

The Scattering of Fast Nucleons from Nuclei*

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The formal theory of the scattering of high-energy nucleons by nuclei is developed in terms of the nucleon-nucleon scattering amplitude. The most important approximations necessary to make numerical calculation feasible are then examined. The optical model potential is derived on this basis and compared with the optical model parameters found from experiment. The elastic scattering and polarization of nucleons from light nuclei is predicted and compared with experiment. The effect of nuclear correlations is discussed. The polarization of inelastically scattered nucleons is discussed and predictions compared with experiments. To within the validity of the approximations the experimental data on the scattering of nucleons from nuclei at energies above ~ 100 Mev appears to be consistent with the theory.

1. INTRODUCTION

Much data has accumulated in recent years on the scattering of fast nucleons (100–300 Mev) by nuclei. The most striking feature that has emerged from this data and its analysis has been that for elastic scattering, at least, nuclei behave as larger or smaller samples of nuclear material, the major differences among nuclei arising from their different sizes. This has made it possible to organize the elastic scattering data in terms of an optical potential, which, apart from radius, has much the same parameters for different elements.

Watson (1-3) and his collaborators have supplied the theoretical justification

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for the use of the optical model. In a series of papers on the theory of scattering, Watson has indicated the manner in which the optical potential for high-energy nucleon-nucleus scattering could be calculated, presupposing a knowledge of the nucleon-nucleon interaction. However, until fairly recently it was not possible to calculate these optical model potentials because of our lack of a detailed knowledge of the two nucleon interaction. A short time ago, however, Bethe (4) was able to demonstrate that the forward angle elastic scattering and polarization data of nucleons on C¹² at 310 Mev was entirely consistent with the nucleon-nucleon scattering amplitude at that energy. This success was a consequence of the very detailed study, both experimental and theoretical (5, 6), that had been made of the nucleon-nucleon (especially, the proton-proton) scattering amplitude at that energy.

The new nucleon-nucleon data, at 310 Mev and at other energies, have also made possible new analyses of the nucleon-nucleon interaction, and some confidence has now been gained that the most important features of this interaction have been understood in terms of a phenomenological potential (7). In the light of these developments a further calculation (8, 9) demonstrated that the forward angle elastic scattering data of nucleons on C¹² at 90 Mev, 150 Mev, as well as 310 Mev were consistent with the two-nucleon potential. Thus there is a strong reason to believe that the elastic scattering and polarization of the nucleons on nuclei can be understood in terms of the two nucleon interaction. However, the calculations referred to above were for an even-even, zero spin nucleus (C¹²), a case for which special simplicities enter. One of the aims of this paper is to investigate in greater detail, and for a wider class of target nuclei, the connection between the nucleon-nucleon force and high-energy nucleon-nucleus scattering.

In addition to the elastic scattering data, there is now data (10) on the inelastic scattering of nucleons which excite a low lying level in the target nucleus. The outstanding characteristic indicated by these experiments is that at high energies the angular distribution of polarization for inelastic scattering from low excited states is very much like that for elastic scattering. This feature is present in Born approximation calculations for a nonspherical optical model potential, and has led several authors to speculate that the data supports the hypothesis that the levels excited in these experiments are collective (11). However, further investigation shows that this feature is a much more general one.

Quasi-elastic scattering, in which elastic scattering and inelastic scattering from low lying levels cannot be resolved, and inelastic scattering from unresolved groups of levels will also be discussed.

2. FORMAL THEORY

We want to describe the scattering, elastic or inelastic, of a nucleon from a target nucleus, in terms of scattering from nucleons in the target, using the methods of Watson (1-3).

Consequently we split the complete Hamiltonian, H , of the system into

$$H = H_0 + V, \quad H_0 = H_N + K_0, \quad (2.1)$$

where H_N is the Hamiltonian of the target nucleus,¹ K_0 the kinetic energy operator of the incident particle and

$$V = \sum_{i=1}^N v(r_0, r_i) = \sum_i v_i \quad (2.2)$$

is the sum of the two-body interactions between the incident particle (r_0) and the target nucleons (r_i), N being the number of nucleons in the target nucleus.

We shall treat V as a perturbation and will always work in the representation of the eigenstates of H_0 ,

$$H_0 \chi_i = E_i \chi_i \quad (2.3)$$

$$\chi_i = \frac{1}{(2\pi)^{3/2}} \exp(i\mathbf{k} \cdot \mathbf{r}_0) \Phi_i(r_1, \dots, r_n), \quad (2.4)$$

where the first factor in (2.4) is the wave function of the incident, or scattered particle and the Φ_i are the eigenstates, completely antisymmetrized, of the target nucleus including its center of mass wave function. The functions Φ_i include, as well as the bound states, all the continuum states in which one or more particles are unbound, and which are the solutions of the equation

$$H_N \Phi_n = \varepsilon_n \Phi_n. \quad (2.5)$$

For convenience ε_0 , the energy of the ground state, is taken equal to zero. Thus

$$E_n = K_0 + \varepsilon_n, \quad (2.6)$$

where K_0 is the kinetic energy of the incident particle and ε_i is the excitation energy of the particular target excited state, and includes the nuclear recoil energy.

At energies above ~ 100 Mev, compound states of the target plus incident nucleon cannot play any role, as the life time of compound states in this energy region is of the order of the transit time across the nucleus. The essential approximation we make is to neglect the antisymmetrization between the incident particle and the target nucleus. The argument in favor of this neglect at the energies here considered has been presented by Takeda and Watson (12), who show that it is reasonable to make this approximation, provided one uses the properly symmetrized two-body scattering matrices, which are in fact those involved in two-body scattering experiments.

The Schrödinger equation we have to solve can be written

$$(H_0 - E)\psi = -V\psi \quad (2.7)$$

¹ For convenience the recoil energy of the target is included in H_N .

or, in integral form

$$\psi = \chi_0 + \frac{1}{E - H_0 + i\epsilon} V\psi, \quad (2.8)$$

where the second term in (2.8) includes the boundary condition of outgoing scattered waves only, and χ_0 is the initial state, consisting of an incident plane wave and the ground state of the target nucleus.

The transition probability for the system depends on the matrix element

$$T = \langle \chi_f | V | \psi \rangle. \quad (2.9)$$

The wave-matrix Ω , introduced by Møller (13) is defined by

$$\psi = \Omega \chi_0, \quad (2.10)$$

or written out in full

$$\psi_{kn} = \frac{1}{(2\pi)^{3/2}} \sum_{n'} \int d\mathbf{k}' \exp(i\mathbf{k}' \cdot \mathbf{r}_0) \Phi_{n'}(\mathbf{x}_i) \langle n'k' | \Omega | kn \rangle, \quad (2.10a)$$

where the labels kn on ψ refer to the initial conditions, i.e., particle of momentum \mathbf{k} incident on target nucleus in state n . Then (2.9) becomes

$$T = \langle \chi_f | V\Omega | \chi_0 \rangle \equiv \langle n'k' | V\Omega | kn \rangle, \quad (2.11)$$

defining the scattering matrix

$$T = V\Omega. \quad (2.12)$$

Equation (2.8) for the wave function can then be written in the condensed form

$$\Omega = 1 + \frac{1}{E - H_0 + i\epsilon} V\Omega = 1 + \frac{1}{E - H_0 + i\epsilon} T \quad (2.13)$$

and an integral equation for the scattering matrix is obtained,

$$T = V + V \frac{1}{E - H_0 + i\epsilon} T \quad (2.14)$$

or written out in full:

$$\langle n'\mathbf{k}' | T | \mathbf{k}n \rangle = \langle n'\mathbf{k}' | V | \mathbf{k}n \rangle + \sum_m \int \frac{d\mathbf{p}}{E - E_m(\mathbf{p}) + i\epsilon} \frac{\langle n'\mathbf{k}' | V | \mathbf{p}m \rangle \langle m\mathbf{p} | T | \mathbf{k}n \rangle}{E - E_m(\mathbf{p}) + i\epsilon}, \quad (2.14a)$$

where

$$E_m(\mathbf{p}) = \epsilon_m + K_0(\mathbf{p}). \quad (2.15)$$

In the nonrelativistic approximation, the scattering amplitude G , defined so

that,

$$\frac{d\sigma}{d\Omega} = |G|^2 \quad (2.16)$$

is given in terms of the scattering matrix T , in the center-of-mass system, by the equation

$$G_{n0}(\mathbf{k}', \mathbf{k}) = -(2\pi)^2 \frac{m}{\hbar^2} \left(\frac{N}{N+1} \right) \sqrt{\frac{k'}{k}} \langle n\mathbf{k}' | T | \mathbf{k}_0 \rangle \quad (2.17)$$

where N is the number of nucleons in the target and \mathbf{k}, \mathbf{k}' are the particle momenta before and after collision in the center-of-mass system. The scattering matrix T is an invariant, and can be calculated in any convenient Galilean frame.

Another useful relation, imposed on the scattering matrix by the conservation of probability, is

$$\langle \mathbf{k}_0 | T^+ \delta(E - H_0) T | \mathbf{k}_0 \rangle = -\frac{1}{\pi} \text{Im}\{\langle \mathbf{k}_0 | T | \mathbf{k}_0 \rangle\} \quad (2.18)$$

or, translated into a relation between cross sections,

$$\sigma_T = \frac{4\pi}{k} \text{Im}\{G_{00}(k, k)\} \quad (2.19)$$

the optical theorem relation between the total cross section and the forward elastic scattering amplitude.

We return now to the solution of the Eq. (2.14) for the scattering matrix neglecting the antisymmetry between incident particle and target nucleons. Using antisymmetrized target wave functions, the matrix element $\langle |v(r_0, r_i)| \rangle$ is independent of the label i , so that Eq. (2.14) can be written (see 2.2)

$$T = Nv \left(1 + \frac{1}{\alpha} T \right). \quad (2.20)$$

This is possible because, although the antisymmetrized wave functions of the target do not form a complete set, the propagator

$$\frac{1}{\alpha} = \frac{\mathcal{Q}}{E - H_0 + i\epsilon} = \frac{\mathcal{Q}}{E - K_0 - H_N + i\epsilon} \quad (2.21)$$

only connects members of the sub set. The operator \mathcal{Q} is a projection operator for completely antisymmetrical nuclear states.

We notice that to first order the solution of (2.20) is

$$T \approx V = Nv, \quad (2.22)$$

i.e., the sum of the Born approximation scattering amplitudes from the nucleons in the target. Intuitively the first approximation would be expected to be something like the sum of the actual two-body scattering amplitudes. Therefore we define a scattering operator τ by

$$\tau = v \left(1 + \frac{1}{\alpha} \tau \right) = v \left\{ 1 + \frac{\alpha}{E - K_0 - H_N + i\epsilon} \tau \right\}. \quad (2.23)$$

For a more general definition see Appendix IV.

The scattering operator τ is not the free two nucleon scattering operator t , which obeys the equation

$$t = v \left(1 + \frac{1}{h} t \right) = v \left(1 + \frac{1}{E - K_0 - K_1 + i\epsilon} t \right), \quad (2.24)$$

where in this case the intermediate states must include states of arbitrary symmetry and K_1 is the kinetic energy operator of a target nucleon. The impulse approximation, as used here, consists in putting

$$\tau \cong t. \quad (2.25)$$

We shall return to this question later, but here we merely note that at high energies this is a reasonable approximation (14).

Equation (2.20) has the formal solution

$$T = \frac{1}{1 - Nv \frac{1}{\alpha}} Nv = Nv \frac{1}{1 - \frac{1}{\alpha} Nv}. \quad (2.26)$$

Note that the projection operator for antisymmetric states α has been included in the definition of $(1/\alpha)$. Eliminating v with the aid of the solution of (2.23), viz.,

$$v = \tau \frac{1}{1 + \frac{1}{\alpha} \tau} = \frac{1}{1 + \tau \frac{1}{\alpha}} \tau, \quad (2.27)$$

we find that T is given by an expression whose denominator, apart from c -numbers, is a function only of the operator $(1/\alpha)\tau$. The denominator can then be re-ordered and we find

$$T = N\tau \frac{1}{1 - (N - 1) \frac{1}{\alpha} \tau} \quad (2.28)$$

or defining

$$T' = \frac{N - 1}{N} T, \quad (2.29)$$

we obtain the integral equation

$$T' = U^{(0)} \left(1 + \frac{1}{\alpha} T' \right), \quad (2.30)$$

$$U^{(0)} = (N - 1)\tau, \quad (2.31)$$

which is of the same form as (2.14) but is now written in terms of the modified scattering matrix τ , so that the first approximation to T' , $T' \approx U^{(0)}$, is, in the impulse approximation ($\tau = t$), simply proportional to the sum of the free scattering amplitudes from the target nucleons.

Equation (2.30) is still a set of coupled integral equations involving all the target excited states. A perturbation solution is

$$T' = U^{(0)} + U^{(0)} \frac{\alpha}{E - H_0 + i\epsilon} U^{(0)} + \dots, \quad (2.32)$$

which breaks up the scattering matrix into contributions from single scattering, double scattering, etc. However this series is not necessarily rapidly convergent. For small momentum transfer the matrix elements of $U^{(0)}$ diagonal in the nuclear states are $\sim N$ times the off-diagonal elements. In the series (2.32) each term contains in general matrix elements diagonal in the nuclear states. It would seem more profitable, then, to separate out the terms of T' which come from matrix elements of $U^{(0)}$ diagonal in the nuclear states, from the terms which involve only nondiagonal elements.

To look at this another way, consider the ground-state elastic scattering part of T' . The contribution of the second term in 2.32 consists of a part which involves only diagonal elements of $U^{(0)}$. In addition there is a part which consists of going to an excited state of the nucleus via excitation of a nucleon by the first factor $U^{(0)}$, followed by a transition to the ground state via a particle transition induced by the second factor $U^{(0)}$; an effect whose magnitude depends on the two particle correlation function of the nucleus. Thus the separation into contributions from diagonal and nondiagonal elements of $U^{(0)}$, i.e., the separation between elastic and inelastic scattering in intermediate states, corresponds approximately to picking out the part of the scattering matrix which depends only on the nucleon density distribution from the part which depends on the 2, 3 etc., particle correlation functions of the nucleus.

Let us look at Eq. (2.30) for the case of ground-state elastic scattering. If we keep only nucleon state labels, i.e., write T'_{00} for $\langle 0k' | T' | k0 \rangle$, then (2.14) can be written

$$T'_{00} = U_{00}^{(0)} + U_{00}^{(0)} \frac{1}{\alpha_0} T'_{00} + \dots, + \text{off diagonal terms} \quad (2.33)$$

where

$$\frac{1}{\alpha_0} = \frac{\alpha}{E - K_0 + i\epsilon} \quad (2.34)$$

as $\varepsilon_0 = 0$.

Neglecting the off-diagonal terms gives an equation which is recognizable as the Schrödinger equation for scattering from a single-particle potential $U_{00}^{(0)}$. For if $\Omega_{00}^{(0)}$ is the wave matrix for elastic scattering from a potential $U_{00}^{(0)}$, it obeys the Schrödinger equation,

$$(E - K_0 - U_{00}^{(0)})\Omega_{00}^{(0)} = 0 \quad (2.35)$$

or

$$\Omega_{00}^{(0)} = 1 + \frac{1}{\alpha_0} U_{00}^{(0)} \Omega_{00}^{(0)} \quad (2.36)$$

and hence the scattering matrix $T'^{(0)}$ for this single particle potential

$$T'^{(0)} = U_{00}^{(0)} \Omega_{00}^{(0)} \quad (2.37)$$

is given by the series (2.33) with neglect of the off-diagonal terms. Therefore in this approximation the elastic scattering is given by the scattering from a single particle potential, $U_{00}^{(0)}$, where

$$U_{00}^{(0)} = (N - 1)\tau_{00} \approx (N - 1)t_{00}. \quad (2.38)$$

We can systematize the separation of T' into parts depending on diagonal and nondiagonal part of $U^{(0)}$ in two stages. First, define the projection operators P_0 and Q_0 , of which P_0 projects onto the nuclear ground state, and Q_0 off it.

$$P_0 = |0\rangle\langle 0|, \quad Q_0 = \sum_{n \neq 0} |n\rangle\langle n|, \quad (2.39)$$

$$P_0 + Q_0 = 1. \quad (2.40)$$

Equation (2.30) can then be written

$$T' = U^{(0)} \left(1 + \frac{1}{\alpha} (P_0 + Q_0) T' \right) \quad (2.41)$$

$$= U \left(1 + \frac{1}{\alpha} P_0 T' \right) = \left(1 + T' \frac{1}{\alpha} P_0 \right) U, \quad (2.41a)$$

where

$$U = \frac{1}{1 - U^{(0)} \frac{1}{\alpha} Q_0} U^{(0)} = U^{(0)} \frac{1}{1 - \frac{1}{\alpha} Q_0 U^{(0)}}, \quad (2.42)$$

and obeys the integral equation

$$U = U^{(0)} \left(1 + \frac{1}{\alpha} Q_0 U \right) = \left(1 + U \frac{1}{\alpha} Q_0 \right) U^{(0)}, \quad (2.43)$$

for which (2.42) is a solution. The matrix U will be called the potential matrix.

For elastic scattering in the ground state (2.41a) gives

$$T'_{00} = U_{00} \left(1 + \frac{1}{\alpha_0} T'_{00} \right) = U_{00} \left(1 + \frac{\alpha}{E - K_0 + i\epsilon} T'_{00} \right), \quad (2.44)$$

which represents a Schrödinger equation for the scattering from a single-particle potential U_{00} . This is now an exact equation showing that elastic scattering can always be reduced to the scattering from an “optical” potential. This potential U_{00} is in general nonlocal in configuration space, and energy dependent, as can be seen from its defining equation (2.43).²

For inelastic scattering to a definite final nucleon state (label n) (2.41) becomes

$$T'_{n0} = U_{n0} + U_{n0} \frac{1}{\alpha_0} T'_{00} = U_{n0} \Omega_{00}^R, \quad (2.45)$$

where

$$\Omega_{00}^R = 1 + \frac{1}{\alpha_0} T'_{00} = 1 + \frac{1}{\alpha_0} U_{00} \Omega_{00}^R. \quad (2.46)$$

is simply the distorted wave corresponding to ground-state elastic scattering, and is a solution of the wave equation for the potential U_{00} with the usual boundary condition of outgoing scattered waves only. For some purposes it is more convenient to use the distorted wave rather than the scattering matrix, which can be written in the same form as (2.45)

$$T'_{00} = U_{00} \Omega_{00}^R, \quad (2.47)$$

and is obtained from (2.44) and (2.46).

Alternatively we can define the wave matrix Ω_{00}^L by

$$T'_{00} = \Omega_{00}^L U_{00}. \quad (2.48)$$

This follows from the second form of (2.41a) which gives

$$T'_{00} = U_{00} + T'_{00} \frac{1}{\alpha_0} U_{00} = \left(1 + T'_{00} \frac{1}{\alpha_0} \right) U_{00} = \Omega_{00}^L U_{00}, \quad (2.49)$$

where

$$\Omega_{00}^L = 1 + \Omega_{00}^L U_{00} \frac{1}{\alpha_0}. \quad (2.50)$$

² Of course it will only be an optical potential in the usual sense if the energy dependence is not too violent and the nonlocality not too wild.

Thus Ω_{00}^L also obeys a Schrödinger equation for elastic scattering. It represents an operator with incoming spherical waves which are enhanced in scattering to produce a plane wave, in contradistinction to Ω_{00}^R which represents an incoming plane wave which is partially absorbed and produces outgoing spherical waves. In fact Ω_{00}^L is the time reverse of Ω_{00}^R and occurs naturally in considering the reverse transition from an excited state to the ground state, for which the scattering matrix is

$$T'_{0n} = U_{0n} + T'_{00} \frac{1}{\alpha_0} U_{0n} = \Omega_{00}^L U_{0n}, \quad (2.51)$$

which is suitably symmetrical with respect to (2.45).

The scattering matrix T' is completely determined by the potential matrix U . In accordance with the previous discussion we seek to expand U in terms of the diagonal and nondiagonal parts of $U^{(0)}$ by writing Eq. (2.43) in terms of the diagonal and nondiagonal parts of $U^{(0)}$.

$$U = U^{(0)} + U_D^{(0)} \frac{1}{\alpha} Q_0 U + U_N^{(0)} \frac{1}{\alpha} Q_0 U, \quad (2.51a)$$

or

$$U = U^{(0)} + U \frac{1}{\alpha} Q_0 U_N^{(0)} + U \frac{1}{\alpha} Q_0 U_D^{(0)}, \quad (2.51b)$$

where the subscripts D , N refer, respectively, to diagonal and nondiagonal elements. We take the term involving $U_D^{(0)}$ to the left-hand side of the above equations, with the result that

$$U = \omega^L U^{(0)} + \omega^L U_N^{(0)} \frac{1}{\alpha} Q_0 U, \quad \text{or}, \quad U = U^{(0)} \omega^R + U \frac{1}{\alpha} Q_0 U_N^{(0)} \omega^R, \quad (2.52a)$$

where

$$\omega^L = \frac{1}{1 - U_D^{(0)} \frac{1}{\alpha} Q_0}, \quad \omega^R = \frac{1}{1 - \frac{1}{\alpha} Q_0 U_D^{(0)}}, \quad (2.53a)$$

or

$$\omega^L = 1 + \omega^L U_D^{(0)} \frac{Q_0}{\alpha}, \quad \omega^R = 1 + \frac{Q_0}{\alpha} U_D^{(0)} \omega^R. \quad (2.53b)$$

The matrices ω^L and ω^R have simple properties. Since they are inverses of diagonal matrices they are themselves diagonal. By virtue of the projection operator Q_0

$$\omega_{00}^L = \omega_{00}^R = 1. \quad (2.53c)$$

In excited states they represent elastic scattering with the approximate optical potential $U_D^{(0)}$, and ω^L and ω^R are related by time reversal.

We can now express U as a product of three factors, two of which are known, and the third of which can be expressed as a perturbation series involving non-diagonal matrix elements of $U^{(0)}$ only, i.e., as purely inelastic scattering in intermediate states. Write

$$U = \omega^L W^L \quad \text{or} \quad U = W^R \omega^R. \quad (2.54)$$

Substituting in (2.52) we have,

$$W^L = U^{(0)} + U_N^{(0)} \frac{1}{\alpha} Q_0 \omega^L W^L, \quad W^R = U^{(0)} + W^R \omega^R \frac{1}{\alpha} Q_0 U_N^{(0)}. \quad (2.55)$$

From the definition (2.53) of ω , and remembering that

$$(AB)^{-1} = B^{-1}A^{-1}, \quad (2.56)$$

we find

$$\frac{1}{\alpha} Q_0 \omega^L = \frac{Q_0}{\alpha - U_D^{(0)} Q_0} = \omega^R \frac{1}{\alpha} Q_0 \quad (2.57)$$

so that

$$\begin{aligned} W^L &= U^{(0)} + U_N^{(0)} Q_0 \frac{1}{\alpha - U_D^{(0)} Q_0} W^L, \\ W^R &= U^{(0)} + W^R \frac{1}{\alpha - U_D^{(0)} Q_0} Q_0 U_N^{(0)}. \end{aligned} \quad (2.58)$$

Then the perturbation series for W is³

$$\begin{aligned} W^L &= U^{(0)} + U_N^{(0)} Q_0 \frac{1}{e} U^{(0)} + U_N^{(0)} Q_0 \frac{1}{e} U_N^{(0)} Q_0 \frac{1}{e} U^{(0)} + \dots, \\ W^R &= U^{(0)} + U^{(0)} \frac{1}{e} U_N^{(0)} + \dots, \end{aligned} \quad (2.59)$$

where

$$e = \alpha - U_D^{(0)} Q_0. \quad (2.60)$$

This propagator gives the propagation of a nucleon in an average nuclear potential.

³ The matrices W are related to the matrices F of Watson (1, 2) by

$$W^L = F^L U^{(0)},$$

$$W^R = U^{(0)} F^R.$$

The perturbation series expansion for F^L is defined by Eq. (2.59).

tial and is equivalent to free propagation via α followed by multiple scattering due to ω^L (see Eq. 2.57). The series (2.59) for W contains only inelastic scattering in intermediate states with transitions to the ground state forbidden. Elastic scattering in intermediate states is now all contained in the new propagator $e = \alpha - U_B^{(0)}Q_0$, which gives the propagation of a nucleon in an average nuclear potential $U_B^{(0)}$. The matrix W is then in a form suitable for perturbation treatment.

For elastic scattering we have from (2.54)

$$U_{00} = \omega_{00}^L W_{00}^L = W_{00}^L \quad (2.61)$$

$$= W_{00}^R \omega_{00}^R = W_{00}^R \quad (2.62)$$

or

$$U_{00} = U_{00}^{(0)} + \sum_{n \neq 0} U_{0n}^{(0)} \frac{1}{\alpha_n - U_{nn}^{(0)}} U_{n0}^{(0)} + \dots \quad (2.63)$$

For inelastic scattering, we may write (2.54) and (2.53c)

$$U_{n0} = \omega_{nn}^L W_{n0}^L, \quad (2.64)$$

or as a perturbation series (2.59),

$$U_{n0} = \omega_{nn}^L \left\{ U_{n0}^{(0)} + \sum_{\substack{m, n \neq 0 \\ m \neq n}} U_{nm}^{(0)} \frac{1}{\alpha_m - U_{mm}^{(0)}} U_{m0}^{(0)} + \dots \right\}. \quad (2.65)$$

Similarly

$$U_{0n} = W_{0n}^R \omega_{nn}^R \quad (2.66)$$

and

$$U_{0n} \approx \left\{ U_{0n}^{(0)} + \sum_{\substack{m \neq n \\ m, n \neq 0}} U_{0m}^{(0)} \frac{1}{\alpha_m - U_{mm}^{(0)}} U_{mn}^{(0)} + \dots \right\} \omega_{nn}^R. \quad (2.67)$$

Collecting results (2.47), (2.61), (2.45), (2.51), (2.64), (2.66) we now have for the scattering matrix

$$T'_{00} = W_{00}^L \Omega_{00}^R = \Omega_{00}^L W_{00}^R, \quad (2.68)$$

$$T'_{n0} = U_{n0} \Omega_{00}^R = \omega_{nn}^L W_{n0}^L \Omega_{00}^R, \quad (2.69)$$

$$T'_{0n} = \Omega_{00}^L U_{0n} = \Omega_{00}^L W_{0n}^R \omega_{nn}^R. \quad (2.70)$$

To summarize, the calculation of elastic scattering proceeds in principle as follows:

Beginning with

(1) the modified scattering operator τ defined by Eq. (2.23), we construct the matrix elements of

(2) the potential operator $U^{(0)} = (N - 1)\tau$. From $U^{(0)}$ we may then construct

(3) the matrix W using the series Eq. (2.59). For elastic scattering,

(4) the optical potential is given by $U_{00} = W_{00}^R = W_{00}^L$. (In first approximation $U_{00} \approx U_{00}^{(0)}$.) From U_{00} ,

(5) the elastic scattering matrix T'_{00} and the wave matrix Ω_{00} may be obtained by solving the integral equations (Eqs. 2.44 and 2.46) or the equivalent Schrödinger equation.

The inelastic scattering matrices T'_{n0} and T'_{0n} are given by Eqs. (2.69) and (2.70). For this we must compute the off diagonal elements of W and the wave matrix ω . The wave matrix ω is given by the Schrödinger equation (2.53). This equation is the Schrödinger equation for scattering from the approximate optical potential $U^{(0)}$.

The matrices for inelastic scattering (2.69), (2.70) have the form of a distorted wave matrix for elastic scattering in the initial state, an inelastic scattering matrix, which contains all the effects of intermediate state inelastic scattering, and a distorted wave matrix corresponding to elastic scattering in the final state.⁴

It will be noticed that no approximations of the order of $1/N$ have been made. The conventional scattering matrix T is obtained from T' by multiplying T by $(N/N - 1)$; see Eq. (2.29).

Taking the lowest term in the expression for ω , W and Ω gives the simple result

$$T' \approx U^{(0)}, \quad (2.71a)$$

or

$$T \approx N\tau, \quad (2.71b)$$

which is the intuitively plausible result at high energies.

3. DISCUSSION OF THE IMPORTANT APPROXIMATIONS

The approximation we shall mostly use is the one in which $W = U^{(0)}$, i.e., the neglect of virtual nuclear excitation in intermediate states, which corresponds roughly to neglecting two body and higher nuclear correlation functions.⁵ This approximation is usually called the multiple scattering approximation. In addition we also use the impulse approximation in which we set

$$\tau \cong t \text{ (impulse approximation).} \quad (3.1)$$

Both these approximations are expected to be reasonable for high energies.

⁴ Notice that the distortion of the wave in the *final* state is given by the scattering in an approximate optical potential $U_{nn}^{(0)}$, while the distortion of the initial wave is given by the exact elastic scattering.

⁵ It should be noted that the approximation $W = U^{(0)}$ contains all corrections due to multiple elastic scattering, the neglected higher terms in the expansion of W give the effect of nuclear correlations.

From the multiple scattering approximation, obtained by setting $W = U^{(0)}$, we obtain (2.54), (2.53c)

$$U_{00} \cong U_{00}^{(0)} = (N - 1) \tau_{00}, \quad (3.2)$$

and with the impulse approximation as well,

$$U_{00} \cong (N - 1) t_{00}, \quad (3.3)$$

$$U_{n0} \cong \omega_{nn}^L U_{n0}^{(0)} \cong (N - 1) \omega_{nn}^L t_{n0}, \quad (3.4)$$

$$T'_{n0} \cong (N - 1) \omega_{nn}^L t_{n0} \Omega_{00}^R. \quad (3.5)$$

The last expression is what is sometimes called the distorted wave Born approximation for inelastic scattering. Reading the factors from right to left we see that the incoming wave is distorted by Ω_{00}^R as though it were going to be elastically scattered, then the nucleus is excited to state n by t_{n0} , and finally the outgoing wave is distorted by the wave matrix ω_{nn}^L . Notice that the distortion on the way out is not given by the exact elastic scattering in state n but by the approximate scattering in an optical potential $U_{nn}^{(0)}$ as determined by ω_{nn}^L .

Our remaining task is to express the nuclear matrix elements of the free two-body scattering matrix t , in terms of the free two-body scattering amplitude.

The matrix elements of the free two-nucleon scattering operator t for nucleon-nucleon scattering is written as:

$$\begin{aligned} \langle \mathbf{k}', \mathbf{k}' | t | \mathbf{k}, \mathbf{k}_1 \rangle &= \langle \mathbf{k}' - \mathbf{k}_1' | t^{(0)} | \mathbf{k} - \mathbf{k}_1 \rangle \delta(\mathbf{k} - \mathbf{k}' + \mathbf{k}_1 - \mathbf{k}_1') \\ &= \langle \mathbf{k}' - \mathbf{k}_1 + \mathbf{q} | t^{(0)} | \mathbf{k} - \mathbf{k}_1 \rangle \delta(\mathbf{k} - \mathbf{k}' + \mathbf{k}_1 - \mathbf{k}_1'), \end{aligned} \quad (3.6)$$

where the δ function expresses conservation of momentum. The essential approximation in the calculation of matrix elements of t between nuclear states is to neglect the dependence of $\langle \mathbf{k}' - \mathbf{k}_1 + \mathbf{q} | t^{(0)} | \mathbf{k} - \mathbf{k}_1 \rangle$ on \mathbf{k}_1 , i.e., in the lab system $\mathbf{k}_1 \sim 0$. This represents neglect of the kinetic energy of the struck particle in the nucleus before the collision.

In order to take a nuclear matrix element of t is it necessary to use the Fourier transform on \mathbf{k}_1 , which now only appears in the δ function, and then the neglect of \mathbf{k}_1 in $t^{(0)}$ leads to the result⁶:

$$t_{n0} = \langle \mathbf{k}' | t^{(0)} | \mathbf{k} \rangle \langle n | \exp(-i\mathbf{q} \cdot \mathbf{r}) | 0 \rangle \equiv \langle \mathbf{k}' | t^{(0)} | \mathbf{k} \rangle F_{n0}(\mathbf{q}) \quad (3.7)$$

$$\equiv \langle \mathbf{k}' | t^{(0)} | \mathbf{k} \rangle \left\langle n \left| \frac{1}{N} \sum_i \exp(-i\mathbf{q} \cdot \mathbf{r}_i) \right| 0 \right\rangle, \quad (3.7a)$$

⁶ It is the neglect of \mathbf{k}_1 that makes possible the factorization given in Eq. (3.7). The dependence on \mathbf{k}_1 may be expressed as a power series, introducing nuclear matrix elements more complicated than $F(q)$. The neglect of the higher order terms in this series is a good approximation because t depends primarily on the momentum transfer at high energy.

Equation (3.7) is strictly correct only when we neglect the dependence of $t^{(0)}$ on spin and isotopic spin. In the actual case we obtain a sum of such products. This will be discussed in Section 5.

where

$$F_{n0}(\mathbf{q}) = \int \rho_{n0}(\mathbf{r}) \exp(-i\mathbf{q} \cdot \mathbf{r}) d\mathbf{r} \quad (3.8)$$

with

$$\rho_{n0}(\mathbf{r}_1) = \int \Phi_n^*(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n) \Phi_0(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n) d\mathbf{r}_2, \dots, d\mathbf{r}_n \quad (3.9)$$

$$\int \rho_{n0}(\mathbf{r}) d\mathbf{r} = \delta_{n0} \quad (3.10)$$

and

$$\mathbf{q} = \mathbf{k}' - \mathbf{k} \quad (3.11)$$

so that $F(\mathbf{q})$ is a nuclear matrix element,⁷ its magnitude depending on the particu-

⁷ The expression $\langle n | (1/N) \sum_i \exp(-i\mathbf{q} \cdot \mathbf{r}_i) | 0 \rangle$ in (3.7a) is in fact of the form

$$\left\langle K'n \left| \frac{1}{N} \sum_i \exp(-i\mathbf{q} \cdot \mathbf{r}_i) \right| K0 \right\rangle, \quad (3.7b)$$

where K, K' are target momenta.

If the variables are changed to

$$\xi_i = \mathbf{r}_i - \mathbf{R}, \quad \mathbf{R} = \frac{1}{N} \sum \mathbf{r}_i,$$

then the N variables r_i go over to the $N + 1$ variables ξ_i, R which are not independent. They can be treated as independent if a δ -function is included in the integration, i.e.,

$$\int d\mathbf{r}, \dots, d\mathbf{r}_N \rightarrow \int d\xi_1, \dots, d\xi_N d\mathbf{R} \delta\left(\frac{1}{N} \sum \xi_i\right)$$

and the matrix element (3.7b) becomes, apart from a momentum conservation δ function factor which has already been taken account of in phase space factors,

$$F_{n0}(q) = \int \rho_{n0}(\mathbf{r}) \exp(-i\mathbf{q} \cdot \mathbf{r}) d\mathbf{r}$$

with

$$\rho_{n0}(\mathbf{r}) = \int \Phi_n^*(r, z_1, \dots, z_{n-1}) \Phi_0(r, z_1, \dots, z_{n-1}) \delta\left[\frac{1}{N} \delta(r + \sum z)\right] dz_1, \dots, dz_{n-1}. \quad (3.9b)$$

Similarly

$$C_{00}(x, y) = \int dz_1, \dots, dz_{n-2} |\Phi(x, y, z_1, \dots, z_{n-2})|^2 \delta\left[\frac{1}{N} \delta(x + y + [\sum z])\right] \quad (3.13b) \\ - \rho_{00}(x) \rho_{00}(y).$$

These changes in definition make no difference in most applications. In the deuteron case,

lar states involved. For elastic scattering the matrix element $F_{00}(\mathbf{q})$ is simply the nuclear form factor, as measured for instance by electron scattering experiments, and $\rho_{00}(\mathbf{x}) \equiv \rho(\mathbf{x})$ is simply the nuclear density distribution normalized according to Eq. (3.10). The matrix elements F obey a sum rule, which is conveniently derived using Eq. (3.7a) and closure. The result is

$$\sum_n F_{0n}(\mathbf{q})F_{n0}(\mathbf{q}') = \frac{1}{N} F_{00}(\mathbf{q} + \mathbf{q}') + \left(1 - \frac{1}{N}\right) F_{00}(\mathbf{q})F_{00}(\mathbf{q}') \\ + \left(1 - \frac{1}{N}\right) \int d\mathbf{x} d\mathbf{y} C_{00}(\mathbf{x}, \mathbf{y}) \exp(-i\mathbf{q} \cdot \mathbf{x} - i\mathbf{q}' \cdot \mathbf{y}), \quad (3.12)$$

where C is the nuclear pair correlation function defined by the equation

$$C_{00} = \int d\mathbf{z}_1 d\mathbf{z}_2 \cdots d\mathbf{z}_{N-2} |\Phi_0^*(\mathbf{x}, \mathbf{y}, \mathbf{z}_1, \dots, \mathbf{z}_{N-2})|^2 - \rho_{00}(\mathbf{x})\rho_{00}(\mathbf{y}). \quad (3.13)$$

Thus

$$\sum_n |F_{n0}(\mathbf{q})|^2 = \sum_n F_{0n}(\mathbf{q})F_{n0}(-\mathbf{q}) \\ = \frac{1}{N} + \left(1 - \frac{1}{N}\right) F_{00}(\mathbf{q})F_{00}(-\mathbf{q}) + \left(1 - \frac{1}{N}\right) C_{00}(\mathbf{q}, -\mathbf{q}), \quad (3.12a)$$

where $C_{00}(\mathbf{k}, \mathbf{k}')$ is the Fourier transform of the pair correlation function. The two body scattering matrix $\langle \mathbf{k}' | t^{(0)} | \mathbf{k} \rangle$ which is an invariant, is proportional to the free two nucleon scattering amplitude M . As M is usually given in terms of the phase shifts in the two-nucleon center-of-mass system we have (2.14) the relation

$$M(\mathbf{q}) = -\frac{(2\pi)^2 m}{\hbar^2} \frac{1}{2} \langle \mathbf{k}' | t^{(0)} | \mathbf{k} \rangle. \quad (3.14)$$

For the free two-nucleon case, however, the relation between $|\mathbf{k}'|$ and $|\mathbf{k}|$ is fixed by the two-particle energy-momentum conservation laws and changes with the coordinate system. In our case the relation between $|\mathbf{k}'|$ and $|\mathbf{k}|$ is fixed by the kinematics of the nucleon-nucleus system, and so we are off the two-body

however, where everything is expressed in terms of the internal wave function of the deuteron, we find that

$$F_{00}^D(\mathbf{q}) = \int |\Phi_D(\mathbf{r})|^2 \exp(-i\mathbf{q} \cdot \mathbf{r}/2) d\mathbf{r}, \quad (3.13c)$$

$$C_{00}^D(\mathbf{q}, \mathbf{q}') = \int C_{00}^D(\mathbf{x}, \mathbf{y}) \exp(i\mathbf{q} \cdot \mathbf{x}) \exp(-i\mathbf{q}' \cdot \mathbf{y}) d\mathbf{x} d\mathbf{y} = F_{00}^D(\mathbf{q} + \mathbf{q}') - F_{00}^D(\mathbf{q})F_{00}^D(\mathbf{q}'), \quad (3.13d)$$

where $\Phi_D(\mathbf{r})$ is the deuteron wave function in relative coordinates.

energy shell. Put another way $\langle t^{(0)} \rangle$ is a function of k^2, q^2, k'^2 , where k'^2 is a certain function of k^2, q^2 for free two-body scattering, and a different function of these variables for bound scattering. In relating $\langle t^{(0)} \rangle$ to the free scattering amplitude we are ignoring the dependence on k'^2 . This is strictly correct at small forward angles.

Consequently in the impulse approximation, and with the multiple scattering approximation Eqs. (3.4), (3.7), and (3.14) give

$$U_{n0} \cong -\frac{1}{(2\pi)^2} \frac{2\hbar^2}{m} (N-1) \omega_{nn}^L M(\mathbf{q}) F_{n0}(\mathbf{q}) \quad (3.15)$$

and

$$U_{00} \simeq U_{00}^{(0)} \cong -\frac{1}{(2\pi)^2} \frac{2\hbar^2}{m} (N-1) M(\mathbf{q}) F_{00}(\mathbf{q}), \quad (3.16)$$

which gives the potential matrix in terms of known quantities.

The Born approximation for the scattering amplitude T will be denoted by G_B . Using relation (2.17) for the connection between the scattering amplitude, and the scattering matrix, we find that, for elastic scattering in the center-of-mass system,

$$G_B(q) = \frac{N}{N+1} 2NM(q)F(q), \quad (3.17)$$

where we have used $T = (N/N-1)T'$. We shall sometimes use $G_B(q)$ in place of the approximate potential matrix $U_{00}^{(0)}$, to which it is proportional.

We will now discuss briefly corrections to the impulse and multiple scattering approximations. First of all let us look at the impulse approximation. From Eq. (2.24) we have

$$v = t \frac{1}{1 + \frac{1}{\hbar} t} = \frac{1}{1 + t \frac{1}{\hbar}} t, \quad (3.18)$$

where

$$\hbar = E - K_0 - K_1 + i\epsilon \quad (3.19)$$

We can now substitute the second of these relations in Eq. (2.23), eliminating v in the integral equation for τ , and obtain

$$\tau = t + t \left(\frac{1}{\alpha} - \frac{1}{\hbar} \right) \tau, \quad (3.20)$$

which is a new integral equation for τ . In general it is as difficult to solve as the original equation for the scattering matrix. However, for high energies it is suffi-

cient to use perturbation theory. The lowest order corrections to τ are then

$$\tau \cong t + t_1 + t_2 = t + \Delta\tau, \quad (3.21)$$

where

$$t_1 = P \left\{ t \left(\frac{1}{\alpha} - \frac{1}{h} \right) t \right\} \quad (3.22)$$

and

$$t_2 = -i\pi \{ t\delta(\alpha)t - t\delta(h)t \}, \quad (3.23)$$

where P stands for principal value. The propagators α and h differ in two respects. First $1/\alpha$ includes the antisymmetrization operator α (2.21).

Further, H_N appears in α , whereas K_1 appears in h . In order to estimate the effect of the antisymmetrization on $\Delta\tau$, we shall assume $H_N \approx K_1$. In the limit of high energy both H_N and K_1 can probably be neglected, but the conditions under which this approximation holds are not well understood. Confining ourselves to the term t_2 , which is on the energy shell, and likely to be of more importance at high energies, we note that if this term had been of the form

$$t'_2 = -i\pi \{ t^+ \delta(\alpha)t - t^+ \delta(h)t \}, \quad (3.24)$$

it would be immediately interpretable. For by the optical theorem (2.18)

$$\langle 0 | t^+ \delta(h)t | 0 \rangle \cong -\frac{1}{\pi} \operatorname{Im} t_{00} \quad (3.25)$$

for small momentum transfer, and is then proportional to the free two-nucleon total cross section. To estimate t'_2 we will approximate the nucleus by a Fermi gas. The first term in Eq. (3.24), differs from the second only by virtue of the fact that the free particle propagator is in the presence of a Fermi gas. The only difference is then that small momentum transfers in intermediate states are forbidden by the Pauli principle and it has been shown by Goldberger (15) that in this case

$$t'_2 \approx \frac{7}{5} \frac{E_F}{E} i \operatorname{Im} t, \quad (3.26)$$

where E_F is the Fermi energy of nuclear matter and E is the incident energy. As t is not Hermitian, t_2 is not precisely of this form (3.24), but will be of the same order to magnitude, and with a phase factor which has to be determined by calculation.⁸ Thus

$$\frac{\Delta\tau}{\tau} \sim \frac{\langle K_i \rangle}{E}, \quad (3.27)$$

⁸ In fact, in the particular case where the integration on angles implied by the δ function is not important t_2 and t'_2 differ *only* by a phase.

where $\langle K_i \rangle$ is the average kinetic energy of a particle in the nucleus. The ratio $\langle K_i \rangle / E$ is of the order of a few percent at high energies, so that $\Delta\tau$ is probably amenable to perturbation treatment.

Let us look at the correction to τ in a little more detail. We have

$$\begin{aligned} \Delta\tau = t \left\{ \frac{1}{\alpha} - \frac{1}{h} \right\} t &= \frac{2m}{\hbar^2} \int d\mathbf{k}'' t^{(0)}(\mathbf{k}', \mathbf{k}'') t^{(0)}(\mathbf{k}'', \mathbf{k}) \\ &\times \left\{ \sum_n \frac{\langle 0 | \exp\{-i(\mathbf{k}' - \mathbf{k}'') \cdot \mathbf{r}_1\} | n \rangle \langle n | \exp\{-i(\mathbf{k}'' - \mathbf{k}) \cdot \mathbf{r}_1\} | 0 \rangle}{k^2 - k''^2 - p_n^2 + i\epsilon} \right. \\ &\left. - \int d\mathbf{p} \frac{\langle 0 | \exp\{-i(\mathbf{k}' - \mathbf{k}'') \cdot \mathbf{r}_1\} | \mathbf{p} \rangle \langle \mathbf{p} | \exp\{-i(\mathbf{k}'' - \mathbf{k}) \cdot \mathbf{r}_1\} | 0 \rangle}{k^2 - k''^2 - p^2 + i\epsilon} \right\}, \end{aligned} \quad (3.28)$$

where $|\mathbf{p}\rangle$ stands for the plane wave final states of the target particle, which form a complete set for the operator $1/h$, and $k^2 = 2mE/\hbar^2$, $p_n^2 = 2m\varepsilon_n/\hbar^2$.

In the second term particles in intermediate states are distinguishable, so that $|\mathbf{p}\rangle$ refers to the state of particle 1.

Let us examine the expression in brackets when p_n^2 and p^2 are neglected compared to k^2 . Then the numerator of the first term can be written,

$$\sum_n F_{0n}(\mathbf{k}' - \mathbf{k}'') F_{n0}(\mathbf{k}'' - \mathbf{k})$$

and becomes, using (3.12),

$$\begin{aligned} \frac{1}{N} F_{00}(\mathbf{k}' - \mathbf{k}) + \left(1 - \frac{1}{N}\right) F_{00}(\mathbf{k}' - \mathbf{k}'') F_{00}(\mathbf{k}'' - \mathbf{k}) \\ + \left(1 - \frac{1}{N}\right) C_{00}(\mathbf{k}' - \mathbf{k}'', \mathbf{k}'' - \mathbf{k}'). \end{aligned} \quad (3.29)$$

Here $C_{00}(\mathbf{p}, \mathbf{p}')$ is the Fourier transform of the nuclear pair correlation function

$$C_{00}(\mathbf{p}, \mathbf{p}') = \int \exp(-i\mathbf{p} \cdot \mathbf{x}) \exp(-i\mathbf{p}' \cdot \mathbf{y}) C_{00}(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y}. \quad (3.30)$$

The pair correlation function is defined by Eq. (3.13), or more exactly (3.13b) and obeys the integral condition

$$\int C_{00}(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y} = 0. \quad (3.30a)$$

On the other hand the numerator of the second expression becomes, because of the distinguishability of particles in intermediate states

$$\begin{aligned} \int d\mathbf{p} \langle 0 | \exp\{-i(\mathbf{k}' - \mathbf{k}'') \cdot \mathbf{r}_1\} | \mathbf{p}(1) \rangle \langle \mathbf{p}(1) | \exp\{-i(\mathbf{k}'' - \mathbf{k}) \cdot \mathbf{r}_1\} | 0 \rangle \\ = F_{00}(\mathbf{k}' - \mathbf{k}). \end{aligned} \quad (3.31)$$

Consequently in closure approximation (3.28) becomes

$$\begin{aligned} \Delta\tau \cong & \frac{2m}{\hbar^2} \int d\mathbf{k}'' \frac{t^{(0)}(\mathbf{k}', \mathbf{k}'') t^{(0)}(\mathbf{k}'', \mathbf{k})}{(k^2 + i\epsilon - k''^2)} \\ & \times \left(\frac{N-1}{N} \right) \{ F_{00}(\mathbf{k}' - \mathbf{k}'') F_{00}(\mathbf{k}'' - \mathbf{k}) - F_{00}(\mathbf{k}' - \mathbf{k}) \quad (3.32) \\ & + C_{00}(\mathbf{k}' - \mathbf{k}'', \mathbf{k}'' - \mathbf{k}) \}. \end{aligned}$$

It would be of interest to investigate this correction carefully with Fermi gas wave functions, but also, for example, with shell model wave functions in a light nucleus like C^{12} .

A somewhat more accurate estimate could be made using closure on the integral equation (3.20) for τ and the anzats $\tau_{nn'} = F_{nn'}\tau^{(0)}$. Then

$$\begin{aligned} F_{00}(\mathbf{k}' - \mathbf{k})\tau^{(0)}(\mathbf{k}', \mathbf{k}) = & F_{00}(\mathbf{k}' - \mathbf{k})t^{(0)}(\mathbf{k}', \mathbf{k}) \\ & + \frac{2m}{\hbar^2} \int d\mathbf{k}'' \frac{t^{(0)}(\mathbf{k}', \mathbf{k}'')\tau^{(0)}(\mathbf{k}'', \mathbf{k})}{(k^2 + i\epsilon - k''^2)} \quad (3.33) \\ & \times \left(\frac{N-1}{N} \right) \{ F_{00}(\mathbf{k}' - \mathbf{k}'') F_{00}(\mathbf{k}'' - \mathbf{k}) - F_{00}(\mathbf{k}' - \mathbf{k}) \} \\ & + C_{00}(\mathbf{k}' - \mathbf{k}'', \mathbf{k}'' - \mathbf{k}) \}. \end{aligned}$$

The correction $\Delta\tau$ gives a first-order correction to the potential matrix, δ_U (3.2)

$$\delta_U = (N-1)\Delta\tau. \quad (3.34)$$

It is interesting to note that, using (3.13d), this vanishes for the deuteron. In general we can expect it to be small compared with corrections to the potential matrix due to virtual nuclear excitation in intermediate states (corrections to the multiple scattering approximation) (see 3.39 below).

Let us consider the corrections to the multiple scattering approximation. To first order we have from (2.63) for the elastic potential matrix

$$\delta U_{00} = \sum_{n \neq 0} U_{0n}^{(0)} \frac{1}{\alpha_n - U_{nn}^{(0)}} U_{n0}^{(0)} + \dots \quad (3.35)$$

Using a closure approximation, together with the relation

$$U_{n0}^{(0)} \cong (N-1)t_{n0} = (N-1) \langle \mathbf{k}' | t^{(0)} | \mathbf{k} \rangle \left\langle n \left| \frac{1}{N} \sum_i \exp(-i\mathbf{q} \cdot \mathbf{r}_i) \right| 0 \right\rangle, \quad (3.36)$$

where $\mathbf{q} = \mathbf{k}' - \mathbf{k}$, we find that

$$\begin{aligned} \delta U_{00} = \delta U_{00}(\mathbf{k}, \mathbf{q}) = & (N-1)^2 \int d\mathbf{k}'' \frac{t^{(0)}(\mathbf{k}', \mathbf{k}'') t^{(0)}(\mathbf{k}'', \mathbf{k})}{\langle \alpha - U_{00}^{(0)} \rangle} \quad (3.37) \\ & \cdot G_{00}(\mathbf{k}' - \mathbf{k}'', \mathbf{k}'' - \mathbf{k}), \end{aligned}$$

where $\langle \alpha - U^{(0)} \rangle$ represents an average defined by closure, and

$$\begin{aligned} G_{00}(\mathbf{k}' - \mathbf{k}'', \mathbf{k}'' - \mathbf{k}) &= \left(1 - \frac{1}{N}\right) C_{00}(\mathbf{k}' - \mathbf{k}'', \mathbf{k}'' - \mathbf{k}) \\ &\quad + \frac{1}{N} \{F_{00}(\mathbf{k}' - \mathbf{k}) - F_{00}(\mathbf{k}' - \mathbf{k}'')F_{00}(\mathbf{k}'' - \mathbf{k})\}. \end{aligned} \quad (3.38)$$

If we neglect the influence of the potential $U^{(0)}$ and also the nuclear excitation in the denominator of (3.37), i.e., write

$$\langle \alpha - U_D^{(0)} \rangle \approx (\hbar^2/2m)(k^2 - k''^2 + i\epsilon),$$

then this correction is very similar in form to that given by the term (3.34) arising from corrections to the impulse approximation. In fact if we look only at the term arising from the pair correlation function, then in the forward direction

$$\delta U_{00} \approx (N - 1)\delta_\tau U_{00} \quad (3.39)$$

and so is in general much the larger correction. [The parts depending on the form factor F in δU_{00} and $\delta_\tau U$ largely cancel.]

In point of fact $U_D^{(0)}$ is not a diagonal matrix in anything except the nuclear states, and the denominator in (3.37) is more easily handled by using the relation

$$\frac{1}{\alpha - U_D^{(0)}} = \omega^R \frac{1}{\alpha}. \quad (2.57)$$

Then (3.37) becomes

$$\begin{aligned} \delta U_{00} &= (N - 1)^2 \int \frac{d\mathbf{k}'' d\mathbf{k}''' t^{(0)}(\mathbf{k}', \mathbf{k}'') \tilde{\omega}_{nn}^R(\mathbf{k}'', \mathbf{k}''') t^{(0)}(\mathbf{k}''', \mathbf{k})}{E - \bar{E}(\mathbf{k}'''^2) + i\epsilon} \\ &\quad \cdot G_{00}(\mathbf{k}' - \mathbf{k}'', \mathbf{k}''' - \mathbf{k}), \end{aligned} \quad (3.40)$$

where $\tilde{\omega}_{nn}^R$ is the wave matrix for elastic scattering in an average excited nuclear state, and $\bar{E}(k'''^2)$ includes some average nuclear excitation energy.

And so δU_{00} is proportional, apart from terms $\sim \sigma(1/N)$ in G_{00} , to the two-body correlation term as expected. Again this term is proportional to $1/E$, but its actual magnitude depends on $t^{(0)}$ and the two particle correlation function. Numerical estimates will be given later (Section 7).

A better estimate of δU_{00} might possibly be obtained using closure on the integral equation for W^L (2.58) with the approximation

$$W_{nn'}^L(\mathbf{k}'', \mathbf{k}) \simeq F_{nn'}(\mathbf{k}'' - \mathbf{k}) u(\mathbf{k}'', \mathbf{k}). \quad (3.41)$$

Then using (2.54) and (2.53c) we have

$$\begin{aligned} U_{00}(\mathbf{k}', \mathbf{k}) &\cong (N - 1)F_{00}(\mathbf{k}' - \mathbf{k})u(\mathbf{k}', \mathbf{k}) \\ &= (N - 1)F_{00}(\mathbf{k}' - \mathbf{k})t^{(0)}(\mathbf{k}', \mathbf{k}) + (N - 1)^2 \int d\mathbf{k}'' d\mathbf{k}''' \\ &\quad \cdot \frac{t^{(0)}(\mathbf{k}', \mathbf{k}'')\bar{\omega}_{nn}^R(\mathbf{k}'', \mathbf{k}''')u(\mathbf{k}''', \mathbf{k})}{E - \bar{E}(k''^2) + i\epsilon} G_{00}(\mathbf{k}' - \mathbf{k}'', \mathbf{k}''' - \mathbf{k}), \end{aligned} \quad (3.42)$$

an integral equation for the defined function $u(\mathbf{k}', \mathbf{k})$ which then gives U_{00} .

4. ELASTIC SCATTERING AND THE OPTICAL POTENTIAL

We shall first investigate the elastic scattering of nucleons on nuclei at high energies. Calculations were performed at the typical laboratory energies of 310, 156, and 90 Mev primarily because both $p-p$ and $n-p$ phase shifts are available in the literature at these energies. The two-nucleon scattering phase shifts employed in the calculations reported below are given in Table I, and from these the two-body scattering amplitude $M(q)$ in the two-body center-of-mass system was calculated. The two-body scattering amplitude is a function of the spin and isotopic spin of the particles. Assuming charge independence and invariance against space and time reflection (16)

$$M = A + B\sigma_1\hat{n}\sigma_2\hat{n} + C(\sigma_1\hat{n} + \sigma_2\hat{n}) + E\sigma_1\hat{q}\sigma_2\hat{q} + F\sigma_1\hat{p}\sigma_2\hat{p}, \quad (4.1)$$

where $[\hat{q}, \hat{n}, \hat{p}]$ are unit vectors which form a right-handed coordinate system, defined by

$$\begin{aligned} \hat{q} &= \mathbf{q}/|\mathbf{q}| & q &= \mathbf{k}_0' - \mathbf{k}_0 \\ \hat{n} &= \mathbf{n}/|\mathbf{n}| & n &= \mathbf{k}_0 \times \mathbf{k}_0' \\ \hat{p} &= \hat{q} \times \hat{n} \end{aligned} \quad (4.2)$$

and \mathbf{k}_0 and \mathbf{k}_0' are the momenta in the two-body center-of-mass system. The scattering coefficients A, B, C, E, F depend strongly on the momentum transfer \mathbf{q}^2 and are also functions of isotopic spin. For example,

$$A = \frac{1}{4}(3A_1 + A_0) + \frac{1}{4}(A_1 - A_0)\mathbf{\tau}_1 \cdot \mathbf{\tau}_2, \quad (4.3)$$

where A_1 is the coefficient for the triplet isotopic spin state and A_0 is the coefficient for the singlet state.

The scattering coefficients actually depend on three variables, q^2 , $(p^2 + q^2) = (2m/\hbar^2)(E' + E)$ and $\mathbf{q} \cdot \mathbf{p} = (m/\hbar^2)(E' - E)$, where $\mathbf{p} = (\mathbf{k}_0 + \mathbf{k}_0')$. Note that $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$ only on the energy shell where $\mathbf{q} \cdot \mathbf{p} = 0$ and $(p^2 + q^2)$ is pro-

TABLE I
NUCLEON-NUCLEON PHASE SHIFTS^a

	$E = 40$ Mev	$E = 90$ Mev	$E = 156$ Mev	$E = 310$ Mev
δ_0	0.6865	0.422	0.1875	-0.1763
δ_2	0.0230	0.084	0.1585	0.2243
δ_4	0	0.004	0.0145	0.01745
δ_{11}	-0.07685	-0.237	-0.3185	-0.4651
δ_{33}	.0	-0.033	-0.053	-0.07679
δ_{55}	0	-0.006	-0.015	0.00175
δ_{10}	0.4358	0.1705	0.0465	-0.2496
δ_{12}	0.08715	0.1815	0.2435	0.282
δ_{32}	0	0.0055	0.0175	0.0130
δ_{34}	0	0.0235	0.0425	0.0651
δ_{54}	0	-0.0055	-0.007	0.0152
ϵ_2	-0.2699	-0.308	-0.2045	-0.0625
ϵ_4	0	-0.6135	-0.586	-0.467
δ_1	-0.1589	-0.240	-0.3145	-0.550
δ_3	-0.0088	-0.027	-0.0505	-0.129
δ_5	0	-0.003	-0.0105	-0.0387
δ_{00}	0	0	0	0
δ_{22}	0.1443	0.3265	0.4545	0.389
δ_{44}	0.0120	0.021	0.054	0.0975
δ_{01}	1.1865	0.6455	0.2715	-0.258
δ_{21}	-0.1249	-0.230	-0.3475	-0.547
δ_{23}	-0.0447	0.067	0.1415	0.217
δ_{43}	0.0204	-0.0475	-0.078	-0.105
δ_{45}	-0.0071	-0.0035	-0.0685 ^b	-0.01635
ϵ_1	-0.0212	0.0995	0.161	0.610
ϵ_3	-0.726	0.609	0.505	0.363

^a These phase shifts are taken from Ref. 7. The phase shifts are nuclear Blatt-Biedenharn phase shifts. The triplet phase shifts are given as δ_{LJ} . The coupling constants are given as ϵ_{J0} . The notation is completely defined in Ref. 6.

^b The value of δ_{45} at 156 Mev appears to be anomalously large. This phase shift has been arbitrarily set equal to zero in the subsequent calculations.

portional to the energy.⁹ In the numerical calculations we shall only need the scattering coefficients A , etc., on the energy shell. The dependence of the scattering coefficients on the energy is weak compared to their variation with q^2 . For the time being we shall consider only the dependence on q^2 .

⁹ Actually there is a sixth term in the general expression (4.1) for M which has the form

$$D(\mathbf{q} \cdot \mathbf{p})(\sigma_1 \hat{\mathbf{p}} \sigma_2 \hat{\mathbf{q}} + \sigma_1 \hat{\mathbf{q}} \sigma_2 \hat{\mathbf{p}}). \quad (4.1a)$$

This vanishes on the energy shell because of the factor $(\mathbf{q} \cdot \mathbf{p})$ which is necessary in order that the term be time reversal invariant.

The scattering coefficients can be expressed in terms of the amplitudes M_{ij} , which refer to the scattering of a nucleon-nucleon system from triplet spin component j to component i , and of the singlet scattering amplitude M_{ss} . These in turn are given as a function of the phase shifts by Stapp (6). In the representation that the axis of quantization is given by the direction of the incident beam,

$$\begin{aligned} A &= \frac{1}{4}(2M_{11} + M_{00} + M_{ss}), \\ B &= \frac{1}{4}(M_{00} - M_{ss} - 2M_{1-1}), \\ C &= \frac{1}{4}i\sqrt{2}(M_{10} - M_{01}) \\ E + F = G &= \frac{1}{2}(M_{11} - M_{ss} + M_{1-1}), \\ E - F = H &= -\frac{1}{2 \sin \theta_0} \sqrt{2}(M_{10} + M_{01}) = \frac{1}{2 \cos \theta_0} (M_{00} - M_{11} + M_{1-1}), \end{aligned} \quad (4.4)$$

where θ_0 is the scattering angle in the two-body center-of-mass system. Table I contains the nucleon-nucleon phase shifts, calculated from the Gammel-Thaler potential (7). Tables II and III contain the corresponding scattering amplitudes M_{ij} and the scattering coefficients A , B , etc., in units of 10^{-13} cm as a function of the two-body center-of-mass scattering angle θ_0 .¹⁰ These do not contain the contributions from the Coulomb potential. The procedure adopted here is to calculate the optical model potential from the nuclear force part of the two-nucleon scattering amplitude, and then to add the Coulomb potential of the nuclear charge distribution. This procedure ignores the interference between the Coulomb amplitude and the nuclear amplitude in individual collisions.

Because M is spin dependent, $M(q)$ in Eq. (3.15) for the potential matrix must be replaced by $\bar{M}(q)$, where $\bar{M}(q)$ indicates a suitable average over the spin and isotopic spin of the nucleons in the nucleus. This will be discussed in Section 8. For the moment we may note that for an even-even nucleus with $J = 0$, and $T = 0$, as for instance C^{12} and O^{16} , terms linear in the spin and isotopic spin of the target nucleons average to zero, and we have

$$\bar{M}(q) = \bar{A}(q) + \bar{C}(q) \mathbf{s} \cdot \hat{\mathbf{n}}, \quad (4.5)$$

where

$$\begin{aligned} \bar{A} &= \frac{1}{4}(3A_1 + A_0), \\ \bar{C} &= \frac{1}{4}(3C_1 + C_0). \end{aligned} \quad (4.6)$$

¹⁰ On the energy shell p^2 and q^2 are related to θ_0 and k_0^2 by the equations

$$\begin{aligned} q^2 &= 4k_0^2 \sin^2 \frac{\theta_0}{2}, \\ p^2 &= 4k_0^2 \cos^2 \frac{\theta_0}{2}. \end{aligned} \quad (4.2a)$$

The vector $\hat{\mathbf{n}}$ is the unit vector normal to the scattering plane. For other nuclei, and for elastic scattering (4.5) and (4.6) are still approximately true, owing to the insensitivity of the coefficients \bar{A} and \bar{C} to neutron excess (see Table VIII for \bar{A} at zero angle). The potential matrix then becomes, following (3.15) and using the impulse and multiple scattering approximations,

$$U_{00}(\mathbf{k}', \mathbf{k}) \cong -\frac{1}{(2\pi)^2} \frac{2\hbar^2}{m} (N-1) F(q) \{ \bar{A}(q) + \bar{C}(q) \mathbf{s} \cdot \hat{\mathbf{n}} \}. \quad (4.7)$$

The potential matrix is now written as an operator in the spin of the scattered particle. The scattering matrix also will contain a spin-dependent term, and for an initially unpolarized beam the polarization normal to the scattering plane \mathbf{P} can be calculated from the relation

$$\mathbf{P} = \frac{\text{Tr}(T^+ \mathbf{s} \cdot \hat{\mathbf{n}} T)}{\text{Tr}(T^+ T)}. \quad (4.8)$$

In the Born approximation, i.e., $T' \approx U^{(0)}$, the scattering amplitude is (3.17)

$$G_B = \frac{2N^2}{(N+1)} F(q) \{ \bar{A}(q) + \bar{C}(q) \mathbf{s} \cdot \hat{\mathbf{n}} \}. \quad (4.9)$$

This gives the scattering amplitude in terms of momentum transfer, which is translated into the angle in the center of mass (θ) by the relation

$$q = 2k \sin(\theta/2). \quad (4.10)$$

In Table III, A and C are given in terms of the two-body center-of-mass angle θ_0 , which can be converted into (θ) by the equation

$$k_0 \sin(\theta_0/2) = k \sin(\theta/2), \quad \frac{k}{k_0} = \frac{2N}{N+1}. \quad (4.11)$$

The explicit relation between two-body center of mass, center of mass, and laboratory scattering angles, and the momentum transfer is given in Table XIV for several values of the incident energy.

Making this transformation, we have the Born scattering amplitude as a function of the center-of-mass scattering angle θ . It displays the effect of the nuclear form factor and of the variation of the two-nucleon potential with angle.

The polarization in the Born approximation is given by (4.8).

$$P_B = \frac{2\text{Re } \bar{A}\bar{C}^*}{|\bar{A}|^2 + |\bar{C}|^2}, \quad (4.12)$$

a result which is independent of the nuclear form factor, and can be expressed as a function of scattering angle using Table III, Eq. (4.11), and Table XIV. Thus in this approximation the polarization should be independent of atomic

TABLE II

The two nucleon scattering amplitudes M_{ij} . The M_{ij} are as defined (4.4) in Ref. 6 and represent the scattering amplitude for spin projection i in the incoming beam to spin projection j in the scattered wave.

θ_0	M_{ss}		M_{00}		M_{11}		M_{01}		M_{10}		M_{1-1}	
	Re	Im	Re	Im								
T = 0												E = 40 Mev
2	-1.526	0.218	0.517	2.429	1.532	2.741	0.0118	0.0086	0.0133	-0.0009	-0.0024	0.0001
4	-1.522	0.217	0.518	2.429	1.530	2.740	0.0237	0.0171	0.0269	-0.0018	-0.0097	0.0002
6	-1.514	0.217	0.520	2.429	1.528	2.737	0.0358	0.0255	0.0408	-0.0026	-0.0216	0.0005
8	-1.504	0.216	0.522	2.428	1.524	2.734	0.0483	0.0337	0.0551	-0.0034	-0.0381	0.0009
10	-1.491	0.214	0.524	2.428	1.520	2.730	0.0612	0.0418	0.0702	-0.0042	-0.0588	0.0014
20	-1.387	0.204	0.551	2.422	1.478	2.697	0.135	0.0770	0.159	-0.0076	-0.214	0.0050
30	-1.227	0.188	0.608	2.415	1.402	2.648	0.224	0.1006	0.269	-0.0095	-0.412	0.0095
40	-1.030	0.166	0.704	2.408	1.287	2.591	0.312	0.1101	0.382	-0.0097	-0.588	0.0134
60	-0.597	0.107	0.995	2.397	0.974	2.480	0.380	0.0889	0.475	-0.0065	-0.719	0.0158
80	-0.191	0.0372	1.251	2.393	0.714	2.414	0.177	0.0331	0.224	-0.0021	-0.637	0.0131
100	+0.191	-0.0372	1.251	2.393	0.714	2.414	-0.177	-0.0331	-0.224	+0.0021	-0.637	0.0131
120	0.597	-0.107	0.995	2.397	0.974	2.480	-0.380	-0.0889	-0.475	0.0065	-0.719	0.0158
140	1.030	-0.166	0.704	2.408	1.287	2.591	-0.312	-0.1101	-0.382	0.0097	-0.588	0.0134
160	1.387	-0.204	0.551	2.422	1.478	2.697	-0.135	-0.0770	-0.159	0.0076	-0.214	0.0050
T = 1												E = 40 Mev
2	1.743	1.165	1.305	0.531	0.192	0.0711	-0.0092	0.0005	-0.0183	-0.0119	0.0003	0.0000
4	1.741	1.165	1.303	0.530	0.191	0.0709	-0.0185	0.0011	-0.0366	-0.0238	0.0012	0.0001
6	1.738	1.165	1.301	0.528	0.189	0.0705	-0.0279	0.0016	-0.0560	-0.0356	0.0026	0.0002
8	1.734	1.165	1.297	0.526	0.186	0.0701	-0.0375	0.0021	-0.0734	-0.0474	0.0046	0.0004
10	1.728	1.164	1.293	0.524	0.182	0.0695	-0.0474	0.0025	-0.0919	-0.0592	0.0072	0.0006
20	1.685	1.163	1.255	0.501	0.154	0.0646	-0.1036	0.0041	-0.187	-0.117	0.0265	0.0023
30	1.619	1.162	1.186	0.465	0.114	0.0571	-0.174	0.0040	-0.285	-0.172	0.0522	0.0046
40	1.538	1.160	1.082	0.414	0.0710	0.0478	-0.260	0.0020	-0.386	-0.223	0.0762	0.0067
60	1.371	1.156	0.750	0.274	0.0058	0.0277	-0.449	-0.0059	-0.573	-0.305	0.0903	0.0079
80	1.261	1.154	0.270	0.0960	-0.0072	0.0088	-0.584	-0.0132	-0.691	-0.351	0.0406	0.0035
100	1.261	1.154	-0.270	-0.0960	+0.0072	-0.0088	-0.584	-0.0132	-0.691	-0.351	-0.0406	-0.0035
120	1.371	1.156	-0.750	-0.274	-0.0058	-0.0277	-0.449	-0.0059	-0.573	-0.305	-0.0903	-0.0079
140	1.538	1.160	-1.082	-0.414	-0.0710	-0.0478	-0.260	0.0020	-0.386	-0.223	-0.0762	-0.0067
160	1.685	1.163	-1.255	-0.501	-0.154	-0.0646	-0.1036	-0.0041	-0.187	-0.117	-0.0265	-0.0023

θ_0	M_{ss}				M_{00}				M_{11}				M_{01}				M_{10}				M_{1-1}				
	Re		Im		Re		Im		Re		Im		Re		Im		Re		Im		Re		Im		
	T = 0				E = 90 Mev				T = 1				E = 90 Mev				T = 0				E = 90 Mev				
2	-1.753	0.335	1.231	1.125	1.860	1.171	-0.0021	0.0132	0.0177	-0.0120	-0.0029	-0.0003	2	1.587	0.390	0.510	0.0985	-0.0025	0.306	-0.0204	0.0043	0.0077	0.0009	0.0017	0.0001
4	-1.745	0.335	1.227	1.122	1.855	1.168	-0.0040	0.0263	0.0356	-0.0240	-0.0117	-0.0011	4	1.582	0.389	0.512	0.0986	-0.0033	0.305	-0.0409	0.0085	0.0151	0.0017	0.0067	0.0003
6	-1.732	0.333	1.221	1.118	1.846	1.164	-0.0053	0.0393	0.0537	-0.0357	-0.0262	-0.0024	6	1.572	0.389	0.515	0.0987	-0.0047	0.304	-0.0620	0.0127	0.0222	0.0025	0.0149	0.0008
8	-1.713	0.332	1.212	1.113	1.834	1.157	-0.0060	0.0520	0.0723	-0.0473	-0.0462	-0.0042	8	1.564	0.381	0.517	0.0996	-0.0056	0.301	-0.0836	0.0167	0.0287	0.0033	0.0261	0.0014
10	-1.690	0.330	1.201	1.106	1.819	1.148	-0.0056	0.0645	0.0915	-0.0585	-0.0714	-0.0065	10	1.543	0.387	0.526	0.0990	-0.0089	0.299	-0.106	0.0206	0.0344	0.0040	0.0401	0.0021
20	-1.508	0.312	1.123	1.053	1.694	1.077	0.0162	0.120	0.198	-0.1074	-0.263	-0.0255	20	1.412	0.378	0.564	0.0998	-0.0263	0.276	-0.235	0.0370	0.0471	0.0064	0.140	0.0077
30	-1.256	0.235	1.034	0.975	1.497	0.970	0.0769	0.160	0.322	-0.139	-0.514	-0.0549	30	0.983	0.249	0.977	0.886	1.245	0.841	0.164	0.179	0.442	-0.151	-0.755	-0.0916
40	-0.983	0.249	0.977	0.886	1.245	0.841	0.164	0.179	0.442	-0.151	-0.755	-0.0916	60	-0.512	0.158	1.041	0.723	0.692	0.580	0.284	0.152	0.527	-0.119	-1.011	-0.169
80	-0.159	0.0542	1.214	0.633	0.306	0.415	0.148	0.0587	0.244	-0.0437	-1.006	-0.222	80	-0.159	-0.0542	1.214	0.633	0.306	0.415	-0.148	-0.0587	-0.244	+0.0437	-1.006	-0.222
100	0.159	-0.0542	1.214	0.633	0.306	0.415	0.415	-0.148	-0.0587	-0.244	-0.0437	-1.006	120	0.512	-0.158	1.041	0.723	0.692	0.580	-0.284	-0.152	-0.527	0.119	-1.011	-0.169
120	0.983	-0.249	0.977	0.886	1.245	0.841	-0.164	-0.179	-0.442	0.151	-0.755	-0.0916	140	0.983	-0.249	0.977	0.886	1.245	0.841	-0.164	-0.179	-0.442	0.151	-0.755	-0.0916
160	1.508	-0.312	1.123	1.053	1.694	1.077	-0.162	-0.120	-0.198	0.1074	-0.263	-0.0255	160	1.412	0.378	0.564	-0.564	-0.564	-0.276	-0.235	-0.370	-0.471	-0.064	0.249	0.0147

TABLE II (Continued)

θ_0	M_{ss}		M_{00}		M_{11}		M_{01}		M_{10}		M_{1-1}		
	Re	Im	Re	Im									
T = 0												$E = 310 \text{ Mev}$	
2	-2.739	0.984	0.0894	0.553	0.530	0.867	-0.0246	0.0203	0.0356	-0.0125	-0.0036	0.0000	
4	-2.714	0.980	0.0876	0.549	0.524	0.863	-0.0488	0.0405	0.0711	-0.0249	-0.0144	0.0000	
6	-2.674	0.975	0.0846	0.541	0.513	0.856	-0.0723	0.0604	0.1066	-0.0370	-0.0322	0.0001	
8	-2.618	0.967	0.0805	0.531	0.498	0.847	-0.0948	0.0800	0.142	-0.0487	-0.0567	0.0002	
10	-2.548	0.958	0.0752	0.518	0.479	0.834	-0.116	0.0992	0.177	-0.0599	-0.0875	0.0003	
20	-2.033	0.881	0.0337	0.419	0.331	0.736	-0.191	0.184	0.348	-0.1034	-0.314	0.0022	
30	-1.400	0.769	-0.0272	0.281	0.118	0.592	-0.201	0.245	0.500	-0.1207	-0.586	0.0080	
40	-0.852	0.639	-0.0970	0.133	-0.118	0.426	-0.151	0.273	0.612	-0.1112	-0.798	0.0201	
60	-0.327	0.373	-0.225	-0.0915	-0.524	0.117	0.0086	0.230	0.617	-0.0504	-0.810	0.0622	
80	-0.135	0.122	-0.296	-0.181	-0.728	-0.0585	0.0398	0.0882	0.264	-0.0077	-0.508	0.1021	
100	0.135	-0.122	-0.296	-0.181	-0.728	-0.0585	-0.0398	-0.0882	-0.264	0.0077	-0.508	0.1021	
120	0.327	-0.373	-0.225	-0.0915	-0.524	0.117	-0.0086	-0.230	-0.617	0.0504	-0.810	0.0622	
140	0.852	-0.639	-0.070	0.133	-0.118	0.426	0.151	-0.273	-0.612	0.1112	-0.798	0.0201	
160	2.033	-0.881	0.0337	0.419	0.331	0.736	0.191	-0.184	-0.348	0.1034	-0.314	0.0022	
T = 1												$E = 310 \text{ Mev}$	
2	1.103	0.290	0.438	0.204	-0.135	0.484	-0.0286	0.0057	0.0144	0.0012	0.0013	0.0000	
4	1.093	0.289	0.435	0.204	-0.137	0.483	-0.0574	0.0114	0.0288	0.0024	0.0051	0.0001	
6	1.078	0.286	0.430	0.204	-0.141	0.480	-0.0865	0.0170	0.0433	0.0035	0.0114	0.0003	
8	1.058	0.283	0.423	0.203	-0.145	0.477	-0.116	0.0225	0.0577	0.0046	0.0200	0.0006	
10	1.031	0.279	0.415	0.202	-0.150	0.473	-0.146	0.0279	0.0722	0.0057	0.0307	0.0009	
20	0.823	0.244	0.349	0.195	-0.184	0.440	-0.309	0.0525	0.145	0.0100	0.1068	0.0027	
30	0.526	0.192	0.264	0.181	-0.205	0.391	-0.483	0.0719	0.214	0.0120	0.189	0.0040	
40	0.196	0.128	0.182	0.162	-0.189	0.332	-0.634	0.0869	0.271	0.0118	0.240	0.0035	
60	-0.366	-0.0010	0.0748	0.1068	-0.0609	0.202	-0.728	0.1113	0.326	0.0071	0.193	-0.0017	
80	-0.646	-0.0838	0.0224	0.0370	0.0112	0.0677	-0.600	0.1301	0.316	0.0030	0.0597	-0.0025	
100	-0.646	-0.0838	-0.0224	-0.0370	-0.0112	-0.0677	-0.600	0.1301	0.316	0.0030	-0.0597	0.0025	
120	-0.366	-0.0010	-0.0748	-0.1068	0.0609	-0.202	-0.728	0.1113	0.326	0.0071	-0.193	0.0017	
140	0.196	0.128	-0.182	-0.162	0.189	-0.332	-0.634	0.0869	0.271	0.0118	-0.240	-0.0035	
160	0.823	0.244	-0.349	-0.195	0.184	-0.440	-0.309	0.0525	0.145	0.0100	-0.107	-0.0027	

TABLE III

The scattering coefficients A, B, C, E, F for T = 0, T = 1 are given as a function of scattering angle in the two-body center of mass system. No Coulomb effects have been included. The scattering coefficients are as defined (4.1) in Ref. 16, and are given in units of 10⁻¹³ cm.

θ_0	A		B		C		E		F	
	Re	Im	Re	Im	Re	Im	Re	Im	Re	Im
[T = 0]										
2	0.514	2.032	0.512	0.553	0.0033	0.0006	0.509	0.553	1.018	0.709
4	0.514	2.031	0.515	0.553	0.0067	0.0011	0.504	0.553	1.017	0.708
6	0.515	2.030	0.519	0.553	0.0099	0.0018	0.496	0.553	1.014	0.708
8	0.517	2.028	0.525	0.553	0.0132	0.0024	0.485	0.553	1.010	0.707
10	0.518	2.025	0.533	0.553	0.0163	0.0032	0.470	0.553	1.006	0.706
20	0.530	2.005	0.592	0.552	0.0299	0.0082	0.359	0.553	0.966	0.696
30	0.546	1.975	0.665	0.552	0.0389	0.0158	0.206	0.553	0.902	0.682
40	0.562	1.939	0.727	0.554	0.0423	0.0246	0.0502	0.554	0.814	0.665
60	0.586	1.866	0.758	0.565	0.0337	0.0337	-0.136	0.563	0.562	0.631
80	0.622	1.814	0.679	0.582	0.0124	0.0164	-0.0769	0.586	0.211	0.609
100	0.717	1.796	0.583	0.601	-0.0124	-0.0164	0.116	0.627	-0.172	0.605
120	0.885	1.812	0.459	0.618	-0.0337	-0.0337	0.263	0.684	-0.435	0.617
140	1.077	1.856	0.213	0.637	-0.0423	-0.0246	0.299	0.748	-0.465	0.637
160	1.244	1.903	-0.102	0.654	-0.0299	-0.0082	0.273	0.798	-0.334	0.655
[T = 1]										
2	0.858	0.459	-0.110	-0.159	0.0044	-0.0032	-0.1092	-0.158	-0.666	-0.388
4	0.856	0.459	-0.110	-0.159	0.0088	-0.0064	-0.1081	-0.158	-0.666	-0.388
6	0.854	0.459	-0.111	-0.159	0.0132	-0.0096	-0.1064	-0.158	-0.667	-0.389
8	0.851	0.458	-0.111	-0.160	0.0175	-0.0127	-0.1040	-0.158	-0.668	-0.389
10	0.846	0.457	-0.112	-0.161	0.0218	-0.0157	-0.1009	-0.158	-0.669	-0.389
20	0.812	0.448	-0.121	-0.167	0.0429	-0.0293	-0.0762	-0.157	-0.676	-0.391
30	0.759	0.435	-0.134	-0.177	0.0623	-0.0392	-0.0386	-0.156	-0.688	-0.394
40	0.690	0.417	-0.152	-0.190	0.0796	-0.0446	+0.0071	-0.155	-0.702	-0.398
60	0.533	0.371	-0.200	-0.224	0.1059	-0.0439	0.0986	-0.153	-0.736	-0.407
80	0.379	0.317	-0.268	-0.266	0.1194	-0.0379	0.1510	-0.155	-0.765	-0.416
100	0.251	0.260	-0.363	-0.311	0.1194	-0.0379	0.1343	-0.161	-0.782	-0.422
120	0.152	0.207	-0.485	-0.354	0.1059	-0.0439	0.0506	-0.171	-0.784	-0.425
140	0.0784	0.163	-0.617	-0.390	0.0796	-0.0446	-0.0665	-0.182	-0.776	-0.425
160	0.0305	0.133	-0.722	-0.415	0.0429	-0.0293	-0.167	-0.191	-0.766	-0.424

E = 40 Mev

θ_0	A		B		C		E		F	
	Re	Im	Re	Im	Re	Im	Re	Im	Re	Im
			$[3/4(T+1) + 1/4(T=0)]$				$E = 40 \text{ Mev}$			
$[T = 0]$										
2	0.772	0.853	0.0458	0.0194	0.0041	-0.0023	0.0454	0.0194	-0.245	-0.114
4	0.771	0.852	0.0462	0.0191	0.0082	-0.0045	0.0450	0.0194	-0.246	-0.114
6	0.769	0.851	0.0469	0.0188	0.0124	-0.0068	0.0442	0.0195	-0.247	-0.115
8	0.767	0.850	0.0478	0.0183	0.0164	-0.0089	0.0432	0.0195	-0.248	-0.115
10	0.764	0.849	0.0490	0.0178	0.0204	-0.0110	0.0419	0.0196	-0.250	-0.115
20	0.742	0.838	0.0573	0.0130	0.0396	-0.0199	0.0326	0.0202	-0.265	-0.119
30	0.706	0.820	0.0655	0.0056	0.0565	-0.0254	0.0226	0.0212	-0.290	-0.125
40	0.658	0.798	0.0678	-0.0039	0.0703	-0.0273	0.0179	0.0226	-0.323	-0.132
60	0.546	0.745	0.0392	-0.0272	0.0878	-0.0245	0.0399	0.0261	-0.411	-0.148
80	0.440	0.691	-0.0314	-0.0540	0.0926	-0.0243	0.0940	0.0305	-0.521	-0.160
100	0.368	0.644	-0.126	-0.0828	0.0864	-0.0325	0.1296	0.0362	-0.629	-0.165
120	0.335	0.608	-0.249	-0.1106	0.0710	-0.0413	0.1038	0.0430	-0.697	-0.164
140	0.328	0.586	-0.410	-0.133	0.0491	-0.0396	0.0250	0.0504	-0.698	-0.160
160	0.329	0.576	-0.567	-0.148	0.0247	-0.0241	-0.0566	0.0566	-0.658	-0.155
$[T = 90]$										
2	0.799	0.951	0.748	0.197	0.0089	0.0070	0.745	0.197	1.061	0.221
4	0.798	0.948	0.749	0.197	0.0178	0.0140	0.737	0.196	1.057	0.220
6	0.795	0.945	0.751	0.197	0.0265	0.0209	0.724	0.195	1.052	0.219
8	0.792	0.940	0.754	0.197	0.0351	0.0277	0.707	0.193	1.044	0.217
10	0.787	0.933	0.759	0.197	0.0435	0.0343	0.684	0.191	1.034	0.215
20	0.751	0.880	0.789	0.198	0.0804	0.0644	0.513	0.172	0.957	0.198
30	0.693	0.800	0.830	0.200	0.1058	0.0866	0.277	0.143	0.841	0.172
40	0.621	0.704	0.867	0.205	0.1166	0.0982	+0.0354	0.110	0.702	0.140
60	0.478	0.511	0.894	0.226	0.0960	0.0858	-0.293	0.0498	+0.379	0.0764
80	0.377	0.379	0.846	0.256	0.0362	0.0337	-0.181	0.0294	-0.129	0.0402
100	0.494	0.352	0.766	0.283	-0.0362	-0.0337	-0.0807	0.0673	-0.350	0.0565
120	0.734	0.431	0.638	0.305	-0.0960	-0.0858	+0.123	0.156	-0.539	0.129
140	1.113	0.580	0.376	0.329	-0.1166	-0.0982	0.211	0.265	-0.457	0.234
160	1.505	0.724	0.0351	0.354	-0.0804	-0.0644	0.203	0.354	-0.241	0.328

TABLE III (Continued)

θ_0	A		B		C		E		F	
	Re	Im	Re	Im	Re	Im	Re	Im	Re	Im
[T = 1]										
							E = 90 Mev			
2	0.523	0.275	-0.270	-0.0729	0.0012	0.0099	-0.268	-0.0728	-0.525	0.0310
4	0.522	0.275	-0.271	-0.0729	0.0024	0.0198	-0.264	-0.0727	-0.525	0.0307
6	0.520	0.274	-0.272	-0.0729	0.0036	0.0298	-0.256	-0.0724	-0.525	0.0302
8	0.517	0.272	-0.273	-0.0730	0.0047	0.0397	-0.245	-0.0721	-0.525	0.0295
10	0.513	0.271	-0.274	-0.0730	0.0059	0.0497	-0.232	-0.0717	-0.524	0.0286
20	0.481	0.257	-0.282	-0.0734	0.0108	0.0997	-0.131	-0.0684	-0.519	0.0212
30	0.429	0.237	-0.282	-0.0736	0.0142	0.148	+0.0012	-0.0638	-0.511	+0.0101
40	0.363	0.212	-0.264	-0.0733	0.0159	0.192	0.126	-0.0589	-0.500	-0.0035
60	0.218	0.155	-0.178	-0.0716	0.0160	0.248	0.260	-0.0533	-0.462	-0.0336
80	0.113	0.0986	-0.1009	-0.0708	0.0150	0.262	0.265	-0.0602	-0.422	-0.0619
100	0.0724	0.0472	-0.0845	-0.0749	0.0150	0.262	0.236	-0.0839	-0.450	-0.0855
120	0.0808	0.0017	-0.120	-0.0852	0.0160	0.248	0.163	-0.123	-0.559	-0.103
140	0.137	-0.0381	-0.237	-0.1006	0.0159	0.192	-0.0008	-0.170	-0.626	-0.115
160	0.225	-0.0685	-0.424	-0.1156	0.0108	0.0997	-0.187	-0.210	-0.576	-0.121
[3/4 (T = 1) + 1/4 (T = 0)]										
							E = 90 Mev			
2	0.592	0.444	-0.0158	-0.0053	0.0031	0.0092	-0.0152	-0.0053	-0.129	0.0784
4	0.591	0.443	-0.0158	-0.0053	0.0062	0.0184	-0.0136	-0.0054	-0.130	0.0780
6	0.589	0.441	-0.0159	-0.0053	0.0093	0.0275	-0.0110	-0.0056	-0.131	0.0774
8	0.585	0.439	-0.0160	-0.0054	0.0123	0.0367	-0.0074	-0.0058	-0.132	0.0764
10	0.581	0.436	-0.0161	-0.0054	0.0153	0.0458	-0.0028	0.0061	-0.134	0.0752
20	0.548	0.413	-0.0142	-0.0056	0.0280	0.0909	+0.0303	-0.0084	-0.150	0.0654
30	0.495	0.378	-0.0052	-0.0038	0.0371	0.133	0.0703	-0.0121	-0.173	0.0506
40	0.428	0.335	0.0192	-0.0037	0.0411	0.169	0.103	-0.0168	-0.199	0.0324
60	0.283	0.244	0.0897	+0.0028	0.0360	0.207	0.122	-0.0275	-0.251	-0.0061
80	0.179	0.169	0.136	0.0108	0.0203	0.205	0.153	-0.0378	-0.348	-0.0364
100	0.178	0.123	0.128	0.0145	0.0022	0.188	0.157	-0.0461	-0.425	-0.0500
120	0.244	0.109	0.0693	0.0124	-0.0120	0.165	0.153	-0.0535	-0.554	-0.0454
140	0.381	0.116	-0.0836	0.0069	-0.0172	0.120	0.0520	-0.0615	-0.584	-0.0276
160	0.545	0.130	-0.309	0.0018	-0.0120	0.0587	-0.0898	-0.0691	-0.492	-0.0085

θ_0	A		B		C		E		F	
	Re	Im								
$[T = 0]$										
2	0.493	0.743	0.744	0.0380	0.0134	0.0157	0.740	0.0377	0.975	0.221
4	0.491	0.739	0.745	0.0378	0.0268	0.0313	0.729	0.0364	0.970	0.220
6	0.487	0.733	0.746	0.0374	0.0400	0.0467	0.712	0.0344	0.962	0.218
8	0.482	0.725	0.748	0.0367	0.0529	0.0617	0.688	0.0314	0.951	0.216
10	0.476	0.714	0.750	0.0360	0.0654	0.0763	0.657	0.0277	0.936	0.213
20	0.423	0.627	0.768	0.0306	0.120	0.139	0.426	-0.0014	0.826	0.187
30	0.340	0.498	0.788	0.0247	0.155	0.179	0.120	-0.0435	0.677	0.146
40	0.239	0.347	0.799	0.0218	0.167	0.192	-0.178	-0.135	0.520	+0.0956
60	0.0502	0.0549	0.751	0.0327	0.131	0.148	-0.500	-0.161	0.232	-0.0154
80	-0.0189	-0.131	0.631	0.0626	0.0473	0.0531	-0.411	-0.158	-0.0949	-0.102
100	0.0529	-0.164	0.559	0.0959	-0.0473	-0.0531	-0.167	-0.0688	-0.483	-0.125
120	0.267	-0.0442	0.535	0.132	-0.131	-0.148	0.0151	+0.0836	-0.716	-0.0615
140	0.690	0.187	0.348	0.181	-0.167	-0.192	0.0688	0.255	-0.629	+0.0697
160	1.222	0.421	-0.0307	0.236	-0.120	-0.139	0.0272	0.447	-0.372	0.204
$[T = 1]$										
2	0.469	0.287	-0.285	-0.0346	0.0006	0.0142	-0.283	-0.0446	-0.553	0.122
4	0.467	0.286	-0.285	-0.0344	0.0012	0.0284	-0.277	-0.0443	-0.551	0.121
6	0.465	0.285	-0.286	-0.0342	0.0017	0.0426	-0.267	-0.0339	-0.548	0.121
8	0.461	0.283	-0.286	-0.0338	0.0023	0.0567	-0.253	-0.0333	-0.543	0.120
10	0.455	0.280	-0.286	-0.0333	0.0028	0.0708	-0.235	-0.0325	-0.538	0.119
20	0.412	0.259	-0.284	-0.0292	0.0046	0.140	-0.105	-0.0265	-0.494	0.113
30	0.342	0.227	-0.264	-0.0226	0.0052	0.205	+0.0590	-0.0180	-0.434	0.104
40	0.252	0.189	-0.217	-0.0143	0.0046	0.258	0.202	-0.0088	0.372	0.0929
60	0.0642	0.1048	-0.0652	+0.0030	0.0027	0.316	0.320	+0.0037	-0.260	0.0710
80	-0.0422	0.0254	+0.0421	0.0114	0.0025	0.320	0.290	-0.0023	-0.175	0.0476
100	-0.0574	-0.0410	0.0575	0.0042	0.0025	0.320	0.275	-0.0319	-0.191	+0.0179
120	-0.0292	-0.0912	0.0302	-0.0166	0.0027	0.316	0.225	-0.0845	-0.355	-0.0172
140	0.0676	-0.129	-0.1034	-0.0452	0.0046	0.258	0.0531	-0.152	-0.522	-0.0507
160	0.235	-0.158	-0.363	-0.0717	0.0046	0.140	-0.154	-0.242	-0.543	-0.0744

TABLE III (Continued)

θ_0	A		B		C		E		F	
	Re		Im		Re		Im		Re	
	[3/4 ($T = 1$) + 1/4 ($T = 0$)]				E = 156 Mev				Im	
2	0.475	0.401	-0.0279	-0.0164	0.0038	0.0146	-0.0273	-0.0165	-0.171	0.147
4	0.473	0.400	-0.0279	-0.0164	0.0076	0.0291	-0.0253	-0.0166	-0.171	0.146
6	0.470	0.397	-0.0278	-0.0163	0.0113	0.0436	-0.0220	-0.0168	-0.170	0.145
8	0.466	0.393	-0.0276	-0.0161	0.0149	0.0580	-0.0176	-0.0171	-0.170	0.144
10	0.460	0.388	-0.0273	-0.0160	0.0184	0.0722	-0.0120	-0.0174	-0.169	0.143
20	0.415	0.351	-0.0212	-0.0142	0.0334	0.140	+0.0282	-0.0203	-0.164	0.131
30	0.342	0.295	-0.0013	-0.0108	0.0426	0.198	0.0742	-0.0244	-0.157	0.114
40	0.248	0.228	+0.0307	-0.0053	0.0452	0.242	0.1074	-0.0291	-0.149	0.0936
60	0.0607	0.0923	0.139	+0.0104	0.0347	0.274	0.115	-0.0374	-0.137	0.0494
80	-0.0364	-0.0137	0.189	0.0242	0.0137	0.253	0.115	-0.0412	-0.155	+0.0101
100	-0.0298	-0.0718	0.183	0.0272	-0.0100	0.227	0.165	-0.0411	-0.264	-0.0177
120	+0.0448	-0.0794	0.156	0.0205	-0.0306	0.200	0.173	-0.0425	-0.445	-0.0283
140	0.223	-0.0502	0.0101	0.0115	-0.0384	0.146	+0.0570	-0.0504	-0.548	-0.0206
160	0.482	-0.0133	-0.280	0.0053	-0.0264	0.0703	-0.1085	-0.0700	-0.500	-0.0047
			[T = 0]				E = 310 Mev			
2	-0.397	0.818	0.709	-0.108	0.0116	0.0213	0.705	-0.108	0.928	0.0496
4	-0.395	0.814	0.708	-0.108	0.0231	0.0424	0.693	-0.108	0.919	0.0495
6	-0.391	0.807	0.706	-0.108	0.0345	0.0632	0.673	-0.109	0.904	0.0495
8	-0.385	0.798	0.703	-0.109	0.0455	0.0837	0.645	-0.110	0.885	0.0494
10	-0.379	0.786	0.700	-0.110	0.0562	0.1036	0.610	-0.111	0.860	0.0493
20	-0.334	0.693	0.674	-0.117	0.1018	0.190	0.350	-0.119	0.675	0.0481
30	-0.297	0.559	0.636	-0.126	0.129	0.248	0.0218	-0.130	0.444	0.0453
40	-0.296	0.406	0.588	-0.137	0.136	0.270	-0.270	-0.137	0.237	0.0404
60	-0.400	0.129	0.430	-0.147	0.0990	0.215	-0.507	-0.122	-0.0035	0.0248
80	-0.472	-0.0438	0.214	-0.127	0.0339	0.0792	-0.384	-0.486	-0.166	0.0092
100	-0.404	-0.1050	0.146	-0.0656	-0.0339	-0.0792	-0.234	+0.0704	-0.452	0.0126
120	-0.237	-0.0576	0.267	+0.0393	-0.0990	-0.215	-0.160	-0.211	-0.670	0.0649
140	+0.129	+0.0863	0.162	0.183	-0.136	-0.270	-0.189	0.360	-0.696	0.182
160	0.682	0.253	-0.343	0.324	-0.1018	-0.190	-0.341	0.489	-0.667	0.321

θ_0	A		B		C		E	
	Re	Im	Re	Im	Re	Im	Re	Im
	[T = 1]				E = 310 Mev			
2	0.318	0.386	-0.167	-0.0214	0.0016	0.0152	-0.165	-0.0216
4	0.314	0.364	-0.167	-0.0212	0.0032	0.0305	-0.162	-0.0212
6	0.307	0.363	-0.168	-0.0208	0.0048	0.0459	-0.156	-0.0209
8	0.298	0.360	-0.169	-0.0203	0.0063	0.0615	-0.147	-0.0204
10	0.286	0.357	-0.169	-0.0195	0.0079	0.0773	-0.136	-0.0198
20	0.201	0.330	-0.172	-0.0138	0.0150	0.160	-0.0549	-0.0148
30	0.0949	0.289	-0.160	-0.0046	0.0212	0.246	+0.0547	-0.0085
40	0.0002	0.239	-0.123	+0.0068	0.0266	0.320	-0.163	-0.0025
60	-0.1032	0.127	+0.0139	0.0278	0.0368	0.373	0.288	+0.0397
80	-0.150	0.0221	0.137	0.0314	0.0450	0.324	0.281	-0.0105
100	-0.173	-0.0640	0.186	0.0105	0.0450	0.324	0.246	-0.0432
120	-0.0797	-0.128	0.169	-0.0273	0.0368	0.373	0.223	-0.0881
140	+0.0979	-0.175	0.0283	-0.0709	0.0266	0.320	0.138	-0.170
160	0.211	-0.208	-0.240	-0.1083	0.0150	0.160	-0.0162	-0.236
[3/4 (T = 1) + 1/4 (T = 0)]								
							E = 310 Mev	
2	0.139	0.479	0.052	-0.0430	0.0041	0.0167	0.0523	-0.0430
4	0.136	0.477	0.051	-0.0429	0.0082	0.0334	0.0520	-0.0430
6	0.132	0.474	0.050	-0.0427	0.0122	0.0502	0.0515	-0.0429
8	0.127	0.469	0.049	-0.0425	0.0161	0.0670	0.0509	-0.0427
10	0.120	0.464	0.047	-0.0422	0.0200	0.0839	0.0502	-0.0425
20	0.0673	0.421	0.039	-0.0395	0.0367	0.168	0.0464	-0.0411
30	-0.0032	0.356	0.038	-0.0350	0.0482	0.247	0.0465	-0.0389
40	-0.0739	0.280	0.054	-0.0291	0.0539	0.307	0.0548	-0.0362
60	-0.177	0.128	0.118	-0.0160	0.0524	0.333	0.0896	-0.0290
80	-0.231	0.0056	0.156	-0.0081	0.0422	0.263	0.115	-0.0200
100	-0.231	-0.0743	0.176	-0.0086	0.0252	0.223	0.126	-0.0148
120	-0.119	-0.1103	0.194	-0.0107	0.0029	0.226	0.127	-0.0208
140	+0.106	-0.1093	0.0594	-0.0074	-0.0140	0.173	0.0560	-0.0376
160	0.328	-0.0936	-0.266	-0.0003	-0.0142	0.0728	-0.0976	-0.434

weight N . We can also write the scattering amplitude as composed of a spin-independent and spin-dependent part, using Bethe's (4) notation

$$G_B = G + iH \sin \theta \mathbf{d} \cdot \hat{\mathbf{n}}, \quad (4.13)$$

where

$$G = \frac{2N^2}{(N+1)} F(q) \bar{A}(q) \quad (4.14)$$

and

$$H = \frac{-i2N^2}{(N+1)} F(q) \bar{C}/\sin \theta. \quad (4.15)$$

We define \tilde{C} and \hat{C} by¹¹

$$\tilde{C} = \frac{\hat{C}}{kk_0} = \frac{\hat{C}}{k^2 \sin \theta} \quad (4.16)$$

so that

$$H = -iN(k/k_0)^2 F(q) \hat{C}. \quad (4.17)$$

Our definition of \hat{C} is not quite the same as that given by Bethe (4) or Riesenfeld and Watson (3). With the aid of (4.16), (4.12) can be written, for small forward angles,

$$P_B/\theta = \frac{4N}{(N+1)} \left(\frac{\operatorname{Re} \bar{A}^* \hat{C}}{|\bar{A}|^2} \right)_{\theta_0=0}, \quad (4.18)$$

¹¹ The quantity \tilde{C} is defined so that it is an invariant to Galilean transformations in the following sense. In general we can always define relative velocities, which are invariant to changes in coordinate system. Then the quantities

$$\mathbf{q} = \mathbf{k}' - \mathbf{k},$$

$$\mathbf{p} = \mathbf{k}' + \mathbf{k},$$

are also invariants, being expressed in terms of relative velocities \mathbf{k}, \mathbf{k}' before and after the collision. When energy is conserved (elastic collision)

$$k'^2 = k^2, \quad \mathbf{k}' \cdot \mathbf{k} = k^2 \cos \theta,$$

and

$$\begin{aligned} q &= 2k \sin \theta/2 \\ p &= 2k \cos \theta/2 \end{aligned} \quad \mathbf{q} \cdot \mathbf{p} = 0.$$

Equation (4.16) for \tilde{C} is equivalent to saying

$$\tilde{C} = \frac{C(\mathbf{p}, \mathbf{q})}{(1/2 |\mathbf{p} \times \mathbf{q}|)},$$

where all quantities are invariant under changes of coordinate system.

or in the laboratory system

$$P_B/\theta_{\text{lab}} = 4 \left(\frac{\text{Re } \bar{A}^* \tilde{C}}{|\bar{A}|^2} \right)_{\theta_0=0}. \quad (4.19)$$

Thus strictly speaking, it is the polarization at small angles in the *laboratory system* that should be independent of N .¹² In the center-of-mass system there is a difference of purely kinematical origin.

We do not expect the Born approximation on the optical potential $T' \approx U$ to be a good one, although, anticipating a little, it does happen that the Born approximation is good for the polarization at small forward angles, so that Eqs. (4.12) or (4.18) can be used in the forward direction. The polarization at small forward angles is thus the quantity which is most directly related to the two-nucleon scattering parameters. In general it is necessary to take into account the effects of the optical potential by using Eqs. (2.29), (2.44), and (2.63)

$$T_{00} = \frac{N}{N-1} T'_{00} \quad (2.29)$$

and

$$T'_{00} \cong U_{00}^{(0)} + U_{00}^{(0)} \frac{1}{\alpha_0} T'_{00}, \quad (2.44)$$

which is equivalent to the solution of the Schrödinger equation

$$(E - H_0 - U_{00}^{(0)}) \Omega_{00} = 0 \quad (4.20)$$

¹² This can also be seen using the general definition of \tilde{C} above and the following fact. For two-body scattering the parameters A , etc. are functions of \mathbf{p} and \mathbf{q} only, although up to now we have emphasized the dependence on \mathbf{q} . In taking the matrix element of the scattering operator t between nuclear states it can be shown that our approximation leads to the function $t^{(0)}$ as in (3.7), but with \mathbf{p} and \mathbf{q} replaced by the same quantities constructed with the relative velocities relevant to the N body scattering. If we call these \mathbf{Q} and \mathbf{P} it is possible to see that

$$\mathbf{Q} = \mathbf{q} = (k_0' - k_0) = -(K' - K),$$

$$\mathbf{P} = \frac{1}{2} \left\{ \left(k_0 - \frac{K}{N} \right) + \left(k_0' - \frac{K'}{N} \right) \right\},$$

where k_0, k_0' are the velocities of the scattered nucleon and K, K' are those of the nucleus before and after collision. In terms of these,

$$P \cong |\mathbf{P} \times \mathbf{Q}| \frac{\text{Re } A^* \tilde{C}}{|\bar{A}|^2}.$$

It is only in the laboratory system, where $K = 0$, that this is independent of N for small scattering angles.

so that we find the approximate scattering amplitude by calculating that due to the approximate potential matrix $U_{00}^{(0)}$ and then multiplying by $N/(N - 1)$.

In any case we cannot carry the Born approximation out to large angles because we expect large angle scattering to be fed mostly by multiple scattering through small angles, which is given by the solution of Eq. (4.20).

We will now consider the effect of multiple scattering as embodied in the integral Eq. (4.20) in the approximation that the potential matrix $U_{00}^{(0)}$ is considered to be a local potential in configuration space. We note that the Born scattering amplitude (4.9) could be interpreted as the scattering from a local potential $U_{00}^{(0)}(\mathbf{k}', \mathbf{k})$ given by Eq. (4.7) which can be written,

$$U_{00}^{(0)}(\mathbf{k}'\mathbf{k}) \sim \tilde{M}(E, \mathbf{q}) F(\mathbf{q}) \approx U_E(\mathbf{q}) \quad (4.21)$$

leaving out inessential proportionality factors. Here $\tilde{M}(E, \mathbf{q})$ is the two-body scattering amplitude as a function of momentum transfer at the incident energy E , corresponding to momentum \mathbf{k} , and $F(\mathbf{q})$ is the nuclear form factor. Treating the potential matrix as a local potential assumes that it is a function of momentum transfer only, depending on the energy E parametrically. This local potential in configuration space, $U_E(\mathbf{x})$, is given by inverting the relation

$$U_{00}^{(0)}(\mathbf{k}', \mathbf{k}) \equiv U_E(\mathbf{q}) = \frac{1}{(2\pi)^3} \int \exp(-i\mathbf{q}\cdot\mathbf{x}) U_E(\mathbf{x}) d\mathbf{x}, \quad (4.22)$$

where

$$\mathbf{q} = \mathbf{k}' - \mathbf{k}, \quad (4.23)$$

so that

$$\begin{aligned} U_E(\mathbf{x}) &= \int \exp(i\mathbf{q}\cdot\mathbf{x}) U_E(\mathbf{q}) d\mathbf{q} \\ &= -\frac{1}{(2\pi)^2} \frac{2\hbar^2}{m} (N - 1) \int \exp(i\mathbf{q}\cdot\mathbf{x}) F(\mathbf{q}) [\bar{A}(E, \mathbf{q}) + \bar{C}(E, \mathbf{q}) \mathbf{s}\cdot\hat{\mathbf{n}}] d\mathbf{q}, \end{aligned} \quad (4.24)$$

a potential which in fact gives rise to the Born scattering amplitude (4.9), when treated in Born approximation. However if we look at the integral equation (4.24) for the scattering matrix, it is

$$\langle \mathbf{k}' | T' | \mathbf{k} \rangle = \langle \mathbf{k}' | U_{00} | \mathbf{k} \rangle + \int d\mathbf{p} \frac{\langle \mathbf{k}' | U_{00} | \mathbf{p} \rangle \langle \mathbf{p} | T' | \mathbf{k} \rangle}{E - E'(\mathbf{p}) + i\epsilon} \quad (4.25)$$

with

$$\mathbf{k}' - \mathbf{k} = \mathbf{q}, \quad \mathbf{k}' - \mathbf{p} = \mathbf{q}', \quad \mathbf{p} - \mathbf{k} = \mathbf{q} - \mathbf{q}'. \quad (4.26)$$

Writing

$$\langle \mathbf{p} | T' | \mathbf{k} \rangle = T'_{EE'}(\mathbf{q} - \mathbf{q}'), \quad (4.27)$$

$$\langle \mathbf{k}' | U_{00} | \mathbf{p} \rangle \equiv U_{EE'}(\mathbf{q}'), \quad (4.28)$$

where E' is the energy corresponding to momentum $\mathbf{p} = \mathbf{k}' - \mathbf{q}'$, Eq. (4.25) becomes

$$T_E(\mathbf{q}) = U_E(\mathbf{q}) + \int d\mathbf{q}' \frac{U_{EE'}(\mathbf{q}') T'_{E'E}(\mathbf{q} - \mathbf{q}')}{E - E' + i\epsilon}, \quad (4.29)$$

where

$$T_E(\mathbf{q}) \equiv T'_{EE}(\mathbf{q}) \quad \text{and} \quad U_E(\mathbf{q}) \equiv U_{EE}(\mathbf{q}). \quad (4.29a)$$

Since $U_{EE'}(\mathbf{q}') \sim \bar{M}(EE'\mathbf{q}')F(\mathbf{q}')$, Eq. (4.29) involves a knowledge of M off the energy shell. On the other hand, using the local potential $U_E(\mathbf{x})$ defined by Eq. (4.24) we have the integral equation for the scattering matrix

$$T_E(\mathbf{q}) = U_E(\mathbf{q}) + \int d\mathbf{q}' \frac{U_E(\mathbf{q}') T_E(\mathbf{q} - \mathbf{q}')}{E - E' + i\epsilon}, \quad (4.30)$$

where again E' is the energy corresponding to momentum $\mathbf{k}' - \mathbf{q}'$. This equation is not the same as Eq. (4.29) as it involves knowledge of the two-nucleon scattering only at the incident energy considered. Writing,

$$\frac{1}{E - E' + i\epsilon} = P \frac{1}{E - E'} - i\pi\delta(E - E'), \quad (4.31)$$

we see that the δ -function part of this integral, as it is on the nucleon-nucleus energy shell, is the same for both expressions. The only difference is in the principal value integral, i.e., in off the energy shell scatterings in intermediate states. If the two-body scattering matrix M is a slowly varying function of energy, then $U_{EE'}(\mathbf{q}')$ will be a slowly varying function of $(E' - E)$, but because of the nuclear form factor as well as the two-body dependence on momentum transfer, a rapidly varying function of \mathbf{q}' , so that contributions from E' very different from E , which necessitate a large momentum transfer, are cut down by the nuclear form factor $F(\mathbf{q}')$ and are less important the larger the nucleus.

Let us examine this a little more closely for $\mathbf{q} = 0$ (in the forward direction). Because of the rapid fall off of $F(\mathbf{q}')$ and $T(\mathbf{q}')$ as a function of \mathbf{q}' , which is characterized by the nuclear radius R , we expect the integrand in Eq. (4.29) to be small when $q' \sim 1/R$. Since $E \sim k^2$ and

$$E' \sim (\mathbf{k}' - \mathbf{q}')^2, \quad \frac{\Delta E}{E} \equiv \frac{(E' - E)}{E} = \frac{q'^2 - 2\mathbf{k}' \cdot \mathbf{q}'}{k'^2} \sim (1/kR) \sim 5A^{-1/3}E^{-1/2},$$

with E in Mev, which for the energies considered here is a small quantity. On the energy shell over an energy interval of the order of 10–20 % of the total energy, the variation of the two nucleon scattering amplitude is small. Therefore, over a similar energy interval we do not expect significant variation off the energy shell. This means that Eqs. (4.29) and (4.30) are essentially equivalent. That is, the optical potential may be considered to be local.

At values of \mathbf{q} near a diffraction minimum of $T(\mathbf{q})$, the contributions of small values of \mathbf{q}' in the integrand of Eq. (4.29) will be discriminated against, so that the effect of nonlocality should be more pronounced. Consequently we may expect that for the scattering at small forward angles, and hence for the total cross section, the local approximation to be a good one, the nonlocality being apparent only in the fine details of the differential cross section diffraction pattern, and we expect the main features of the scattering to be given by the optical potential (4.24).

For another discussion of the nonlocality of the potential, starting from the definition

$$U(\mathbf{x}, \mathbf{x}') = \frac{1}{(2\pi)^3} \int \exp(-i\mathbf{k}' \cdot \mathbf{x}) \exp(i\mathbf{k} \cdot \mathbf{x}') \langle \mathbf{k}' | U_{00} | \mathbf{k} \rangle d\mathbf{k} d\mathbf{k}', \quad (4.32)$$

see the article by Feshbach (17).

5. CONSTRUCTION OF THE OPTICAL POTENTIAL

We can construct the local potential for a spin zero target from Eq. (4.24) by comparing with a local optical potential of the standard form.

$$V(r) = V_c(r) + \frac{1}{r} \frac{d}{dr} V_{s0}(r) \mathbf{l} \cdot \mathbf{s} \equiv \int \exp(i\mathbf{q} \cdot \mathbf{r}) V(\mathbf{q}) d\mathbf{q}. \quad (5.1)$$

Then the corresponding potential matrix defined by,

$$U_E(q) \equiv \langle \mathbf{k}' | V(r) | \mathbf{k} \rangle = \frac{1}{(2\pi)^3} \int \exp(-i\mathbf{k}' \cdot \mathbf{r}) V(\mathbf{r}) \exp(i\mathbf{k} \cdot \mathbf{r}) d\mathbf{r} = V(\mathbf{q}) \quad (5.2)$$

is given by the expression

$$U_E(\mathbf{q}) = V_c(\mathbf{q}) - iV_{s0}(\mathbf{q}) \mathbf{d} \cdot \mathbf{k} \times \mathbf{k}'. \quad (5.3)$$

Equation (5.3) is to be compared with the potential matrix given by Eq. (4.7). To facilitate the comparison we rewrite the term $\tilde{C}(\mathbf{d} \cdot \hat{\mathbf{n}})$.

$$\tilde{C}(\mathbf{d} \cdot \hat{\mathbf{n}}) = \tilde{C} \frac{\mathbf{d} \cdot \mathbf{k} \times \mathbf{k}'}{|\mathbf{k} \times \mathbf{k}'|} = \frac{\tilde{C}}{k^2 \sin \theta} \mathbf{d} \cdot \mathbf{k} \times \mathbf{k}' \quad (5.4)$$

or, using Eq. (4.16)

$$\tilde{C}(\mathbf{d} \cdot \hat{\mathbf{n}}) = \tilde{C} \mathbf{d} \cdot \mathbf{k} \times \mathbf{k}' \quad (5.5)$$

and then, comparing the coefficients of Eqs. (5.3) and (4.7), we have

$$V_c(\mathbf{q}) = -\frac{2\hbar^2}{(2\pi)^2 m} (N - 1) \bar{A}(\mathbf{q}) F(\mathbf{q}), \quad (5.6)$$

$$V_{s0}(\mathbf{q}) = \frac{i}{(2\pi)^2} \frac{2\hbar^2}{m} (N - 1) \tilde{C}(\mathbf{q}) F(\mathbf{q}), \quad (5.7)$$

or in configuration space

$$V_c(r) = -\frac{2\hbar^2}{m} (N-1) \frac{1}{(2\pi)^2} \int \exp(i\mathbf{q} \cdot \mathbf{r}) F(\mathbf{q}) \bar{A}(\mathbf{q}) d\mathbf{q}, \quad (5.8)$$

$$V_{s0}(r) = \frac{2i\hbar^2}{m} (N-1) \frac{1}{(2\pi)^2} \int \exp(i\mathbf{q} \cdot \mathbf{r}) F(\mathbf{q}) \tilde{C}(\mathbf{q}) d\mathbf{q}, \quad (5.9)$$

which is simply a rewriting of (4.24). To obtain an exact expression for the potential $V(r)$ we would have to extend the two-nucleon scattering parameters for all values of the momentum transfer. However if we consider the potential integrated over all space, this depends only on the potential matrix for zero momentum transfer, so we have, remembering that $F(0) = 1$

$$U = -\frac{1}{N} \int V_c(r) d\mathbf{r} = 4\pi \frac{\hbar^2 N - 1}{m N} \bar{A}(0), \quad (5.10)$$

$$W_{s0} = -\frac{1}{N} \int V_{s0}(r) d\mathbf{r} = -4\pi \frac{\hbar^2 N - 1}{m N} i\tilde{C}(0), \quad (5.11)$$

which is given in terms of measurable quantities. These values of integrated potentials per target nucleon are then independent of assumptions about the behavior of the two-particle scattering at nonforward angles and can be compared with the experimental values found by optical model analysis.

For a heavy nucleus in which $F(\mathbf{q})$ is a rapidly varying function of momentum transfer, the variation of the two-body amplitude with momentum transfer can be neglected to first order. Using

$$\int F(\mathbf{q}) \exp(i\mathbf{q} \cdot \mathbf{r}) d\mathbf{r} = (2\pi)^3 \rho(r), \quad \int \rho(r) d\mathbf{r} = F(0) = 1, \quad (5.12)$$

where $\rho(r)$ is the nucleon density in the target, we have

$$V_c(r) \approx -4\pi \frac{\hbar^2}{m} (N-1) \bar{A}(0) \rho(r), \quad (5.13)$$

$$V_{s0}(r) \approx 4\pi \frac{\hbar^2}{m} (N-1) i\tilde{C}(0) \rho(r), \quad (5.14)$$

so that the radial dependence of the spin orbit potential, $(1/r)(d/dr)V_{s0}(r)$, is obtained by differentiating the central potential. In this case a theorem due to Köhler and others (see Bethe (4), p. 205) shows that the polarization is given to good accuracy by the Born approximation so that it is determined by the scattering coefficients and little affected by the form factor. One condition for this theorem to hold is that the destructive interference between the contribution of various partial waves to the scattering should not be too marked, a condition which will break down near diffraction minima of the elastic scatter-

ing.¹³ Beyond this point the geometrical effects associated with the nuclear form factor dominate, as discussed in some detail by Bethe (4). The polarization at small forward angles and the integrated potentials (5.10) and (5.11) are then the quantities most directly connected with the two nucleon scattering coefficients.

The variation of the two-body amplitudes can be taken care of to first order by adjusting the mean square radius of the density distribution. The mean square radius of the optical potential is given by

$$\langle R^2 \rangle = -3 \left(\frac{1}{U_E(q)} \frac{d^2 U_E(q)}{dq^2} \right)_{q=0} \quad (5.15)$$

or

$$\langle R^2 \rangle = R_c^2 + r_{\text{nuc}}^2, \quad (5.16)$$

where

$$R_c^2 = -3 \left(\frac{1}{F} \frac{d^2 F}{dq^2} \right)_{q=0} \quad (5.17)$$

is the mean square radius of the charge distribution as given by the Stanford experiments (18), and

$$r_{\text{nuc}}^2 = -3 \left(\frac{1}{M} \frac{d^2 \bar{M}}{dq^2} \right)_{q=0} \quad (5.18)$$

is the contribution due to the variation of the two-body scattering amplitude. As there are four different parts to the potential, the real and imaginary parts of the central potential and the real and imaginary parts of the spin orbit potential, there will be correspondingly four radii r_1 to r_4 . If we write

$$\begin{aligned} \bar{A} &= \bar{A}_R + i\bar{A}_I, \\ \hat{C} &= \hat{C}_R + i\hat{C}_I, \end{aligned} \quad (5.19)$$

then,

$$\begin{aligned} r_1^2 &= -3 \left(\frac{1}{A_R} \frac{d^2 \bar{A}_R}{dq^2} \right)_0, & r_2^2 &= -3 \left(\frac{1}{A_I} \frac{d^2 \bar{A}_I}{dq^2} \right)_0, \\ r_3^2 &= -3 \left(\frac{1}{\hat{C}_I} \frac{d^2 \hat{C}_I}{dq^2} \right)_0, & r_4^2 &= -3 \left(\frac{1}{\hat{C}_R} \frac{d^2 \hat{C}_R}{dq^2} \right)_0, \end{aligned} \quad (5.20)$$

where r_1, r_3 refer to the real, and r_2, r_4 to the imaginary parts of the central and spin orbit potentials, respectively. The change in role of the real and imaginary parts of \hat{C} with respect to A comes from the factor i in (5.9).

¹³ The diffraction minima show up more strongly in the left and right scattering of a polarized beam than in the unpolarized scattering.

We shall take the resultant effective density to be of the same shape as the charge density distribution but with a different radius. The change in slope at the edge of the distribution would depend on the fourth moment $\langle R^4 \rangle$ of the potential which would enable us to find the surface thickness a . In fact the information obtained by electron scattering gives very little information about shape except that given by knowledge of the second and fourth moments of the distribution, so that if we could determine the fourth moment, we would have about as much information as the experiments warrant. Each successive moment, however, becomes less reliable, and the detailed shape of the effective local potential as contrasted with the true nonlocal potential is probably not too meaningful.

If we write

$$\begin{aligned} V_c(r) &= V_1 + iV_2, \\ V_{s0}(r) &= V_3 + iV_4, \end{aligned} \quad (5.21)$$

we have from (5.13)

$$\begin{aligned} V_1 &= -4\pi \frac{\hbar^2}{m} (N - 1) \bar{A}_R(0) \bar{\rho}_1(r), \\ V_2 &= -4\pi \frac{\hbar^2}{m} (N - 1) \bar{A}_I(0) \bar{\rho}_2(r), \end{aligned} \quad (5.22)$$

and

$$\begin{aligned} V_3 &= -4\pi \frac{\hbar^2}{m} (N - 1) \bar{C}_I(0) \bar{\rho}_3(r), \\ V_4 &= +4\pi \frac{\hbar^2}{m} (N - 1) \bar{C}_R(0) \bar{\rho}_4(r), \end{aligned} \quad (5.23)$$

where $\bar{\rho}_i(r)$ is taken to be a function of the same form as the charge distribution but with a mean square radius $\langle R_i^2 \rangle$ given by

$$\langle R_i^2 \rangle = R_c^2 + r_i^2. \quad (5.24)$$

We can also express the optical potential in terms of the depth of the potential at the center of the nucleus. For this purpose we write the optical potential in the form

$$V = -V_{cR} f_1(r) - iV_{cI} f_2(r) - \left(\frac{\hbar}{\mu c}\right)^2 \left\{ V_{sR} \frac{1}{r} \frac{df_3}{dr} + iV_{sI} \frac{1}{r} \frac{df_4}{dr} \right\} \mathbf{e} \cdot \mathbf{l}, \quad (5.25)$$

where the factor $(\hbar/\mu c)^2$ is introduced to give the four constants the dimensions of energy, and $\hbar/\mu c$ is the π -meson Compton radian wavelength

$$\left(\frac{\hbar}{\mu c}\right)^2 \sim 2(\text{fermi})^2.$$

The potential shape factors f_i are defined by

$$\bar{\rho}_i(r) = \bar{\rho}_i(0)f_i(r) \quad (5.26)$$

so that

$$f_i(0) = 1 \quad (5.27)$$

and the volume for the potential is (5.12)

$$\Omega_i = \int f_i(r) dr = 1/\bar{\rho}_i(0). \quad (5.28)$$

Then

$$\begin{aligned} V_{eR} &= \frac{4\pi\hbar^2}{m} \left(\frac{N-1}{\Omega_1} \right) \bar{A}_R(0), \\ V_{eI} &= \frac{4\pi\hbar^2}{m} \left(\frac{N-1}{\Omega_2} \right) \bar{A}_I(0), \\ V_{sR} &= \frac{4\pi\hbar^2}{m} \left(\frac{\mu c}{\hbar} \right)^2 \left(\frac{N-1}{\Omega_3} \right) \bar{C}_I(0), \\ V_{sI} &= - \frac{4\pi\hbar^2}{m} \left(\frac{\mu c}{\hbar} \right)^2 \left(\frac{N-1}{\Omega_4} \right) \bar{C}_R(0). \end{aligned} \quad (5.29)$$

Table IV contains the spin-independent and spin-dependent two nucleon forward scattering coefficients together with the contribution to the mean square radii from the q^2 term of the two-nucleon amplitudes. Table V gives the mean square radii of the charge distribution for various nuclei, together with the mean square radius of the central potential (5.24) deduced from this and the two-

TABLE IV

THE CONTRIBUTION TO THE MEAN SQUARE RADIUS ARISING FROM FINITE RANGE OF THE NUCLEON-NUCLEON INTERACTION. THE COEFFICIENTS \bar{A} AND \bar{C} ARE GIVEN FOR $q = 0$

Energy (Mev)	310	156	90
$\bar{A}(0)$ (f)	$0.139 + 0.479i$	$0.475 + 0.401i$	$0.592 + 0.444i$
$\bar{C}(0)$ (f)	$0.117 + 0.479i$	$0.109 + 0.417i$	$0.089 + 0.264i$
$k_0(f^{-1})$	1.93	1.37	1.04
$r_1^2(f^2)$	7.5	3.5	3.6
r_2^2	1.7	3.6	3.5
r_3^2	—	1.4	2.0
r_4^2	2.5	5.7	7.4
$f = \text{fermi} = 10^{-13} \text{ cm}$			

TABLE V

THE MEAN SQUARE RADII FOR THE CENTRAL PART OF THE OPTICAL POTENTIAL COMPARED TO THE ELECTROMAGNETIC SIZE OF THE NUCLEUS. THE UNITS
ARE fermi² = 10⁻²⁶ cm²

Element	He ⁴	C ¹²	Al ²⁷	Ca ⁴⁰
$\langle R_e^2 \rangle$ (fermi ²)	2.59	5.76	9.55	12.39
$\langle R^2 \rangle$ 310 Mev	4.28	7.40	11.2	14.1
$\langle R^2 \rangle$ 156, 90 Mev	6.10	9.24	13.0	15.9

TABLE VI

COMPARISON OF THEORETICAL AND EXPERIMENTAL ROOT MEAN SQUARE RADII

Element	He ⁴	C ¹²	Al ²⁷	Bi
$R(310 \text{ Mev})$	2.07	2.72	3.35	5.66
Exp (19)		2.83	3.70	6.44
$R(156 \text{ Mev})$	2.47	3.04	3.61	5.82
Exp (19)		2.92	3.63	6.78

body amplitude. For the α -particle a correction has been made for the electromagnetic size of the proton. An attempt to get independent values of the radius by analysis of experimental data has been made by Riese (19). These are compared in Table VI with the values calculated from the charge radius plus nuclear force radius (Table V). Agreement is good except for heavy nuclei.

Table VII contains values calculated for the integrated potentials U (5.10) in units Mev fermi³, W_{s0} (5.11) in units Mev fermi⁵, and the potential strengths at the center of the nucleus V_{cR} , etc. (5.29) in Mev together with values deduced from analysis of scattering experiments. These latter are taken mostly from the tables compiled by Feshbach (17) where full references to the experimental work are given. Because $r_1^2 \sim r_2^2$ for 156 and 90 Mev, and $V_{cI} \gg V_{cR}$ for 310 Mev (see Tables IV and VII) it is convenient to take the same radius for the real and imaginary parts of the central potential in the calculation of V_{cR} , V_{cI} . A value $N/\Omega = 0.127$ was chosen, corresponding to a heavy nucleus of Saxon profile with $R_0 = 1.2A^{1/3}$ fermi and $a = 0.5$ fermi. Relativistic kinematical corrections are included (see Appendix I).

The "experimental" values in Table VII obtained by optical model analysis of the data show considerable fluctuations—see for instance the tables of values given in Feshbach (17) and Bethe (4). Bearing this in mind the agreement between the calculated and experimental values of the real part of the central potential is quite good. For 156 and 90 Mev the experimental value of the imaginary part of the central potential is much lower than the theoretical value. Of the

TABLE VII

COMPARISON OF THEORETICAL AND EXPERIMENTAL OPTICAL PARAMETERS. THE INTEGRATED POTENTIALS U AND W_{s0} ARE GIVEN IN UNITS OF Mev fermi³ AND fermi⁵, RESPECTIVELY. THE POTENTIAL STRENGTHS V ARE GIVEN IN Mev

Energy (Mev)	310	"156"	90	Ref
$U(\text{Th})$	$59 + 203i$	$221 + 187i$	$289 + 217i$	
$U(\text{Exp})$	$24 + 165i$	$183 + 118i$	$300 + 119i$	a
		$55 + 137i$		b
W_{s0}	$25 - 6.1i$	$50 - 13i$	$58 - 20i$	
$V_{sR} + iV_{sI}(\text{Th})$	$7 + 25.7i$	$28 + 23.7i$	$37 + 27.5i$	c
(Exp)	$0 + 18i$			d
	$14 \pm 8 + 29 \pm 8i$			
$V_{sR} + iV_{sI}(\text{Th})$	$1.59 - 0.39i$	$3.15 - 0.82i$	$367 - 1.24i$	e
(Exp)	$1.22 - 1.5i$	$5 + 0i$		f
	$1.93 + 0i$	$\sim 1.3 + 0i$		
	$\sim 0.7 + 0.2i$			d

^a The values are obtained by rearranging the values in Feshbach (17) Table X. "156" Mev means 130–160 Mev.

^b Results are for 220-Mev protons, Hafner (41).

^c Bjorklund *et al.* (44). Their quoted results are slightly altered to adjust to the standard radius used here.

^d See Ref. 45.

^e See Ref. 41.

^f Sternheimer (46); "156" means 130 Mev.

spin orbit potential, all one can say is that both theoretical and experimental values are quite small, considerably smaller than the value required by the shell model at low energies which requires $|V_{s0}| \sim 9.5$ Mev (17).

The experimental values quoted in Table VII are mostly an average over a series of elements, whereas the theoretical are for nuclei with $A = 2Z$, more appropriate to light nuclei. More exactly, but still neglecting small spin-dependent terms $\sim 1/N$, we should use in place of (4.6) the relation

$$A = \frac{1}{4}(1 \pm \Delta)A_0 + \frac{3}{4}(1 \mp \Delta/3)A_1 \quad (5.30)$$

in (4.5) for $\bar{M}(q)$, where the neutron excess Δ is¹⁴

$$\Delta = \frac{N - 2Z}{N}, \quad (5.31)$$

the upper sign being for incident protons, and the lower for neutrons. The magnitude of this effect can be estimated from Table III and is given in Table VIII for the cases of protons on Pb^{208} (labelled *a*), neutrons or protons on C^{12} (labelled

¹⁴ Note that N is the total number of particles and not the number of neutrons.

TABLE VIII

THE EFFECT OF NEUTRON EXCESS ON THE THEORETICAL OPTICAL PARAMETERS: (a) PROTONS ON Pb^{208} , (b) NEUTRONS OR PROTONS ON C^{12} , (c) NEUTRONS ON Pb^{208} . THE PARAMETERS \bar{A} AND \hat{C} ARE CALCULATED AT ZERO ANGLE, AND f STANDS FOR 10^{-13} cm

Energy (Mev)	310	156	90
$\bar{A}(0)$ (f) (a)	$0.111 + 0.502i$	$0.475 + 0.422i$	$0.605 + 0.478i$
	$0.139 + 0.479i$	$0.475 + 0.401i$	$0.592 + 0.444i$
	$0.167 + 0.456i$	$0.475 + 0.380i$	$0.580 + 0.410i$
$\hat{C}(0)$ (f) (a)	$0.132 + 0.478i$	$0.142 + 0.415i$	$0.100 + 0.258i$
	$0.177 + 0.479i$	$0.109 + 0.417i$	$0.089 + 0.264i$
	$0.107 + 0.480i$	$0.098 + 0.420i$	$0.078 + 0.270i$
U (Mev f^3) (a)	$47 + 2i3i$	$221 + 197i$	$295 + 233i$
	$59 + 203i$	$221 + 187i$	$289 + 217i$
	$71 + 193i$	$221 + 177i$	$283 + 201i$
W_{s0} (Mev f^5) (a)	$28 - 6.1i$	$65 - 13i$	$65 - 20i$
	$25 - 6.1i$	$50 - 13i$	$58 - 20i$
	$23 - 6.1i$	$42 - 13i$	$51 - 20i$
$\Delta = 0.21$			

b : these are the values used in the previous tables) and for neutrons on Pb^{208} (labelled *c*). Quoted are values for the two-body forward scattering amplitudes and the integrated potentials. The effect is quite small and lies well below the sort of fluctuation obtained in estimating optical model parameters from analysis of experiment.

6. ELASTIC SCATTERING AND POLARIZATION IN LIGHT NUCLEI

As a further comparison with the theoretical values, the cross sections and the polarization in the forward direction were calculated for C^{12} ; for this purpose no account was taken of the variation of the effective radius with energy. A Gaussian shape factor was assumed,

$$f(r) = e^{-(r/a)^2}$$

and the parameter a was taken equal to 1.96 fermi, the electron scattering value. The scattering from a Gaussian distribution has been treated by Bethe (4) in the W.K.B. approximation. If we neglect contributions from the spin dependent amplitude (4.9) gives

$$G(q) = \frac{2N^2}{(N+1)} F(q)\bar{A}(0). \quad (6.1)$$

Defining γ as

$$\gamma = -2iG(0)/ka^2, \quad (6.2)$$

the scattering amplitude g for small forward angles, taking account of corrections to the Born approximation, is given as an expansion in γ (Ref. 4, Eq. 4.25) by

$$g(x) = \frac{1}{2} i k a^2 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n! n} \gamma^n \exp(-x/n), \quad (6.3)$$

where

$$qa \equiv x \cong \frac{1}{4} k^2 a^2 \theta^2. \quad (6.4)$$

In the limit $\gamma \rightarrow 0$,

$$g(q) \rightarrow G(q), \quad (6.5)$$

the Born approximation. The total cross section is given by

$$\sigma_T = \frac{4\pi}{k} \operatorname{Im} g(0), \quad (6.6)$$

and the inelastic cross section by the relation

$$\sigma_{\text{inel}} = \pi a^2 [\ln 2\gamma_0 + 0.577 - Ei(-2\gamma_0)], \quad (6.7)$$

where $\gamma_0 = \operatorname{Re} \gamma$.

For the polarization, advantage is taken of Kohler's (20) theorem, and it is calculated using the Born approximation expression (4.12). Table IX lists the Born spin-independent amplitude $G(0)$ (4.14), the Born spin-dependent amplitude $H(0)$ (4.15), the parameter γ , and the spin-independent scattering amplitude $g(0)$ after correction by the W.K.B. method. The values deduced by Bethe (4) and Wilson (21) from experiment appear in brackets. The experimental cross sections are compared with calculation in Table X.

TABLE IX

THE EFFECT OF CORRECTIONS TO THE BORN APPROXIMATION ON THE SPIN INDEPENDENT SCATTERING AMPLITUDES, CALCULATED IN THE W.K.B. APPROXIMATION. THE AMPLITUDES G , g , AND H ARE GIVEN FOR ZERO ARGUMENT, FOR THE CASE OF C^{12}

Energy (Mev)	310	156	90
G (10^{-13} cm)	$3.6 + 12.4i$	$11.9 + 10.0i$	$14.5 + 10.9i$
g (10^{-13} cm)	$1.57 + 9.10i$ ($1.7 + 8.6i$)	$4.74 + 8.87i$ ($6.5 \pm 1.0 + 7.6 \pm 0.2i$)	$6.55 + 9.70i$
H (10^{-13} cm)	$26.8 - 6.5i$ ($29.5 - (9.1)i$)	$22.5 - 7i$ (19.5 ± 4)	$13.3 - 4.5i$
γ	$1.55 - 0.45i$	$1.83 - 2.17i$	$2.67 - 3.55i$

TABLE X

CARBON SCATTERING DATA COMPARED WITH THE THEORY. THE TABLE CONTAINS THE CALCULATED POLARIZATION AND CROSS SECTIONS TOGETHER WITH THE EXPERIMENTAL VALUE FOR THESE QUANTITIES (IN BRACKETS)

Energy (Mev)	310	156	135	90
$\frac{d\sigma}{d\Omega_{lab}^{(\theta)}} \left(\frac{\text{barns}}{\text{sterad}} \right)$	0.85 (1.0) ^a	1.0 (1.2)	(1.18 ± 0.08)	1.4 $(\sim 1.7)^b$
σ_T (barns)	0.27 (0.29) ^c	0.39 (0.31) ^c	$(0.36 \pm .01)$	0.57 (0.54) ^c
σ_{inel} (barns)	0.21 (0.20) ^d	0.23 (0.22) ^d	$(0.27 \pm .01)$	0.27 (0.23) ^d
$(P/\theta_L) \left(\frac{\text{percent}}{\text{degree}} \right)$	7.4 (8.4) ^e	4.1 (5.0) ^f		2.2 (2.1) ^e

^a See Ref. 47.

^b See Ref. 48.

^c See Ref. 51.

^d See Ref. 52.

^e See Ref. 49.

^f See Ref. 50.

The agreement in Table X is good, so that for light nuclei there is no disagreement between the predictions and experiment. It has, however, to be borne in mind that the above calculation for the cross sections neglects the effect of the spin-dependent part of the potential. Taking this into account gives a correction¹⁵

¹⁵ This correction is obtained by taking the integral equation for T and retaining only the δ -function in the free propagator, i.e., neglecting virtual scattering. The approximation is also made that, as a function of q , T has its Born approximation form, with a constant multiplier which is determined by solving the resultant algebraic equation when the integrations have been performed. Neglecting spin-dependent terms, for instance, this gives

$$g(0) = G(0)/(1 + \frac{1}{4}\gamma), \quad (6.8a)$$

which is equivalent to Eq. (6.3) for small γ . Including spin-dependent terms this gives the coupled equations for the forward scattering amplitudes:

$$g(0) = G - \frac{1}{4}\gamma \left\{ g + \frac{2h}{k^2 a^2} \left(\frac{H}{G} \right) \right\}, \quad (6.8b)$$

$$h(0) = H - \frac{1}{4}\gamma \left\{ \frac{1}{2}h + \frac{1}{2}g \left(\frac{H}{G} \right) + \frac{4}{k^2 a^2} h \left(\frac{H}{G} \right) \right\}. \quad (6.8c)$$

If the terms in $1/k^2 a^2$ are neglected, the above equations result in

$$\frac{\delta g(0)}{g(0)} \sim -\frac{1}{2} \frac{\gamma}{k^2 a^2} \left(\frac{H}{G}\right)^2, \quad (6.8)$$

which causes an increase in the forward scattering amplitude $\sim 5\%$ at 310 Mev and $\sim 10\%$ at 156 and 90 Mev. Also as only the charge radius has been used, no allowance has been made for increase of the radius due to the variation with momentum transfer of the two-nucleon scattering. These effects are small at 310 Mev but non-negligible at 156 and 90 Mev.

The scattering data from carbon have recently been analyzed by Wilson (21) applying Bethe's analysis (4) to the 313-Mev proton and 350-Mev neutron data, and the neutron and proton data at 135 and 155 Mev. Wilson finds the variation of the two-nucleon scattering with momentum transfer, as evinced by the effective carbon radius, to be an important effect. He finds $\langle R^2 \rangle \sim 7.3$ fermi² at 300 Mev, and $\langle R^2 \rangle \sim 9$ fermi² at 150 Mev, in rough agreement with Table V. However he finds it necessary to use a smaller radius for the imaginary part of the potential to fit the absorption cross section. A more careful analysis would be of interest in establishing the best experimental value of the optical potential for the small angle scattering observed.

The good agreement for the polarization holds only for forward angles as is shown in Fig. 1. In Fig. 1 is given the polarization as a function of scattering angle, calculated from Eq. (4.12) which takes into account the variation of the two body scattering with angle. The polarization is also given as a function of scattering angle in Table XI. At large angles diffraction effects due to nuclear size and shape come into play. On the curve for 156 Mev are shown Harding's (22) experiments on neutron polarization which fit the calculated polarization excluding Coulomb effects. A W.K.B. calculation of the polarization at small forward angles, including the effects of Coulomb interference, and using phase shifts obtained from the Signel-Marshak (23), as well as the Gammel-Thaler (7) potential, has been published by Ohnuma (9).

The corrections to the Born approximation as shown in Table IX affect mostly the real part of the forward scattering amplitude. As one goes to lighter elements, the parameter γ , which serves as a measure of the importance of the W.K.B. corrections, diminishes so that for nucleon-deuteron and helium scattering, the

$$h \cong H/(1 + \frac{1}{4}\gamma), \quad g \cong G/(1 + \frac{1}{4}\gamma); \quad h/g \cong H/G, \quad (6.8d)$$

which is Köhler's theorem (20). Taking these terms into account to first order, the correction terms are given by

$$\frac{\delta g}{g} \simeq -\frac{\gamma}{2k^2 a^2} \left(\frac{H}{G}\right)^2 \frac{1}{(1 + \frac{1}{4}\gamma)} \quad \frac{\delta h}{h} \simeq -\frac{\gamma}{k^2 a^2} \left(\frac{H}{G}\right) \frac{1}{(1 + \frac{1}{8}\gamma)}. \quad (6.8e)$$

These expressions give an order of magnitude estimate of the coupling between spin-dependent and spin-independent terms arising from multiple scattering.

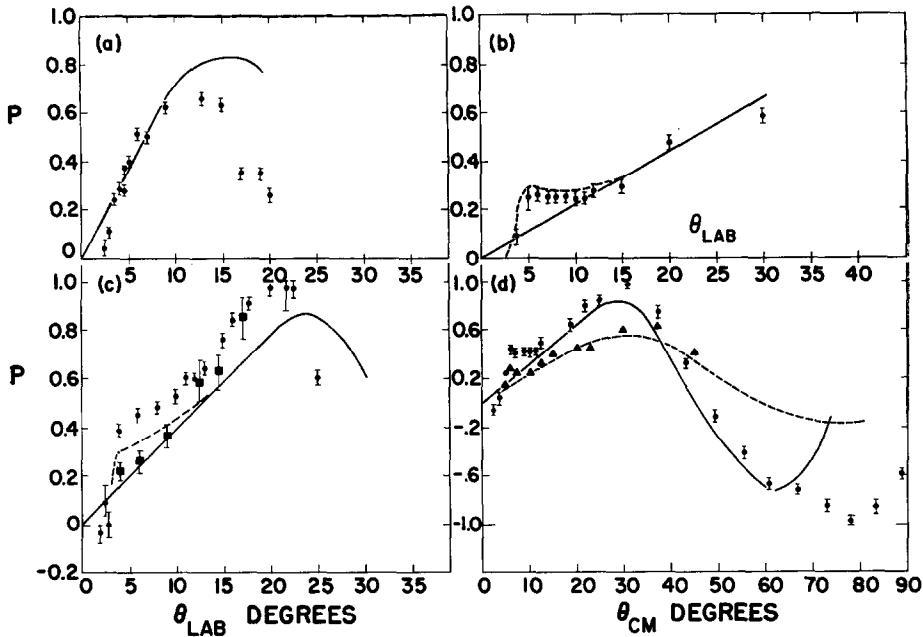


FIG. 1. Polarization in elastic scattering. In panel (a) is given the theoretical prediction for polarization of neutrons scattered from a spin zero nucleus at 310 Mev. The experimental points are for proton-carbon scattering (49). In panel (b) results for 90 Mev are given. The dashed curve represents the Coulomb correction for protons taken from Ohnuma (9). The experimental points are for proton-carbon scattering and are due to Dickson and Salter (51). In panel (c) results for 156 Mev are given. The Coulomb correction is again due to Ohnuma (9). The experimental points indicated by squares are for neutron-carbon scattering (22). The experimental points indicated by dots are for proton-carbon scattering (50). In panel (d) are given results for neutron-deuteron and neutron-alpha scattering at 156 Mev. The solid curve represents the theoretical prediction for neutron-alpha scattering, the dashed curve the prediction for neutron-deuteron scattering. The experimental points indicated by dots are for proton-alpha scattering (24). The triangles represent the experimental points for proton-deuteron scattering (21).

Born approximation should be fairly good. For small N , the difference between T' and T is not negligible, so that the parameter γ to be used in Eq. (6.3) should be defined as

$$\gamma = -2i \frac{(N-1)}{N} G/ka^2, \quad (6.9)$$

and g is equal to $N/(N-1)$ times the right-hand side of Eq. (6.3). This change for instance decreases γ in the case of the deuteron by a factor 2. Furthermore, for the lightest nuclei, the scattering is more nearly on the two-nucleon energy

TABLE XI

THE POLARIZATION FOR NUCLEONS SCATTERED BY A SPIN ZERO NUCLEUS FOR WHICH
 $Z = N/2$. $P = 2 \operatorname{Re}(A^*C)/(|A|^2 + |C|^2)$

θ^0	$E = 40$ Mev	$E = 90$ Mev	$E = 156$ Mev	$E = 310$ Mev
2	0.0019	0.0217	0.0396	0.0689
4	0.0038	0.0433	0.0791	0.138
6	0.0057	0.0651	0.119	0.208
8	0.0076	0.0869	0.159	0.278
10	0.0096	0.1088	0.198	0.348
20	0.0202	0.220	0.399	0.694
30	0.0323	0.337	0.597	0.923
40	0.0456	0.456	0.762	0.906
60	0.0690	0.661	0.619	0.412
80	0.0704	0.743	-0.120	-0.133
100	0.0389	0.754	-0.556	-0.411
120	-0.0054	0.304	-0.701	-0.653
140	-0.0311	0.0846	-0.423	-0.766
160	-0.0261	0.0066	-0.155	-0.187

shell and so the q dependence is reliable for larger values of q . We cannot now neglect the variation of the two nucleon amplitude with momentum transfer in estimating the multiple scattering correction. So in (6.9) a^2 should be replaced by a value which takes this variation into account.

For the α -particle, from electron scattering measurements (18),

$$F(q) = e^{-1/4a^2q^2}, \quad (6.10)$$

where $a = 1.45$ fermi, corresponding to a mean square radius $\langle R^2 \rangle = 3.15$ fermi 2 , a number of the same order of magnitude as the increments to the mean square radii (r_i^2) coming from the variation of the two-nucleon scattering amplitude. More accurately, a correction should be made for the electromagnetic size of the proton, so that if

$$\langle R^2 \rangle = \langle R_0^2 \rangle + \langle r_p^2 \rangle, \quad (6.11)$$

where $\langle r_p^2 \rangle$ is the electromagnetic size of the proton, ~ 0.64 fermi 2 , then

$$\langle R_0^2 \rangle = 2.51 \text{ fermi}^2,$$

or $a = 1.29$ fermi. As the Born approximation should be reasonably good for such a small target we have from (4.9) and (2.16)

$$\frac{d\sigma}{d\Omega} \cong |G_B|^2 = N^2 \left(\frac{2N}{N+1} \right)^2 \{ |A(q)|^2 + |C(q)|^2 \} |F(q)|^2 \quad (6.12)$$

or

$$\frac{d\sigma}{d\Omega}/|F|^2 = N^2 \left(\frac{2N}{N+1} \right)^2 \{ |A(q)|^2 + |C(q)|^2 \}. \quad (6.12a)$$

TABLE XII

PARAMETERS REQUIRED FOR CALCULATION OF $n - \alpha$ SCATTERING. Fermi = 10^{-13} cm

<i>E</i>	310	156	90
$k(\text{fermi}^{-1})^a$	3.18	2.23	1.68
$\bar{a}^2(\text{fermi}^2)$	2.8	4.0	4.0
$G(\text{fermi})$	$0.92 + 3.16i$	$3.08 + 2.60i$	$3.84 + 2.88i$
γ	$0.52 - 0.15i$	$0.43 - 0.51i$	$0.63 - 0.84i$
$g(\text{fermi})$	$0.72 + 2.81i$	$2.48 + 2.63i$	$2.77 + 2.90i$

^a The values of k are taken from Table XIV.

Corrections to the Born approximation were taken into account by renormalizing expression (6.12) by the W.K.B. correction to the forward scattering amplitude (6.8a) (6.9). Table XII gives a list of the parameters involved in estimating this correction, which is $\sim 20\%$ at 310 and 156 Mev. A comparison of the calculated and experimental cross sections is shown in Fig. 2 for 310 and 147 Mev. The difference between the two curves at 310 Mev shows the effect of the electromagnetic size of the proton, no correction for this having been made in the dotted curve. Even with corrections to the Born approximation included, the calculated scattering is larger than the experimental, though the variation with momentum transfer is given fairly well. For this latter the variation of the two-nucleon scattering amplitude is important as is demonstrated by a comparison of the experimental values of the left-hand side of Eq. (6.12a) with the theoretical values of the right-hand side [Cormack *et al.* (24)].

The deuteron has spin 1, which changes some of the previous analysis. In this case (see Section VIII and also Tamor (25) in the Born approximation). The differential cross section is given by

$$\frac{d\sigma}{d\Omega} = N^2 \left(\frac{2N}{N+1} \right)^2 |F_D(q)|^2 |M|^2, \quad (6.13)$$

and the polarization by

$$P_D = 2 \operatorname{Re}\{A^+ C + \frac{2}{3}B^+ C\} / |M|^2, \quad (6.14)$$

where

$$|M|^2 = |A|^2 + \frac{5}{3}|C|^2 + \frac{2}{3}\{|E|^2 + |F|^2 + |B|^2\}, \quad (6.15)$$

and $F_D(q)$ is the form factor for the deuteron;

$$F_D(q) = \int \exp(i\mathbf{q} \cdot \mathbf{r}/2) |\Phi_D(\mathbf{r})|^2 d\mathbf{r}. \quad (6.16)$$

The cross section is shown in Fig. 3 compared with the Harvard 147-Mev results. The effect of the deuteron *D*-state was ignored and the deuteron wave

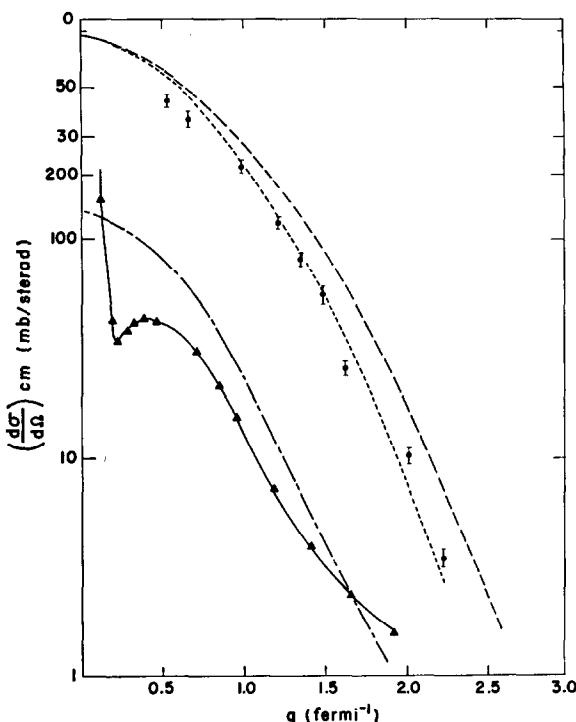


FIG. 2. The nucleon-alpha elastic scattering angular distribution. The corrections to the Born approximation have been included as described in Section 6. The upper curves are for neutron-alpha scattering at 310 Mev. The experimental points are for proton-alpha scattering (49). The two curves illustrate the effect of the electromagnetic size of the nucleon. The curve given by the short dashes has not been corrected for the finite size of the nucleon. The lower curves are for 147 Mev. The solid curve represents the experimental data (24). The curve represented by alternate long and short dashes gives the theoretical prediction at 156 Mev.

function taken to be $u = r\psi = e^{-\gamma r} - e^{-\delta r}$ with $\gamma = 0.23$ and $\delta = 1.2 \text{ fermi}^{-1}$. The effects of Coulomb interference have been calculated by Postma (26), and by Singh and Sawicki (27). The predicted polarization for the deuteron is given in Table XIII, and is shown for 156 Mev together with the polarization for helium in Fig. 1, panel *d*, and with the Harvard experimental values (28).

7. THE EFFECT OF NUCLEAR CORRELATIONS

For the light nuclei the forward scattering shows reasonable agreement with the two-nucleon phase shifts except for the discrepancy in the magnitude of the He^4 cross sections. On the other hand, in the average behaviour of nuclear scattering as evinced by fits to the optical model parameters, there is a discrepancy

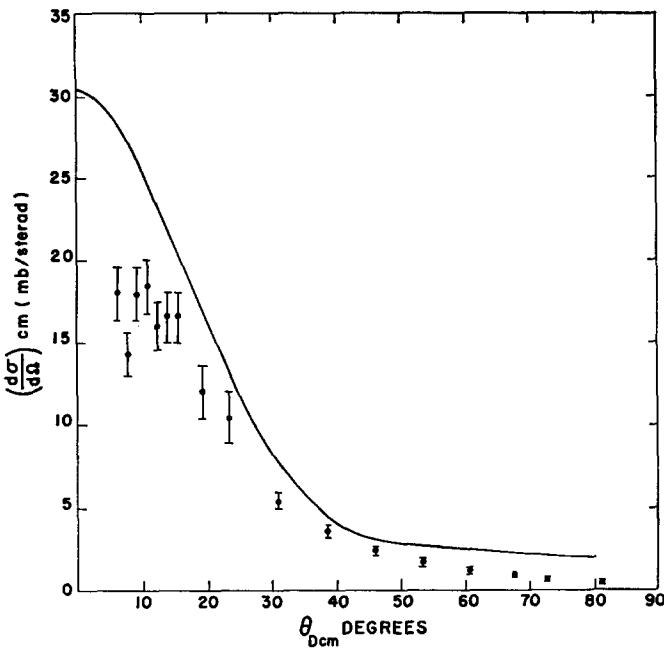


FIG. 3. The 156-Mev nucleon-deuteron elastic scattering angular distribution. The solid curve represents the theoretical prediction for neutron-deuteron scattering at 156 Mev. The experimental points are for 156-Mev proton-deuteron scattering (21).

TABLE XIII
POLARIZATION OF NUCLEONS SCATTERED FROM DEUTERIUM

θ^0	$E = 40 \text{ Mev}$	$E = 90 \text{ Mev}$	$E = 156 \text{ Mev}$	$E = 310 \text{ Mev}$
2	0.0020	0.0208	0.0352	0.0611
4	0.0039	0.0416	0.0703	0.122
6	0.0059	0.0625	0.105	0.182
8	0.0079	0.0833	0.140	0.242
10	0.0100	0.104	0.175	0.301
20	0.0231	0.207	0.340	0.552
30	0.0343	0.306	0.475	0.655
40	0.0477	0.390	0.544	0.587
60	0.0651	0.467	0.369	+0.263
80	0.0512	0.343	0.0233	-0.0528
100	0.0123	0.199	-0.145	-0.231
120	-0.0225	0.0921	-0.144	-0.282
140	-0.0367	0.0422	-0.102	-0.256
160	-0.0259	0.0130	-0.0360	-0.0580

in the imaginary part of the central potential. It is hard to say how real this is without repeating the optical model calculations, giving greater weight to the small angle data, but it is clear that at some energy the approximations used here must break down. Thus at 40 Mev, for a heavy nucleus, we obtain a value for the central potential at the origin, $V_{e0} = 49 + 54i$ Mev, which has too large an imaginary part. We must therefore look at the corrections to the multiple scattering and impulse approximations. Let us first consider the correction to the multiple scattering approximation which depends on the nuclear two-particle correlation function. This correction is given by Eq. (3.39). Dropping terms $\sim O(1/N)$ and making the lowest order approximation to the wave matrix $\tilde{\omega}_{nn}^R$, i.e.,

$$\tilde{\omega}_{nn}^R \approx \delta(\mathbf{k}'' - \mathbf{k}''), \quad (7.1)$$

we have from (3.40), (3.38), changing the variable of integration \mathbf{k}'' to

$$\begin{aligned} \mathbf{k}'' - \mathbf{k} &= \mathbf{q}' \\ \delta U_E(\mathbf{q}) \equiv \delta U_{00}(E, \mathbf{q}) &\approx N^2 \int d\mathbf{q}' d\mathbf{x} d\mathbf{y} \frac{t^{(0)}(E', \mathbf{q} - \mathbf{q}') t^0(E, \mathbf{q}')}{E - \bar{E}(\mathbf{k} + \mathbf{q}') + i\epsilon} \\ &\cdot C(\mathbf{x}, \mathbf{y}) \exp\{-i(\mathbf{q} - \mathbf{q}') \cdot \mathbf{x}\} \exp\{-i\mathbf{q}' \cdot \mathbf{y}\}, \end{aligned} \quad (7.2)$$

where $C(\mathbf{x}, \mathbf{y}) \equiv C_{00}(\mathbf{x}, \mathbf{y})$ is the pair correlation function of the nucleus obeying the integral condition

$$\int C(\mathbf{x}, \mathbf{y}) d\mathbf{x} d\mathbf{y} = 0, \quad (3.30a)$$

and where \bar{E} is

$$\bar{E}(\mathbf{k} + \mathbf{q}') = \bar{E}_n + \hbar^2/2m (\mathbf{k} + \mathbf{q}')^2. \quad (7.3)$$

Here \bar{E}_n is the average nuclear excitation involved in the closure approximation. If we use the closure approximation for each momentum transfer, \mathbf{q}' , separately, \bar{E}_n will be equal to the recoil energy of the struck particle in a free two nucleon collision. Thus

$$E - \bar{E}(\mathbf{k} + \mathbf{q}') \approx -(\hbar^2/m)(q'^2 + \mathbf{k} \cdot \mathbf{q}'). \quad (7.4)$$

The pair correlation function is essentially unknown, though from the work of Brueckner (29) and Weisskopf (30) we expect the correlation properties of nuclear matter to be governed mainly by the repulsive core in the interaction potential with radius ~ 0.4 fermi, and the characteristic correlation distance of the Fermi distribution $\lambda_f \sim 0.7$ fermi. As the mean distance between nucleons is ~ 1.7 fermi, nuclear matter is not dense in the sense of a liquid, so the correlation

TABLE XIV

Two-body center-of-mass angle (θ_0) versus center-of-mass angle (θ) and momentum transfer (q) for various targets (N) and incident energies (E_L), together with two-body center-of-mass momentum (k_0) and center-of-mass momentum (k).

$E_L = 310 \text{ Mev}$		$k_0 = 1.93 \text{ fm}^{-1}$	
θ_0 degrees	θ degrees	$q(\text{fm}^{-1})$	
		$N = 2$ $k = 2.60$	$N = 4$ $k = 3.18$
2	1.48	1.20	1.04
6	4.46	3.64	3.06
10	7.42	6.06	5.14
20	14.80	12.10	10.24
30	22.14	18.06	15.26
40	29.40	23.96	20.22
60	43.56	35.34	29.74
80	57.00	45.94	38.54
100	69.30	55.40	46.30
120	80.00	63.44	52.78
140	88.46	69.54	57.66
160	93.94	73.40	60.74
180	95.86	74.74	61.76

$E_L = 156 \text{ Mev}$		$k_0 = 1.37 \text{ fm}^{-1}$	
θ_0 degrees	θ degrees	$q(\text{fm}^{-1})$	
		$N = 2$ $k = 1.84$	$N = 4$ $k = 2.23$
2	1.48	1.24	1.06
6	4.46	3.68	3.16
10	7.41	6.14	5.26
20	14.86	12.24	10.50
30	22.22	18.30	15.68
40	29.50	24.26	20.76
60	43.70	35.78	30.54
80	57.20	46.52	39.60
100	69.54	56.14	47.60
120	80.30	64.28	54.30
140	88.80	70.52	59.36
160	94.34	74.46	62.52
180	96.24	75.80	63.60

θ_0 degrees	θ degrees	$q(\text{fm}^{-1})$	
		$N = 2$ $k = 2.60$	$N = \infty$ $k = 2.85$
2	1.48	1.24	1.06
6	4.46	3.68	3.16
10	7.41	6.14	5.26
20	14.86	12.24	10.50
30	22.22	18.30	15.68
40	29.50	24.26	20.76
60	43.70	35.78	30.54
80	57.20	46.52	39.60
100	69.54	56.14	47.60
120	80.30	64.28	54.30
140	88.80	70.52	59.36
160	94.34	74.46	62.52
180	96.24	75.80	63.60

TABLE XIV (Continued)

θ_0 degrees	$E_L = 90$ Mev		$k_0 = 1.04 f^{-1}$		$q(f^{-1})$
	$N = 2$ $k = 1.39$	$N = 4$ $k = 1.68$	$N = 12$ $k = 1.96$	$N = \infty$ $k = 2.13$	
2	1.50	1.24	1.06	0.96	0.036
6	4.50	3.72	3.18	2.94	0.109
10	7.48	6.22	5.30	4.86	0.181
20	14.94	12.34	10.56	9.74	0.361
30	22.34	18.44	15.78	14.52	0.538
40	29.66	24.44	20.90	19.22	0.711
60	43.94	36.06	30.76	28.26	1.04
80	57.50	46.90	39.88	36.58	1.34
100	69.94	56.62	47.96	43.94	1.59
120	80.76	64.84	54.72	50.04	1.80
140	89.34	71.14	59.82	54.64	1.95
160	94.94	75.14	63.00	57.48	2.05
180	96.86	76.50	64.10	58.44	2.08

function might not be too different from a Debye distribution. Thus we may write

$$C(\mathbf{x}, \mathbf{y}) \sim \rho(\mathbf{x})\rho(\mathbf{y})F_c(\mathbf{x} - \mathbf{y}) \quad (7.5)$$

with

$$\begin{aligned} F_c(r) &= -1, & r < r_0, \\ &= 0, & r > r_0, \end{aligned} \quad (7.6)$$

where r_0 is a distance characterizing the nuclear pair correlation function. Due to the effect of the long range attractive forces between nucleons, one would expect F_c in practice to rise rapidly to zero around a distance r_0 , to a positive maximum to ensure the proper normalization, and behave as a damped oscillation at large distances. Most of the effects should be given by a function of the form (7.6).

With the above form for the correlation function, using the fact that for a large nucleus where $r_0 \ll R$,

$$\rho(\mathbf{y}) = \rho(\mathbf{x} - \xi) \simeq \rho(\mathbf{x}), \quad (7.7)$$

where $\mathbf{x} - \mathbf{y} = \xi$, and $\rho^2(\mathbf{x}) \sim (1/\Omega)\rho(\mathbf{x})$, where Ω is the nuclear volume (5.28), Eq. (7.2) reduces to

$$\delta U_B(\mathbf{q}) \cong -(N^2 m / \hbar^2) t^2(0) / \Omega$$

$$\int d\mathbf{q}' d\mathbf{x} d\xi \frac{F_c(\xi) \rho(\mathbf{x}) \exp\{-i(\mathbf{q}' \cdot \xi + \mathbf{q} \cdot \mathbf{x})\}}{q'^2 + \mathbf{k} \cdot \mathbf{q}' - i\epsilon}. \quad (7.8)$$

In the above we have made the approximation that $t_0(\mathbf{q})$ is sufficiently slowly varying with energy and with momentum transfer to be replaced by $t(0)$ in the integrand. This substitution will somewhat over-estimate the effect.

If we look mainly at small momentum transfer, $\mathbf{q} \sim 0$, we can express the result as a change in V_{c0} , the central potential at $r = 0$ (5.25). Since [using (5.29) and (3.14)]

$$t(0) = -\frac{\Omega}{N} \frac{1}{(2\pi)^3} V_{c0} \quad (7.9)$$

and using (5.2), (5.25), and (5.28)

$$V_{c0} = -(2\pi)^3 U_E(\mathbf{q})/\Omega = V_{cR} + iV_{cI}, \quad (7.10)$$

(7.8) can be written

$$\delta V_{c0} = \frac{V_{c0}^2}{(2\pi)^3} \left(\frac{m}{\hbar^2} \right) \int d\mathbf{q} d\xi \frac{F_c(\xi) \exp(-i\mathbf{q} \cdot \xi)}{q^2 + \mathbf{k} \cdot \mathbf{q} - i\epsilon}. \quad (7.11)$$

Using $F_c(\xi)$ of the form (7.6), Eq. (7.11) becomes

$$\delta V_{c0} = \frac{-V_{c0}^2}{2E} f(kr_0), \quad (7.12)$$

where

$$f(kr_0) = (1 - \cos kr_0) + i(kr_0 - \sin kr_0). \quad (7.13)$$

The off the energy shell contribution, $(1 - \cos kr_0)$ is a purely oscillatory function. At sufficiently high energies the dominating term in $f(kr_0)$ is ikr_0 coming from the energy shell contribution so that (7.12) becomes

$$\delta V_{c0} \approx \frac{-iV_{c0}^2}{2E} kr_0, \quad (7.14)$$

which is the form discussed by Watson.

To see whether the condition

$$\int F(\xi) d\xi = 0 \quad (7.15)$$

has any effect on the results $F(\xi)$ was modified by making it a positive constant from $\xi = r_0$ to $\xi = 2r_0$, choosing the constant so that (7.15) was obeyed. The result is again Eq. (7.12) but with $f(kr_0) \rightarrow 8/7f(kr_0) - 1/7f(2kr_0)$. The difference was negligible except at 90 Mev.

Table XV lists δV_{c0} for $r_0 = 0.5, 0.75$, and 1.0 fermi; the sign of δV_{c0} is adjusted to have the same sign convention as $(V_{cR} + iV_{cI})$ in Table VII.

Much of the contribution to the imaginary part comes from the off the energy shell contribution at 156 and 90 Mev, but the trend with energy is independent

TABLE XV
ESTIMATE OF THE CORRECTION TO THE CENTRAL POTENTIAL AT THE ORIGIN
DUE TO TWO-BODY NUCLEAR CORRELATIONS

Energy (Mev)	δV_{c0} (Mev)		
	$r_0 = 0.5$	0.75	1.0
310	$1.9 - 0.2i$	$3.8 + 1.3i$	$4.2 + 3.7i$
156	$-3.1i$	$3.4 - 7.0i$	$8.1 - 10.1i$
90	$-1 - 4.3i$	$-10.8i$	$3.0 - 20i$

of this. If we neglect the off the energy shell contribution and drop the oscillatory part of the rest, we may write from (7.14) and (7.10);

$$\delta V_{c0} = \frac{-kr_0}{2E} \{ -2V_{cR}V_{cI} + i(V_{cR}^2 - V_{cI}^2) \}. \quad (7.16)$$

Thus if $|V_{cR}| > |V_{cI}|$, the correction to the imaginary part of the potential results in a net decrease in magnitude (remembering the sign convention). On the other hand if the magnitude of imaginary part of the potential is larger than the real part, the correction increases the magnitude of the imaginary part of the potential. This change-over occurs (Table VII) somewhere between 156 and 310 Mev, and is reflected in Table XII. At higher energies as V_{cR} changes sign but remains small while $|V_{cI}|$ increases, the correlation contribution continues to give an increase in the magnitude of the imaginary part of the potential. This is the effect discussed by Glauber (31).

Looking at Table XV it is clear that the correlation contribution could be sizeable at lower energies, but the actual numerical estimate is poor, as we have neglected the momentum dependence of the two-body scattering, and used only the lowest order approximation to the wave-matrix appearing in (3.40). This term, which represents the effect of distorted waves in intermediate states, can be estimated by writing (see Eqs. 2.46, 2.53)

$$\omega^R = 1 - \frac{1}{E - H_0} + i\epsilon T_D, \quad (7.17)$$

where T_D is the scattering matrix for elastic scattering in an excited state. The net effect is to introduce into the integrand of Eq. (7.8) an extra factor

$$\begin{aligned} \int \exp(-i\mathbf{q}'' \cdot \mathbf{x}) \rho(\mathbf{x}) d\mathbf{x} d\mathbf{q}'' \left\{ \delta(\mathbf{q}'') - \frac{1}{2\pi^2} \frac{1}{E(\mathbf{q}'') + i\epsilon} g(\mathbf{q}'') \right\} \\ = \int d\mathbf{q}'' F(\mathbf{q}'') \left\{ \delta(\mathbf{q}'') - \frac{1}{2\pi^2} \frac{1}{E(\mathbf{q}'') + i\epsilon} g(\mathbf{q}'') \right\}, \end{aligned} \quad (7.18)$$

where $g(\mathbf{q}'')$ is the scattering amplitude for elastic scattering in an average excited state with momentum transfer \mathbf{q}'' , $F(\mathbf{q}'')$ is the nuclear form factor, and

$$E(\mathbf{q}'') = -2q'^2 + 2\mathbf{k} \cdot \mathbf{q}' - \mathbf{q}''^2 + 2\mathbf{q}'' \cdot (\mathbf{k} + \mathbf{q}'). \quad (7.19)$$

Apart from the term $-2q'^2 + 2\mathbf{k} \cdot \mathbf{q}'$ in the propagator, (7.18) is simply the ratio of the scattering amplitude to the Born scattering amplitude at zero momentum transfer. As most of the contribution to the integral in (7.8) comes from a small solid angle around the region where this term vanishes, the effect of the factor (7.19) is approximately to multiply the expression (7.12) by this ratio, so that

$$\delta V_{e0} \sim \frac{V_{e0}^2}{2E} f(kr_0) \frac{g(0)}{G(0)}. \quad (7.20)$$

The correction factor $g(0)/G(0)$ is not independent of nuclear size and shape, and simply gives the effect of the shielding of nucleons in the interior of the nucleus due to refraction and attenuation. It can be roughly estimated on the lines of Bethe's calculation with the result for $N \sim 100$, and for $r_0 = 1$ fermi.

E	$g(0)/G(0)$	δV_{e0} (Mev)
310	$\sim 0.55 - 0.10i$	$\sim 2.5 + 1.5i$
156	$\sim 0.40 - 0.20i$	$\sim 1.0 - 6.0i$
90	$\sim 0.30 - 0.20i$	$\sim -3.0 - 6.5i$

Comparison of (7.21) with the last column of Table XV illustrates the effect of distorted waves. In summary, the effect of pair correlations in the nucleus is to decrease the magnitude of the imaginary part of the potential at 90 and 156 Mev and to increase it at energies $\gtrsim 300$ Mev, which is roughly what is required by the data. The above approximations do not obtain for light nuclei. For the tightly bound alpha particle one would expect these corrections to be especially large. This may account for the discrepancy in the magnitude of the nucleon-alpha scattering cross section.

For the deuteron Glauber (32) has indicated that the correlation correction may in fact have the opposite sign to that in other nuclei.

There is some hope of getting more direct experimental evidence for light elements about the pair correlation function. For in the Born approximation, the differential cross section without restriction of the energy of the scattered nucleon becomes

$$\frac{d\sigma}{d\Omega} \approx \sum_n |G_{0n}(\mathbf{q}_n)|^2 = N^2 \sum_n (k_n^2/k_0^2) |\bar{M}(\mathbf{q}_n)|^2 F_{0n}(\mathbf{q}_n) F_{n0}(-\mathbf{q}_n).$$

With the aid of the sum rule (3.12), and closure, this becomes

$$\frac{d\sigma}{d\Omega} \approx N^2 k^2 / k_0^2 | \bar{M}(\bar{\mathbf{q}}) |^2 \cdot \left[\frac{1}{N} + \left(1 - \frac{1}{N} \right) F_{00}^2(\bar{\mathbf{q}}) + \left(1 - \frac{1}{N} \right) C_{00}(\bar{\mathbf{q}}, -\bar{\mathbf{q}}) \right]. \quad (7.22)$$

Here \bar{q} is an average momentum transfer for fixed scattering angle defined by closure. To first approximation $\bar{\mathbf{q}}$ will be the momentum defined by free nucleon-nucleon scattering. If elastic scattering is excluded

$$\begin{aligned} \left(\frac{d\sigma}{d\Omega} \right)_{n.e.} &\approx \sum_{n \neq 0} | G_{0n}(\mathbf{q}_n) |^2 \\ &\approx N^2 (k^2 / k_0^2) | \bar{M}(\bar{\mathbf{q}}) |^2 \left[\left(1 - \frac{1}{N} \right) C_{00}(\bar{\mathbf{q}}, -\bar{\mathbf{q}}) + \frac{1}{N} (1 - F_{00}^2(\bar{\mathbf{q}})) \right], \end{aligned} \quad (7.23)$$

which is more directly related to the pair correlation function. Distorted wave effects will obscure the relationship, but to a lesser extent in very light nuclei. A discussion of the effect of distorted waves and an improved closure approximation has been given by Fowler (3).

For the deuteron, using (3.13d), the cross section, (7.22) becomes simply proportional to the average free nucleon-nucleon cross section, as expected.

8. POLARIZATION IN INELASTIC SCATTERING

Let us consider inelastic scattering, leaving the target in a definite excited state. In the lowest possible approximation, we may write

$$T_{n0} = - \frac{1}{2\pi^2} \frac{\hbar^2}{m} N \langle n | M(\mathbf{q}) \exp(-i\mathbf{q} \cdot \mathbf{r}) | 0 \rangle. \quad (8.1)$$

Using (4.1) for the spin-dependence of $M(q)$ and remembering the isotopic spin dependence (4.3) we find (see Appendix III) that the differential cross section I_0 , and polarization P , for an initially unpolarized beam, are given by the expressions

$$\begin{aligned} I_0 &= \sum_l \frac{1}{2l+1} \left\{ | \mathbf{A} \cdot \mathbf{N}_l |^2 + | \mathbf{C} \cdot \mathbf{N}_l |^2 + \frac{1}{2} \sum_k | \mathbf{B} \cdot \mathbf{Q}_{kl} |^2 + | \mathbf{C} \cdot \mathbf{Q}_{kl} |^2 \right. \\ &\quad \left. + | \mathbf{F} \cdot \mathbf{Q}_{kl} |^2 \right\} + \sum_{l' \neq k} \frac{1}{(2k+1)} (1l00 | k0) (1l'00 | k0) \\ &\quad \cdot \left\{ (\mathbf{E} \cdot \mathbf{Q}_{kl})(\mathbf{E} \cdot \mathbf{Q}_{kl'})^* - \frac{1}{2} [(\mathbf{B} \cdot \mathbf{Q}_{kl})(\mathbf{B} \cdot \mathbf{Q}_{kl'})^* + (\mathbf{C} \cdot \mathbf{Q}_{kl})(\mathbf{C} \cdot \mathbf{Q}_{kl'})^* \right. \\ &\quad \left. + (\mathbf{F} \cdot \mathbf{Q}_{kl})(\mathbf{F} \cdot \mathbf{Q}_{kl'})^*] \right\}, \end{aligned} \quad (8.2)$$

$$\begin{aligned} I_0 P = & \sum_l \frac{1}{2l+1} \{ 2 \operatorname{Re}(\mathbf{A} \cdot \mathbf{N}_l)(\mathbf{C} \cdot \mathbf{N}_l)^* + \sum_k \operatorname{Re}(\mathbf{B} \cdot \mathbf{Q}_{kl})(\mathbf{C} \cdot \mathbf{Q}_{kl})^* \} \\ & - \sum_{l'k} \frac{1}{2k+1} (1l00 | k0)(1l'00 | k0) \operatorname{Re}(\mathbf{B} \cdot \mathbf{Q}_{kl})(\mathbf{C} \cdot \mathbf{Q}_{kl'})^* \end{aligned} \quad (8.3)$$

Here the products $\mathbf{A} \cdot \mathbf{N}_l$, etc., come from the isotopic spin dependence of the two-nucleon amplitudes (4.3)

$$\mathbf{A} \cdot \mathbf{N}_l \equiv A_\alpha N_{l\alpha} + A_\beta N_{l\beta}, \quad (8.4)$$

where

$$A_\alpha = \frac{1}{4}(3A_1 + A_0) \quad A_\beta = \frac{1}{4}(A_1 - A_0) \quad (8.5)$$

and

$$N_{l\alpha} = \langle JT \parallel \rho_l Y_l \parallel J_0 T_0 \rangle, \quad (8.6)$$

$$N_{l\beta} = f(T, T_0) \langle JT \parallel \rho_l Y_l \tau^1 \parallel J_0 T_0 \rangle \quad (8.7)$$

with

$$\rho_l = [l] \sqrt{4\pi} (-i)^l j_l(qr), \quad [l] = \sqrt{2l+1} \quad (8.8)$$

and

$$f(T, T_0) = - \sum_{q'=1}^{+1} \frac{\sqrt{3}}{[T]} (-1)^{q'} \left(1\frac{1}{2} - q' t_{03} \left| \frac{1}{2} t_3 \right. \right) (1T_0 q' T_{03} | TT_3), \quad (8.9)$$

where the last two factors are vector addition coefficients and (J_0, T_0, T_{03}) ; (J, T, T_3) are the initial and final total angular momentum and isotopic spin states of the target, τ^1 is a tensor operator of rank 1 in the isotopic spin of the target particle, $(\frac{1}{2}, t_{03})$; $(\frac{1}{2}, t_3)$ are the initial and final isotopic spin states of the incident nucleon. Finally $N_{l\beta}$, $N_{l\alpha}$, are the reduced nuclear matrix elements for a nonspin-flip transition, with and without isotopic spin flip, respectively. Similarly,

$$Q_{kl\alpha} = \langle JT \parallel \rho_l T_k(ls) \parallel J_0 T_0 \rangle, \quad (8.10)$$

$$Q_{kl\beta} = f(T, T_0) \langle JT \parallel \rho_l T_k(ls) \tau^1 \parallel J_0 T_0 \rangle \quad (8.11)$$

are the corresponding reduced nuclear matrix elements for spin-flip transitions.¹⁶

¹⁶ For small q many of these matrix elements are just the same as those encountered in nuclear β and γ decay.

As an example the matrix element for excitation of rotational states of deformed nuclei is given by

$$N_{l\alpha} \sim (J_0 K_0, K = K_0 | JK) (\chi_K | j_l(qr) Y_{l,K-K_0} | \chi_{K\delta}),$$

where the first factor is the vector addition coefficient which always appears in rotational

Here $T_k(ls)$ is a tensor operator of rank k , composed of operators in the product space (l,s) of the orbital and spin angular-momentum operators of a target nucleon. The reduced nuclear matrix elements can be calculated for shell model states via the method of fractional parentage coefficients. Formulas are given in Appendix III.

The isotopic spin-flip terms include charge exchange collisions. If charge exchange is excluded, then

$$f(T, T_0) = \pm \frac{1}{[T]} (1T_0 0T_{03} | TT_3), \quad (8.12)$$

where the positive sign is for incident protons, the negative for incident neutrons.

The complexity of the expressions (8.2) (8.3) is due almost entirely to the interference between isotopic spin-flip and nonspin-flip terms. If only one of these terms is allowed the expressions greatly simplify. In the case of elastic scattering from a nucleus for which $J = T = 0$ only $N_{0\alpha}$ is nonzero and the polarization is simply

$$P = \frac{2 \operatorname{Re}(AC)}{|A|^2 + |C|^2}, \quad (8.13a)$$

with $A = \frac{1}{4}(3A_1 + A_0)$, etc., a well-known result.

In the case where one of the states has isotopic spin 0, then for $\Delta T = 0$ only the terms labelled α enter, i.e., the combination $(\frac{3}{4}A_1 + \frac{1}{4}A_0)$, etc., in the coefficients. If on the other hand $\Delta T = 1$, only the term β enters, corresponding to combinations $\frac{1}{4}(A_1 - A_0)$. In either case the polarization can be put in the simple form.

$$P = \frac{2 \operatorname{Re}(AC^*) + 2\lambda \operatorname{Re} BC^*}{|A|^2 + |C|^2 + \lambda(|B|^2 + |C|^2 + |F|^2) + 2\mu |E|^2} \quad (8.13)$$

where

$$\lambda = \frac{L_2 - L_3}{2L_1} \quad \mu = \frac{L_3}{2L_1} \quad (8.14)$$

and

$$\begin{aligned} L_1 &= \sum_l \frac{1}{(2l+1)} |N_l|^2 & L_2 &= \sum_{lk} \frac{1}{(2l+1)} |Q_{kl}|^2 \\ L_3 &= \sum_{ll'k} \frac{1}{(2k+1)} (1l'00 | k0)(1l'00 | k0) Q_{kl} Q_{kl'}^* \end{aligned} \quad (8.15)$$

problems and leads to intensity rules, and the second factor is an intrinsic matrix element. For excitation of the first rotational state in an even-even nucleus, this matrix element is just the intrinsic quadrupole moment (for small q). Similarly the spin-flip matrix element Q_{10} is related to the factor $(g_K - g_R)$ which comes into magnetic dipole matrix elements for odd-even rotating nuclei. [See Bohr and Mottelson (33) and Kerman (33).]

Here

$$A = A_\alpha, \quad N_l = N_{l\alpha}, \quad \Delta T = 0; \quad (8.16)$$

or,

$$A = A_\beta, \quad N_l = N_{l\beta}, \quad \Delta T = 1. \quad (8.17)$$

In general the selection rules for change of isotopic spin are,

$$\Delta T = 0,$$

for the nonisotopic spin-flip matrix elements and

$$\Delta T = 0, \pm 1, \text{ no } 0 \rightarrow 0$$

for the isotopic spin-flip matrix elements. Consequently if a change of isotopic spin results from the transition, the simple formulas (8.13), (8.17) always apply.

The selection rules for the parameters l, l', k appearing in (8.2) (8.3) are, for the nonspin-flip part

$$|J_0 - J| < l < J_0 + J$$

and for the spin-flip part

$$|J_0 - J| < k < J_0 + J \quad l = k, k \pm 1.$$

In both cases the parity change $\Delta\pi$ is

$$\Delta\pi = (-1)^l.$$

For the last term in the expressions (8.2) (8.3) there is the additional restriction that $l + k + 1$ must be even and that

$$l' = l, l \pm 2.$$

For comparatively small angle scattering, in most cases the lowest allowed value of l will predominate, due to the q -dependence (8.8) of the radial integrals in the reduced nuclear matrix elements, and for such cases the polarization can be written in the form

$$P \approx \frac{2 \operatorname{Re}(\bar{A}\bar{C}^*) + 2\lambda \operatorname{Re}(\tilde{B}\tilde{C}^*)}{|\bar{A}|^2 + |\bar{C}|^2 + \lambda(|\tilde{B}|^2 + |\tilde{C}|^2 + |\tilde{F}|^2) + 2\mu|\tilde{E}|^2}, \quad (8.18)$$

where \bar{A} , \tilde{B} , etc., are appropriate averages of A_α and A_β (or A_1 and A_0). The polarization then depends on four parameters, the two isotopic spin averages of the two nucleon scattering coefficients and the two ratios λ, μ of reduced nuclear matrix elements. For the case here where only one value of l contributes, these ratios are essentially constants independent of momentum transfer.

Thus in most cases, the formulae for the polarization reduce to simple expres-

sions. Polarization in elastic scattering from the deuteron is included as a special case. If we neglect the D component of the wave function, only $l = 0$ and $k = 1$ contribute. Also as we have a $T = 0$ to $T = 0$ transition, formulas (8.13) (8.16) apply. The reduced nuclear matrix elements are trivial in this case,¹⁷ yielding $\lambda = \frac{2}{3}$, $\mu = \frac{1}{3}$ so that we have

$$P_D = \frac{2 \operatorname{Re}(AC^* + \frac{2}{3}BC^*)}{|A|^2 + \frac{5}{3}|C|^2 + \frac{2}{3}(|B|^2 + |E|^2 + |F|^2)}, \quad (8.19)$$

where A , etc., = $\frac{3}{4}A_1 + \frac{1}{4}A$. This is just the expression previously given by Tamor (25). For inelastic scattering leading to breakup of the deuteron into the 1S resonance at zero energy, only $Q_{10\beta}$ is nonzero and the polarization is given by (neglecting D state again)

$$\begin{aligned} P &= \frac{2 \operatorname{Re}(BC^*)}{\{|B|^2 + |C|^2 + |E|^2 + |F|^2\}}, \\ B &= \frac{1}{4}(B_1 - B_0), \text{etc.} \end{aligned} \quad (8.19a)$$

The polarization in inelastic scattering from a nucleus of spin and isotopic spin zero is especially simple: it depends primarily on whether the final state corresponds to "normal," i.e., $\Delta\pi = (-1)^J$ or "abnormal" ($\Delta\pi = (-1)^{J+1}$) parity change, and whether it has isotopic spin change 0 ($\Delta T = 0$) or 1 ($\Delta T = 1$). As the ground-state isotopic spin is zero the polarization is given by Eq. (8.13). The isotopic spin dependence of the scattering coefficients is given by (8.16) for $\Delta T = 0$ and by (8.17) for $\Delta T = 1$. The selection rules are $l = J$ for the nonspin-flip part and $k = J$ for the spin-flip part.

For "normal" parity change, $l = k = J$ for the spin-flip part and hence L_3 vanishes, $\mu = 0$, and the polarization depends on a single nuclear parameter λ , which is independent of momentum transfer, and which is a measure of the spin-flip/nonspin-flip matrix element ratio. Elastic scattering from even-even isotopic spin zero nuclei corresponds to this case with $\Delta T = 0$ and $\lambda = 0$, (compare (8.13) and (8.13a)) so that polarization in inelastic scattering with "normal" parity change and $\Delta T = 0$ should resemble that for elastic scattering (8.13a) if λ is small. This case covers most of the low-lying states of light elements whose polarizations have been measured by the Uppsala group, (10) viz., the 4.4-Mev and 9.6-Mev levels in C^{12} and the 6.1-Mev group in O^{16} . The 4.4-Mev level in C^{12} is 2^+ , $T = 0$ and the 9.63-Mev level is 1^- or 2^+ , $T = 0$ (34). The 6.1-Mev group in O^{16} is composed of contributions from the known levels (34) at 6.06

¹⁷ The case of nucleon-nucleon scattering is of course included in the general expression and corresponds to $\lambda = 1$, $\mu = \frac{1}{2}$ so that

$$P_{NN} = \frac{2 \operatorname{Re}\{AC^* + BC^*\}}{\{|A|^2 + |B|^2 + 2|C|^2 + |E|^2 + |F|^2\}}.$$

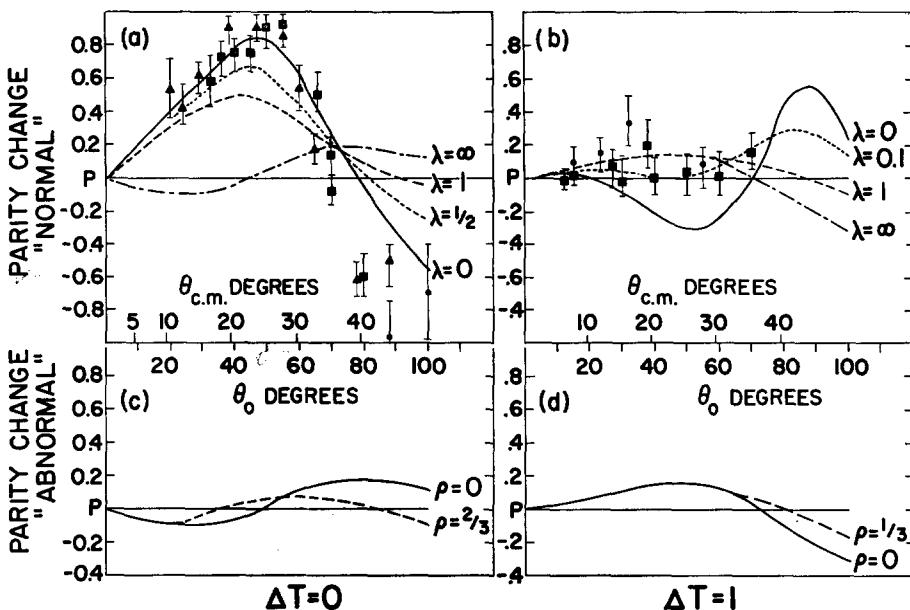


FIG. 4. Polarization in inelastic neutron scattering at 156 Mev for an even-even nucleus with spin and isotopic spin zero. In panel (a) are given theoretical curves for $\Delta T = 0$ and experimental points for the polarization of a proton which excites the 6.0-Mev level in O^{16} (triangles) (10), the 4.43-Mev level in C^{12} (10). In panel (b) the experimental points represent the excitation (10) of the "20-Mev" groups in C^{12} (squares) and O^{16} (dots), and the theoretical curves are for $\Delta T = 1$. Panels (c) and (d) contain theoretical curves for abnormal parity change and $\Delta T = 0$ and 1, respectively (8,20).

($0^+, T = 0$), 6.14 ($3^-, T = 0$), 6.92 ($2^+, T = 0$) and 7.12 ($1^-, T = 0$) Mev; the principal contribution is believed to come from the 6.14-Mev 3^- level. The measured polarization from these levels is plotted in Fig. 4(a) together with the theoretical values for 156 Mev and various values of λ . Table XVI contains values for other energies. The theoretical distribution is quite stable to small admixtures of the spin flip matrix element. Comparison with experiment clearly demands that $\lambda \sim 0$, corresponding to the observation that the polarization from these levels is identical within experimental error and very much the same as for the elastic polarization.

Now in $L \cdot S$ coupling for the C^{12} 4.4-Mev level, $\lambda = 0$, and for the 3^- state in O^{16} a shell model calculation by Elliott and Flowers (35) indicates a situation close to $L \cdot S$ coupling with mainly $S = 0$ and very small triplet admixture, so that $\lambda \sim 0$. The results then are not in disagreement with shell theory. Rather detailed calculations for the polarization from the 4.43-Mev state in C^{12} for both $L \cdot S$ and $j \cdot j$ coupling have been made by Squires (36).

For "normal" parity change but with $\Delta T = 1$, the resulting angular distribu-

tions for 156 Mev are shown in Fig. 4(b) and Table XVII for several values of λ . A marked dependence on λ is seen for small values of λ and no relation to elastic scattering is found. This case corresponds to the polarization from the group of levels in the 19–21 Mev region in C^{12} and O^{16} which from photonuclear data (37, 38) consists mainly of levels with $J = 1^-$, and $T = 1$. The polarization in

TABLE XVI

The polarization for nucleons inelastically scattered from nuclei of spin 0. In the case calculated here the transition is for a spin zero ground state with $T = 0$ to an excited state with spin J and isotopic spin $T = 0$. The parity change is taken to be normal, $\Delta \pi = (-1)^J$. The polarization is calculated according to Eq. (8.13) with $\mu = 0$.

θ_0	λ				
	0	1/2	1	2	∞
	$[3/4 (T = 1) + 1/4 (T = 0)]$			E = 40 Mev	
2	0.0019	0.0020	0.0021	0.0023	0.0588
4	0.0038	0.0040	0.0042	0.0046	0.118
6	0.0057	0.0060	0.0064	0.0070	0.178
8	0.0076	0.0081	0.0086	0.0094	0.238
10	0.0096	0.0102	0.0108	0.0120	0.299
20	0.0202	0.0218	0.0233	0.0264	0.583
30	0.0323	0.0352	0.0381	0.0438	0.780
40	0.0456	0.0499	0.0541	0.0625	0.876
60	0.0690	0.0732	0.0774	0.0855	0.639
80	0.0704	0.0670	0.0636	0.0572	-0.139
100	0.0389	0.0231	0.0087	-0.0169	-0.332
120	-0.0054	-0.0294	-0.0495	-0.0814	-0.280
140	-0.0311	-0.0526	-0.0677	-0.0875	-0.154
160	-0.0261	-0.0356	-0.0409	-0.0466	-0.0595
<hr/>					
	$[3/4 (T = 1) + 1/4 (T = 0)]$			E = 90 Mev	
	2	0.0217	0.0215	0.0213	0.0209
4	0.0433	0.0429	0.0426	0.0418	-0.451
6	0.0651	0.0645	0.0638	0.0626	-0.461
8	0.0869	0.0860	0.0851	0.0834	-0.421
10	0.1088	0.1076	0.1064	0.1041	-0.373
20	0.220	0.216	0.212	0.204	-0.176
30	0.337	0.325	0.314	0.294	-0.0687
40	0.456	0.429	0.405	0.365	+0.0076
60	0.661	0.575	0.513	0.428	0.112
80	0.743	0.558	0.458	0.351	0.116
100	0.574	0.413	0.331	0.247	0.0765
120	0.304	0.244	0.206	0.162	0.0411
140	0.0846	0.0904	0.0954	0.103	0.162
160	0.0066	0.0158	0.0227	0.0321	0.0682

TABLE XVI (Continued)

θ_0	λ				
	0	1/2	1	2	∞
	$[3/4 (T = 1) + 1/4 (T = 0)]$		E = 156 Mev		
2	0.0396	0.0386	0.0376	0.0356	-0.302
4	0.0791	0.0771	0.0750	0.0709	-0.480
6	0.119	0.116	0.112	0.1059	-0.534
8	0.159	0.154	0.149	0.140	-0.518
10	0.198	0.192	0.186	0.174	-0.473
20	0.399	0.377	0.356	0.319	-0.239
30	0.597	0.536	0.485	0.404	0.0926
40	0.762	0.629	0.536	0.414	+0.0032
60	0.619	0.435	0.353	0.277	0.139
80	-0.120	0.0065	0.0524	0.0907	0.151
100	-0.556	-0.241	-0.135	-0.0510	0.0758
120	-0.701	-0.359	-0.245	-0.153	-0.0140
140	-0.423	-0.341	-0.281	-0.201	+0.0895
160	-0.115	-0.0679	-0.0349	+0.0085	0.154
$[3/4 (T = 1) + 1/4 (T = 0)]$			E = 310 Mev		
2	0.0689	0.0657	0.0525	0.0565	-0.107
4	0.138	0.131	0.125	0.112	-0.198
6	0.208	0.197	0.186	0.167	-0.265
8	0.278	0.262	0.247	0.219	-0.306
10	0.348	0.327	0.306	0.269	-0.326
20	0.693	0.615	0.549	0.442	-0.284
30	0.923	0.750	0.624	0.451	-0.195
40	0.906	0.676	0.531	0.358	-0.114
60	0.412	+0.293	+0.229	0.161	+0.0126
80	-0.133	-0.0675	-0.0326	+0.0039	0.0819
100	-0.411	-0.268	-0.192	-0.114	-0.0353
120	-0.653	-0.403	-0.297	-0.201	-0.0353
140	-0.766	-0.593	-0.492	-0.380	-0.110
160	-0.187	-0.115	-0.0726	-0.0260	+0.0846

this case is observed to be small and is plotted in Fig. 4(b) together with the theoretical values.

For "abnormal" parity change, the nonspin-flip matrix element vanishes and the polarization becomes

$$P_\rho = \frac{2(1 - \rho) \operatorname{Re} BC^*}{[(1 - \rho)(|B|^2 + |C|^2 + |F|^2 + 2\rho|E|^2)]} \quad (8.20)$$

and so depends on a single nuclear parameter $\rho = L_3/L_2$ whose values are restricted to lie between 0 and 1. The angular distribution in this case is shown for several values of ρ for $\Delta T = 0$ in Fig. 4(c) and Table XVII and for $\Delta T = 1$ in Fig. 4(d) and Table XVII. In both cases the polarization in the forward direction is

small. The polarization from groups at 12.5 Mev in O¹⁶ and 15 Mev in C¹² probably correspond to these two cases, as in O¹⁶ there are known (34) levels 1⁻, T = 0 at 12.43, 2⁻ at 12.52, 0⁻, T = 1 at 12.78 and 2⁻, T = 1 at 12.96 Mev, which are mostly of "abnormal" parity, and the 15-Mev group in C¹² could be the 15.11-Mev 1⁺, T = 1 level. The observed polarization is small.

Elastic polarization and polarization from the 2.43-Mev level of Be⁹ have been

TABLE XVII

The polarization in inelastic scattering of 156 Mev nucleons from nuclei as a function of momentum transfer. The transition is from a spin zero ground state with T = 0. For case A, $\Delta T = 1, \Delta \pi = (-1)^J$. For case B, $\Delta T = 1, \Delta \pi = (-1)^{J+1}$. For case C, $\Delta T = 0, \Delta \pi = (-1)^{J+1}$. The case $\Delta T = 0, \Delta \pi = (-1)^J$ is given in Table XVI.

A.	P_λ	$(\Delta T = 1, \Delta \pi = (-1)^J)$		156 Mev	
θ_0 degrees	$q(f^{-1})$	$\lambda = 0$	$\lambda = 0.1$	$\lambda = 1.0$	$\lambda = \infty$
2	0.05	0.010	0.008	0.008	0.008
6	0.14	0.026	0.025	0.024	0.023
10	0.24	0.038	0.039	0.040	0.040
20	0.48	0.012	0.060	0.081	0.085
30	0.71	-0.152	0.050	0.121	0.132
40	0.94	-0.449	0.031	0.157	0.175
60	1.37	0.278	0.231	0.211	0.208
80	1.76	0.872	0.615	0.202	0.056
100	2.10	0.478	0.314	-0.056	-0.233
120	2.37	-0.380	-0.388	-0.410	-0.425
160	2.70	-0.406	-0.412	-0.456	-0.628

B.	P_ρ	$(\Delta T = 1, \Delta \pi = (-1)^{J+1})$		156 Mev	
θ_0 degrees	$\rho = 0$	$\rho = 1/3$	$\rho = 2/3$		
2	0.008	0.006	0.004		
6	0.023	0.018	0.011		
10	0.040	0.032	0.020		
20	0.085	0.077	0.061		
30	0.132	0.132	0.131		
40	0.175	0.161	0.130		
60	0.208	0.121	0.053		
80	0.056	0.026	0.010		
100	-0.233	-0.168	-0.092		
120	-0.425	-0.382	-0.293		
160	-0.628	-0.279	-0.105		

TABLE XVII - 2

C.	P_ρ	$(\Delta T = 0, \Delta \pi = (-1)^{J+1})$	156 Mev
θ_0 degrees		$\rho = 0$	$\rho = 1/3$
2		-0.013	-0.013
6		-0.039	-0.038
10		-0.060	-0.059
20		-0.082	-0.081
30		-0.056	-0.052
40		0.002	0.002
60		0.131	0.117
80		0.140	0.125
100		0.056	0.047
120		-0.005	-0.005
160		0.047	0.044
			$\rho = 2/3$

measured by Hafner (39) for 220-Mev protons. Here the ground state has $J_0 = \frac{3}{2} -, T_0 = \frac{1}{2}, T_{03} = -\frac{1}{2}$ and the 2.43-Mev state has $J = \frac{5}{2} -, T = \frac{1}{2}, T_3 = -\frac{1}{2}$. Shell model calculations by French *et al.* (40) indicate that this state and the ground state lie close to $L \cdot S$ coupling. In the $L \cdot S$ coupling limit the ground state is $^{22}P(41)$, the excited state $^{22}D(41)$. Only one value of l is likely to contribute, so that expression (8.18) for the polarization can be applied. Calculation shows that for the ground state

$$\bar{A} = (\frac{13}{18}A_1 + \frac{5}{18}A_0), \text{etc.}, \quad \bar{B} = (\frac{1}{2}B_1 + \frac{1}{2}B_0), \text{etc.}, \quad (8.21)$$

$$\lambda = 0.026, \quad \mu = 0.0034.$$

The spin-flip contribution to the elastic polarization is thus small. The isotopic spin averages show that for spin-flip scattering it is essentially the odd neutron that contributes, as the combination $\frac{1}{2}(B_1 + B_0)$ is just that for neutron-proton scattering. For the nonspin-flip contribution \bar{A}, \bar{C} are just the scattering amplitudes for the scattering of a proton by the average of 5 neutrons and 4 protons. Thus ignoring the small spin-flip contribution, the polarization is not quite the same as the polarization from a spin zero $T = 0$ nucleus which has a slightly different weighting of the two-body isotopic singlet and triplet contributions, i.e., $A = \frac{3}{4}A_1 + \frac{1}{4}A_0$. However, as can be seen from Table VIII, the coefficients A and C at small angles are relatively insensitive to proton or neutron excess. This fact and the fact that the spin-flip contribution tends to be small are undoubtedly responsible for the observed fact that elastic polarization at these energies is independent of atomic weight in the laboratory system at small forward angles.

For the inelastic scattering from the 2.43-Mev level of Be⁹ we find that

$$\bar{A} = \frac{1}{15}A_1 + \frac{4}{15}A_0, \quad \bar{B} = \frac{1}{2}(B_1 + B_0), \quad (8.22)$$

and

$$\lambda = 0.005, \quad \mu \approx 0.002.$$

Here again the spin-flip contribution is very small and again comes from the odd neutron. The nonspin-flip contribution is again quite close to the scattering of a proton by the average of 5 neutrons and 4 protons (the actual numbers from (8.22) corresponding to 4.8 neutrons and 4.2 protons).

Again, owing to the small spin flip contribution, and the relative insensitivity of the nonspin-flip amplitude to neutron or proton excess, the polarization in inelastic scattering from the 2.43-Mev level in Be⁹ should strongly resemble the polarization in elastic scattering from a spin 0, $T = 0$ nucleus, which is in fact observed. Using the wave vectors as tabulated by French (40) as a function of the intermediate coupling parameter, it would be an easy matter to find out how sensitive these results are to this parameter.

In conclusion the polarization in scattering from individual levels in light nuclei correlates well with what is known from shell theory and the two nucleon data. The formalism adopted here makes it difficult to generalize but it is clear that the results are rather independent of the details of the shell model, and depend mainly on the symmetry properties of the nuclear wave functions.

We have here neglected the effect of distorted waves which enter in the next approximation (3.5)

$$T_{n0} = N\omega_{nn}^L t_{n0} \Omega_{00}^R. \quad (8.23)$$

However, Kohler (41) has shown that as long as the spin-orbit part of the potential giving rise to the distorted waves can be considered as a perturbation, the polarization for forward angles is given by the Born approximation, apart from a correction term which depends on the difference of the ratios of the spin dependent part to the spin independent part of the Born inelastic and elastic scattering. If these are the same, the Born approximation should be very good. This has been verified by direct calculation by Brown and McCauley (42) in the case of polarization from the 4.43-Mev level of C¹². Squires (36) has calculated the polarization from the same level, but using spin-independent distorted waves, and found that the polarization was again that given by the Born approximation. The indications are then that for light nuclei the correction term is very small and that the Born approximation should be a good one. Because the differential cross section for inelastic scattering tends to show less interference effects at forward angles than for elastic scattering, the Born approximation should hold out to larger angles than in the elastic case. Experimentally the

polarization for inelastic scattering from low-lying nuclear levels follows that given by the q -dependence of the two-nucleon scattering amplitude out past the change of sign of the polarization in a region in which the elastic polarization is dominated by diffraction effects.

APPENDIX I. RELATIVISTIC CORRECTION TERMS

Only relativistic corrections of a purely kinematical origin will be considered. Møller (13) gives as the invariant form for the cross section

$$d\sigma = (2\pi)^4 |I|^2 / B_0 \iint \delta(\mathbf{K} - \mathbf{K}') \delta(W - W') \frac{d\mathbf{k}_1'}{E_1'} \frac{d\mathbf{k}_2'}{E_2'}, \quad (I.1)$$

where I , B_0 are the invariants,

$$\langle I \rangle = \sqrt{E_1' E_2'} \langle T \rangle \sqrt{E_1 E_2}, \quad (I.2)$$

$$B_0 = [|\mathbf{k}_1 E_2 - \mathbf{k}_2 E_1|^2 - |\mathbf{k}_1 \times \mathbf{k}_2|^2]^{1/2}. \quad (I.3)$$

Here K , W are the total momentum and energy of the system. E_1 , E_2 , E_1' , E_2' are the total energies of nucleon and target before and after collision, and \mathbf{k}_1 , \mathbf{k}_2 etc. the corresponding momenta.

It is easily deduced that, in the center-of-mass system

$$G_{n0}(\mathbf{k}', \mathbf{k}) = \frac{(2\pi)^2}{\hbar^2 c^2} \sqrt{\frac{k'}{k}} \frac{E_1 E_2}{E_1 + E_2} \langle T \rangle \quad (I.4)$$

dropping terms $\sim \bar{\epsilon}/mc^2$, where $\bar{\epsilon}$ is the nuclear excitation resulting from the collision, and where the quantities E_1 , E_2 , \mathbf{k}, \mathbf{k}' are understood as evaluated in the center-of-mass system.

In the nonrelativistic (N.R.) approximation $E_1 \rightarrow mc^2$, $E_2 \rightarrow N mc^2$, and (I.4) reduces to (2.17). If the mass of the target is taken as infinite, then (I.4) becomes

$$G_{n0}(\mathbf{k}', \mathbf{k}) = -(2\pi)^2 \sqrt{\frac{k_L'}{k_L}} \frac{E_L}{\hbar^2 c^2} \langle T \rangle, \quad (I.5)$$

which is also true in the laboratory system for any target mass for small forward angles. E_L , k_L are the incident energy and momenta in the laboratory system.

The scattering matrix is a function of three invariants which could be taken as (with units such that $\hbar c = 1$),

$$(\Delta_1)^2 = (E_1 + E_2)^2 - (\mathbf{k}_1 + \mathbf{k}_2)^2, \quad (I.6)$$

describing the initial state;

$$(\Delta_2)^2 = (E_1 - E_1')^2 - (\mathbf{k}_1 - \mathbf{k}_1')^2, \quad (I.7)$$

the invariant momentum transfer; and

$$(\Delta_3)^2 = (E_1' - E_2')^2 - (\mathbf{k}_1' - \mathbf{k}_2')^2, \quad (I.8)$$

describing the final state. These can be calculated in any convenient Lorentz frame. In the center-of-mass system $(\Delta_2)^2 = -q^2$. The scattering matrix T is not itself an invariant. However, the quantity $\langle I \rangle$ defined by Eq. (I.2) is, so that it is easy to calculate T starting from any convenient coordinate system.

As we deal mainly with the center-of-mass system, we want the 2-nucleon scattering matrix in the center-of-mass system, $\langle \mathbf{k}' | t^{(0)} | \mathbf{k} \rangle_{N \text{ c.m.}}$ in terms of the two body scattering amplitude $M(q)$ in the two-body center-of-mass system. Now from (I.4) in the two-nucleon case

$$\langle \mathbf{k}' | t^{(0)} | \mathbf{k} \rangle_{N \text{ c.m.}} = \frac{-2\hbar^2 c^2}{(2\pi)^2 E_0} M(\mathbf{q}) \quad (I.9)$$

and from (I.2)

$$E_1 \bar{E}_2 \langle t^{(0)} \rangle_{N \text{ c.m.}} = E_0^2 \langle t^0 \rangle_{N \text{ c.m.}}. \quad (I.10)$$

Here E_0 is the total energy of the incident nucleon in the two-body center-of-mass system. E_1 is the total energy of the incident nucleon in the center-of-mass system and \bar{E}_2 the total energy of a *target nucleon* in the center-of-mass system. This corresponds to a target nucleon at rest in the laboratory system in accordance with the approximation made in factorizing the two-nucleon scattering matrix in Section 3.

$$t_{n0} = \langle \mathbf{k}' | t^{(0)} | \mathbf{k} \rangle \langle n | \exp(-i\mathbf{q} \cdot \mathbf{r}) | 0 \rangle \dots. \quad (3.7)$$

Consequently we have

$$\langle \mathbf{k}' | t^{(0)} | \mathbf{k} \rangle_{N \text{ c.m.}} = \frac{-2\hbar^2 c^2}{(2\pi)^2} \frac{E_0}{E_1 \bar{E}_2} M(\mathbf{q}), \quad (I.11)$$

and hence, using (3.4), (3.7), (I.11),

$$U_{n0} \approx U_{n0}^{(0)} = -\frac{1}{(2\pi)^2} \frac{2\hbar^2 c^2 E_0}{E_1 \bar{E}_2} (N - 1) \omega^L M(\mathbf{q}) F_{n0}(\mathbf{q}). \quad (I.12)$$

In the Born approximation, $T = Nt$, so that using (I.4), (I.11) we have, in the center-of-mass system,

$$G_B = \frac{2E_0 E_2}{(E_1 + E_2) \bar{E}_2} NM(\mathbf{q}) F_{n0}(\mathbf{q}), \quad (I.13)$$

where E_2 is the total energy of the *target nucleus* in the center-of-mass system. We now have to evaluate E_0 , E_1 , E_2 , and \bar{E}_2 .

For the two-nucleon system we have, using the invariant $(\Delta_1)^2$ (I.6) for this

system

$$(\Delta_1)^2 = 4E_0^2 = 2m(m + E_L) \quad (I.14)$$

so that

$$E_0^2 = \frac{1}{2}m(E_L + m), \quad (I.15)$$

$$k_0^2 = \frac{1}{2}m(E_L - m), \quad (I.16)$$

or

$$E_0 = \frac{1}{2}m \frac{k_L}{k_0}, \quad (I.17)$$

$$m^2 + k_L^2 = E_L^2. \quad (I.18)$$

Here E_L , E_0 are the energies of the incident particle in the laboratory and two-body center-of-mass system, respectively, and k_L , k_0 the corresponding momenta. For the nucleon nucleus case the corresponding invariant $(\Delta_1)^2 = (\Delta_N)^2$ is

$$(\Delta_N)^2 = (E_1 + E_2)^2 = 2NmE_L + m^2(N^2 + 1). \quad (I.19)$$

Using

$$\begin{aligned} E_1^2 &= m^2 + k^2, & E^2 &= N^2m^2 + k^2, \\ \bar{E}_2^2 &= m^2 + k^2/N, \end{aligned} \quad (I.20)$$

where k is the incident nucleon momentum in the center-of-mass system, we find that

$$k = \frac{mN}{\Delta_N} k_L, \quad (I.21)$$

$$E_1 = \frac{mN}{\Delta_N} (E_L + m/N), \quad (I.22)$$

$$E_2 = \frac{mN}{\Delta_N} (E_L + mN), \quad (I.23)$$

$$\bar{E}_2 = E_2/N. \quad (I.24)$$

Finally we have

$$G_B = Nk/k_0 M(\mathbf{q}) F_{n0}(\mathbf{q}), \quad (I.25)$$

which is of the same form as the N.R. approximation (3.17) remembering that,

$$k/k_0 = \frac{2N}{N+1}. \quad (\text{N.R.}) \quad (4.11)$$

Also

$$U_{n0} = -\frac{1}{(2\pi)^2} \frac{\hbar^2 c^2}{E_L} \frac{k_L}{k_0} \eta \omega^L M(\mathbf{q}) F_{n0}(\mathbf{q}), \quad (\text{I.26})$$

where

$$\eta = \frac{1 + \frac{2N}{N^2 + 1} E_L/m}{1 + \frac{N}{N^2 + 1} (E/m + m/E)} \quad (\text{I.27})$$

is a factor which differs from unity only by terms $\sim (1/N)[(E - m)/E]$. Thus the relativistic correction has the effect of multiplying the N.R. result (3.15) for the potential matrix by a factor $(m/2E_L)(k_L/k_0)\eta$. This correction applies to both the central and spin-orbit part of the potential, remembering that factors k, k_0 in the expression for the spin-orbit potential should be given the values (I.21), (I.16) in the relativistic case. Table VII uses the relativistic expressions with $\eta = 1$.

APPENDIX II

The introduction of the optical potential U along with the separation of diagonal and nondiagonal nuclear matrix elements is unnecessarily complicated in view of the approximations which are finally used. The development was carried through in this way in order to elucidate the role of the optical potential. (An approximate optical potential (2.38) has already been shown to exist for elastic scattering.) However, it is algebraically simpler to separate diagonal and non-diagonal nuclear matrix elements in the equation for the scattering matrix T , and this will be done here. We wish in general to solve the equation

$$T = V \left[1 + \frac{1}{a} T \right] \equiv V \Omega^R. \quad (\text{II.1})$$

To do this we introduce diagonal quantities (wave matrices). (These are not the same as (2.53) because they lack projection operators.)

$$\omega^R = \left(1 - \frac{1}{a} V_D \right)^{-1}, \quad (\text{II.2})$$

$$\omega^L = \left(1 - V_D \frac{1}{a} \right)^{-1}. \quad (\text{II.3})$$

We note that the introduction of ω^L and ω^R is essentially equivalent to working in a new representation (distorted waves in the potential V_D) for the scattered particle whose wave functions are

$$|x^{(+)}\rangle = \omega^R |\varphi\rangle, \quad (\text{II.4})$$

$$\langle x^{(-)} | = \langle \varphi | \omega^L. \quad (\text{II.5})$$

In this representation the separation of V_D and V_N can be done using a general theorem of Gell-Mann and Goldberger. In our notation the theorem states

$$(\varphi | T | \varphi) = (\chi^{(-)} | V_D | \varphi) + (\chi^{(-)} | V_N | \psi^{(+)}) \quad (\text{II.6})$$

or in operator notation

$$T = \omega^L V_D + \omega^L V_N \Omega^R. \quad (\text{II.7})$$

In operator form the proof goes as follows

$$\Omega^R = 1 + \frac{1}{a} V \Omega^R, \quad (\text{II.8})$$

$$T = V_D \Omega^R + V_N \Omega^R \equiv V \Omega^R \quad (\text{II.9})$$

Using (II.3)

$$1 = \omega^L - \omega^L V_D \frac{1}{a}, \quad (\text{II.10})$$

we have

$$\begin{aligned} T &= \omega^L V_D \Omega^R + \omega^L V_N \Omega^R - \omega^L V_D \frac{1}{a} V \Omega^R \\ &= \omega^L V_D \Omega^R + \omega^L V_N \Omega^R - \omega^L V_D (\Omega^R - 1) \\ &= \omega^L V_D + \omega^L V_N \Omega^R, \end{aligned} \quad (\text{II.11})$$

so that the theorem is proved.

The separation of V_D and V_N can now be carried one stage further by considering the equation for Ω^R

$$\Omega^R = 1 + \frac{1}{a} V \Omega^R = 1 + \frac{1}{a} V_D \Omega^R + \frac{1}{a} V_N \Omega^R, \quad (\text{II.12})$$

or

$$\left(1 - \frac{1}{a} V_D\right) \Omega^R = 1 + \frac{1}{a} V_N \Omega^R, \quad (\text{II.13})$$

and

$$\begin{aligned} \Omega^R &= \omega^R + \omega^R \frac{1}{a} V_N \Omega^R \\ &= \left(1 - \omega^R \frac{1}{a} V_N\right)^{-1} \omega^R = \omega^R \left(1 - V_N \frac{1}{a} \omega^R\right)^{-1} \\ &= \omega^R + \Omega^R V_N \frac{1}{a} \omega^R. \end{aligned} \quad (\text{II.14})$$

Now define F^R by

$$\Omega^R = F^R \omega^R. \quad (\text{II.15})$$

Then

$$F^R = 1 + \omega^R \frac{1}{a} V_N F^R. \quad (\text{II.16})$$

But

$$\omega^R \frac{1}{a} = \left(1 - \frac{1}{a} V_D\right)^{-1} a^{-1} = (a - V_D)^{-1} = \frac{1}{e} \quad (\text{II.17})$$

so that

$$F^R = 1 + \frac{1}{e} V_N F^R, \quad (\text{II.18})$$

which is related to the F used by Watson. Notice that e is a propagator in the average potential V_D . With result (II.15), we have finally from (II.11)

$$T = \omega^L V_D + \omega^L V_N F^R \omega^R. \quad (\text{II.19})$$

The approximation used to obtain results like (2.68–2.70) is just to neglect the off-diagonal elements in F , i.e., $F \simeq 1$

$$T \approx \omega^L V_D + \omega^L V_N \omega^R \quad (\text{II.20})$$

so that

$$\begin{aligned} T_D &\approx \omega^L V_D, \\ T_N &\approx \omega^L V_N \omega^R. \end{aligned} \quad (\text{II.21})$$

APPENDIX III

Using (4.1) (4.3) for the spin and isotopic spin dependence of $M(q)$, we can write, choosing the axis of quantization along the momentum transfer direction \mathbf{q} ,

$$\begin{aligned} M &= M(\alpha) + M(\beta)(\tau_0 \cdot \tau), \\ M(\alpha) &= M_0(\alpha) + \sum_{\mu} (-1)^{\mu} M_{\mu}^1(\alpha) \sigma_{\mu}^1, \\ M(\beta) &= M_0(\beta) + \sum_{\mu} (-1)^{\mu} M_{\mu}^1(\beta) \sigma_{\mu}^1, \end{aligned} \quad (\text{III.1})$$

where σ_{μ}^1 are tensor operators of rank one in the spin of a target nucleon, formed in the standard way

$$\sigma_0^1 = \sigma_{\hat{n}}, \quad \sigma_1^1 = -\frac{1}{\sqrt{2}} (\sigma_{\hat{n}} + i\sigma_{\hat{p}}), \quad \sigma_{-1}^1 = \frac{1}{\sqrt{2}} (\sigma_{\hat{n}} - i\sigma_{\hat{p}}), \quad (\text{III.2})$$

and τ is a tensor operator of rank 1 in the isotopic spin of a nucleon.

The quantities $M_0(\alpha)$, $M_\mu^1(\alpha)$ are defined by

$$\begin{aligned} M_0(\alpha) &= A_\alpha + C_\alpha \sigma_{\hat{n}}, & M_0(\beta) &= A_\beta + C_\beta \sigma_{\hat{n}}, \\ M_0^1 &= E \sigma_{\hat{q}}, & M_1^1 &= -\frac{1}{\sqrt{2}} (C + B \sigma_{\hat{n}} - i F \sigma_{\hat{p}}), \\ M_{-1}^1 &= \frac{1}{\sqrt{2}} (C + B \sigma_{\hat{n}} + i F \sigma_{\hat{p}}), \end{aligned} \quad (\text{III.3})$$

and are operators in the spin of the incident nucleon. The quantities A_α , A_β are the isotopic spin combinations defined by (8.5).

Then the matrix element we wish to find is

$$\begin{aligned} \bar{M}(q) &= \langle n | M(q) \exp(-i\mathbf{q} \cdot \mathbf{r}) | 0 \rangle \\ &= \langle JT | \exp(-i\mathbf{q} \cdot \mathbf{r}) | J_0 T_0 \rangle M_0(\alpha) \\ &\quad + \sum_{\lambda'=-1}^{+1} (-1)^{\lambda'} \langle JT | \exp(-i\mathbf{q} \cdot \mathbf{r}) \tau_{\lambda'}^1 | J_0 T_0 \rangle \tau_{-\lambda'}^1 M_0(\beta) \\ &\quad + \sum_{\mu=-1}^{+1} (-1)^\mu \{ \langle JT | \exp(-i\mathbf{q} \cdot \mathbf{r}) \sigma_\mu^1 | J_0 T_0 \rangle M_\mu^1(\alpha) \\ &\quad + \sum_{\lambda'=-1}^{+1} (-1)^{\lambda'} \langle JT | \exp(-i\mathbf{q} \cdot \mathbf{r}) \sigma_\mu^1 \tau_{\lambda'}^1 | J_0 T_0 \rangle \tau_{-\lambda'}^1 M_\mu^1(\beta) \}. \end{aligned} \quad (\text{III.4})$$

The target particle operators $\exp(-i\mathbf{q} \cdot \mathbf{r})$, $\exp(-i\mathbf{q} \cdot \mathbf{r}) \sigma_\mu^1$ can be expressed in standard tensor form

$$\begin{aligned} \exp(-i\mathbf{q} \cdot \mathbf{r}) &= \sum_l \rho_l Y_{l0}, \\ \exp(-i\mathbf{q} \cdot \mathbf{r}) \sigma_\mu^1 &= \sum_{lk} \rho_l (1l\mu 0 | k\mu) T_k^{\mu}(ls), \\ \rho_l &= [l] \sqrt{4\pi} (-i)^l j_l(qr), \quad [l] = \sqrt{2l+1}, \end{aligned} \quad (\text{III.4})$$

where $T_k^{\mu}(ls)$ is a tensor operator of rank k composed of operators in the product space ls

$$T_k^{\mu}(ls) = \sum_{\mu'\mu''} (ls\mu'\mu'' | k\mu) Y_{l\mu'} \sigma_{\mu''}^1.$$

Using the standard formulas for reduced matrix elements we have, for instance

$$\begin{aligned} \sum_{\lambda} (-1)^\lambda \langle JT | \exp(-i\mathbf{q} \cdot \mathbf{r}) \sigma_\mu^1 \tau_{\lambda}^1 | J_0 T_0 \rangle \tau_{-\lambda}^1 \\ = \sum_{lk} f(T T_0) \frac{1}{[J]} (1l\mu 0 | k\mu) (k J_0 \mu M_0 | JM) Q_{kl}, \end{aligned} \quad (\text{III.5})$$

where $f(TT_0)$ is given by (8.3) and Q_{kl} is the reduced nuclear matrix element

$$Q_{kl} = \langle JT \parallel \rho_l T^k (ls) \tau^l \parallel J_0 T_0 \rangle. \quad (\text{III.6})$$

The expression for $f(T, T_0)$ follows from our normalization of the spin operators which gives for the reduced matrix elements

$$\langle \frac{1}{2} \parallel \sigma^1 \parallel \frac{1}{2} \rangle = \langle \frac{1}{2} \parallel \tau^1 \parallel \frac{1}{2} \rangle = -\sqrt{6}. \quad (\text{III.7})$$

The relations (8.2) (8.3) of the text follow immediately from the definitions

$$I_0 = \frac{1}{4} \text{Tr } \bar{M} \bar{M}^*, \quad I_0 P = \frac{1}{4} \text{Tr} (\bar{M} \sigma \cdot n \bar{M}^*),$$

and from the relations, implicit in the definitions, (III.3), with $\sigma_{\hat{n}} \sigma_{\hat{p}} = i \sigma_{\hat{q}}$,

$$\begin{aligned} \frac{1}{4} \text{Tr}(M_0 M_0^*) &= |A|^2 + |C|^2, & \frac{1}{4} \text{Tr}(M_0^1 M_0^{1*}) &= |E|^2, \\ \frac{1}{4} \text{Tr}(M_1^1 M_1^{1*}) &= \frac{1}{4} \text{Tr}(M_{-1}^1 M_{-1}^{1*}) = \frac{1}{2}(|B|^2 + |C|^2 + |F|^2), & (\text{III.8}) \\ \frac{1}{4} \text{Tr}(M_0 \sigma_n M_0^*) &= 2 \text{Re } AC^*, & \frac{1}{4} \text{Tr}(M_0^1 \sigma_n M_0^{1*}) &= 0, \\ \frac{1}{4} \text{Tr}(M_1^1 \sigma_n M_0^{1*}) &= \frac{1}{4} \text{Tr}(M_{-1}^1 \sigma_n M_{-1}^{1*}) = \text{Re } BC^*. \end{aligned}$$

The reduced matrix elements can be written in terms of shell model states if, for instance, we expand the nuclear states in terms of LS coupled wave functions with coefficients $A(LST)$

$$\begin{aligned} |J_0 T_0\rangle &= \sum A_0(L_0 S_0 T_0) |L_0 S_0 T_0 ; J_0\rangle, \\ |JT\rangle &= \sum A(LST) |LST; J\rangle, \end{aligned} \quad (\text{III.9})$$

and expand the $L - S$ wave functions $|L_0 S_0 T_0\rangle$, $|LST\rangle$ in terms of the $L - S$ states of $N - 1$ particles plus the odd target particle

$$|L_0 S_0 T_0\rangle = \langle L_0 S_0 T \alpha_0 \{L'_0 S'_0 T'_0 \alpha'_0, \lambda_0 s_0 t_0\} \rangle |\psi(L'_0 S'_0 T'_0 \alpha'_0), \Phi(\lambda_0 s_0 t_0)\rangle, \quad (\text{III.10})$$

where $\langle L_0 S_0 T \alpha_0 \{L'_0 S'_0 T'_0 \alpha'_0, \lambda_0 s_0 t_0\} \rangle$ is the fractional parentage coefficient for decomposition of the state $L_0 S_0 T_0$, with symmetry α_0 into the states $L'_0 S'_0 T'_0 \alpha'_0$ of $(N - 1)$ particles, and states $\lambda_0 s_0 t_0$ of the odd particle and $|\psi, \Phi\rangle$ represents the vector coupled wave function. Then using the formulas for reduced matrix elements for vector coupled wave functions of a tensor operator acting on a subspace of the wave functions, and writing

$$\mathbf{N}_l = N_{l\alpha} \mathbf{a} + N_{l\beta} \mathbf{b}, \quad (\text{III.11})$$

we find that

$$\begin{aligned} \mathbf{N}_l &= \sum_{L_0 S_0 L S} A_0(L_0 S_0 T_0) A^*(LST) (-1)^{l+s-J_0-L} [J_0][J] W(L_0 J_0 L J, S L), \\ &\quad \sum_{L' S' T' \alpha} \langle L_0 S_0 T_0 \alpha_0 \{L' S' T' \alpha', \lambda_0 s_0 t_0\} \rangle \\ &\quad \cdot \langle LST \alpha \{L' S' T' \alpha', \lambda s t\} \rangle \Lambda(l L') \times \delta_{ss_0} \mathbf{T}(T') \end{aligned} \quad (\text{III.12})$$

and that

$$Q_{kl} \sum_{L_0 S_0 L S} A(LST) A^*(LST') [J][J][k] \times \begin{pmatrix} L_0 & S_0 & J_0 \\ L & S & J \\ l & 1 & k \end{pmatrix} \quad (\text{III.13})$$

$$\sum_{L' S' T' \alpha} \langle L_0 S_0 T_0 \alpha_0 \{L' S' T' \alpha', \lambda_0 s_0 t_0\} \rangle \langle LST \alpha \{L' S' T' \alpha, \lambda st\} \rangle^* \Lambda(lL') \sum(S') \mathbf{T}(T'),$$

where

$$\begin{aligned} \Lambda(lL') &= (-1)^{l+l'-L-\lambda_0} [L_0][L] W(\lambda_0 L_0 \lambda L, L'l), \\ \sum(S') &= (-1)^{S'-S-(1/2)} \sqrt{6} [S_0][S] W(\tfrac{1}{2} S_0 \tfrac{1}{2} S, S' 1), \\ \mathbf{T}(T) &= \alpha \delta_{TT_0} \delta_{T_0 T_0} + \beta (-1)^{[T'-T-(1/2)]} \sqrt{6} [T_0][T] W(\tfrac{1}{2} T_0 \tfrac{1}{2} T, T' 1). \end{aligned} \quad (\text{III.14})$$

For Be⁹ these matrix elements can be calculated using the vectors tabulated by French (40) and the fractional percentage coefficients for (*p*)⁵ in the tables of Jahn *et al.* (43).

APPENDIX IV

In Sections 2 and 3, unnecessary emphasis has been put on the scattering operator τ defined by Eq. (2.23). In fact we can define a whole series of scattering operators t_σ ,

$$t_\sigma = v + v \frac{1}{h_\sigma} t_\sigma \quad (\text{IV.1})$$

where h_σ is left undefined for the moment.

Then

$$t_\sigma = t + t \left(\frac{1}{h_\sigma} - \frac{1}{h} \right) t_\sigma \quad (\text{IV.2})$$

where $1/h$ is the free propagator (2.24).

If, for instance we choose

$$\frac{1}{h_\sigma} = \frac{\alpha}{E - H_0 + i\epsilon} = \frac{1}{\alpha} \quad (\text{IV.3})$$

then,

$$t_\sigma = \tau. \quad (\text{IV.4})$$

The equation for the scattering matrix becomes, for any choice of h_σ

$$T' = (N - 1) t_\sigma \left(1 + \frac{1}{\alpha_\sigma} T' \right) \quad (\text{IV.5})$$

where

$$\frac{1}{\alpha_\sigma} = \frac{N}{N - 1} \frac{\alpha}{E - H_0 + i\epsilon} - \frac{1}{N - 1} \frac{1}{h_\sigma} \quad (\text{IV.6})$$

The equation defining elastic scattering via an optical potential (2.44) becomes

$$T'_{00} = U_{00} \left(1 + \left\langle \frac{1}{\alpha_\sigma} \right\rangle_{00} \right) T'_{00} \quad (\text{IV.7})$$

For this to be the single particle Schrödinger equation in a potential U_{00} , it is sufficient that

$$\left\langle \frac{1}{\alpha_\sigma} \right\rangle_{00} = \left\langle \frac{1}{\alpha'} \right\rangle_{00} \quad (\text{IV.8})$$

that is,

$$\left\langle \frac{1}{h_\sigma} \right\rangle_{00} = \left\langle \frac{1}{\alpha'} \right\rangle_{00} \quad (\text{IV.9})$$

Thus we can choose, for instance

$$\frac{1}{h_\sigma} = \frac{\Lambda}{E - H_0 + i\epsilon} \quad (\text{IV.10})$$

where Λ is any projection operator which does not affect the ground state diagonal matrix element.

If $\Lambda = \alpha$, the antisymmetrization operator, then $t_\sigma = \tau$. On the other hand if we choose

$$\Lambda = P_j \quad (\text{IV.11})$$

where

$$\begin{aligned} P_j &= 1 && \text{for collision partner } j \\ &= 0 && \text{otherwise,} \end{aligned} \quad (\text{IV.12})$$

we have a propagator $1/h_\sigma$ which is closer to the free particle propagator $1/h$ which also singles out a definite collision partner. We then have

$$t_\sigma = t'$$

where

$$t' = t + t \left(\frac{P_j}{E - H_0 + i\epsilon} - \frac{1}{h} \right) t'. \quad (\text{IV.13})$$

Here $h = E - K_1 - K_j + i$ and

$$t \frac{P_j}{E - H_0 + i\epsilon} \leftarrow t' \quad \text{means} \quad t_j \rightarrow \frac{1}{E - H_0 + i\epsilon} t'_j$$

i.e., successive scattering via t_j, t'_j from the same target particle j . In this case t' differs from t by an amount $\Delta t'$.

$$\Delta t' \sim t \frac{1}{E - H_0 + i\epsilon} (H_0 - K_j) \frac{1}{E - H_0 + i\epsilon} t. \quad (\text{IV.14})$$

This is just the binding energy correction discussed by Chew (14) and Wick (14).

The perturbation expansion in the general case for the optical potential goes as in the text with

$$U^{(0)} = (N - 1)t_\sigma. \quad (\text{IV.15})$$

The second order correction δU_{00} then becomes Eq. (3.35)

$$\delta U_{00} = \sum_{n \neq 0} U_{0n}^{(0)} \left\langle \frac{1}{\alpha_\sigma - U^{(0)}} \right\rangle_{nn} U_{n0}^{(0)} \quad (\text{IV.16})$$

and this depends on the choice of projection operator Λ . If we take $\Lambda = Q$, $t_\sigma = \tau$, then δU_{00} is given by Eqs. (3.38), (3.40) of the text. These involve essentially the quantity

$$\begin{aligned} G_{00}(\mathbf{k}' - \mathbf{k}'', \mathbf{k}''' - \mathbf{k}) &= \left(1 - \frac{1}{N}\right) C_{00}(\mathbf{k}' - \mathbf{k}'', \mathbf{k}''' - \mathbf{k}) \\ &\quad + \frac{1}{N} [F_{00}(\mathbf{k}' - \mathbf{k}) - F_{00}(\mathbf{k}' - \mathbf{k}'')F_{00}(\mathbf{k}'' - \mathbf{k})] \end{aligned} \quad (3.38)$$

To this must be added the correction arising from $\Delta\tau$ given by Eqs. (3.32) and (3.34) which partially cancels the $1/N$ terms in (3.38).

If on the other hand we choose $\Lambda = P_j$, $t_\sigma = t'$, then in a closure approximation using (3.29) for the part of $1/\alpha_\sigma$ involving the antisymmetrization operator Q and completeness for the rest, we find that δU_{00} is given by Eq. (3.40) but with

$$G_{00}(\mathbf{k}' - \mathbf{k}'', \mathbf{k}''' - \mathbf{k}) = C_{00}(\mathbf{k}' - \mathbf{k}'', \mathbf{k}''' - \mathbf{k}) \quad (\text{IV.17})$$

Comparing with (3.38) we see that all $1/N$ terms are now cancelled out, and the corrections due to $\Delta t'$ from (IV.14) now involve only binding energy corrections. This form was in fact used in Section 7, to estimate correlation corrections, and agrees with the expression given by Watson (1-3), apart from the definition of T' and the replacement of N by $N - 1$. In fact in this case the term in P_j appearing in $1/\alpha_\sigma$ has the effect, in conjunction with the antisymmetrized term, of eliminating successive scattering from the same target particle in the perturbation series for the optical potential.

To find the optimum form for t_σ involves looking at the convergence of the series (2.59), i.e., looking at higher terms in the expansion.

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