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Unified Theory of Nuclear Reactions*

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A new formulation of the theory of nuclear reactions based on the properties of a generalized "optical" potential is presented. The real and imaginary part of this potential satisfy a dispersion type relation while its poles give rise to resonances in nuclear reactions. A new derivation of the Breit-Wigner formula is given in which the concept of channel radius is not employed. This derivation is extended to the case of overlapping resonances. These results can then be employed to obtain the complex potential well model for pure elastic scattering. This potential well is shown to become real as the average width of the resonances increases. Reactions as well as elastic scattering are treated. Considering the former process in an isolated resonance, we obtain a nonresonant term analogous to the familiar potential scattering term of elastic scattering. This is just the direct interaction term which thus appears automatically in this formalism. Upon performing the appropriate energy averages over resonances, the complex potential well model is generalized so as to include inelastic scattering. The effects of the identity of nucleons is investigated. It is shown that our formalism is valid as long as the exit channels can at most contain one nucleon.

I. INTRODUCTION

By this time it has become abundantly clear that many differing aspects of the nucleon-nucleus interaction show up in nuclear reactions and scattering. There is consequently a multiplicity of descriptions of nuclear reactions which are brought to mind by the terms compound nucleus, the statistical hypothesis, optical model and direct and surface interaction. Since all the phenomena characterized by these concepts are general properties of the many body system (*1*) it should be possible to develop a theory of nuclear reactions from which each of these phenomena can be abstracted in a natural and straight forward fashion. It is the purpose of this paper to present such a formalism.

This program has been carried out in part by Thomas (*2*), Bloch (*3*) who employ a suitably modified form of the Wigner (*4*) theory of nuclear resonances and by Brown and Dominicis (*5*) who work with the Kapur-Peierls (*6*) formalism. The present paper is based on a new formulation of resonance theory which

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appears (at least to the author!) to have a much simpler conceptual basis and derivation and to lead quite directly to the optical model, direct interactions, etc. Some elements of the procedure we shall use have been employed by Lamarsh and Feshbach (*7*), by Yoshida (*8*) and by Newton (*9*).

It will be useful to devote the remainder of the introduction to a brief review. From our point of view the concept of the compound nucleus holds a central position in the theory of nuclear reactions (*1*, *10–14*). The existence of narrow resonances (for example for slow neutrons) shows directly that long lived compound systems formed from the target nucleus and the incident nucleon exist. The energy dependence of the cross-section in the neighborhood of a resonance is given by the celebrated Breit-Wigner (*13*) formula which fits the experimental data with remarkable precision. For example the cross-section for the elastic scattering of $l = 0$ neutrons by a target nucleus of spin I is:

$$\sigma_e = 4\pi\lambda\!\!\!^{-2} \frac{2J+1}{(2I+1)(2s+1)} \left| \sin\delta\, e^{i\delta} - \frac{\frac{1}{2}\Gamma_n^{(J)} e^{2i\delta}}{E - E_c + \frac{i}{2}\Gamma_c^{(J)}} \right|^2, \qquad (1.1)$$

where $\lambda\!\!\!^{-}$ is the wavelength/2π of the incident neutron, and s ($= \frac{1}{2}$) its spin. The level of the compound nucleus in resonance at the energy E_c has a spin, J, a width, $\Gamma_c^{(J)}$, and a neutron width of $\Gamma_n^{(J)}$. The quantity δ is the phase shift coming from the so-called potential scattering. The corresponding amplitude $e^{i\delta}\sin\delta$ varies slowly with the energy and describes the scattering between resonances.

It is of course necessary to generalize the Breit-Wigner formula to the case of overlapping resonances. The most complete solution of this problem has been given by Wigner, Eisenbud, and Teichmann (*4*) and by Kapur and Peierls (*6*). The results of this formalism are much too complex to be summarized here and in any event have been discussed in some detail by both Blatt and Weisskopf (*10*) and Sachs (*11*) in their books on nuclear theory. It will suffice for our purposes here to indicate some of the principal ideas. These theories divide configuration space into two regions. The first is an internal region in which all the nucleons interact strongly and form nearly stable configurations, the states of the compound nucleus. The second, the external region, consists of channels corresponding to the various ways in which the compound nucleus can disintegrate. In the external region the reaction products are well separated so that their only interactions are non-nuclear (e.g., Coulomb force). The dividing surface between external and internal is taken to be the various channel radii R_α which roughly equal the nuclear radius R. A complete set of wave functions for the internal region are obtained by insisting that they satisfy the many-body Schrödinger equation and homogeneous boundary conditions at the channel

radii. The Wigner and Peierls treatment differ only in their choice of boundary conditions. The wave function describing the actual reaction process may then be described in the internal region by an expansion in this complete set and in the external region by the known wave functions for the disintegrated system in each channel. It is clear that the amplitudes for the various possible reactions will depend upon the channel radii R_α, the energies for the levels of the compound nucleus (these are complex in the Kapur-Peierls formalism) and the overlap $y_{c\alpha}$ between the compound nuclear wave functions and the wave function in the corresponding channel α. The widths Γ_n are proportional to the square of $y_{c\alpha}$. The wide applicability and power of this theory are well known. It gives completely all the "kinematical features" of reaction theory, i.e., those features which are independent of the details of the dynamics of the nucleon-nucleon interaction except for the qualitative property that this interaction is short range. Not only does it describe the resonances but it also forms the starting point for theories of the optical model. There is however one undesirable feature of this formalism and that is the explicit dependence on the channel radii, R_α. The channel radius cannot be so sharply defined and this is reflected in the theory in that there are many different possible choices of R_α all of which must give rise to the identical scattering and reaction amplitudes. However this invariance property is not an obvious consequence of the theory. This difficulty is avoided in the formalism to be presented in this paper.

The Wigner-Peierls theory gives a rigorous expression for the situation where many levels overlap or are simultaneously important because of the inhomogeneity in energy of the incident beam. Under these circumstances, the various cross-sections will have a smooth energy dependence. Obviously the resonance expansion is not a convenient representation. At this point, the *statistical hypothesis* is invoked (*15*, *16*) in order to obtain the sum over resonances. For a complete application of a statistical theory we need a statement of the probability distribution of the values of $y_{c\alpha}$ and the energies of the compound nucleus E_c. Some empirical information on these matters is now becoming available (*17*). The first part of the statistical hypothesis states that positive and negative values of $y_{c\alpha}$ are equally probable. All interference terms then automatically vanish and the transition matrix and cross-sections become a series of resonance terms coming from each resonance E_c (*18*, *19*) separately. The second part of the statistical hypothesis involves an energy average, a consequence of the assumption that the energy levels E_c are randomly distributed. When both parts of the statistical hypothesis are applied to the reaction cross-section the expressions given by Wolfenstein (*20*) and by Hauser and Feshbach (*21*) are obtained. If applied to the diagonal entrance channel components of the transition matrix one obtains the scattering amplitude of the "complex potential model" (*18*); i.e., one obtains a smoothly varying average scattering amplitude for the entrance channel. The

"complex potential" model results when one postulates in addition that there is a wave function describing the motion of the incident nucleon, which satisfies a one-body Schrödinger equation with a complex potential. This potential is adjusted so as to give the average scattering amplitude calculated from reaction theory. Thomas (*19*), Bloch (*22*), Bowcock (*23*), and Brown and Dominicis (*5*) have shown how this connection might be made. A review of this work is contained in an article by this writer now in press (*24*).

These authors generally assume the existence of the complex potential and then go on to determine its properties. In this paper we shall derive an explicit (though not necessarily as practical) expression for the complex potential from which we not only show that the potential exists but we can also make some of its qualitative features apparent.

In summary, the statistical hypothesis leads to both the statistical theory of nuclear reactions and to the "complex potential" model. It is important to remember that these theories in their usual form consider average cross-sections only. For a complete theory it is necessary to predict the fluctuations away from the average (*25*) which, it is easy to see from the Porter-Thomas distribution, can be large.

As is well known (*26*, *27*) the statistical theory of nuclear reactions is not completely in agreement with experiment. For example in inelastic neutron scattering, the statistical theory appears to be quite satisfactory as long as the neutrons lose a considerable fraction of their initial energy. Cranberg and Levin (*28*) measured the angular distribugion of inelastically scattered neutrons for a variety of elements and energies up to several Mev and found substantial agreement with the theory of Ref. *21*. A similar result was obtained by Stewart and Rosen (*29*) who, working at higher energies (14 Mev), found that those neutrons which had lost a good fraction of their energy, about 90 per cent of those inelastically scattered, had an isotropic angular distribution. No measurements of the polarization of inelastically scattered nucleons has been made. The prediction of statistical theory is that on the average there should not be any polarization. When however the neutrons lose only a little energy or in other words when only the low lying levels of the target nucleus are excited, then statistical theory fails. For example in the experiments of Stewart and Rosen, the remaining ten percent of inelastically scattered neutrons were strongly peaked forward. The angular distribution was not symmetric about 90°. For a complete review see Refs. *26* and *27*.

Of course this phenomenon could be in part the fluctuations away from the average as mentioned earlier. A much more plausible explanation, that of the *direct interaction*, was advanced by Austern, Butler, and McManus (*30*). These authors envisage a process in which there is a direct collision between the incident nucleon and a nucleon in the nucleus and in which the latter leaves the nucleus without further interaction except with the nucleus as a whole, i.e., via

the complex well potential. These conditions are most easily met when the struck nucleon is near the target nucleus surface and the target nucleus relatively opaque. Austern *et al.* applied their analysis to this situation. Later authors (*7*, *31*) considered volume effects as well. The very complete analysis carried out by Levinson and Banerjee demonstrated that not only is the direct interaction necessary to understand the inelastic scattering of protons by carbon but that the inelastic protons as well as the elastically scattered ones must be assumed to move in a complex potential. For further discussion the reader is referred to a paper by Butler (*32*) and a monograph by Butler and Hittmair (*33*).

Another type of direct interaction leads to the excitation of the rotational levels of the residual nucleus. Recent treatments of this problem have been given by Yoshida (*8*) and by Chase, Wilets, and Edmonds (*34*).

One might ask why the general theory did not include the direct interaction. The statistical hypothesis must apparently fail, i.e., the assumption of complete randomness is not correct. Sufficient correlation among the $y_{c\alpha}$'s must remain so that the direct interaction is possible. However this seems at best a very inconvenient way of describing direct interactions. A better representation of this situation is one of the goals of this paper.

We now summarize the content of the paper. In Section II a generalized optical potential is derived and a dispersion relation between its real and imaginary parts obtained. In Section III this potential is shown to lead to resonances. The Breit-Wigner formula and its generalization to overlapping levels is obtained. In Section IV the complex well model is derived. Sections II, III, and IV discuss the situation in which only pure elastic scattering can take place. Section V generalizes these results to reactions. A non-resonant contribution to the reaction amplitude is obtained and is interpreted as describing the direct interaction effects. In Section VI we discuss the hard core and identical particle effects. Portions of Section II have been given earlier by the author in a review article (*24*) in which a projection operator formalism was employed. It is worth noting that it is easy (and for this reason it has not been included in this paper) to rewrite the entire discussion of this paper in terms of such a formalism.

II. THE GENERALIZED OPTICAL POTENTIAL

We are concerned with the wave function Ψ describing a system of $A + 1$ nucleons consisting of a nucleon plus the target nucleus of mass number A. Let the wave functions describing the various states i of the target nucleus be $\psi_i(\mathbf{r}_1 \cdots \mathbf{r}_A)$ and let the corresponding energies be ϵ_i. The variables $\mathbf{r}_k$ include position, spin and isotopic spin. The wave functions ψ_i form a complete orthonormal set. Following many authors, e.g., Lane, Thomas and Wigner (*35*) we expand the wave function Ψ in terms of the set ψ_i as follows:

$$\Psi = \Sigma_i \psi_i(\mathbf{r}_1 \cdots \mathbf{r}_A) u_i(\mathbf{r}_0). \tag{2.1}$$

The $i = 0$ term in this sum describes the target nucleus in its ground state plus a nucleon of energy E in the incident channel. The other values of i correspond to possible exit channels in which the emergent particle is a nucleon which differs in some way, e.g., energy, spin, isotopic spin, etc., from the incident nucleon. Expansion (2.1) is convenient for the description of elastic scattering, inelastic scattering, (n, n'), (p, p') as well as for (n, p) and (p, n) reactions. It is not convenient for the discussion of reactions, such as (n, α), in which the emergent particle is composite. However because of the completeness of the set, ψ_i, the sum (2.1) does contain these processes but it would require a regrouping of the terms in the expansion to make them more apparent.

There are two fundamental difficulties with expansion (2.1). First the expansion is not at all appropriate when the nucleon-nucleon potential contains hard cores. The condition that Ψ vanish whenever $\mathbf{r}_0$ is inside the hard core of $\mathbf{r}_1$, $\mathbf{r}_2$ etc. is not automatic. A second difficulty has to do with the identity of nucleons. In particular one has to reckon with the phenomenon of exchange scattering (*36*) in which the incident nucleon is captured by the target nucleus and an identical nucleon in the target nucleus is ejected. Contributions to exchange scattering from expansion (2.1) come from those ψ_i which are in the continuum; that is those ψ_i in which one or more of the particles of the target nucleus is not bound. Because of exchange scattering the amplitudes u_i do not completely describe the various channels. An associated requirement on the wave function Ψ following from the identity of the particles is that Ψ be antisymmetrical. Expansion (2.1) does not satisfy this requirement. It should be noted that the ψ_i are antisymmetrized. For the present we shall neglect these effects. We shall return to them in Section VI where we shall show that the results we obtain below are correct *in form* although the meaning of some of the matrix-elements is modified.

With these reservations in mind, we now insert expansion (2.1) into the Schrödinger equation

$$\mathcal{H}\Psi = E\Psi, \tag{2.2}$$

where

$$\mathcal{H} = H_A(\mathbf{r}_1 \cdots \mathbf{r}_A) + T_0 + V(\mathbf{r}_0, \mathbf{r}_1 \cdots \mathbf{r}_A). \tag{2.3}$$

H_A is the Hamiltonian for the A particles of the target nucleus, T_0 is the kinetic energy operator for the variable $\mathbf{r}_0$, and V is the potential energy of the nucleon in the field of the nucleons of the target nucleus. The wave function ψ_i satisfy

$$H_A \psi_i = \epsilon_i \psi_i . \tag{2.4}$$

Employing the orthonomal properties of the set ψ_i a corresponding set of coupled equations can be obtained for the amplitudes u_i :

$$(T_0 + V_{ii} + \epsilon_i - E)u_i = -\sum_{j \neq i} V_{ij} u_j . \tag{2.5}$$

Here the matrix elements V_{ij} are

$$V_{ij}(\mathbf{r}_0) = (\psi_i, V\psi_j), \tag{2.6}$$

$$V_{ji} = V_{ij}^*, \tag{2.7}$$

where the integrations implied by the round brackets are over the variables $\mathbf{r}_1$ to $\mathbf{r}_A$ excluding $\mathbf{r}_0$. Equations (2.5) have been considered by Breit (*37*) and by Cini and Fubini (*38*).

We shall now derive an uncoupled Schrödinger equation for u_0. For this purpose we need a notation which singles out u_0. We define a uni-columnar matrix $\mathbf{\Phi}$:

$$\mathbf{\Phi} \equiv \begin{pmatrix} u_1 \\ u_2 \\ \vdots \end{pmatrix}. \tag{2.8}$$

We need also the matrix operator $\mathbf{H}$ defined by

$$H_{ij} = T_0\,\delta_{ij} + V_{ij} + \epsilon_i\delta_{ij}, \qquad i, j \neq 0. \tag{2.9}$$

Finally to describe the coupling of $\mathbf{\Phi}$ with u_0 we define $\mathbf{V}_0$ and $\mathbf{V}_0^\dagger$:

$$\mathbf{V}_0 = (V_{01}, V_{02}, \cdots),$$
$$\mathbf{V}_0^\dagger = \begin{pmatrix} V_{01}^* \\ V_{02}^* \\ \vdots \end{pmatrix}. \tag{2.10}$$

Then Eq. (2.5) takes on the following simple appearance:

$$(T_0 + V_{00} - E)\,u_0 = -\mathbf{V}_0\mathbf{\Phi} \tag{2.11a}$$

$$(\mathbf{H} - E)\mathbf{\Phi} = -\mathbf{V}_0^\dagger u_0, \tag{2.11b}$$

where $\mathbf{V}_0\mathbf{\Phi}$ is of course the matrix product of $\mathbf{V}_0$ and $\mathbf{\Phi}$. To obtain an equation for u_0 we need to eliminate $\mathbf{\Phi}$. Solving Eq. (2.11b) formally we obtain $\mathbf{\Phi}$ in terms of u_0

$$\mathbf{\Phi} = \frac{1}{E^{(+)} - \mathbf{H}}\,\mathbf{V}_0^\dagger u_0. \tag{2.12}$$

Here $E^{(+)}$ is defined by

$$E^{(+)} = E + i\eta, \qquad \eta \to 0^{(+)}. \tag{2.13}$$

The term $i\eta$ insures that in expression (2.12) only outgoing waves will be present in the exit channels u_i $(i \geq 1)$. Inserting Eq. (2.12) into (2.11a) we obtain

$$\left[T_0 + V_{00} + \mathbf{V}_0 \frac{1}{E^{(+)} - \mathbf{H}}\,\mathbf{V}_0^\dagger - E\right] u_0 = 0. \tag{2.14}$$

The effective potential, which we shall refer to as the generalized optical potential, $\mathcal{U}$, is

$$\mathcal{U} = V_{00} + \mathbf{V}_0 \frac{1}{E^{(+)} - \mathbf{H}} \mathbf{V}_0^{\dagger}. \tag{2.15}$$

We now discuss some of the properties of $\mathcal{U}$ which follow from this equation. First $\mathcal{U}$ is complex as follows from the $i\eta$ term in the denominator of Eq. (2.15). Physically this complex term comes from processes in which particles leave the entrance channel u_0 because of the coupling $\mathbf{V}_0^{\dagger}$ and are emitted in one of the exit channels u_i contained in $\mathbf{\Phi}$. Of course this does not happen unless the energy E is large enough ($E > \epsilon_1$) so that reactions can occur.

Secondly $\mathcal{U}$ is a nonlocal potential. To see this more explicitly we expand the inverse operator in Eq. (2.15) in terms of the eigenfunctions of $\mathbf{H}$. In general the spectrum of $\mathbf{H}$ will consist of a discrete part and a continuum. Let the eigenfunctions for the states in the discrete part of the spectrum be $\mathbf{\Phi}_n$ with eigenvalue $\mathcal{E}_n$:

$$\mathbf{H}\mathbf{\Phi}_n = \mathcal{E}_n \mathbf{\Phi}_n . \tag{2.16}$$

The eigenfunctions for the continuum states are $\mathbf{\Phi}(\mathcal{E}', \alpha)$ where the eigenvalue is $\mathcal{E}'$ and α labels the various states having a common $\mathcal{E}'$:

$$\mathbf{H}\mathbf{\Phi}(\mathcal{E}', \alpha) = \mathcal{E}'\mathbf{\Phi}(\mathcal{E}', \alpha). \tag{2.17}$$

Since the discrete states $\mathbf{\Phi}_n$ will play an important role throughout this paper we shall digress briefly in order to discuss their physical meaning. This last is easiest to see in the weak coupling limit defined by:

$$V_{ij} = 0, \qquad i \neq j \neq 0 \quad \textit{weak coupling limit.} \tag{2.18}$$

Then $\mathbf{H}$ becomes diagonal so that the eigenfunctions $\mathbf{\Phi}_n$ are of the form:

$$\mathbf{\Phi}_n = \begin{pmatrix} 0 \\ u_i^{(n)} \\ 0 \end{pmatrix} \qquad \textit{weak coupling limit.} \tag{2.19}$$

That is all the u_i ($i \neq 0$) are zero except one, $u_i^{(n)}$, and that one satisfies a simple Schrödinger equation:

$$[T_0 + V_{ii}]u_i^{(n)} = -(\epsilon_i - \mathcal{E}_n)u_i^{(n)}. \tag{2.20}$$

This wave function and therefore $\mathbf{\Phi}_n$ describe a bound state of the nucleon when moving in the field of an excited target nucleus. Clearly such bound states exist only if V_{ii} is an attractive potential; the number of such bound states depends upon the strength and range of V_{ii} . When the coupling is not weak, $V_{ij} \neq 0$, the simple expression for $\mathbf{\Phi}_n$, Eq. (2.19), is no longer valid. Generally all the u_i

will differ from zero. However the description of $\mathbf{\Phi}_n$, as being the wave function describing a nucleon bound to an excited target nucleus still stands.

We can now express $\mathcal{U}$ in terms of $\mathbf{\Phi}_n$ and $\mathbf{\Phi}(\mathcal{E}', \alpha)$ as follows:

$$\mathcal{U} = V_{00} + \sum_n \frac{\mathbf{V}_0\mathbf{\Phi}_n\rangle\langle\mathbf{\Phi}_n\mathbf{V}_0^\dagger}{E - \mathcal{E}_n} + \int d\alpha \int_{\epsilon_1}^{\infty} d\mathcal{E}' \frac{\mathbf{V}_0\mathbf{\Phi}(\mathcal{E}', \alpha)\rangle\langle\mathbf{\Phi}(\mathcal{E}', \alpha)\mathbf{V}_0^\dagger}{E^{(+)} - \mathcal{E}'}. \tag{2.21}$$

The nonlocal character of $\mathcal{U}$ becomes apparent when we let it operate on u_0. Take for example the effect of the sum over n. We obtain

$$\sum_n \frac{\mathbf{V}_0\mathbf{\Phi}_n\rangle\langle\mathbf{\Phi}_n\mathbf{V}_0^\dagger u_0\rangle}{E - \mathcal{E}_n} = \sum_n \sum_{i,j\neq 0} \frac{V_{0i}(\mathbf{r}_0)u_i^{(n)}(\mathbf{r}_0) \int [u_j^{(n)}(\mathbf{r}')]^* V_{j0}u_0(\mathbf{r}')\, d\mathbf{r}'}{E - \mathcal{E}_n}$$

$$= \int K(\mathbf{r}_0 \mid \mathbf{r}')u_0(\mathbf{r}')\, d\mathbf{r}',$$

where K is defined by this equation. These terms then give rise to a potential which in all but exceptional circumstances is nonlocal.

Some general properties of the potential $\mathcal{U}$ can be obtained from Eq. (2.21). First note that the numerators in expansion (2.21) are positive definite, viz.:

$$\langle\chi\mathbf{V}_0\mathbf{\Phi}_n\rangle\langle\mathbf{\Phi}_n\mathbf{V}_0^\dagger\chi\rangle = |\langle\chi\mathbf{V}_0\mathbf{\Phi}_n\rangle|^2 \geq 0.$$

Here the function χ is arbitrary. We now readily obtain the following results:

$$\text{Re}\,\mathcal{U} = V_{00} + \sum_n \frac{\mathbf{V}_0\mathbf{\Phi}_n\rangle\langle\mathbf{\Phi}_n\mathbf{V}_0^\dagger}{E - \mathcal{E}_n} + \mathcal{P}\int \frac{d\mathcal{E}'}{E - \mathcal{E}'} \cdot \int d\alpha \mathbf{V}_0\mathbf{\Phi}(\mathcal{E}', \alpha)\rangle\langle\mathbf{\Phi}(\mathcal{E}', \alpha)\mathbf{V}_0^\dagger, \tag{2.22}$$

$$\begin{aligned} \text{Im}\,\mathcal{U} &= -\pi \int d\alpha\, \mathbf{V}_0\mathbf{\Phi}(E, \alpha)\rangle\langle\mathbf{\Phi}(E, \alpha)\mathbf{V}_0^\dagger \quad \text{if } E > \epsilon_1, \\ &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{if } E < \epsilon_1. \end{aligned} \tag{2.23}$$

$\mathcal{P}$ means Cauchy principal value. Note that Im $\mathcal{U}$ is negative definite as it must be if $\mathcal{U}$ is to be an absorptive potential. The Re $\mathcal{U}$ has poles at $E = \mathcal{E}_n$ as well as a branch line for $E > \mathcal{E}_1$. The Re $\mathcal{U}$ is monotonic:

$$\frac{\partial(\text{Re}\,\mathcal{U})}{\partial E} \leq 0. \tag{2.24}$$

Finally we note that if we substitute Eq. (2.23) into Eq. (2.22) we obtain a dispersion type relation between Re $\mathcal{U}$ and Im $\mathcal{U}$:

$$\text{Re}\,\mathcal{U}(E) = V_{00} + \sum_n \frac{\mathbf{V}_0\mathbf{\Phi}_n\rangle\langle\mathbf{\Phi}_n\mathbf{V}_0^\dagger}{E - \mathcal{E}_n} - \frac{1}{\pi}\mathcal{P}\int_{\epsilon_1}^{\infty} \frac{\text{Im}\,\mathcal{U}(\mathcal{E}')}{E - \mathcal{E}'} d\mathcal{E}' \tag{2.25}$$

For infinite nuclear matter the sum in (2.25) will not appear and

$$\text{Re } \mathfrak{U}(E) = V_{00} - \frac{1}{\pi} \mathcal{P} \int \frac{\text{Im } \mathfrak{U}(\mathcal{E}')}{E - \mathcal{E}'} d\mathcal{E}' \qquad \text{(infinite nuclear matter).} \tag{2.26}$$

Since $\mathfrak{U}$ is nonlocal this dispersion relation between the real and imaginary parts of the generalized optical potential is not a dispersion relation between the real and imaginary parts of the square of the index of refraction for nuclear matter. It is the latter which is presumably linearly related to the empirical optical potential. If $\mathfrak{U}$ were local, then Eq. (2.26) would hold between the real and imaginary parts of the volume integral of $\mathfrak{U}$ or of the empirical optical potential. Deviations of the empirical potential from (2.26) indicate the extent of the nonlocal nature of the potential $\mathfrak{U}$. It would show how much of the energy dependence of the empirical potential arises from nonlocality and how much from the dispersive nature of nuclear matter. (See Ref. *24* for further discussion.)

III. RESONANCE THEORY FOR PURE ELASTIC SCATTERING

The potential $\mathfrak{U}$, Eq. (2.15) or Eq. (2.21) must give rise to the narrow resonances which are described by the Breit-Wigner formula. In this section we shall treat the case where all channels are closed except the incident channel so that no reactions or inelastic scattering are possible. These last cases can be treated as we shall see (Section V) by a simple generalization of the method of this section. We consider first the case of an isolated resonance.

(a) Isolated Resonance

As we shall see, resonances will occur whenever E is close to one of the $\mathcal{E}_n$, that is when E is close to one of the poles of the potential $\mathfrak{U}$ as given in Eq. (2.21). In the limit of weak coupling [Eqs. (2.18)–(2.20)] this reduces to the statement that resonances will occur at the energies at which a nucleon in the absence of coupling with the incident channel could be in a stationary bound state in the field of the target nucleus in an excited state.

Suppose then that E is so close to $\mathcal{E}_n$ that only the singular term in $\mathfrak{U}$, Eq. (2.21) varies with the energy E. Therefore after separating off the n'th term, E can be put equal to $\mathcal{E}_n$ in the remainder.

$$\mathfrak{U} = U_n + \frac{\mathbf{V}_0\boldsymbol{\Phi}_n\rangle\langle\boldsymbol{\Phi}_n\mathbf{V}_0^\dagger}{E - \mathcal{E}_n}, \tag{3.1}$$

where

$$U_n \simeq V_{00} + \int \frac{d\mathcal{E}'}{\mathcal{E}_n - \mathcal{E}'} \int d\alpha \mathbf{V}_0\boldsymbol{\Phi}(\mathcal{E}', \alpha)\rangle\langle\boldsymbol{\Phi}(\mathcal{E}', \alpha)\mathbf{V}_0^\dagger + \sum_{m \neq n} \frac{\mathbf{V}_0\boldsymbol{\Phi}_m\rangle\langle\boldsymbol{\Phi}_m\mathbf{V}_0^\dagger}{\mathcal{E}_n - \mathcal{E}_m}. \tag{2.28}$$

The $i\eta$ term is absent because only elastic scattering can occur. The Schrödinger equation for u_0 when E is close to ε_n becomes

$$(H_0 - E)u_0 = -\Lambda_n \mathbf{V}_0 \Phi_n , \tag{3.2}$$

where the constant Λ_n is

$$\Lambda_n = \langle \Phi_n \mathbf{V}_0^\dagger u_0 \rangle / (E - \varepsilon_n) \tag{3.3}$$

and

$$H_0 = T_0 + U_n . \tag{3.4}$$

Formally solving Eq. (3.2) we obtain

$$u_0 = v_0^{(+)} + \Lambda_n \frac{1}{E^{(+)} - H_0} \mathbf{V}_0 \Phi_n , \tag{3.5}$$

where $v_0^{(+)}$ is the outgoing wave solution of

$$(H_0 - E)v_0^{(+)} = 0. \tag{3.6}$$

Substituting Eq. (3.5) into Eq. (3.3) we obtain a linear equation for Λ_n which we can solve with the result

$$\Lambda_n = \frac{\langle \Phi_n \mathbf{V}_0^\dagger v_0^{(+)} \rangle}{E - \varepsilon_n - \left\langle \Phi_n \mathbf{V}_0^\dagger \frac{1}{E^{(+)} - H_0} \mathbf{V}_0 \Phi_n \right\rangle}. \tag{3.7}$$

The transition matrix $\mathfrak{I}$ can now be obtained from the asymptotic behavior of Eq. (3.5) (*39*):

$$\mathfrak{I} = \mathfrak{I}_p + \Lambda_n \langle v_0^{(-)} \mathbf{V}_0 \Phi_n \rangle, \tag{3.8}$$

where $\mathfrak{I}_p$ is the transition matrix describing the asymptotic behavior of $v_0^{(+)}$; i.e., it gives the scattering arising just from the potential U_n. As we shall see it gives rise to the potential scattering term in the Breit-Wigner formula. The function $v_0^{(-)}$ is the solution of Eq. (3.6) consisting of an incident wave with the final momentum of the nucleon plus a converging wave. Substituting the value of Λ_n given by Eq. (3.7) in Eq. (3.8) we obtain $\mathfrak{I}$:

$$\mathfrak{I} = \mathfrak{I}_p + \frac{\langle v_0^{(-)} \mathbf{V}_0 \Phi_n \rangle \langle \Phi_n \mathbf{V}_0^\dagger v_0^{(+)} \rangle}{E - \varepsilon_n - \left\langle \Phi_n \mathbf{V}_0^\dagger \frac{1}{E^{(+)} - H_0} \mathbf{V}_0 \Phi_n \right\rangle}. \tag{3.9}$$

This expression for $\mathfrak{I}$ exhibits a resonance. The energy denominator in the resonance term is complex. The indicated matrix element contributes a real part which gives rise to a shift in the position of the resonance from ε_n plus an imaginary term which describes the width of the resonance. This formula is some-

what more general than the usual Breit-Wigner formula (though not intrinsically so) because separation according to angular momenta has not yet been made. Moreover, and this is a principal advantage of the above formalism, the channel radius does not make an appearance either in the final result or in the course of the derivation. Nuclear size effects of course are present in the matrix elements $\langle \mathbf{\Phi}_n \mathbf{V}_0^\dagger v_0^{(+)} \rangle$, etc. The range of both the potential U_n and the coupling potential $\mathbf{V}_0$ is of the order of the nuclear radius. The wave functions $v_0^{(\pm)}$ are sensitive to the range of U_n and of course to the extra-nuclear potential. These functions therefore contain both the penetration effects as well as the possibility of nuclear size resonances (*40*) which will occur whenever amplitude of $v_0^{(\pm)}$ is large within the range of U_n .

We now derive the Breit-Wigner formula (1.1). In order to avoid the complicated details of angular momentum calculations (these are given to a considerable extent by Bethe (*15*)) let us consider the simple example of spinless systems and let the only component of v_0 which is coupled to $\mathbf{\Phi}_n$ be the $l = 0$ solution of Eq. (3.6). Then

$$v_0 = e^{i\delta} f(r_0),$$

where

$$f(r_0) \to \left(\frac{2}{\pi} k^2 \frac{dk}{dE}\right)^{1/2} \frac{\sin (kr_0 + \delta)}{kr_0}.$$

Then

$$\mathfrak{I}_p = -\frac{1}{\pi} e^{i\delta} \sin \delta, \tag{3.10}$$

and

$$\left\langle \mathbf{\Phi}_n \mathbf{V}_0^\dagger \frac{1}{E^{(+)} - H_0} \mathbf{V}_0 \mathbf{\Phi}_n \right\rangle = \left\langle \mathbf{\Phi}_n \mathbf{V}_0^\dagger \mathcal{P} \frac{1}{E - H_0} \mathbf{V}_0 \mathbf{\Phi}_n \right\rangle - i\pi \langle \mathbf{\Phi}_n \mathbf{V}_0^\dagger f \rangle \langle f \mathbf{V}_0 \mathbf{\Phi}_n \rangle, \tag{3.11}$$

and

$$\mathfrak{I} = -\frac{1}{\pi} \left[\sin \delta e^{i\delta} - \pi \frac{e^{2i\delta} \langle f \mathbf{V}_0 \mathbf{\Phi}_n \rangle \langle \mathbf{\Phi}_n \mathbf{V}_0^\dagger f \rangle}{E - \mathcal{E}_n - \Delta_n + i\pi \langle f \mathbf{V}_0 \mathbf{\Phi}_n \rangle \langle \mathbf{\Phi}_n \mathbf{V}_0^\dagger f \rangle} \right]. \tag{3.12}$$

Where Δ_n is the energy shift

$$\Delta_n = \left\langle \mathbf{\Phi}_n \mathbf{V}_0^\dagger \mathcal{P} \frac{1}{E - H_0} \mathbf{V}_0 \mathbf{\Phi}_n \right\rangle.$$

This is in the Breit-Wigner form. We can identify the $l = 0$ width Γ as

$$\Gamma = 2\pi \mid \langle \Phi_n \mathbf{V}_0^\dagger f \rangle \mid^2. \tag{3.13}$$

We have already discussed the nuclear size and penetrability effects as they appear in $\mathbf{V}_0$ and f_0. In addition we note that Γ is proportional to k in virtue of the normalization of f as given in Eq. (3.10).

(b) MANY OVERLAPPING RESONANCES

The above analysis is of course no longer correct if the resonances overlap. In terms of our formalism, we need to separate out a group of the singular terms in $\mathfrak{V}$ before the remainder can be considered as constant over the group. Suppose N resonances overlap. Then

$$\mathfrak{V} = U^{(N)} + \sum_\nu \frac{\mathbf{V}_0 \Phi_\nu \rangle \langle \Phi_\nu \mathbf{V}_0^\dagger}{E - \mathcal{E}_\nu}.$$

The sum over ν contains N terms. The equation satisfied by u_0 is now

$$(H_0^{(N)} - E) u_0 = -\sum \Lambda_\nu \mathbf{V}_0 \Phi_\nu, \tag{3.14}$$

where

$$H_0^{(N)} = T_0 + U^{(N)}$$

and Λ_ν is defined by Eq. (3.3). We now repeat the analysis of part (a). Solving Eq. (3.14) for u_0 we obtain

$$u_0 = v_0^{(+)} + \sum_\nu \Lambda_\nu \frac{1}{E^{(+)} - H_0^{(N)}} \mathbf{V}_0 \Phi_\nu, \tag{3.15}$$

where $v_0^{(+)}$ is the appropriate solution of

$$(H_0^{(N)} - E) v_0^{(+)} = 0.$$

The transition matrix $\mathfrak{I}$ is

$$\mathfrak{I} = \mathfrak{I}_p^{(N)} + \sum_\nu \Lambda_\nu \langle v_0^{(-)} \mathbf{V}_0 \Phi_\nu \rangle. \tag{3.16}$$

Substituting Eq. (3.15) into the definition of Λ_ν leads to a set of simultaneous equations determining Λ_ν. These are

$$\sum_\nu [(E - \mathcal{E}_\mu)\delta_{\mu\nu} - W_{\mu\nu}]\Lambda_\nu = \langle \Phi_\mu \mathbf{V}_0^\dagger v_0^{(+)} \rangle, \tag{3.17}$$

where

$$W_{\mu\nu} = \left\langle \Phi_\mu \mathbf{V}_0^\dagger \frac{1}{E^{(+)} - H_0^{(N)}} \mathbf{V}_0 \Phi_\nu \right\rangle. \tag{3.17'}$$

The solution of the finite set of equations (3.17) can be obtained by the usual determinantal methods and indeed all the results we shall derive below can be obtained in this way. It is however more convenient to use another method. Let the ith solution of the eigenvalue problem

$$\sum_{\nu} [(E - \mathcal{E}_{\mu})\delta_{\mu\nu} - W_{\mu\nu}]X_{\nu} = 0 \tag{3.18}$$

be $X_{\nu}^{(i)}$ and let the corresponding eigenvalue be E_i. Since $W_{\mu\nu}$ is complex E_i is generally also complex. We can also define a vector adjoint to $X_{\nu}^{(i)}$, $(X_{\nu}^{(i)})^{\dagger}$ with the following orthogonality and normalization properties:

$$\sum_{\nu} (X_{\nu}^{(i)})^{\dagger} X_{\nu}^{(j)} = \delta_{ij}\,; \qquad \sum_{i} (X_{\nu}^{(i)})^{\dagger} X_{\mu}^{(i)} = \delta_{\nu\mu}\,. \tag{3.19}$$

The solution of Eq. (3.17) has then the familiar Green's function form (*41*):

$$\Lambda_{\nu} = \sum_{\mu,i} \frac{X_{\nu}^{(i)}(X_{\mu}^{(i)})^{\dagger}}{E - E_i} \langle \mathbf{\Phi}_{\mu} \mathbf{V}_0^{\dagger} v_0^{(+)} \rangle. \tag{3.20}$$

$\mathfrak{Z}$ becomes

$$\mathfrak{Z} = \mathfrak{Z}_p^{(N)} + \sum_{i} \frac{A_i}{E - E_i}\,, \tag{3.21}$$

where

$$A_i = \sum_{\nu,\mu} X_{\nu}^{(i)}(X_{\mu}^{(i)})^{\dagger} \langle \mathbf{\Phi}_{\mu} \mathbf{V}_0^{\dagger} v_0^{(+)} \rangle \langle v_0^{(-)} \mathbf{V}_0 \mathbf{\Phi}_{\nu} \rangle. \tag{3.22}$$

We see that $\mathfrak{Z}$ consists of a potential scattering term plus a sum over resonances of N terms. However each of these resonance terms is not in the Breit-Wigner form (3.12) in which the numerator of the resonance term is proportional (in Eq. (3.12) the proportionality constant is $-(e^{2i\delta}/\pi)$) to the imaginary part of the denominator. However we shall now show that

$$\sum_{i} A_i = \sum_{\mu} \langle \mathbf{\Phi}_{\mu} \mathbf{V}_0^{\dagger} v_0^{(+)} \rangle \langle v_0^{(-)} \mathbf{V}_0 \mathbf{\Phi}_{\mu} \rangle \tag{3.23}$$

and that in the $l = 0$ case leading to Eq. (3.12) that

$$\sum_{i} A_i = -\frac{1}{\pi} e^{2i\delta} \sum_{i} \operatorname{Im} E_i\,. \tag{3.24}$$

Equation (3.24) tells us that the Breit-Wigner form holds on the average; that is the average of the numerators equals $[-(1/\pi) \exp (2i\delta)]$ times the average of the imaginary part of the denominators. We shall later find that Eq. (3.23) is of great importance in the discussion of the complex potential model.

The proof follows. To obtain (3.23) we need only sum Eq. (3.22) over i and use the second of the two orthogonality properties given in Eq. (3.19). To obtain

Eq. (3.24) we first use the diagonal sum rule for the eigenvalues of Eq. (3.18):

$$\sum_i E_i = \sum_\mu (\varepsilon_\mu + W_{\mu\mu}).$$

Therefore

$$\sum_i \text{Im } E_i = \sum_\mu \text{Im } W_{\mu\mu} = -\pi \sum_\mu \langle \mathbf{\Phi}_\mu \mathbf{V}_0^\dagger \delta(E - H_0^{(N)}) \mathbf{V}_0 \mathbf{\Phi}_\mu \rangle. \quad (3.25)$$

To obtain Eq. (3.24) we need only insert Eq. (3.10) for v_0 in Eq. (3.23) and in Eq. (3.25). This completes the proof.

It will be convenient for our later work to introduce the concept of the "average width" $\langle \Gamma \rangle$ which is

$$\langle \Gamma \rangle = \frac{2\pi}{N} \sum_\mu \langle \mathbf{\Phi}_\mu \mathbf{V}_0^\dagger \delta(E - H_0^{(N)}) \mathbf{V}_0 \mathbf{\Phi}_\mu \rangle, \quad (3.26)$$

which in the $l = 0$ case of Eq. (3.12) reduces to

$$\langle \Gamma \rangle = \frac{2\pi}{N} \sum | \langle \mathbf{\Phi}_\mu \mathbf{V}_0^\dagger f \rangle |^2. \quad (3.27)$$

We note again that the average width is equal to identical expressions whether one is averaging over N overlapping or nonoverlapping levels.[1]

IV. COMPLEX POTENTIAL MODEL FOR PURE ELASTIC SCATTERING

We shall now use the results of the preceding section to derive the complex potential model. We still assume that the energy E is low enough so that only elastic scattering is possible.

The experimental quantities of interest are the energy average, $\langle \sigma \rangle$, of the various possible cross-sections σ:

$$\langle \sigma \rangle = \frac{1}{\Delta E} \int \sigma \, dE. \quad (4.1)$$

The integration is taken over a narrow energy range ΔE which is nevertheless sufficiently broad that ΔE includes many resonances. In this section we shall be concerned only with the average total cross-section $\langle \sigma_t \rangle$ and the average elastic cross-section $\langle \sigma_e \rangle$. These energy averages are particularly appropriate when the energy spread in the incident beam is large compared to the distance D between resonances, and when the resonances overlap.

[1] One other place where Eq. (3.24) is useful should be mentioned. Empirically a single resonance requires only two constants, the position of the resonance and the width. When two levels overlap then according to Eq. (3.21) eight constants are needed, since four complex numbers A_1, A_2, E_1, and E_2 are required. Relation (3.24) introduces two equations between these constants reducing the number to six.

The complex potential model (*18*) considers the average transition matrix, $\langle \mathfrak{I} \rangle$:

$$\langle \mathfrak{I} \rangle = \frac{1}{\Delta E} \int \mathfrak{I} \, dE. \tag{4.2}$$

A complex potential $\mathfrak{U}$ is introduced. Its properties are specified in terms of a wave function χ satisfying the corresponding Schrödinger equation:

$$(T_0 + \mathfrak{U} - E)\chi = 0. \tag{4.3}$$

Then $\mathfrak{U}$ must be so defined that the transition matrix for the scattering described by this equation is just $\langle \mathfrak{I} \rangle$. In other words, for large distances χ must asymptotically approach the energy average $\langle u_0 \rangle$ of the wave function u_0 :

$$\chi \to \langle u_0 \rangle \quad \text{as} \quad r_0 \to \infty. \tag{4.4}$$

The complex model transition matrix is just $\langle \mathfrak{I} \rangle$.

There is both a total and an elastic cross-section associated with Eq. (4.3). Because the total cross-section is directly proportional to the imaginary part of the transition matrix for zero degree scattering, it immediately follows from Eq. (4.4) that the total cross-section predicted by Eq. (4.3) is just $\langle \sigma_t \rangle$. However the complex model elastic cross-section which is usually referred to as the shape elastic cross-section, σ_{se}, is proportional to $[\langle \mathfrak{I} \rangle]^2$ whereas the average cross-section $\langle \sigma_e \rangle$ is proportional to $\langle \mathfrak{I}^2 \rangle$. These are not equal and in fact

$$\langle \sigma_e \rangle - \sigma_{se} \sim \langle \mathfrak{I}^2 \rangle - \langle \mathfrak{I} \rangle^2 \geq 0. \tag{4.5}$$

The shape elastic cross-section and the average elastic cross-section approach each other only when the fluctuations in the transition matrix $\mathfrak{I}$ go to zero. This will happen when there is sufficient overlap among the resonances. Then the complex potential model will predict both $\langle \sigma_t \rangle$ and $\langle \sigma_e \rangle$ correctly.

Let us now determine the complex potential $\mathfrak{U}$. Suppose that the energy band ΔE encompasses N resonances. Then $\mathfrak{I}$ is given by Eq. (3.21) and $\langle \mathfrak{I} \rangle$ by

$$\langle \mathfrak{I} \rangle = \mathfrak{I}_p^{(N)} + \frac{1}{\Delta E} \int \sum \frac{A_i}{E - E_i} \, dE. \tag{4.6}$$

We assume that ΔE is sufficiently small so that $\mathfrak{I}_p$, A_i, and E_i are constant over the range of integration. We shall also assume that N is large and finally that the imaginary part of E_i is small compared to ΔE. Then to a good approximation

$$\langle \mathfrak{I} \rangle = \mathfrak{I}_p^{(N)} - \frac{i\pi}{\Delta E} \sum A_i. \tag{4.7}$$

Inserting Eq. (3.23) for $\sum A_i$,

$$\langle \mathfrak{I} \rangle = \mathfrak{I}_p^{(N)} - \frac{i\pi}{\Delta E} \sum_{\mu} \langle v_0^{(-)} \mathbf{V}_0 \boldsymbol{\Phi}_\nu \rangle \langle \boldsymbol{\Phi}_\nu \mathbf{V}_0^\dagger v_0^{(+)} \rangle. \tag{4.8}$$

Notice that in this derivation that no statistical assumptions are required nor are any assumptions made about the ratio $\langle\Gamma\rangle/D$.

We now need to find the potential $\mathfrak{U}$ which will produce the transition matrix $\langle\mathfrak{z}\rangle$ of Eq. (4.8). From the form of Eq. (4.8) it follows that χ must satisfy the following equation

$$\chi = v_0^{(+)} - (i\pi/\Delta E) \sum_\nu \frac{1}{E^{(+)} - H_0^{(N)}} \mathbf{V}_0\Phi_\nu\rangle\langle\Phi_\nu\mathbf{V}_0^\dagger v_0^{(+)}\rangle \tag{4.9}$$

or

$$(H_0^{(N)} - E)\chi = \frac{i\pi}{\Delta E} \sum_\nu \mathbf{V}_0\Phi_\nu\langle\Phi_\nu\mathbf{V}_0^\dagger v_0^{(+)}\rangle. \tag{4.10}$$

Hence

$$[H_0^{(N)} + \sum \alpha_\nu \mathbf{V}_0\Phi_\nu\rangle\langle\Phi_\nu\mathbf{V}_0^\dagger - E]\,\chi = 0, \tag{4.11}$$

where

$$\alpha_\nu\langle\Phi_\nu\mathbf{V}_0^\dagger\chi\rangle = -\frac{i\pi}{\Delta E}\langle\Phi_\nu\mathbf{V}_0^\dagger v_0^{(+)}\rangle.$$

Inserting χ from Eq. (4.9) we obtain

$$\alpha_\nu = -\frac{i\pi}{\Delta E}\left[1 - \frac{i\pi}{\Delta E}\sum_\mu W_{\nu\mu}\frac{\langle\Phi_\mu\mathbf{V}_0^\dagger v_0^{(+)}\rangle}{\langle\Phi_\nu\mathbf{V}_0^\dagger v_0^{(+)}\rangle}\right]^{-1}. \tag{4.12}$$

By inserting this expression for α_ν into Eq. (4.11) we can now obtain the "compound nuclear" part of the complex potential $\mathfrak{U}$. $\mathfrak{U}$ then can be written as follows:

$$\mathfrak{U} = U^{(N)} + \mathfrak{U}_{cN}, \tag{4.13}$$

where

$$U_{cN} = -\frac{i\pi}{\Delta E}\sum\left[1 - \frac{i\pi}{\Delta E}\sum_\mu W_{\nu\mu}\frac{\langle\Phi_\mu\mathbf{V}_0^\dagger v_0^{(+)}\rangle}{\langle\Phi_\nu\mathbf{V}_0^\dagger v_0^{(+)}\rangle}\right]^{-1}\mathbf{V}_0\Phi_\nu\rangle\langle\Phi_\nu\mathbf{V}_0^\dagger, \tag{4.14}$$

while $U^{(N)}$ is

$$U^{(N)} = V_{00} + \sum_{\nu\neq n}\frac{\mathbf{V}_0\Phi_n\rangle\langle\Phi_n\mathbf{V}_0^\dagger}{E - \mathcal{E}_n} + \int\frac{d\mathcal{E}}{E - \mathcal{E}}\int d\alpha\mathbf{V}_0\Phi(\mathcal{E}, \alpha)\rangle\langle\Phi(\mathcal{E}, \alpha)\mathbf{V}_0^\dagger, \tag{4.15}$$

where E is some energy in the interval ΔE.

The compound nuclear potential $\mathfrak{U}_{cN}$ arises out of transitions from the open incident channel to closed channels and then back to incident channel. Note that for small overlap among the levels ($W_{\nu\mu}$ small) that the denominator in Eq. (4.12) or Eq. (4.14) is unity and $\mathfrak{U}_{cN}$ is pure negative imaginary as expected (*18*). A useful description of the origin of this imaginary term has been given by

Adair (*42*) and Friedman and Weisskopf (*12*). $\langle u_0 \rangle$ is essentially a wave packet in energy space. We can therefore employ a sequential description of the scattering process. The incident wave packet upon scattering divides into two parts. One corresponding to the "shape" elastic scattering suffers no distortion or time delay in crossing the target nucleus. The other, is considerably delayed because of the formation of the compound nuclear states of a relatively long lifetime. The solution of (4.3) cannot include the time-delayed part of the wave packet with the result that the amplitude of the wave packet appears reduced, and the effective potential must be complex.[2] It is also clear that as the lifetime of the compound nuclear states decreases, the average delay time will also decrease with a corresponding decrease in the apparent absorption. We therefore expect that as the width of the nuclear resonances increases that the imaginary part of $\mathfrak{U}_{cN}$ will decrease and eventually vanish. At this point, the shape elastic and average elastic cross-sections should be the same or according to Eq. (4.5) the fluctuation in the transition matrix $\mathfrak{z}$ should vanish. $\mathfrak{z}$ and the average $\langle \mathfrak{z} \rangle$ should then equal. This of course occurs because when the width of the resonances become large enough they overlap and the fluctuations in the cross-sections are reduced.

This behavior of $\mathfrak{U}_{cN}$ is explicitly contained in expression (4.14). We can readily demonstrate it for the case where v_0 describes a wave of a particular angular momentum. Introducing the definition of $W\nu\mu$, Eq. (3.17′) and of $\langle \Gamma \rangle$ Eq. (3.27) we readily obtain:

$$\alpha_\nu = -\frac{i\pi}{\Delta E}\left[1 - \frac{\pi}{2}\frac{\langle \Gamma \rangle}{D} - \frac{i\pi}{\Delta E}\sum_\mu \left\langle \mathbf{\Phi}_\nu \mathbf{V}_0^\dagger \mathcal{P}\frac{1}{E - H_0^{(N)}}\mathbf{V}_0\mathbf{\Phi}_\mu \right\rangle \frac{\langle \mathbf{\Phi}_\mu \mathbf{V}_0^\dagger v_0^{(+)} \rangle}{\langle \mathbf{\Phi}_\nu \mathbf{V}_0^\dagger v_0^{(+)} \rangle}\right]^{-1}. \quad (4.16)$$

We now show that the sum over μ in Eq. (4.16) is real. First since the set of functions $\mathbf{\Phi}_\mu$ describe bound states they can be chosen real. Therefore the first bracket in the sum is real. Second in the ratio multiplying this bracket phase factors will come in from $v_0^{(+)}$. However if we are dealing with a particular angular momentum this phase will be common to both the numerator and denominator of the ratio concluding the proof.

A number of conclusions about $\mathfrak{U}_{cN}$ may now be drawn. From the condition that the imaginary part of $\mathfrak{U}_{cN}$ must be negative we obtain the familiar inequality

$$\frac{\langle \Gamma \rangle}{D} \leq \frac{2}{\pi}. \quad (4.17)$$

[2] Another point of view which leads to the same qualitative results starts with the observation that $\langle u_0 \rangle$ obviously does not furnish a complete description of the wave function u_0. Not only do we require the average $\langle u_0 \rangle$ but also higher moments of u_0 as well. These higher moments and the average $\langle u_0 \rangle$ are all coupled so that the scattering of the initial wave pocket will generate these higher moments with a corresponding reduction in the amplitude of $\langle u_0 \rangle$. This "absorption" leads us once again to the conclusion that $\mathfrak{U}$ must be complex.

Moreover when $\langle\Gamma\rangle/D$ is equal to $(2/\pi)$ the potential $\mathfrak{U}_{cN}$ is real. We thus see that the conjectures about the behavior of $\mathfrak{U}_{cN}$ as a function of resonance width given just above are completely verified.

At the other end of the scale of $\langle\Gamma\rangle/D$, i.e., when this ratio is very small, $\mathfrak{U}_{cN}$ is pure imaginary. Moreover, as can be seen by comparing Eq. (4.8) for $\langle\mathfrak{z}\rangle$ and Eq. (4.14) for $\mathfrak{U}_{cN}$, the effects of $\mathfrak{U}_{cN}$, in this limit, can be calculated using perturbation theory in which the unperturbed wave function is v_0.

This concludes this section on the complex potential model. Our discussion also suggests the possibility of deriving equations for the higher energy moments of u_0 which would permit a more complete discussion than that provided by the complex potential model. It should be noted that the complex well derived is nonlocal. Statistical assumptions nor assumptions about the value of Γ/D are not required in the derivation.

V. INELASTIC SCATTERING

Suppose that the energy of the incident nucleon is sufficient to excite one or more levels of the target nucleus. Note that, in virtue of the essential identity of nucleons as expressed here by the isotopic spin formalism, possible processes include not only inelastic scattering, e.g. (n, n') and (p, p'), but also the charge exchange (n, p) and (p, n) reactions. It is now convenient to group the channel wave functions u_i into those which are open u_0, $u_1 \cdots u_p$ and into those which are closed u_{p+1}, etc. We therefore define two unicolumnar matrices as follows:

$$\mathbf{\Phi}_O = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_p \end{pmatrix}, \tag{5.1}$$

$$\mathbf{\Phi}_C = \begin{pmatrix} u_{p+1} \\ u_{p+2} \\ \vdots \\ \vdots \end{pmatrix}. \tag{5.2}$$

The subscripts O and C refer to open and closed, respectively. We also need to redefine the coupling potential matrices. These are now rectangular:

$$\mathbf{V}(O \mid C) = \begin{pmatrix} V_{0,p+1} & V_{0,p+2} \cdots \\ V_{1,p+1} & V_{1,p+2} \cdots \\ \vdots & \vdots \\ V_{p,p+1} & V_{p,p+2} \cdots \end{pmatrix}. \tag{5.3}$$

The Hermitian adjoint to $\mathbf{V}(O \mid C)$, $\mathbf{V}^\dagger(C \mid O)$ is defined as follows:

$$[\mathbf{V}^\dagger(C \mid O)]_{ik} = [\mathbf{V}(O \mid C)]_{ki}^* \tag{5.4}$$

Finally we need the square matrices $\mathbf{V}(O \mid O)$ and $\mathbf{V}(C \mid C)$:

$$\mathbf{V}(O \mid O) = \begin{pmatrix} V_{00} & V_{01} & \cdots & V_{0p} \\ V_{10} & V_{11} & & \\ \vdots & \vdots & & \\ V_{p0} & V_{p,1} & \cdots & V_{p,p} \end{pmatrix}, \tag{5.5}$$

$$\mathbf{V}(C \mid C) = \begin{pmatrix} V_{p+1,p+1} & V_{p+1,p+2} & \cdots \\ V_{p+2,p+1} & V_{p+2,p+2} & \cdots \\ \vdots & \vdots & \end{pmatrix}. \tag{5.6}$$

In terms of this notation the fundamental equations (2.5) become

$$\begin{aligned} [\mathbf{H}_o' - E]\mathbf{\Phi}_o &= -\mathbf{V}(O \mid C)\mathbf{\Phi}_C , \\ [\mathbf{H}_C - E]\mathbf{\Phi}_C &= -\mathbf{V}^\dagger(C \mid O)\mathbf{\Phi}_o , \end{aligned} \tag{5.7}$$

where

$$\begin{aligned} (\mathbf{H}_o')_{ij} &= (T_0 + \epsilon_i)\delta_{ij} + V_{ij} , & i, j \leq p; \\ (\mathbf{H}_C)_{ij} &= (T_0 + \epsilon_i)\delta_{ij} + V_{ij} , & i, j > p. \end{aligned} \tag{5.8}$$

Finally the products such as $\mathbf{V}(O|C)\mathbf{\Phi}_C$ are matrix products:

$$[\mathbf{V}(O \mid C)\mathbf{\Phi}_C]_i = \sum_{j=p+1} V_{ij}u_j \qquad i \leq p. \tag{5.9}$$

The form of Eq. (5.7) is identical to that of Eq. (2.11) and we can adopt the same method of solution:

$$\mathbf{\Phi}_C = \frac{1}{E - \mathbf{H}_C}\mathbf{V}^\dagger(C \mid O)\mathbf{\Phi}_0 . \tag{5.10}$$

so that

$$\left[\mathbf{H}_o' + \mathbf{V}(O \mid C)\frac{1}{E - \mathbf{H}_C}\mathbf{V}^\dagger(C \mid O) - E\right]\mathbf{\Phi}_o = 0, \tag{5.11}$$

an equation which is analogous to Eq. (2.14). The analogy can be continued up to Eq. (2.21). The corresponding equation is

$$\begin{aligned} \mathbf{V}(O \mid C)\frac{1}{E - \mathbf{H}_C}\mathbf{V}^\dagger(C \mid O) = &\sum_n \frac{\mathbf{V}(O \mid C)\mathbf{\Phi}_{Cn}\rangle\langle\mathbf{\Phi}_{Cn}\mathbf{V}^\dagger(C \mid O)}{E - \mathcal{E}_n} \\ &+ \int d\alpha \int_{\epsilon_{p+1}} \frac{\mathbf{V}(O \mid C)\mathbf{\Phi}_C(\mathcal{E}, \alpha)\rangle\langle\mathbf{\Phi}_C(\mathcal{E}, \alpha)\mathbf{V}^\dagger(C \mid O)}{E - \mathcal{E}} d\mathcal{E} \end{aligned} \tag{5.12}$$

The functions $\mathbf{\Phi}_{Cn}$ and $\mathbf{\Phi}_C(\varepsilon, \alpha)$ satisfy the eigenvalue equations

$$\begin{aligned} \mathbf{H}_C\mathbf{\Phi}_{Cn} &= \mathcal{E}_n\mathbf{\Phi}_{Cn} , \\ \mathbf{H}_C\mathbf{\Phi}_C(\mathcal{E}, \alpha) &= \mathcal{E}\mathbf{\Phi}_C(\mathcal{E}, \alpha). \end{aligned} \tag{5.13}$$

We can now derive the expressions for the transition matrix by essentially paraphrasing the discussion of Section III. Let us consider first the case of isolated resonances.

(a) Isolated Resonances

We again suppose E is close to $\mathcal{E}_n$ and rewrite Eq. (5.12) as follows:

$$\mathbf{V}(O \mid O) + \mathbf{V}(O \mid C)\frac{1}{E - \mathbf{H}_C}\mathbf{V}^\dagger(C \mid O) = \mathbf{U}_n + \frac{\mathbf{V}(O \mid C)\mathbf{\Phi}_{Cn}\rangle\langle\mathbf{\Phi}_{Cn}\mathbf{V}^\dagger(C \mid O)}{E - \mathcal{E}_n}, \tag{5.14}$$

where the definition of $\mathbf{U}_n$ is in precise analogy to Eq. (2.28). Then $\mathbf{\Phi}_O$ satisfies the equation

$$(\mathbf{H}_O - E)\mathbf{\Phi}_O = -\Lambda_n \mathbf{V}(O \mid C)\mathbf{\Phi}_{Cn}, \tag{5.15}$$

where

$$(\mathbf{H}_O)_{ij} = (T_0 + \epsilon_i)\delta_{ij} + (\mathbf{U}_n)_{ij},$$

and Λ_n is a pure number:

$$\Lambda_n = \frac{\langle\mathbf{\Phi}_{Cn}\mathbf{V}^\dagger(C \mid O)\mathbf{\Phi}_O\rangle}{E - \mathcal{E}_n}. \tag{5.16}$$

We can again determine Λ_n using the procedure described below Eq. (3.4) and obtain

$$\Lambda_n = \frac{\langle\mathbf{\Phi}_{Cn}\mathbf{V}^\dagger(C \mid O)\mathbf{v}_0^{(+)}\rangle}{E - \mathcal{E}_n - \left\langle\mathbf{\Phi}_{Cn}\mathbf{V}^\dagger(C \mid O)\frac{1}{E^{(+)} - \mathbf{H}_O}\mathbf{V}(O \mid C)\mathbf{\Phi}_{Cn}\right\rangle}. \tag{5.17}$$

The matrix $\mathbf{v}_0^{(+)}$ is a solution of

$$(\mathbf{H}_O - E)\mathbf{v}_0^{(+)} = 0 \tag{5.18}$$

subject to the condition that $\mathbf{v}_0^{(+)}$ at large distances consits of the incident wave plus diverging waves. The incident plane wave in the incident channel $\mathbf{\Pi}_0$ is of the form (spinless case again)

$$\mathbf{\Pi}_0 = \begin{pmatrix} e^{i\mathbf{k}\cdot\mathbf{r}_0} \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \tag{5.19}$$

The outgoing wave has generally entries corresponding to each of the possible open channels. Thus the potential $\mathbf{U}_n$ induces transitions from the incident channel to other open channels; i.e., gives rise to inelastic scattering. Let β number the open channels. Then the transition matrix $\mathfrak{I}_p(\beta \mid 0)$ describing transitions from the incident channel to channel β because of potential $\mathbf{U}_n$ is

$$\mathfrak{I}_p(\beta \mid 0) = \langle \mathbf{\Pi}_\beta \mathbf{U}_n \mathbf{v}_0^{(+)} \rangle. \tag{5.20}$$

The reason for the subscript p will become clear later. The function $\mathbf{\Pi}_\beta$ describes a plane wave in channel β with the appropriate energy, $\mathbf{\Pi}_0$ [Eq. (5.19)] being the case where the channel is the incident one.

Returning now to the general problem we can now give the transition matrix $\mathfrak{I}(\beta \mid 0)$ for Eq. (5.15) employing the value of Λ_n obtained in Eq. (5.17). [See discussion including and following Eq. (3.8).] We find

$$\mathfrak{I}(\beta \mid 0) = \mathfrak{I}_p(\beta \mid 0) + \frac{\langle \mathbf{v}_\beta^{(-)} \mathbf{V}(O \mid C) \mathbf{\Phi}_{Cn} \rangle \langle \mathbf{\Phi}_{Cn} \mathbf{V}^\dagger(C \mid O) \mathbf{v}_0^{(+)} \rangle}{E - \mathcal{E}_n - \left\langle \mathbf{\Phi}_{Cn} \mathbf{V}^\dagger(C \mid O) \frac{1}{E^{(+)} - \mathbf{H}_O} \mathbf{V}(O \mid C) \mathbf{\Phi}_{Cn} \right\rangle}. \tag{5.21}$$

$\mathfrak{I}_p(\beta \mid 0)$ has been defined in Eq. (5.20). The matrix $\mathbf{v}_\beta^{(-)}$ is the solution of Eq. (5.18) which at large distances consists of the incident wave in channel β, $\mathbf{\Pi}_\beta$, with the final momentum in that channel plus a converging wave. The transition matrix for elastic scattering is:

$$\mathfrak{I}(0 \mid 0) = \mathfrak{I}_p(0 \mid 0) + \frac{\langle \mathbf{v}_0^{(-)} \mathbf{V}(O \mid C) \mathbf{\Phi}_{Cn} \rangle \langle \mathbf{\Phi}_{Cn} \mathbf{V}^\dagger(C \mid O) \mathbf{v}_0^{(+)} \rangle}{E - \mathcal{E}_n - \left\langle \mathbf{\Phi}_{Cn} \mathbf{V}^\dagger(C \mid O) \frac{1}{E^{(+)} - \mathbf{H}_O} \mathbf{V}(O \mid C) \mathbf{\Phi}_{Cn} \right\rangle}. \tag{5.22}$$

We observe the typical appearance of resonance denominators. Perhaps more noteworthy is the appearance of a nonresonant inelastic scattering, $\mathfrak{I}_p(\beta \mid 0)$. The role of $\mathfrak{I}_p(\beta \mid 0)$ in inelastic scattering is identical with that of the potential scattering here given by $\mathfrak{I}_p(0 \mid 0)$ in elastic scattering. Hence the subscript p. $\mathfrak{I}_p(\beta \mid 0)$ contains the so-called direct inelastic scattering. Since all the possible levels which can be excited by inelastic scattering are included in the original expansion (2.1) $\mathfrak{I}_p(\beta \mid 0)$ describes the excitation of rotational levels, single particle levels, as well as direct (n, p) and (p, n) reactions. In fact Eq. (5.18) is quite similar to the equations usually employed in the direct interaction theory. We can see this in more detail if we take a special case. Suppose there is only one channel beside the incident channel which is open. Then Eq. (5.18) becomes

$$\begin{aligned} [T_0 + (\mathbf{U}_n)_{00} - E] v_{00} &= -(\mathbf{U}_n)_{01} v_{01}, \\ [T_0 + (U_n)_{11} - E] v_{01} &= -(\mathbf{U}_n)_{10} v_{00}, \end{aligned} \tag{5.23}$$

where we have placed

$$v_0 = \begin{pmatrix} v_{00} \\ v_{01} \end{pmatrix}. \tag{5.24}$$

Equations (5.23) are similar to those employed in Refs. *7* and *31*. These authors of course make specific assumptions about $\mathbf{U}_n$ in particular that it is complex. We shall discuss this later when we average over resonances as we did in Section IV. The present description is appropriate for the discussion of direct inelastic scattering in the presence of resonance scattering.

We consider now the resonance terms. The total width, Γ, can be obtained from the imaginary part of the resonance denominator in Eq. (5.21). It is

$$\Gamma = 2\pi \sum_{\beta} | \langle \Phi_{Cn} \mathbf{V}^{\dagger}(C \mid O) \mathbf{v}_{\beta}^{(+)} \rangle |^2 \tag{5.25a}$$

or

$$\Gamma = 2\pi \sum_{\beta} | \langle \mathbf{v}_{\beta}^{(-)} \mathbf{V}(O \mid C) \Phi_{Cn} \rangle |^2. \tag{5.25b}$$

This suggests the definition of the partial widths

$$\Gamma_{\beta} = 2\pi | \langle \Phi_{Cn} \mathbf{V}^{\dagger}(C \mid O) \mathbf{v}_{\beta}^{(+)} \rangle |^2 = 2\pi | \langle \mathbf{v}_{\beta}^{(-)} \mathbf{V}(O \mid C) \Phi_{Cn} \rangle |^2. \tag{5.26}$$

We see that Eq. (5.21) and Eq. (5.22) are then in the Breit-Wigner form (*43*) except of course for the addition of the terms $\mathfrak{I}_p(\beta \mid 0)$ ($\beta \neq 0$) and the unknown phase between these direct interaction and the resonance terms.

(b) Many Overlapping Resonances

It should be clear by now that the development of the formalism for this case will follow the discussion of Section IIIb precisely. It will therefore suffice to just give the results. In analogy to Eq. (3.21) we have

$$\mathfrak{I}(\beta \mid 0) = \mathfrak{I}_p(\beta \mid 0) + \sum_i \frac{A_i(\beta \mid 0)}{E - E_i}, \tag{5.27}$$

where following Eq. (3.22):

$$A_i(\beta \mid 0) = \sum_{\nu,\mu} X_{\nu}{}^{i} (X_{\mu}{}^{i})^{\dagger} \langle \mathbf{v}_{\beta}^{(-)} \mathbf{V}(O \mid C) \mathbf{\Phi}_{C\nu} \rangle \langle \mathbf{\Phi}_{C\mu} \mathbf{V}^{\dagger}(O \mid C) \mathbf{v}_0{}^{(+)} \rangle. \tag{5.28}$$

The $X_{\nu}{}^{(i)}$ are the eigen solutions and E_i the corresponding complex eigenvalues of the equations [see Eq. (3.18)]

$$\sum_{\nu} [(E - \varepsilon_{\mu})\delta_{\mu\nu} - W_{\mu\nu}] X_{\nu} = 0, \tag{5.29}$$

where $W_{\mu\nu}$ is now

$$W_{\mu\nu} = \left\langle \mathbf{\Phi}_{C\mu} \mathbf{V}^{\dagger}(C \mid O) \frac{1}{E^{(+)} - \mathbf{H}_O{}^{(N)}} \mathbf{V}(O \mid C) \mathbf{\Phi}_{C\nu} \right\rangle. \tag{5.30}$$

The functions $\mathbf{v}_{\beta}$ are the appropriate solutions of

$$(\mathbf{H}_O{}^{(N)} - E)\mathbf{v}_{\beta} = 0, \tag{5.31}$$

where

$$(\mathbf{H}_O^{(N)})_{ij} = (T_0 + \epsilon_i)\delta_{ij} + \mathbf{U}_{ij}^{(N)}. \tag{5.32}$$

Finally $\mathbf{U}^{(N)}$ consists of $\mathbf{V}(O \mid O)$ plus the terms on the right-hand side of Eq. (5.12) from which N terms containing $\mathbf{\Phi}_{C\nu}$ which appear in Eq. (5.30) and Eq. (5.28) have been omitted.

Equation (5.27) consists as expected of the nonresonant scattering plus typical resonance terms which are not in the Breit-Wigner form. However as in Section IIIb we have the theorems:

$$\sum A_i(\beta \mid 0) = \sum_\mu \langle \mathbf{v}_\beta^{(-)} \mathbf{V}(O \mid C)\Phi_{C\mu}\rangle\langle \Phi_{C\mu}\mathbf{V}^\dagger(C \mid O)\mathbf{v}_0^{(+)}\rangle. \tag{5.33}$$

Also from the diagonal sum rule

$$\text{Im} \sum E_i = \text{Im} \sum W_{\mu\mu} = -\pi \sum_{\mu,\beta} |\, \langle \Phi_{C\mu}\mathbf{V}^\dagger(O \mid C)\mathbf{v}_\beta^{(+)}\rangle \,|^2.$$

Hence we again have the result that the Breit-Wigner form holds on the average.

(c) Complex Potential Model

We intend to develop here the theory of the complex potential which applies when inelastic scattering is present. The procedure will be for the most part a duplicate of that employed in Section IV. We first define the average transition matrix, $\langle \mathfrak{I}(\beta \mid 0)\rangle$:

$$\langle \mathfrak{I}(\beta \mid 0)\rangle = \frac{1}{\Delta E}\int \mathfrak{I}(\beta \mid 0)\, dE. \tag{5.34}$$

The conditions on ΔE are identical with those described in Section IV. Again $\langle \mathfrak{I}(\beta \mid 0)\rangle$ is generally useful only for the calculation of the average total cross-section. Since both the elastic and inelastic cross-sections depend quadratically on $\mathfrak{I}(\beta \mid 0)$, the average $\langle \mathfrak{I}(\beta \mid 0)\rangle$ will be useful only when fluctuations of $\mathfrak{I}(\beta \mid 0)$ in its dependence on energy are negligible. This will presumably occur at high energies where the resonances overlap considerably. We calculate $\langle \mathfrak{I}(\beta \mid 0)\rangle$ and employing Eq. (5.33) we find

$$\langle \mathfrak{I}(\beta \mid 0)\rangle = \mathfrak{I}_p(\beta \mid 0) - \frac{i\pi}{\Delta E}\sum_\mu \langle \mathbf{v}_\beta^{(-)}\mathbf{V}(O \mid C)\mathbf{\Phi}_{C\mu}\rangle\langle \mathbf{\Phi}_{C\mu}\mathbf{V}^\dagger(C \mid O)\mathbf{v}_0^{(+)}\rangle. \tag{5.35}$$

We now ask for the potential $\mathbf{U}_{cN}$ such that the transition matrix described by the equation

$$[\mathbf{H}_O^{(N)} + \mathbf{U}_{cN} - E]\chi = 0 \tag{5.36}$$

is just $\langle \mathfrak{I}(\beta \mid 0)\rangle$.

The determination of $\mathbf{U}_{cN}$ is identical with that described in Section IV. For example it follows from Eq. (5.35) that χ must satisfy an equation like Eq. (4.10):

$$(\mathbf{H}_O^{(N)} - E)\chi = \frac{i\pi}{\Delta E}\sum_\nu \mathbf{V}(O \mid C)\mathbf{\Phi}_{C\nu}\rangle\langle\mathbf{\Phi}_{C\nu}\mathbf{V}^\dagger(C \mid O)\mathbf{v}_0^{(+)}\rangle. \tag{5.37}$$

The discussion then proceeds as indicated below Eq. (4.10) and we finally obtain

$$\mathbf{U}_{cN} = -\frac{i\pi}{\Delta E}\sum_\nu \left[1 - \frac{i\pi}{\Delta E}\sum_\mu W_{\nu\mu}\frac{\langle\mathbf{\Phi}_{C\mu}\mathbf{V}^\dagger(C \mid O)\mathbf{v}_0^{(+)}\rangle}{\langle\mathbf{\Phi}_{C\nu}\mathbf{V}^\dagger(C \mid O)\mathbf{v}_0^{(+)}\rangle}\right]^{-1} \cdot\mathbf{V}(O \mid C)\mathbf{\Phi}_{C\nu}\rangle\langle\mathbf{\Phi}_{C\nu}\mathbf{V}^\dagger(C \mid O), \tag{5.38}$$

where $W_{\nu\mu}$ is defined by Eq. (5.30). The properties of $\mathbf{U}_{cN}$ are of course so similar to those of $\mathbf{U}_{cN}$ that there is no reason to repeat them here.

Equation (5.36) is not the complex potential model. To obtain the latter it is necessary to eliminate all the open channels but the incident one. For this purpose we rewrite Eq. (5.36) as follows:

$$(T_0 + \mathbf{U} - E)\chi = 0, \tag{5.39}$$

where

$$U_{ij} = (\mathbf{U}^{(N)})_{ij} + (\mathbf{U}_{cN})_{ij} + \epsilon_i\delta_{ij}\,. \tag{5.40}$$

We now split off the incident channel wave function χ_0 from χ:

$$\chi = \begin{pmatrix}\chi_0\\ \chi_I\end{pmatrix}.$$

Then Eq. (5.39) becomes

$$(T_0 + U_{00} - E)\chi_0 = -\mathbf{U}_{0I}\chi_I\,, \tag{5.41}$$

$$(T_0 + \mathbf{U}_{II} - E)\chi_I = -\mathbf{U}_{I0}\chi_0\,, \tag{5.42}$$

where

$$\mathbf{U} = \begin{pmatrix}U_{00} & \mathbf{U}_{0I}\\ \mathbf{U}_{I0} & \mathbf{U}_{II}\end{pmatrix}. \tag{5.43}$$

The complex potential $\mathfrak{U}$ can be obtained from Eq. (5.41) by eliminating χ_I. We then find that

$$(T_0 + \mathfrak{U} - E)\chi_0 = 0, \tag{5.44}$$

where

$$\mathfrak{U} = U_{00} + \mathbf{U}_{0I}\frac{1}{E^{(+)} - T_0 - \mathbf{U}_{II}}\mathbf{U}_{I0}\,. \tag{5.45}$$

The term U_{00} contains the potential V_{00} plus the effects of virtual transitions from the incident channel to the closed channels and back. Those which gave rise to resonances in the energy region ΔE have been averaged and form the imaginary part of U_{00}. The second term in Eq. (5.45) contains several effects. There are those virtual transitions through the closed channels in which the open inelastic channels, I, were intermediate states either for the transition from the incident channel to the closed channel or for the transition from the closed back to the incident channel or both. In addition, because of the existence of inelastic scattering there is a contribution coming from that part of the Green's function

$$[E^{(+)} - T_0 - \mathbf{U}_{II}]^{-1}$$

on the energy shell. If the effect of the resonant state is small ($\mathbf{U}$ real) then this term is pure imaginary and of course represents a real absorption.

We can also employ these equations to describe inelastic scattering. From Eq. (5.42) we have the transition matrix

$$\langle \mathfrak{I}_I(\beta \mid 0) \rangle = \langle \mathbf{v}_I{}^{(-)} \mathbf{U}_{I0} \chi_0{}^{(+)} \rangle, \tag{5.46}$$

where $\chi_0{}^{(+)}$ is to be obtained from Eq. (5.44) and $(\mathbf{v}_{I\beta}{}^{(-)})^*$ is that solution of

$$(T_0 + \mathbf{U}_{II} - E)(\mathbf{v}_{I\beta}{}^{(-)})^* = 0, \tag{5.47}$$

which asymptotically consists of a plane wave in channel β with momentum $(-\mathbf{k}_\beta)$ and a diverging wave. The quantity $\mathbf{k}_\beta$ is just the momentum of the inelastically scattered particle in channel β. Another form of Eq. (5.46) is

$$\langle \mathfrak{I}_I(\beta \mid 0) \rangle = \langle \mathbf{w}_{I\beta}{}^{(-)} \mathbf{U}_{I0} v_0{}^{(+)} \rangle, \tag{5.48}$$

where

$$\left[T_0 + \mathbf{U}_{II} + \mathbf{U}_{I0} \frac{1}{E^{(+)} - T_0 - U_{00}} \mathbf{U}_{0I} - E \right] (\mathbf{w}_{I\beta}{}^{(-)})^* = 0$$

and

$$(T_0 + U_{00} - E) v_0{}^{(+)} = 0.$$

The boundary conditions on $(\mathbf{w}_{I\beta}{}^{(-)})^*$ are identical with those for $(\mathbf{v}_{I\beta}{}^{(-)})^*$. Equations (5.46) and (5.48) must of course be identical with Eq. (5.35). They are however considerably more convenient because the effect of transitions to and from the resonance producing closed channels on the wave functions have been explicitly exhibited. Equations of the form of Eq. (5.46) together with Eq. (5.44), Eq. (5.41) and Eq. (5.42) were employed in Refs. *7* and *31*.

In conclusion, we remind the reader of the limitations of the development of this section, V(c). Expressions like (5.46) describe the average transition matrix for inelastic scattering. The average inelastic or reaction cross-section can be computed from the average only if the fluctuations in the inelastic cross-section

are small. This will occur for two circumstances. In one the direct inelastic scattering, that is the nonresonant cross-section dominates. In the other the resonances are so broad that they blend together to form a smooth cross-section. On the other hand the average total cross-section may be obtained correctly from Eq. (5.44).

The average transition matrix combines the effects of direct and compound nuclear inelastic scattering. An exact separation of $\langle \mathfrak{I}_I \rangle$ into independent terms corresponding to each process does not seem possible. However if it is permissible to apply perturbation theory to either $(\mathbf{U}^{(N)})_{I0}$ or to $(\mathbf{U}_{cN})_{I0}$ then the transition matrix for direct scattering is

$$\langle \mathbf{v}_{I\beta}{}^{(-)}(\mathbf{U}^N)_{I0}\chi_0{}^{(+)} \rangle \simeq \langle \mathbf{w}_{I\beta}{}^{(-)}(\mathbf{U}^N)_{I0}v_0{}^{(+)} \rangle, \tag{5.49}$$

while the average transition matrix for compound nuclear inelastic scattering is

$$\langle \mathbf{v}_{I\beta}{}^{(-)}(\mathbf{U}_{cN})_{I0}\chi_0{}^{(+)} \rangle \simeq \langle \mathbf{w}_{I\beta}{}^{(-)}(\mathbf{U}_{cN})_{I0}v_0{}^{(+)} \rangle. \tag{5.60}$$

VI. EFFECT OF HARD CORE POTENTIALS AND IDENTITY OF PARTICLES; HIGH ENERGY APPROXIMATION

As we stated early in Section II the theory developed in Sections II to V is not appropriate when the potential V of Eq. (2.3) contains hard cores and when the incident nucleon is identical with the particles of the target nucleus. It is the purpose of this section to show that the results of Sections II to Sections V are correct in form although the meaning of the matrix elements which appear are modified. The discussion is based on the realization that the results of Section II to V depend principally on the existence of a set of coupled equations of the form of Eq. (2.5). If therefore after taking hard cores and the exclusion principle into account we can transform the resulting equations into a set identical in form with Eq. (2.5) and if the interpretation at large distances of the various wave functions analogous to u_i remains then the entire development of the preceding sections can be repeated. Let us illustrate these remarks with an important example.

It has been recently shown (*44*) on the basis of Brueckner's theory that hard core potentials and the exclusion principle give rise to short range correlation between nucleons. These short range correlations could be included *ab initio* by modifying expansion (2.1). For example we could take

$$\Psi = \sum_i \psi_i(\mathbf{r}_1 \cdots \mathbf{r}_A)u_i(\mathbf{r}_0) \prod_{k=1}^{A} g_i(\mathbf{r}_0 - \mathbf{r}_k), \tag{6.1}$$

where the g's differ from one only when $|\mathbf{r}_0 - \mathbf{r}_k|$ is sufficiently small. If such a wave function is inserted into Schrödinger equation (2.2) the structure of Eq. (2.5) is modified as follows:

$$(T_0 + V_{ii}' + \epsilon_i \mathfrak{N}_{ij} - EN_{ij})u_i = -\sum_{j \neq i} V_{ij}'u_j. \tag{6.2}$$

The matrix elements V_{ij}' include not only the matrix elements of the potential V with respect to the set of functions $\psi_i \Pi g_i$ but also terms coming from the kinetic energy operators T_i. The coefficients $\mathfrak{N}_{ij}$ and N_{ij} deviate from δ_{ij} because this set is not orthogonal. They are functions of $\mathbf{r}_0$. It is important to notice that because of the short range character of the functions g_i that for large values of $\mathbf{r}_0$, both $\mathfrak{N}_{ij}$ and N_{ij} approach δ_{ij} rapidly. Therefore the interpretation of u_0 as the wave function for the entrance channel, etc., is still valid. We need only reduce Eq. (6.2) to the form of Eq. (2.5) for the results of Sections II to V to remain valid.

Let $\mathbf{\Psi}'$ be defined as follows:

$$\mathbf{\Psi}' \equiv \begin{pmatrix} u_0 \\ \mathbf{\Phi} \end{pmatrix} \equiv \begin{pmatrix} u_0 \\ u_1 \\ \vdots \end{pmatrix}. \tag{6.3}$$

Then Eq. (6.2) becomes

$$(\mathbf{H}' - E\mathbf{N})\mathbf{\Psi}' = 0, \tag{6.4}$$

where

$$(\mathbf{N})_{ij} = N_{ij}.$$

Since $\mathbf{N}$ is Hermitian we can diagonalize it with a unitary transformation $\mathbf{S}$:

$$\mathbf{SNS}^{-1} = \mathbf{\Delta}. \tag{6.5}$$

$\mathbf{\Delta}$ is diagonal and real. $\mathbf{S}$ as well as $\mathbf{\Delta}$ is a function of $\mathbf{r}_0$. Both approach the unit matrix as $\mathbf{r}_0$ becomes large. Under transformation $\mathbf{S}$, $\mathbf{\Psi}'$ is transformed to $\mathbf{\Psi}''$:

$$\mathbf{\Psi}'' = \mathbf{S}^{-1}\mathbf{\Psi}'.$$

Equation (6.4) becomes

$$(\mathbf{H}'' - E\mathbf{\Delta})\mathbf{\Psi}'' = 0. \tag{6.6}$$

Finally letting

$$\mathbf{\Psi}''' = \mathbf{\Delta}^{1/2}\mathbf{\Psi}'', \tag{6.7}$$

we obtain

$$(\mathbf{H}''' - E)\mathbf{\Psi}''' = 0, \tag{6.8}$$

where

$$\mathbf{H}''' = \mathbf{\Delta}^{-1/2}\mathbf{H}'\mathbf{\Delta}^{-1/2}.$$

We have now reduced Eq. (6.4) to the form of Eq. (2.5). Moreover the interpretation of the various channel wave functions u_i is the same as in Eq. (2.5). Of

course transformation (6.5) does introduce a formidable obstacle in the deduction of matrix elements such as widths from nucleon-nucleon forces given by V. Note that both $\mathbf{S}$ and $\mathbf{\Delta}$ are independent of the energy E.

(a) HARD CORE POTENTIALS; HIGH ENERGY APPROXIMATION

We begin with some introductory remarks. Expansion (2.1) is usually regarded as being not correct when hard cores are present because the matrix elements V_{ij} are said not to be bounded. However the hard core should be simply considered as advice for insuring that the wave function be zero when $|\mathbf{r}_0 - \mathbf{r}_k|$ is less than the range of the hard core potential. Exactly this effect insofar as the dependence of the wave function on $\mathbf{r}_0$ is concerned can be produced by a potential consisting of a suitable combination of a delta function and its derivatives acting at the surface of the core. The matrix elements V_{ij} are then not infinite and it becomes possible to employ exapnsion (2.1) as a trial function in the variational principle for E. However expansion (2.1) is not sufficiently complete so that the resulting equations are no longer exact. Expansion (6.1) is thus to be preferred to (2.1). Suitable functions g_i could presumably be obtained from the solutions of the Bethe-Goldstone equation.

A related procedure which is more suitable for our present purposes and which connects more directly with the Brueckner method consists in transforming the potential V rather than changing the wave function expansion. The possibility of such a method is a direct consequence of a theorem of Kerman (*45*, *24*). The conditions of the theorem are (1) that $V(\mathbf{r}_0, \mathbf{r}_1 \cdots \mathbf{r}_A)$ operate only on wave functions which are antisymmetrical in coordinates $\mathbf{r}_1$ to $\mathbf{r}_A$ and (2) that V can be written as the sum of two particle interactions:

$$V(\mathbf{r}_0; \mathbf{r}_1, \mathbf{r}_2 \cdots \mathbf{r}_A) = \sum_i v(\mathbf{r}_0, \mathbf{r}_i). \tag{6.9}$$

The first condition is met by expansion (2.1) since ψ_i are antisymmetrized. Finally we define the operator τ by the equation

$$\tau = v\left[1 + \frac{1}{E^{(+)} - \mathcal{H}_0}\tau\right], \tag{6.10}$$

where v is $v(\mathbf{r}_0, \mathbf{r}_1)$ and $\mathcal{H}_0$ is

$$\mathcal{H}_0 = H_A + T_0. \tag{6.11}$$

It then becomes possible to define an effective many body potential

$$V_e = (A - 1)\tau \tag{6.12}$$

and a corresponding effective Hamiltonian $\mathcal{H}_e$

$$\mathcal{H}_e = \mathcal{H}_0 + V_e.$$

One can then show that $\mathfrak{I}$, the transition matrix associated with the actual Hamiltonian $\mathfrak{H}$ and $\mathfrak{I}_e$ the transition matrix associated $\mathfrak{H}_e$ are simply related:

$$\mathfrak{I} = \frac{A}{A-1}\mathfrak{I}_e . \tag{6.13}$$

We may therefore employ V_e instead of V in the discussion of Section II to V making only the small correction, $A/(A-1)$ to obtain $\mathfrak{I}$.

Employing V_e instead of V has two advantages. First τ and therefore V_e should be less singular than V. This expectation is based on the close relation betwen τ and t the two body transition matrix which satisfies the equation

$$t = v\left[1 + \frac{1}{E^{(+)} - T_0}\, t\right]. \tag{6.14}$$

When v contains a hard core t is less singular then v. The structure of Eq. (6.14) and (6.10) are so similar that we expect that τ like t will be less singular than v.

A second advantage which accrues when V_e is used instead of V is the consequent connection which can be made to the Watson multiple scattering approximation (*45*, *24*) which is valid for high energy for the incident nucleon. According to Eq. (2.15) the generalized potential describing the effective interaction of the nucleon with the nucleus is $\mathcal{U}^{(e)}$.

$$\mathcal{U}^{(e)} = V_{00}{}^{(e)} + \mathbf{V}_0{}^{(e)} \frac{1}{E^{(+)} - \mathbf{H}^{(e)}} \mathbf{V}_0{}^{(e)\dagger}. \tag{6.15}$$

Here the superscript e indicates the use of V^e where V occurs in Eq. (2.15). The Watson approximation consists of two parts. First the contribution from the excited states is dropped, viz.,

$$\mathcal{U}^{(e)} \simeq V_{00}{}^{(e)} = (A-1)(\psi_0 , \tau\psi_0). \tag{6.16}$$

This might be called the multiple scattering approximation. The second step, consists in replacing τ by t:

$$\mathcal{U}^{(e)} \simeq (A-1)(\psi_0 , t\psi_0) \tag{6.17}$$

For the discussion of the justification of these steps the reader is referred to the original articles (*46*) and to a review article (*24*). $\mathcal{U}^{(e)}$ contains therefore both the description of the low energy resonance phenomena and the high energy multiple scattering process which occur when a nucleon interacts with a nucleus.

(b) Effects of the Identity of Particles

We turn now to the effects of the identity of the incident nucleon with nucleons in the nucleus. In accordance with the Fermi statistics obeyed by the nucleons expansion (2.1) must be antisymmetrized. The set ψ_i is already anti-symmetrized

so that an additional antisymmetrization must be carried out with respect to the pairs $(\mathbf{r}_0, \mathbf{r}_k)$. Upon inserting the resulting wave function into the Schrödinger equation (2.2) we obtain an equation of the form of Eq. (6.4). Characteristic exchange potentials appear and $\mathbf{N}$ is not diagonal because of the lack of orthogonality after antisymmetrization. We shall now prove the following important theorem. If the energy E is sufficiently low so that the exit channels can contain only one nucleon [no $(n, 2n)$, (n, α), etc. reactions] then (1) as $\mathbf{r}_0$ approaches infinity $\mathbf{\Psi}'''$ of Eq. (6.8) approaches $\mathbf{\Psi}'$ of Eq. (6.4) at a rate faster than $(1/r_0)$ and (2) the asymptotic dependence of u_i describe the events in the corresponding channel. By the second part of the theorem we mean that the effects of exchange scattering are contained in u_i as an automatic consequence of the process of antisymmetrization. The two parts of the theorem are intimately connected, as we shall see.

We consider first the coefficient of E in the ith equation in Eq. (6.4). It is

$$(\mathbf{N}\mathbf{\Psi}')_i = u_i(\mathbf{r}_0) + \sum_{j,k} (\pm) \int d\mathbf{r}_1 \cdots d\mathbf{r}_A \psi_i^*(\mathbf{r}_1 \cdots \mathbf{r}_A) u_j(\mathbf{r}_k) \cdot \psi_j(\mathbf{r}_1 \cdots \mathbf{r}_{k+1}, \mathbf{r}_0, \mathbf{r}_{k+1} \cdots \mathbf{r}_A). \tag{6.18}$$

We have only indicated the sign of each term which because of the permutation of $\mathbf{r}_0$ and $\mathbf{r}_k$ may be either positive or negative. We ask now for the behavior of the sum in Eq. (6.18) as $\mathbf{r}_0$ approaches infinity. It is clear that those terms, in which ψ_j describes a state in the discrete spectrum of the target nucleus, approach zero exponentially as r_0 goes to infinity. However when ψ_j is in the continuous spectrum this is no longer true for the individual terms in the sum in Eq. (6.18). This is very closely tied in of course with the possibility of exchange scattering. However the sum over j of these states still goes to zero rapidly.[3] Consider the behavior of ψ_j as r_0 goes to infinity:

$$\psi_j(\mathbf{r}_1 \cdots \mathbf{r}_{k-1}, \mathbf{r}_0, \mathbf{r}_{k+1} \cdots \mathbf{r}_A) \xrightarrow[r_0 \to \infty]{} \varphi_\alpha(\boldsymbol{\kappa}; \mathbf{r}_1 \cdots \mathbf{r}_{k-1}, \mathbf{r}_{k+1} \cdots \mathbf{r}_A) w_\alpha(\boldsymbol{\kappa}, \mathbf{r}_0), \tag{6.19}$$

where we have replaced j by the continuous momentum variable $\boldsymbol{\kappa}$ and by α. Finally

$$w_\alpha(\boldsymbol{\kappa}, \mathbf{r}_0) \xrightarrow[r_0 \to \infty]{} e^{i\boldsymbol{\kappa}\cdot\mathbf{r}_0} + \text{diverging waves}. \tag{6.20}$$

We are thus led to investigate

$$J = \int d\boldsymbol{\kappa} e^{i\boldsymbol{\kappa}\cdot\mathbf{r}_0} \int d\mathbf{r}_1 \cdots d\mathbf{r}_A \psi_i^*(\mathbf{r}_1 \cdots \mathbf{r}_A) u_j(\mathbf{r}_k) \varphi_\alpha(\boldsymbol{\kappa}; \mathbf{r}_1 \cdots \mathbf{r}_{k-1}, \mathbf{r}_{k+1} \cdots \mathbf{r}_A). \tag{6.21}$$

J is a group of terms included in the sum in Eq. (6.18) where however we have obviously replaced w_α by its leading asymptotic term. We shall determine the

[3] This was pointed out to the author by Professor Francis Low.

asymptotic dependence of J. An even stronger statement can then be made for the diverging wave part of Eq. (6.20). We now need the energy condition on the theorem. It implies that for every real value of κ, $u_j(\mathbf{r}_k)$ and φ_α approach zero exponentially as $(\mathbf{r}_1 \cdots \mathbf{r}_A)$ individually go to infinity. If this were not so it would be possible to find two free particles at infinity in contradiction to the energy condition. As a consequence the integral over $\mathbf{r}_1$ to $\mathbf{r}_A$ in expression for J [Eq. (6.21)] converges for all real values of κ and for large values of κ goes to zero. It immediately follows that for large values of r_0, that J approaches zero more rapidly than $(1/r_0)$. Thus that part of the sum in Eq. (6.18) which comes from the continuous spectrum of the target nucleus goes to zero more rapidly than $(1/r_0)$. This result means that $\mathbf{N}$ is effectively diagonal for large values of r_0 and consequently that Eq. (6.4) and Eq. (6.8) are identical for large values of r_0. Thus the first part of the theorem that $\mathbf{\Psi}'$ approaches $\mathbf{\Psi}'''$ for large values of r_0 is proven and it becomes possible to determine the asymptotic dependence of u_i directly from the solution of Eq. (6.8).

We turn now to the second part of the theorem. To obtain the amplitude for the process in which the target nucleus is left in the state i it is necessary to evaluate the following.

$$\lim_{r_0 \to \infty} \int \psi_i^*(\mathbf{r}_1 \cdots \mathbf{r}_A)\Psi(\mathbf{r}_0 \cdots \mathbf{r}_A)\, d\mathbf{r}_1 \cdots d\mathbf{r}_A\,, \tag{6.22}$$

where Ψ is just the antisymmetrized version of expansion (2.1). Expression (6.22) is just

$$\lim_{r_0 \to \infty} (N\Psi)_i \to \lim_{r_0 \to \infty} u_i(\mathbf{r}_0) \tag{6.23}$$

as we have shown above.

We now summarize the discussion of this section on the effect of the identity of the incident (or emergent) nucleon with the nucleons in the target nucleus. Our results hold only when the energy E is sufficiently small that the process consists of one nucleon incident and one nucleon emerging. Under these conditions we show that upon antisymmetrization of expansion (2.1) for Ψ, that the equations for u_i can be transformed to the form (6.8) to which the analysis of Section II to V can be applied. Secondly for large values of r_0

$$\Psi''' \to \Psi' = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \end{pmatrix}.$$

Finally we show that the asymptotic dependence of u_i gives the complete amplitude of the process in which the target nucleus is left in state i. Thus the results of Sections II to V apply even when the exclusion principle and exchange scattering are taken into account subject of course to the condition on E stated above.

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NOTE ADDED IN PROOF

F. Coester an H. Kümmel discuss in a preprint the complex potential model employing a time dependent theory. Their results are contained in part in Section II of this paper. Their proof of the validity of the model when the Pauli principle is included is more general than the proof of Section VI.

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