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Homework Problems
Author: Tawheed Abdul-Raheem
Sets, Relations, Functions, and Countability

1. Suppose that $A \subseteq B \subseteq C \subseteq D \subseteq E$ and $A \neq C$, $B \neq D$, $C \neq E$. Two of the four statements $A \neq B$, $B \neq C$, $C \neq D$, $D \neq E$ must be true, but there are three acceptable combinations of two. Show which possibilities are valid and why (ex. argue that the truth of $X \neq Y$ and $W \neq Z$ is compatible with the facts for three pairs of the form ($X \neq Y$, $W \neq Z$)).

   The three acceptable combinations of two are ($A \neq B$ and $C \neq D$) this is true because $B = C$ and $D = E$, the second pair that is acceptable is ($B \neq C$ and $C \neq D$) this is true because $A = B$ and $D = E$, the third acceptablecombination is ($B \neq C$ and $D \neq E$), this is because $A = B$ and $C = D$.

   The other three combinations are not acceptable because they happen to contradict our statement. The pairs are ($A \neq B$ and $B \neq C$) this means that $C = D$ and $D = E$ which happens to contradict the statement $C \neq E$. Secondly ($A \neq B$ and $D \neq E$, this combination will mean that $C = D$ and $B = C$ which happens to contradict the statement $D \neq B$. Finally the pairs that are not acceptable are ($C \neq D$ and $D \neq E$) this because, for that to be true will mean that $A = B$ and $B = C$ which happens to contradict the statement $A \neq C$

2. Consider $f : A \to B$, where A and B are finite sets with the same cardinality and f is a total function. Prove that f is surjective if and only if it is injective and that in this case there exists an inverse function $g : B \to A$ such that $g \circ f$ is the identity function.

   To prove that if f is surjective then it is injective, we know that A and B are finite sets with the same cardinality, this means that we have the same number of arrows pointing out of A as we have elements in B. For this relationship to be injective it must be that all the elements in B has at most one arrow that maps to it. If we were to have two arrows pointing to one element in B, this will mean that the relationship is not injective. Also having two arrows point to one item in B will mean that there is at least one element that does not have an arrow pointing to it. By definition for a relationship to be surjective all the elements in the codomain must have at least one arrow coming to it. Also to prove that if f is injective then it is surjective, by definition of injection all the elements in the codomain must have at most one arrow mapping to it. If we have two arrows mapping onto one item in our codomain this will mean that it is not injective. Because we know that both sets have the same cardinality and also per our definition of injection this implies that the relationship is also surjective. Finally due to our one-to-one correspondence (bijection), we can say for $g \circ f$ has an identity because we can't have a correspondence from g to f that has two arrows.

3. Show that for any finite set S, $|\mathcal{P}(S)| = 2^{|S|}$. (*Hint: use induction on* $|S|$.)

   Base case: S = 0
   $\mathcal{P}(S) = \{\emptyset\}$
   $|\mathcal{P}(S)| = 1$
   $2^0 = 1$
   Inductive hypothesis: For any set of n elements $|s| = n$, $n \geq 0$
   $|\mathcal{P}(S)| = 2^{|S|} = 2^n$
   $|S| = n + 1$
   We need to show that:
   $|\mathcal{P}(S)| = 2^{|n+1|} = 2^{|n|} \cdot 2^1 = 2^{|S|} \cdot 2^1$
   Inductive Step: We have a bag of n presents from this bag we can create all possible different little bags. Now lets say that we added a new present into the bag, we take all the little old bags, we make duplicate of every one of them and then add our new present to it. In relation to the big bag we still have our old little bags and their duplicate + the new present. We have 2^n old little bags (by the inductive hypothesis) and the same amount of new little bags. That is in total
   $2^n + 2^n$
   $2^n \cdot 2^1$
   2^{n+1}

4. Show that if A and B are countable, then $A \times B$ is countable.

   To show that if A and B are countable, then $A \times B$, there are three cases that we are intrested in exploring for the verification that it is indeed true. First is if both sets are finite, by definition we know that sets A and B are finite sets then the cross product of the two sets are also finite therefore it must be countable.

   The second case is when one of the set is finite and the other is infinite, to prove that this is countable we pair each infite element with our finite set, this association is possible because we know the end of our finite set and also know its cardinality. An example would be to take the first infinite element $\times$ the elements in our finite set. We dont move to the next infinite number until we exhaust all the possible numbers. See below for image.

The final case that we need to examine is when both sets are infinite, proving the countablility of an infinite set is a little bit harder than the finite set because it does not have an end and also we wont know when to move to the next pair. The approach we can use is to not try to exhaust all the possible cross products on the first pass through.

5. Show that the set of all functions $f : \{0, 1\} \to \mathbb{N}$ is countable. (*Hint: show that there is a bijection between the set of such functions and the set* $\mathbb{N} \times \mathbb{N}$*.*)

   .... Once we prove that there is a bijection between the sets we can map it in reverse order from $\mathbb{N}$ to our domain domain which is $f : \{0, 1\}$ and then apply the diagonal rule for mapping our sets. This would gurantee us that each set is going to get covered