a_2x^3 + \dots \infty) + 2(x^1 + x^2 + x^3 + \dots)$$

$$\Rightarrow 3\sum_{r=1}^{\infty} x^r a_{r-1} + 2\sum_{r=1}^{\infty} x^r = 3x(a_0x^0 + a_1x^1 + a_2x^2 + \dots) + x2(1\cdot x^0 + 1\cdot x^1 + 1\cdot x^2 + 1\cdot x^3 + \dots)$$

$$\Rightarrow 3\sum_{r=1}^{\infty} x^r a_{r-1} + 2\sum_{r=1}^{\infty} x^r = 3x[A(x)] + 2x\left(\frac{1}{1-x}\right)$$

Equating LHS and RHS,

$$[A(x)] - a_0x^0 = 3x[A(x)] + \left(\frac{2x}{1-x}\right) \quad (\because a_0 = 1)$$

$$\Rightarrow A(x) = \left(\frac{2x}{1-x} + 1\right)\left(\frac{1}{1-3x}\right) = \frac{(1+x)}{(1-x)(1-3x)}$$

Decomposing $A(x)$ into partial fractions

$$\Rightarrow \frac{(1+x)}{(1-x)(1-3x)} = \frac{A}{(1-x)} + \frac{B}{(1-3x)}$$

or, $(1+x) = A(1-3x) + B(1-x)$

for $x = 1$,

$\Rightarrow 2 = -2A + 0$

$\Rightarrow A = -1$

for $x = 1/3$

$\Rightarrow \frac{\cancel{2}4}{\cancel{3}} = 0 + \frac{\cancel{2}}{\cancel{3}}B$

$\Rightarrow B = 2$

Thus, $A(x) = \left(\frac{2}{1-3x}\right) + \left(\frac{-1}{1-x}\right)$

or, $a_r = 2(3)^r - (1)^r$

$\therefore \boxed{a_r = 2(3)^r - 1}$

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Section B

Q2(a): A jigsaw puzzle contains of a number of pieces. Two or more pieces with matched boundaries can be put together to form a big pieces finally, when all pieces are put together as one single block, the jigsaw puzzle is set to be solved. Putting 2 blocks with matched boundaries together is counted as one move. Use PMI to prove that for a jigsaw puzzle with n pieces it will take $n-1$ moves to solve the puzzle.

Ans Basis:

- A puzzle with 1 piece, requires 0 moves to solve it.
- A puzzle with 2 pieces, requires 1 moves to solve it.
- Induction hypothesis is that a puzzle with n pieces, requires $(n-1)$ moves to solve it.
- for $n=1$, moves required $= 1-1 = 0$
- for $n=2$, moves required $= 2-1 = 1$

$\therefore$ Induction for $P(1)$ and $P(2)$ holds.

Induction:

Assume $P(n)$ is true for $n=K$,

$\Rightarrow$ a puzzle with K pieces requires $(K-1)$ moves solve it.

~~Assume P~~ Proving for puzzle with $(K+1)$ pieces,

- Suppose that final two pieces of the puzzle are made up of K_1 and K_2 pieces such that, $(K_1 + K_2) = (K+1)$.
- Puzzle moves required to form the last two pieces is given by $(K_1 - 1) + (K_2 - 1)$
- Final move joins the two pieces

$\therefore$ Total moves required to put together a puzzle with $(K+1)$ pieces $= (K_1 - 1) + (K_2 - 1) + 1$

$= (K+1) - 2 + 1 = \underline{K}$ moves.

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∴ Induction holds for $P(k+1)$

As the hypothesis is true for $n = k+1$, it is true for all n in $[1, k]$ i.e. $1 \leq n \leq k$.

Hence proved!

Q2(b). What is a poset? Draw a Hasse Diagram for the given poset:

$$(\{2, 4, 5, 10, 12, 20, 25\}, |)$$

Ans

- A set 'A' along with a binary relation on A that is reflexive, antisymmetric and transitive, i.e. a partial ordering relation, is called a partially ordered set (A, R) commonly abbreviated as poset.
- For ordered pair (a, b) in R, operation that corresponding to the pair is written ~~ator~~ inplace of R, eg. $(A, \leq)$.
- For $(\{2, 4, 5, 10, 12, 20, 25\}, |)$,

  $A = \{2, 4, 5, 10, 12, 20, 25\}$

  $R =$ divisibility operator $(|)$

or, $R = \{(a, b) : a$ completely divides $b\}$

i.e.
$$M_R = \begin{array}{c} \\ 2 \\ 4 \\ 5 \\ 10 \\ 12 \\ 20 \\ 25 \end{array} \begin{array}{c} \begin{array}{ccccccc} 2 & 4 & 5 & 10 & 12 & 20 & 25 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

The digraph and the Hasse diagram for the poset can be drawn as follows:

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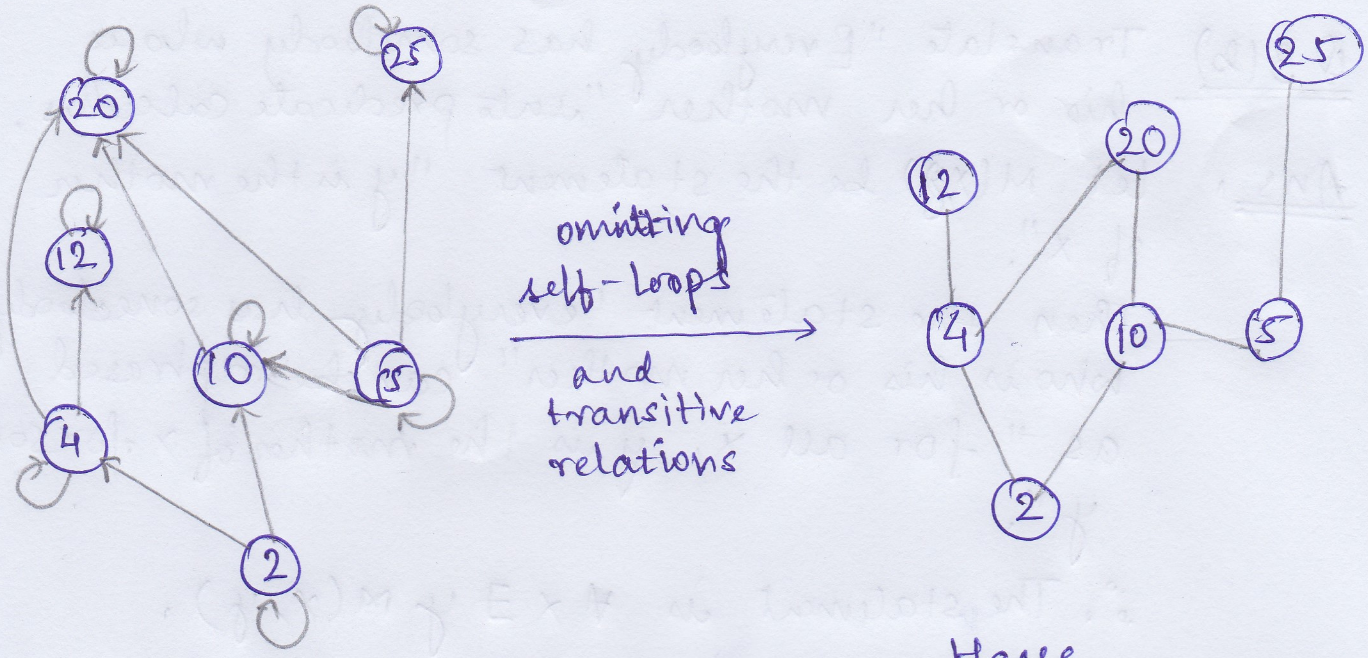

Digraph

Hasse Diagram

Q3(a) Consider two sets A and B, $A = \{1,2,3,4\}$ and $B = \{3,4,5,6\}$. Find the elements of each of the relation R stated below. Also, find the domain and range of R

(i) $a \in A$ is related to $b \in B$ ie aRb iff $a < b$

(i) $R = \{(1,3),(1,4),(1,5),(1,6),(2,3),(2,4),(2,5),(2,6),(3,4),(3,5),(3,6),(4,5),(4,6)\}$

domain(R) $= \{a \in A : (a,b) \in R \text{ for some } b \in B\}$
$= \{1,2,3,4\}$

range(R) $= \{b \in B : (a,b) \in R \text{ for some } a \in A\}$
$= \{3,4,5,6\}$

(ii) $a \in A$ is related to $b \in B$ i.e. aRb iff a and b are both odd numbers.

(ii) $R = \{(1,3),(1,5),(3,5),(1,1),(3,3)\}$

domain (R) $= \{1,3\}$

range (R) $= \{1,3,5\}$

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Q 3(b) Translate "Everybody has somebody who is his or her mother" into predicate calculus.

Ans. Let $M(x,y)$ be the statement "y is the mother of x".

Then, the statement "everybody has somebody who is his or her mother" can be rephrased as "for all x, y is the mother of x for some y".

$\therefore$ The statement is $\forall x \, \exists y \, M(x,y)$.

Q 3(c) Give Big O notation for the factorial function.

Ans. Big O notation is defined as follows:

$$O(g(n)) = \{ f(n) : \exists \text{ +ve constants } c \text{ and } n_0 \text{ such that } 0 \le f(n) \le c \cdot g(n) \ \forall n \geq n_0$$

Here, $f(n) = n! = n \cdot (n-1) \cdot (n-2) \ldots 3 \cdot 2 \cdot 1$

We know,

$\Rightarrow 0 \le n! \le n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$

$\Rightarrow 0 \le n! \le n \cdot n \cdot n \cdots n$

$\Rightarrow 0 \le n! \le n^n$

Also, considering $n! \le n^n$ inequality,

$\Rightarrow \log(n!) \le \log(n^n)$

$\Rightarrow \log(n!) \le n \log(n)$

$\therefore n! = O(n \cdot \log(n))$

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Q4(a) Show all the steps of Bubble Sort to put the following list of items in increasing order:

4 9 3 6 2 5 1

Ans

							Iteration
4	9 ↔	3 ↔	6 ↔	2 ↔	5 ↔	1	
4 ↔	3	6 ↔	2 ↔	5 ↔	1 |	9	1
3	4 ↔	2	5 ↔	1 |	6	9	2
3 ↔	2	4 ↔	1 |	5	6	9	3
2	3 ↔	1 |	4	5	6	9	4
2 ↔	1 |	3	4	5	6	9	5
1	2	3	4	5	6	9	6

Q4(b) A factory makes custom sports cars at an increasing rate. In the first month, only one car is made, in the second month, two cars are made, and so on, with n cars made in the nth month.

(i) Set up a recurrence relation for the number of cars produced in the first n months by this factory.

(i) For the rth month, the production is given by the recurrence relation:

$$\{a_r = a_{r-1} + r\} \text{ with } (a_0 = 0)$$

(ii) Use the recurrence relation to solve how many cars are produced in the first year.

(ii)

$a_1 = 0+1 = 1$	$a_4 = 6+4 = 10$	$a_7 = 21+7 = 28$
$a_2 = 1+2 = 3$	$a_5 = 10+5 = 15$	$a_8 = 28+8 = 36$
$a_3 = 3+3 = 6$	$a_6 = 15+6 = 21$	$a_9 = 36+9 = 45$
$a_{10} = 45+10 = 55$	$a_{11} = 55+11 = 66$	$a_{12} = 66+12 = 78$

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$\therefore$ No. of cars produced in the first year is $\underline{78}$

(iii) Find an explicit formula for the number of cars produced in the first n months by the factory

(iii) No. of cars produced in n months $(n \geq 0)$

$$= (n) + (n-1) + (n-2) + \dots + 3 + 2 + 1 + 0$$

$$= \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

$$\therefore \boxed{a_r = \frac{r(r+1)}{2}} \text{ with } a_0 = 0$$

Q5.(a) Show that the following argument is valid:
If Mohan is a lawyer, then he is ambitious.
If Mohan is an early riser, then he does not like rice.
If Mohan is ambitious, then he is early riser.
Then if Mohan is a lawyer, then he does not like rice.

Ans. Assume the following ~~premises~~ statements,

L : Mohan is a lawyer
A : Mohan is ambitious
E : Mohan is an early riser
R : Mohan does not like rice.

The premises are as follows:

$L \to A$
$E \to R$
$A \to E$

Conclusion is $L \to R$.

To prove:

$$(L \to A) \land (E \to R) \land (A \to E) \Rightarrow (L \to R)$$

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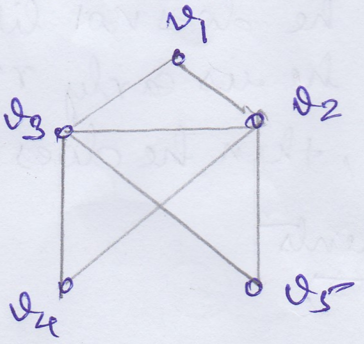
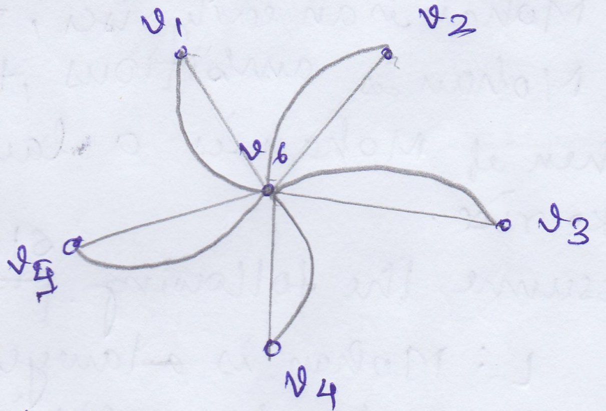
		Rule
(1)	$A \to E$	P
(2)	$E \to R$	P
(3)	$A \to R$	T, $I_{13}(1,2)$: $P \to Q, Q \to R \Rightarrow P \to R$
(4)	$L \to A$	P
(5)	$L \to R$	T, $I_{13}(4,5)$: $P, Q, Q \to R \Rightarrow P \to R$

$\therefore$ $L \to R$ i.e. if Mohan is a lawyer, he does not like rice is a valid conclusion from the given premises.

Q5(b) State the condition for Eulerian path and Eulerian circuit. Determine whether the given graphs G_1 and G_2 have Eulerian circuit or Eulerian path





Ans
- Condition for existence of Eulerian circuit

  if $\forall v \in V$, $\deg(v) = 2n \ \forall n \in \mathbb{N}$, that is, if every vertex has an even degree, there exists an Eulerian circuit.

- Condition for existence of Eulerian path

  if exactly and only 2 vertices have odd degrees and all other vertices in the path have even degrees, there exists an Eulerian path.

- In Graph 1,

  $\deg(v_1) = 2$; $\deg(v_2) = 4$; $\deg(v_3) = 4$; $\deg(v_4) = 2$; $\deg(v_5) = 2$

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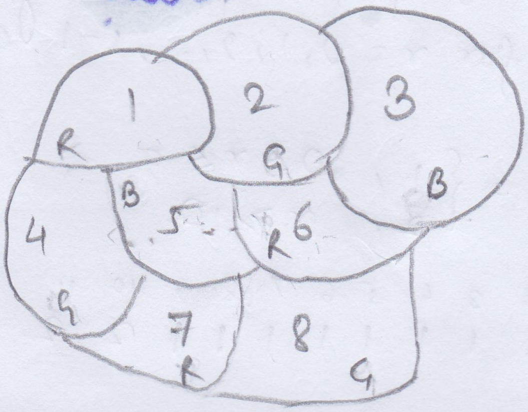
As all vertices have even degrees,
the graph has an Eulerian circuit.

- <u>In Graph 2</u>,

$deg(v_1) = 2$; $deg(v_2) = 2$; $deg(v_3) = 2$; $deg(v_4) = 2$
$deg(v_5) = 2$; $deg(v_6) = 10$

As all vertices have even degrees,
the graph has an Eulerian circuit.

<u>Q5(c).</u> Define chromatic number for a graph. Determine χ(chi) for the given graph G

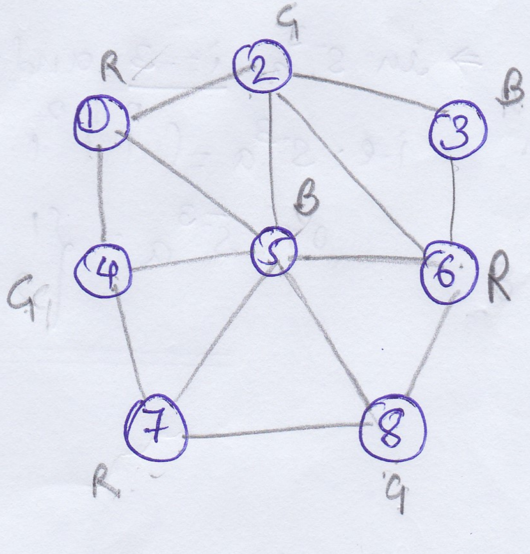


<u>Ans</u>

- The chromatic number $\chi(G)$ is defined as the least number of identifiers to each vertex of the graph such that no two adjacent vertices have the same colour.

- Considering the graph G,
the following connections exist:

→ (1,2), (1,4), (1,5)
→ (2,3), (2,6), (2,5)
→ (4,5), (4,7)
→ (5,7), (5,8), (5,6)
→ (7,8)
→ (6,8)
→ (3,6)



Therefore, a possible combination could be as follows:

Identifier	Regions
R	1, 6, 7
G	2, 4, 8
B	3, 5
~~6~~	~~6~~

∴ Chromatic number $\chi(G) = \underline{3}$

Q6(a) Find $s^6 a$ and $s^{-3} a$ for the following numeric function where r is zero for $r = 0, 1, 2, \ldots, i-1$ and is a_{r-1} for $r \geq i$.

$$a_r = \begin{cases} 1, & 0 \leq r \leq 9 \\ 2, & r \geq 10 \end{cases}$$

Ans.

$$a_r = \begin{array}{ccccccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & & \\ (1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & \ldots &) \end{array}$$

Now,

→ in $s^6 a$, $\underline{i = 6}$ and thus function postponed by 6 values

i.e. $s^6 a = \begin{array}{cccccccccccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & \\ (0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & \ldots) \end{array}$

or, $s^6 a = \begin{cases} 0, & 0 \leq r \leq 5 \\ 1, & 6 \leq r \leq 15 \\ 2, & r \geq 16 \end{cases}$

→ in $s^{-3} a$, $\underline{i = -3}$ and thus function preponed by 3 values

i.e. $s^{-3} a = \begin{array}{ccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \\ (1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & \ldots) \end{array}$

or, $s^{-3} a = \begin{cases} 1, & 0 \leq r \leq 6 \\ 2, & r \geq 7 \end{cases}$

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Q6(b) Consider $H_0 = 0$, $H_1 = 1$ and $H_n = H_{n-1} + 2H_{n-2}$. Give an explicit solution for H_n.

Ans Given recurrence relation is,

$$H_n = H_{n-1} + 2H_{n-2} \quad \text{with } H_0 = 0,\ H_1 = 1$$

Solving using generating function method,

→ Multiplying both sides by x^n.

$$\Rightarrow x^n H_n = x^n H_{n-1} + 2H_{n-2}x^n$$

→ Summing for $n \geq 2$,

$$\Rightarrow \sum_{n=2}^{\infty} x^n H_n = \sum_{n=2}^{\infty} x^n H_{n-1} + 2\sum_{n=2}^{\infty} x^n H_{n-2}$$

→ Considering LHS,

$$\Rightarrow \sum_{n=2}^{\infty} x^n H_n = (x^2H_2 + x^3H_3 + \dots + x^nH_n + \dots)$$

$$= A(x) - x^0H_0 - x^1H_1 \quad (\because H_0 = 0, H_1 = 1)$$

$$= \underline{A(x) - x}$$

→ Considering RHS,

$$\Rightarrow \sum_{n=2}^{\infty} x^n H_{n-1} + 2\sum_{n=2}^{\infty} x^n H_{n-2} = (x^2H_1 + x^3H_2 + \dots + x^nH_{n-1} + \dots) + 2(x^2H_0 + x^3H_1 + \dots)$$

$$= x(x^1H_1 + x^2H_2 + \dots) + 2x^2(x^0H_0 + x^1H_1 + \dots)$$

$$= x(A(x) - x^0H_0) + 2x^2(A(x))$$

$$= \underline{x\,A(x) + 2x^2 A(x)} \quad (\because H_0 = 0)$$

→ Equating LHS and RHS,

$$(A(x)) - x = x(A(x)) + 2x^2(A(x))$$

$$\Rightarrow A(x)\left[2x^2 + x - 1\right] = -x$$

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$$\Rightarrow A(x) = \frac{(-x)}{(2x^2+x-1)} = \frac{-x}{2x(x+1)-1(x+1)} = \frac{-x}{(x+1)(2x-1)}$$

$$\Rightarrow A(x) = \frac{-x}{(1+x)(2x-1)}$$

→ Decomposing $A(x)$ into partial fractions,

$$\Rightarrow \frac{-x}{(1+x)(2x-1)} = \frac{A}{(1+x)} + \frac{B}{(2x-1)}$$

$$\Rightarrow -x = A(2x-1) + B(1+x)$$

Let $x=-1$,

$$\Rightarrow \frac{-1}{3} = A$$

Let $x = 1/2$

$$\Rightarrow \frac{-1}{2} = B\left(\frac{3}{2}\right)$$

$$\Rightarrow B = \frac{-1}{3}$$

→ Simplifying,

$$\Rightarrow A(x) = \frac{-x}{(1+x)(2x-1)} = \frac{-1}{3}\left(\frac{1}{1+x}\right) - \frac{1}{3}\left(\frac{1}{2x-1}\right)$$

$$\Rightarrow A(x) = \frac{-1}{3}\left(\frac{1}{1-(-x)}\right) + \frac{1}{3}\left(\frac{1}{1-2x}\right)$$

$$\therefore \boxed{H_n = \frac{-1}{3}(-1)^n + \frac{1}{3}(2)^n}$$

Q7(a) Draw a directed graph of the following relations R defined on the set $\{1,2,3,4\}$. Decide whether the relation is reflexive, symmetric or transitive.
$R = \{(1,1),(2,2),(3,3),(4,4),(1,2),(2,3),(1,3),(3,2)\}$

Ans The directed graph of the relation R is given as,

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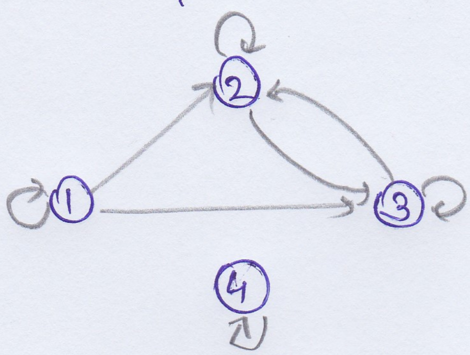

- **Reflexivity**: for R to be reflexive, $(a,a) \in R \ \forall a$

As, $(2,2), (1,1), (3,3), (4,4) \in R$,

$\therefore$ R is reflexive.

- **Symmetry**: for R to be symmetric, $(a,b) \in R \Leftrightarrow (b,a) \in R$

As, $(2,1), (3,1) \notin R$ but $(1,2), (1,3) \in R$,

$\therefore$ R is not symmetric.

- **Transitivity**: for R to be transitive, $(a,b), (b,c) \in R \rightarrow (a,c) \in R$,

As, $(1,2), (2,3), (1,3), (2,2), (3,3), (4,4),$
$(3,2),$ ~~(2,3)~~ $(1,1) \in R$,

$\therefore$ R is transitive.

Q7(b). Let f be the function from set $X = \{2,3,4,5,6,7\}$ into set $Y = \{0,1,2,3,4\}$ defined by $f(x) = 2x \bmod 5$. Write f as set of ordered pairs. Is f one-one or onto?

Ans. Let $f(X) = Y$,

then $f(X) = \{(2,2), (3,3), (4,4), (5,0), (6,1), (7,2)\}$

As $2 \in Y$, but has two distinct pre-images in X which are not unique,

$\therefore$ f is not injective or one-one function.

As $\forall y \in Y$ there exists a pre image in X,

$\therefore$ f is surjective or onto function