Graph Basics for Data Structures

GRAPH

A graph is a non-empty set of points called vertices and a set of line segments joining pairs of vertices called edges.

GRAPH

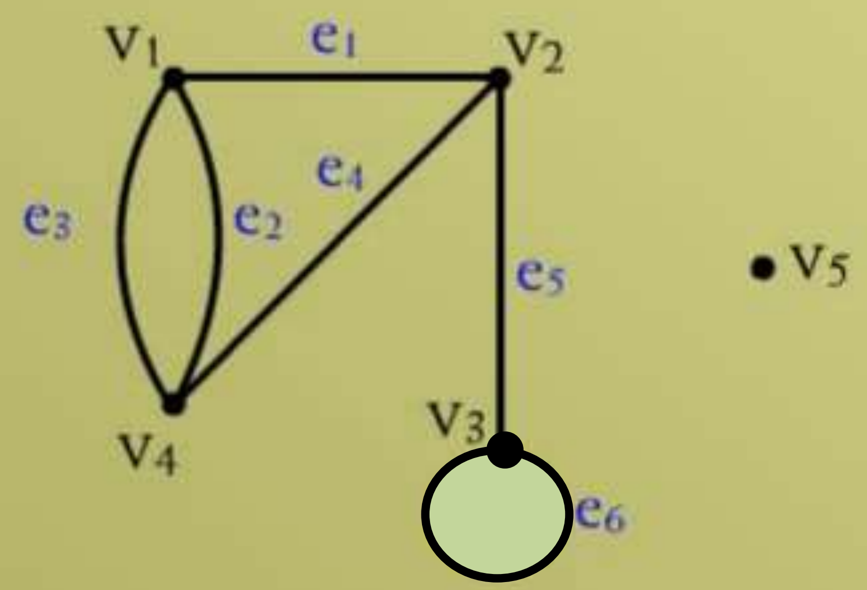
Formally, a graph G consists of two finite sets:

(1) A set V=V(G) of vertices (or points or nodes)

(2) A set E=E(G) of edges.

Where each edge corresponds to a pair of vertices.

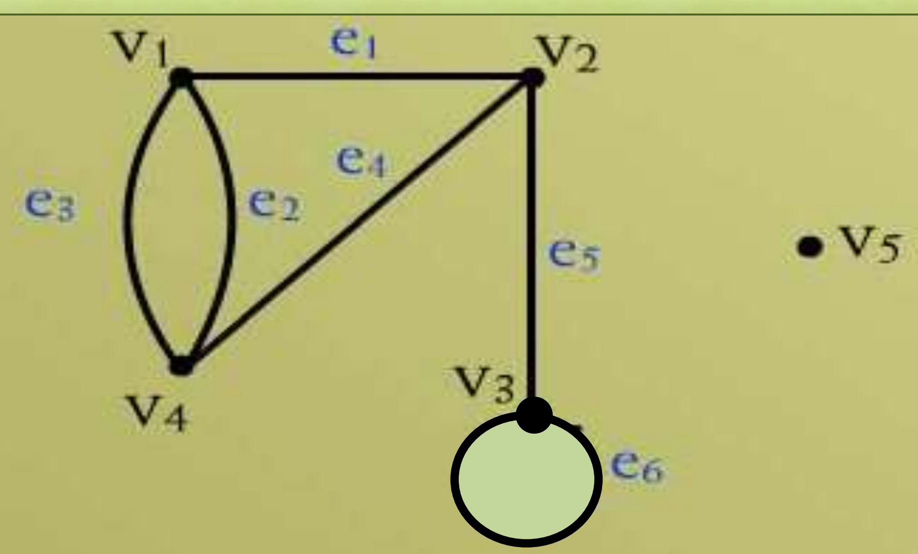
EXAMPLE



We have five vertices labeled by v_1, v_2, v_3, v_4, v_5.

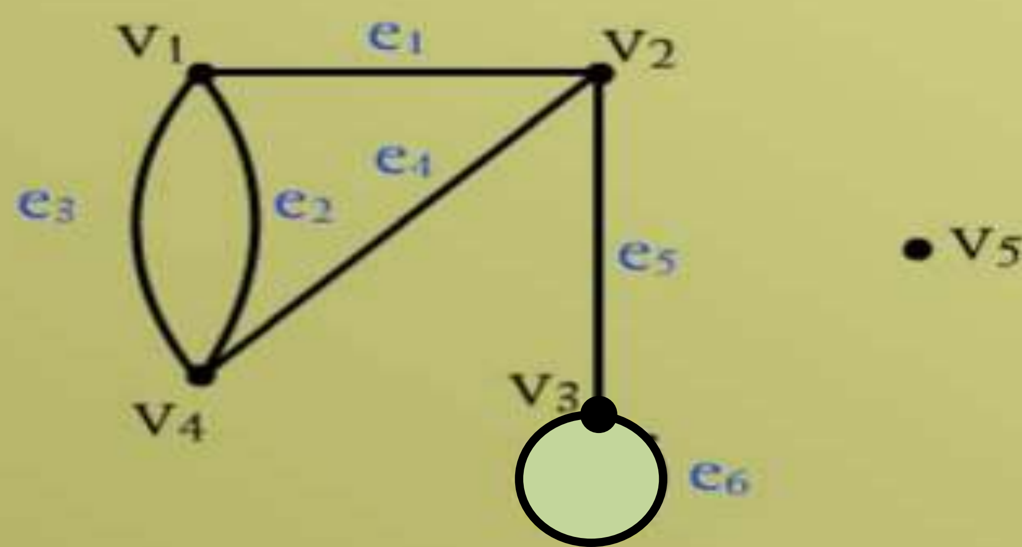
We have edges $e_1, e_2, \ldots\ldots, e_6$.

EXAMPLE



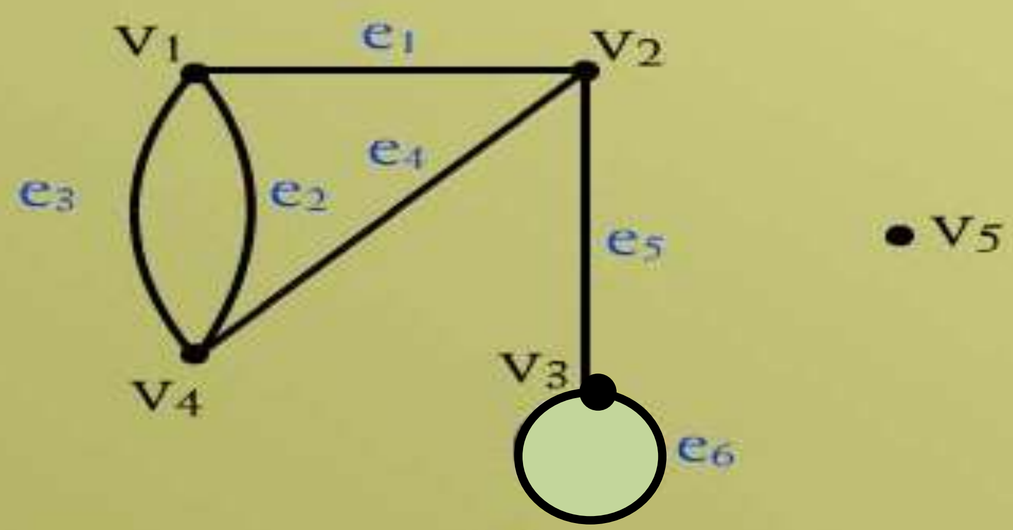
- e_1 edge is for vertices v_1 and v_2.
- e_2 and e_3 end points v_1 and v_4.
- e_4 has end points v_2 and v_4.

EXAMPLE



- e_5 has end points v_2 and V3
- e_6 is a loop.
- v_5 is isolated vertex.

SOME TERMINOLOGY



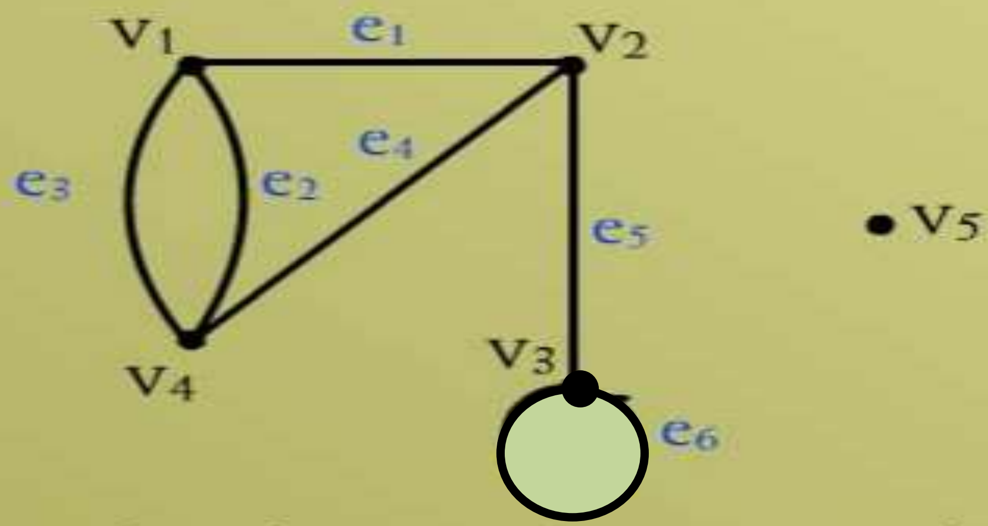
1- An edge connects either one or two vertices called its endpoints (edge e_1 connects vertices v_1 and v_2 described as $\{v_1, v_2\}$).

SOME TERMINOLOGY

2- An edge with just one endpoint is called a loop. Thus a loop is an edge that connects a vertex to itself (e.g., edge e_6)

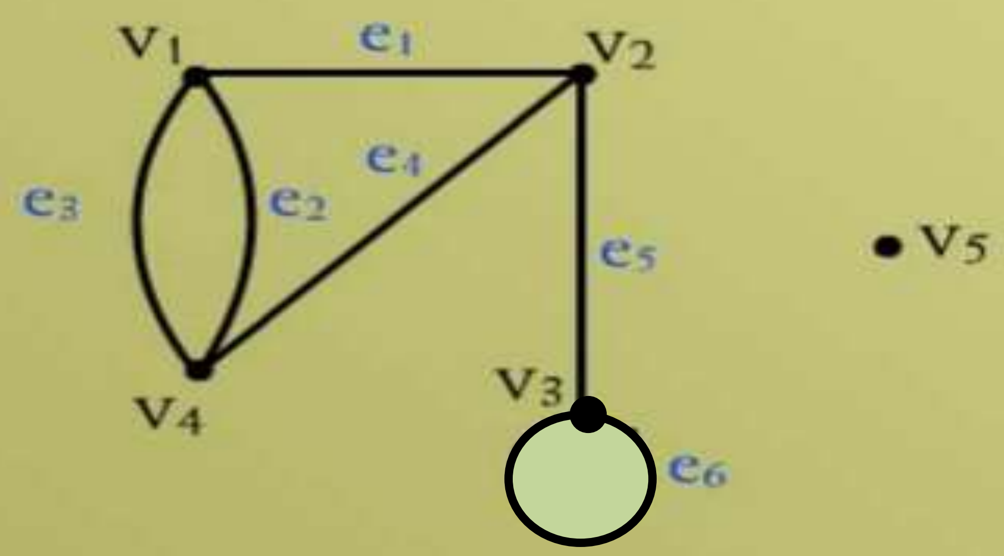
3- Two vertices that are connected by an edge are called adjacent, and a vertex that is an endpoint of a loop is said to be adjacent to itself.

SOME TERMINOLOGY



- An edge is said to be incident on each of its endpoints.
- A vertex on which no edges are incident is called isolated (e.g., v_5)

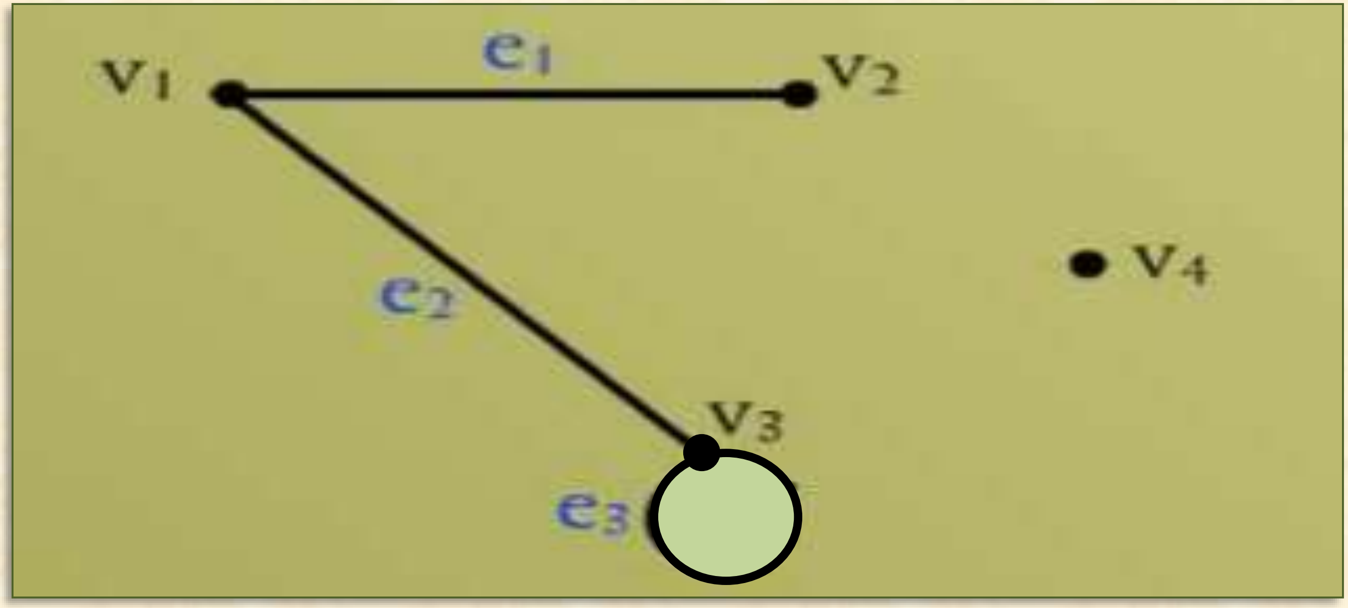
SOME TERMINOLOGY



- Two distinct edges with the same set of end points are said to be parallel.
  (e_2 & e_3 are parallel).

EXAMPLE

Define the following graph formally by specifying its vertex set, its edge set, and a table giving the edge endpoint function.



SOLUTION

Vertex Set = $\{v_1, v_2, v_3, v_4\}$

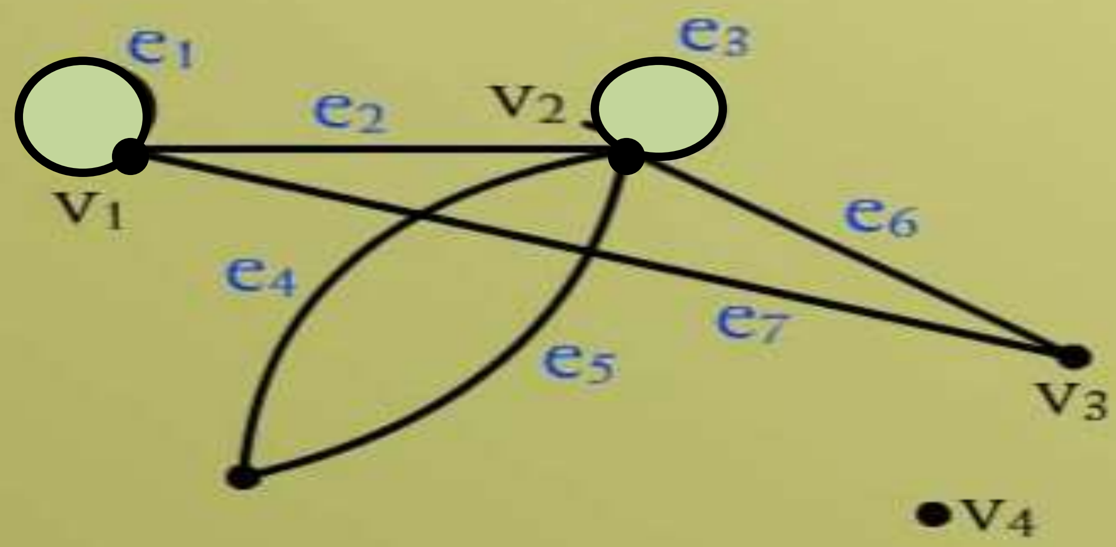
Edge Set = $\{e_1, e_2, e_3\}$

Edge - endpoint function:

Edge	Endpoint
e_1	$\{v_1, v_2\}$
e_2	$\{v_1, v_3\}$
e_3	$\{v_3\}$

EXAMPLE

For the graph shown below:



- Find all edges that are incident on v_1.

EXAMPLE

- Find all vertices that are adjacent to v_3.

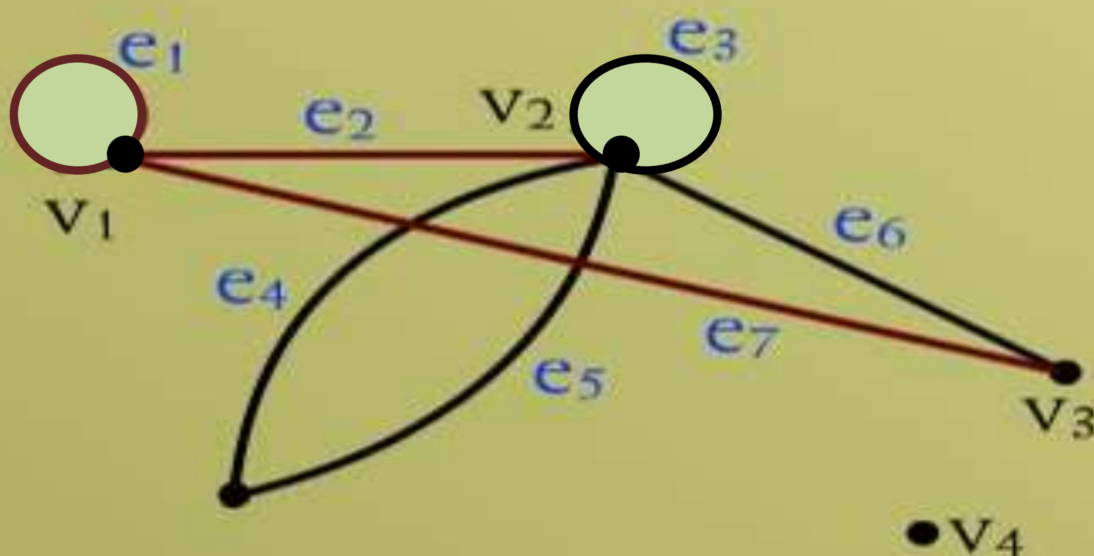
- Find all loops.

- Find all parallel edges.

- Find all isolated vertices.

SOLUTION

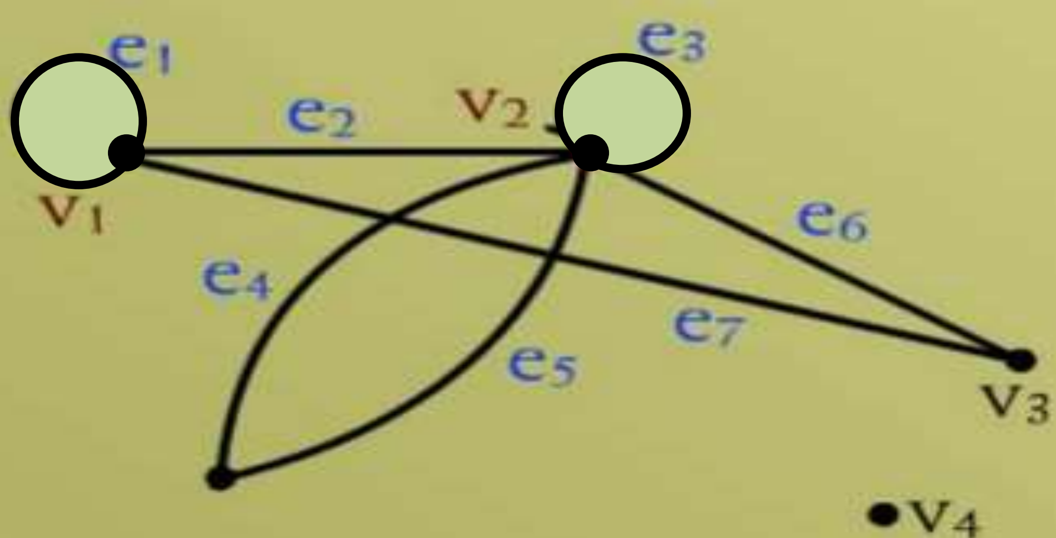
- Find all edges that are incident on v_1.



v_1 is incident with edges e_1, e_2 and e_7.

SOLUTION

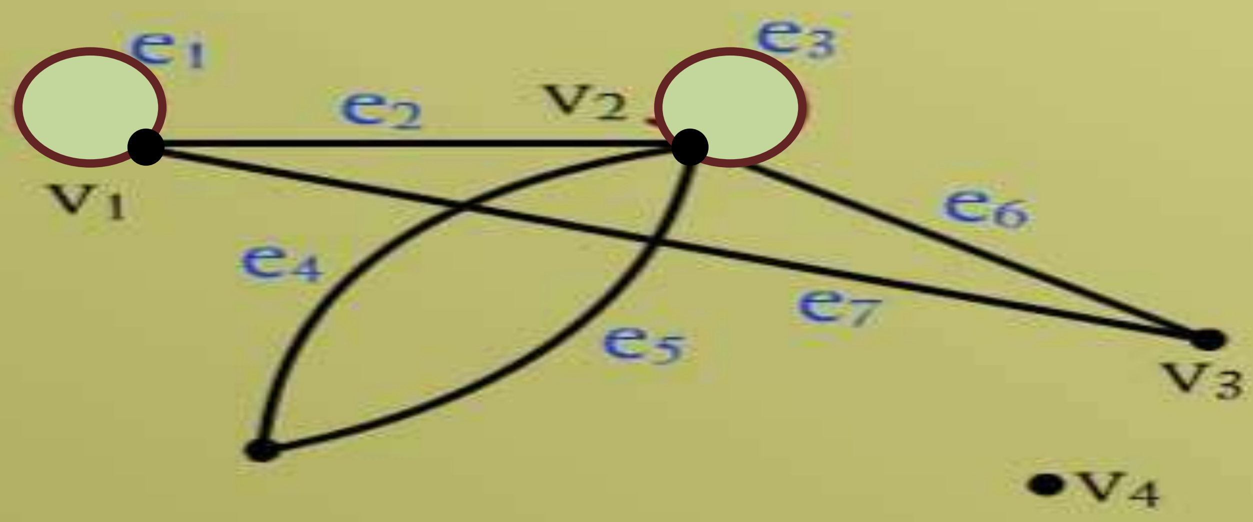
- Find all vertices that are adjacent to v_3.



Vertices adjacent to v_3 are v_1 and v_2 .

SOLUTION

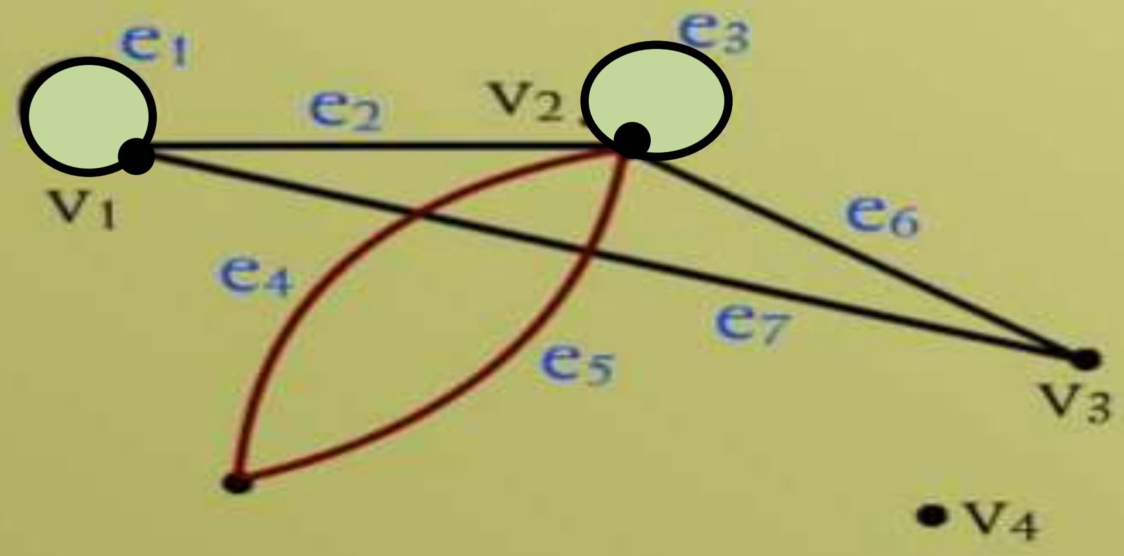
- Find all loops.



Loops are e_1 and e_3.

SOLUTION

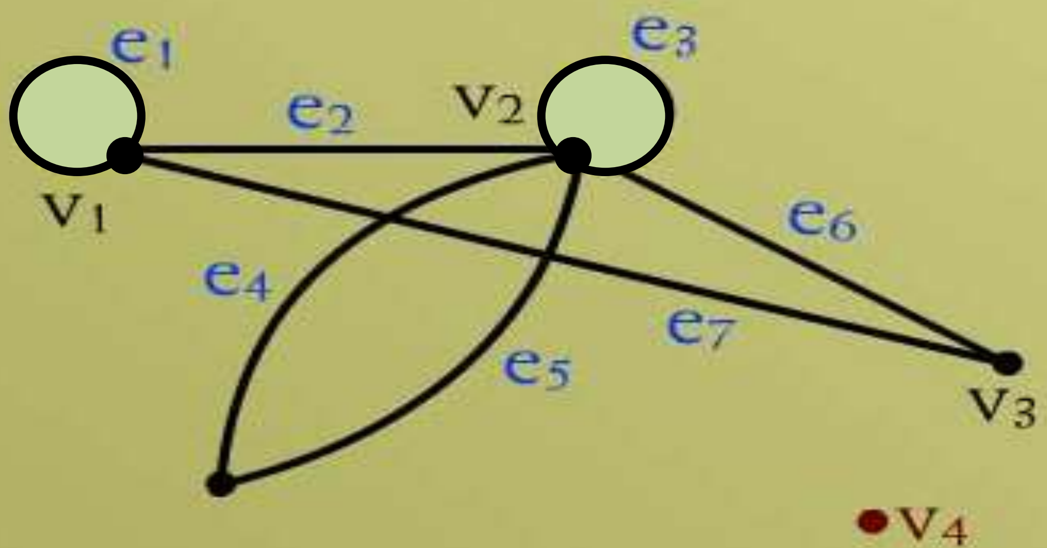
- Find all parallel edges.



Only edges e_4 and e_5 are parallel.

SOLUTION

- Find all isolated vertices.



The only isolated vertex is v_4 in this Graph.

EXAMPLE

Draw picture of Graph H having vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ and edge set $\{e_1, e_2, e_3, e_4\}$ with edge endpoint function.

Edge	Endpoint
e_1	$\{v_1\}$
e_2	$\{v_2, v_3\}$
e_3	$\{v_2, v_3\}$
e_4	$\{v_1, v_5\}$

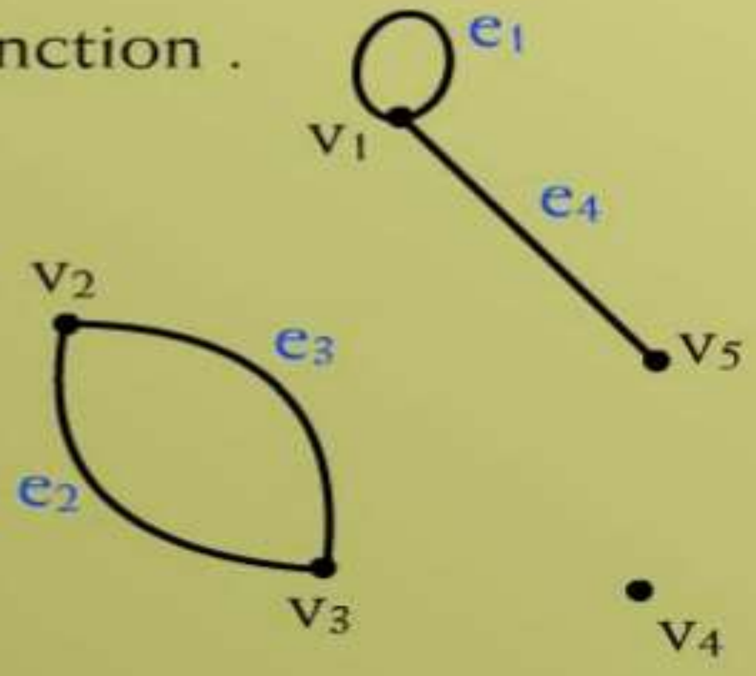
SOLUTION

$$V(H) = \{v_1, v_2, v_3, v_4, v_5\}$$

and $\quad E(H) = \{e_1, e_2, e_3, e_4\}$

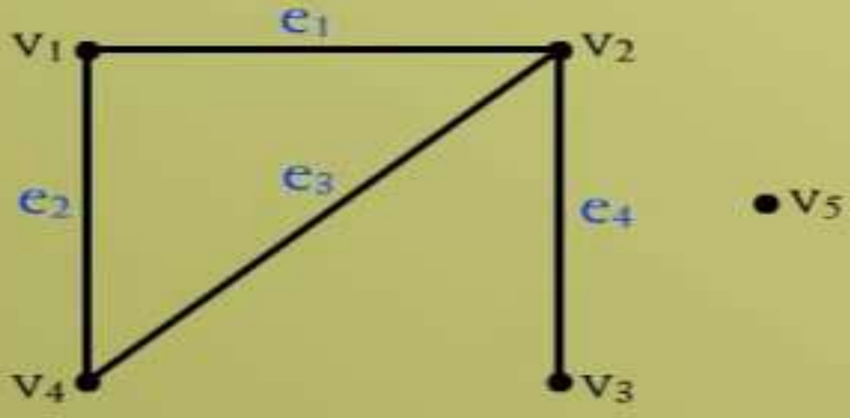
with edge endpoint function .

Edge	Endpoint
e_1	$\{v_1\}$
e_2	$\{v_2, v_3\}$
e_3	$\{v_2, v_3\}$
e_4	$\{v_1, v_5\}$



SIMPLE GRAPH

A simple graph is a graph that does not have any loop or parallel edges.



$V(H) = \{v_1, v_2, v_3, v_4, v_5\}$
$E(H) = \{e_1, e_2, e_3, e_4\}$

DEGREE OF A VERTEX

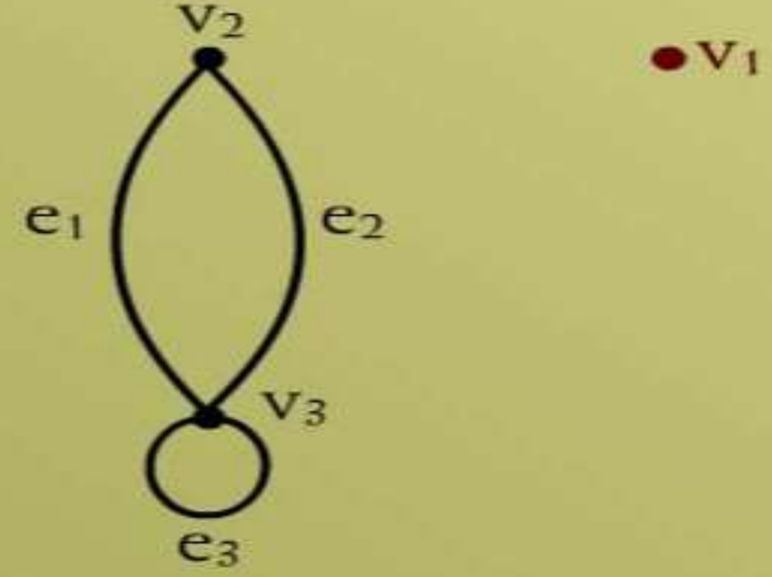
Let G be a graph and "v" a vertex of G. The degree of "v", denoted deg(v), equal the number of edges that are incident on "v", with an edge that is a loop counted twice.

The total degree of G is the sum of the degrees of all the vertices of G.

$$\deg(G) = \sum_{i=1}^{n} \deg(v_i) = \deg(v_1) + \deg(v_2) + \cdots + \deg\,(v_n)$$

EXAMPLE

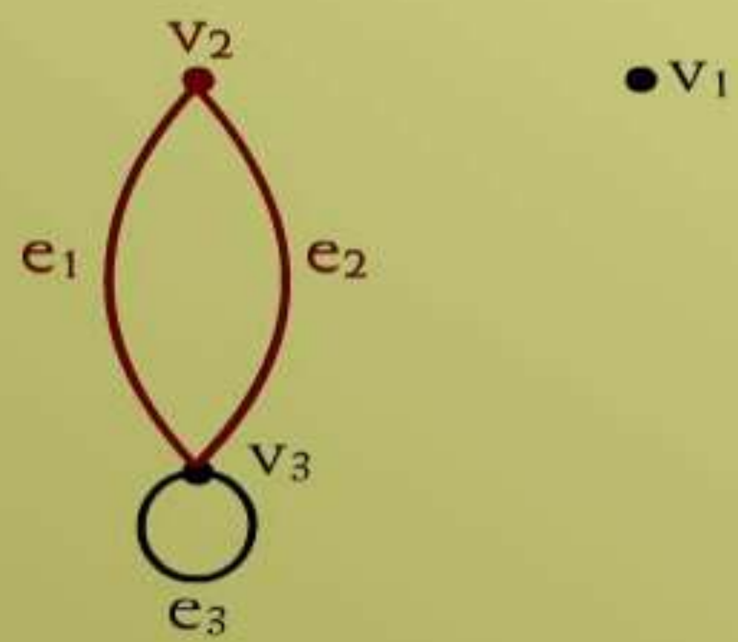
For the graph given below



deg $(v_1) = 0$, since v_1 is isolated vertex.

EXAMPLE

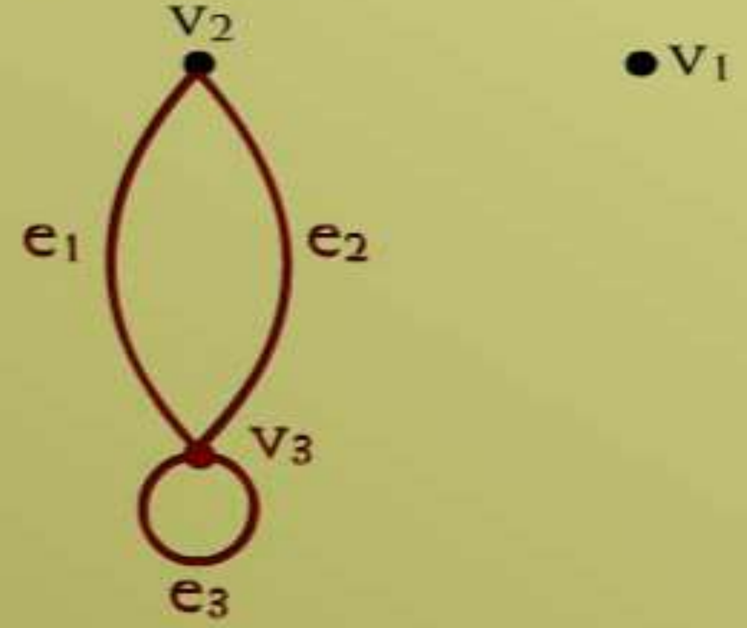
For the graph given below



$\deg(v_2) = 2$, since v_2 is incident on e_1 and e_2 .

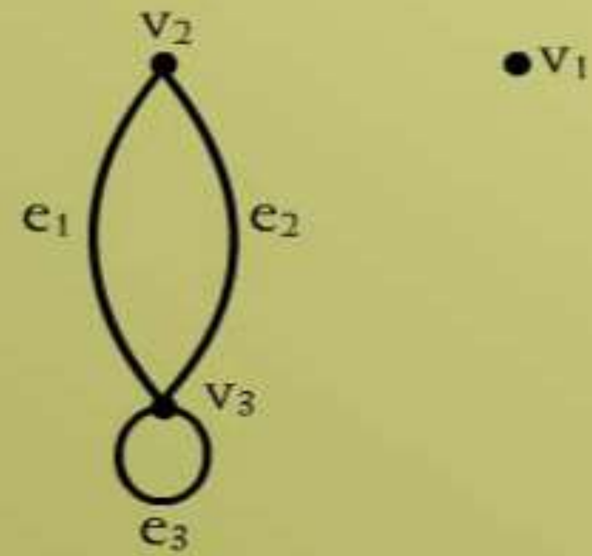
EXAMPLE

For the graph given below



deg $(v_3) = 4$, since v_3 is incident on e_1 , e_2 and the loop e_3.

EXAMPLE



$$\text{Total degree of } G = \deg(v_1) + \deg(v_2) + \deg(v_3)$$
$$= 0 + 2 + 4$$
$$= 6$$

HANDSHAKING THEOREM

If G is any graph, then the sum of the degrees of all the vertices of G equals twice the number of edges of G.

Specifically, if the vertices of G are v_1, v_2, …, v_n, where n is a positive integer, then

The Total degree of
G = deg(v1) + deg(v2) + … + deg(vn)
= 2 . (the number of edges of G)

EXAMPLE

Draw a graph with the specified properties or explain why no such graph exists.

(i) Graph with four vertices of degrees 1, 2, 3 and 3.

(ii) Graph with four vertices of degrees 1, 2, 3 and 4.

(iii) Simple graph with four vertices of degrees 1, 2, 3 and 4.

SOLUTION

(i) Graph with four vertices of degrees 1, 2, 3 and 3.

$$\text{Total degree of graph} = 1 + 2 + 3 + 3$$
$$= 9 \text{ an odd integer}$$

Hence by Hand-Shaking Theorem, first graph is not possible .

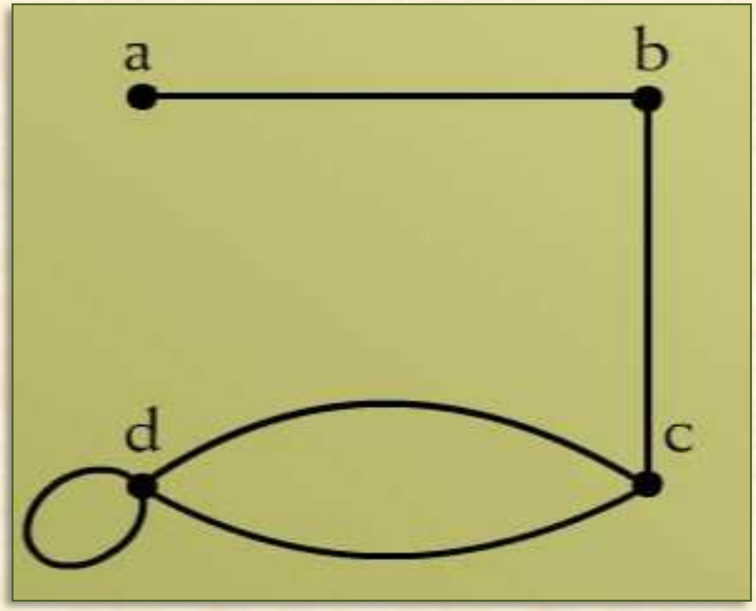
SOLUTION

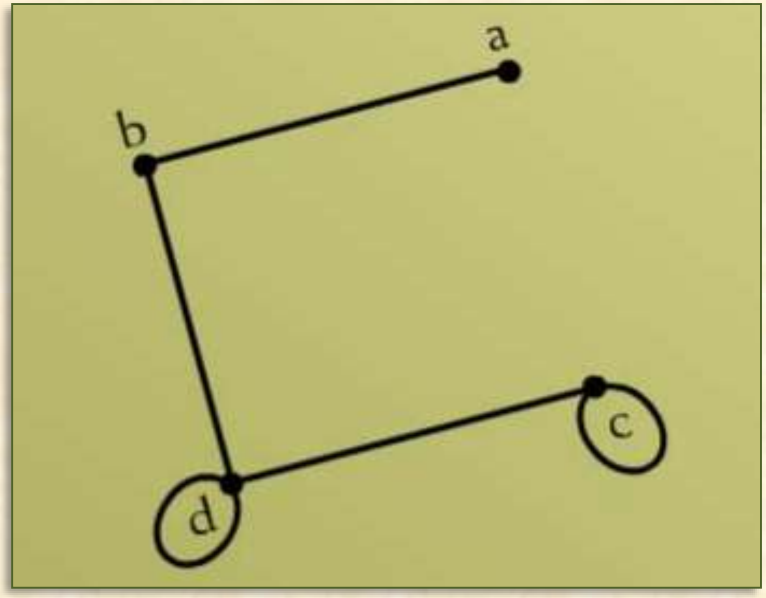
(ii) Graph with four vertices of degrees 1, 2, 3 and 4.

$$\text{Total degree of graph} = 4 + 3 + 2 + 1$$
$$= 10 \text{ an even integer}$$

There are many solutions two of them are given.

SOLUTION





deg (a) = 1 deg (b) = 2
deg (c) = 3 deg (d) = 4

deg (a) = 1 deg (b) = 2
deg (c) = 3 deg (d) = 4

EXAMPLE

Suppose a graph has vertices of degrees 1, 1, 4, 4 and 6. How many edges does the graph have ?

SOLUTION

The total degree of graph

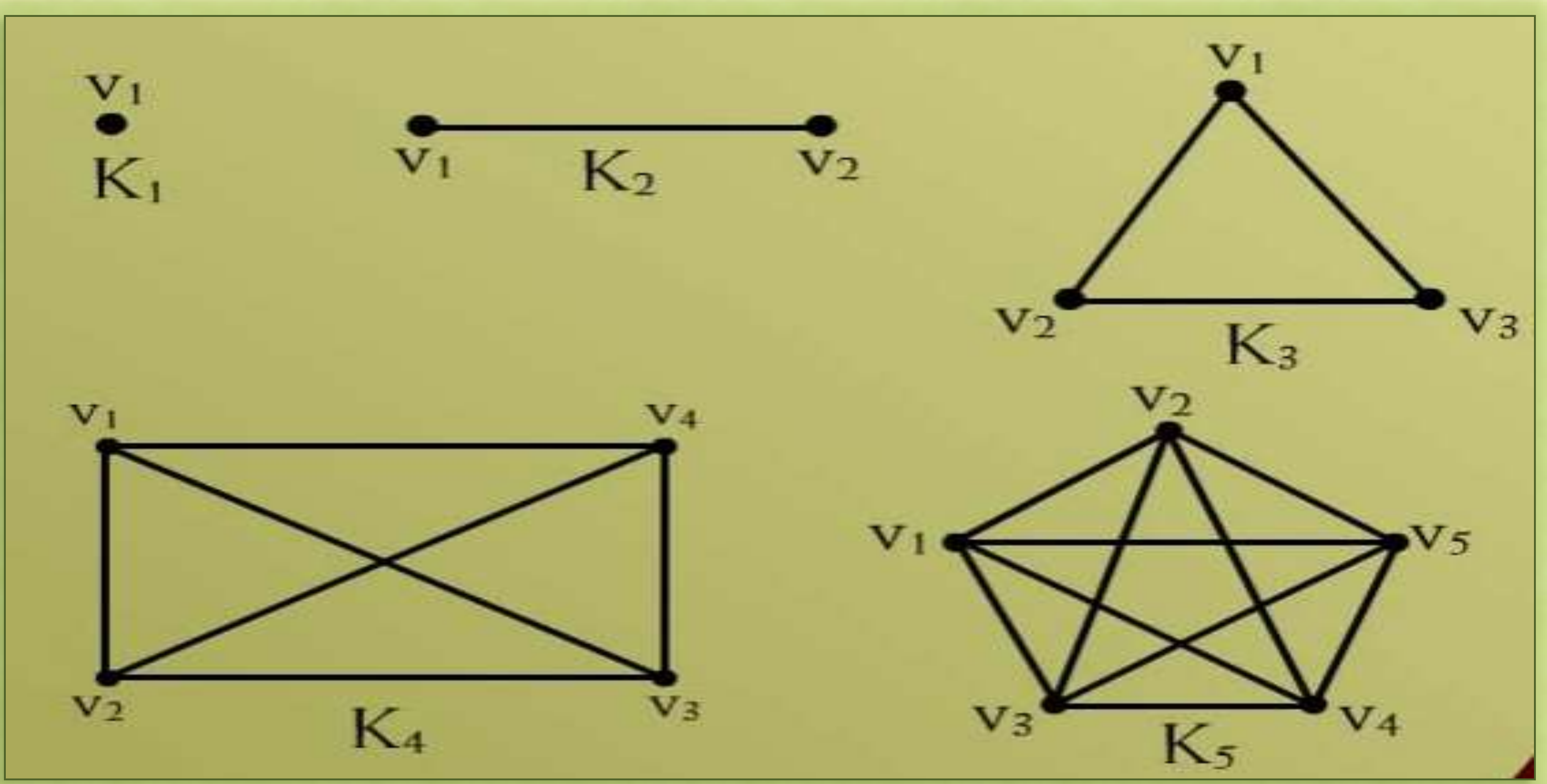
$$= 1 + 1 + 4 + 4 + 6$$
$$= 16$$

Number of edges of graph $= 16/2 = 8$

COMPLETE GRAPH

A complete graph on "n" vertices is a simple graph in which each vertex is connected to every other vertex and is denoted by K_n.

EXAMPLE



EXERCISE

For the complete graph K_n, find

(i) The degree of each vertex.

(ii) The total degrees.

(iii) The number of edges.

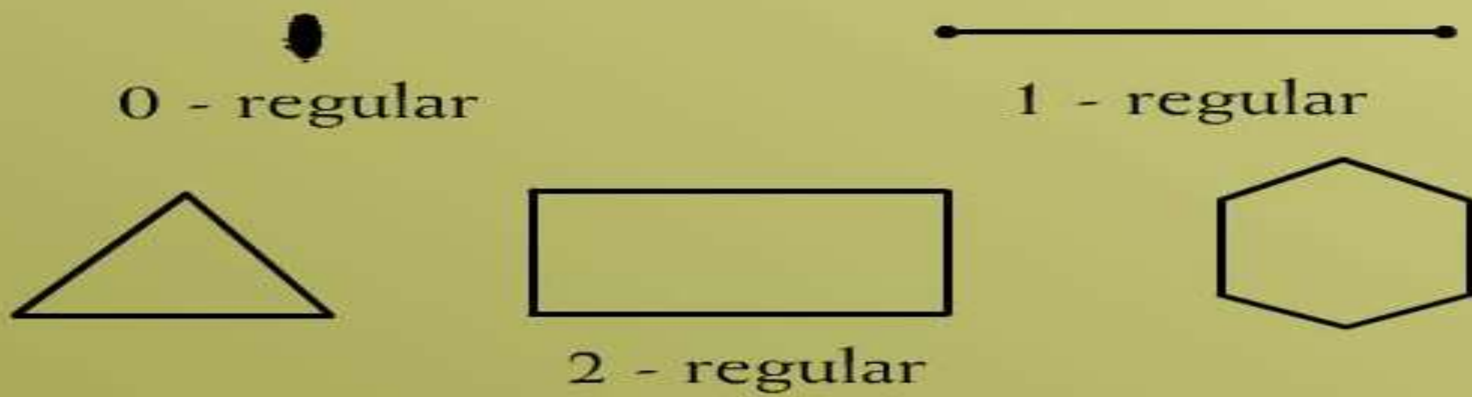
i. Degree of each vertex is n-1

ii. $\deg(K_n) = n(n-1) = 2m$

iii. No. of edges = $m = n(n-1)/2$

REGULAR GRAPH

A graph G is regular of degree k or k-regular if every vertex of G has degree k.
In other words, a graph is regular if every vertex has the same degree.



i. K_n are (n-1)-regular graphs.

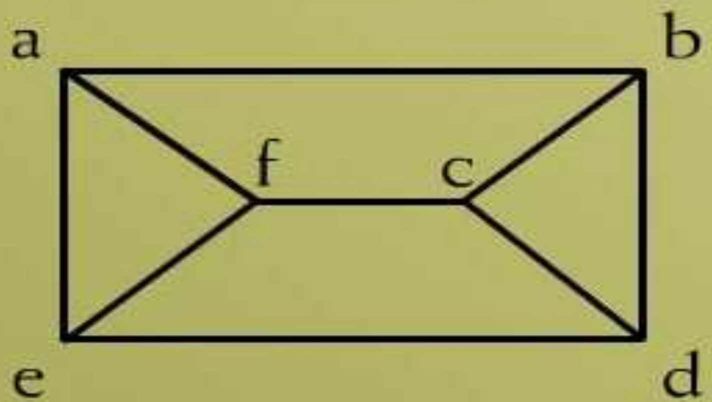
ii. Also, from the handshaking theorem, a regular graph of odd degree will contain an even number of vertices.

iii. A 3-regular graph is known as a cubic graph.

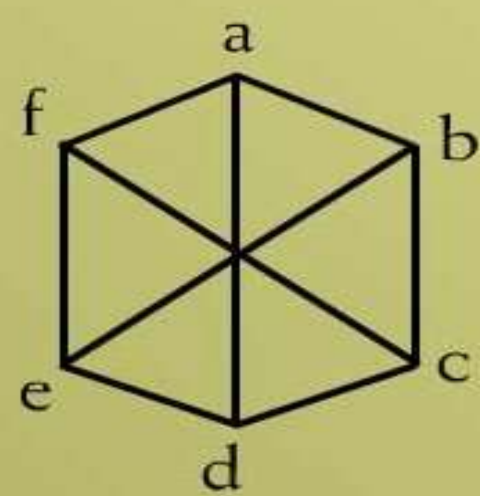
EXAMPLE

Draw two 3-regular graphs with six vertices.

SOLUTION



3-regular graph

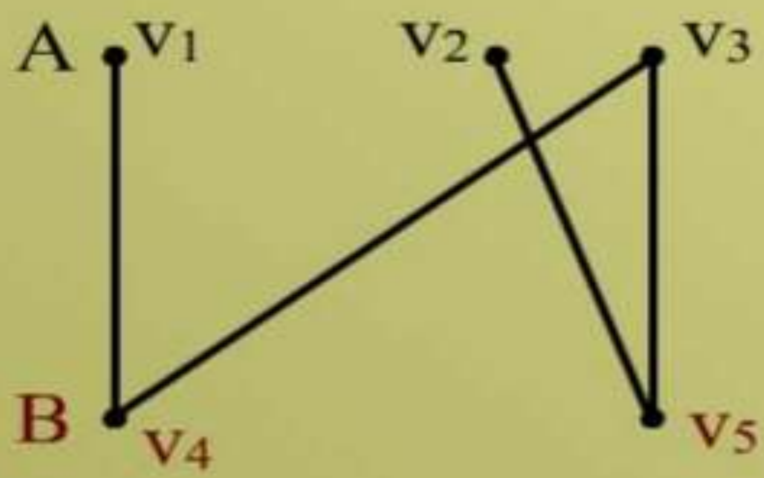


3-regular graph

BIPARTITE GRAPH

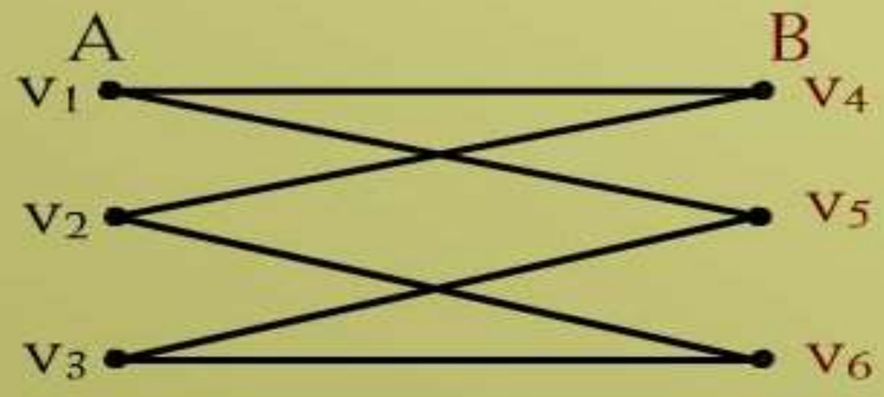
A bipartite graph G is a simple graph whose vertex set can be partitioned into two mutually disjoint non empty subsets A and B such that the vertices in A may be connected to vertices in B, but no vertices in A are connected to vertices in A and no vertices in B are connected to vertices in B.

EXAMPLE



$A = \{ v_1, v_2, v_3 \}$ $B = \{ v_4, v_5 \}$

EXAMPLE



$A = \{ v_1, v_2, v_3 \}$ $B = \{ v_4, v_5, v_6 \}$

DETERMINING BIPARTITE GRAPH

The following labeling procedure determines whether a graph is bipartite or not.

1 - Label any vertex "a".

2 - Label all vertices adjacent to "a" with the label "b".

3 - Label all vertices that are adjacent to "a" vertex just labeled "b" with label "a".

DETERMINING BIPARTITE GRAPH

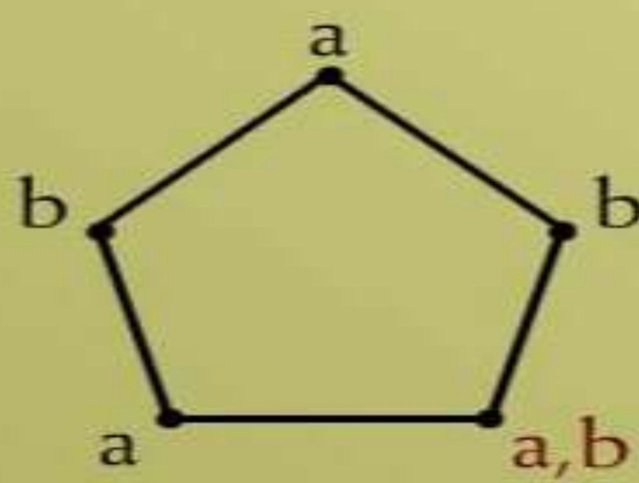
4 - Repeat steps 2 and 3 until all vertices got a distinct label (a bipartite graph).

If there is a conflict i.e., a vertex is labeled with "a" and "b" (not a bipartite graph).

EXAMPLE

Find which of the following graphs are bipartite. Redraw the bipartite graph so that its bipartite nature is evident.

SOLUTION



(conflict)

The graph is not bipartite.

SOLUTION

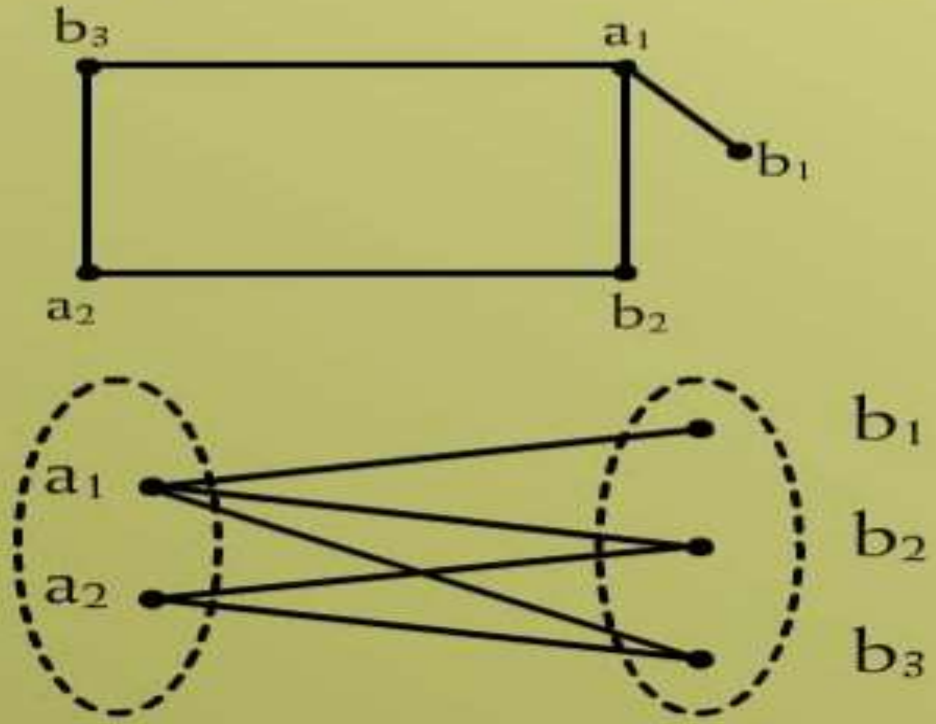
There is no conflict that is there are no adjacent vertex which have same label.

SOLUTION

$A = \{ a_1 , a_2 \}$ $B = \{ b_1, b_2, b_3 \}$

SOLUTION

$A = \{ a_1 , a_2 \}$ $B = \{ b_1, b_2, b_3 \}$

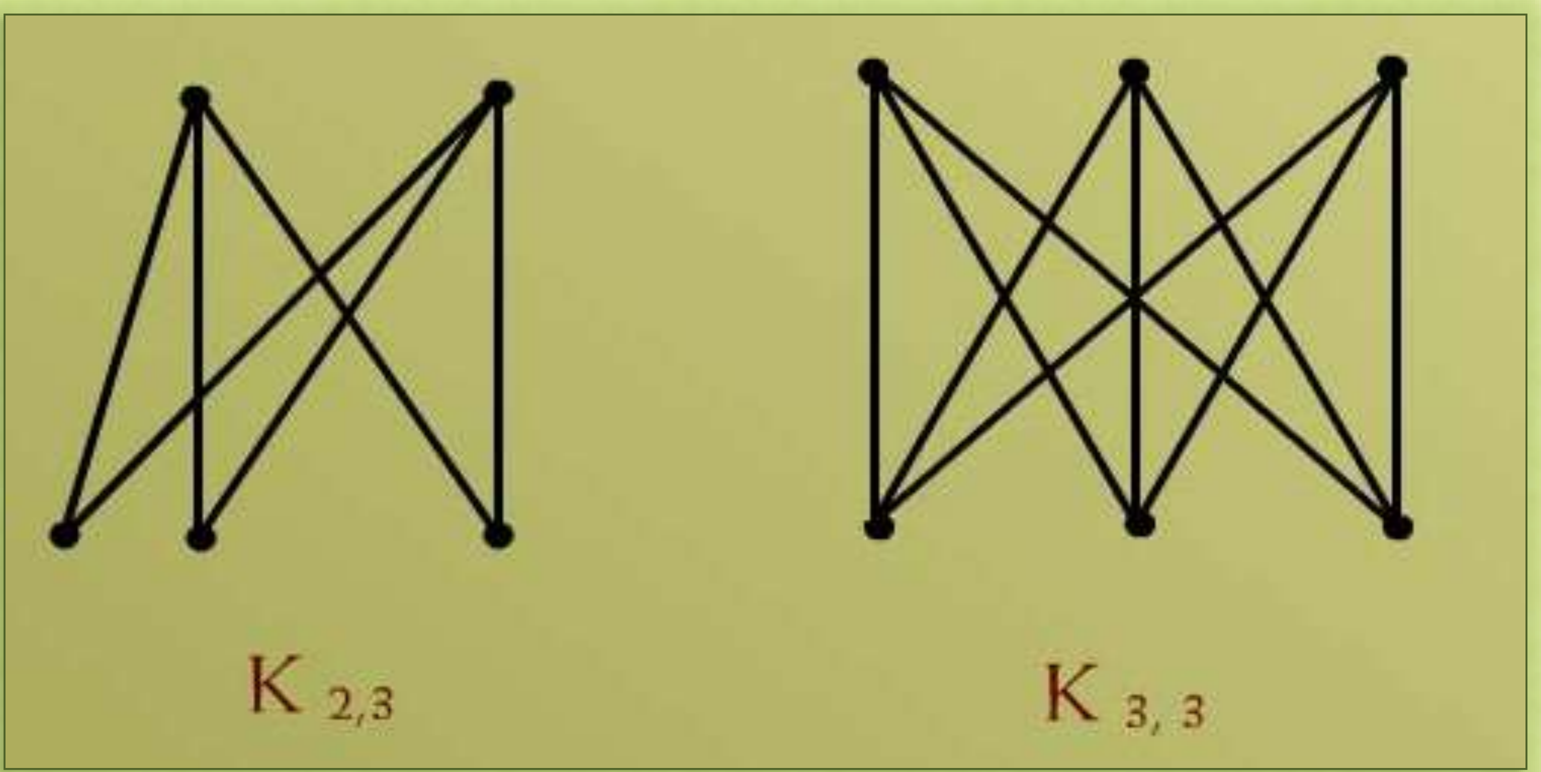


COMPLETE BIPARTITE GRAPH

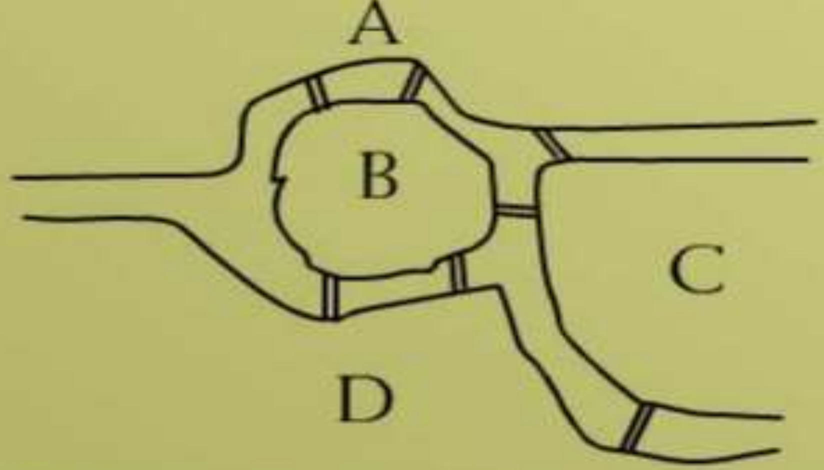
A complete bipartite graph on $(m+n)$ vertices denoted $K_{m,n}$ is a simple graph whose vertex set can be partitioned into two mutually disjoint non empty subsets A and B containing m and n vertices respectively, such that each vertex in set A is connected (adjacent) to every vertex in set B, but the vertices within a set are not connected.

No. of edges in $K_{m,n}$ is given by mn.

COMPLETE BIPARTITE GRAPH



KONIGSBERG BRIDGES PROBLEM

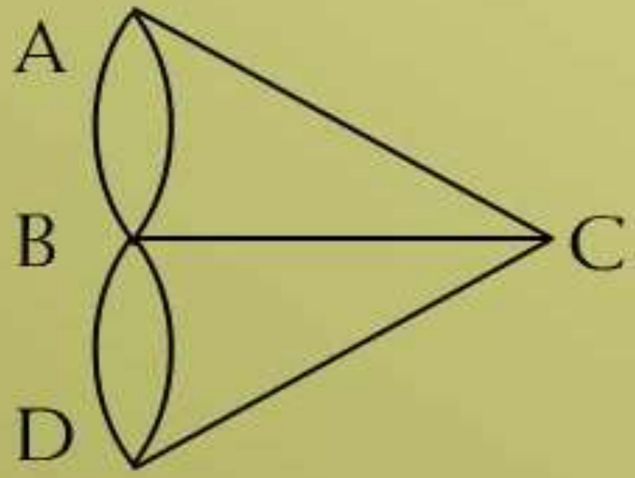


Is it possible for **a person** to take a walk around town, **starting and ending** at the **same location** and crossing each of the **seven bridges** exactly once?

SOLUTION

Is it possible to find **a route** through the **graph** that **starts and ends** at some **vertex A, B, C** or **D** and **traverses** each edge exactly once?

EQUIVALENT FORM OF BRIDGE PROBLEM



Is it possible to trace this graph, starting and ending at the same point, without ever lifting your pencil from the paper?

TERMINOLOGY

Let G be a graph and let v and w be vertices in graph G.

1. WALK

A walk from v to w is a finite alternating sequence of adjacent vertices and edges of G.

TERMINOLOGY

Thus a **walk** has the form

$$v_0e_1v_1e_2 \ldots v_{n-1}e_nv_n$$

where the **v's** represent vertices, the **e's** represent edges $v_0=v, v_n=w$, and for all $i = 1, 2 \ldots n$, v_{i-1} and v_i are **endpoints** of e_i.

The **trivial walk** from v to v consists of the **single vertex** v.

TERMINOLOGY

2. CLOSED WALK

A closed walk is a walk that starts and ends at the same vertex.

3. CIRCUIT

A circuit is a closed walk that does not contain a repeated edge.

Thus a circuit is a walk of the form

$$v_0 e_1 v_1 e_2 \ldots v_{n-1} e_n v_n$$

where $v_0 = v_n$ and all the e_i's are distinct

TERMINOLOGY

4. SIMPLE CIRCUIT

A simple circuit is a circuit that does not have any other repeated vertex except the first and last.

Thus a simple circuit is a walk of the form

$$v_0 e_1 v_1 e_2 \ldots v_{n-1}\ e_n\ v_n$$

where all the e_i's are distinct and all the v_j's are distinct except that $v_0 = v_n$

TERMINOLOGY

5. PATH

A **path** from **v** to **w** is a **walk** from **v** to **w** that does not contain a **repeated edge**.

Thus a path from **v** to **w** is a **walk** of the form

$$v = v_0 e_1 v_1 e_2 \ldots v_{n-1}\ e_n\ v_n = w$$

where all the e_i's are **distinct** (that is $e_i \neq e_k$ for any $i \neq k$).

TERMINOLOGY

6. SIMPLE PATH

A **simple path** from v to w is a path that does not contain a **repeated vertex**.

Thus a **simple path** is a **walk** of the form

$$v = v_0 e_1 v_1 e_2 \ldots v_{n-1}\, e_n\, v_n = w$$

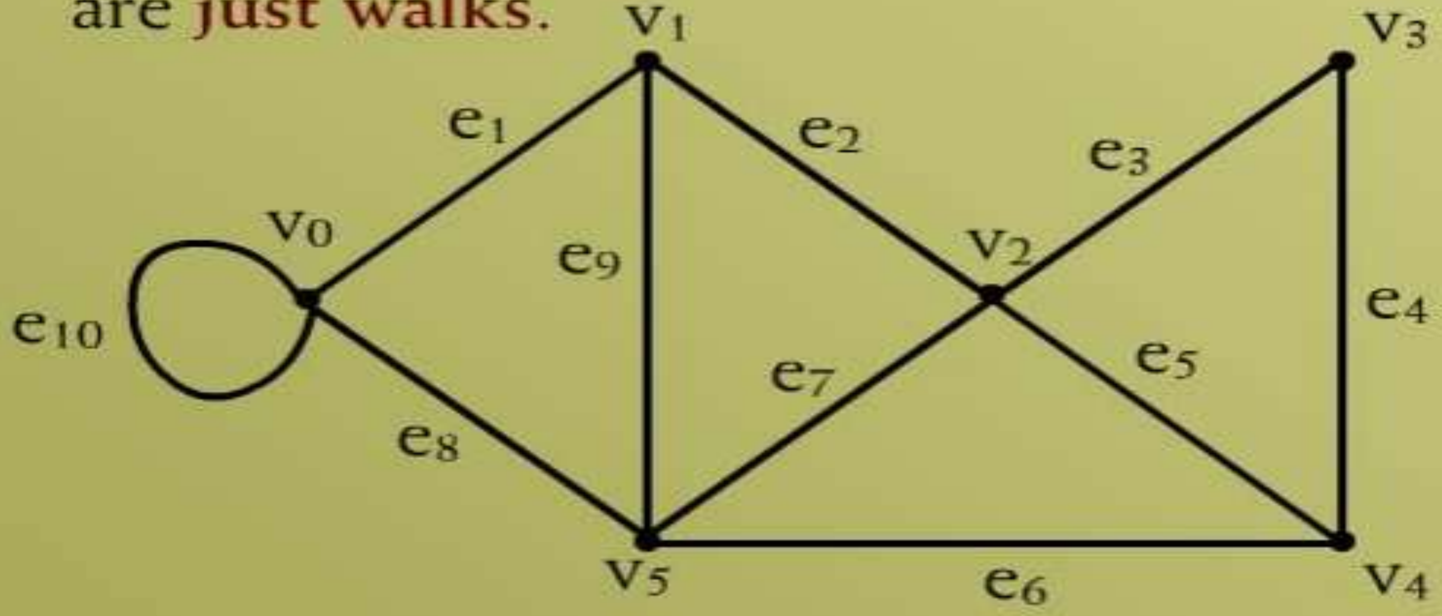
where all the e_i's are **distinct** and all the v_j's are also **distinct** (that is, $v_j \neq v_m$ for any $j \neq m$).

SUMMARY

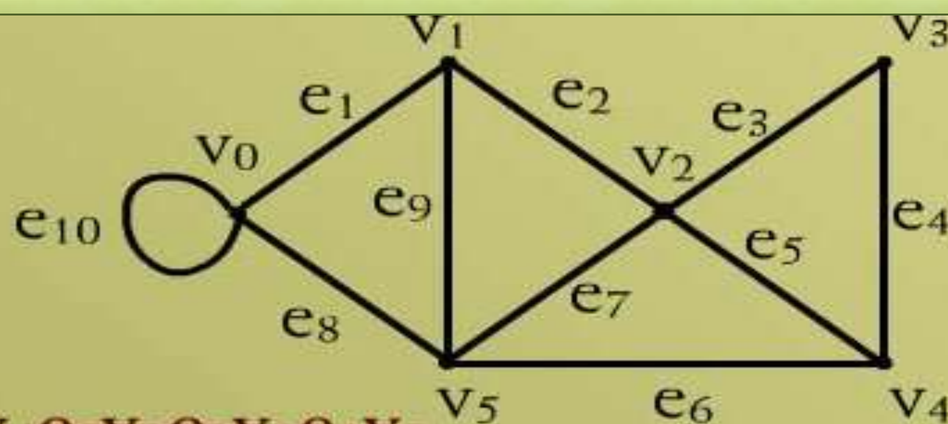
Criteria / Terms	Repeated Edge	Repeated Vertex	Starts and Ends at Same Point
walk	allowed	allowed	allowed
closed walk	allowed	allowed	yes
circuit	no	allowed	yes
simple circuit	no	first and last only	yes
path	no	allowed	no
simple path	no	no	no

PROBLEM

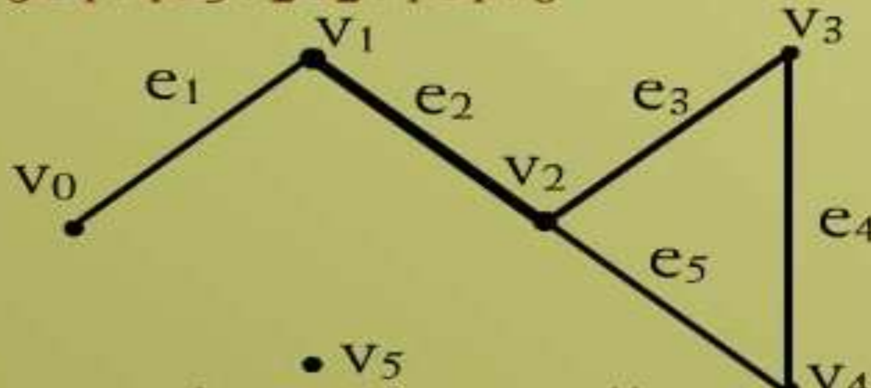
In the **graph** below, determine whether the following **walks** are **paths**, **simple paths**, **closed walks**, **circuits**, **simple circuits**, or are **just walks**.



SOLUTION

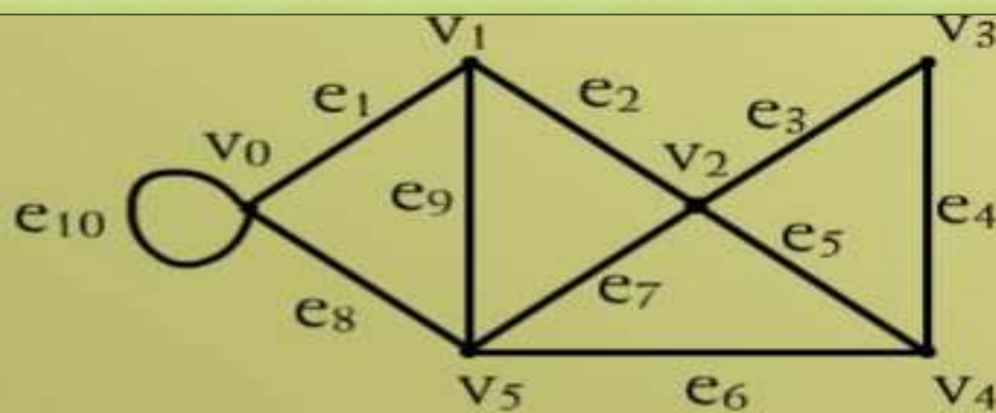


(a) $v_1 e_2 v_2 e_3 v_3 e_4 v_4 e_5 v_2 e_2 v_1 e_1 v_0$

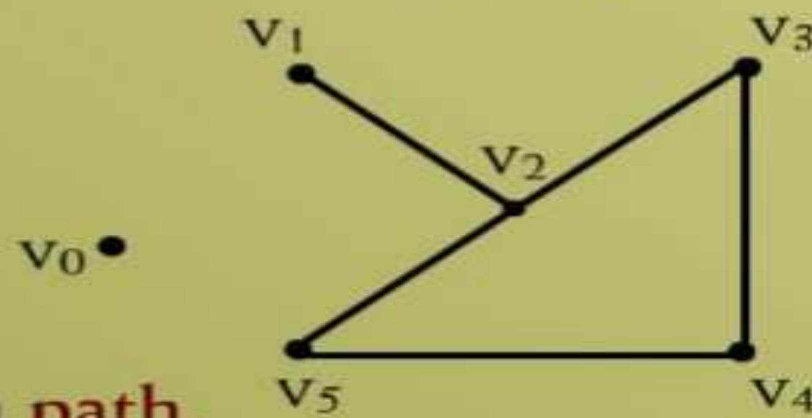


The **graph** is just a **walk**.

SOLUTION

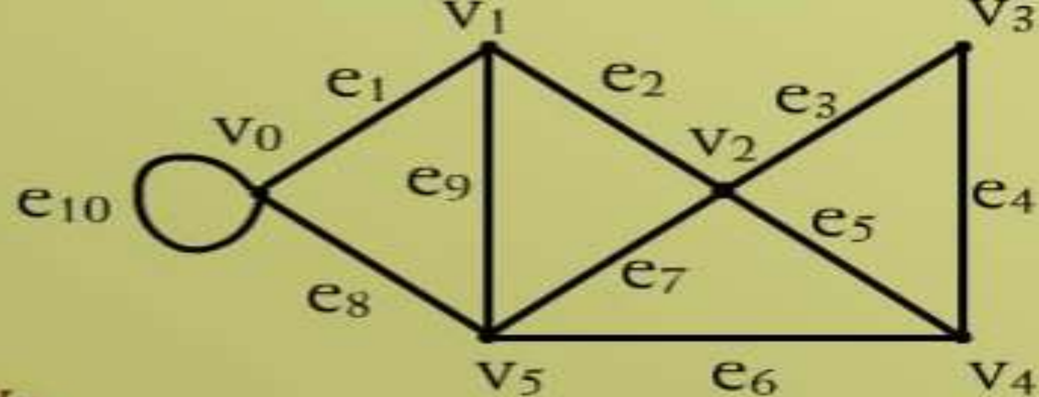


(b) $v_1v_2v_3v_4v_5v_2$

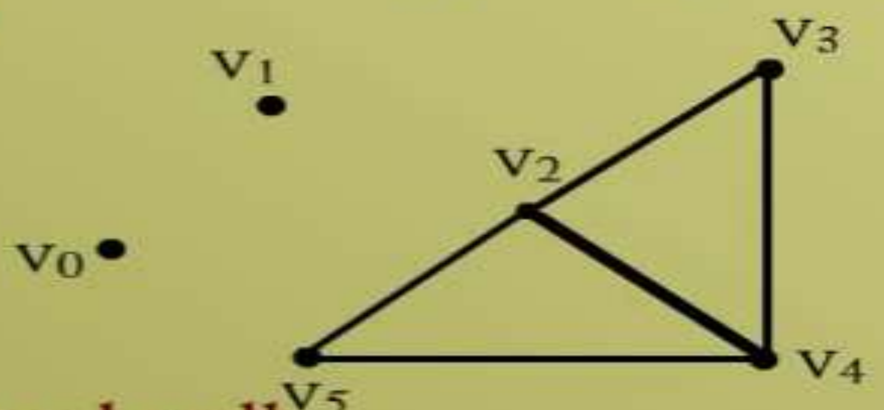


The graph is a path.

SOLUTION

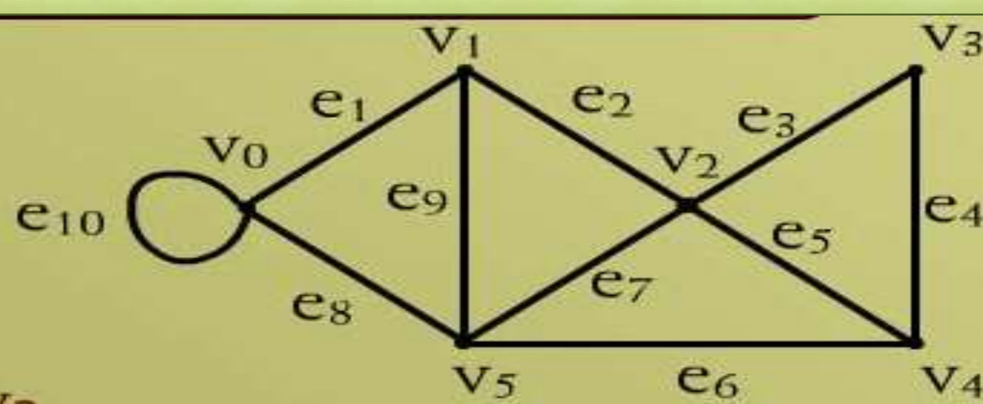


(c) $v_4v_2v_3v_4v_5v_2v_4$

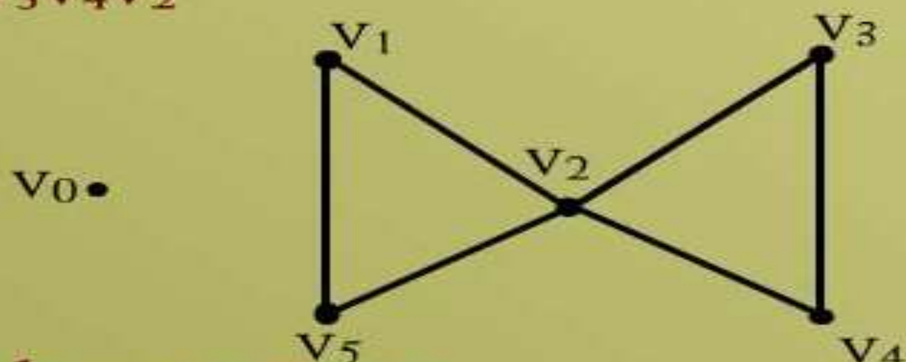


The graph is a closed walk.

SOLUTION

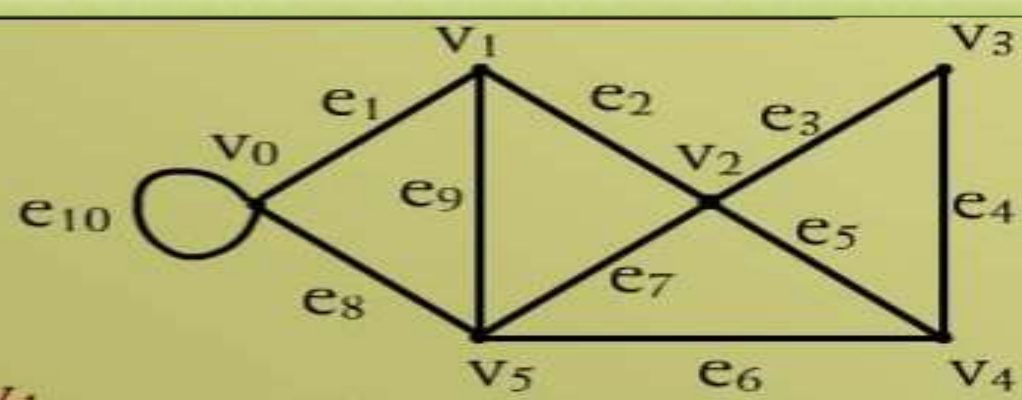


(d) $v_2v_1v_5v_2v_3v_4v_2$

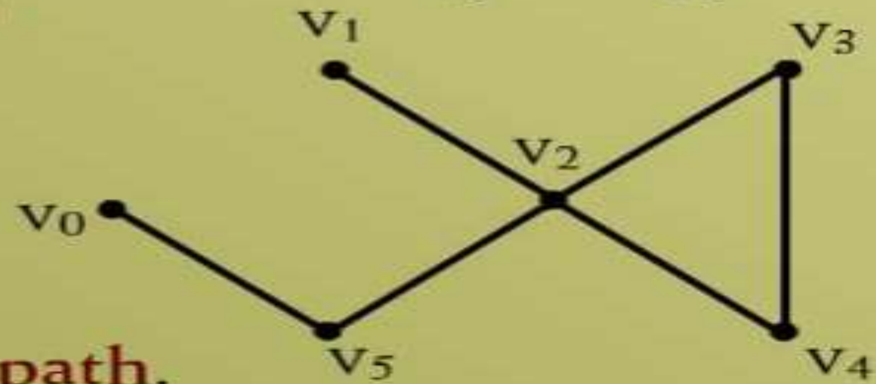


The **graph** is a **circuit**.

SOLUTION

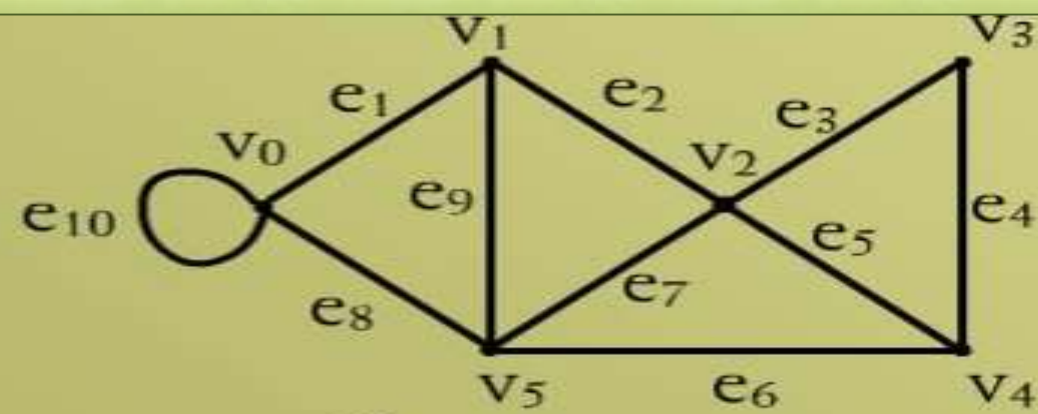


(e) $v_0v_5v_2v_3v_4v_2v_1$

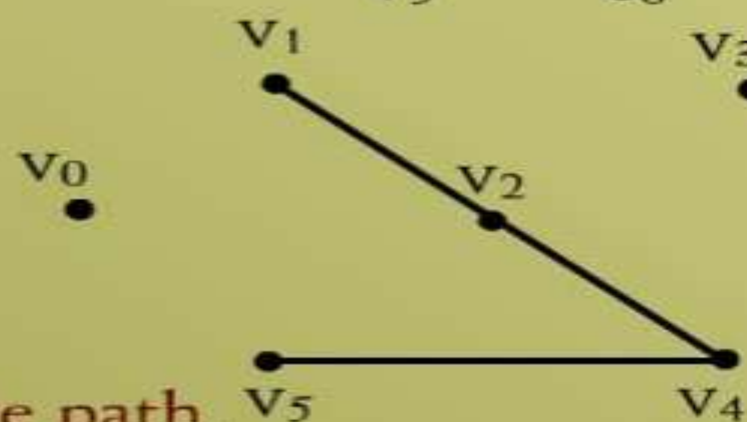


The graph is a path.

SOLUTION



(f) $v_5v_4v_2v_1$



The graph is a simple path.

CONNECTEDNESS

Let G be a graph. Two vertices v and w of G are connected if, and only if, there is a walk from v to w.

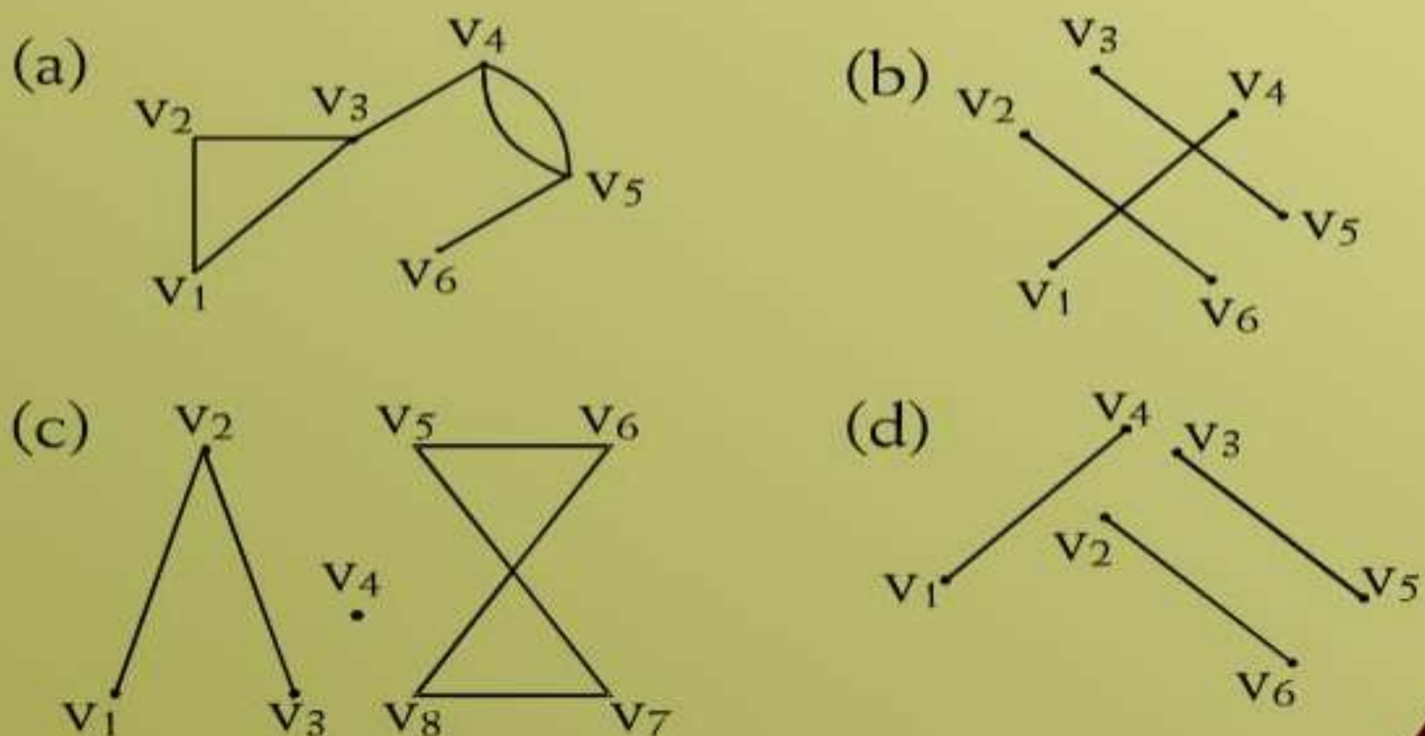
The graph G is connected if, and only if, given any two vertices v and w in G, there is a walk from v to w.

Symbolically:

G is connected $\Leftrightarrow \forall$ vertices $v, w \in V(G)$, $\exists$ a walk from v to w.

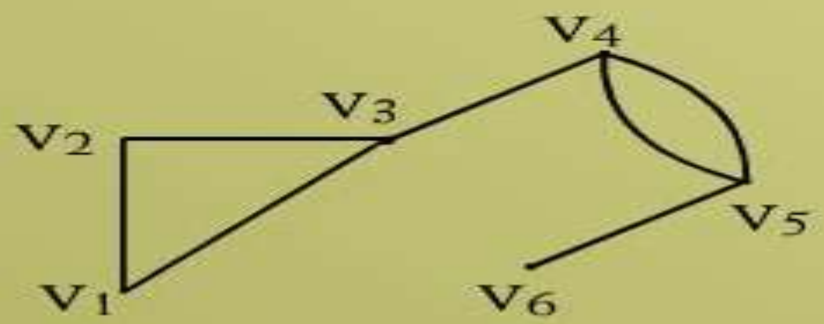
EXAMPLE

Which of the following graphs are connected?



EXAMPLE

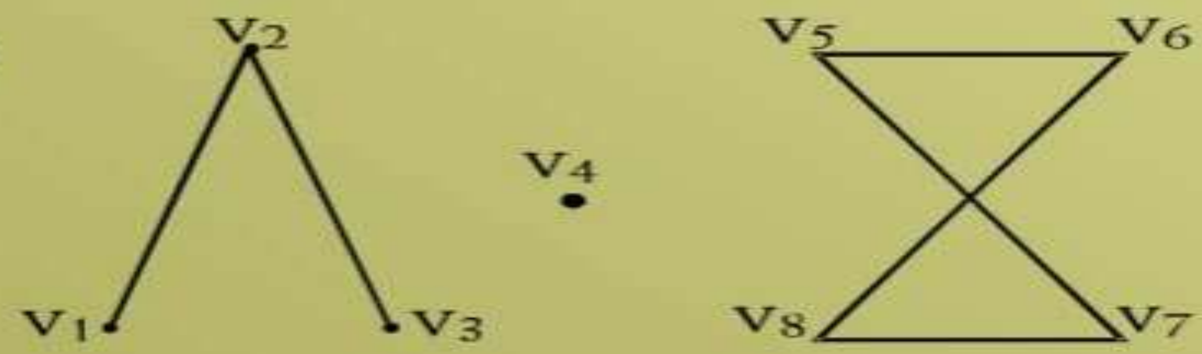
(a)



It has six vertices $v_1, v_2, \ldots, v_6$.
The graph is connected.

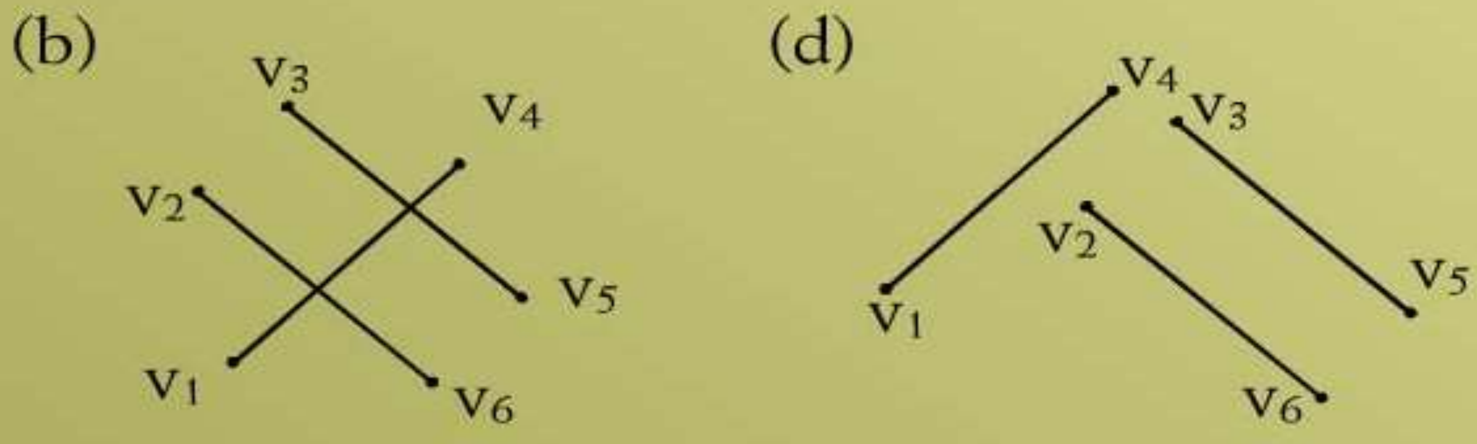
EXAMPLE

(c)



This graph is not connected.

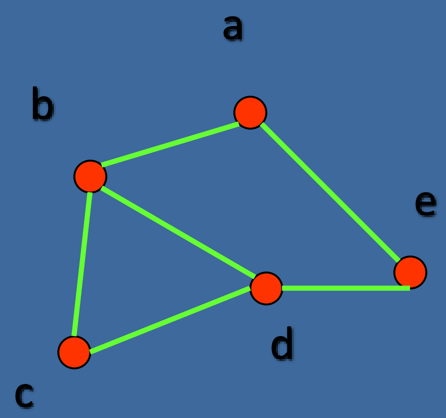
EXAMPLE



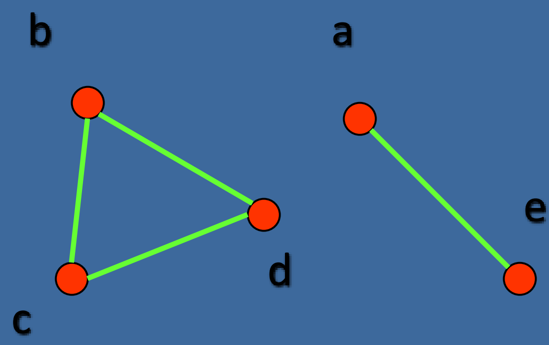
So **graph (b)** and **(d)**are not **connected**.

Connectivity

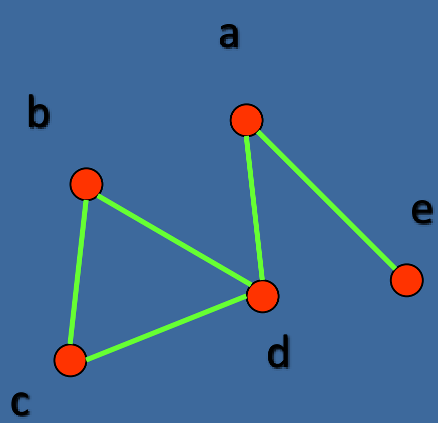
•**Example:** Are the following graphs connected?



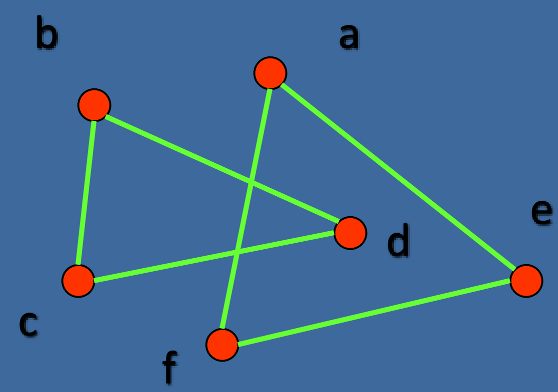
Yes.



No.



Yes.



No.

EULER CIRCUITS

Let G be a graph. An Euler circuit of G is a circuit that contains every vertex and every edge of G.

That is, an Euler circuit of G is sequence of adjacent vertices and edges in G that starts and ends at the same vertex, uses every vertex of G at least once, and uses every edge of G exactly once.

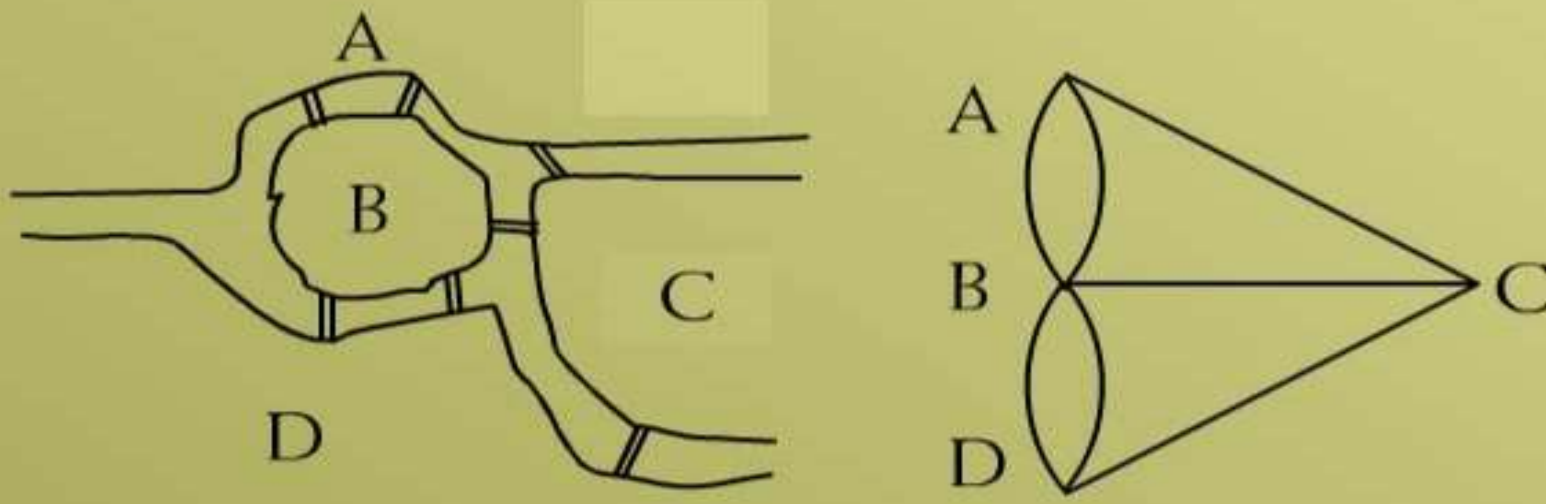
EULER RESULT

A graph G has an Euler circuit if, and only if, G is connected and every vertex of G has an even degree.

KONIGSBERG BRIDGES PROBLEM

$$\deg(a) = 3 = \deg(c)$$
$$\deg(b) = 5, \deg(d) = 3$$

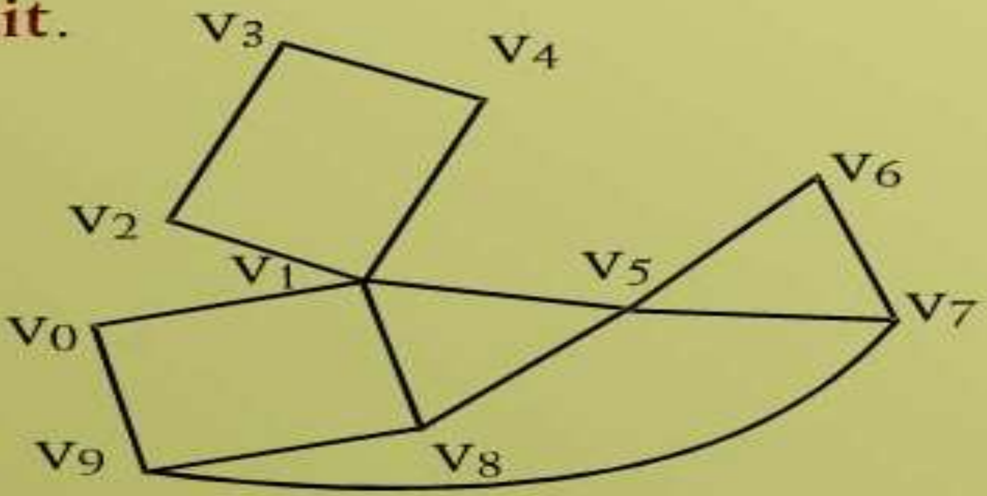
KONIGSBERG BRIDGES PROBLEM



No vertex has even degree so there is no possibility of an Euler circuit.

EXERCISE

Determine whether the following **graph** has **an Euler circuit**.

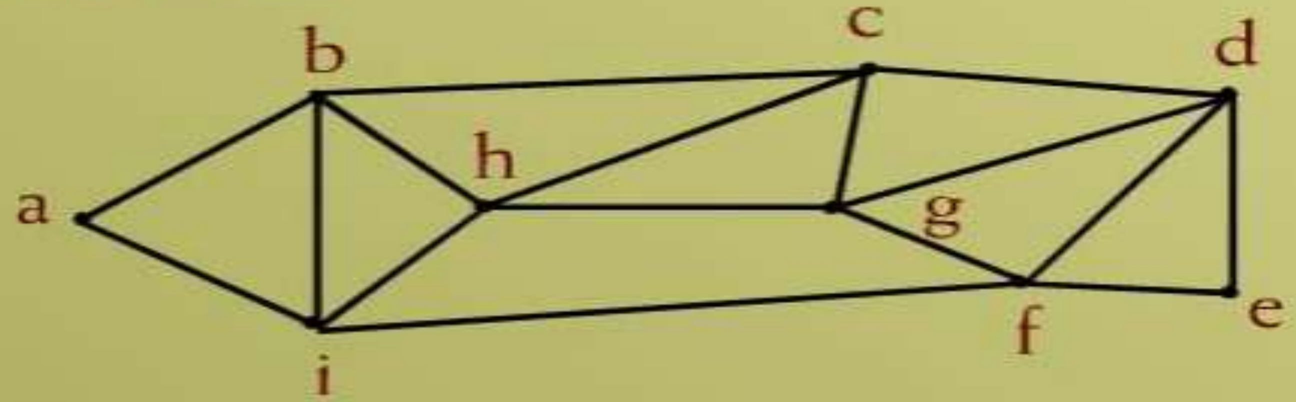


$deg(v_3)=2=deg(v_4)=deg(v_2)$, $deg(v_1)=5$

As v_1 has **odd degree** so this **graph** can't have **an Euler circuit**.

EXERCISE

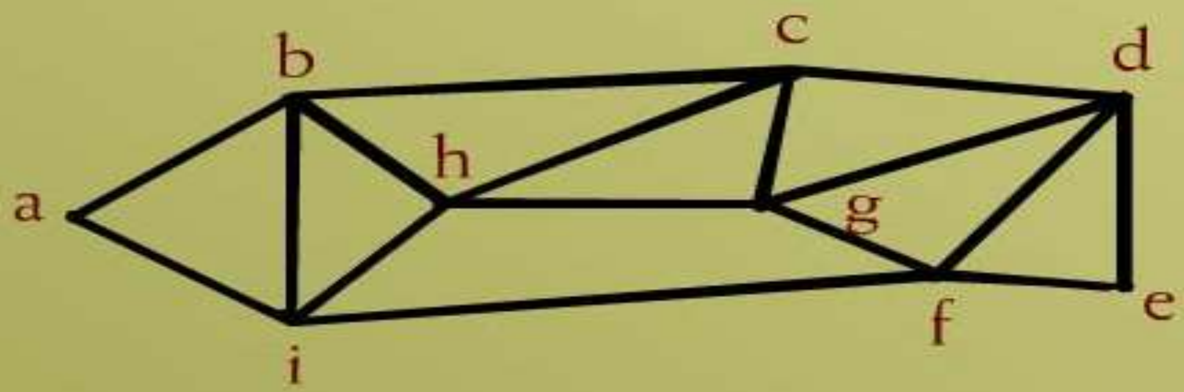
Determine whether the following **graph** has **Euler circuit**.



deg(a)=2, deg(b)=4, deg(c)=4, deg(d)=4,
deg(e)=2, deg(f)=4, deg(g)=4, deg(h)=4,
deg(i)=4

EXERCISE

So the every vertex is of even degree, clearly Euler theorem is applicable. We should be able to find Euler circuit here:



Euler circuit: {a, b, c, d, f, e, d, g, f, i, h, g, c, h, b, i, a}.

EULER PATH

Let G be a graph and let v and w be two vertices of G.

An Euler path from v to w is a sequence of adjacent edges and vertices that starts at v, end at w, passes through every vertex of G at least once, and traverses every edge of G exactly once.

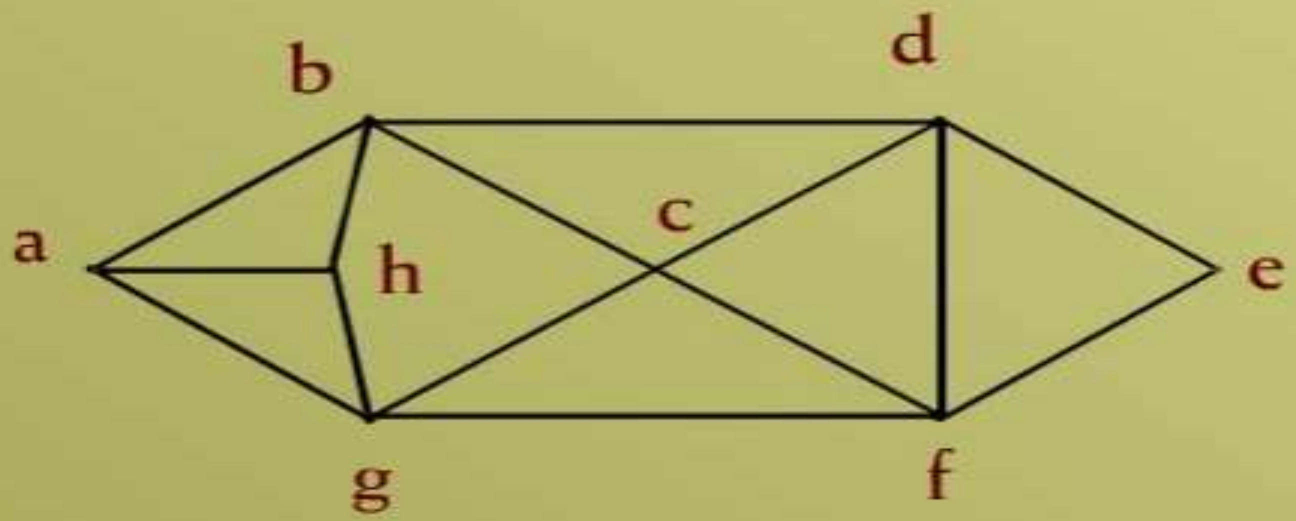
HAMILTONIAN CIRCUITS

Given a graph G, a Hamiltonian circuit for G is a simple circuit that includes every vertex of G.

That is, a Hamiltonian circuit for G is a sequence of adjacent vertices and distinct edges in which every vertex of G appears exactly once.

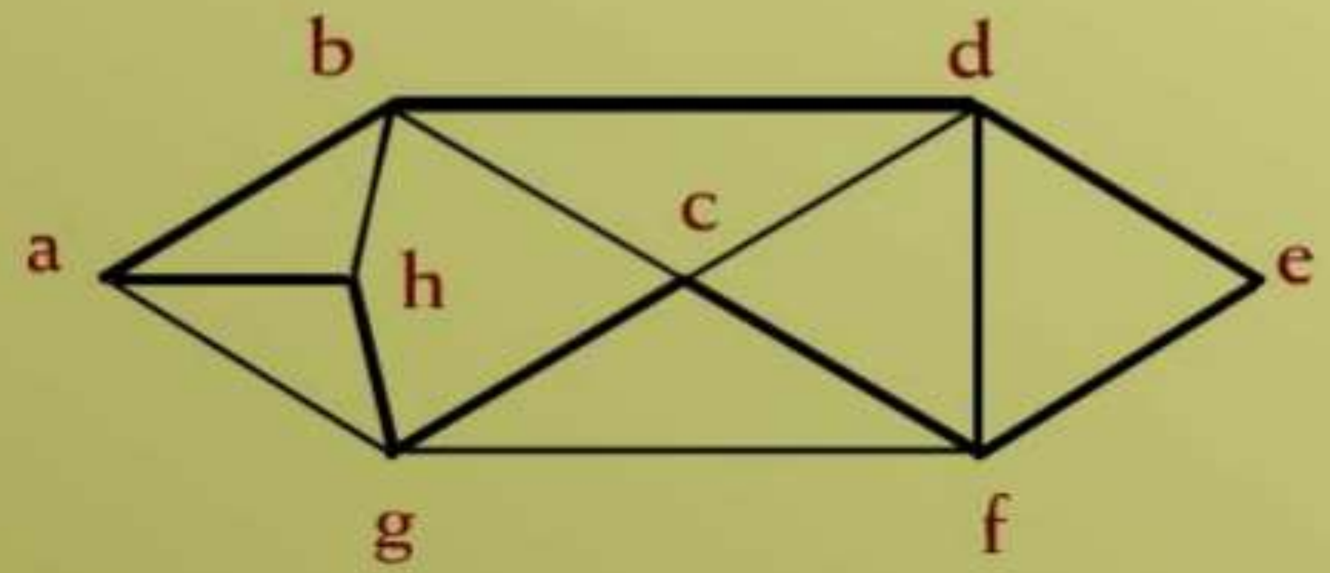
EXERCISE

Find Hamiltonian Circuit for the following graph.



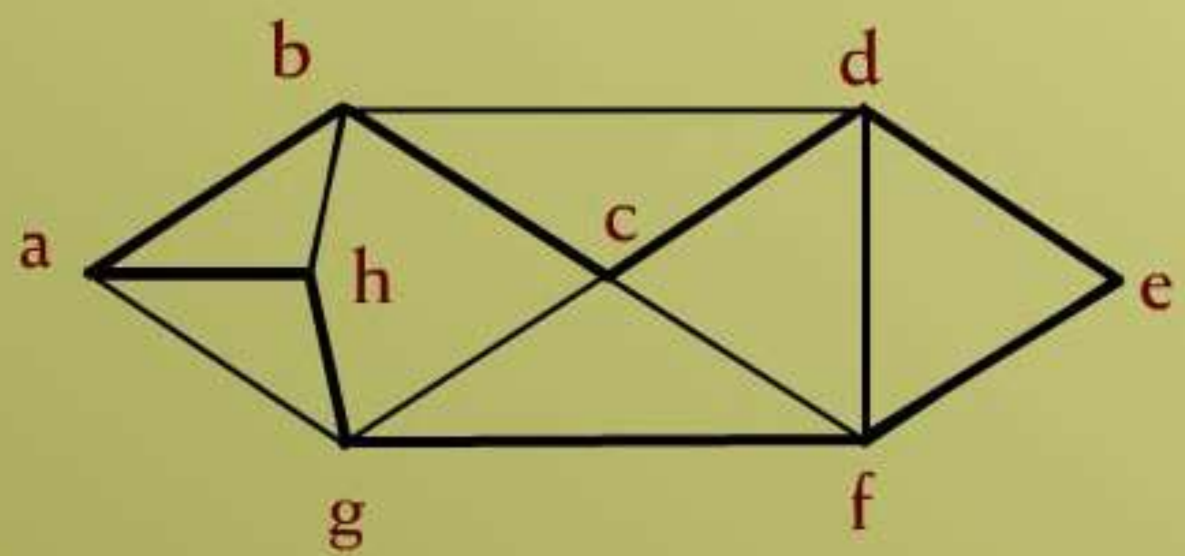
SOLUTION

The Hamiltonian Circuit is:



SOLUTION

Another Hamiltonian Circuit could be:



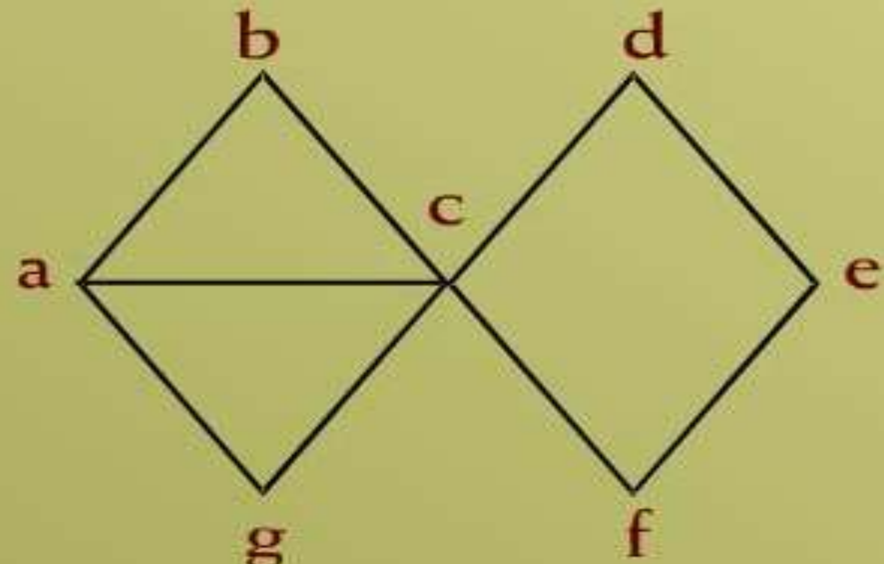
PROPERTIES

If a graph G has a Hamiltonian circuit then G has a sub-graph H with the following properties:

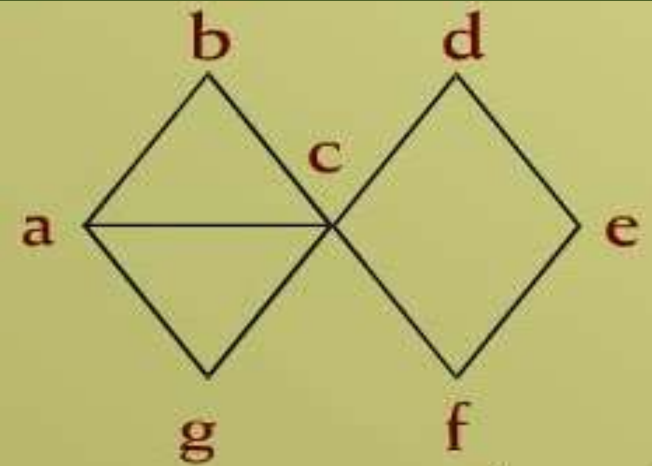
1. H contains every vertex of G.
2. H is connected.
3. H has the same number of edges as vertices.
4. Every vertex of H has degree 2.

EXAMPLE

Show that whether the Hamiltonian circuit is possible or not?



EXAMPLE



$\deg(c)=5$, if we remove 3 edges from vertex c then $\deg(a)<2, \deg(b)<2, \deg(g)<2, \deg(d)<2$

It means that this graph does not have a subgraph with the desired properties, so the Hamiltonian circuit is not possible.

Is the following graph a Hamiltonian graph? Give the explicit reason.