COMPUTER SCIENCE 20, SPRING 2014
Module #18 (Digraphs and Relations)

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Readings from Meyer

- Meyer section 9.1, 9.3, 9.4, 9.10, 9.11

Executive Summary

1. Properties of binary relations
   - *Transitive*: A binary relation R on the set A is transitive iff $uRv \wedge vRw \implies uRw$ for all $u, v, w \in A$.
   - *Reflexive*: uRu for all $u \in A$.
   - *Irreflexive*: $\neg(uRu)$ for all $u \in A$
   - *Symmetric*: $uRw \implies wRu$ for all $u, w \in A$.
   - *Antisymmetric*: $uRw \implies \neg(wRu)$ for all $u, w \in A$, $u \neq w$.
   - *Asymmetric*: $uRw \implies \neg(wRu)$ for all $u, w \in A$.
2. Recall that G is a binary relation on V, where uGw means that there is an edge from u to w.
   - G^+ is transitive and is the *transitive closure* of G. This means that G^+ is the minimal transitive relation that includes G (i.e. $G \subseteq G^+$).
   - G^* is reflexive, transitive, and the *reflexive transitive closure* of G.
3. The vertices $u, v \in V$ are *strongly connected* iff $uG^*v \wedge vG^*u$. That is, if there exists a walk from u to v and a walk back from v to u.
4. Special types of relations
   - *Strict partial orders*: transitive and irreflexive
   - *Weak partial orders*: transitive, reflexive, and antisymmetric
   - *Equivalence relations*: transitive, reflexive, and symmetric
   - A relation R is a weak partial order iff $R = D^*$ for some DAG D
   - A relation R is a equivalence relation iff R is the strongly connected relation of some digraph
5. An equivalence relation R decomposes the domain into subsets called *equivalence classes*, where aRb iff a and b are in the same equivalence class.

Check-in problem

1. Let R be a binary relation on Harvard students such that aRb iff a and b are people in the same house. Then R is (check all that apply):
   (a) Reflexive
   (b) Transitive
   (c) Symmetric
   (d) Antisymmetric

Small group problems

1. Draw a directed graph with 3 vertices A, B, C representing a relationship that is:
   (a) Reflexive
   (b) Symmetric
   (c) Antisymmetric
   (d) Transitive
2. Prove that if a relation R is transitive and irreflexive, then it is asymmetric.
3. Say that a string x overlaps a string y if there exist strings p, q, r such that $x = pq$ and $y = qr$, with $q \neq \epsilon$. For example, *abcde* overlaps *cdefg*, but does not overlap *bcd* or *cdab*. Answer each of the following questions and prove your answer.
   (a) Is the overlap relation reflexive?
   (b) Is it symmetric?
   (c) Is it transitive?
4. Determine what properties each of the following relations have. For those that are equivalence relations, briefly describe what the equivalence classes are in the relation.
   (a) The relation "shares a class with", where two people share a class if there is a class they are both enrolled in this semester.
   (b) The relation R on $\mathbb{Z}$, where aRb if b is a multiple of a.
   (c) The relation R on $\mathbb{Z} \times \mathbb{Z}$, with $(a, b)\ R\ (c, d)$ if $ad = bc$.