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Recap on pairing... Unless noted, everything true in 1D and 3D.

• We observed that two particles in the medium always can form a "bound state" with less energy than they would have in the Fermi sea ⇒ Cooper pairs when there is a short-range attraction.
⇒ motivated us to look for collective many-body pairing.
  ◦ consider pairs of spin up - spin down with back-to-back momenta (k and $-k$)

• Variational calculation using $|BCS\rangle$ ansatz

$$\boxed{|BCS\rangle = \prod_{k_1}\left(u_{k_1} + v_{k_1} a^\dagger_{k_1\uparrow} a^\dagger_{-k_1\downarrow}\right)|0\rangle}$$

with $\boxed{u_{k_1}^2 + v_{k_1}^2 = 1}$

or introduce canonical transformation

$$\boxed{\alpha_k = u_k a_{k\uparrow} - v_k a^\dagger_{-k\downarrow}} \qquad \boxed{\beta_{-k} = u_k a_{-k\downarrow} + v_k a_{k\uparrow}}$$

and require $\boxed{\alpha_k|BCS\rangle = \beta_k|BCS\rangle = 0}$ for all k.

$\alpha_k^\dagger, \beta_k^\dagger$ and α_k, β_k are quasiparticle creation and destruction operator. The "Hamiltonian" takes the form after minimization in the variational case or requiring part of $\hat{K}$ to vanish:

$$\boxed{\hat{K} = \hat{H} - \mu\hat{N} = U + \hat{H}_1 + \cancel{\hat{H}_2}^{\,0} + N(\hat{V})} \leftarrow \text{treated as perturbation}$$

with $\boxed{U = \sum_k (\epsilon_k^0 - \mu)\left(1 - \xi_k/E_k\right) - \frac{|\lambda|\rho}{4}N - \frac{L}{|\lambda|}\Delta^2}$

$$\boxed{\hat{H}_1 = \sum_k E_k\left(\alpha_k^\dagger\alpha_k + \beta_k^\dagger\beta_k\right)}$$

with $\boxed{E_k \equiv \sqrt{\xi_k^2 + \Delta^2}}$ and $\boxed{\xi_k = \epsilon_k^0 - \frac{|\lambda|}{2}\rho - \mu = \epsilon_k^{HF} - \mu}$

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The coefficients u_k and v_k are related to the gap Δ and ξ_k by

$$\boxed{\Delta = \frac{|\lambda|}{L}\sum_k u_k v_k = \frac{|\lambda|}{2L}\sum_k \frac{\Delta}{\sqrt{\Delta^2+\xi_k^2}}}$$

and $\boxed{v_k^2 = (1-u_k^2) = \frac{1}{2}\left(1 - \frac{\xi_k}{\sqrt{\Delta^2+\xi_k^2}}\right)}$ and $\boxed{N = 2\sum_k v_k^2}$

- In a nucleus, the lowest energy state is found by adding states paired in the spherical analog of $k\uparrow$ to $-k\downarrow$, which is coupled to $J=0$.
  - More next week when we consider finite systems!
    $\Rightarrow$ we'll see how the excited state spectrum works out.
  - For now, note that $\alpha_k^\dagger|BCS\rangle$ or $\beta_k^\dagger|BCS\rangle$ is an odd number of particles so we need at least two to generate an excited state $\Rightarrow$ excitation $\geq 2E_k \geq 2\Delta$.
    $\Rightarrow$ gap in spectrum of even-even nuclei (as in figure).

- For an odd nucleus, if we take $|\alpha\rangle = \alpha_k^\dagger|BCS\rangle$ as the ground state, then $\boxed{E_{N+1}^{GS} = \langle\alpha|\hat{K}+\mu\hat{N}|\alpha\rangle = E_N^{GS} + \mu + E_k}$ when E_k comes from $\hat{H}_1$.
  - But if we add another particle, the ground state energy is $\boxed{E_{N+2}^{GS} \doteq E_N^{GS} + 2\mu}$

$$\Rightarrow \quad \frac{1}{2}\left(E_{N+2}^{GS} - E_{N+1}^{GS}\right) - \frac{1}{2}\left(E_{N+1}^{GS} - E_N^{GS}\right) = -E_k \simeq -\Delta$$

$$\text{or} \quad \underbrace{\frac{1}{2}\left(E_{N+2}^{GS} - E_N^{GS}\right)}_{\text{average energy of even-even nuclei}} - \underbrace{E_{N+1}^{GS}}_{\text{energy of odd-even nucleus}} \simeq -\Delta$$

$\Rightarrow$ consistent with the figure with odd-even energies
  $\Rightarrow$ figure out the gap Δ from nearby nuclei
- excited states of odd: compare $\alpha_a^\dagger|BCS\rangle$ to $\alpha_b^\dagger|BCS\rangle \Rightarrow (\xi_b^2+\Delta_b^2)^{1/2} \simeq (\xi_a^2+\Delta_a^2)^{1/2}$
  $\Rightarrow$ no gap in excitation spectrum.

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Let's compare the energy of the normal ("n") and superconducting ("s") states.

Recall that we always have a solution with $\Delta = 0$

$$\Rightarrow \boxed{E_k = \sqrt{\xi_k^2 + \Delta^2} = |\xi_k| \geq 0}$$

The u_k and v_k coefficients satisfy in this case

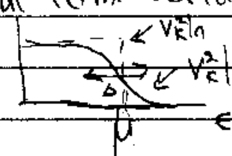
$$\left.\begin{aligned} u_k v_k|_n &= 0 \\ u_k^2|_n &= \tfrac{1}{2}\left(1 + \xi_k/|\xi_k|\right) = \theta(\epsilon_k - \mu) \\ v_k^2|_n &= \tfrac{1}{2}\left(1 - \xi_k/|\xi_k|\right) = \theta(\mu - \epsilon_k) \end{aligned}\right\}$$ canonical transformation to particles and holes

· We'll label the corresponding BCS coefficients $u_k^2|_s$ and $v_k^2|_s$.

· First compare the number of particles. Fix μ, then

$$\boxed{\begin{aligned} N_s &= 2\sum_k v_k^2|_s \\ N_n &= 2\sum_k v_k^2|_n \end{aligned}}$$

difference is only for $\sim\Delta$ of μ at Fermi surface



$$\Rightarrow N_s - N_n = \frac{L}{2\pi}\int dk\, \xi_k \left[\frac{1}{|\xi_k|} - \frac{1}{\sqrt{\xi_k^2 + \Delta^2}}\right]$$

$$\doteq N(0)\, L \int d\xi\, \xi \left[|\xi|^{-1} - \left(\xi^2 + \Delta^2\right)^{-1/2}\right]$$

$$= 0 \quad \text{since odd in } \xi$$

if mostly happening near Fermi surface and $\mu \gg \Delta$ (not always true in 1D, but true in nuclear case)

· Here $N(0)$ is the "density of states" at the Fermi surface $\boxed{\left.\dfrac{d^D k}{(2\pi)^D}\right|_\mu = N(0)\, d\xi}$

so in 3-D ($D=3$), $\boxed{N(0) = \left(\dfrac{k^2}{2\pi^2}\dfrac{dk}{d\epsilon_k}\right)_{\epsilon_k = \mu}}$

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- $N_s \doteq N_n$ tells us that the mean number of particles is unchanged at the same μ
  - or if we fix N, the difference in μ is small.
  - define

$$\boxed{\delta \equiv \frac{\mu_s - \mu_n}{\mu_n} \ll 1}$$

- Now find the change in ground state energy from normal to superconducting:

$$E_s - E_n = U_s(\mu_s) - U_n(\mu_n) + (\mu_s - \mu_n)N$$
$$= U_s(\mu_n) + (\mu_s - \mu_n)\left(\frac{\partial U_s}{\partial \mu}\right)_{\mu_n} - U_n(\mu_n) + \delta \mu_n N$$
$$= U_s(\mu_n) - U_n(\mu_n) + \delta \cdot \mu_n \underbrace{\left[\left(\frac{\partial U_s}{\partial \mu}\right)_{\mu_n} + N\right]}_{\to 0} + O(\delta^2)$$
$$\doteq U_s(\mu_n) - U_n(\mu_n)$$

vanishes by thermodynamics to this order

so we can calculate the energy difference at fixed μ

$$E_s = 2\sum_k \xi_k V_k^2\Big|_s - \sum_k \Delta (U_k V_k)\Big|_s + \frac{\lambda}{L}\sum_{kk'} V_k^2 V_{k'}^2\Big|_s$$

$$E_n = 2\sum_k \xi_k V_k^2\Big|_n + 0 + \frac{\lambda}{L}\sum_{kk'} V_k^2 V_{k'}^2\Big|_n$$

$$\Rightarrow E_s - E_n = -\sum_k \left(\frac{\xi_k^2}{(\xi_k^2+\Delta^2)^{1/2}} - \frac{\xi_k^2}{|\xi_k|}\right) - \frac{1}{2}\sum_k \frac{\Delta^2}{(\Delta^2+\xi_k^2)^{1/2}} + \frac{\lambda}{4L}\sum_{kk'}\left[\left(1-\frac{\xi_k}{E_k}\right)\left(1-\frac{\xi_{k'}}{E_{k'}}\right) - \left(1-\frac{\xi_k}{|\xi_k|}\right)\left(1-\frac{\xi_{k'}}{|\xi_{k'}|}\right)\right]$$

same trick with Fermi surface

$$\doteq L N(0)\int d\xi \left[\frac{\xi^2}{|\xi|} - \frac{\xi^2}{(\xi^2+\Delta^2)^{1/2}} - \frac{1}{2}\frac{\Delta^2}{(\xi^2+\Delta^2)^{1/2}}\right] + \frac{\lambda}{4}L[N(0)]^2 \times \iint d\xi\, d\xi' \left[\left(1-\frac{\xi}{(\xi^2+\Delta^2)^{1/2}}\right)\left(1-\frac{\xi'}{(\xi'^2+\Delta^2)^{1/2}}\right) - \left(1-\frac{\xi}{|\xi|}\right)\left(1-\frac{\xi'}{|\xi'|}\right)\right]$$

$$\doteq \frac{1}{2} L N(0)\Delta^2 \doteq \Omega_s - \Omega_n$$

(binding energy/pair) $\times$ (# of pairs within Δ of Fermi surface). Small effect on nuclear energy. Change in V_k^2 matters!

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Comments on the Mathematica Notebook

- The notebook bcs_dimreg.nb, titled "One-Dimensional BCS equations," is available from the 880 web page.

- It was written to illustrate how to solve the 1D BCS equations using Mathematica, and to test the analytic evaluation of the integrals.

- In the 2nd reference there is a table of results (with a few typos) for comparison. Note that they were unable to solve the integrals analytically (apparently). This is relevant only because there are some numerical problems in the integrals for very large and very small values of rho.

- Note that the right sides of the equations given in the Overview section are the ones used in the numerical evaluations. They are all finite.
  - However, we evaluate them analytically in dimensional regularization using the formulas:

$$\boxed{\int_0^\infty \frac{z^\alpha}{\sqrt{z^2+2\beta z+1}}\,dz = -\frac{\pi}{\sin\pi\alpha}P^0_\alpha(\beta)}$$

from Gradshteyn + Rezhik (3.252.11)

or

$$\boxed{\int_0^\infty \frac{\epsilon^\alpha\,d\epsilon}{\sqrt{(\epsilon-a)^2+b^2}} = (a^2+b^2)^{\alpha/2}\left(\frac{-\pi}{\sin\pi\alpha}\right)P^0_\alpha\left(\frac{-a}{\sqrt{a^2+b^2}}\right)}$$

where we use the right side of the integral even for α's such as $1/2$ and $3/2$ for which the integral diverges.
  - For example, in the ρ equation $\rho = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(1 - f_k/\sqrt{\Delta^2+f_k^2}\right)dk$ we drop the first term and evaluate the second using the formula.
    - Mathematica verifies that it works!

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- When doing a computational physics problem, it is always good to solve at least some subset of the problem in at least two ways.
  - Always test your methods for numerical accuracy.
  - The methods here differ at something less to 10^{-8}, but I'm not sure which is more accurate (or if both have that error)!

- You might imagine many ways to solve the gap and number equations self-consistently for Δ and μ, given ρ. I chose to solve the gap equation with a rootfinder for Δ given a value of μ. Then this is used to find the μ that goes with a given ρ.
  - Note that $\rho = \rho(\mu, \Delta(\mu))$, so we can't invert to find ρ until we know $\Delta(\mu)$.

- The rootfinder FindRoot has numerical problems at small or large ρ. There is a "Tests" section (not expanded in the printed version) to explore this.
  - It happens using either the Newton's Method (one starting value specified) or the Secant Method (two starting values).
  - There is still some precision, but it seems ~~to~~ be as low as 2 digits for some value. NEVER IGNORE THE WARNINGS ABOUT LACK OF PRECISION. Your answer may be total garbage!

- How might we improve the results?

- The last section includes some Mathematica tricks with tables that you might find useful.

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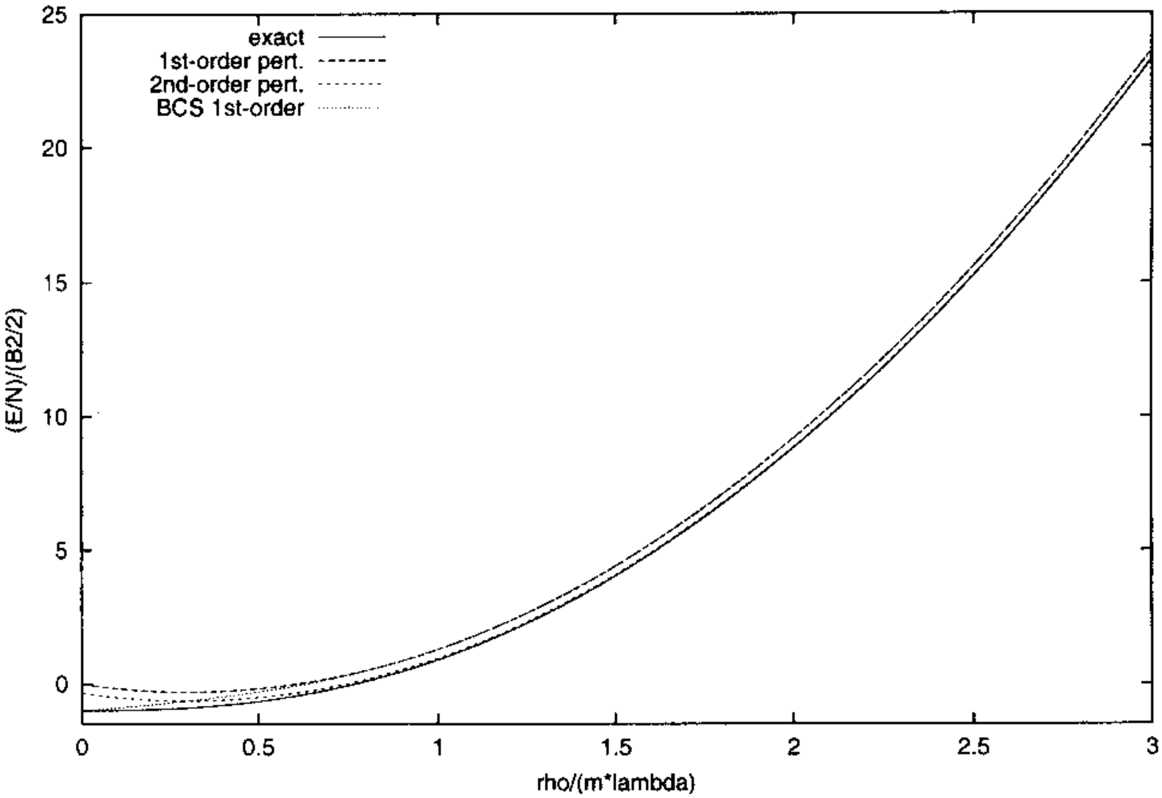
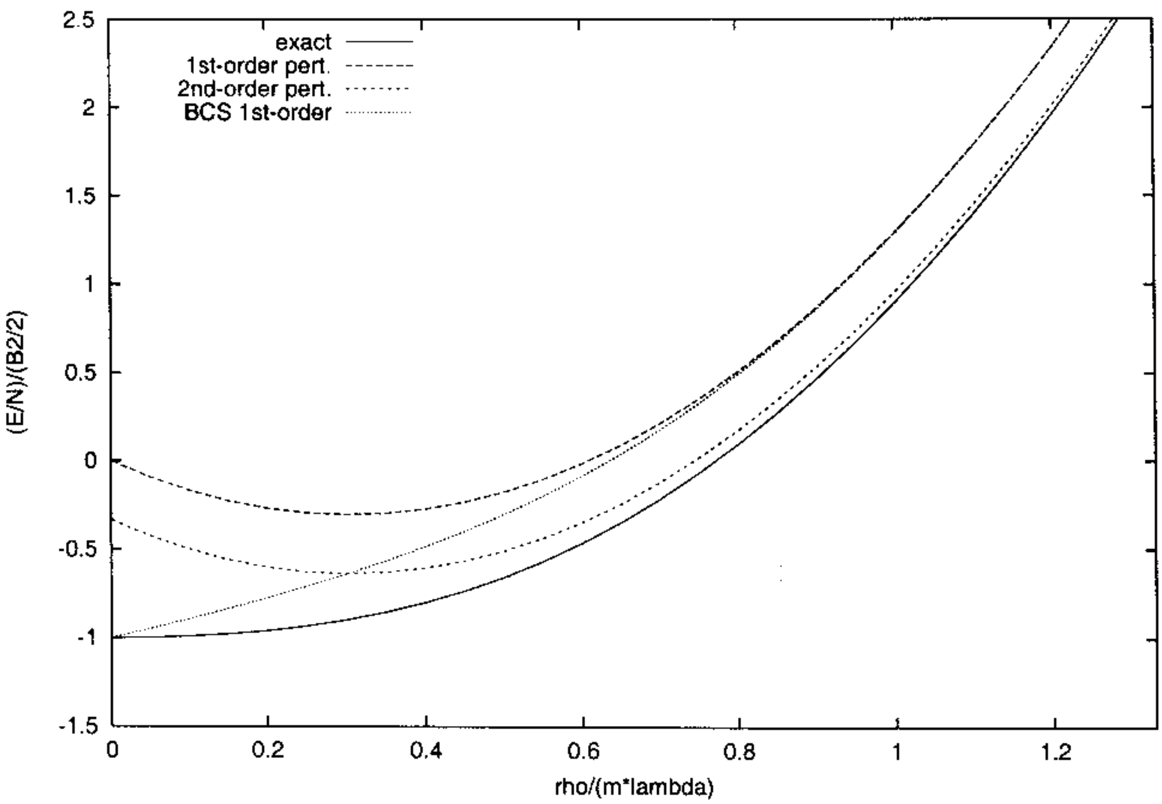
One-Dimensional Attractive Delta Function



One-Dimensional Attractive Delta Function



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Effective Action and Pairing

- The treatment of pairing in nuclei is typically not much more sophisticated than the formalism we've considered (extended to finite systems, of course).

- We saw we could get these results from a variational ansatz (wave function) or a canonical transformation $\Rightarrow$ 2nd quantization.

- We'd like to extend our effective field theory approach to include pairing.
  - Systematic power counting is the goal.
  - Should be present even in the dilute, natural system we have considered.
  - Particularly important in large scattering length $k_F a_s \gtrsim 1$ problem.

- So how do we find pairing in a field theory / path integral framework? Answer: In the effective action formalism.
  - Here: Show the basic idea and highlight the only questions (of which there are many!)

- Recall analogy of effective action and spin systems 224 - 226
  - A lattice of spins $s_i = \pm 1$ with Hamiltonian in external magnetic field H (we'll sum the interaction over all pairs here):

$$\boxed{\mathcal{H} = -\frac{1}{2N} J \sum_{\{i,j\}} s_i s_j - H \sum_i s_i}$$

with partition function

$$\boxed{Z(\beta, H, N) = \sum_{\{s_i\}} e^{\beta\left(\frac{1}{2}\frac{J}{N}\sum_{\{i,j\}} s_i s_j + H\sum_i s_i\right)} \longrightarrow \int \mathcal{D}s\, e^{-\beta\int dx\,[\mathcal{H}(s) - H s(x)]}}$$

4/4/03

One relevant question is whether the ground state at zero external magnetic field ($H=0$) has a non-zero magnetization.

- The magnetization M is the expectation value of $\sum_i s_i$:

$$\boxed{M = \frac{1}{Z} \mathrm{Tr}\left\{ \left(\sum_i s_i\right) e^{-\beta\left(\frac{1}{2}\frac{J}{N}\sum_{i,j} s_i s_j + H \sum_i s_i\right)} \right\} = -\frac{\partial F(H)}{\partial H}}$$

where F is the Helmholtz free energy:

$$\boxed{F = -\frac{1}{\beta} \ln Z} \quad \text{or} \quad \boxed{Z = e^{-\beta F(H)}}$$

- We could calculate $F(H)$ in a perturbative expansion, but it will always predict $M=0$ if $H=0$.
  - But, we can do a Legendre transformation to the Gibb's free energy: invert $M = -\frac{\partial F(H)}{\partial H}$ to find $H(M)$, then

$$\boxed{G(M) = F(H(M)) + M H(M)}$$

- Note that

$$\boxed{\frac{\partial G}{\partial M} = \underbrace{\frac{\partial F}{\partial H}}_{-M} \frac{\partial H}{\partial M} + H + M \frac{\partial H}{\partial M} = H}$$

so G as a function of M is minimized $\frac{\partial G}{\partial M}=0$ when there is no external field.

** A perturbative approximation to $G(M)$ can have nontrivial (that is $M \neq 0$) solutions to $\partial G/\partial M = 0$.

- Similarly, our expansion of the dilute Fermi gas in EFT counting (powers of k_F/Λ) did not reveal pairing.
  - $\Rightarrow$ do analog of magnetic Legendre transformation.

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- The magnetization example is one of spontaneously broken symmetry: in particular, the global symmetry of rotational invariance is broken by the magnetization in the ground state picking out a direction.
  - (Put back the vectors — $m \to \vec{m}$ and $\vec{H} \cdot \sum \vec{S}_i$ to see this)
  - The external field H acts as a source term to break the symmetry. Then we see from $G(m)$ whether it survives as $|\vec{H}| \to 0$.

- What is the corresponding symmetry in the fermion case with delta-function interaction?

$$\boxed{\mathcal{L} = \psi_\alpha^\dagger \left[i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m} + \mu \right] \psi_\alpha - \frac{1}{2} C_0 \left(\psi_\alpha^\dagger \psi_\alpha \right)^2}$$

has the usual Gaililean invariance and parity and time reversal symmetries.

- But there is also a global (perform the same transformation at every space-time point) $U(1)$ symmetry:

$$\boxed{\psi_\alpha \to e^{-i\theta} \psi_\alpha} \quad \boxed{\psi_\alpha^\dagger \to e^{i\theta} \psi_\alpha^\dagger}$$

since ψ_α always appears with $\psi_\alpha^\dagger$.

- Noether's theorem says that, to each symmetry of a local Lagrangian, there corresponds a conserved current. (This is the classical version; see Peskin + Schroeder.)
  - We can find the current (use $\psi_\alpha^\dagger \frac{\nabla^2}{2m} \psi_\alpha \to -\frac{1}{2m} (\nabla \psi^\dagger)(\nabla \psi)$)

$$\Rightarrow \boxed{j_0(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi^\dagger)} \Delta \psi^\dagger + \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} \Delta \psi = \psi^\dagger \psi}$$

$$\boxed{j_i(x) = -\frac{i}{2m} \left(\psi^\dagger \nabla_i \psi - (\nabla_i \psi^\dagger) \psi \right)}$$

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so the conserved charge is the fermion number $N = \int \psi^\dagger \psi \, dx$.

- This will be a broken symmetry if we have a non-zero expectation value for $\langle \psi_\uparrow \psi_\downarrow \rangle$.

- Let's show one way to proceed, following M. Stone's "The Physics of Quantum Fields" discussion.
  - We'll work in Euclidean space at temperature $1/\beta$ (with $\hbar = 1$) and take $\beta \to \infty$ early on.
  - We'll consider spin-1/2 only, and attractive $C_0 = -|\lambda| < 0$.

- The partition function is (in 1-D, the 3-D generalization is immediate):

$$\boxed{\begin{aligned} Z &= \mathrm{Tr}\left(e^{-\beta \hat{H} - \mu \hat{N}}\right) \\ &= \int \mathcal{D}(\psi^\dagger \psi)\, e^{-\int_0^\beta dx\, d\tau \left[\sum_{\alpha=1}^{2} \psi_\alpha^\dagger \left(\frac{\partial}{\partial \tau} - \frac{1}{2m}\frac{d^2}{dx^2} - \mu\right)\psi_\alpha - |\lambda| \psi_1^\dagger \psi_2^\dagger \psi_2 \psi_1\right]} \end{aligned}}$$

where $\alpha = 1, 2$ correspond to $\uparrow, \downarrow$.

- If we think about Grassmann anticommuting fields, we don't need to worry about anticommutators so we can see we have simply rearranged the usual $\frac{1}{2}(\psi_\alpha^\dagger \psi_\alpha)^2$ term: (all at x, t)

$$\boxed{\begin{aligned} \tfrac{1}{2}\psi_\alpha^\dagger \psi_\alpha \psi_\beta^\dagger \psi_\beta &= \tfrac{1}{2}(\psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2)(\psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2) \\ &= \tfrac{1}{2}(\psi_2^\dagger \psi_2 \psi_1^\dagger \psi_1 + 1 \leftrightarrow 2) \\ &= \psi_1^\dagger \psi_2^\dagger \psi_2 \psi_1 \end{aligned}}$$ (2 interchanges $\Rightarrow$ same sign)

- Note that the Minkowski $e^{iS} = e^{i\int dx\,dt\, \mathcal{L}(x,t)}$ becomes the Euclidean $Z = e^{-S_E} = e^{-\int dx\, d\tau\, \mathcal{L}_E(x,t)}$ where

$\boxed{t \to -i\tau}$ and $\boxed{\mathcal{L}_E(x,\tau) = -\mathcal{L}(x, -i\tau)}$ (so $i\frac{\partial}{\partial t} \to -\frac{\partial}{\partial \tau}$)

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The basic plan is to eliminate the $\psi_\uparrow^\dagger\psi_\downarrow^\dagger\psi_\downarrow\psi_\uparrow$ term by introducing an auxiliary field, just as we did for the $(\psi^\dagger\psi)^2$ term in the large N discussion.

- What can we say about the field?
- $\psi^\dagger\psi$ is Hermitian, but $\psi_\downarrow\psi_\uparrow$ is not so we can expect to need a charged scalar. We'll call it Δ and Δ^*.

- We note the Gaussian integral over Δ and Δ^* can be shifted, so we can write

$$\boxed{1 = \frac{\int \mathcal{D}(\Delta,\Delta^*)\, e^{-\frac{1}{|\lambda|}\int dx\,d\tau\,(\Delta^* - |\lambda|\psi_1^\dagger\psi_2^\dagger)(\Delta - |\lambda|\psi_2\psi_1)}}{\int \mathcal{D}(\Delta,\Delta^*)\, e^{-\frac{1}{|\lambda|}\int dx\,d\tau\,|\Delta|^2}}}$$

where we have a nice convergent integral at each x,τ.

- The sign of the quartic term in the exponent is just right to kill the corresponding term in Z.

So we can write (absorbing the constant denominator in the equation above into the measure):

$$\boxed{Z = \int \mathcal{D}(\psi^\dagger,\psi)\mathcal{D}(\Delta,\Delta^*)\, e^{-\int_0 dx\,d\tau \left\{ \psi_\alpha^\dagger\left(\frac{\partial}{\partial\tau} - \frac{1}{2m}\frac{\partial^2}{\partial x^2} - \mu\right)\psi_\alpha - \Delta^*\psi_2\psi_1 - \Delta\psi_1^\dagger\psi_2^\dagger + \frac{1}{|\lambda|}|\Delta|^2 \right\}}}$$

Note that the equation of motion for Δ is $\boxed{\Delta = |\lambda|\psi_2\psi_1}$

$\left(\frac{\delta S_E}{\delta\Delta^*} = 0 \Rightarrow \frac{1}{|\lambda|}\Delta - \psi_2\psi_1 = 0\right)$

vs.

We can verify that the new "interaction term" is Hermitian:

$$\boxed{\left(\Delta^*\psi_2\psi_1 + \Delta\psi_1^\dagger\psi_2^\dagger\right)^\dagger = +\Delta^*\psi_2\psi_1 + \Delta\psi_1^\dagger\psi_2^\dagger}$$

noting that Δ and Δ^* commute with the Grassman variables.

- Also the sign of the interaction is not relevant, since $\Delta \leftrightarrow -\Delta$ and $\Delta^* \leftrightarrow -\Delta^*$ leaves Z invariant.

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To consider the same global U(1) symmetry when Δ and Δ^* have been added, we need to take

$$\boxed{\Delta \to e^{+2i\theta}\Delta} \text{ and } \boxed{\Delta^* \to e^{-2i\theta}\Delta^*}$$

to keep each term individually invariant.

- $|\Delta|^2$ doesn't change since it involves the magnitude.

- So now if Δ gets an expectation value, that picks out a phase and we no longer have the invariance.
  - Just as in the magnetization case, we can add a source to single out a direction (an external magnetic field in that case) and then take it to zero at the end to see if the symmetry breaking persists.

- This implies we should add sources coupled to Δ and Δ^*

$$\Rightarrow \boxed{Z \to Z[j,j^*] = \int \mathcal{D}\psi^{1-4}\mathcal{D}(\Delta,\Delta^*)\, e^{-\tilde{S}_E + \int dx\, d\tau\, (j^*(x)\Delta(x) + j(x)\Delta^*(x))}}$$

where $\tilde{S}_E$ includes the Δ, Δ^* fields.

- At this stage Stone shifts to $\beta \to \infty$, which we can accomplish by simply taking the τ limits to be from $-\infty$ to $+\infty$.

- The usual plan at this point is to identify a classical expectation value and then to expand about it.
  - What is the Legendre transform?

If $\boxed{\Gamma[\Delta_c] \equiv W[j,j^*] - \int dx\, d\tau\, (\Delta_c^* j + j^*\Delta_c)}$ then $\boxed{\Delta_c^*(x) = \frac{\delta W}{\delta j(x)}}$ and $\boxed{\Delta_c(x) = \frac{\delta W}{\delta j^*(x)}}$.

So find $\Delta_c[j(x)]$ and $\Delta_c^*[j^*(x)]$. Note that $\frac{\delta \Gamma}{\delta \Delta_c} = -j^* \to 0$ when $j \to 0$.

4/14/03

In our previous effective action example, $\sigma \propto \psi^\dagger_\alpha \psi_\alpha$ and we expanded $\sigma(x,t) = \sigma_c(x) + \eta(x,t)$.

Here we write

$$\boxed{\Delta(x,\tau) = \Delta_c(x) + \eta(x,\tau)}$$

and expand in fluctuations η about Δ_c.

- For our initial discussion we'll just need the "classical" piece. So we take $\Delta \to \Delta_c$ and $\Delta^* \to \Delta_c^*$ in $Z[j,j^*]$ and do the Gaussian $\psi^\dagger_\alpha, \psi_\alpha$ integrations.

- Previously, when we introduced σ, we had (for $g=2$)

$$\boxed{\begin{pmatrix} \psi_1^\dagger & \psi_2^\dagger \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu - |\lambda|\sigma_c(x) & 0 \\ 0 & \frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu - |\lambda|\sigma_c(x) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}}$$

and so the determinant of the matrix (from the Gaussian integral) simply picked up a factor of 2 (or g in general) from the 2×2 submatrix.

- The present matrix mixes ψ_1 and ψ_2, so the above form doesn't work. But we can swap ψ_2 and $\psi_2^\dagger$ (remembering they are Grassmann!):

$$\boxed{\begin{pmatrix} \psi_1^\dagger & \psi_2 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu & -\Delta_c \\ -\Delta_c^* & \frac{\partial}{\partial\tau} + \frac{\nabla^2}{2m} + \mu \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2^\dagger \end{pmatrix}}$$

- We can just redefine $\psi_2 \to \tilde{\psi}_2^\dagger$ and $\psi_2^\dagger \to \tilde{\psi}_2$ and we have a more ordinary looking Gaussian integral, with a non diagonal submatrix.

4/14/03

Plan: Work in a basis that diagonalizes the $\frac{\partial}{\partial\tau} \pm \frac{\nabla^2}{2m}$ parts and then just calculate the determinant of the 2×2 matrix.

· You might imagine introducing quasi particle operators instead to diagonalize it!

· So we work in the Fourier basis

· Then we need

minus sign

$$\boxed{\begin{aligned}\Gamma_{\text{fermion}}[\Delta_c] &= -\ln \prod_\omega \prod_k \det -\begin{pmatrix} i\omega + \frac{k^2}{2m} - \mu & \Delta_c \\ \Delta_c^* & i\omega - \frac{k^2}{2m} + \mu \end{pmatrix} \\ &= -\ln \prod_\omega \prod_k \left[\omega^2 + \left(\frac{k^2}{2m} - \mu\right)^2 + |\Delta_c|^2 \right] \\ &= -\sum_\omega \sum_k \ln\left[\omega^2 + \left(\frac{k^2}{2m}-\mu\right)^2 + |\Delta_c|^2\right] \\ &\longrightarrow -LT \int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} \ln\left[(\omega^2 + \xi_k^2) + |\Delta_c|^2\right]\end{aligned}}$$

with $\boxed{\xi_k \equiv \frac{k^2}{2m} - \mu}$ [note: no Hartree-Fock part!]

· Now we can use our derivative trick again:

· let $|\Delta_c|^2 \to \gamma|\Delta_c|^2$ and take $\int_0^1 d\gamma \frac{d}{d\gamma}[\quad]$ which will give us $\Gamma[\Delta_c] - \Gamma[0]$.

$$\Rightarrow \boxed{\begin{aligned}&\int_0^1 d\gamma \int \frac{d\omega}{2\pi} \frac{d}{d\gamma}\left(\ln\left[\omega^2 + \xi_k^2\right) + \gamma|\Delta_c|^2\right]\right) \\ &= \int_0^1 d\gamma \int \frac{d\omega}{2\pi} \frac{|\Delta_c|^2}{\omega^2 + \xi_k^2 + \lambda|\Delta_c|^2} \\ &= \int_0^1 d\gamma\, \frac{1}{2} \frac{|\Delta_c|}{\sqrt{\xi_k^2 + \lambda|\Delta_c|^2}} \\ &= \sqrt{\xi_k^2 + \Delta_c^2} - \xi_k\end{aligned}}$$

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So the leading order (LO) effective action is

$$\boxed{\Gamma_{LO}[\Delta_c] - \Gamma_{LO}[0] = -LT\left(\frac{|\Delta_c|^2}{|\lambda|} - \int \frac{dk}{2\pi}\left(\sqrt{\xi_k^2 + |\Delta_c|^2} - \xi_k\right)\right)}$$

Minimize with respect to Δ_c to get the ground state:

$$\boxed{\begin{aligned}\frac{\partial \Gamma_{LO}}{\partial \Delta_c} &= -LT\left(\frac{2\Delta_c^*}{|\lambda|} - 2\Delta_c^* \int \frac{dk}{2\pi} \frac{1}{\sqrt{\xi_k^2 + |\Delta_c|^2}} \cdot \frac{1}{2}\right) \\ &= -LT\,\Delta_c^*\left(\frac{2}{|\lambda|} - \int \frac{dk}{2\pi} \frac{1}{\sqrt{\xi_k^2 + |\Delta_c|^2}}\right)\end{aligned}}$$

Setting $\partial\Gamma_{LO}/\partial\Delta_c = 0$ and looking for a non-zero Δ_c^* yields the gap equation:

$$\frac{2}{|\lambda|} - \int \frac{dk}{2\pi} \frac{1}{\sqrt{\xi_k^2 + |\Delta_c|^2}} = 0$$

or $\boxed{1 = \frac{|\lambda|}{4\pi}\int_{-\infty}^{\infty} dk \frac{1}{\sqrt{\xi_k^2 + |\Delta_c|^2}}}$ as before.

The thermodynamic potential follows from

$\boxed{\Omega[\Delta_c] = -\Gamma[\Delta_c]/T}$ and $\boxed{E = \Omega + \mu N = \Omega - \mu\left(\frac{\partial\Omega}{\partial\mu}\right)_L}$

So
$$\frac{N}{L} = \frac{1}{LT}\frac{\partial\Gamma[\Delta_c]}{\partial\mu} = -\int \frac{dk}{2\pi}\left(\frac{\xi_k}{\sqrt{\xi_k^2 + |\Delta_c|^2}} - 1\right) \quad \text{as before}$$

and
$$\frac{E}{L} = \frac{-\Gamma[\Delta_c]}{LT} + \mu\int d^3k\left(1 - \frac{\xi_k}{\sqrt{\xi_k^2 + |\Delta_c|^2}}\right)$$

again yields the same result.