Theory
Finite-Difference Time-Domain

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1 Maxwell Equations

1.1 Equations

$$\frac{\partial \vec{E}}{\partial t} = \frac{1}{\epsilon}\left(\nabla \times \vec{H} - \sigma^e \vec{E} - \vec{J}_i\right) \tag{1}$$

$$\frac{\partial E_x}{\partial t} = \frac{1}{\epsilon_x}\left(\left[\nabla \times \vec{H}\right]\cdot \hat{x} - \sigma^e_x E_x - J_{ix}\right) = \frac{1}{\epsilon_x}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - \sigma^e_x E_x - J_{ix}\right) \tag{2}$$

$$\frac{\partial E_y}{\partial t} = \frac{1}{\epsilon_y}\left(\left[\nabla \times \vec{H}\right]\cdot \hat{y} - \sigma^e_y E_y - J_{iy}\right) = \frac{1}{\epsilon_y}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} - \sigma^e_y E_y - J_{iy}\right) \tag{3}$$

$$\frac{\partial E_y}{\partial t} = \frac{1}{\epsilon_z}\left(\left[\nabla \times \vec{H}\right]\cdot \hat{z} - \sigma^e_z E_z - J_{iz}\right) = \frac{1}{\epsilon_z}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - \sigma^e_z E_z - J_{iz}\right) \tag{4}$$

$$\frac{\partial \vec{H}}{\partial t} = \frac{1}{\mu}\left(-\nabla \times \vec{E} - \sigma^m \vec{H} - \vec{J}_i\right) \tag{5}$$

$$\frac{\partial H_x}{\partial t} = \frac{1}{\mu_x}\left(-\left[\nabla \times \vec{E}\right]\cdot \hat{x} - \sigma^m_x H_x - J_{ix}\right) = \frac{1}{\mu_x}\left(\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y} - \sigma^m_x H_x - J_{ix}\right) \tag{6}$$

$$\frac{\partial H_y}{\partial t} = \frac{1}{\mu_y}\left(-\left[\nabla \times \vec{E}\right]\cdot \hat{y} - \sigma^m_y H_y - J_{iy}\right) = \frac{1}{\mu_y}\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} - \sigma^m_y H_y - J_{iy}\right) \tag{7}$$

$$\frac{\partial H_y}{\partial t} = \frac{1}{\mu_z}\left(-\left[\nabla \times \vec{E}\right]\cdot \hat{z} - \sigma^m_z H_z - J_{iz}\right) = \frac{1}{\mu_z}\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} - \sigma^m_z H_z - J_{iz}\right) \tag{8}$$

1.2 Discretization

Electric Field:

$$\begin{aligned}\frac{E_x^{n+1}(i,j,k) - E_x^n(i,j,k)}{\Delta t} &= \frac{\left[\nabla \times \vec{H}\right]^{n+\frac{1}{2}}(i,j,k)\cdot \hat{x}}{\epsilon_x(i,j,k)} \\ &\quad - \frac{\sigma^e_x(i,j,k)}{\epsilon_x(i,j,k)} E_x^{n+\frac{1}{2}}(i,j,k) - \frac{1}{\epsilon_x(i,j,k)} J_{ix}^{n+1}(i,j,k)\end{aligned} \tag{9}$$

$$E_x^{n+\frac{1}{2}}(i,j,k) = \frac{E_x^{n+1}(i,j,k) + E_x^n(i,j,k)}{2} \tag{10}$$

$$\begin{aligned}\frac{E_x^{n+1}(i,j,k) - E_x^n(i,j,k)}{\Delta t} &= \frac{\left[\nabla \times \vec{H}\right]^{n+\frac{1}{2}}(i,j,k)\cdot \hat{x}}{\epsilon_x(i,j,k)} \\ &\quad - \frac{\sigma^e_x(i,j,k)}{\epsilon_x(i,j,k)} \frac{E_x^{n+1}(i,j,k) + E_x^n(i,j,k)}{2} - \frac{1}{\epsilon_x(i,j,k)} J_{ix}^{n+1}(i,j,k)\end{aligned} \tag{11}$$

$$\begin{aligned}E_x^{n+1}(i,j,k) - E_x^n(i,j,k) &= \frac{\Delta t\left[\nabla \times \vec{H}\right]^{n+\frac{1}{2}}(i,j,k)\cdot \hat{x}}{\epsilon_x(i,j,k)} \\ &\quad - \frac{\Delta t \sigma^e_x(i,j,k)}{2\epsilon_x(i,j,k)} E_x^{n+1}(i,j,k) - \frac{\Delta t \sigma^e_x(i,j,k)}{2\epsilon_x(i,j,k)} E_x^n(i,j,k) \\ &\quad - \frac{\Delta t}{\epsilon_x(i,j,k)} J_{ix}^{n+1}(i,j,k)\end{aligned} \tag{12}$$

$$
\begin{aligned}
E_x^{n+1}(i,j,k) - E_x^n(i,j,k) = & \frac{\Delta t \left[\nabla \times \vec{H}\right]^{n+\frac{1}{2}}(i,j,k) \cdot \hat{x}}{\epsilon_x(i,j,k)} \\
& - \frac{\Delta t \sigma_x^e(i,j,k)}{2\epsilon_x(i,j,k)} E_x^{n+1}(i,j,k) - \frac{\Delta t \sigma_x^e(i,j,k)}{2\epsilon_x(i,j,k)} E_x^n(i,j,k) \\
& - \frac{\Delta t}{\epsilon_x(i,j,k)} J_{ix}^{n+1}(i,j,k)
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
\left(1 + \frac{\Delta t \sigma_x^e(i,j,k)}{2\epsilon_x(i,j,k)}\right) E_x^{n+1}(i,j,k) = & \frac{\Delta t \left[\nabla \times \vec{H}\right]^{n+\frac{1}{2}}(i,j,k) \cdot \hat{x}}{\epsilon_x(i,j,k)} \\
& + \left(1 - \frac{\Delta t \sigma_x^e(i,j,k)}{2\epsilon_x(i,j,k)}\right) E_x^n(i,j,k) \\
& - \frac{\Delta t}{\epsilon_x(i,j,k)} J_{ix}^{n+1}(i,j,k)
\end{aligned} \tag{14}
$$

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\boxed{f_e = \frac{\Delta t \sigma_x^e(i,j,k)}{2\epsilon_x(i,j,k)})} \tag{15}
$$

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\boxed{
\begin{aligned}
\left[\nabla \times \vec{H}\right]^{n+\frac{1}{2}}(i,j,k) = & + \left[\frac{H_z^n(i,j,k) - H_z^n(i,j-1,k)}{\Delta y} - \frac{H_y^n(i,j,k) - H_y^n(i,j,k-1)}{\Delta z}\right] \hat{x} \\
& + \left[\frac{H_x^n(i,j,k) - H_x^n(i,j,k-1)}{\Delta z} - \frac{H_z^n(i,j,k) - H_z^n(i-1,j,k)}{\Delta x}\right] \hat{y} \\
& + \left[\frac{H_y^n(i,j,k) - H_y^n(i-1,j,k)}{\Delta x} - \frac{H_x^n(i,j,k) - H_x^n(i,j-1,k)}{\Delta y}\right] \hat{z}
\end{aligned}
} \tag{16}
$$

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\boxed{
\begin{aligned}
E_x^{n+1}(i,j,k) = & \frac{1-f_e}{1+f_e} E_x^n(i,j,k) \\
& + \frac{\Delta t}{(1+f_e)\epsilon_x(i,j,k)} \left[\nabla \times \vec{H}\right]^{n+\frac{1}{2}}(i,j,k) \cdot \hat{x} \\
& - \frac{\Delta t}{(1+f_e)\epsilon_x(i,j,k)} J_{ix}^{n+1}(i,j,k)
\end{aligned}
} \tag{17}
$$

$$
\boxed{
\begin{aligned}
E_y^{n+1}(i,j,k) = & \frac{1-f_e}{1+f_e} E_y^n(i,j,k) \\
& + \frac{\Delta t}{(1+f_e)\epsilon_y(i,j,k)} \left[\nabla \times \vec{H}\right]^{n+\frac{1}{2}}(i,j,k) \cdot \hat{y} \\
& - \frac{\Delta t}{(1+f_e)\epsilon_y(i,j,k)} J_{iy}^{n+1}(i,j,k)
\end{aligned}
} \tag{18}
$$

$$
\boxed{
\begin{aligned}
E_z^{n+1}(i,j,k) = & \frac{1-f_e}{1+f_e} E_z^n(i,j,k) \\
& + \frac{\Delta t}{(1+f_e)\epsilon_z(i,j,k)} \left[\nabla \times \vec{H}\right]^{n+\frac{1}{2}}(i,j,k) \cdot \hat{z} \\
& - \frac{\Delta t}{(1+f_e)\epsilon_z(i,j,k)} J_{iz}^{n+1}(i,j,k)
\end{aligned}
} \tag{19}
$$

Magnetic Field:

$$
\begin{aligned}
\frac{H_x^{n+\frac{1}{2}}(i,j,k) - H_x^{n-\frac{1}{2}}(i,j,k)}{\Delta t} = & \frac{\left[-\nabla \times \vec{E}\right]^n(i,j,k) \cdot \hat{x}}{\mu_x(i,j,k)} \\
& - \frac{\sigma_x^m(i,j,k)}{\mu_x(i,j,k)} H_x^n(i,j,k) - \frac{1}{\mu_x(i,j,k)} M_{ix}^n(i,j,k)
\end{aligned} \tag{20}
$$

$$
H_x^n(i,j,k) = \frac{H_x^{n+\frac{1}{2}}(i,j,k) + H_x^{n-\frac{1}{2}}(i,j,k)}{2} \tag{21}
$$

$$
\begin{aligned}
\frac{H_x^{n+\frac{1}{2}}(i,j,k) - H_x^{n-\frac{1}{2}}(i,j,k)}{\Delta t} = & \frac{\left[-\nabla \times \vec{E}\right]^n (i,j,k) \cdot \hat{x}}{\mu_x(i,j,k)} \\
& - \frac{\sigma_x^m(i,j,k)}{\mu_x(i,j,k)} \frac{H_x^{n+\frac{1}{2}}(i,j,k) + H_x^{n-\frac{1}{2}}(i,j,k)}{2} - \frac{1}{\mu_x(i,j,k)} M_{ix}^n(i,j,k)
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
H_x^{n+\frac{1}{2}}(i,j,k) - H_x^{n-\frac{1}{2}}(i,j,k) = & \frac{\Delta t \left[-\nabla \times \vec{E}\right]^n (i,j,k) \cdot \hat{x}}{\mu_x(i,j,k)} \\
& - \frac{\Delta t \sigma_x^m(i,j,k)}{2\mu_x(i,j,k)} H_x^{n+\frac{1}{2}}(i,j,k) - \frac{\Delta t \sigma_x^m(i,j,k)}{2\mu_x(i,j,k)} H_x^{n-\frac{1}{2}}(i,j,k) \\
& - \frac{\Delta t}{\mu_x(i,j,k)} M_{ix}^n(i,j,k)
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
H_x^{n+\frac{1}{2}}(i,j,k) - H_x^{n-\frac{1}{2}}(i,j,k) = & \frac{\Delta t \left[-\nabla \times \vec{E}\right]^n (i,j,k) \cdot \hat{x}}{\mu_x(i,j,k)} \\
& - \frac{\Delta t \sigma_x^m(i,j,k)}{2\mu_x(i,j,k)} H_x^{n+\frac{1}{2}}(i,j,k) - \frac{\Delta t \sigma_x^m(i,j,k)}{2\mu_x(i,j,k)} H_x^{n-\frac{1}{2}}(i,j,k) \\
& - \frac{\Delta t}{\mu_x(i,j,k)} M_{ix}^n(i,j,k)
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
\left(1 + \frac{\Delta t \sigma_x^m(i,j,k)}{2\mu_x(i,j,k)}\right) H_x^{n+\frac{1}{2}}(i,j,k) = & \frac{\Delta t \left[-\nabla \times \vec{E}\right]^n (i,j,k) \cdot \hat{x}}{\mu_x(i,j,k)} \\
& + \left(1 - \frac{\Delta t \sigma_x^m(i,j,k)}{2\mu_x(i,j,k)}\right) H_x^{n-\frac{1}{2}}(i,j,k) \\
& - \frac{\Delta t}{\mu_x(i,j,k)} M_{ix}^n(i,j,k)
\end{aligned} \tag{25}
$$

$$
\boxed{f_m = \frac{\Delta t \sigma_x^m(i,j,k)}{2\mu_x(i,j,k)})} \tag{26}
$$

$$
\boxed{
\begin{aligned}
\left[\nabla \times \vec{E}\right]^n (i,j,k) = & \left[\frac{H_z^n(i,j+1,k) - H_z^n(i,j,k)}{\Delta y} - \frac{H_y^n(i,j,k+1) - H_y^n(i,j,k)}{\Delta z}\right] \hat{x} \\
& + \left[\frac{H_x^n(i,j,k+1) - H_x^n(i,j,k)}{\Delta z} - \frac{H_z^n(i+1,j,k) - H_z^n(i,j,k)}{\Delta x}\right] \hat{y} \\
& + \left[\frac{H_y^n(i+1,j,k) - H_y^n(i,j,k)}{\Delta x} - \frac{H_x^n(i,j+1,k) - H_x^n(i,j,k)}{\Delta y}\right] \hat{z}
\end{aligned}
} \tag{27}
$$

$$
\boxed{
\begin{aligned}
H_x^{n+\frac{1}{2}}(i,j,k) = & \frac{1 - f_m}{1 + f_m} H_x^{n-\frac{1}{2}}(i,j,k) \\
& + \frac{\Delta t}{(1 + f_m)\mu_x(i,j,k)} \left[-\nabla \times \vec{E}\right]^n (i,j,k) \cdot \hat{x} \\
& - \frac{\Delta t}{(1 + f_m)\mu_x(i,j,k)} M_{ix}^n(i,j,k)
\end{aligned}
} \tag{28}
$$

$$
\boxed{
\begin{aligned}
H_y^{n+\frac{1}{2}}(i,j,k) = & \frac{1 - f_m}{1 + f_m} H_y^{n-\frac{1}{2}}(i,j,k) \\
& + \frac{\Delta t}{(1 + f_m)\mu_y(i,j,k)} \left[-\nabla \times \vec{E}\right]^n (i,j,k) \cdot \hat{y} \\
& - \frac{\Delta t}{(1 + f_m)\mu_y(i,j,k)} M_{iy}^n(i,j,k)
\end{aligned}
} \tag{29}
$$

$$
\boxed{
\begin{aligned}
H_z^{n+\frac{1}{2}}(i,j,k) &= \frac{1-f_m}{1+f_m} H_z^{n-\frac{1}{2}}(i,j,k) \\
&+ \frac{\Delta t}{(1+f_m)\mu_z(i,j,k)} \left[-\nabla \times \vec{E} \right]^n (i,j,k) \cdot \hat{z} \\
&- \frac{\Delta t}{(1+f_m)\mu_z(i,j,k)} M_{iz}^n(i,j,k)
\end{aligned}
}
\tag{30}
$$