Note on low-momentum universality and eigenvectors from eigenvalues

Here are some very loose ideas of how one might prove[1] low-momentum universality using the recent theorem of Denton et al. that expresses the eigenvectors of a Hermitian matrix entirely in terms of its eigenvalues and the eigenvalues of the principle minors. Recall the statement of the theorem. Consider a n-by-n Hermitian matrix H with eigenvalues E_i and eigenvector components $\langle j|\psi_i\rangle$. Then for the i^{th} eigenstate (here I limit myself to the non-degenerate case for simplicity):

$$|\langle j|\psi_i\rangle|^2 = \frac{\prod_{k=1}^{n-1}\{E_i - \lambda_k(M_j)\}}{\prod_{k\neq i}^{n}\{E_i - E_k\}}\,, \tag{1}$$

where $\lambda_k(M_j)$ is the k^{th} eigenvalue of the principle minor M_j, which is defined as the matrix obtained from H by striking out the j^{th} column and row. Note that any SRG scale dependence of the eigenvectors is isolated in the eigenvalues of the principle minors, since the energy eigenvalues should be invariant under SRG transformations.

Now let's consider two Hamiltonians, H^A and H^B that give the same phase shifts over some finite low energy range, but that differ at higher energies. I assume we can extract phase shifts (analogous to Lüscher or Busch formulas) from the energy eigenvalues obtained by diagonalizing the discretized momentum space Schrödinger equation. Therefore, let's translate the phase equivalence of H^A and H^B over some finite energy range to the statement that their discretized eigenvalues are equivalent over (say) the first $n_p < n$ eigenvalues. Presumably E_{n_p} will be similar to the COM energy where the phase shifts of the two potentials begins to differ substantially. Therefore, we have

[1] This is not at all rigorous, as my manipulations assume I'm working with the discretized k-space Hamiltonian. The handwaving at the end about taking the continuum limit is particularly iffy.

$$E_i^A = E_i^B \equiv E_i \qquad (i \le n_p)\,, \tag{2}$$

and

$$E_i^A \ne E_i^B \qquad (i > n_p)\,. \tag{3}$$

Now assume we do a block-diagonal decoupling for each starting Hamiltonian, where Λ is equal to the n_p^{th} momentum mesh point. Then we can write the decoupled low momentum blocks of the $H^{A,B}(\Lambda)$ as:

$$\langle i|H^{A,B}(\Lambda)|k\rangle = \sum_{j=1}^{n_p} E_j \langle i|\psi_j^{A,B}(\Lambda)\rangle\langle\psi_j^{A,B}(\Lambda)|k\rangle\,. \tag{4}$$

Since the first n_p eigenvalues are equal for the two models, a sufficient condition for $H^{A,B}(\Lambda)$ to be the same is that their low energy wave functions, which live entirely in the low-momentum subspace, are the same.[2]

Now let's apply the eigenvector identity of Denton et al. to a low energy (i.e., one of the first n_p) eigenvector of one of the block diagonal Hamiltonians $H^A(\Lambda)$ or $H^B(\Lambda)$. The first thing to note is that, due to the block diagonal structure of the evolved Hamiltonian, the principle minor M_j with $j \le n_p$ is also block diagonal. The eigenvalues of the $(n_p-1)\times(n_p-1)$ low momentum sub-block of M_j interlace with the n_p lowest eigenvalues of $H^{A,B}(\Lambda)$ due to the well-known result of Cauchy,

$$E_k \le \lambda_k(M_j) \le E_{k+1} \qquad (1 \le k \le n_p - 1)\,, \tag{5}$$

while the high-momentum block of M_j is (for $j \le n_p$) exactly equal to the high-momentum block of $H(\Lambda)$. Therefore, its eigenvalues obey

$$\lambda_k(M_j) = E_{k+1} \qquad (n_p \le k \le n-1)\,. \tag{6}$$

If we now make use of these features in Eq. 1 for a low-energy eigenstate (which only has non-zero low-momentum components) of $H(\Lambda)$, we get the simplified form

$$|\langle j|\psi_i^{A,B}(\Lambda)\rangle|^2 = \frac{\prod\limits_{k=1}^{n_p-1}\{E_i - \lambda_k(M_j^{A,B}(\Lambda))\}}{\prod\limits_{k\ne i}^{n_p}\{E_i - E_k\}}\,. \tag{7}$$

[2]This is certainly consistent with what we've seen in our calculations to date. It would be interesting to see if we can show that the low-momentum components of the initial unevolved low-energy wave functions are identical and hardly changed under SRG evolution. I haven't looked at this yet. For now let's stick with trying to show that the $|\psi_j^A(\Lambda)\rangle = |\psi_j^B(\Lambda)\rangle$ for $j \le n_p$.

Therefore, the ratio of the SRG-evolved wave functions for the 2 different models is

$$\frac{|\langle j|\psi_i^A(\Lambda)\rangle|^2}{|\langle j|\psi_i^B(\Lambda)\rangle|^2} = \frac{\prod_{k=1}^{n_p-1}\{E_i - \lambda_k(M_j^A(\Lambda))\}}{\prod_{k=1}^{n_p-1}\{E_i - \lambda_k(M_j^B(\Lambda))\}} \,. \tag{8}$$

Combining Eq. 5 together with the fact that as we add more and more mesh points to our discretization (i.e., approach the continuum limit), the spacing between adjacent eigenvalues E_i and E_{i+1} approaches zero, the above ratio should (?) approach 1. Therefore, the two low-momentum blocks of $H^A(\Lambda)$ and $H^B(\Lambda$ should become equal.