

COMPUTER SCIENCE E-20, SPRING 2014

Homework Problems

Pigeonhole, Proofs, Induction I

**Due Thursday, February 5, 2014 at 8:45 PM. Submit electronically at <https://canvas.harvard.edu/courses/417/assignments/1083>**

1. Gauss's tomb is in the shape of a 17-sided star. You learn that he left proofs of theorems in seven different vertices. Prove that there must exist a sequence of five consecutive vertices that contains at least three theorems.

Cryptic hint: each theorem is five pigeons, and  $35 > 34$ .

2. The gamekeeper of a wild animal park has a square of area 4.5 square miles in which he would like to place his ten leopards. These cats are highly territorial, and if two of them are within a mile or less of one another, they will fight. Show that there is no way to place the leopards without causing a fight to occur.
3. Prove by contradiction that if  $ab = n$  where  $a$ ,  $b$ , and  $n$  are nonnegative integers, then  $a$  or  $b$  (or both) must be less than or equal to  $\sqrt{n}$ .
4. Prove by contradiction that  $\sqrt{3}$  is irrational. Then explain why your proof is no longer valid if you replace 3 by an arbitrary positive integer  $n$ .
5. (a) Prove that for all nonnegative integers  $n$

$$\sum_{i=0}^n i^3 = \left( \sum_{i=0}^n i \right)^2$$

*Hint: the following identity may be useful*

$$\sum_{i=0}^n i = \frac{n(n+1)}{2}$$

- (b) An application: you receive an email from Klingon that invites you to help in diversifying the Klingon economy by making an investment in a new company that is commercializing the first Klingon-invented cryptographic algorithm. A document describing the algorithm is attached. It begins:

“Choose a large prime number  $p$  that is the sum of the cubes of seven consecutive integers.”

What is a good mathematical reason for not investing?