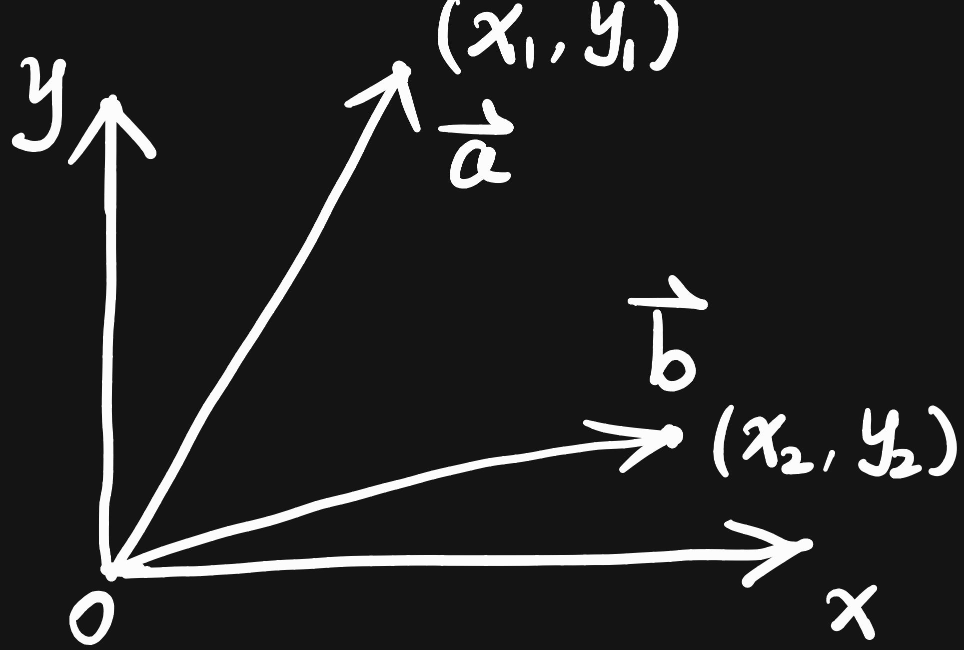
Vector 向量

$$\vec{a} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$$\vec{a}^T = \begin{bmatrix} x_1 & y_1 \end{bmatrix} \quad \vec{b}^T = \begin{bmatrix} x_2 & y_2 \end{bmatrix}$$



$$\vec{a}^T \cdot \vec{b} = \begin{bmatrix} x_1 & y_1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = x_1 x_2 + y_1 y_2$$

$$\vec{b}^T \cdot \vec{a} = \begin{bmatrix} x_2 & y_2 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = x_1 x_2 + y_1 y_2$$

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} = x_1 x_2 + y_1 y_2$$

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} = \vec{a}^T \cdot \vec{b} = \vec{b}^T \vec{a}$$

Inner Product

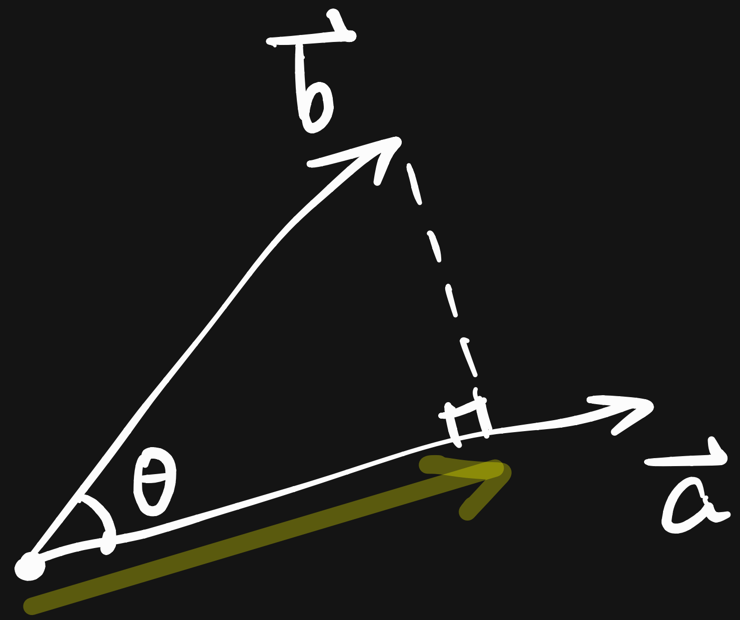
内积

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} = \vec{a}^T \vec{b} = \vec{b}^T \vec{a}$$

$$= a_1 b_1 + \cdots + a_n b_n = \sum_{i=1}^{n} a_i b_i$$

Projection 投影 & Perpendicular/Orthogonal 垂直



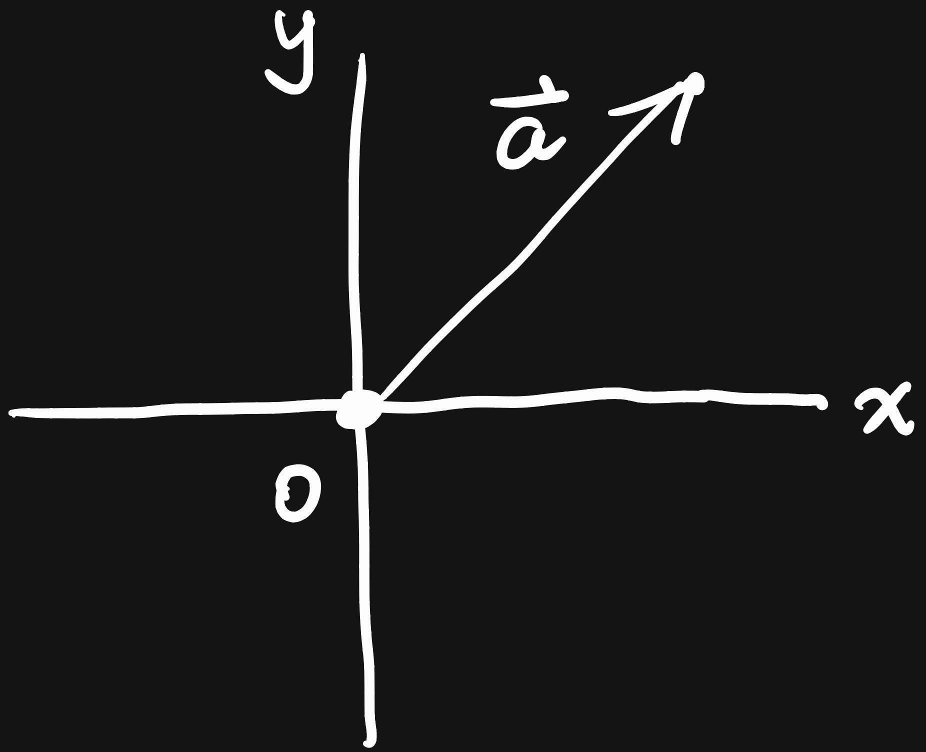
$$\vec{a}\cdot\vec{b} = \|\vec{a}\|\cdot\|\vec{b}\|\cdot\cos\langle\vec{a},\vec{b}\rangle$$

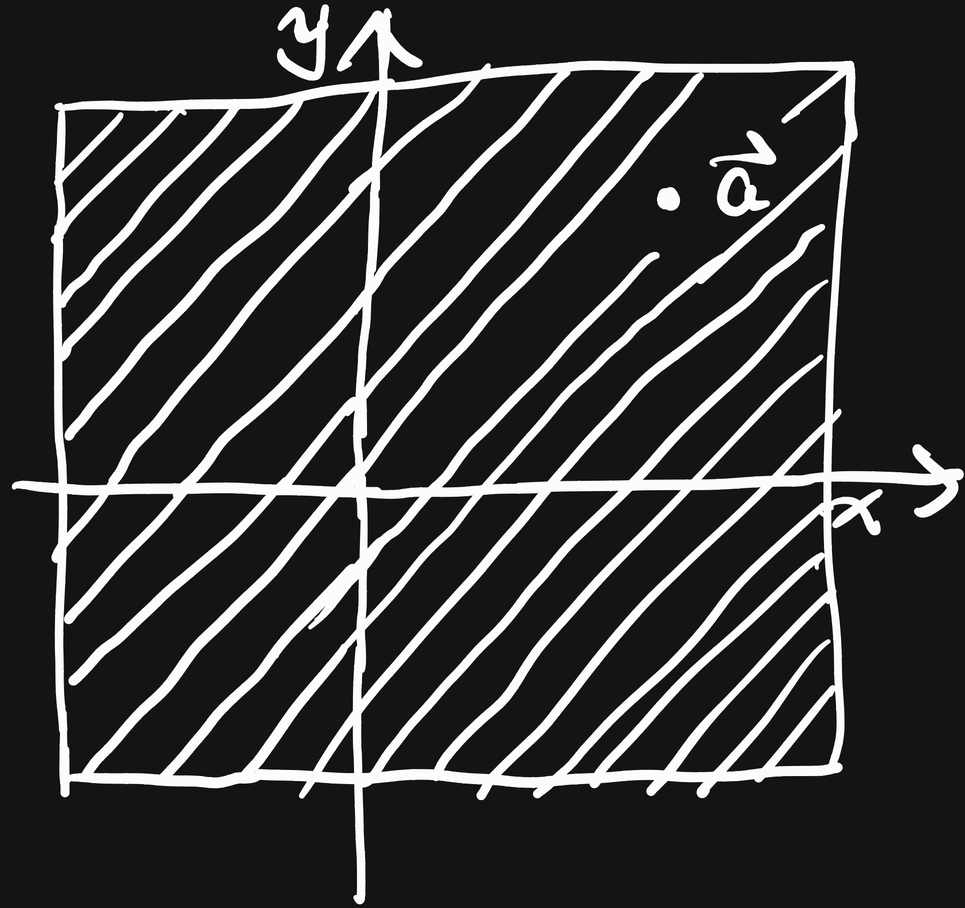
$$= \|\vec{a}\|\cdot\left(\|\vec{b}\|\cdot\cos\theta\right)$$

$$\theta = \langle\vec{a},\vec{b}\rangle$$

If $\vec{a}\cdot\vec{b} = 0$, $\|\vec{a}\| > 0$, $\|\vec{b}\| > 0$

then $\cos\theta = 0$ $\quad \theta = 90^\circ \Leftrightarrow \vec{a} \perp \vec{b}$





$\vec{a} \in \mathbb{R}^2$ $\quad \left\{ \vec{a} = \begin{bmatrix} x \\ y \end{bmatrix}, \forall\, x \in \mathbb{R}, y \in \mathbb{R} \right\} \subset \mathbb{R}^2$

same idea for n dimensional space $\mathbb{R}^n$

Matrix 矩阵

$$A_{3\times4} = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_{3\times 4} \cdot \vec{x}_{4\times 1} = \vec{b}_{3\times 1}$$

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

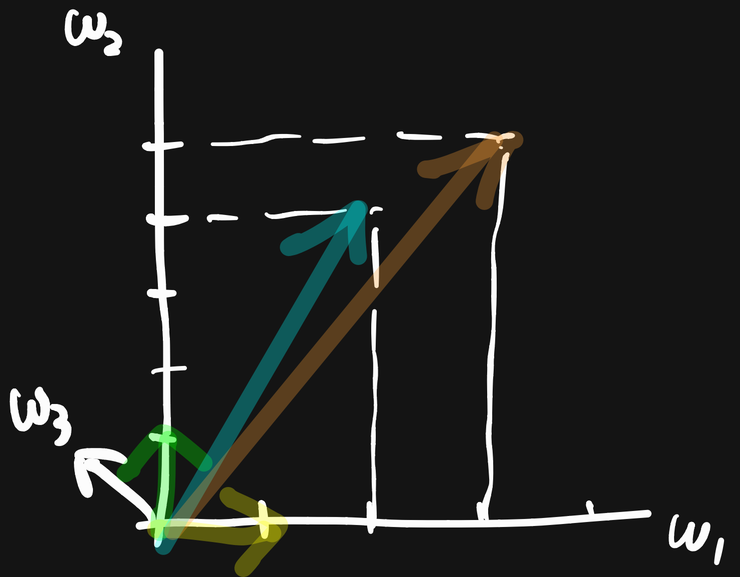
$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot x_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot x_2 + \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} \cdot x_3 + \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix} \cdot x_4$$

$$f_1(\vec{x}_{4\times 1}) = \vec{b}_{3\times 1} \qquad f_1 : \mathbb{R}^4 \rightarrow \mathbb{R}^3$$

$$x_i \in \mathbb{R},\ i \in \{1, 2, 3, 4\}$$

Linear span of colum vectors

线性生成空间　　　　列向量



$$\begin{bmatrix}1&0&2&3\\0&1&4&5\\0&0&0&0\end{bmatrix}\cdot x$$

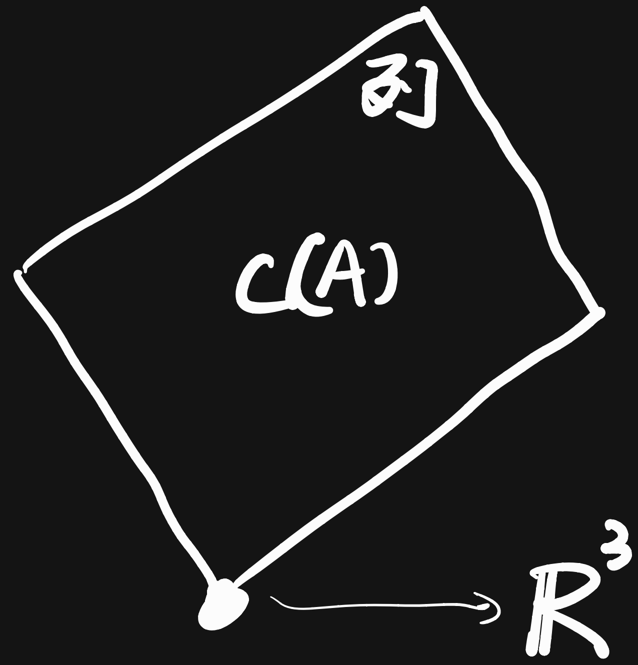
$$= \begin{bmatrix}1\\0\\0\end{bmatrix}\cdot x_1 + \begin{bmatrix}0\\1\\0\end{bmatrix}x_2 + \begin{bmatrix}2\\4\\0\end{bmatrix}x_3 + \begin{bmatrix}3\\5\\0\end{bmatrix}x_4$$

This 2-dim plane forms a subspace, which is named as $\boxed{\text{Colum Space } C(A)}$ $\dim C(A) = 2$

$\forall \vec{x} \in \mathbb{R}^4$, $\boxed{\text{the set of all possible } \vec{b}}$

$\Downarrow$

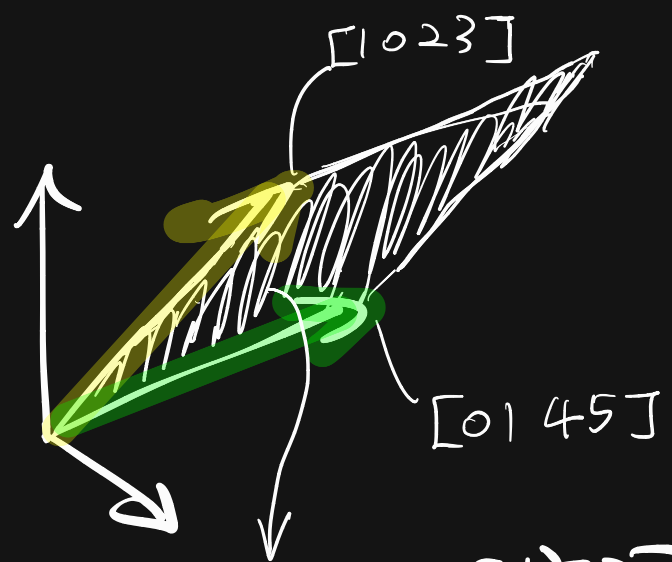
a subspace of $\mathbb{R}^3$: $C(A) \subset \mathbb{R}^3$


列
C(A)
$\mathbb{R}^3$

$$\vec{y}^T \cdot A = \vec{c}^T$$

$$[y_1 \quad y_2 \quad y_3] \cdot \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} = [c_1 \quad c_2 \quad c_3 \quad c_4]$$

$y_i \in \mathbb{R},\ i \in \{1, 2, 3\}$



Row Space 行空间

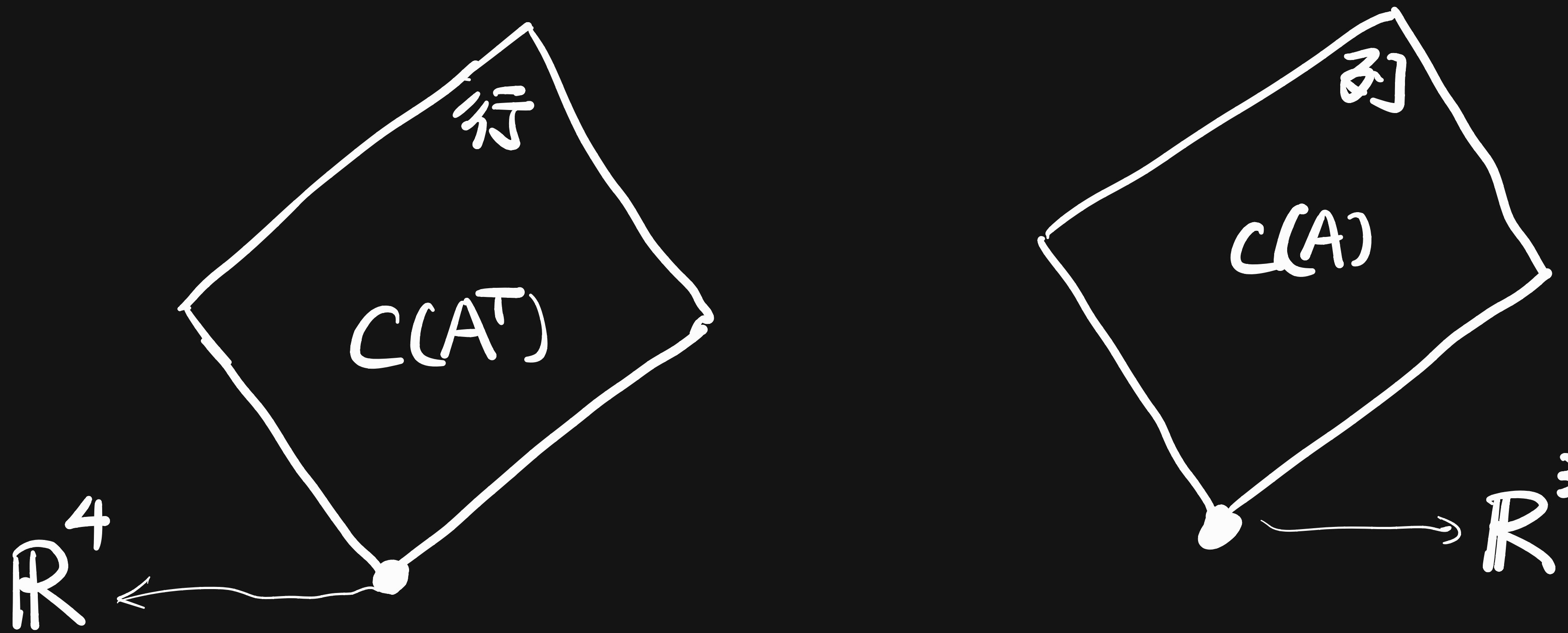
$C(A^T)$

$$F_2(\vec{y}_{1\times 3}) = \vec{c}_{1\times 4} \qquad F_2 : \mathbb{R}^3 \to \mathbb{R}^4$$

A的行空间 = A^T的列空间

$$A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 0 \end{bmatrix}$$

$$((\vec{y}^T) \cdot A)^T$$
$$= A^T \cdot \vec{y}^{TT}$$
$$= A^T \cdot \vec{y}$$



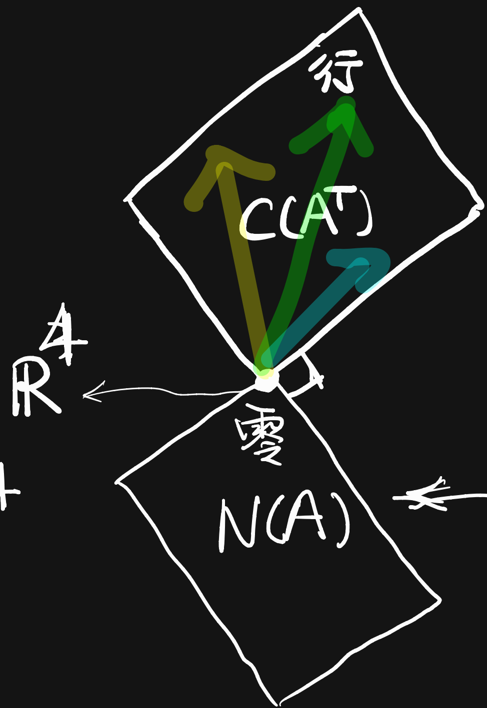
行
$C(A^T)$
$\mathbb{R}^4$
列
$C(A)$
$\mathbb{R}^3$

$$A\vec{x}=\vec{0}$$

$$\overset{A}{\begin{bmatrix}1&0&2&3\\0&1&4&5\\0&0&0&0\end{bmatrix}}\cdot\overset{\vec{x}}{\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}}=\overset{\vec{0}}{\begin{bmatrix}0\\0\\0\end{bmatrix}}$$

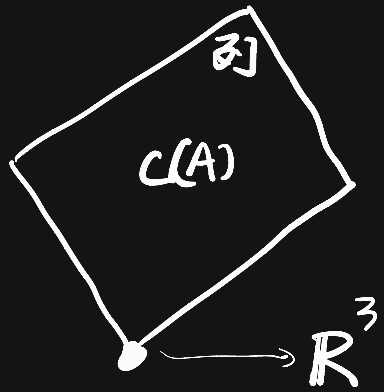
$$\vec{a}_{1\cdot}\cdot\vec{x}=0$$
$$\vec{a}_{2\cdot}\cdot\vec{x}=0$$
$$\vec{a}_{3\cdot}\cdot\vec{x}=0$$

$$\vec{x}\perp\vec{a}\in\{\vec{a}_1,\vec{a}_2,\vec{a}_3\}$$



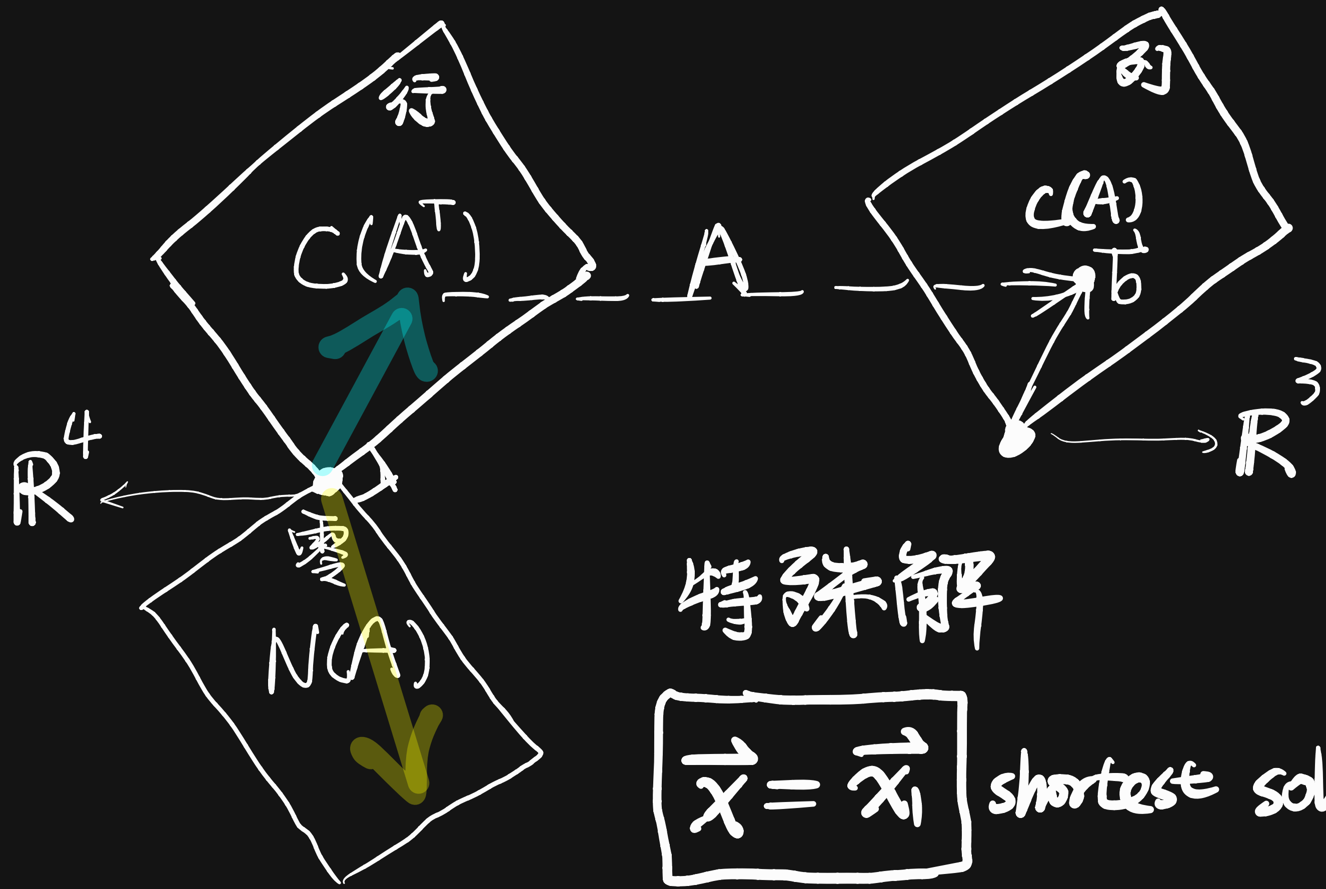
$x\in\mathbb{R}^4$

Null Space N(A)
零空间



$$A\vec{x} = \vec{b}$$
$$= \vec{b} + \vec{0}$$

$$\vec{x} = \vec{x_1} + \lambda \vec{x_2}, \lambda \in \mathbb{R}$$



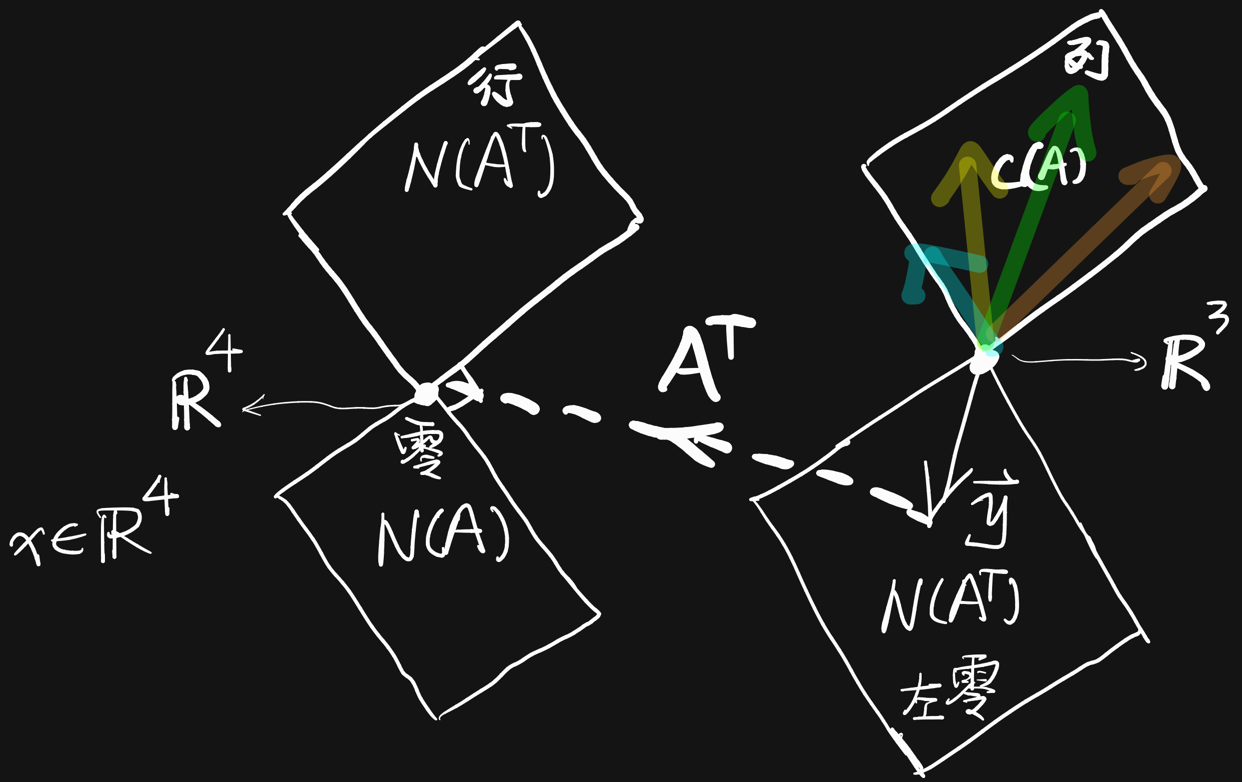
特殊解

$\boxed{\vec{x} = \vec{x_1}}$ shortest solution that minimized $\|\vec{x}\|$

$$\vec{y}^T \cdot A = \vec{0}^T$$

$$[y_1\ \ y_2\ \ y_3] \cdot \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} = [0\ \ 0\ \ 0\ \ 0]$$

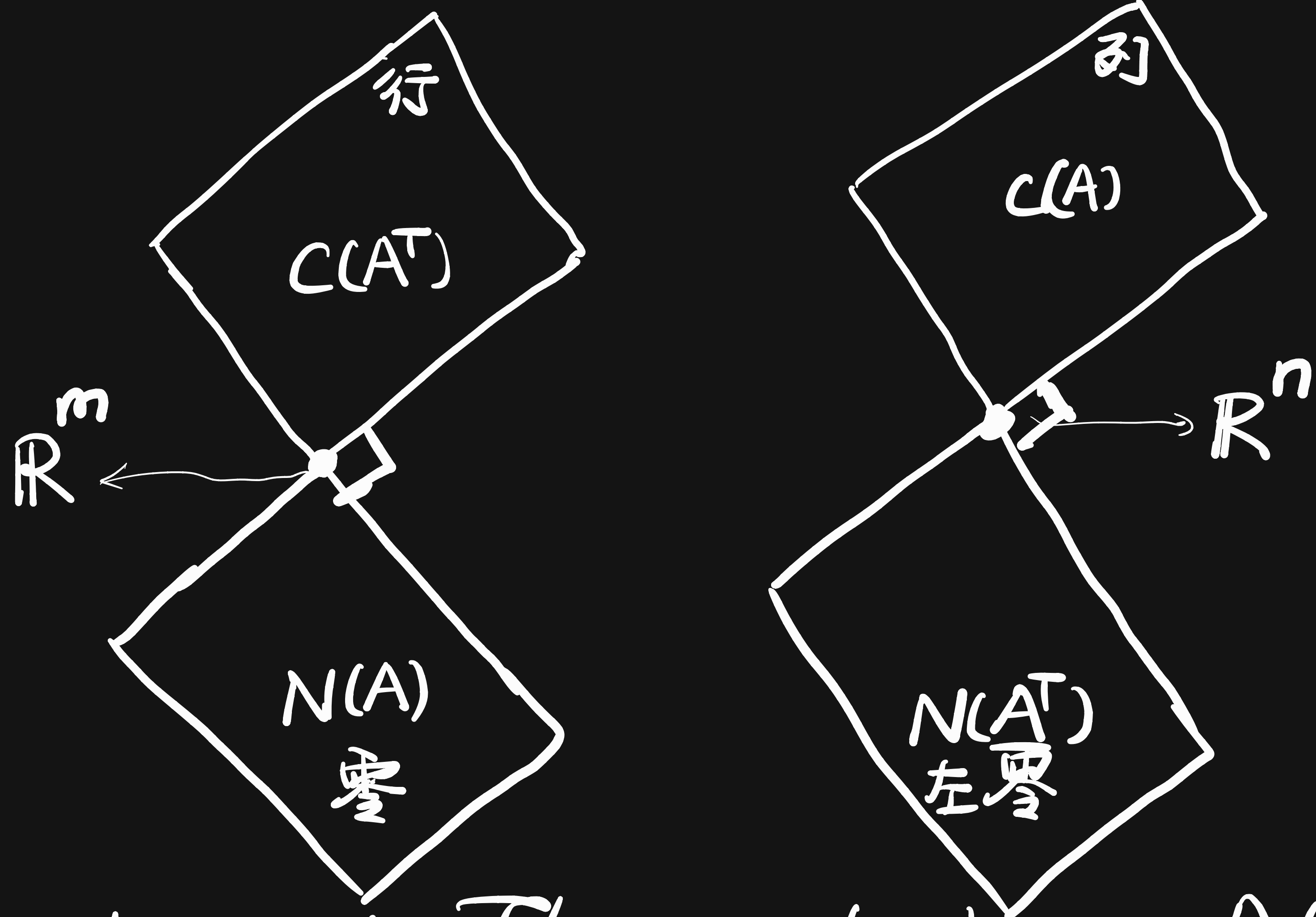
$$y \perp \forall a \in \{a_1, a_2, a_3, a_4\}$$



$$(\vec{y}^T A)^T = (\vec{0}^T)^T$$

$$A^T \vec{y} = \vec{0}$$

$$A_{m\times n}\cdot\vec{x}_{n\times 1}=\vec{b}_{m\times 1}$$ Four Spaces 四种空间



"Fundamental Theory of Linear Algebra"

SVD: Singular Value Decomposition 奇异值分解

$$A_{m\times n} \cdot V_{n\times n} = U_{m\times m} \cdot \Sigma_{m\times n}$$

$$A_{m\times n} \cdot \begin{bmatrix} | & | & & | & & | \\ v_1 & v_2 & \cdots & v_r & \cdots & v_n \\ | & | & & | & & | \end{bmatrix}_{n\times n} = \begin{bmatrix} | & | & & | & & | \\ u_1 & u_2 & \cdots & u_r & \cdots & u_m \\ | & | & & | & & | \end{bmatrix}_{m\times m} \cdot \begin{bmatrix} \sigma_1 & & & & 0 & \cdots \\ & \sigma_2 & & & & \\ & & \ddots & & & \\ & & & \sigma_r & & \\ & 0 & & & 0 & \\ & & & & & \ddots \\ & & & & & 0 \; \cdots \end{bmatrix}_{m\times n}$$

$(n>m)$

$$\Rightarrow \quad A \cdot \vec{v_1} = \vec{u_1} \cdot \sigma_1$$

$$A \cdot \vec{v_2} = \vec{u_2} \cdot \sigma_2$$

$$\vdots$$

$$A \cdot \vec{v_r} = \vec{u_r} \cdot \sigma_r$$

$$A \cdot \vec{v}_{r+1} = 0$$

$$\vdots$$

$$A \cdot \vec{v}_n = 0$$

V, U are Orthogonal Matrix 正交矩阵

$$V = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$$

正交基 Orthogonal Basis:

$\{\vec{v}_1, \ldots, \vec{v}_n\}$

$\|\vec{v}_i\| = 1$, $i=1,\cdots,n$

$\vec{v}_i \cdot \vec{v}_j = 0$, $i \neq j$

→ 类比于单位坐标系

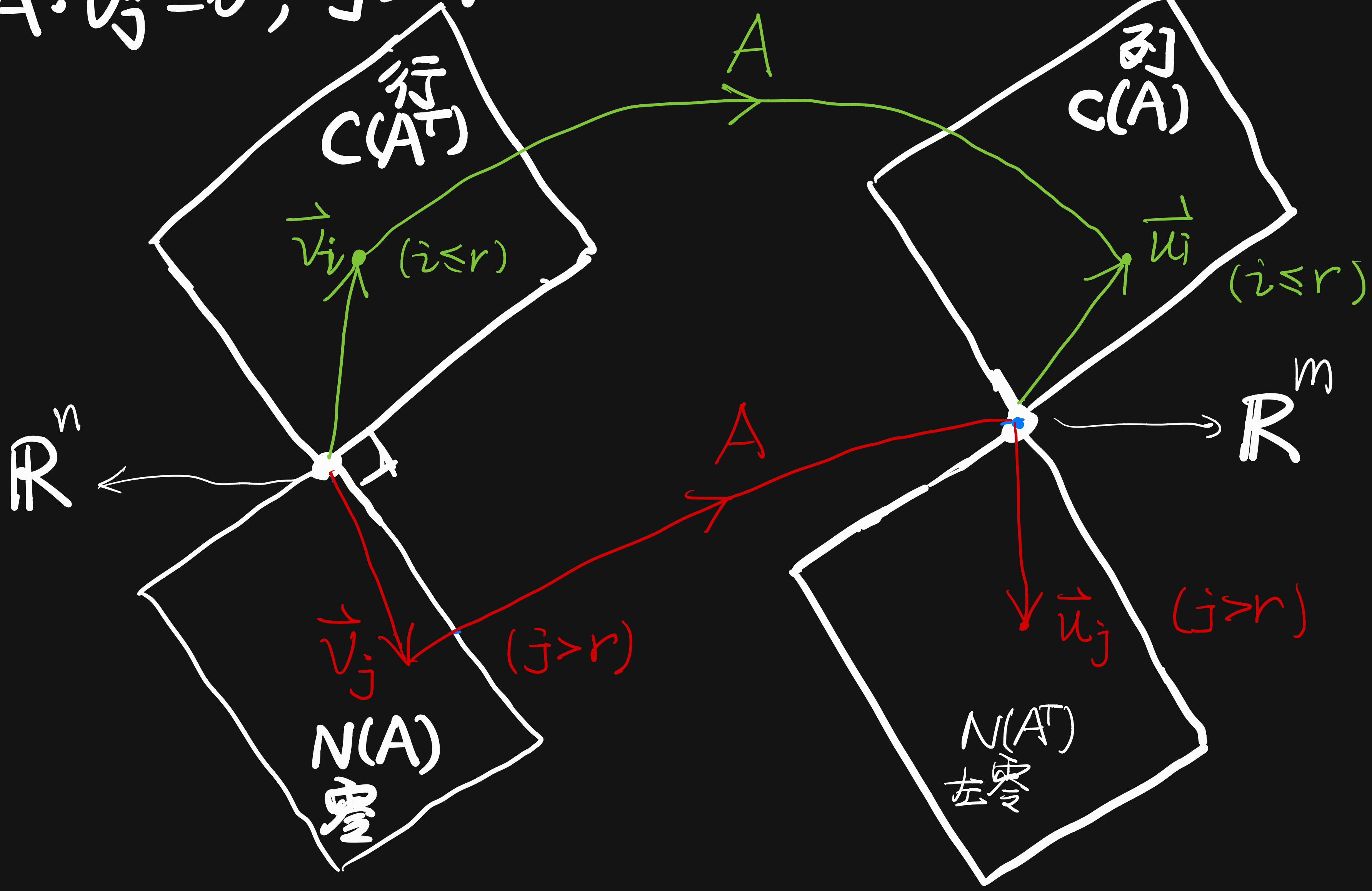
$$V^T V = V V^T = I$$

$$I_{n\times n} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}_{n\times n}$$

Identity Matrix 单位矩阵

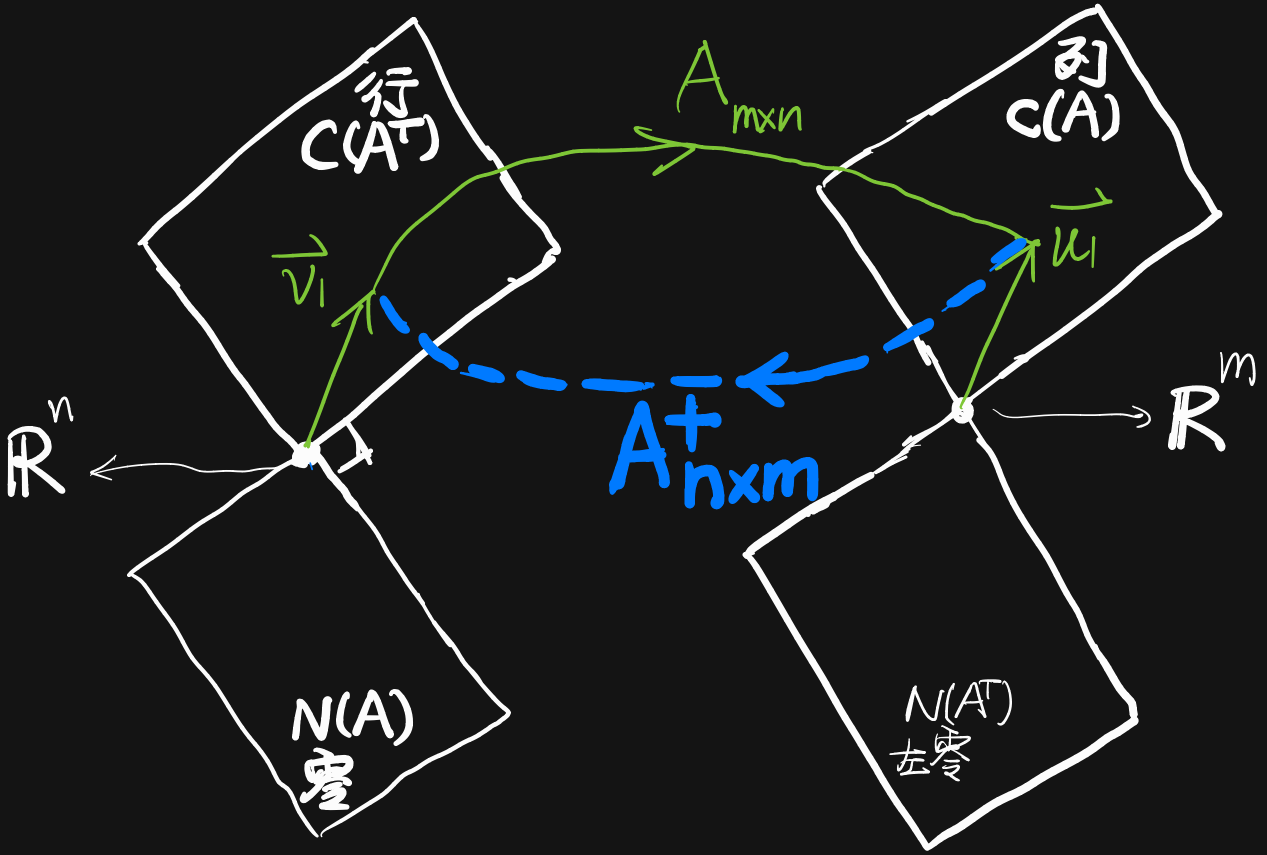
$$A \cdot \vec{v}_i = \vec{u}_i \cdot \sigma_i \quad i=1,\ldots,r \quad r=\mathrm{rank}(A)$$

$$A \cdot \vec{v}_j = 0, \quad j>r$$



Pseudo-inverse A^+ 伪逆矩阵

$$A\underbrace{\boxed{A^+b}}_{\vec{x}}=b$$



$$\boxed{\begin{aligned} A^+\vec{u}_1 &= \vec{v}_1/\sigma_1 \\ &\vdots \\ A^+\vec{u}_r &= \vec{v}_r/\sigma_r \\ \hline A^+\vec{u}_{r+1} &= 0 \\ &\vdots \\ A^+\vec{u}_m &= 0 \end{aligned}}$$

$\Downarrow$ solve A^+

A^+ 伪逆矩阵

$A(A^+b)=b$

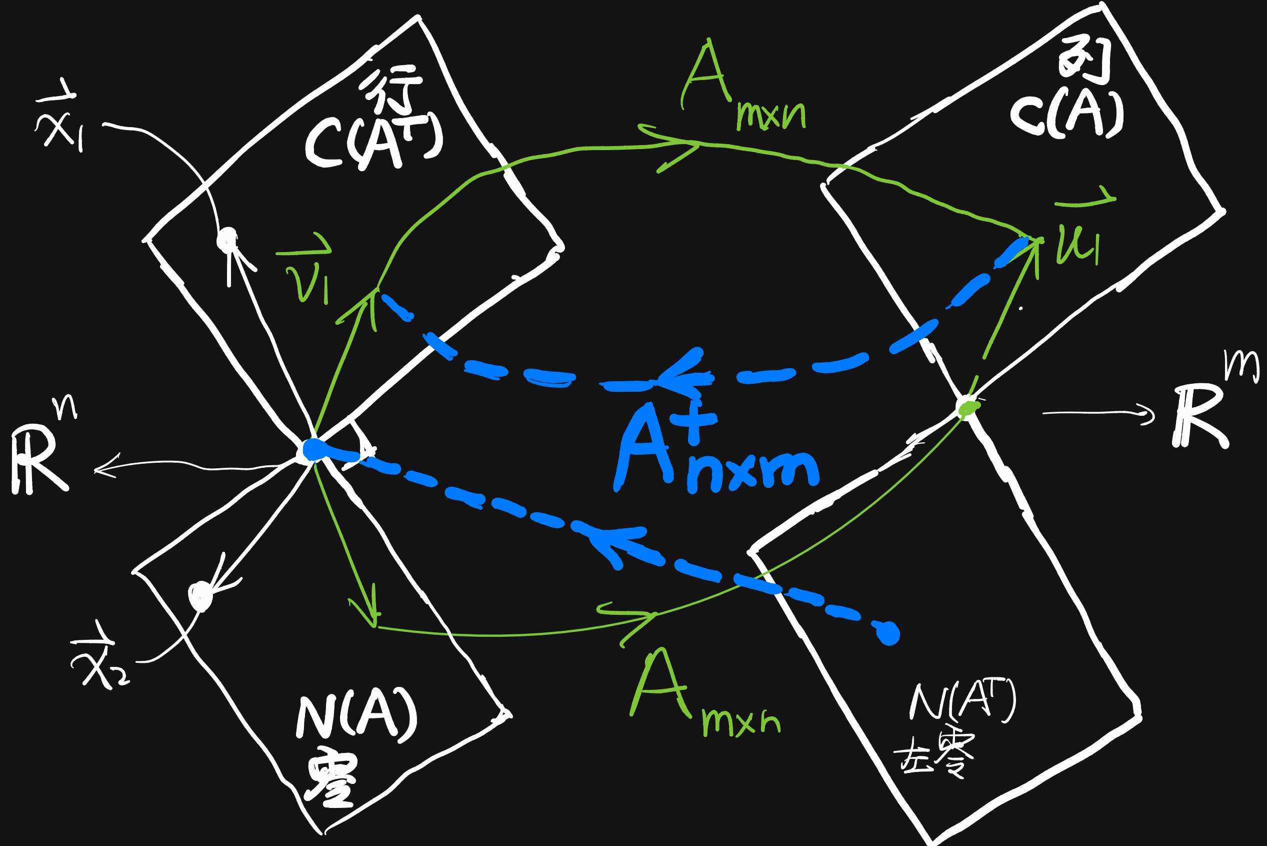
① $N(A^T) = N(A^+)$

② $\tilde{x} = A^+b$ shortest solution to $Ax=b$

③ A^+Ax

no $\vec{x}_2$ any more

$\vec{x}$ is projected to $\vec{x}_1 \in C(A^T)$



③

$\vec{x} = \vec{x}_1 + \vec{x}_2$

$\vec{x}_1 \in C(A^T)$

$\vec{x}_2 \in N(A)$

$A^+Ax = \vec{x}_1$

Computational Benefits 利用SVD简化计算

$$A = U \Sigma V^T$$

$$\begin{aligned} AA^T &= U \Sigma V^T \cdot (U \Sigma V^T)^T \\ &= U \Sigma \cancel{(V^T \cdot V)} \Sigma U^T \\ &= U \Sigma^2 U^T \end{aligned}$$

Similarly, $AA^T = V \Sigma^2 V^T$

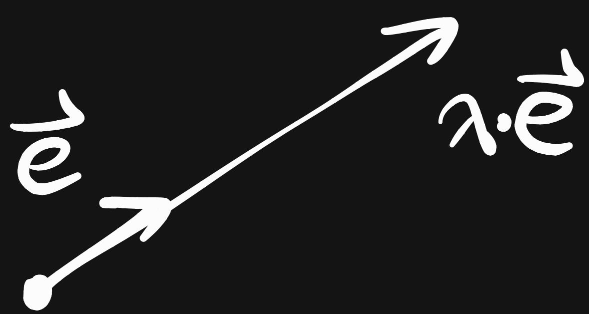
$$\begin{aligned} (AA^T)^{100} &= (V \Sigma^2 V^T)^{100} \\ &= V \cdot \Sigma^{200} \cdot V^T \end{aligned}$$

$$(A^T A)^{100} = U \cdot \Sigma^{200} \cdot U^T$$

Eigen-values / Eigenvectors

$$A \cdot \vec{e} = \lambda \cdot \vec{e}$$

$$\therefore A^t \cdot \vec{e} = \lambda^t \cdot \vec{e}$$



If $\lambda > 1$, $t \to \infty$, $\|A^t \vec{e}\| = \infty$

If $0 < \lambda < 1$, $t \to \infty$, $\|A^t \vec{e}\| = 0$

$$\underset{n\times n}{A}\ \underset{n\times n}{S} = \underset{n\times n}{\Lambda}\ \underset{n\times n}{S} \Rightarrow A = S^{-1}\Lambda S$$

S^{-1} → not computationally stable

$$S = \begin{bmatrix} | & | & & | \\ \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \\ | & | & & | \end{bmatrix}_{n\times n}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}_{n\times n}$$

compare between SVD & Eigendecomposition
(SVD much better)

	SVD	Eigen-decomposition
A's shape	rectangular ($m \times n$)	must be square ($n \times n$)
computational stable ?	Yes	No (S^{-1} hard to compute sometimes)

In the paper, SVD, Eigendecomposition, QR decomposition are compared using a small 2×2 matrix example