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Wednesday 880.05

• Questions on PS#3?

• Recap: We are exploring an EFT ⇒ $\boxed{\mathcal{L}_E^{EFT} = \psi^\dagger\left[\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2M}\right]\psi + \frac{C_0}{2}(\psi^\dagger\psi)^2 + \dots}$

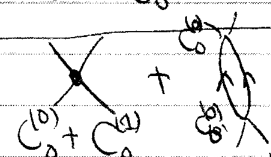
• We calculated  contribution to energy density from the Feynman rules and found

$$\boxed{\begin{aligned}\mathcal{E}_2 = 4M\nu(\nu-1)\left(C_0^{(0)}\right)^2 k_F^7 \int\frac{d^3s}{(2\pi)^3}\int\frac{d^3t}{(2\pi)^3}\int\frac{d^3u}{(2\pi)^3}\,\frac{1}{t^2-u^2}\\ \times\left[\theta(1-|\vec{s}+\vec{t}|)\,\theta(1-|\vec{s}-\vec{t}|)\,\theta(|\vec{s}+\vec{u}|-1)\,\theta(|\vec{s}-\vec{u}|-1)\right]\end{aligned}}$$

⟹ ∞! Consider scattering to understand

• In the EFT, we found the same divergence in  but we could fix by "cutting off" the integral at Λ_c and choosing

$$\boxed{C_0 = C_0^{(0)} + C_0^{(1)} = \frac{4\pi a_s}{M} + \left(C_0^{(0)}\right)^2\frac{M}{2\pi^2}\Lambda_c = \frac{4\pi a_s}{M}\left(1 + \frac{2a_s\Lambda_c}{\pi}\right)}$$

so that 

= finite! ⇒ $\boxed{-T(k,\cos\theta) = \frac{4\pi a_s}{M}\left[1 - i a_s k + O(k^2)\right]}$

scattering amplitude

⇒ shift from loop integral to vertex!

• We noted that using Λ_c is awkward and left all orders in k/Λ_c
⇒ motivation for dimensional regularization.

Moving on:

① briefly check that this fix really fixes $\mathcal{E}_2$ (173)-(174)

② complete dim. reg. discussion + power counting (175)-(183) ← (in part)

③ discuss effective action in context of Bose limit (185)-(198)

and then continue...

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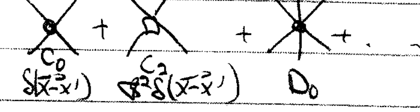
- In nuclei, despite powerful short-range forces between protons and neutrons (like an incompressible liquid in many ways), the nucleons behave as if they move independently of each other
    - shell model prediction of orbitals
    - electron scattering evidence (last week)

  $\Rightarrow$ not bare nucleons, but quasinucleons.

- Although quasiparticles are good degrees of freedom to deal with, their derivation from the underlying forces (as specified by the Hamiltonian or Lagrangian) may be prohibitively difficult.
    - But Fermi liquid theory indicates that we don't need to derive the properties of the quasiparticles and their interactions.
    - Instead we parametrize our ignorance systematically
        - $\Rightarrow$ effective mass, parameters F_0, F_1, G_0, F_1,
        - $\Rightarrow$ write most general form, fit parameters with one set of observables, then predict other observables.

$\Rightarrow$ a form of effective field theory

cf. using most general Lagrangian with 

replacing the actual, complicated interaction.

- You'll gain familiarity with Fermi liquid parameters in homework problem $\Rightarrow$ apply to 1-d.
  (maybe)

- Now we switch gears and go back to ground state properties
    - nonperturbative systems
    - spontaneous symmetry breaking

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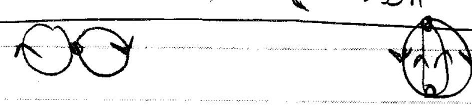
- The "tool" that we'll use to study these things is the <u>effective action</u>.
  - As an introduction we'll apply the effective action formalism in a rather unexpected way: deriving the boson ground state energy for a dilute gas from the dilute fermion calculation.

- We found that the energy density (energy per particle) for a low density, uniform system of fermions with degeneracy ν is (for spin-independent interactions):

$$\boxed{\mathcal{E} = \rho \frac{k_F^2}{2m}\left[\frac{3}{5} + (\nu-1)\frac{2}{3\pi}(k_F a_s) + (\nu-1)\frac{4}{35\pi^2}(11-2\ln 2)(k_F a_s)^2 + \dots\right]}$$

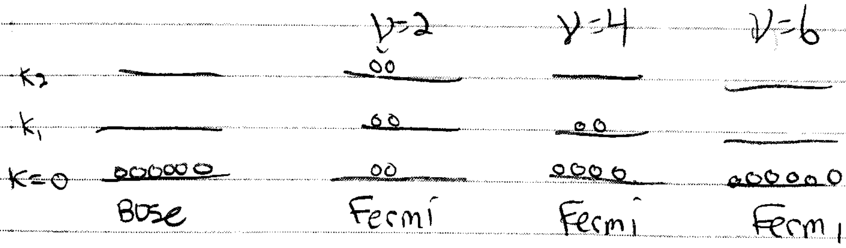


- The corresponding answer for a dilute (spinless) Bose system (again, with spin-independent interaction) is:

$$\boxed{\mathcal{E} = \frac{2\pi a_s \rho^2}{m} + \frac{2\pi a_s}{m}\rho^2 \frac{128}{15\sqrt{\pi}}\sqrt{\rho a_s^3} + \dots}$$

- <u>Is</u> there a connection? $k_F \propto \rho^{1/3}$, but Bose $\mathcal{E}$ has $\rho^{5/2}$! Where does that exponent come from?

- Let's start by considering Bose and Fermi <u>noninteracting</u> ground states (i.e. $T=0$). Suppose there are 6 particles



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So the Bose ground state with N particles has the same occupation numbers as a Fermi system with $\nu \geq N$.

- How are the wave functions related? Fermi wave function must be totally antisymmetric under particle exchange
  - $\Rightarrow$ same spatial wave function $*$ totally antisymmetric spin (or flavor) wave function.
- e.g. $N=2, \nu=2$ $\quad \Psi_{\text{Fermi}}(x_1, x_2) = \varphi_{\text{Bose}}(\vec{x}_1, \vec{x}_2) \times (\uparrow\downarrow - \downarrow\uparrow)$

$\Rightarrow$ for a noninteracting Bose system, introduce an artificial "flavor" or "color" degeneracy ν and make it bigger than the number of particles.

- Treat like Fermi system and ignore flavor wavefunction at the end.

- Then turn on the interaction $\Rightarrow$ if the state that evolves adiabatically is the ground state (true for a dilute system), then we generate the Bose interacting ground state!

Plan: Calculate the dilute Fermi system for arbitrary g.
Take $\nu \to \infty$ with ρ constant (thermodynamic limit)

$$\Rightarrow \boxed{\rho = \frac{\nu k_F^3}{6\pi^2}} \text{ constant } \Rightarrow \boxed{k_F \to 0} \Rightarrow \text{Bose ground state}$$

- Let's try it on the diagrams of the Fermi series.
- Recall that this is a systematic expansion in powers of $(k_F a_s)$.
- Every additional $\bullet$ means another power of k_F
- We can count maximum powers of ν easily
  - $\Rightarrow$ find $\nu^i k_F^j$ for any diagram.

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Try out the first Three Fermi terms...

$$\boxed{\mathcal{E}_0 = \frac{3}{5}\frac{k_f^2}{2m}\rho \xrightarrow[k_f\to 0]{} 0}$$

Ok, in noninteracting Bose system condensed with $k=0$ $\Rightarrow$ no kinetic energy

$$\boxed{\mathcal{E}_1 = \frac{k_f^2}{2m}\left((\nu-1)\frac{2}{3\pi}(k_f a_s)\right)\rho = \left(1-\frac{1}{\nu}\right)\frac{2\pi a_s \rho^2}{m} \xrightarrow[\nu\to\infty]{} \frac{2\pi a_s \rho^2}{m}}$$

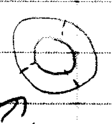
no exchange!

✓ Hartree (but not Fock) survives. Gives known answer.

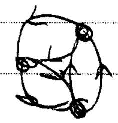
$$\boxed{\mathcal{E}_2 = \frac{k_f^2}{2m}(\nu-1)\frac{4}{35\pi^2}(11-2\ln 2)(k_f a_s)^2\rho \propto \nu^2 k_f^7 \propto (\nu k_f)^3 k_f \xrightarrow{k_f\to 0} 0}$$

$\Rightarrow$ no beach ball contribution.

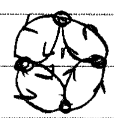
What about particle-particle (hole-hole) rings?



maximum ν's



$$\boxed{\mathcal{E} \propto \nu^2 k_f^8 \propto (\nu k_f^3)^2 k_f^2 \xrightarrow[k_f\to 0]{} 0}$$

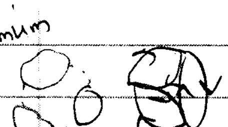


$$\boxed{\mathcal{E} \propto \nu^2 k_f^9 \propto (\nu k_f^3)^2 k_f^3 \xrightarrow[k_f\to 0]{} 0}$$

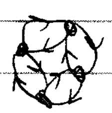
and so on. $\Rightarrow$ all zero!

But what about particle-hole rings?

maximum ν's



$$\boxed{\mathcal{E} \propto \nu^3 k_f^8 \propto (\nu k_f^3)^3/k_f \xrightarrow[k_f\to 0]{} \infty}$$ oops!



$$\boxed{\mathcal{E} \propto \nu^4 k_f^{10} \propto (\nu k_f^3)^4/k_f^2 \xrightarrow[k_f\to 0]{} \infty}$$ even worse!

<u>Plan</u>: Problem is $\nu\to\infty$, so figure out how to solve the system in large ν limit.

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Hint of what to do by considering

$$\boxed{Z = \int \mathcal{D}(\psi^\dagger,\psi)\, e^{i\int d^4x\left[\psi^\dagger_\alpha\left(i\frac{\partial}{\partial t}+\frac{\nabla^2}{2m}+\mu\right)\psi_\alpha - \frac{1}{2}C_0\left(\psi^\dagger_\alpha\psi_\alpha\right)^2\right]}}$$

- Heuristically, $\psi^\dagger_\alpha\psi_\alpha \longrightarrow$ Factor of $\nu \Rightarrow$ "$\nu(\psi^\dagger\psi)$"
- let $C_0 = c_0/\nu$

$$\Rightarrow \boxed{Z = \int \mathcal{D}(\psi^\dagger\psi)\, e^{i\nu\int d^4x\left[\psi^\dagger\left(i\frac{\partial}{\partial t}+\frac{\nabla^2}{2m}+\mu\right)\psi - \frac{1}{2}c_0\left(\psi^\dagger\psi\right)^2\right]}}$$

so there is an overall factor of ν. Solve this problem in the large ν limit.

- Recall our model partition function and introduce "l"

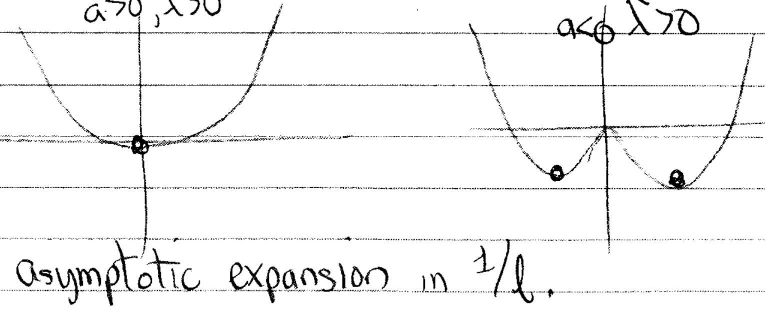
$$\boxed{Z = \int_{-\infty}^{\infty}\frac{dx}{\sqrt{2\pi a}}\, e^{-\frac{x^2}{2a}}\, e^{-\frac{\lambda}{4}x^4}} \quad \leftarrow a \text{ or } \frac{1}{a}?$$

- let $\lambda = \tilde{\lambda}/l$, $x = \sqrt{l}\,t \Rightarrow dx = \sqrt{l}\,dt$

$$\boxed{Z(l) = \sqrt{\frac{l}{2\pi a}}\int_{-\infty}^{\infty} dt\; e^{-l\left(\frac{t^2}{2a}+\frac{\tilde{\lambda}}{4}t^4\right)}}$$

what happens as l gets large?

$\Rightarrow$ more and more of the contribution to $Z(l)$ comes from region in integrand near stationary points of integral (where the derivative of the integrand vanishes).



$\Rightarrow$ asymptotic expansion in $1/l$.

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Consider the more general integral

$$\boxed{I(\ell) = \int_{-\infty}^{\infty} dt \; e^{-\ell f(t)}}$$

where $f(t)$ has an absolute minimum at $t = t_0$ (just one, to keep things simple).

- Expand $f(t)$ about $t = t_0$

$$\boxed{\begin{aligned} f(t) &= f(t_0) + (t-t_0) f'(t_0) + \tfrac{1}{2}(t-t_0)^2 f''(t_0) + \sum_{n=3}^{\infty} \tfrac{1}{n!}(t-t_0)^n f^{(n)}(t_0) \\ &= f_0 + (t-t_0)\cancelto{0}{f'(t_0)} + \tfrac{1}{2}(t-t_0)^2 f_0'' + \sum_{n=3}^{\infty} \tfrac{1}{n!}(t-t_0)^n f_0^{(n)} \end{aligned}}$$

- Since t_0 is a minimum, $\boxed{f_0' = 0}$ and $\boxed{f_0'' > 0}$

So

$$\boxed{I(\ell) = e^{-\ell f_0} \int_{-\infty}^{\infty} dt \; e^{-\frac{\ell}{2} f_0'' (t-t_0)^2 - \ell \sum_{n=3}^{\infty} \frac{(t-t_0)^n}{n!} f_0^{(n)}}}$$

As ℓ gets large, the dominant part of the integral comes from $t \sim t_0$ $\Rightarrow$ shift variables to

$$\boxed{\tau \equiv \sqrt{\ell}\,(t-t_0)}$$

$$\Rightarrow \boxed{I(\ell) = e^{-\ell f_0} \sqrt{\frac{2\pi}{\ell}} \int_{-\infty}^{\infty} d\tau \; e^{-\frac{f_0'' \tau^2}{2}} \, e^{-\sum_{n=3}^{\infty} \frac{\tau^n}{n!} \frac{f_0^{(n)}}{\ell^{(n/2-1)}}}}$$

as $\ell \to \infty$, we can expand $I(\ell)$ (or $\ln I(\ell)$, which is really what we want) in powers of $1/\ell$.

$\Rightarrow$ expand exponent and do Gaussian integrals

$$\Rightarrow \boxed{I(\ell) = e^{-\ell f_0 + \frac{1}{2}\ln\left(\frac{2\pi}{\ell f_0''}\right) + \frac{1}{\ell}\left(\frac{5}{24}\frac{(f_0^{(3)})^2}{(f_0'')^3} - \frac{1}{8}\frac{f_0^{(4)}}{(f_0'')^2}\right) + \mathcal{O}\left(\frac{1}{\ell^2}\right)}}$$

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- Note that the Gaussian $\left(\frac{1}{\ell}\right)^0$ integrand gives a determinant (1×1 in this case!), which we exponentiate to get a ln correction to the leading ℓf_0 term

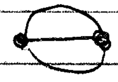
- We can apply our diagrammatic expansion approach (introduce a source term $j\tau$, remove the interaction term using $\frac{\partial}{\partial j}$, complete the square, apply the linked cluster theorem — replica method!)

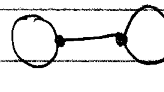
$$\Rightarrow \boxed{\ln I(\ell) = -\ell f_0 + \frac{1}{2}\ln\left(\frac{2\pi}{\ell f_0''}\right) + \sum(\text{all linked diagrams})}$$

where the "Feynman rules" are

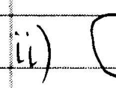
1) $1/f_0''$ for each "propagator"
2) $-\frac{f_0^{(n)}}{\ell^{(n/2-1)}} =$  for a vertex with n legs
3) same symmetry factor as before

- Let's reproduce the $1/\ell$ terms. To get $1/\ell$ either:
  i) $2\times(n=3)$ vertices or ii) $1\times(n=4)$ vertex.

i)  $\left(-\frac{f_0^{(3)}}{\ell^{1/2}}\right)^2\left(\frac{1}{f_0''}\right)^3\left(1\cdot\frac{1}{3!}\cdot\frac{1}{2}\right) = \frac{1}{12}\frac{(f_0^{(3)})^2}{\ell(f_0'')^3}$

 $\left(-\frac{f_0^{(3)}}{\ell^{1/2}}\right)^2\left(\frac{1}{f_0''}\right)^3\left(\left(\frac{1}{2}\right)^2\cdot1\cdot\frac{1}{2}\right) = \frac{1}{8}\frac{(f_0^{(3)})^2}{\ell(f_0'')^3}$

$\Big\}\ \frac{1}{\ell}\frac{5}{24}\frac{(f_0^{(3)})^2}{(f_0'')^3}$

ii)  $\left(-\frac{f_0^{(4)}}{\ell}\right)^1\left(\frac{1}{f_0''}\right)^2\left(\left(\frac{1}{2}\right)^2\cdot\frac{1}{2}\cdot1\right) = -\frac{1}{8}\frac{f_0^{(4)}}{\ell(f_0'')^2}$

which agrees with our previous expression.

- We can generalize to the complex exponent case (see Negele + Orland).
- We'll see the same structure: "classical" + "trace ln" + diagrams.

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- The effective action is found as a functional Legendre transformation of $\ln Z[J]$ (where J is a source).
  - It can be minimized to obtain the ground state energy and is particularly useful when we have "spontaneous symmetry breaking"
  - We'll expand $Z[J]$ in a "loop expansion" (this is the saddle-point / stationary phase expansion) that generates the large ν expansion of the effective action

- Recall that we've already reviewed Legendre transformations in thermodynamics; which relate

$$E \to F \to G \to \Omega$$

- Consider another example, which will be a relevant analogy when we consider pairing: a spin system with Hamiltonian $\mathcal{H}(s)$.
  - For example:

external magnetic field "source" — spin operator

$$\boxed{\mathcal{H}(s) = -\frac{1}{2N} J \sum_{\{ij\}} \vec{S}_i \cdot \vec{S}_j - \vec{H} \cdot \sum_i \vec{S}_i}$$

where the sums are over lattice sites.
(actually just the first term is $\mathcal{H}(s)$.)

- This is a highly contrived example, because we sum the interaction over all pairs of spins (unlike the Ising model, where only nearest neighbors interact)
  - We take the exchange energy to be $-J/N$, so there is a finite $N \to \infty$ limit
  - What is the physical origin of the exchange energy? Why is it unrealistic to say it is long ranged?

- The external magnetic field $\vec{H}$ acts like a source J.

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The partition function Z is (dropping vectors and let $S_i = \pm 1$)

$$\boxed{Z(\beta, H, N) = \sum_{\{s_i\}} e^{\beta\left(\frac{1}{2}\frac{J}{N}\sum_{\{ij\}} S_i S_j + H\sum_i S_i\right)} \longrightarrow \int \mathcal{D}\vec{s}\, e^{-\beta\int dx\,[\mathcal{H}(s) - H s(x)]}}$$

configurations of spin – 2^N of them!

Now the Helmholtz free energy is obtained as

$$\boxed{F = -\frac{1}{\beta}\ln Z} \quad \text{or} \quad \boxed{Z = e^{-\beta F(H)}}$$

The magnetization $\overline{m}$ is the expectation value of $\sum_i s_i$:

$$\boxed{m = \frac{1}{Z}\mathrm{Tr}\left\{\left(\sum_i s_i\right) e^{-\beta(\mathcal{H}(s) + H\sum_i s_i)}\right\} = -\frac{\partial F(H)}{\partial H}}$$

$\Rightarrow$ find $m = m(H)$

· Now we can invert this equation to find $H = H(m)$ and define the Gibb's free energy by the Legendre transformation:

$$\boxed{G(m) = F(H(m)) + mH(m)}$$

note that

$$\boxed{\frac{\partial G}{\partial m} = \overbrace{\frac{\partial F}{\partial H}}^{-m}\frac{\partial H}{\partial m} + H + m\frac{\partial H}{\partial m} = H}$$

so that G as a function of m is minimized at $H=0$.

· That is, if $H=0$, the most stable state is the minimum of $G(m)$.

$\Rightarrow$ use this to study ferromagnetism!

· $F(H)$ and $G(m)$ in principle have the same physical information.

· But $G(m)$ is much better to approximate.

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- A perturbative approximation to $F(H)$ never predicts a ferromagnetic phase with $H \to 0$. But a perturbative approximation to $G(m)$ does!

- $G(m)$ is the analog of the effective action.
  - We'll come back to this example soon, when we do pairing.

- For now simply ask how we would deal with evaluating Z with the inconvenient spin sum $\sum_{\{S_i\}} e^{\sum S_i S_j}$. A very useful technique is to introduce an auxiliary field μ using the Gaussian identity

$$\boxed{e^{\beta \frac{J}{2N}\left(\sum_i S_i\right)^2} = \int_{-\infty}^{\infty} \frac{d\mu}{\sqrt{2\pi/N\beta J}}\, e^{-\frac{N\beta J}{2}\mu^2 + \beta J \mu \sum_{i=1}^{N} S_i}} \qquad (S_i = \pm 1)$$

Since

$$\boxed{\sum_{\{S_i\}} e^{\beta(J\mu + H)\sum_i S_i} = \prod_{i=1}^{N} \left(e^{\beta(J\mu+H)} + e^{-\beta(J\mu+H)}\right)}$$

we get Z as an integral over μ:

$$\boxed{Z(\beta, H, N) = \int_{-\infty}^{\infty} \frac{d\mu}{\sqrt{2\pi/N\beta J}}\, e^{-\frac{N\beta J}{2}\mu^2 + N \log 2\operatorname{ch}\beta(H+\mu J)}}$$

which yields the magnetization and susceptibility from derivatives of $\log Z$ with respect to H.

- Note the overall factor of $N \Rightarrow$ large N limit as <u>saddlepoint expansion</u>

- We'll return to this later: For now let's do the Fermi effective field theory.

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Start with our EFT Lagrangian: (with μ) shorthand here!

$$\boxed{\mathcal{L}_{EFT} = \psi^\dagger_\alpha\left[i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m} + \mu\right]\psi_\alpha - \frac{1}{2}C_0(\psi^\dagger_\alpha\psi_\alpha)^2 + \frac{C_2}{16}\left[(\psi\psi)^\dagger(\psi\overleftrightarrow{\nabla}^2\psi + h.c.\right] + \frac{C_2'}{8}(\psi\overleftrightarrow{\nabla}\psi)^\dagger\cdot(\psi\overleftrightarrow{\nabla}\psi) + \ldots}$$

with $\overleftrightarrow{\nabla} = \overleftarrow{\nabla} - \overrightarrow{\nabla}$.

- We'll only use C_0 here (δ-function potential)
  $\Rightarrow$ appropriate for a dilute system.

- By matching to 2-to-2 scattering, $\boxed{C_0 = \frac{4\pi a_s}{m}}$
  (with dimensional regularization / minimal subtraction).

- The "partition function" (in Minkowski space, so not really!)

$$\boxed{Z = \int \mathcal{D}(\psi^\dagger,\psi)\, e^{i\int d^4x\left[\psi^\dagger_\alpha\left(i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m} + \mu\right)\psi_\alpha - \frac{1}{2}C_0(\psi^\dagger_\alpha\psi_\alpha)^2\right]}}$$

has the same complication of a quartic term as we just saw in the spin system.
$\Rightarrow$ use the HS auxiliary field technique to replace $\psi^\dagger_\alpha\psi_\alpha$ by an auxiliary scalar (bosonic) field σ.

We can do this easily using

$$\boxed{1 = \frac{\int\mathcal{D}\sigma\, e^{\frac{i}{2}C_0\int d^4x\,(\sigma(x) - \psi^\dagger_\alpha(x)\psi_\alpha(x))(\sigma(x) - \psi^\dagger_\beta(x)\psi_\beta(x))}}{\int\mathcal{D}\sigma\, e^{\frac{i}{2}C_0\int d^4x\,[\sigma(x)]^2}}}$$

which follows simply by shifting the $\sigma(x)$ integration in the numerator.

Note that the term proportional to $(\psi^\dagger_\alpha\psi_\alpha)^2$ is equal and opposite to the one in Z.

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So now

$$\boxed{Z = \int \mathcal{D}(\psi^\dagger,\psi)\,\mathcal{D}\sigma\; e^{i\left[\int d^4x\, \psi^\dagger_\alpha(x)\left(i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m} + \mu - C_0\sigma(x)\right)\psi_\alpha(x) + \frac{1}{2}C_0(\sigma(x))^2\right]}}$$

$\Rightarrow$ we can do the $\mathcal{D}(\psi^\dagger,\psi)$ integral (with $\sigma(x)$ fixed)

- It is a Gaussian integral $\Rightarrow$ determinant of the operator between $\psi^\dagger_\alpha$ and ψ_α.

- We can identify the operator as an inverse fermion propagator:

$$\boxed{G^{-1}(x,y)\,\delta_{\alpha\beta} \equiv \left[i\frac{\partial}{\partial t} + \frac{\nabla_x^2}{2m} + \mu - C_0\sigma(x)\right]\delta^4(x-y)\,\delta_{\alpha\beta}}$$

  - still depends on $\sigma(x)$, which is integrated over.

Now use the identity $\boxed{\det A = e^{\mathrm{Tr}\ln A}}$

(discussed long ago in the notes on Gaussian integrals).

$$\Rightarrow \boxed{Z[J] = e^{iW[J]} = \int \mathcal{D}\sigma\; e^{g\,\mathrm{Tr}\ln G^{-1}(x,y)}\, e^{\frac{i}{2}C_0\int d^4x\,(\sigma(x))^2}\, e^{i\int d^4x\, J(x)\sigma(x)}}$$

- we've introduced a source term $J(x)\sigma(x)$ and defined $W[J]$ so we can do a perturbative expansion.
- Note that the path integral looks something like a sample path integral over σ we introduced earlier in the quarter, but with a strange $e^{g\,\mathrm{Tr}\ln G^{-1}(x,y)}$ term.
- The g comes from the spin/flavor trace and Tr means a trace over space-time (think in terms of discretizing space and time $\Rightarrow$ matrices).

- If we scale $C_0 = c_0/g$ and $\sigma = g\sigma'$, then there is a single overall g factor in the exponent
  $\Rightarrow$ stationary phase approximation as $g \to \infty$

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- $W[J]$ is analogous to $F(H)$, so do a Legendre transformation analogously

- We define the classical field $\sigma_c(x)$ [cf. magnetization] in the presence of $J(x)$ to be the ground state expectation value of $\sigma(x)$:

$$\boxed{\sigma_c(x) \equiv \langle \Psi_0 | \sigma(x) | \Psi_0 \rangle_J = -i \frac{1}{Z} \frac{\delta}{\delta J(x)} Z[J] = -i \frac{\delta}{\delta J(x)} \ln Z[J] = \frac{\delta W[J]}{\delta J(x)}}$$

- Just like m, $\sigma_c(x)$ is a weighted average over all configurations.

- Then the "effective action" $\Gamma[\sigma_c]$ is defined by the <u>functional</u> Legendre transformation:

$$\boxed{\Gamma[\sigma_c] \equiv W[J] - \int d^4x \, J(x) \sigma_c(x)}$$

where we have inverted $\sigma_c = \frac{\delta W}{\delta J}$ to obtain $J[\sigma_c(x)]$.

Note that

$$\boxed{\frac{\delta \Gamma[\sigma_c]}{\delta \sigma_c(x)} = \int d^4y \, \underbrace{\frac{\delta W}{\delta J(y)}}_{\sigma_c(y)} \frac{\delta J[\sigma_c(y)]}{\delta \sigma_c(x)} - \int d^4y \, \frac{\delta J[\sigma_c(y)]}{\delta \sigma_c(x)} \sigma_c(y) - J(x) = -J(x)}$$

so for a vanishing source $J=0$, which is the physical state, we have $\boxed{\frac{\delta \Gamma[\sigma_c]}{\delta \sigma_c(x)} = 0}$

and solutions to this equation represent the stable quantum states.

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At the minimum σ_c^0 [with $J(x)=0$] of a uniform system, the energy density $\mathcal{E}$ of the ground state is related to the effective action by

$$\boxed{\Gamma[\sigma_c^0]\Big|_{J=0} = -VT\mathcal{E}}$$

where VT is the space-time volume.

- more generally, at finite density we must examine spatially dependent σ_c to find the absolute ground state.

- Ok, so what do we do? We can't carry out the Legendre transformation on the full W.
  $\Rightarrow$ carry out the saddle point evaluation.

- Write $\boxed{\sigma = \sigma_c + \eta}$ and expand in quantum fluctuations η about the classical field (cf. $t = t_0 + \tau$)

- We'll derive this result next time:

$$\boxed{\begin{aligned}\Gamma[\sigma_c] = \frac{g}{i}\,\mathrm{Tr}\ln\left[G_H^{-1}(x,y)\right] + \frac{c_0}{2}\int d^4x\,(\sigma_c(x))^2 \\ + \frac{1}{2}\mathrm{Tr}\ln\left[D_\sigma^{-1}(x,y)\right] + (\text{connected 1PI diagrams in } D_\sigma)\end{aligned}}$$

where

$$\boxed{\begin{aligned}G_H^{-1}(x,y) &\equiv \left[i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m} + \mu - c_0\sigma_c(x)\right]\delta^4(x-y) \\ D_\sigma^{-1}(x,y) &\equiv -ic_0\delta^4(x-y) + g\,c_0^2\,G_H(y,x)\,G_H(x,y)\end{aligned}}$$