COMPUTER SCIENCE 20, SPRING 2014
Module #28 (Bayes Theorem & the Monty Hall problem)

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Executive Summary

1. Conditional probability
   - From last class: $P(A|B) = \frac{P(A \cap B)}{P(B)}$, provided $P(B) > 0$.
   - Bayes' Rule (version 1): a tautology, given our definition:
   $$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$
   - Bayes' Rule (version 2):
   $$P(A|B) = \frac{P(A)P(B|A)}{P(B|A)P(A) + P(B|\overline{A})P(\overline{A})}$$
2. Conditioning on a partition - useful when conditional probabilities are given
   - If events $B_1, B_2, \cdots B_n$ are disjoint, and their union is the entire sample space S, then
   $P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \cdots + P(A|B_n)P(B_n)$.
   - Useful special case: $P(A) = P(A|B)P(B) + P(A|\overline{B})(1 - P(B))$.
   - Finding the right partition is often the secret to solving a conditional probability problem.
   - Conditioning on a set of events whose union is not the entire sample space is a celebrated way to go wrong.
3. Generalized Monty Hall problems
   - Behind some of the doors are prizes; behind the others are booby prizes. All arrangements of the prizes and booby prizes are assumed equally likely.
   - The contestant chooses a door.
   - Monty Hall, who knows where the prizes are located, opens a door, different from the one chosen by the contestant, behind which he knows there is a booby prize. (If there were multiple prizes, he could choose to reveal a prize).

- Monty Hall then invites the contestant to switch his choice to one of the unopened doors. The question is whether making the switch increases the contestant's chance of winning a prize.
- Conditioning on the event $C =$ "there is a prize behind the contestant's original door" usually makes these problems trivial.

Small group problems

1. Wisdom of Solomon

   In ancient Jerusalem, true prophets tell the truth 90% of the time, while false prophets tell the truth half the time. Solomon's servant has brought him a group of three prophets, two true, one false. Each of the prophets knows which is which, but Solomon does not. Solomon asks prophet 1, "Is prophet 2 a true prophet?"

   (a) If the answer is "No," what is the conditional probability that prophet 3 is a true prophet?

   (b) If the answer is "Yes," what is the conditional probability that prophet 2 is a true prophet?

   Hint for part a.

   As events, use $A_3 =$ "prophet 3 is the false prophet" and $B =$ "Solomon receives the answer 'No."' When calculating $P(B)$, partition the sample space into A_3 and the other two equally probable events
   $A_2 =$"prophet 2 is the false prophet" and
   $A_1 =$"prophet 1 is the false prophet."
   Once you know $P(A_3|B)$, the problem is solved.

2. An ambitious preschool, whose aim is to prepare toddlers for Harvard, has 8 students, 4 boys and 4 girls. Following the example of Harvard, it divides its students randomly into two groups of 4, which it names "Lowell House" and "Eliot House."

   (a) What is the probability P_4 that all four boys end up in Eliot House?

   (b) What is the probability P_3 that three boys and one girl end up in Eliot House, with the other boy and three girls in Lowell House?

   (c) What is the probability P_2 that optimal diversity is achieved, with two boys in Eliot House and two in Lowell House?

   (d) Verify that the sum of probabilities for all the ways of dividing the class between the houses is correct.

(e) A newly hired teacher meets one student, chosen at random, from Eliot House. The student is a boy. Calculate the conditional probabilities, given this event, that there are 4, 3, 2, 1, or 0 boys in Eliot House. (These conditional probabilities should also sum to 1.)

(f) Next the teacher meets a randomly chosen student from Lowell House, who turns out to be a girl. Calculate the conditional probabilities, given both this event and the previous one, that there are 4, 3, 2, 1, or 0 boys in Eliot House.

3. In the admissions office at Monty Hall University there are four interviewers. Three of them are friendly, while the fourth is unfriendly. Every morning the Dean of Admissions assigns them randomly to offices 1, 2, 3, and 4, with an equal probability for each possible assignment. A student arrives for an interview and is asked to select which office he wants to be interviewed in. He chooses office 1 and learns that the interviewer in there is busy for the next half hour. "While you are waiting," says the Dean to the student, "I would like you to meet one of our friendly interviewers. He opens an office door (but not office 1) and introduces the student to a friendly interviewer. He then continues, "Rather than waiting, would you prefer to be interviewed by someone in one of the other offices?"

   If the student accapts this offer, does his probability of getting a friendly interviewer increase or decrease?

4. After the sinking of the Titanic, many unidentified bodies were brought to Halifax for burial. The policy used was that anyone who was wearing a crucifix or holding a rosary was buried in the Catholic cemetery, while everyone else was buried in the non-Catholic cemetery.

   Assume the following:

   - 20% of unidentified Titanic victims were Catholic.
   - The conditional probability that a Catholic on a sinking ship will have a crucifix or rosary is 90%.
   - The conditional probability that a non-Catholic on a sinking ship will have a crucifix or rosary is 15%.

   (a) Calculate the conditional probability that someone buried under a Titanic grave marker in the Catholic cemetery is in fact Catholic.

   (b) Calculate the conditional probability that someone buried under a Titanic grave marker in the non-Catholic cemetery is a Catholic.