COMPUTER SCIENCE 20, SPRING 2014
Homework Problems
Recursive Definitions, Structural Induction, States and Invariants
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Note: Use the following definition of the set of valid integer binary trees T.

- **Base Cases:**

$$\epsilon \in T$$

- **Constructor Case:** If $x \in \mathbb{Z}$, $l \in T$, $r \in T$, then

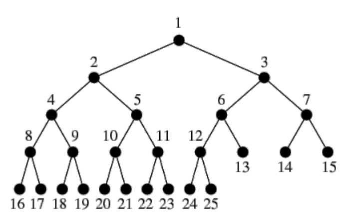
$$(x, l, r) \in T$$

- A *node* of a tree is any element in T that is a three-tuple, that is, it is of the form (x, l, r). This means that $\epsilon \in T$ is not a node.
- If $z = (x, l, r)$ is a node, then l and r are child nodes of z, if they are non-empty.

1. The nodes in a tree obey the *heap property* if, for every node z in the tree, the value in z, that is x, is at least as big as the value in each of z's children (the tree in module 15 obeys the heap property).

   Prove that if a binary tree has the heap property, then the value in the root of the tree is at least as large as the value in any node of the tree.

2. A *complete* binary tree (example below) is a binary tree in which every level, except possibly the last level, is completely filled, and all nodes are as far left as possible.



   Also, remember that an *internal node* is a node that has children.

   Prove that the number of internal nodes in a complete binary tree with n nodes is $\lfloor n/2 \rfloor$ where $\lfloor x \rfloor$ equals the largest integer that is not greater than x.

3. (From Meyer, problem 6.6) Let $m, n \in \mathbb{Z}$ where $m \neq 0$ and $n \neq 0$. Then, let's define a set of integers, $L_{m,n}$, recursively as follows:

   - **Base cases:** $m, n \in L_{m,n}$
   - **Constructor cases:** If $j, k \in L_{m,n}$, then
     (a) $-j \in L_{m,n}$. and
     (b) $j + k \in L_{m,n}$

   Let L be an abbreviation for $L_{m,n}$ for the rest of this problem.

   (a) Prove by structural induction that every common divisor of m and n also divides every member of L.
   (b) Prove that any integer multiple of an element of L is also in L.
   (c) Show that if $j, k \in L$ and $k \neq 0$ then the remainder of j divided by k is also in L; that is, that $\mathsf{rem}(j, k) \in L$.

4. (From Sipser, exercise 1.6) Draw state machines that only accept strings in the following set. Assume that the alphabet is $\Sigma = \{0, 1\}$; that is, that all strings $s \in \Sigma^*$.
   (Bonus point(s) if you draw your raw state machines in LaTeX; check out the TikZ package to do this.)

   $\{w : w$ contains the substring 0101, i.e. $w = x0101y$ for some x and $y\}$

5. (From Meyer, problem 5.29) A robot named Wall-E wanders around a two-dimensional grid. He starts at (0,0) and is allowed to take four different types of steps:

   (a) (+2,-1)
   (b) (+1, -2)
   (c) (+1, +1)
   (d) (-3, 0)

   For example, Wall-E might take the following stroll. The types of his steps are denoted by each arrow's subscript:

   $$(0, 0) \rightarrow^a (2, -1) \rightarrow^c (3, 0) \rightarrow^b (4, -2) \rightarrow^d (1, -2) \rightarrow \ldots$$

   Wall-E's true love, the fashionable and high-powered robot, Eve, awaits at (0,2).

   (a) Describe a state machine model of this problem.
   (b) Will Wall-E ever find his true love? Either find a path from Wall-E to Eve or use the Invariant Principle to prove that no such path exists.