Landau Fermi liquid theory I: phenomenological approach

In this lecture, we develop Landau's Fermi liquid theory (1957)

Fermi liquid theory is the effective theory of low-energy, low-momentum excitations (the quasiparticles (holes) discussed in the previous lecture) in interacting Fermi systems at low temperatures. The microscopic derivation of Fermi liquid theory will follow in the next lecture

The quasiparticle excitations are long-lived in the vicinity of the Fermi surface $\tau_{\vec{k}}^{-1} \sim (\varepsilon_{qp}(\vec{k}) - \mu)^2$ (or $\sim T^2$), thus at low temperatures the (effective) degrees of freedom are quasiparticles / holes in the vicinity of the Fermi surface, and we expand the quasiparticle propagator around $\omega \gtrsim \mu$ and $|\vec{k}| \gtrsim k_F$

will follow

$$G_{qp}(\vec{k},\omega) = \frac{z_{\vec{k}}}{\omega - \varepsilon_{qp}(\vec{k}) + \frac{i}{\tau_{\vec{k}}}} = \frac{z_{\vec{k}}}{\omega - \underbrace{\varepsilon_{\vec{k}} - \Sigma(\vec{k}, \varepsilon_{qp}(\vec{k}))}_{\varepsilon_{qp}(\vec{k}) + \text{Im part}}}$$

with $$z_{\vec{k}} = \frac{1}{1 - \frac{\partial \Sigma}{\partial \omega}\Big|_{\omega = \varepsilon_{qp}(\vec{k})}}$$

$$\varepsilon_{\vec{k}} + \Sigma(\vec{k}, \varepsilon_{qp}(\vec{k})) = \varepsilon_{qp}(k_F) + \frac{k_F}{m}(k-k_F) + (k-k_F)\left(\frac{\partial \Sigma}{\partial k} + \frac{\partial \Sigma(\vec{k}, \varepsilon_{qp}(\vec{k}))}{\partial \varepsilon_{qp}(\vec{k})} \frac{d\varepsilon_{qp}(\vec{k})}{dk}\right)$$

$k = k_F$ and $\varepsilon_{qp}(k_F)$

This requires some special care.

We also drop $\frac{1}{\tau_{\vec{k}}}$, since $\sim (\varepsilon_{qp}(\vec{k}) - \mu)^2$ and thus it goes faster to 0 as $k \to k_F$ as the $(k-k_F)$ term.

$$\varepsilon_{qp}(\vec{k}) - \varepsilon_{qp}(k_F) = \left.\frac{d\varepsilon_{qp}(\vec{k})}{dk}\right|_{k_F} (k-k_F)$$

$$= \frac{k_F}{m}(k-k_F) + (k-k_F)\left(\left.\frac{\partial\Sigma}{\partial k}\right|_{\substack{k=k_F\\ \omega=\mu}} + \left.\frac{\partial\Sigma}{\partial\omega}\right|_{\substack{k=k_F\\ \omega=\mu}} \left.\frac{d\varepsilon_{qp}(\vec{k})}{dk}\right|_{k_F} \right)$$

$$\Longrightarrow \left.\frac{d\varepsilon_{qp}(\vec{k})}{dk}\right|_{k_F} = \frac{k_F}{m}\left(1 + \frac{m}{k_F}\left.\frac{\partial\Sigma}{\partial k}\right|_{\substack{k=k_F\\ \omega=\mu}}\right) \underbrace{\left(1 - \left.\frac{\partial\Sigma}{\partial\omega}\right|_{\substack{k=k_F\\ \omega=\mu}}\right)^{-1}}_{Z_{k_F}}$$

We define the <u>effective masses</u>

"k-mass" $\dfrac{m_k}{m} = \dfrac{1}{1 + \dfrac{m}{k_F}\left.\dfrac{\partial\Sigma}{\partial k}\right|_{\substack{k=k_F\\ \omega=\mu}}}$ $\longrightarrow$ reflects the spatial non-locality of the self energy.

"ε-mass" $\dfrac{m_\varepsilon}{m} = \dfrac{1}{Z_{k_F}}$

and $\dfrac{m^*}{m} = \dfrac{m_\varepsilon}{m}\,\dfrac{m_k}{m}$

Then, we have $\left.\dfrac{d\varepsilon_{qp}(\vec{k})}{dk}\right|_{k=k_F} = \dfrac{k_F}{m}\,\dfrac{1}{\frac{m_k}{m}\frac{m_\varepsilon}{m}} = \dfrac{k_F}{m^*}$

$\Longrightarrow$ quasiparticle propagator in the vicinity of the Fermi surface (quasihole $\to -i\eta$)

$$\boxed{G_{qp}(\vec{k},\omega) = \frac{Z_{k_F}}{\omega - \underbrace{\varepsilon_{qp}(k_F)}_{\mu} - \frac{k_F}{m^*}(k-k_F) + i\eta}\,, \quad \begin{matrix}\omega \gtrsim \mu \\ k \approx k_F\end{matrix}}$$

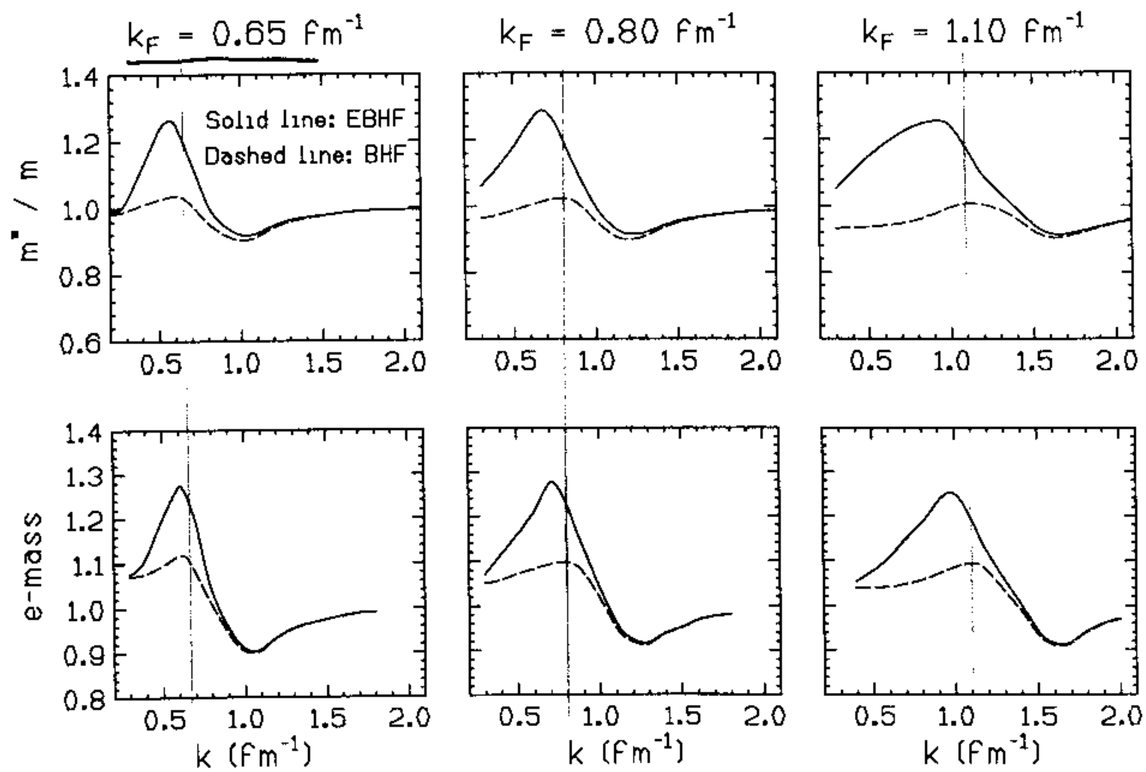
One widely uses a quadratic dispersion relation to approximate away from the Fermi surface to general k: $\varepsilon_{qp}(\vec{k}) \approx \frac{k^2}{2m^*}$

With this the qp Green's function is given by

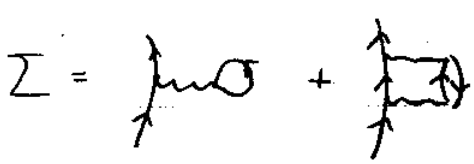
$$G_{qp} \simeq \frac{z_{k_F}}{\omega - \frac{k^2}{2m^*} + i\eta}$$ and $\mu = \frac{k_F^2}{2m^*}$ as in free case, but $m \to m^*$ and $z_{k_F} < 1$.

One generally finds that m_ε is peaked at the Fermi surface, whereas m_k is smooth.

Results for neutron matter, e.g., Lombardo et al., PR C64 (2001) 021301.



EBHF: $\Sigma =$



Landau's approach to interacting Fermi liquids

In the last lecture, we found that the elementary excitations in the interacting system are similar to that of a free Fermi gas. This followed from the analytic structure of the Green's function and the constraints imposed by symmetry.

We use the one-to-one correspondence of states to classify the states of the interacting system by the distribution of particles corresponding to the free Fermi gas.

Fermi systems which have a one-to-one correspondence between the states of the free Fermi gas and the interacting system, i.e., if one takes a non-interacting Fermi gas in a particular state and adiabatically turns on the interaction between particles, one obtains a state of the interacting system, are called normal Fermi liquids. Not a normal Fermi liquid are e.g., superfluids, superconductors, ...

We do not know from first principles, whether a given system is a Fermi liquid, but Landau's Fermi liquid theory makes testable predictions.

We introduce the quasiparticle distribution function

$n_{\vec{p}\sigma}$ = qp dist. function

σ = ±½ spin of qp

A normal Fermi liquid is characterized by $n_{\vec{p}\sigma}$, and therefore the energy of the system is a functional of $n_{\vec{p}\sigma}$:

$E = E[n_{\vec{p},\sigma}]$ (think of a free Fermi gas with $\frac{k^2}{2m^*}$)

If we add a qp to an unoccupied state $|\vec{k}| > k_F$ (or add a quasihole, i.e., remove a qp occupying the state $|\vec{k}| < k_F$), the total energy increases by an amount, the quasiparticle energy $\varepsilon_{qp}(\vec{k})$

For $\delta n_{\vec{k}\sigma} = \begin{cases} +1 & |\vec{k}| > k_F \\ -1 & |\vec{k}| < k_F \end{cases}$, the qp energy is defined as the first variation of the energy

$$\boxed{\delta E = \sum_{\vec{k},\sigma} \varepsilon_{qp}(\vec{k},\sigma)\, \delta n_{\vec{k}\sigma}}$$

For any variation about thermodynamic equilibrium, we have

$$\delta E = T\,\delta S + \mu\, \delta N$$

variation in entropy

variation of particle number

The entropy is given by combinatorical considerations, and since the states of the interacting system are in one-to-one correspondence with the states of the free Fermi gas, the entropy density must have the same form as free gas

$$S = -\sum_{\vec{p}\sigma} \left[n_{\vec{p}\sigma} \ln(n_{\vec{p}\sigma}) + (1-n_{\vec{p}\sigma}) \ln(1-n_{\vec{p}\sigma}) \right] \qquad (k_B = 1)$$

The number of quasiparticles in the interacting system equals the number of particles in the corresponding case of the free Fermi gas, [1]

$$N = \sum_{\vec{p}\sigma} n_{\vec{p}\sigma}$$

$$\Rightarrow \quad \delta S = -\sum_{\vec{k}\sigma} \Big[\delta n_{\vec{k}\sigma} \ln n_{\vec{k}\sigma} + n_{\vec{k}\sigma} \frac{1}{n_{\vec{k}\sigma}} \delta n_{\vec{k}\sigma} - \delta n_{\vec{k}\sigma} \ln(1-n_{\vec{k}\sigma}) + (1-n_{\vec{k}\sigma}) \frac{1}{1-n_{\vec{k}\sigma}} (-\delta n_{\vec{k}\sigma})$$

$$= -\sum_{\vec{k}\sigma} \delta n_{\vec{k}\sigma} \ln \frac{n_{\vec{k}\sigma}}{1-n_{\vec{k}\sigma}}$$

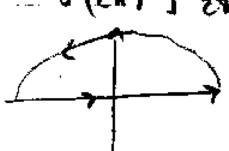
$$\delta N = \sum_{\vec{k}\sigma} \delta n_{\vec{k}\sigma}$$

[1] This may again be understood from the pole structure of the Green's function.

We have $\rho(\vec{x}) = -i \,\mathrm{tr}\, G(\vec{x},t;\vec{x},t^+)$

$$= -i \,\mathrm{tr}\, \delta_{\alpha\beta} \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} e^{i\vec{k}(\vec{x}-\vec{x})} e^{-i\omega(t-t^+)} G(\vec{k},\omega)$$

$$= -i\, 2 \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} e^{-i\omega(t-t^+)} \left[\int_0^\infty \frac{d\omega'}{2\pi} \left(\frac{\rho^+(\vec{k},\omega')}{\omega-\mu-\omega'+i\eta} + \frac{\rho^-(\vec{k},\omega')}{\omega-\mu+\omega'-i\eta} \right) \right]$$



ω Im $\omega > 0$ and we close this way

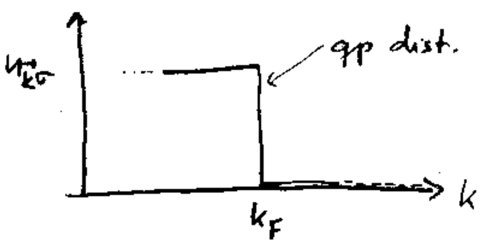
$$\Rightarrow \quad \varepsilon_{qp}(\vec{k}\sigma) = \mu + T \ln\left(n_{\vec{k}\sigma}^{-1} - 1\right)$$

Or $\quad n_{\vec{k}\sigma} = \dfrac{1}{e^{\frac{\varepsilon_{qp}(\vec{k}\sigma)-\mu}{T}} + 1}$ Fermi-Dirac distribution

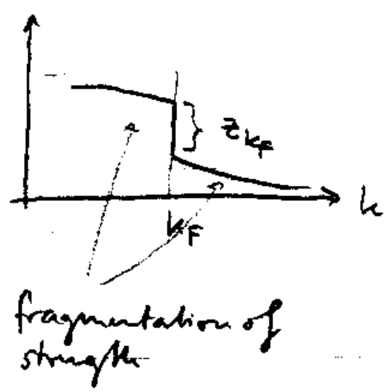
(Note, $\varepsilon_{qp}(\vec{k}\sigma)$ depends on $n_{\vec{k}\sigma}$ itself, so complicated distribution)

At $T=0$: $\quad n_{\vec{k}\sigma} = \begin{cases} 1 & \varepsilon_{qp}(\vec{k}\sigma) < \mu \\ 0 & \text{otherwise} \end{cases}$ $\rightarrow$ Fermi sea occupied up to k_F.

Thus, close to the Fermi surface



one can show that this corresponds to a bare-particle distribution



$$\Rightarrow \rho(\vec{x}) = -2i \int \frac{d^3k}{(2\pi)^3} \int_0^\infty \frac{d\omega'}{2\pi} \frac{2\pi i}{2\pi} \underbrace{g^-(\vec{k}, \mu - \omega')}_{\substack{N \text{ simple} \\ \text{free particle poles for } |\vec{k}| < k_F}} = 2 \int \frac{d^3k}{(2\pi)^3} \, \Theta(k_F - |\vec{k}|)$$

$\Rightarrow$ Since every free particle pole turns into a qp pole as we turn on the interactions, the number of particles = number of quasiparticles

Because of the interaction between quasiparticles, $\varepsilon_{qp}(\vec{k},\sigma)$ depends on the qp distribution. The quasiparticle interaction is defined by

$$\boxed{\underbrace{\delta\varepsilon_{qp}(\vec{k}\sigma)}_{\text{change in qp energy due to the presence of } \vec{k}',\sigma' \text{ qp}} = \frac{1}{V}\sum_{\vec{k}',\sigma'} \underbrace{f_{\vec{k}\sigma,\vec{k}'\sigma'}}_{\text{qp interaction}}\, \delta n_{\vec{k}'\sigma'}}$$

Thus, qp interaction is a second variation of E

$$\boxed{f_{\vec{k}\sigma,\vec{k}'\sigma'} = V\frac{\delta^2 E}{\delta n_{\vec{k}\sigma}\,\delta n_{\vec{k}'\sigma'}}}$$

obviously sym in $\vec{k}\sigma;\vec{k}'\sigma'$

And the variation of the energy due to $\delta n_{\vec{k}\sigma}$ close to the Fermi surface can be written as

$$\boxed{E = E_0 + \sum_{\vec{k}\sigma}\varepsilon_{qp}(\vec{k},\sigma) + \frac{1}{2V}\sum_{\vec{k}\sigma,\vec{k}'\sigma'} f_{\vec{k}\sigma,\vec{k}'\sigma'}\,\delta n_{\vec{k}\sigma}\,\delta n_{\vec{k}'\sigma'}}$$

The quasiparticle interaction is constrained by symmetries:

(i) rotational invariance in space

(ii) " " in spin space

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*(Aside: if we have tensor or spin-orbit forces, only inv. under combined rotations)

[nuclear case (iii) " " " isospin space]

$$\boxed{f_{\vec{k}\sigma,\vec{k}'\sigma'} = \underbrace{f_{\vec{k}\alpha_1\beta_1,\vec{k}'\alpha_2\beta_2}}_{\text{if not all qp are eigenstates of } \sigma_z} = f_{\vec{k}\vec{k}'}\,\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2} + g_{\vec{k}\vec{k}'}\,\vec{\sigma}_{\alpha_1\beta_1}\cdot\vec{\sigma}'_{\alpha_2\beta_2}}$$

At low T, need to consider interactions of quasiparticles both basically on the Fermi surface. Therefore, we can expand f and g in terms of the angle between $\vec{k}$ and $\vec{k}'$: $\hat{k}\cdot\hat{k}' = \cos\theta$.

$$f_{\vec{k}\vec{k}'} = \sum_{l=0}^{\infty} f_l\, P_l(\cos\theta) \quad \text{and} \quad g_{\vec{k}\vec{k}'} = \sum_{l=0}^{\infty} g_l\, P_l(\cos\theta)$$

f_l, g_l are called Fermi liquid parameters (assumed spherical Fermi surface)
Dimensionless Fermi liquid parameters $F_l = N(0)\, f_l$, $G_l = N(0)\, g_l$, where $N(0) = \frac{k_F m^*}{\pi^2}$ is the density of states at the Fermi surface, provide a measure of the strength of the qp interaction on the Fermi surface.

<u>Observable properties of a normal Fermi liquid</u>