COMPUTER SCIENCE 20, SPRING 2014

Module #28 (Random Variables and Expectation)

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Reading from Meyer

- Section 18 discusses Random Variables.

Executive Summary

1. Random Variables
   - Definition: A random variable R on a probability space is a total function whose domain is the sample space (Meyer Definition 18.1.1).
   - A random variable maps outcomes to values (they are actually functions).
   - An indicator random variable is a random variable that maps every outcome to either 0 or 1.
   - Random variables X and Y are *independent* if events $X = a_i$ and $Y = b_j$ are independent: if $P((X = a_i) \cap (Y = b_j)) = P(X = a_i)P(Y = b_j)$.
   - For independent random variables, $E(XY) = E(X)E(Y)$, which is not true in general.
2. Probability Density Function
   - Definition: Let R be a random variable with codomain V . The probability density function of R is a function $PDF_R : V \to [0, 1]$ defined by:
     $$PDF_R = \begin{cases} Pr(R = x) & : x \in range(R) \\ 0 & : x \notin range(R) \end{cases}$$
   - If the codomain is a subset of the real numbers, then the cumulative distribution function is the function $CDF_R : \mathbb{R} \to [0, 1]$ defined by:
     $$CDF_R = Pr(R \le x)$$
   - It follows that $CDF_R(x) = \sum_{y \le x} PDF_R(y)$
3. Expectation and variance

(a) The *expectation* of simple random variable X is defined as

$$E(X) = \sum_{i=1}^{n} a_i P(X = a_i).$$

If X is the payoff in a game of chance, $E(x)$ is the fair amount to pay for the right to play the game.

(b) Whether or not X and Y are independent, $E(X+Y) = E(X)+E(Y)$. This result extends by induction to any finite sum of random variables.

(c) The variance of X with expectation μ is defined as

$$\mathrm{Var}(X) = E[(X-\mu)^2], \text{ easy to calculate as } E(X^2) - \mu^2.$$

(d) If X and Y are independent, $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$.

4. Examples of Random Variables

   - If X is a "Bernoulli" random variable, for which $P(x=1) = p$, $P(x = 0) = q = 1-p$, then $E(X) = p$ and $\mathrm{Var}(X) = pq$.
   - If X is a "geometric" random variable , for which $P(x=i) = p(1-p)^{i-1}$, then $E(X) = \frac{1}{p}$ and $\mathrm{Var}(X) = \frac{1-p}{p^2}$.

Small group problems

1. You roll ten unloaded dice. Random variable Y is the total number of dots on the top faces. Calculate the expectation and variance of Y.

2. You are invited to play a game where you roll a die and collect the number of dollars equal to the number you roll. After your first roll, you have the choice of taking the payout or rolling again and taking the second payout. What is the expected payoff?

3. A couple decides to keep having children until they have at least one boy and at least one girl, and then stop. Assume they never have twins, that the "trials" are independent with probability $\frac{1}{2}$ of a boy, and that they are fertile enough to keep producing children indefinitely. What is the expected number of children?[1]

4. Randomly, k distinguishable balls are placed into n distinguishable boxes, with all possibilities equally likely. Find the expected number of empty boxes.[2]

5. What is the expected number of times a die must be rolled until the numbers 1 through 6 have all shown up at least once?

[1] Taken from Stat 110 2011 homework

[2] Taken from Stat 110 2011 homework