COMPUTER SCIENCE 20, SPRING 2014
Module #29 (Convergent and Divergent Series)

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Executive Summary

1. Common series

   (a) The geometric series: $\sum_{i=0}^{\infty} q^i = \frac{1}{1-q}$, provided $|q| < 1$.

   (b) The negative binomial series: $\sum_{i=0}^{\infty} \binom{i+r-1}{r-1} q^i = \frac{1}{(1-q)^r}$ if $|q| < 1$.

   (c) The "harmonic series" $\sum_{i=1}^{\infty} \frac{1}{i}$ is divergent, but $\sum_{i=1}^{n} \frac{1}{i^2} = \frac{\pi^2}{6}$.

   (d) The "exponential series" $\sum_{i=0}^{\infty} \frac{\alpha^i}{i!} = e^{\alpha}$.

Small group problems

1. Paul offers to let you play a game. He'll flip a fair coin until he flips tails, then pay you 2^k dollars, where k is the number of heads he flipped. For instance, the sequence $HHHT$ would earn you \$8, while a tails on the first flip would leave you with just \$1. What is the expected value of your winnings if you play this game once? How much would you be willing to pay Paul to play this game one time?

2. Simplify $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{99\cdot 100}$.

3. Prove that the "harmonic series" $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ is divergent.

   Hint: try to show that $\forall N > 0, \exists m$ such that $\sum_{i=N+1}^{m} \frac{1}{i} > \frac{1}{2}$ (why does this prove divergence?)

4. Simplify $\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{a^2}\right)\left(1 + \frac{1}{a^4}\right) \cdots \left(1 + \frac{1}{a^{2^{100}}}\right)$.

   Hint: What could you multiply the given product by on the left that would help you simplify this expression? The formula for the "difference of two squares" may help you here.

5. **Challenge:** Let S be the set $\{1, 2, 3, \ldots, 9, 11, 12, \ldots, 19, 21, \ldots\}$ consisting of all natural numbers that do not contain the digit zero. Do you think that the sum of the reciprocals of the elements of S converges or diverges? Justify your answer.