

## A Unified Theory of Nuclear Reactions. II\*

HERMAN FESHBACH

*Department of Physics and Laboratory for Nuclear Science, Massachusetts Institute of Technology, Cambridge, Massachusetts*

The effective Hamiltonian method for nuclear reactions described in an earlier paper with the same title, part I, is generalized so as to include all possible reaction types, as well as the effects arising from the identity of particles.

The principal device employed, as in part I, is the projection operator which selects the open channel components of the wave function. It is found that the formal structure of part I providing a unified description for direct and compound nuclear reactions including the coupled equation description for direct reactions remains valid in this wider context. A Kapur-Peierls expansion may also be readily obtained. The concept of channel radii is not needed nor is any decomposition of the wave function for the system into angular momentum eigenstates required, so that the expressions for transition amplitudes and widths are invariant with respect to the angular momentum coupling scheme. Since the open channels can only be defined in an asymptotic sense, the corresponding projection operators are not unique. As a consequence the projection operator method has a flexibility which in the first place is consonant with the wide range of phenomena which can occur in nuclear reactions and in the second place can effectively exploit an intuitive understanding of the phenomena. Example of projection operators are obtained including one which leads to the Wigner-Eisenbud formalism, another which is appropriate for the stripping reaction, and, finally, one which takes the Pauli exclusion principle into account. Note that explicit representations of the projection operators are not required for the development of general formal results but are necessary if, eventually, quantitative calculations are made.

### I. INTRODUCTION

In this paper the formalism developed in "A Unified Theory of Nuclear Reactions" (I)<sup>1</sup> is generalized and improved. In I we were principally concerned with developing a theory of nuclear reactions from which a description of direct and compound nuclear processes, as well as the complex potential model, would be both easily and naturally abstracted. This inescapably led to a derivation

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<sup>1</sup> To be referred to as I.

of resonance reactions which did not involve the concept of the channel radius, nor did it require the decomposition of the wave function of the system into orbital angular momentum eigenstates.

Although our treatment was successful, there were a number of limitations. We considered only situations in which the mass number of the incident projectile and that of the emergent light particle were the same as they would be in an  $(n, p)$  or  $(n, n')$  process. Stripping and similar processes were not treated. Secondly, although it was possible to show that the effects of the Pauli exclusion principle and exchange scattering would not change the structure of the formalism, the discussion was rather indirect and did not provide a method for the evaluation of these effects.<sup>2</sup> Finally the connection of resonance formalism in I with the boundary condition methods of Wigner and Eisenbud (2) and Kapur and Peierls (3) was not established.

It is possible to remove these limitations so that the resulting theory has a universal applicability; a direct procedure for evaluating the effects stemming from the identity of the particles is provided and finally the relation to the formalisms referred to earlier can be readily determined. Gamma ray emission (or absorption) does not require any special considerations but can be described in the same terms as those used in the discussion of heavy particle reactions.

The method employed in this paper is identical with that of I (see also (4)). Applying a projection operator which selects out the open channel configurations it is possible to partition the total wave function into closed and open channel segments. A Schrodinger equation for the open channel part can be obtained by eliminating the closed channels. One thus derives an effective Hamiltonian for the open channels and it is by analysis of its structure that one can obtain the various aspects of reaction theory alluded to above. By means of a very simple generalization of this projection operator technique it has been possible in this paper to extend the methods of I so as to include all reaction types as well as the effects of the exclusion principle and exchange within one general yet quite elementary formal structure. This structure depends only upon the existence of the appropriate projection operator (its explicit realization is not needed), with the consequence that the formal results derived in this paper are identical with those obtained in I.

As it turns out there are a number of ways in which the equations for the closed and open channels may be discussed. In the procedure used in I, the effective Hamiltonian is broken up into a part which varies rapidly with energy and a part which varies slowly. The part which varies slowly is responsible for

<sup>2</sup> As was pointed out to the author by J. S. Bell, the discussion in I was incomplete. The missing elements of the argument are given in Appendix B of "Radiation of Low Energy Quanta in Nuclear Reactions" written with D. Yennie and submitted for publication in Nuclear Physics.

the direct reactions, for potential scattering and for "single particle" resonances. The rapidly varying part gives rise to the narrow compound nuclear resonances; the widths for these are expressible as matrix elements involving the "single particle" wave functions of the direct reaction Hamiltonian. In another treatment the effective Hamiltonian is not broken up into a fast and slowly varying parts. Instead the transition matrix is expanded in terms of the eigen solutions of the effective Hamiltonian for the closed channels obtained by eliminating the open channels in the absence of any incident wave; the open channel part of these eigenfunctions satisfy an outgoing wave boundary condition. The eigenfunctions have complex eigenvalues and are clearly of the Kapur-Peierls type and the resulting expansion of the transition matrix becomes an expansion over resonances.<sup>3</sup> There are however several significant differences between this result and that of the standard Kapur-Peierls formalism. These all stem from the fact that in the present work not even the concept of nuclear radius is used. Our results hold therefore even when the potential involved does not have a sharp cutoff. As a consequence there is no need to decompose the wave function into eigenstates of the orbital angular momentum. Finally, the widths are expressed in terms of matrix elements of the interaction rather than in terms of overlap integrals between inside and outside wave functions at the nuclear radius.

These two expressions for the transition matrix, that of I which we shall call the effective Hamiltonian method and the method of expansion in complex eigenvalues, are of course equivalent. However it seems to us that the effective Hamiltonian method is most convenient for nuclear reactions in that it differentiates between the narrow compound nuclear resonances and the "single particle" giant resonances; i.e., between compound and direct nuclear reactions. In the complex eigenvalue expansion both kinds of resonances are lumped together indiscriminately.

The projection operator which selects open channels is not unique since it is possible to define open channels only in terms of the asymptotic behavior of the wave function when the reaction products are far apart. This gives the formalism an additional flexibility which is extremely useful since it allows one to choose that projection operator which is most convenient for the problem under investigation. In the present paper we are for the most part concerned with the type of projection operator employed in I in which the open channel wave functions are defined by extending the form of the wave function which is valid asymptotically to all of configuration space. This means that our open channel

<sup>3</sup> These eigenvalues are functions of the energy,  $E$ , of the system. An expansion of the transition employing complex eigenvalues which are independent of  $E$  is referred to as the Siegert-Hamblet series. Such a series is derived in the appendix to this paper. For a recent discussion see Humblet and Rosenfeld (5).

wave functions are linear superpositions of the possible residual target nuclear states. The class of these states are restricted of course by energy conservation and other conservation rules which may be pertinent.

We also briefly discuss another projection operator which introduces the channel radius into the theory. The projection operator here is unity for regions in configuration space where the particles are separated by distances greater than the channel radius, and is zero elsewhere. If the channel radii are chosen so that except for Coulomb or other long range potentials the interaction vanishes where the projection operator is unity, we are directly led to the Wigner-Eisenbud (2) boundary conditions.

These two types of projection operators have essentially complementary domains of usefulness. The Wigner-Eisenbud projection is most convenient when details of the internal region are not under scrutiny. There is no difficulty with the Pauli principle and exchange scattering. The resonance energies have a definition independent of the incident energy. The projection operator used in I is appropriate when details of the interaction are of interest. It is the natural extension to the continuum of methods employed for bound state problems and at the same time it can be easily joined on to the high energy multiple scattering limit of Kerman *et al.* (6). There is thus no need to change the formalism as one's attention changes from negative to positive and finally to high energies. The close relation of the shell model potential and the real part of the complex potential becomes manifest and one realizes that the residual potential is responsible for the variegated phenomena which occur in nuclear reactions. On the other hand with this projection operator type the treatment of the Pauli principle (discussion Section III) is not simple although it does seem manageable.

Several writers have contributed to the equivalent Hamiltonian method since the publication of I.<sup>4</sup> We mention Brenig (7), Newton and Fonda (8), Agodi and Eberle (9), and most recently Lipperheide (10). Each of these have developed material which overlaps some of our work as we shall duly note below. Lane and Thomas (11) and G. Breit (12) give excellent reviews of the Wigner  $R$  matrix theory and reaction theories in general, while the most recent treatment of the expansion in resonances is given by Rosenfeld and Humblet (5).

We conclude this introduction with a brief description of the contents of the paper. In Section II the general theory is developed in terms of projection operators. Specific forms of these operators are derived in Section III for the stripping reaction, for incorporating the Pauli principle, and for the Wigner formalism. In Section IV, resonance theory is developed and an expansion in terms of resonances of a Kapur-Peierls type is derived. Section IV does not depend upon the details of Section III so that it can be read directly after II. In an Appendix, the Seigert-Humblet description is derived. There will be some review of the

<sup>4</sup> See I for the earlier references.

earlier paper on this subject, I, both for the purpose of completeness as well as to permit some further remarks which round out the earlier discussion. The interaction of gamma rays with nuclei will be discussed in another communication.

## II. THE EFFECTIVE HAMILTONIAN

When a projectile  $a$  strikes a target nucleus  $X$ , a variety of reactions can occur. Elastic scattering in which the final products are again  $a$  and  $X$  will always take place. In addition there might be various transmutations in which the final products  $b$  and  $Y$  ( $b$  can itself be composite) differ in some intrinsic respect from  $a$  and  $X$  respectively. Those reactions (including elastic scattering) which are energetically allowed are referred to as open channels; the others are closed channels. Let the wave functions describing the energetically possible<sup>5</sup> residual nuclei be denoted by  $\phi_i$ . Note that these functions are generally not mutually orthogonal nor do they necessarily involve the same number of coordinates. Asymptotically, when the reaction products are well separated, the wave function for the system will take on the antisymmetrized form:

$$\Psi \rightarrow \mathcal{A} \sum f_i \phi_i \quad (2.1)$$

where  $f_i$  are functions of the coordinates not contained in  $\phi_i$  and of course satisfy appropriate asymptotic boundary conditions.  $\mathcal{A}$  is the antisymmetrization operator.

It is now possible to define the projection operator which selects the open channels. It is any projection operator  $P$  which, operating on any antisymmetrical function  $\chi$ , satisfying the same asymptotic boundary conditions as  $\Psi$ , but otherwise arbitrary, yields a function which asymptotically is of the form (2.1):

$$P\chi \rightarrow \mathcal{A} \sum v_i \phi_i \quad (2.2)$$

The quantities  $v_i$  are functions of the coordinates not contained in  $\phi_i$  satisfying the same boundary conditions as  $f_i$ , as well as conditions imposed by invariance principles.

A simple example of such an operator is the one  $P_{WE}$  which, as we shall show in Section III, leads to the Wigner-Eisenbud formalism:

$$\begin{aligned} P_{WE} &= 1 \text{ in the region in configuration space in which nuclear inter-} \\ &\quad \text{actions (except for Coulomb and other long range poten-} \\ &\quad \text{tials) vanish} \\ P_{WE} &= 0 \text{ otherwise} \end{aligned} \quad (2.3)$$

<sup>5</sup> It may sometimes be convenient to exclude one or more of the open channel residual nuclei states from the set  $\{\phi_i\}$ . This is particularly true near the threshold energy at which a closed channel becomes an open channel.

Another example is furnished by the projection operator  $P_I$  employed in I which we shall now suitably generalize. Consider the set of all antisymmetrized wave functions  $\Phi$  satisfying the same asymptotic boundary conditions as  $\Psi$  which have the form:

$$\Phi = \alpha \sum u_i \phi_i \quad (2.4)$$

The  $u_i$  approach the  $v_i$  of Eq. (2.2) asymptotically. The set  $\{\Phi\}$  subtends a portion of Hilbert space (let us call it the "open-channel subspace"), associated with the Hamiltonian and the other pertinent operators of the system. Moreover a projection operator  $P_I$  must exist which projects on to this open channel subspace. In other words if  $\chi$  is an arbitrary state vector,  $P_I \chi$  belongs to the set  $\Phi$ ; that is,  $P_I \chi$  can be expressed as a sum over the open channel residual state  $\phi_i$ . We shall obtain explicit expression for  $P_I$  in Section III.

Note that  $P_{WE}$  and  $P_I$  are only two examples of possible projection operators which satisfy condition (2.2). Many others are possible; which one is to be used will be determined by the physics of the phenomenon under investigation. It is important to realize that the remainder of the discussion in this section, as well as that in Section IV, requires only the existence of the projection operator. If  $\Psi$  is the wave function for the system it is clear that we need only consider  $P\Psi$  in order to obtain the asymptotic behavior of  $\Psi$  and thus the scattering and transition amplitudes. It was this fact which was employed in I for the case considered there and it will be useful principally for completeness to rephrase some of the results of I in terms of an explicit projection operator formalism.<sup>6</sup> We note that we have already obtained the generalization of I to situations in which the final reactions products can be arbitrarily complex. There is no limitation on the number of open channels or on the nature of the emergent particles. All we need to do is construct the appropriate projection operator.

Let us rephrase the calculation in I up to the point where we obtain the equivalent Hamiltonian (called generalized optical potential in I).  $\Psi$  satisfied the equation

$$\mathcal{H}\Psi = E\Psi \quad (2.5)$$

Then operating on Eq. (2.5) with  $P$ , and with  $Q \equiv 1 - P$ , we obtain

$$(E - \mathcal{H}_{PP})P\Psi = \mathcal{H}_{PQ}Q\Psi \quad (2.6a)$$

$$(E - \mathcal{H}_{QQ})Q\Psi = \mathcal{H}_{QP}P\Psi \quad (2.6b)$$

where

$$\begin{aligned} \mathcal{H}_{PQ} &= P\mathcal{H}Q \\ \mathcal{H}_{PP} &= P\mathcal{H}P \quad \text{etc.}^7 \end{aligned} \quad (2.7)$$

<sup>6</sup> This option was mentioned in I, was partially described in ref. 4 and has been employed in refs. 8 and 9.

<sup>7</sup> For  $P_{WE}$  these symbols need to be defined with some care. This is discussed in Section III.

Since  $Q\Psi$  projects into the closed channel subspace we can solve (2.6b) as follows<sup>8</sup>

$$Q\Psi = \frac{1}{E - \mathcal{H}_{QQ}} \mathcal{H}_{QP} P\Psi \quad (2.8)$$

Substituting Eq. (2.8) in Eq. (2.6a) yields the Schroedinger equation for  $P\Psi$ :

$$(E - H)P\Psi = 0 \quad (2.9)$$

where  $H$ , the effective Hamiltonian, is

$$H = \mathcal{H}_{PP} + \mathcal{H}_{PQ} \frac{1}{E - \mathcal{H}_{QQ}} \mathcal{H}_{QP} \quad (2.10)$$

Equation (2.9) is identical in form to Eq. (17) in ref. 4 and Eq. (2.15) of I but has a wider applicability as a consequence of the generalization in the definition of  $P$  given above. Moreover, because the form of Eq. (2.9) is identical to that of Eq. (2.15) of I, the deductions in I, including the resonance formulas, the complex potential model, and the direct reactions based on Eq. (2.15) of I, remain valid in the present context. We shall discuss these at greater length in Section IV. For the present let us render Eq. (2.10) less formal by determining  $P$  for some examples of nuclear reactions. However let us emphasize that the formal results of Section IV do not require explicit expressions for  $P$ . The need for such expressions occurs when evaluations of the various matrix elements for various nuclear models or eventually from nucleon-nucleon forces are being made.

### III. PROJECTION OPERATORS

In this section the projection operators of the type  $P_I$  corresponding to various possible final states are explicitly given. We have chosen to express these in configuration space. However it is also possible, and it is often more convenient, to use the language of second quantization.<sup>9</sup> The translation, configuration space to second quantization is quite straightforward for the cases considered here, so that we will leave it to the reader to rewrite our results in the latter form. We conclude this section with a discussion of  $P_{WE}$  and the Wigner-Eisenbud formalism.

(a) We consider first the case where the incident and emergent particle are same but are not identical with the particles in the target nucleus. Let the wave

<sup>8</sup> If  $P$  does not include all the open channels, the  $E$  in Eq. (2.8) should be replaced by  $E^{(\pm)} = E + i\eta$ ,  $\eta \rightarrow 0^+$ .

<sup>9</sup> For example, if we wish to project out a state  $\Phi$  consisting of  $N$  fermions in single particle states

$$P\Psi = \Phi \langle \text{vac} | \prod_1^N a_k | \Psi \rangle$$

where  $a_k$  are the destruction operators corresponding to the various single particle states.

function for the ground state of the target nucleus be  $\phi_0$  and suppose that the incident particle is sufficiently energetic, as to be able to excite the first  $n$  excited states of the target whose wave functions are  $\phi_j$ ,  $j$  running from one to  $n$ . Then

$$P = \sum_{j=1}^n \phi_j \langle \phi_j | \quad Q = \sum_{n+1} \phi_j \langle \phi_j | \quad (3.1)$$

Inserting this result into Eqs. (2.9) and (2.10) and letting

$$u_i(\mathbf{r}_0) \equiv \langle \phi_i(\mathbf{r}_1 \cdots \mathbf{r}_A) \Psi(\mathbf{r}_0 \mathbf{r}_1 \cdots \mathbf{r}_A) \rangle \quad (3.2)$$

we obtain

$$[E - H_{ii}]u_i = \sum_{i \neq j} H_{ij}u_j \quad i, j \leq n \quad (3.3)$$

where

$$H_{ij}(\mathbf{r}_0) = \langle \phi_i(\mathbf{r}_1 \cdots \mathbf{r}_A) H(\mathbf{r}_0 \mathbf{r}_1 \cdots \mathbf{r}_A) \phi_j(\mathbf{r}_1 \cdots \mathbf{r}_A) \rangle \quad (3.4)$$

Equations (3.3) were discussed in I.

(b) We turn our attention here to pick up (or stripping) such as the (n, d) process. We assume for purposes of simplifying the discussion that there are only two final open channels. One corresponds to the elastic scattering of the neutrons in which the target nucleus remains in its ground state  $\phi_0(\mathbf{r}_1 \cdots \mathbf{r}_A)$ . In the second the residual nucleus has  $(A - 1)$  particles and we shall assume that it may be left in the ground state  $\psi_0(\mathbf{r}_2, \cdots \mathbf{r}_A)$ . The extension to the situation in which many states in both possible residual nuclei can be excited is simple. We shall not include the Pauli principle, the effects of which shall be shortly discussed for another case.

We consider first the simple case where  $\phi_0$  is proportional to  $\psi_0$ , as would occur in an independent particle<sup>10</sup> description of these nuclei, viz:

$$\phi_0(\mathbf{r}_1 \cdots \mathbf{r}_A) = f(\mathbf{r}_1) \psi_0(\mathbf{r}_2 \cdots \mathbf{r}_A) \quad (3.5)$$

Then  $P$  is simply

$$P = \psi_0 \langle \psi_0 | \quad (3.6)$$

Equation (2.8) becomes

$$\begin{aligned} [E - H_{00}(\mathbf{r}_0, \mathbf{r}_1)]w(\mathbf{r}_0, \mathbf{r}_1) &= 0 \\ w &= \langle \psi_0, \Psi \rangle \\ H_{00} &= \langle \psi_0 H \psi_0 \rangle \end{aligned} \quad (3.7)$$

<sup>10</sup> The particle to be "picked up" is not antisymmetrized with respect to the particles in the residual  $(A-1)$  nuclear wave function  $\psi_0$ . This ansatz implicitly assumes that  $A-2$  particles act essentially as a spectator particle, not participating in the reaction.



This is the form from which most direct interaction theories of the pick up process start. There is one important difference. Equation (3.7) contains the resonance phenomena as well for the incident neutrons or if we are discussing the inverse process, stripping, the resonance phenomena for the incident deuteron. We shall discuss the resonance terms in Section IV.

Our discussion is not yet complete. We should continue the projection procedure one step further so as to isolate that part of the wave function  $w(\mathbf{r}_0, \mathbf{r}_1)$  which describes the process in which the residual nucleus wave function is  $\phi_0(\mathbf{r}_1 \cdots \mathbf{r}_A)$ . Toward this end we write

$$w(\mathbf{r}_0, \mathbf{r}_1) = P_N w + Q_N w$$

where  $P_N$  is the projection operator

$$P_N = |f\rangle\langle f|$$

and

$$Q_N = 1 - P_N$$

The function  $w$  then assumes the desired form

$$w(\mathbf{r}_0, \mathbf{r}_1) = u(\mathbf{r}_0)f(\mathbf{r}_1) + Q_N w \quad (3.8)$$

where

$$u \equiv \langle f(\mathbf{r}_1)w(\mathbf{r}_0, \mathbf{r}_1) \rangle$$

and

$$\langle f, Q_N w \rangle = 0 \quad (3.9)$$

Note that  $u$  will asymptotically describe the incident (if any) and emergent nucleon waves, while  $Q_N w$ , the remainder of  $w$ , will provide a similar description of the deuteron. The functions  $u$  and  $Q_N w$  satisfy the following pair of coupled equations

$$[E - H'(\mathbf{r}_0)]u(\mathbf{r}_0) = \langle f(\mathbf{r}_1), H_{00}(\mathbf{r}_0, \mathbf{r}_1)Q_N w \rangle \quad (3.10)$$

$$[E - Q_N H_{00} Q_N]Q_N w = H_{00}(\mathbf{r}_0, \mathbf{r}_1)f(\mathbf{r}_1)u(\mathbf{r}_0) - f(\mathbf{r}_1)H'(\mathbf{r}_0)u(\mathbf{r}_0)$$

where

$$H'(\mathbf{r}_0) = \langle f(\mathbf{r}_1), H_{00}(\mathbf{r}_0, \mathbf{r}_1)f(\mathbf{r}_1) \rangle \quad (3.11)$$

Equation (3.10) is very similar in structure to Eq. (3.3) and suggests that as a general principle it will be possible to describe a reaction involving the amplitudes for various open channels in terms of coupled Schroedinger type equations involving effective Hamiltonians.

We consider next the more realistic situation in which  $\phi_0$  is not proportional to  $\psi_0$ ; i.e., Eq. (3.5) is not valid. Then

$$P = \phi_0'(\mathbf{r}_1 \cdots \mathbf{r}_A)\langle\phi_0'(\mathbf{r}_1 \cdots \mathbf{r}_A) + \psi_0(\mathbf{r}_2 \cdots \mathbf{r}_A)\rangle\langle\psi_0(\mathbf{r}_2 \cdots \mathbf{r}_A) \quad (3.12)$$

where

$$\phi_0' = N[\phi_0 - \psi_0\langle\psi_0, \phi_0\rangle]; \quad \langle\phi_0', \psi_0\rangle = 0 \quad (3.13)$$

$N$  is just a normalization chosen so that

$$\langle \phi_0', \phi_0' \rangle = 1$$

Note that projection operator (3.12) reduces to (3.6) when  $\phi_0$  is proportional to  $\psi_0$ .

With (3.12), and with suitable rearrangements Eq. (2.9) becomes an equation for

$$P\Psi = u(\mathbf{r}_0)\phi_0(\mathbf{r}_1 \cdots \mathbf{r}_A) + Q_N w(\mathbf{r}_0, \mathbf{r}_1)\psi_0(\mathbf{r}_2 \cdots \mathbf{r}_A) \quad (3.14)$$

where

$$Q_N = 1 - \phi_0 \langle \phi_0 | \quad (3.15)$$

The equations for  $u$  and  $Q_N w$  are identical in form with Eq. (3.10):

$$\begin{aligned} [E - \langle \phi_0, PHP\phi_0 \rangle]u &= \langle \phi_0 PHP Q_N w \psi_0 \rangle \\ [E - Q_N PHP Q_N]w\psi_0 &= Q_N PHP u \phi_0 \end{aligned} \quad (3.16)$$

where  $H$  is given in Eq. (2.10).

(c) The inclusion of the Pauli principle into the theory of reactions has been the subject of several papers (10, 11, 14-16). In the present treatment we avoid the subtleties and complexities of the time dependent treatment, in which the behavior at infinite times must be examined with considerable care. Our effective potential will be hermitian; some treatments deduce nonhermitian effective potentials.

For simplicity we again consider the case where the incident particle is a nucleon. Moreover we assume that it is not energetic enough to excite the target nucleus so that the only process energetically allowed is elastic scattering. If the ground state of the target nucleus is the antisymmetrized wave function  $\phi_0(\mathbf{r}_1 \cdots \mathbf{r}_A)$  the projection operator we require has the following property

$$P\Psi(\mathbf{r}_0, \mathbf{r}_1 \cdots \mathbf{r}_A) = \alpha u(\mathbf{r}_0)\phi_0(\mathbf{r}_1 \cdots \mathbf{r}_A) \quad (3.17)$$

where  $\alpha$  is the antisymmetrization operator and  $\Psi$  is antisymmetrized. If  $\Psi$  is the wave function of the system, then the function  $u(\mathbf{r}_0)$  asymptotically yields both the direct and exchange scattering amplitude. The operator  $P$  selects out of  $\Psi$  that part which describes the target nucleus in the ground state.

The equation<sup>11</sup> determining  $u$  is

$$\langle \phi_0(\mathbf{r}_1 \cdots \mathbf{r}_A), [\Psi(\mathbf{r}_0 \mathbf{r}_1 \cdots \mathbf{r}_A) - \alpha u(\mathbf{r}_0)\phi_0(\mathbf{r}_1 \cdots \mathbf{r}_A)] \rangle = 0 \quad (3.18)$$

where the integration is over the coordinates which are common to both sides of the inner product in (3.18). That condition (3.18) is sufficient can be shown as follows. From (3.18) we have

<sup>11</sup> In the discussion to follow,  $\Psi$  is an arbitrary function except that it satisfies appropriate boundary conditions at infinity.

$$\langle u(\mathbf{r}_0)\phi_0(\mathbf{r}_1 \cdots \mathbf{r}_A), [\Psi(\mathbf{r}_0\mathbf{r}_1 \cdots \mathbf{r}_A) - \mathcal{Q}u(\mathbf{r}_0)\phi_0(\mathbf{r}_1 \cdots \mathbf{r}_A)] \rangle = 0 \quad (3.19)$$

and since the quantity in the square brackets is antisymmetrical, it follows that

$$\langle \mathcal{Q}u(\mathbf{r}_0)\phi_0(\mathbf{r}_1 \cdots \mathbf{r}_A), [\Psi(\mathbf{r}_0\mathbf{r}_1 \cdots \mathbf{r}_A) - \mathcal{Q}u(\mathbf{r}_0)\phi_0(\mathbf{r}_1 \cdots \mathbf{r}_A)] \rangle = 0 \quad (3.20)$$

From this last equation we see that once (3.18) is solved, we can break up  $\Psi$  into two mutually orthogonal parts of which one is the desired form (3.17). It is still necessary to show that the resultant form  $\mathcal{Q}(u\phi_0)$  can be written as a projection operator on  $\Psi$ . But for this we need first to solve Eq. (3.18). Performing the indicated operations we obtain

$$U(\mathbf{r}_0) = u(\mathbf{r}_0) - A \langle \phi_0(\mathbf{r}_1\mathbf{r}_2 \cdots \mathbf{r}_A), \phi_0(\mathbf{r}_0\mathbf{r}_2 \cdots \mathbf{r}_A)u(\mathbf{r}_1) \rangle \quad (3.21)$$

where

$$U(\mathbf{r}_0) \equiv \langle \phi_0(\mathbf{r}_1 \cdots \mathbf{r}_A), \Psi \rangle \quad (3.22)$$

Equation (3.21) is an inhomogeneous integral equation of the Fredholm type with the kernel

$$K(\mathbf{r}_0 | \mathbf{r}_1) = A \langle \phi_0(\mathbf{r}_0\mathbf{r}_2 \cdots \mathbf{r}_A), \phi_0(\mathbf{r}_1\mathbf{r}_2 \cdots \mathbf{r}_A) \rangle \quad (3.23)$$

Equation (3.21) becomes

$$U(\mathbf{r}_0) = u(\mathbf{r}_0) - \langle K(\mathbf{r}_0 | \mathbf{r}_1)u(\mathbf{r}_1) \rangle \quad (3.21')$$

The kernel  $K$  has several important properties. It is hermitian. Since  $\phi_0$  is the wave function for a bound state,  $K$  decreases exponentially as either  $r_1$  or  $r_0$  become infinite.  $K$  is bounded:

$$\int d\mathbf{r}_1 \int d\mathbf{r}_0 |K(\mathbf{r}_0 | \mathbf{r}_1)|^2 \leq A$$

The trace of  $K$  is finite and equal to  $A$ . It will be useful to keep in mind as an *example* the kernel  $K$  we obtain when  $\phi_0$  is a Slater determinant composed of mutually orthogonal elements  $w_i$ . Then

$$K = \sum_1^A w_i^*(\mathbf{r}_1)w_i(\mathbf{r}_0) \quad (3.24)$$

Before solving the inhomogeneous equation (3.21) it is convenient to consider the eigenvalue problem of the corresponding homogeneous form:

$$u_\alpha(\mathbf{r}_0) = \lambda_\alpha \langle K(\mathbf{r}_0 | \mathbf{r}_1)u_\alpha(\mathbf{r}_1) \rangle \quad (3.25)$$

From the properties of  $K$  it follows that  $\lambda_\alpha$  are real and

$$|\lambda_\alpha| \geq | \quad \text{and} \quad \sum \frac{1}{\lambda_\alpha} = A \quad (3.26)$$

The eigenfunction  $u_\alpha$  will form an orthonormal set which is not necessarily complete. However,  $K$  can be expanded in terms of normalized  $u_\alpha$  as follows

$$K(\mathbf{r}_0 | \mathbf{r}_1) = \sum \frac{u_\alpha^*(\mathbf{r}_0) u_\alpha(\mathbf{r}_1)}{\lambda_\alpha} \quad (3.27)$$

For example, (3.24),  $\{u_\alpha\} = \{w_i\}$  and  $\lambda_\alpha = 1$  for the entire set.

The case where the  $\lambda_\alpha$  equal unity requires some special considerations. Denoting these eigenfunctions by  $u_{\alpha 1}$ , we note that they satisfy

$$u_{\alpha 1} = \langle K(\mathbf{r}_0 | \mathbf{r}_1) u_{\alpha 1} \rangle$$

or

$$\langle \phi_0(\mathbf{r}_1 \cdots \mathbf{r}_A), \mathcal{A} u_{\alpha 1}(\mathbf{r}_0) \phi_0(\mathbf{r}_1 \cdots \mathbf{r}_A) \rangle = 0$$

From this equation we have

$$\langle \mathcal{A} u_{\alpha 1}(\mathbf{r}_0) \phi_0(\mathbf{r}_1 \cdots \mathbf{r}_A), \mathcal{A} u_{\alpha 1}(\mathbf{r}_0) \phi_0(\mathbf{r}_1 \cdots \mathbf{r}_A) \rangle = 0$$

or

$$\mathcal{A} u_{\alpha 1}(\mathbf{r}_0) \phi_0(\mathbf{r}_1 \cdots \mathbf{r}_A) = 0 \quad (3.28)$$

For example (3.24); this is just the determinantal theorem which states that the determinant vanishes when two columns are identical. More generally it may be shown that if condition (3.28) is satisfied,  $\phi_0$  itself is given by

$$\phi_0(\mathbf{r}_1 \cdots \mathbf{r}_A) = \mathcal{A} u_{\alpha 1}(\mathbf{r}_1) \psi(\mathbf{r}_2 \cdots \mathbf{r}_A)$$

where  $\psi$  is antisymmetrical. Hence by examining  $\phi_0$  one can pick out those  $u_\alpha$ ,  $u_{\alpha 1}$ , whose eigenvalue  $\lambda_\alpha = 1$ . Another consequence of immediate interest is that

$$\langle u_{\alpha 1}, U \rangle = 0 \quad (3.29)$$

To prove this, simply note that the above bracket is identical with

$$\langle u_{\alpha 1}(\mathbf{r}_0) \phi_0(\mathbf{r}_1 \cdots \mathbf{r}_A), \Psi \rangle$$

and since  $\Psi$  is antisymmetrical it is proportional to

$$\langle \mathcal{A} u_{\alpha 1}(\mathbf{r}_0) \phi_0(\mathbf{r}_1 \cdots \mathbf{r}_A), \Psi \rangle$$

which in virtue of (3.28) is zero, proving (3.29).

We now return to Eq. (3.21'). Employing the expansion (3.27) for  $K$ , and the orthonormality of the set  $\{u_\alpha\}$  we find

$$\langle u_\alpha, u \rangle = \frac{\lambda_\alpha}{\lambda_\alpha - 1} \langle u_\alpha, U \rangle \quad \lambda_\alpha \neq 1 \quad (3.30)$$

Finally,

$$u = \sum u_{\alpha 1} \langle u_{\alpha 1}, u \rangle + U + \sum_{\lambda_\alpha \neq 1} \frac{u_\alpha}{\lambda_\alpha - 1} \langle u_\alpha, U \rangle \quad (3.31)$$

As indicated, the terms in  $u_\alpha$ —the amplitude of the eigenfunctions whose eigenvalue  $\lambda_\alpha$  are unity—are not determined by Eq. (3.21). However, because of (3.28) the function of interest,  $\mathcal{A}(u\phi_0)$ , does not contain these terms. In fact:

$$\mathcal{A}u(\mathbf{r}_0)\phi_0(\mathbf{r}_1 \cdots \mathbf{r}_A) = \mathcal{A}u'(\mathbf{r}_0)\phi_0(\mathbf{r}_1 \cdots \mathbf{r}_A) \quad (3.32)$$

where

$$u'(\mathbf{r}_0) = U + \sum_{\lambda_\alpha \neq 1} \frac{u_\alpha}{\lambda_\alpha - 1} \langle u_\alpha, U \rangle \quad (3.33)$$

Since  $U$  is just the projection of  $\Psi$  on  $\phi_0$  we can rewrite (3.32) as follows

$$\mathcal{A}(u\phi_0) = \mathcal{A} \left\{ \phi_0 \rangle \langle \phi_0 + \sum_{\lambda_\alpha \neq 1} \frac{u_\alpha \phi_0 \rangle \langle u_\alpha \phi_0}{\lambda_\alpha - 1} \right\} \Psi \rangle \quad (3.34)$$

Finally, making use of the antisymmetry of  $\Psi$  we have

$$\mathcal{A}(u\phi_0) = P\Psi \quad (3.35)$$

where<sup>12</sup>

$$P = \frac{\mathcal{A}}{\sqrt{A+1}} \left\{ \phi_0 \rangle \langle \phi_0 + \sum_{\lambda_\alpha \neq 1} \frac{u_\alpha \phi_0 \rangle \langle u_\alpha \phi_0}{\lambda_\alpha - 1} \right\} \frac{\mathcal{A}}{\sqrt{A+1}} \quad (3.36)$$

$P$  is the desired projection operator which projects out of a function that part which can be written  $\mathcal{A}u\phi_0$  (cf. Eq. (3.17)). To prove that it is a projection operator we need only note that  $P$  is hermitian, and from Eq. (3.20) that

$$\langle P\Psi, (1 - P)\Psi \rangle = 0$$

These two properties have the consequence that  $P^2 = P$ , proving that  $P$  is a projection operator.

It is amusing to note that when  $\phi_0$  is a Slater determinant  $\mathcal{A}(u\phi_0)$  is just  $\mathcal{A}\phi_0 \rangle \langle \phi_0$  according to (3.35). This result tells us that in that example, to project out the dependence on  $\phi_0$  we need only employ the naive projection operator  $\phi_0 \rangle \langle \phi_0$  and then antisymmetrize. This result is no surprise from the second quantization point of view since the appropriate projection operators in this case can be written simply as a product of creation operators for the states corresponding to  $w_i$  acting on the "vacuum." The more general operator (3.36) corresponds to the creation operator in which the  $A$  particles are created in the state  $\phi_0$ . Although the simplicity of this result for the simple determinantal  $\phi_0$  cannot be maintained for more complicated  $\phi_0$  there appears to be very little difficulty in solving Eq. (3.25) for  $u_\alpha$  and  $\lambda_\alpha$  as long as  $\phi_0$  may be written as a linear combination of Slater determinants. We have therefore provided a direct and practical method for the determination of the projection operator which

<sup>12</sup> In the first term of this expression  $\phi_0(\mathbf{r}_1 \dots \mathbf{r}_A) \rangle \langle \phi_0(\mathbf{r}'_1 \dots \mathbf{r}'_A)$  a factor, the unit operator  $\delta(\mathbf{r}_0 - \mathbf{r}'_0)$ , is not explicitly given.

selects out that part of  $\Psi$  which represents the target nucleus in its ground state.

Equation (2.9) now becomes an equation for  $\alpha(u\phi_0)$ . The existence of such an equation is all that we need for the derivation of the resonance formulas, the optical model, etc. The antisymmetrization operators give rise to exchange interactions in the effective Hamiltonian. Note that the effective Hamiltonian operator is hermitian and that from the asymptotic behavior of  $u(\mathbf{r}_0)$  we can obtain the elastic scattering amplitude which includes both the direct and exchange scattering amplitudes.

It may be convenient to derive an equation involving only one coordinate, say  $\mathbf{r}_0$ . This equation can be obtained by multiplying Eq. (2.9) from the left by  $\phi_0^*(\mathbf{r}_1 \cdots \mathbf{r}_A)$  and integrating over coordinates  $\mathbf{r}_1 \cdots \mathbf{r}_A$ . The properties of the projection operator permit some important simplifications in the resultant equation. Note that Eq. (3.18) can be rewritten

$$U(\mathbf{r}_0) \equiv \langle \phi_0(\mathbf{r}_1 \cdots \mathbf{r}_A), \Psi \rangle = \langle \phi_0(\mathbf{r}_1 \cdots \mathbf{r}_A), P\Psi \rangle \quad (3.18')$$

This equation is valid for *any antisymmetrical* function  $\Psi$ . Because of this it is most convenient to consider the equation for  $U$  which, in virtue of (3.18') and the symmetry of  $H$  becomes:

$$EU(\mathbf{r}_0) = \left\langle \phi_0(\mathbf{r}_1 \cdots \mathbf{r}_A) \left[ \mathcal{H} + \mathcal{H}Q \frac{1}{E - \mathcal{H}Q} Q\mathcal{H} \right] \right. \\ \left. \cdot \mathcal{A} \left[ \phi_0(\mathbf{r}_1 \cdots \mathbf{r}_A) U(\mathbf{r}_0) + \sum_{\lambda_\alpha \neq 1} \frac{u_\alpha(\mathbf{r}_0) \phi_0(\mathbf{r}_1 \cdots \mathbf{r}_A) \langle u_\alpha U \rangle}{\lambda_\alpha - 1} \right] \right\rangle \quad (3.37)$$

This equation has the important virtue that the modifications arising directly from the inclusion of the Pauli principle do not introduce any additional explicit energy dependence.

(d) In a certain sense the simplest projection operator, which projects out from the exact solution  $\Psi$  a part which has the same asymptotic behavior as  $\Psi$ , is one which is unity outside the region of interaction and zero inside said region. In other words,  $P\Psi$  is just the asymptotic dependence of  $\Psi$  outside of the region in configuration space in which interactions occur.<sup>13</sup> As we have asserted earlier this is the projection operator employed by the Wigner-Eisenbud (2) formalism for nuclear reactions as we shall now proceed to show.

To keep the discussion simple we shall only consider the situations in which the only possible reaction is elastic scattering. The procedure is easy enough to generalize and we shall indicate how this can be done but shall not carry out the details. In the elastic scattering case

$$P(\mathbf{r}_0, \mathbf{r}_1 \cdots \mathbf{r}_A) = 1 \quad \text{if} \quad |\mathbf{R}_T - \mathbf{r}_\alpha| > a \\ = 0 \quad \text{if} \quad |\mathbf{R}_T - \mathbf{r}_\alpha| < a \quad \text{for all } \alpha \quad (3.38)$$

<sup>13</sup> Interactions here refer to specifically nuclear interactions. Coulomb and other long range forces are, as is customary, included in the description of the motion of the particles in the asymptotic region.

Here  $\mathbf{R}_T$  is the center of mass of the residual nucleus, and  $\mathbf{r}_\alpha$  the coordinate of the emerging light particle. If we are dealing with the case of many final channels, then

$$P = \Sigma P_\sigma \quad (3.39)$$

where  $P_\sigma$  is a projection operator as defined in Eq. (3.38) appropriate to channel  $\sigma$ ; e.g.,  $P_\sigma$  is 1 as long as the relative coordinate between the residual nucleus and light fragment is greater than some fixed value  $a_\sigma$ . The radius  $a$  in Eq. (3.38) (and  $a_\sigma$  in the more complicated case of (3.39)) is chosen so that in the region  $|\mathbf{R}_T - \mathbf{r}_\alpha| > a$  the wave functions for the reaction products do not overlap. As a consequence the Pauli principle is trivially satisfied, a very important advantage of the boundary condition formulation.

The application of the kinetic energy operator will generally give rise to singularities. We shall avoid this difficulty by defining  $\nabla_{\rho_\alpha}^2 P\Psi$  where  $\rho_\alpha$  is the relative coordinate  $\mathbf{R}_T - \mathbf{r}_\alpha$  as the limit  $\nabla^2 P\Psi$  as  $\rho_\alpha - a$  approaches zero from the positive side, while  $\nabla_{\rho_\alpha}^2 Q\Psi$  is the limit of  $\nabla^2 Q\Psi$  as  $\rho_\alpha - a$  approaches zero from the negative side. However we now need to insure the continuity of  $\Psi$ , i.e.,  $P\Psi$  and  $Q\Psi$  must have the same value and normal slope at  $\rho_\alpha = a$ . This would have been a consequence of the equations if we had allowed the singularities in  $\nabla^2 P\Psi$  to remain; but it is more convenient to have these joining conditions explicitly accounted for. This can be achieved by introducing the appropriate coupling between  $P\Psi$  and  $Q\Psi$ . Let  $B$ , the boundary condition operator, be defined as follows:

$$B \equiv \sum_\alpha B_\alpha; \quad B_\alpha \equiv \left( \mu + \nu \frac{\partial}{\partial n_\alpha} \right) \delta(a - \rho_\alpha) \quad (3.40)$$

Here  $\mu$  and  $\nu$  are arbitrary constants while  $\partial/\partial n_\alpha$  is the normal derivative to the surface  $\rho_\alpha = a$ . Then Eq. (2.6) is replaced by<sup>14</sup>

$$[E - P\mathcal{K}P - B]P\Psi = -BQ\Psi \quad (3.41)$$

$$[E - Q\mathcal{K}Q - B]Q\Psi = -BP\Psi \quad (3.42)$$

The fact that singularities on both sides of each equation must match leads immediately to the continuity of slope and values in crossing the dividing surface  $\rho_\alpha = a$ . The effective Hamiltonian for  $P\Psi$  replacing Eq. (2.10) is

$$H = P\mathcal{K}P + B + B \frac{1}{E - Q\mathcal{K}Q - B} B \quad (3.43)$$

It can be readily verified that this effective Hamiltonian leads directly to the correct joining equation. For this purpose, we need the eigenfunctions of the homogeneous form of Eq. (3.42).

$$[E_\lambda - Q\mathcal{K}Q - B]X_\lambda = 0 \quad \rho_\alpha < a$$

<sup>14</sup> Singular boundary value operators are also used by Bloch (17a).

By integrating over an infinitesimal region enclosing the surface  $\rho_\alpha = a$  one can immediately show that  $X_\lambda$  satisfied the equation and boundary conditions

$$\begin{aligned} (E_\lambda - Q\mathcal{H}Q)X_\lambda &= 0 \\ \nu \frac{\partial X_\lambda}{\partial n_\alpha} &= \mu X_\lambda \quad \text{at} \quad \rho_\alpha = a \end{aligned} \quad (3.44)$$

In the same way, Hamiltonian (3.43) corresponds to the following continuity equation

$$\Psi(a) = \sum_\lambda \frac{X_\lambda(a) \langle X_\lambda B \Psi \rangle}{E_\lambda - E} \quad (3.45)$$

where  $\Psi(a)$  just means that all  $\rho_\alpha$  equal  $a$ . The expansion for  $(E - Q\mathcal{H}Q)^{-1}$  in terms of  $X_\lambda$  is not sufficiently convergent to be used to obtain  $\partial\Psi/\partial n_\alpha$ . Actually, since  $B$  involves just boundary operators there is no need to do so. Indeed Eq. (3.45) is just the fundamental equation of the Wigner-Eisenbud theory and leads directly to the  $R$  matrix upon separation of  $\Psi$  into angular momentum eigenstates.

There is of course no particular advantage in obtaining the results of the Wigner-Eisenbud theory in this fashion. We have shown that it forms a rather natural procedure within our formalism. Within the framework of the boundary condition model another insight into the direct reaction process becomes possible. Perhaps, most important of all, the separation of a "surface" region as suggested by Thomas (17) is easily managed by reducing  $a$ , with the consequence that interactions will occur in the region beyond  $a$ . This does introduce complications because of the possibility of channel overlap. We shall not enter into these here, but refer the reader to the discussion in the Thomas paper.

#### IV. RESONANCES

We can now rephrase the derivations of the transition amplitude for resonance reactions given in I in terms of the projection operator formalism. We shall first consider the isolated resonance case in which the width of the level  $\Gamma$  is much smaller than the energy difference,  $D$ , between resonances.

##### A. ISOLATED RESONANCE

Our analysis is based on the observation that when the compound state has a very long lifetime the probability of a particle re-entering an open channel is very small. Therefore, to a first approximation, the wave function describing the compound state is a bound state solution of the homogeneous form of Eq. (2.6b)

$$(\mathcal{E}_\lambda - \mathcal{H}_{QQ})\Phi_\lambda = 0 \quad (4.1)$$



$\Phi_s$  is of course not the exact compound nuclear state since it has an infinite lifetime, a consequence of dropping the  $\mathcal{H}_{QP}P\Psi$  terms of (2.6b) which permitted the decay of  $Q\Psi$  into  $P\Psi$ . However, because the compound state has a very long lifetime we may expect that the compound state wave function will be closely approximated by  $\Phi_s$  and its resonance energy by  $\varepsilon_s$ .

These considerations lead to the following discussion of the effective Hamiltonian,  $H$ , Eq. (2.8) of the open channel wave function  $P\Psi$  satisfying Eq. (2.9). We expand the operator  $(E - \mathcal{H}_{QQ})^{-1}$  in the complete set of wave functions defined by Eq. (4.1) so that<sup>15</sup>

$$H = \mathcal{H}_{PP} + \sum_n [\mathcal{H}_{PQ}\Phi_n]\langle\Phi_n\mathcal{H}_{QP}\rangle/(E - \varepsilon_n) + \int d\alpha \int d\varepsilon [\mathcal{H}_{PQ}\Phi(\varepsilon, \alpha)]\langle\Phi(\varepsilon, \alpha)\mathcal{H}_{QP}\rangle/(E - \varepsilon) \quad (4.2)$$

where  $\varepsilon$  is the continuum energy eigenvalue of (4.1) and where  $\alpha$  denotes the eigenvalue which, in addition to the energy, is required to describe the state  $\Phi(\varepsilon, \alpha)$ . Examine now the behavior of  $H$  in the neighborhood of  $\varepsilon_s$ . It is quite clear that we can break up  $H$  into two parts, one of which varies slowly in this neighborhood the other rapidly, viz:

$$H = H' + [\mathcal{H}_{PQ}\Phi_s]\langle\Phi_s\mathcal{H}_{QP}\rangle/(E - \varepsilon_s) \quad (4.3)$$

$$H' \equiv \mathcal{H}_{PP} + \sum_{n \neq s} [\mathcal{H}_{PQ}\Phi_n]\langle\Phi_n\mathcal{H}_{QP}\rangle/(E - \varepsilon_n) + \int d\alpha \int d\varepsilon [\mathcal{H}_{PQ}\Phi(\varepsilon, \alpha)]\langle\Phi(\varepsilon, \alpha)\mathcal{H}_{QP}\rangle/(E - \varepsilon) \quad (4.4)$$

Equation (2.9) now may be written

$$(E - H')P\Psi = [\mathcal{H}_{PQ}\Phi_s]\langle\Phi_s\mathcal{H}_{QP}\rangle/(E - \varepsilon_s) \quad (4.5)$$

and a solution using the methods of I may be easily obtained in terms of the eigenfunctions of  $H'$ . If  $\psi_0^{(+)}$  is the outgoing wave solution of

$$(E - H')\psi_0^{(+)} = 0 \quad (4.6)$$

then

$$P\Psi = \psi_0^{(+)} + \frac{1}{E^+ - H'} \left[ \frac{\mathcal{H}_{PQ}\Phi_s\langle\Phi_s\mathcal{H}_{QP}\psi_0^{(+)}\rangle}{E - \varepsilon_s - \left\langle \Phi_s\mathcal{H}_{QP} \frac{1}{E^+ - H'} \mathcal{H}_{PQ}\Phi_s \right\rangle} \right] \quad (4.7)$$

<sup>15</sup> If  $P\Psi$  does not include all the open channels, replace  $E$  in the continuum integral by  $E^+$ .

The transition matrix,  $T(\beta | 0)$ , giving the amplitude for a transition to a final state  $\beta$  is

$$T(\beta | 0) = T_P(\beta | 0) + \frac{\langle \psi_\beta^{(-)} \mathcal{H}_{PQ} \Phi_s \rangle \langle \Phi_s \mathcal{H}_{QP} \psi_0^{(+)} \rangle}{E - \mathcal{E}_s - \left\langle \Phi_s \mathcal{H}_{QP} \frac{1}{E^+ - H'} \mathcal{H}_{PQ} \Phi_s \right\rangle} \quad (4.8)$$

Here  $\psi_\beta^{(-)}$  is the solution of (4.6) with incoming wave boundary condition in which the incident wave is in channel  $\beta$ .  $T_P(\beta | 0)$  is the transition amplitude which would follow from Eq. (4.6). It is the so-called direct reaction amplitude while the second term is the resonant amplitude in the Breit-Wigner form as shown in I.  $T_P$  varies smoothly with energy in the region of the isolated resonance, the rapid variation being described by the resonant term. If the resonance can be characterized, beside the energy, by a specific set of characteristic numbers such as total angular momentum,  $J$ , helicity, etc., then the second term just reduces to the customary Breit-Wigner form. This is shown in I and need not be repeated here.

It should be emphasized that in deriving the expression (4.8) it was not only not necessary to introduce a sharp "channel radius," but it was not necessary to carry out a decomposition into particular angular momentum states or to use any specific coupling scheme. Formula (4.8) is valid in any coupling scheme.

In addition (4.8) suggests what seems to us to be a useful extension of the width concept. We are led to this most directly if we evaluate the bracket in the denominator, particularly its imaginary part more explicitly. In this connection note that the solutions of (4.6) for different channels having the same energy are orthogonal (18):

$$\langle \psi_\beta^{(\pm)}, \psi_\mu^{(\pm)} \rangle = \delta_{\mu\beta} \quad (4.9)$$

It therefore follows that  $\Delta^{(s)}$  and  $\Gamma^{(s)}$  defined by

$$\left\langle \Phi_s \mathcal{H}_{QP} \frac{1}{E^+ - H'} \mathcal{H}_{PQ} \Phi_s \right\rangle \equiv \Delta^{(s)} - i\Gamma^{(s)}/2 \quad (4.10)$$

are given by

$$\Delta^{(s)} = \left\langle \Phi_s \mathcal{H}_{QP} \mathcal{O} \frac{1}{E - H'} \mathcal{H}_{PQ} \Phi_s \right\rangle \quad (4.11)$$

$$\begin{aligned} \Gamma^{(s)} &= 2\pi \langle \Phi_s \mathcal{H}_{QP} \delta(E - H') \mathcal{H}_{PQ} \Phi_s \rangle = 2\pi \sum_\mu \int d\Omega |\langle \Phi_s \mathcal{H}_{QP} \psi_\mu^{(+)} \rangle|^2 \\ &= 2\pi \sum_\mu \int d\Omega |\langle \psi_\mu^{(-)} \mathcal{H}_{PQ} \Phi_s \rangle|^2 \end{aligned} \quad (4.12)$$

where the last expressions we have chosen to make the integration over the possible final angles,  $\Omega$ , of emission explicit. These results suggest the definition

of a *differential partial width* which measures the relative probability of emission into a given solid angle for a given channel  $\mu$ :

$$\frac{d\Gamma_{\mu}^{(s)}}{d\Omega} = 2\pi |\langle \Phi_s | \mathcal{H}_{QP} | \psi_{\mu}^{(+)} \rangle|^2 = 2\pi |\langle \psi_{\mu}^{(-)} | \mathcal{H}_{PQ} | \Phi_s \rangle|^2 \quad (4.13)$$

By expanding  $\langle \Phi_s | \mathcal{H}_{QP} | \psi_{\mu}^{(+)} \rangle$  in terms of the spin, angular momentum eigenfunction appropriate to a given coupling scheme, the relation of (4.13) to the more customary expression for the width may be readily established.

Finally let us point out that the derivation of (4.8) does not involve any approximations, contrary to what might have been indicated by I.<sup>16</sup> However the statement that  $T_P$  is slowly varying with energy is correct only when the resonance is isolated. When the resonances start to overlap  $T_P$  is no longer slowly varying, and Eq. (4.8), although correct, is no longer convenient. The results which apply in this situation will be discussed shortly. Returning to  $T_P$  we note that even when  $T_P$  does vary slowly over the width  $\Gamma$  of a narrow isolated resonance, it may itself be undergoing a "single particle" resonance with a width very much larger than  $\Gamma$ . In the case of nuclear reactions, this corresponds to a large amplitude of  $\psi_{\mu}$  inside the nucleus leading immediately to a qualitative understanding of the so-called giant resonance. However, when the density of fine structure resonances is small, as it can well be in systems which are simpler than the compound nuclear case in the sense that  $Q\Psi$  can be reasonably well represented by just a few modes, the single particle and fine structure resonance may not overlap.<sup>17</sup> Hence we would find that there are resonances which are closely associated with the bound state wave functions of the closed channels and "single particle" resonances which are a consequence of the action of a potential which at most varies slowly with energy.

## B. OVERLAPPING RESONANCES

When  $T_P$  varies rapidly with energy over the width of the resonance into  $T_R$ , the separation of  $T$  into these two parts is no longer convenient. In this case it will not suffice to make explicit the effect of the state  $\Phi_s$  whose energy  $\varepsilon_s$  is close to  $E$ . One or more neighboring states must be included; the number involved being that required to remove the rapid energy dependence of  $T_P$ . Equation (4.5) becomes

$$(E - H'')P\Psi = \sum [\mathcal{H}_{PQ}\Phi_{\mu}]\langle \Phi_{\mu} | \mathcal{H}_{QP} | \rangle / (E - \varepsilon_{\mu}) \quad (4.14)$$

<sup>16</sup> This has been verified by direct calculation by L. Zamick. Compare with remarks in Ref. (7).

<sup>17</sup> These remarks are particularly relevant when we are dealing with a few coupled equations such as occur in the description of direct nuclear reactions, or their counterparts in baryon reactions of "elementary" particle physics.

where the sum over  $\mu$  is *finite* and  $H''$  is given by (4.4), except that in the sum over  $n$ , all the terms on the right-hand side of (4.14) are omitted. Equation (4.14) has been discussed in I and there is no need, in view of the example afforded by the preceding section, to rephrase this calculation in our present projection operator notation. The results are that

$$T = T''_P + T''_R \quad (4.15)$$

where  $T''_P$  is the transition matrix which follows from the Hamiltonian  $H''$ .  $T''_R$  has the form

$$T''_R(\beta|0) = \sum_i \frac{\langle \psi_\beta^{(-)} \mathcal{H}_{PQ} \omega_i \rangle \langle \omega_i \mathcal{H}_{QP} \psi_\beta^{(+)} \rangle}{E - E_i} \quad (4.16)$$

Here  $\psi_0$  and  $\psi_\beta$  are appropriate scattering eigenfunctions of  $H''$ ;  $\omega_i$  are linear combinations of

$$\omega_i = \sum X_\mu^{(i)} \Phi_\mu \quad (4.17)$$

where the  $X_\mu^{(i)}$  are solutions of the secular equation which follows from Eq. (4.14):

$$\sum_\nu [(E_i - \epsilon_\mu) \delta_{\mu\nu} - W_{\mu\nu}] X_\nu^{(i)} = 0 \quad (4.18)$$

where

$$W_{\mu\nu} = \left\langle \Phi_\mu \mathcal{H}_{QP} \frac{1}{E^{(+)} - H''} \mathcal{H}_{PQ} \Phi_\nu \right\rangle \quad (4.19)$$

The  $E_i$  in (4.16) and (4.18) are the complex eigenvalues of the secular determinant following from Eq. (4.18).

Note that the expression for that part of  $T$ ,  $T''_R$  which varies rapidly with energy is expressed as a sum over resonances. The numerators are factorable products in which one factor depends only on the initial state (is independent of the final state), the second depends only on the final state. However, each of the resonance terms is not in the Breit-Wigner form. For example, for pure elastic scattering (no reaction processes possible) the imaginary part of  $E_i$  is not simply proportional to the corresponding numerator. Instead certain sum rules are obeyed. These are given in ref. 19. One of these will be needed here for later discussion:

$$\sum_i \langle \psi_\beta^{(-)} \mathcal{H}_{PQ} \omega_i \rangle \langle \omega_i \mathcal{H}_{QP} \psi_0^{(+)} \rangle = \sum_\mu \langle \psi_\beta^{(-)} H_{PQ} \Phi_\mu \rangle \langle \Phi_\mu H_{QP} \psi_0^{(+)} \rangle \quad (4.20)$$

There are a minimal number of terms which must be included in the finite sum over  $\mu$  in Eq. (4.14). However, there is no limit on the number of terms greater than this minimum, so that it is possible to obtain an expansion of  $T$  in

terms of "resonances," which is identical to that discussed below in Section IV, C. For most purposes, however, it is more convenient to keep the number of resonance terms to a minimum and to express the remainder of  $T$ ,  $T_P$ , as the consequence of the action of a potential as is implied by its slow variation with energy.

### C. KAPUR-PEIERLS EXPANSION

Expansions of  $T$  into an infinite resonance series have been obtained by a method suggested by Siegert (20) and investigated by Humblet (21), as well as by Kapur and Peierls (5) with improvements by Brown *et al.* (22). For a review of these results see Lane and Thomas (11).

In the Siegert-Humblet type of expansion the resonance energies do not depend upon  $E$ , the energy, whereas in the Kapur-Peierls case they do. We shall develop the Kapur-Peierls type here since it is intimately connected with the discussion in Section IV, B and shall relegate a derivation of the Siegert-Humblet expansion to an appendix.

The Kapur-Peierls type of expansion can be readily obtained in the present formalism by appropriate analysis of the fundamental equations (2.3) and (2.4). Toward this end we shall obtain an exact expression for the transition matrix. First solve (2.6a) for  $P\Psi$ :

$$P\Psi = \psi_0^{(+)} + \frac{1}{E^+ - \mathfrak{H}_{PP}} \mathfrak{H}_{PQ} Q\Psi \quad (4.21)$$

where  $\psi_0^{(+)}$  is the solution of the homogeneous form of (2.6a) chosen so as to satisfy appropriate boundary conditions. Inserting this equation into (2.6b) leads to an effective Hamiltonian equation for  $Q\Psi$  of a form similar to (2.9):

$$\left[ E - \mathfrak{H}_{QQ} - \mathfrak{H}_{QP} \frac{1}{E^+ - \mathfrak{H}_{PP}} \mathfrak{H}_{PQ} \right] Q\Psi = \mathfrak{H}_{QP} \psi_0^{(+)} \quad (4.22)$$

We can now obtain  $Q\Psi$  and substitute for  $Q\Psi$  in (4.21) and so obtain  $P\Psi$ . The resulting expression for the transition matrix is

$$T = T_P + \left\langle \psi_\beta^{(-)} \mathfrak{H}_{PQ} \frac{1}{E - \mathfrak{H}_{QQ} - \mathfrak{H}_{QP} \frac{1}{E^+ - \mathfrak{H}_{PP}} \mathfrak{H}_{PQ}} \psi_0^{(+)} \right\rangle \quad (4.23)$$

where  $T_P$  is the scattering associated with the homogeneous form of Eq. (2.6a).

It is natural to expand  $T$  in terms of the eigenfunctions  $\Xi$  of the effective Hamiltonian for  $Q\Psi$ . The corresponding eigenvalues  $E_s$  are complex with a negative imaginary part as we shall shortly show.  $T$  then becomes

$$T = T_P + \sum_i \frac{\langle \psi_\beta^{(-)} \mathfrak{H}_{PQ} \Xi_i \rangle \langle \Xi_i^A \mathfrak{H}_{QP} \psi_0^{(+)} \rangle}{E - E_i} \quad (4.24)$$

$\Xi_t^A$  and  $\Xi_t$  form a biorthogonal set. These functions are solutions of

$$\left[ E_t - \mathcal{H}_{QQ} - \mathcal{H}_{QP} \frac{1}{E^+ - \mathcal{H}_{PP}} \mathcal{H}_{PQ} \right] \Xi_t = 0 \quad (4.25)$$

In terms of Eq. (2.3) and (2.4)  $\Xi_t$  is the solution for  $Q\Psi$  while

$$P\Psi = \frac{1}{E^+ - \mathcal{H}_{PP}} \mathcal{H}_{PQ} \Xi_t \quad (4.26)$$

Note there is no incident wave.  $P\Psi$  thus satisfies the Kapur-Peierls type of boundary condition (3); we have been able to specify this without the use of a "radius." In this formalism, however, one has to pay the price of shifting the boundary condition as  $E$  changes according to (4.36), thus using a different orthogonal set as  $E$  changes. For a further discussion of this point see Brown (22). The main difference between this development and that of Section IV, B is that in the latter the energy dependence is carried by the Hamiltonian  $H'$ . But of course the two methods are completely equivalent. The principal advantage of development in Section IV, B is the explicit separation of fine and gross structure resonances.

The function  $\Xi_t^A$  is given by:

$$\Xi_t^A = \Xi_t^* \quad (4.27)$$

To obtain the imaginary part of  $E_t$ , multiply (4.25) from the left by  $(\Xi_t)^*$  and integrate

$$E_t - E_t^* = \left\langle \Xi_t \mathcal{H}_{QP} \left( \frac{1}{E^+ - \mathcal{H}_{PP}} - \frac{1}{E^- - \mathcal{H}_{PP}} \right) \mathcal{H}_{PQ} \Xi_t \right\rangle \quad (4.28)$$

$$\text{Im } E_t = -\pi \langle \Xi_t \mathcal{H}_{QP} \delta(E - \mathcal{H}_{PP}) \mathcal{H}_{PQ} \Xi_t \rangle$$

Note that  $\text{Im } E_t$  is less than zero. We have thus obtained the major results of the Kapur-Peierls formalism using a very simple almost algebraic procedure. No radius or separation into partial waves was required.

#### D. THE COMPLEX POTENTIAL MODEL<sup>18</sup>

As described in I, the potential must account for the average transition amplitude  $\langle T \rangle$ :

$$\begin{aligned} \langle T \rangle &= T_P - \frac{i\pi}{\Delta E} \sum_i \langle \psi_\beta^{(-)} \mathcal{H}_{PQ} \omega_i \rangle \langle \omega_i \mathcal{H}_{QP} \psi_0^{(+)} \rangle \\ &= T_P - \frac{i\pi}{\Delta E} \sum_\mu \langle \psi_\beta^{(-)} \mathcal{H}_{PQ} \Phi_\mu \rangle \langle \Phi_\mu \mathcal{H}_{QP} \psi_0^{(+)} \rangle \end{aligned} \quad (4.29)$$

<sup>18</sup> The result, Eq. (4.33), obtained in this section was first derived by M. Baranger (private communication).

where we have made use of Eq. (4.16) and Eq. (4.20).  $\Delta E$  is the energy interval over which  $T$  is averaged. Let the complex potential wave function which is to give rise to (4.29) be  $\varphi$ . Then  $\varphi$  must satisfy the equation:

$$\varphi = \chi_0^{(+)} + \frac{1}{E^+ - H''} \{ -i\pi \mathfrak{C}_{PQ} \Lambda \mathfrak{C}_{QP} \chi_0^{(+)} \} \quad (4.30)$$

where

$$\Lambda \equiv \frac{1}{\Delta E} \sum_{\mu} \Phi_{\mu} \langle \Phi_{\mu} \quad (4.31)$$

and  $(E - H'')\chi_0^{(+)} = 0$ . We now solve Eq. (4.30) for  $\chi_0^{(+)}$

$$\chi_0^{(+)} = \left[ 1 - i\pi \frac{1}{E^+ - H''} \mathfrak{C}_{PQ} \Lambda \mathfrak{C}_{QP} \right]^{-1} \varphi$$

Finally, operating with  $E - H''$  on both sides we obtain

$$(E - H'') \frac{1}{1 - i\pi \frac{1}{E^+ - H''} \mathfrak{C}_{PQ} \Lambda \mathfrak{C}_{QP}} \varphi = 0$$

or

$$\left[ E - H'' + i\pi \mathfrak{C}_{PQ} \Lambda \mathfrak{C}_{QP} \frac{1}{1 - i\pi \frac{1}{E^+ - H''} \mathfrak{C}_{PQ} \Lambda \mathfrak{C}_{QP}} \right] \varphi = 0$$

or

$$[E - H'' - H_{CN}]\varphi = 0 \quad (4.32)$$

$$H_{CN} = -i\pi \mathfrak{C}_{PQ} \frac{1}{1 - i\pi \Lambda \mathfrak{C}_{QP} \frac{1}{E^+ - H''} \mathfrak{C}_{PQ}} \Lambda \mathfrak{C}_{QP} \quad (4.33)$$

$H''$  gives rise to the direct processes and the additional term gives the averaged influence of the compound nuclear processes on the direct.  $H''$  is of course very closely related to the shell model potential. Note that this suggests that complex potential models can be employed not only for the scattering of nucleons where it has been so extensively applied, but also to other reactions such as, for example, the entire syndrome of reactions which occur when deuterons are the incident or emerging projectile including elastic, inelastic, and stripping and pickup processes. An approximate evaluation for neutron induced reactions will be discussed in a paper in preparation.

## V. CONCLUDING REMARKS

Our final remarks are concerned with the Hamiltonian  $\mathfrak{H}$  of Eq. (2.5). Because<sup>19</sup> of the singular hard core interactions between two nucleons  $\mathfrak{H}$  is *not* the

<sup>19</sup> Even when the interactions are not singular it is usually necessary to employ the self-consistent method.

many-body Hamiltonian but a suitably modified nonsingular version, the precise determination of which involves a self-consistent procedure of the Brueckner type. For the present purposes it is sufficient to note that in the limits of low and high energy good estimates of  $\mathcal{H}$  are available. At low energies,  $\mathcal{H}$  consists of the shell model Hamiltonian plus the residual potential. The latter will generally include two-body potentials as well as collective model interactions.<sup>20</sup> At suitably high energies it is permissible to neglect exchange scattering and consequently to treat the incident nucleon (or nucleons) as distinguishable from those of the target nucleus. Then  $\mathcal{H}$  can be written as a sum of the Hamiltonian for the target nucleus and an effective potential describing the interaction of the incident nucleon and the target nucleus where the latter is determined via the multiple scattering approximation as developed for example in ref. 6. Once the effective Hamiltonian is given, a quantitative treatment of reactions becomes conceivable and it is for this reason that the explicit expression for the projection operators was obtained in Section III.

It is probably not necessary to add that the formalism developed in this paper has a wide range of applicability extending beyond just nuclear reactions upon which our attention has been focused. The nature of the system involved has been kept open except for the assumption that a Hamiltonian exists, the Hamiltonian itself remaining unspecified. The formalism can consequently be applied to reactions involving many-body systems. The Hamiltonian may be given in configuration space and describe a system of interacting nucleons or it may be a field theoretic Hamiltonian in which case our procedure is closely related to the Tamm-Dancoff method.

#### APPENDIX

The Kapur-Peierls type of expansion suffers from the difficulty that the complex eigenvalues  $E_\mu$  are functions of the incident energy. Rigorously the resonances of a physical system correspond to poles in the  $S$ -matrix which occur at energies (we shall call these *complex resonance energies*) with a negative imaginary part corresponding to the half width of the resonance. These complex eigenvalues are of course independent of the energy  $E$  of the system. However it may be shown (22) that the complex eigenvalues of the Kapur-Peierls expansion are close to the complex resonance energies as long as we are concerned with the complex resonance energies whose real part is near the energy  $E$  and whose imaginary part is small. This is in fact the merit of the treatment in Section IV, B since the resonance description is reserved for only these terms.

It may be occasionally convenient to express the resonance terms directly in

<sup>20</sup> Actually the residual interaction may very well include many-body potentials; but these are usually neglected.



terms of the complex resonance energies and the related wave functions. This we shall now proceed to do.

The complex resonance energies are eigenvalues of the total Hamiltonian  $\mathcal{H}$  of the system, the wave functions being subject to the radiation boundary condition:

$$\partial\Psi/\partial\rho_\alpha \rightarrow ik_\alpha\Psi \quad \text{as} \quad \rho_\alpha \rightarrow \infty \quad (\text{A.1})$$

where  $k_\alpha$  has the usual relation to the channel energy and  $\rho_\alpha$  is the radial coordinate for channel  $\alpha$ . The eigenvalues of  $\mathcal{H}$  subject to the boundary condition (A.1) include the bound states in which case  $k_\alpha$  is pure imaginary,  $\text{Im}k_\alpha > 0$ . The complex eigenvalues of the system correspond to  $k_\alpha$  with  $\text{Im}k_\alpha < 0$  while  $\text{Re}k_\alpha \leq 0$ , the two signs corresponding to ingoing and outgoing waves. In spite of the fact that the eigenfunctions corresponding to these complex eigenvalues grow exponentially as  $\rho_\alpha$  approaches infinity, they form an orthonormal set.<sup>21</sup> If the eigenvalues are  $E_\mu$  and the eigenfunctions  $\Omega_\mu$ , the orthogonality and normalization conditions are:

$$\lim_{\rho_\alpha \rightarrow \infty} \left\{ \int \Omega_\mu \Omega_\nu d\tau + \frac{1}{E_\mu - \mathcal{H}} \sum_\alpha \int \left[ \Omega_\mu \frac{\partial \Omega_\nu}{\partial \rho_\alpha} - \Omega_\nu \frac{\partial \Omega_\mu}{\partial \rho_\alpha} \right] dS_\alpha \right\} = \delta_{\mu\nu} \quad (\text{A.2})$$

The operator  $\mathcal{H}$  operates only on  $\Omega_\nu$ . There is no difficulty in evaluating the inverse operator  $(E_\mu - \mathcal{H})^{-1}$  for large values of  $\rho_\alpha$  since here the interaction vanishes. The volume element  $d\tau$  signifies the volume element in configuration space while  $dS_\alpha$  is the surface element. Note that the surface integral which has the function of canceling the divergent part of the usual normalization integral is evaluated at infinite separation, that is, in the region where the asymptotic forms of  $\Omega_\mu$  and  $\Omega_\nu$  are valid. The normalization and orthogonality given by Eq. (A.2) permit us to project out of any function that part which is proportional to a particular  $\Omega_\mu$  and orthogonal to the other  $\Omega_\nu$  as well as the bound states.

With this development in hand it becomes possible to obtain a complex eigenvalue representation of the resonance terms in the transition matrix. We write the complete wave function for the system,  $\Psi$ , in terms of an "incident" wave  $\psi_0$  and a scattered wave  $\Psi_s$ :

$$\Psi = \psi_0 + \Psi_s$$

where  $\psi_0$  is defined to be the solution of the equation

$$\begin{aligned} (E - \mathcal{H}_{PP})P\Psi &= 0 \\ (E - \mathcal{H}_{QQ})Q\Psi &= 0 \end{aligned} \quad (\text{A.3})$$

<sup>21</sup> We shall not discuss their completeness a moot property as we shall not need it in the following.

subject to the condition that  $\psi_0$  asymptotically approach an incident plane wave in one of the open channels. Since there is no coupling to the closed channels  $Q\psi_0 = 0$ ,  $\psi_0$  will of course contain outgoing waves which describe scattering and reactions. We shall call the corresponding transition amplitude  $T_0(\beta | 0)$  where  $\beta$  denotes a possible open exit channel.  $\Psi_s$  satisfies the equation

$$(E - \mathcal{H}) \Psi_s = \mathcal{H}_{QP} P\psi_0 \quad (\text{A.4})$$

The transition amplitude  $T(\beta | 0)$  is the sum of  $T_0(\beta | 0)$  and  $T_1(\beta | 0)$ :

$$T_1(\beta | 0) = \langle \Psi_{s\beta}^{(-)} \mathcal{H}_{QP} P\psi_0 \rangle$$

or

$$T_1(\beta | 0) = \left\langle \psi_{\beta}^{(-)} \mathcal{H}_{PQ} \frac{1}{E^+ - \mathcal{H}} \mathcal{H}_{QP} \psi_0 \right\rangle \quad (\text{A.5})$$

Employing the orthonormal conditions (A.2) it is possible to pick out the dependence of the Green's function  $(E^+ - \mathcal{H})^{-1}$  on a particular  $\Omega_{\mu}$ . It is however useful to consider only those states  $\Omega_{\mu}$  whose imaginary part is relatively small. Including all the possible  $\Omega_{\mu}$  would lead to convergence problems since in such a series we expand a bounded function in terms of functions which have exponential growth. We therefore write

$$T_1(\beta | 0) = T_1^{(\infty)} + \sum_{\mu} \frac{\langle \psi_{\beta}^{(-)} \mathcal{H}_{PQ} \Omega_{\mu} \rangle \langle \tilde{\Omega}_{\mu} \mathcal{H}_{QP} \psi_0 \rangle}{E - E_{\mu}} \quad (\text{A.6})$$

$$\tilde{\Omega}_{\mu}^* = \Omega_{\mu} \quad (\text{A.7})$$

No surface terms have been included, an omission which is valid if  $\mathcal{H}_{PQ}$  decreases with increasing  $\rho_{\alpha}$  more rapidly than exponentially. Thus, for a finite range  $\mathcal{H}_{PQ}$ , or one which is given by a gaussian, no surface terms will be present. Note that the sum over  $\mu$  is over a finite number of terms and that  $T_1^{(\infty)}$  represents the remainder. Again we note that the development leading to this result did not require the notion of channel radii or a decomposition in orbital angular momentum eigenstates. An expansion of  $T$  in a complex eigenvalue expansion is also obtained by Rosenfeld and Humblet (5) but only after separating into partial waves.

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