Linear Algebra

Data Science & Business Analytics

Lesson 1

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Lesson Plan

- Context
- Vectors
- Matrixes
- Algebra with python (using numpy)
- Exercises

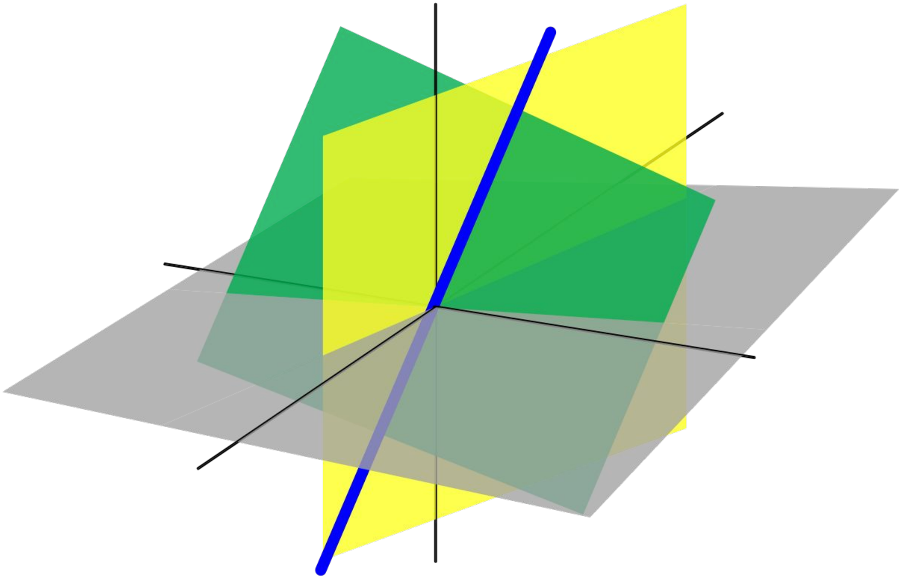
What is linear Algebra?

Linear algebra is a field in mathematics.

Sometimes referred as the language of data, because most of it can be expressed in vectors and matrices.

Linear algebra is all about linear combinations. Using arithmetic on vectors and matrices, to create new vectors and matrices.

Linear algebra is the study of lines and planes.



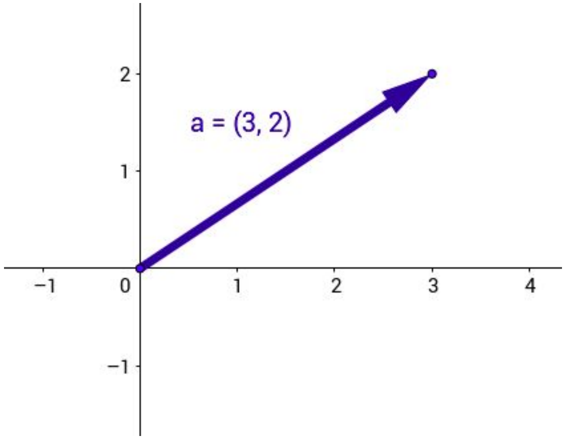
Vectors

A vector is an object that has both a magnitude and a direction. Geometrically, we can picture a vector as a directed line segment, whose length is the magnitude of the vector and with an arrow indicating the direction. The direction of the vector is from its tail to its head.

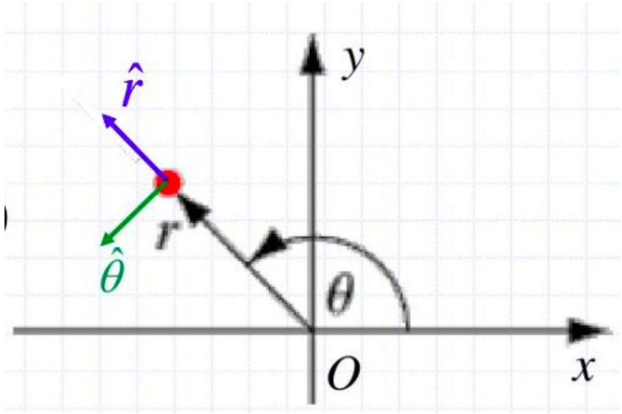
Two vectors are the same if they have the same magnitude and direction. We can apply the following operations: additions, subtraction, multiplication by scalar

Vectors Representation

Cartesian Coordinates



Polar Coordinates



Despite the image be a 2D representation, vectors have n dimensions.

More info here https://mathinsight.org/dot_product

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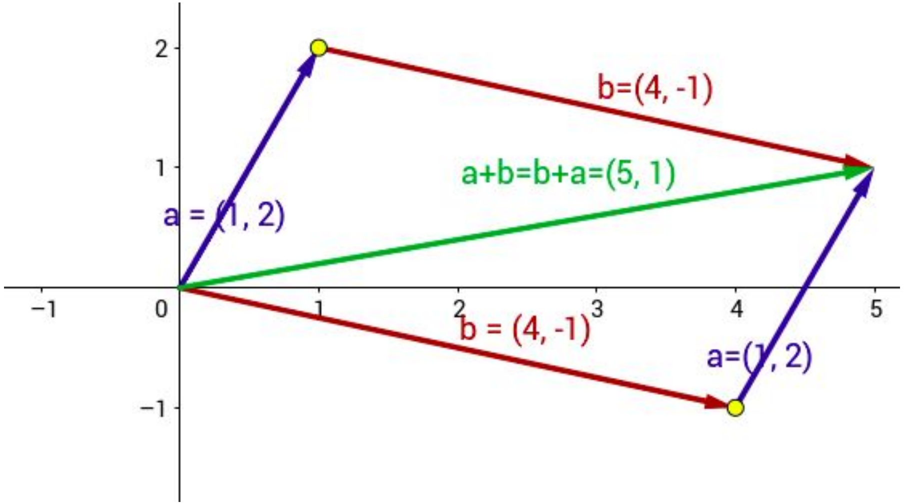
Vectors operations - Additions (subtraction)

A = (1,2)

B = [4, -1]

If C = A + B = [a1 + b1, a2 + b2]

C = [5, 1]



Vectors Norm i.e. Magnitude

Using pythagoras theorem.

$||C|| = \sqrt{||A|| \; + \; ||B||}$

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Vectors properties

- To change direction only need to multiple by -1
- If we multiply by a scalar, only magnitude changes, not direction.
- There are other norms i.e. L1 norm (sum of all values in the vector)
- Vector operations are commutative a + b = b + a
- Normalize a vector, means the is norm = 1

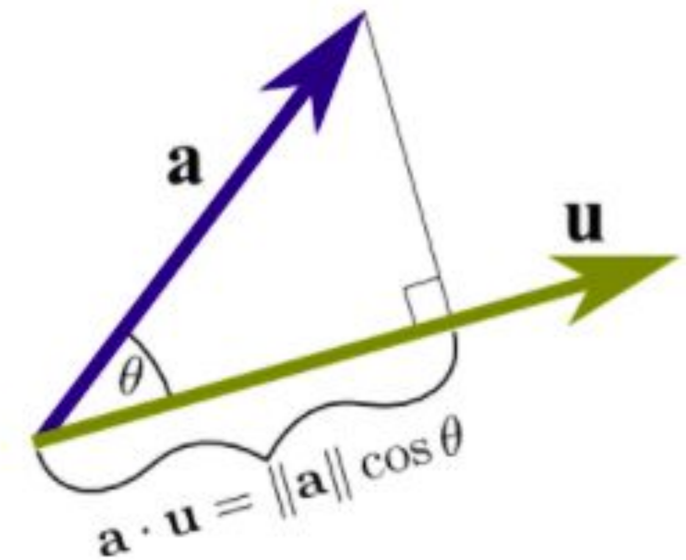
More info on norms here: https://machinelearningmastery.com/vector-norms-machine-learning/

Vectors: Dot product

a . u represents the projection of the u vector.

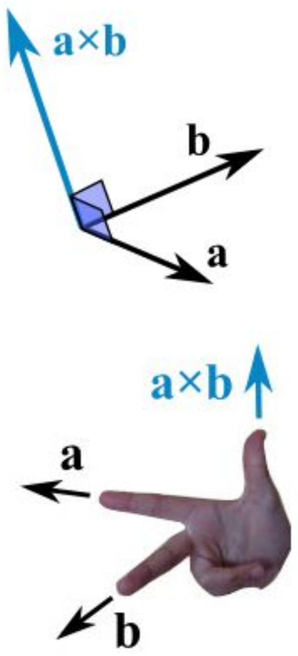
How to compute:

$$a \cdot b = \sum_{i=1}^{n} a_i b_i$$



Vectors: Cross product

The Cross Product a × b of two vectors is another vector that is at right angles to both.



How to calculate:

- cx = ay.bz − az.by
- cy = az.bx − ax.bz
- cz = ax.by − ay.bx

More info: https://www.mathsisfun.com/algebra/vectors-cross-product.html

Matrices

You can view a matrix simply as a generalization of a vector, where we arrange numbers in both rows and columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

If B = [3,2] then

$$B = \begin{bmatrix} 4 & -3 \\ 7 & 9 \\ -5 & 0 \end{bmatrix}$$

Vectors can be seen as a special type of matrices. For example a column or a row of length 1.

By standard Aij where i = rows, j= columns

More info:
https://mathinsight.org/matrix_introduction

Matrix transposition

Matrix transpositions is a very common operation and simply involves flipping the matrix dimensions.

This can also be applied to vectors.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix},$$

$$\mathbf{x}^T = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \end{bmatrix}.$$

More info: https://mathinsight.org/matrix_transpose

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Matrix-Vector Product

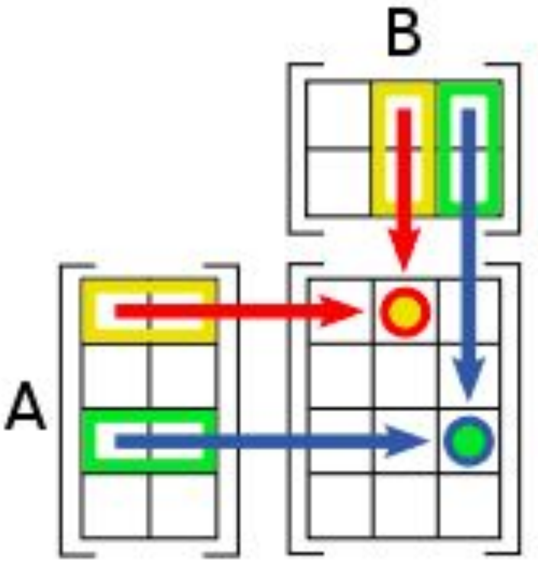
$$\mathbf{A}\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

Matrix to Matrix Multiplication

Rules:

- Dimensions must match row columns count.
- Associative properties
- Not commutative: ABx != (BA)x != (BxA)

Matrix to Matrix Multiplication



$$(AB)_{1,2} = \sum_{r=1}^{2} a_{1,r} b_{r,2} = a_{1,1} b_{1,2} + a_{1,2} b_{2,2}$$

$$(AB)_{3,3} = \sum_{r=1}^{2} a_{3,r} b_{r,3} = a_{3,1} b_{1,3} + a_{3,2} b_{2,3}$$

Matrix sum

- Term by term sum
- Dimension must match Aij = Bij

$$A + B = \begin{bmatrix} 1 & 3 \\ 1 & 0 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 7 & 5 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 3+0 \\ 1+7 & 0+5 \\ 1+2 & 2+1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 8 & 5 \\ 3 & 3 \end{bmatrix}$$

- Broadcasting sum
- Very important concept in computation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + e = \begin{bmatrix} a+e & b+e \\ c+e & d+e \end{bmatrix}$$

Linear Independence

A set of vectors is linearly independent when we can't write one as a linear combination of the others

$$\vec{v}_1 = \begin{bmatrix} 6 \\ 2 \\ 9 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 12 \\ 4 \\ 18 \\ 2 \end{bmatrix}$$

V1 and v2 are not independent because we can obtain v2 by multiplying by 2.

$$c_1 \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 10 \\ 2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -5 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The rank of the matrix is defined by the number of independent rows or columns.

$$\left[\begin{array}{ccc|c} 1 & 10 & -5 & 0 \\ 4 & 2 & 0 & 0 \\ 0 & 1 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

More info: https://www.mathbootcamps.com/linearly-independent-vectors-examples/

Matrix determinant

Scalar value that is function of the matrix values.

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Identity matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ba = 0$$

Matrix decomposition

- Reduces matrix to smaller constituents
- Simplifies complex operations

There are several methods:

- LU decompositions
- Single value decomposition
- . . .

LU decomposition

The idea is to separate the matrix in upper and lower triangle.

This can generalize for rectangle matrixes as well. This helps for example solving equations because we only need to solve parts of the equation.

$$\begin{aligned} \ell_{11} \cdot u_{11} + 0 \cdot 0 &= 4 \\ \ell_{11} \cdot u_{12} + 0 \cdot u_{22} &= 3 \\ \ell_{21} \cdot u_{11} + \ell_{22} \cdot 0 &= 6 \\ \ell_{21} \cdot u_{12} + \ell_{22} \cdot u_{22} &= 3. \end{aligned}$$

$$\begin{bmatrix} 4 & 3 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} \ell_{11} & 0 \\ \ell_{21} & \ell_{22} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}.$$

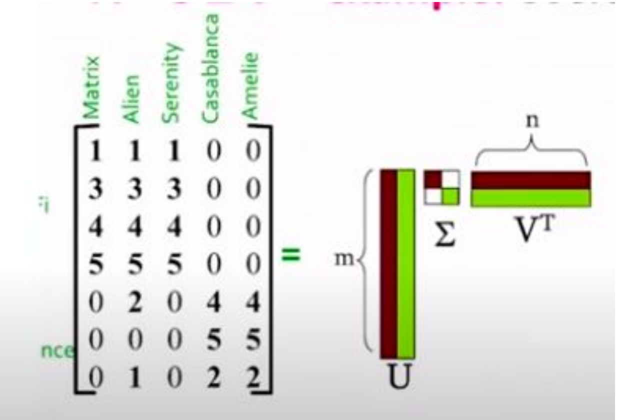
$$\begin{bmatrix} 4 & 3 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1.5 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 0 & -1.5 \end{bmatrix}$$

Singular Value Decomposition (SVD)

- Decomposition into eigenvectors and eigenvalues
- Popular data reduction tool
- Used in Principal Component Analysis and FM (Factorization Machines) i.e. used in recommendation algorithms for example.
- Definition: A = U W V ^ T

Singular Value Decomposition (SVD)

As part of the process we need to calculate the eigenvalues and eigen values. Which basic represent the principal vector of the matrix, after transformation



$$\begin{bmatrix} 0.13 & 0.02 & -0.01 \\ 0.41 & 0.07 & -0.03 \\ 0.55 & 0.09 & -0.04 \\ 0.68 & 0.11 & -0.05 \\ 0.15 & -0.59 & 0.65 \\ 0.07 & -0.73 & -0.67 \\ 0.07 & -0.29 & 0.32 \end{bmatrix} \times \begin{bmatrix} 12.4 & 0 & 0 \\ 0 & 9.5 & 0 \\ 0 & 0 & 1.3 \end{bmatrix} \times \begin{bmatrix} 0.56 & 0.59 & 0.56 & 0.09 & 0.09 \\ 0.12 & -0.02 & 0.12 & -0.69 & -0.69 \\ 0.40 & -0.80 & 0.40 & 0.09 & 0.09 \end{bmatrix}$$

More info
https://www.youtube.com/watch?v=P5mlg91as1c