Ho Chi Minh City
University of Technology and Education
Faculty of Applied Sciences
Department of Mathematics

MATHEMATICAL STATISTICS FOR ENGINEERS (MATH132901E)

PROBABILITY

Nguyễn Ngọc Tứ

Introduction

- **Topic**: Mathematical Statistics for Engineers
- Syllabus on web: www.fhqx.hcmute.edu.vn
- **Email**: tunn@hcmute.edu.vn
- **Text**: Mathematical Statistics for Engineers, Jay L. Devore
  Read the text!
- **Reference:** Introduction to Probability, 2nd Edition
  D. P. Bertsekas and J. N. Tsitsiklis, Athena Scientific, 2008
- **Coursework:**
  - **Quizz 1**: 25%
  - **Quizz 2**: 25%
  - **Final exam**: 50%

Lecture outline

- Probability models
  - sample space
  - The language of sets
  - De Morgan's law
  - probability law
  - Axioms of probability
- Conditional probability
- Three important tools:
  - Multiplication rule
  - Total probability theorem
  - Bayes' rule
- Independence

Sample space

Definition

Consider a trial (or an experiment) that has a random outcome. The set of all possible outcomes is the sample space, Ω.

Example

Rolling a (6-sides) die: $\Omega = \{$⚀, ⚁, ⚂, ⚃, ⚄, ⚅$\}$
Tossing a coin: $\Omega = \{H, T\}$.
Tossing two coins:

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Rolling a (6-sides) die: $\Omega = \{$⚀, ⚁, ⚂, ⚃, ⚄, ⚅$\}$
Tossing a coin: $\Omega = \{H, T\}$.
Tossing two coins: $\Omega = \{(H, H), (H, T), (T, H), (T, T)\}$.

"Events are what we assign a probability to"

Definition

An event A is a subset of Ω. We say that an event A has (not) occurred if the outcome of the trail is (not) contained in A.

Example

Tossing a coin and getting a head $A = \{H\}$.
Tossing two coins and getting at least one head:

"Events are what we assign a probability to"

Definition

An event A is a subset of Ω. We say that an event A has (not) occurred if the outcome of the trail is (not) contained in A.

Example

Tossing a coin and getting a head $A = \{H\}$.
Tossing two coins and getting at least one head:
$A = \{(H, H), (H, T), (T, H)\}$.
Rolling a die and getting an even number: $A = \{$⚁, ⚃, ⚅$\}$.
Rolling two dice and getting a total of 5:

"Events are what we assign a probability to"

Definition

An event A is a subset of Ω. We say that an event A has (not) occurred if the outcome of the trail is (not) contained in A.

Example

Tossing a coin and getting a head $A = \{H\}$.
Tossing two coins and getting at least one head:
$A = \{(H, H), (H, T), (T, H)\}$.
Rolling a die and getting an even number: $A = \{$⚁, ⚃, ⚅$\}$.
Rolling two dice and getting a total of 5:
$A = \{$(⚀, ⚃)(⚁, ⚂), (⚂, ⚁), (⚃, ⚀)$\}$.

The language of sets

Let us consider subsets of a set Ω.

Definition

The complement of $A \subset \Omega$ is denoted $A^c \subset \Omega$:

$$w \in A^c \Leftrightarrow w \notin A.$$

Clearly, $(A^c)^c = A$.

Example – Empty set

The complement of Ω is the empty set $\emptyset$:

$$w \notin \emptyset, \forall w \in \Omega.$$

The language of sets

Definition

The union of A and B is denoted $A \cup B$:

$$w \in A \cup B \Leftrightarrow w \in A \text{ or } w \in B \text{ or both.}$$

The intersection of A and B is denoted $A \cap B$:

$$w \in A \cap B \Leftrightarrow w \in A \text{ and } w \in B.$$

If $A \cap B = \emptyset$, then we say A and B are *disjoint*.

Remark

For all subsets A of Ω, $A \cup \Omega = \Omega, A \cup \emptyset = A.$

$$A \cap \Omega = A, A \cap \emptyset = \emptyset.$$

The language of sets - Exercise

1. Four universities—1, 2, 3, and 4—are participating in a holiday basketball tournament. In the first round, 1 will play 2 and 3 will play 4. Then the two winners will play for the championship, and the two losers will also play. One possible outcome can be denoted by 1324 (1 beats 2 and 3 beats 4 in first-round games, and then 1 beats 3 and 2 beats 4).

a. List all outcomes in Ω.

b. Let A denote the event that 1 wins the tournament. List outcomes in A.

c. Let B denote the event that 2 gets into the championship game. List outcomes in B.

d. What are the outcomes in $A \cup B$ and in $A \cap B$? What are the outcomes in A^c?

Solution.

a. List all outcomes in Ω.

The language of sets - Exercise

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Solution.

a. List all outcomes in Ω.

$$\Omega = \{1324, 1342, 1423, 1432, 2314, 2341, 2413, 2431, \\ 3124, 3142, 4123, 4132, 3214, 3241, 4213, 4231\}$$

The language of sets - Exercise

b. Event A contains the outcomes where 1 is first in the list
$A = \{1324, 1342, 1423, 1432\}$

c. Event B contains the outcomes where 2 is first or second
$B = \{2314, 2341, 2413, 2431, 3214, 3241, 4213, 4231\}$

d. The compound event $A \cup B$ contains the outcomes in A or B or both

$$\begin{aligned} A \cup B = \{&1324, 1342, 1423, 1432, 2314, 2341,\\ &2413, 2431, 3214, 3241, 4213, 4231\} \end{aligned}$$

$A \cap B = \emptyset$, since 1 and 2 can't both get into the championship game.

$$\begin{aligned} A^c = \Omega \setminus A = \{&2314, 2341, 2413, 2431,\\ &3124, 3142, 4123, 4132, 3214, 3241, 4213, 4231\} \end{aligned}$$

De Morgan's law

Definition – De Morgan's law

Let A, B be subsets of Ω. Then

$$(A \cup B)^c = A^c \cap B^c \text{ and } (A \cap B)^c = A^c \cup B^c.$$

Proof.

$$\begin{aligned} w \in (A \cup B)^c &\Leftrightarrow w \notin A \cup B \\ &\Leftrightarrow w \notin A \text{ and } w \notin B \\ &\Leftrightarrow w \in A^c \text{ and } w \in B^c \\ &\Leftrightarrow w \in A^c \cap B^c. \end{aligned}$$

The other equality if proved similarly. □

De Morgan's law - Exercises

2. Give two events A and B. Find the event X from the following equation:

$$(X \cup A)^c \cup (X \cup A^c)^c = B$$

Classical probability

Definition – Probability

Let $\Omega = \{w_1, w_2, \ldots, w_N\}$. Let $A \subseteq \Omega$ be an event. Then the probability of A is

$$P(A) = \frac{\text{Number of outcomes in A}}{\text{Number of outcomes in } \Omega} = \frac{|A|}{N}.$$

Example

You draw a number at random from $\{1, 2, \ldots, 30\}$. What is the probability of the following events:

1. A = the number drawn is even
2. B = the number drawn is divisible by 3
3. C = the number drawn is less than 12

Examples

Example

You draw a number at random from $\{1, 2, \ldots, 30\}$. What is the probability of the following events:

1. A = the number drawn is even
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3. C = the number drawn is less than 12

Solution.

1. $A = \{2, 4, 6, \ldots, 30\}$, so $|A| = 15$ and hence $P(A) = \frac{15}{30} = \frac{1}{2}$.
2. $B = \{3, 6, 9, \ldots, 30\}$, so $|B| = 10$ and hence $P(B) = \frac{10}{30} = \frac{1}{3}$.
3. $C = \{1, 2, 3, \ldots, 11\}$, so $|C| = 11$ and hence $P(C) = \frac{11}{30}$.

σ-field and Probability Axioms

Definition

A family $\mathcal{F}$ of subsets of Ω is a σ-field if

1. $\Omega \in \mathcal{F}$
2. if $A_1, A_2, \ldots \in \mathcal{F}$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$
3. if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$

Axioms

1. **Nonnegativity:** $P(A) \geq 0, \forall A \subset \mathcal{F}.$
2. **Normalization:** $P(\Omega) = 1.$
3. **Additivity:** If $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B).$

Probability law

Definition – Probability measure

A Probability measure on $(\Omega, \mathcal{F})$ is a function $P : \mathcal{F} \to [0,1]$ satisfying

- $P(\Omega) = 1$
- if $A_i \in \mathcal{F}, i = 1, 2, \ldots$ are such that $A_i \cap A_j = \emptyset$ for all $i \neq j$, then

$$P\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

The triple $(\Omega, \mathcal{F}, P)$ is called a probability space.

Basic properties of probability measures

Theorem

1. $P(A^c) = 1 - P(A)$.
2. If $A \subseteq B$, then $P(A) \leq P(B)$.
3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
   $$P(A \cup B \cup C) = P(A) + P(B) + P(C) \\ - P(AB) - P(AC) - P(BC) + P(ABC).$$

Example – Fair coins

Tossing a coin has $\Omega = \{H, T\}$. Let $P(H) = p$ and $P(T) = 1 - p$. The coin is fair if $p = \frac{1}{2}$.

Basic properties of probability measures

13. A computer consulting firm presently has bids out on three projects. Let $A_i = \{\text{awarded project } i\}$, for $i = 1, 2, 3$ and suppose that

$$P(A_1) = 0.22, P(A_2) = 0.25, P(A_3) = 0.28, P(A_1 \cap A_2) = 0.11,$$

$$P(A_1 \cap A_3) = 0.05, P(A_2 \cap A_3) = 0.07, P(A_1 \cap A_2 \cap A_3) = 0.01.$$

Computer the probability of each event:

a. $A_1 \cup A_2$ **b.** $A_1^c \cap A_2^c$ [Hint: $(A_1 \cup A_2)^c = A_1^c \cap A_2^c$]

c. $A_1 \cup A_2 \cup A_3$ **d.** $A_1^c \cap A_2^c \cap A_3^c$

Solution

e. $A_1^c \cap A_2^c \cap A_3$ [Hint: (using Venn diagram)] **f.** $(A_1^c \cap A_2^c) \cup A_3$

Homework

25. The three most popular options on a certain type of new car are a built-in GPS (A), a sunroof (B), and an automatic transmission (C). If 45% of all purchasers request A, 55% request B, 70% request C, 62% request A or B, 76% request A or C, 80% request B or C, and 85% request A or B or C, determine the probabilities of the following events.
[**Hint:** "A or B" is the event that at least one of the two options is requested; try drawing a Venn diagram and labeling all regions.]
a. The next purchaser will request at least one of the three options.
b. The next purchaser will select none of the three options.
c. The next purchaser will request only an automatic transmission and not either of the other two options.
d. The next purchaser will select exactly one of these three options.

Homework

26. A certain system can experience three different types of defects. Let A_i $(i = 1, 2, 3)$ denote the event that the system has a defect of type i. Suppose that

$$P(A_1) = 0.12, P(A_2) = 0.07, P(A_3) = 0.05, P(A_1 \cup A_2) = 0.13$$
$$P(A_1 \cup A_3) = 0.14, P(A_2 \cup A_3) = 0.10, P(A_1 \cap A_2 \cap A_3) = 0.01$$

a. What is the probability that the system does not have a type 1 defect?

b. What is the probability that the system has both type 1 and type 2 defects?

c. What is the probability that the system has both type 1 and type 2 defects but not a type 3 defect?

d. What is the probability that the system has at most two of these defects?

Conditional probability

Definition

Let A, B be events. The conditional probability $P(A|B)$ the event of "A occurs given that B occured" is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) \neq 0.$$

Example. Rolling two dice.

Let A be the event "the second die shows a greater than the first" and B be the event "the first die shows a ⚄". Then

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Example. Rolling two dice.
Let A be the event "the second die shows a greater than the first" and B be the event "the first die shows a ⚄". Then

$$B = \{(\text{⚄}, \text{⚀}), (\text{⚄}, \text{⚁}), (\text{⚄}, \text{⚂}), (\text{⚄}, \text{⚃}), (\text{⚄}, \text{⚄}), (\text{⚄}, \text{⚅})\}$$

$$A \cap B = \{(\text{⚄}, \text{⚅})\}.$$

Hence, $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{|A \cap B|}{|B|} = \dfrac{1}{6}$

Example

45. The population of a particular country consists of three ethnic groups. Each individual belongs to one of the four major blood groups. The accompanying joint probability table gives the proportions of individuals in the various ethnic group–blood group combinations.

		Blood Group			
		O	**A**	**B**	**AB**
Ethnic Group	**1**	0.085	0.106	0.008	0.004
	2	0.135	0.141	0.015	0.006
	3	0.215	0.200	0.065	0.020

Suppose that an individual is randomly selected from the population, and define events by A = "type A selected", B = "type B selected", C = "ethnic group 1 selected".

a. Calculate $P(A), P(C), P(A \cap C)$, $P(A|C)$ and $P(C|A)$.

b. If the selected individual does not have type A blood, what is the probability that he or she is from ethnic group 2?

Example

				Blood Group	
		O	**A**	**B**	**AB**
	1	0.085	0.106	0.008	0.004
Ethnic Group	**2**	0.135	0.141	0.015	0.006
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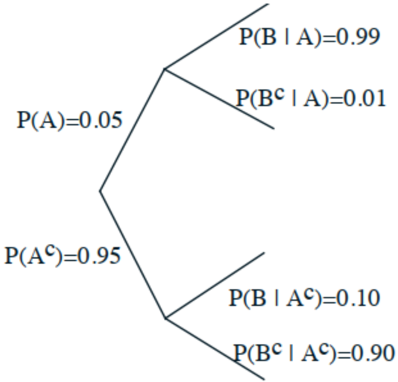
a. Calculate $P(A), P(C), P(A \cap C)$, $P(A|C)$ and $P(C|A)$.

b. If the selected individual does not have type A blood, what is the probability that he or she is from ethnic group 2?

Models based on conditional probabilities

Event A: Airplane is flying above
Event B: Something registers on radar screen



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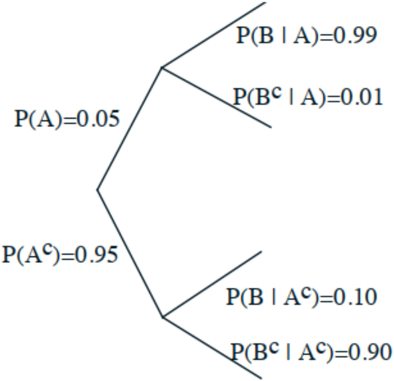

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

$$\begin{aligned} P(A\cap B) &= P(B|A)P(A) \\ &= 0.99 \times 0.05 \end{aligned}$$

$$\begin{aligned} P(B) &= P(A\cap B) + P(A^c \cap B) \\ &= P(B|A)P(A) + P(B|A^c)P(A^c) \\ &= 0.99 \times 0.05 + 0.10 \times 0.95 \end{aligned}$$

Multiplication rule - Total probability theorem

Theorem – Multiplication rule

(i) $P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$
(ii) $P(A \cap B \cap C) = P(A)P(B|A)P(C|AB)$
(iii) Let $A_1, A_2, \ldots, A_n$ be events with $P(A_1 \cap A_2 \cap \ldots \cap A_n) \neq 0$. Then
$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_n|A_{n-1} \ldots A_1) \times \ldots \times P(A_2|A_1)P(A_1)$

Theorem – Total probability theorem

Let $A_1, A_2, \ldots, A_n$ be disjoint events that form a partition of sample space and assume that $P(A_i) > 0$, for all $i = 1, \ldots, n$. Then, for any event B, we have

$$P(B) = P(B|A_1)P(A_1) + \ldots + P(B|A_n)P(A_n)$$

Total probability theorem - Exercise

60. 75% of the light aircraft that disappear while in flight in a certain country are subsequently discovered. Of the aircraft that are discovered, 60% have an emergency locator, whereas 85% of the aircraft not discovered do not have such a locator. Suppose a light aircraft has disappeared.

a. If it has an emergency locator, what is the probability that it will not be discovered?

b. If it does not have an emergency locator, what is the probability that it will be discovered?

Bayes' rule

Theorem – Bayes' rule

Let $A_1, A_2, \ldots, A_n$ be disjoint events that form a partition of sample space and assume that $P(A_i) > 0$, for all $i = 1, \ldots, n$. Then, for any event B such that $P(B) > 0$, we have

$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(B|A_i)P(A_i)}{P(B)} = \frac{P(B|A_i)P(A_i)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}$$

59. At a certain gas station, 55% of the customers use regular gas (A1), 30% use plus gas (A2), and 15% use premium (A3). Of those customers using regular gas, only 70% fill their tanks (event B). Of those customers using plus, 60% fill their tanks, whereas of those using premium, 50% fill their tanks.

a. What is the probability that the next customer will request plus gas and fill the tank?

b. What is the probability that the next customer will fill the tank?

c. If the next customer fills the tank, what is the probability that regular gas is requested? Plus? Premium?

Bayes' rule - Exercise

Homework

103. A certain company sends 35% of its overnight mail parcels via express mail service E_1, 50% of the overnight parcels are sent via express mail service E_2 and the remaining 15% are sent via E_3. Of these parcels sent via E_1, 1% arrive after the guaranteed delivery time (denote the event "late delivery" by L). Of those sent via E_2, only 4% arrive late, whereas 4% of the parcels handled by E_3 arrive late.

a. What is the probability that a randomly selected parcel arrived late?

b. If a randomly selected parcel has arrived on time, what is the probability that it was not sent via E_2?

Independence of two events

Two events A, B are said to be independent if the chance of one occuring is not altered by the other's occurence; that is, $P(A|B) = P(A)$. The multiplication rule implies that

$$P(A \cap B) = P(A|B)P(B) = P(A)P(B)$$

and, by symmetry, $P(B|A) = P(B)$.

Definition

Two events A, B are independent if

$$P(A \cap B) = P(A)P(B)$$

Independence of two events

Example – (a)

Roll two fair dice.

- A = { 1st roll is a *1* }
- B = { sum of two roll is a *7* }
- $A \cap B$ = { the result of the two rolls *(1,6)* }

$P(A) = ?$ $\quad P(B) = ?$ $\quad P(A \cap B) = ?$

Independence of two events

Example – (a)

Roll two fair dice.

- $A = \{$ 1st roll is a 1 $\}$
- $B = \{$ sum of two roll is a 7 $\}$
- $A \cap B = \{$ the result of the two rolls $(1{,}6)$ $\}$

$P(A) = ?$ $P(B) = ?$ $P(A \cap B) = ?$

- $P(A) = \dfrac{6}{36}$ and $P(B) = \dfrac{6}{36}$
- $P(A \cap B) = \dfrac{1}{36}$

Hence, $P(A \cap B) = P(A)P(B)$, and the independence of A and B is verified.

Independence of two events

Example – (b)

Roll two fair dice.

- A = { maximum of the two rolls is 2 }
- B = { minimum of the two rolls is 2 }
- $A \cap B$ = { the result of the two rolls $(2,2)$ }

$P(A) = ?\quad P(B) = ?\quad P(A \cap B) = ?$

Independence of two events

Example – (b)

Roll two fair dice.

- $A = \{$ maximum of the two rolls is 2 $\}$
- $B = \{$ minimum of the two rolls is 2 $\}$
- $A \cap B = \{$ the result of the two rolls $(2,2)$ $\}$

$P(A) =?$ $P(B) =?$ $P(A \cap B) =?$

- $P(A) = \dfrac{3}{36}$ and $P(B) = \dfrac{9}{36}$
- $P(A \cap B) = \dfrac{1}{36}$

Hence, $P(A \cap B) \neq P(A)P(B)$, and A and B is not independent.

Independence of several events

Definition

There events A_1, A_2, A_3 are independent if and only if

(i) $P(A_1 \cap A_2) = P(A_1)P(A_2)$

(ii) $P(A_1 \cap A_3) = P(A_1)P(A_3)$

(iii) $P(A_2 \cap A_3) = P(A_2)P(A_3)$

(iv) $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$

82. Consider independently rolling two fair dice, one red and the other green. Let A be the event that the red die shows 2 dots, B be the event that the green die shows 5 dots, and C be the event that the total number of dots showing on the two dice is 7.

a. Are these events pairwise independent (i.e., are A and B independent events, are A and C independent, and are B and C independent)?

b. Are the three events mutually independent?

Independence of several events

Independence of several events

Remark

1. Pairwise independent does not imply independence.
2. The equality $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$ is not enough for independence.

Definition

The events $A_1, A_2, \ldots, A_n$ are independent if and only if

$$P(\cap_{i \in I} A_i) = \prod_{i \in I} P(A_i), \text{ for every subsets } I \text{ of } \{1, 2, \ldots, n\}.$$

Homework

84. 80% percent of all vehicles examined at a certain emissions inspection station pass the inspection. Assuming that successive vehicles pass or fail independently of one another, calculate the following probabilities:

a. P(all of the next three vehicles inspected pass)

b. P(at least one of the next three inspected fails)

c. P(exactly one of the next three inspected passes)

d. P(at most one of the next three vehicles inspected passes)

e. Given that at least one of the next three vehicles passes inspection, what is the probability that all three pass (a conditional probability)?

87. Consider randomly selecting a single individual and having that person test drive 3 different vehicles. Define events A_1, A_2, and A_3 by

A_1 = likes vehicle #1; A_2 = likes vehicle #2; A_3 = likes vehicle #3. Suppose that

$$P(A_1) = 0.55, P(A_2) = 0.65, P(A_3) = 0.70, P(A_1 \cup A_2) = 0.80,$$

$$P(A_2 \cap A_3) = 0.40, \text{ and } P(A_1 \cup A_2 \cup A_3) = 0.88.$$

a. Are A_2 and A_3 independent events? Answer in two different ways.

b. If you learn that the individual did not like vehicle #1, what now is the probability that he/she liked at least one of the other two vehicles?