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880.05 Lecture 4

Handouts:

- PS#1. Note that part is due next Monday (MATLAB exercises) and part a week from Friday.
- Office hours Friday 3-5 pm, but also stop by any morning.
- Try the problems early to identify issues.
- Some are bonus — adjust to your workload.

- Return to the "preview" on page (42) of calculating the partition function $\mathrm{Tr}\, e^{-\beta \hat{H}'}$. Let's make another pass at explaining.
  - $\hat{H}'$ is the Hamiltonian plus any source term such as $-\mu\hat{N}$ or more theoretical ones like the j term added for our model partition function.
  - We can't evaluate $e^{\beta \hat{H}'}$ without solving the problem (e.g. diagonalizing $\hat{H}'$) but we can use the exact relation $e^{\hat{A}} = (e^{\hat{A}/m})^m$ for m integer to rewrite
$$\mathrm{Tr}\, e^{-\beta\hat{H}'} = \mathrm{Tr}(\underbrace{e^{-\epsilon\hat{H}'} e^{-\epsilon\hat{H}'} \cdots e^{-\epsilon\hat{H}'}}_{M \text{ terms}}) \quad \text{with } M\epsilon = \beta.$$
  - The idea will be to evaluate each $e^{-\epsilon\hat{H}'}$ by inserting complete sets of states on the left and the right $\Rightarrow$ we need to get the parts of $\hat{H}'$ that we can solve on one side or the other.
    - For example: If $\hat{H}' = \hat{T} + \hat{V}$ and $\hat{T}$ is diagonal in momentum states while $\hat{V}$ is diagonal in position states, then the claim is

Try to check on problem set!

$$e^{-\epsilon\hat{H}'} = e^{-\epsilon(\hat{T}+\hat{V})} = e^{-\epsilon\hat{T}} e^{-\epsilon\hat{V}} + O(\epsilon^2) \quad (\text{with the error} \propto [\hat{T},\hat{V}])$$

$\Rightarrow$ The total error made is (at worst) $\epsilon^2 \cdot M = \beta\epsilon \xrightarrow[\epsilon\to 0]{} 0$ (as long as the rest is finite)

- More generally we normal-order $e^{-\epsilon\hat{H}'}$, denoted $:e^{-\epsilon\hat{H}'}:$ so that operators of one type are all to the left of the other type. More to come!

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Recap of model partition function with source j:

$$\boxed{Z_\lambda[j] = N\int_{-\infty}^{\infty} d\xi \; e^{-\frac{a}{2}\xi^2 - \frac{\lambda}{4}\xi^4 + j\xi} = N e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4}\int_{-\infty}^{\infty} d\xi \; e^{-\frac{a}{2}\xi^2 + j\xi}}$$

and $Z_{\lambda 0} \equiv Z_0$ is

$$Z_0[j] = N\int_{-\infty}^{\infty} d\xi \; e^{-\frac{a}{2}\xi^2 + j\xi}$$

Every $\frac{\partial}{\partial j}$ brings down a power of ξ, so $f(\xi)$ can be replaced by $f\left(\frac{\partial}{\partial j}\right)$. THIS IS IMPORTANT TO UNDERSTAND!

Simplest case: $\langle\xi\rangle = \int_{-\infty}^{\infty} d\xi \; \xi \; e^{-\frac{a}{2}\xi^2 - \frac{\lambda}{4}\xi^4 + j\xi} = \frac{\partial}{\partial j}\int_{-\infty}^{\infty} d\xi \; e^{-\frac{a}{2}\xi^2 - \frac{\lambda}{4}\xi^4 + j\xi}$

Completing the square:

$$N\int_{-\infty}^{\infty} d\xi \; e^{-\frac{a}{2}\xi^2 + j\xi} = N\int_{-\infty}^{\infty} d\xi \; e^{-\frac{a}{2}\left(\xi + \frac{j}{a}\right)^2} \; e^{+\frac{j^2}{2a}} = Z_0 \, e^{\frac{j^2}{2a}}$$

shift to $\xi' = \xi + \frac{j}{a}$

So our master formula is

$$\frac{Z_\lambda}{Z_0} = \left[e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4} e^{\frac{1}{2} j a^{-1} j}\right]\Bigg|_{j=0} \quad \text{at the end.}$$

Now continue with 51 ...

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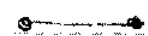
· Let's carry out the first two orders in λ for $\langle \xi^2 \rangle$.

$$\boxed{\begin{aligned}\langle \xi^2\rangle &= \frac{\left[\frac{\partial}{\partial j}\frac{\partial}{\partial j} Z_\lambda[j]\right]_{j=0}}{Z_\lambda[j]_{j=0}} = \frac{\left[\frac{\partial}{\partial j}\frac{\partial}{\partial j}\, e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4} e^{\frac{1}{2} j a^{-1} j}\right]\Big|_{j=0}}{\left[e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4} e^{\frac{1}{2} j a^{-1} j}\right]\Big|_{j=0}} \\ &= \frac{\left\{\frac{\partial}{\partial j}\frac{\partial}{\partial j}\left[1-\frac{1}{4}\lambda\left(\frac{\partial}{\partial j}\right)^4+\ldots\right]\left[1+\frac{1}{2}(j a^{-1} j)+\ldots\right]\right\}\Big|_{j=0}}{\left\{\left[1-\frac{1}{4}\lambda\left(\frac{\partial}{\partial j}\right)^4+\ldots\right]\left[1+\frac{1}{2}(j a^{-1} j)+\ldots\right]\right\}\Big|_{j=0}}\end{aligned}}$$

· The expansion has both a numerator and denominator, so we'll have to be careful to expand consistently.

$$\boxed{\lambda^0: \quad \frac{\frac{\partial}{\partial j}\frac{\partial}{\partial j}[1]\left[\frac{1}{2} j a^{-1} j\right]}{[1][1]} = \frac{1}{a}}$$

just expand far enough so j's exactly eaten by $\frac{\partial}{\partial j}$'s.



$1/a$

This is the "noninteracting propagator." Our "Feynman rule" is to let the $\frac{\partial}{\partial j}$'s represent endpoints on a line.

· Now you write out the next order $(\lambda^0 + \lambda^1)$:

$$\boxed{\begin{aligned}&\frac{1}{a} + \frac{\frac{\partial}{\partial j}\frac{\partial}{\partial j}\left(-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4\right)\frac{1}{3!}\left(\frac{1}{2} j a^{-1} j\right)\left(\frac{1}{2} j a^{-1} j\right)\left(\frac{1}{2} j a^{-1} j\right)}{1-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4\frac{1}{2!}\left(\frac{1}{2} j a^{-1} j\right)\left(\frac{1}{2} j a^{-1} j\right)} \\ &= \left(\frac{1}{a} - \frac{\lambda}{4}\frac{1}{3!}\left(\frac{1}{2}\right)^3\frac{1}{a^3}\,6!\right)\left(1+\frac{\lambda}{4}\frac{1}{2!}\left(\frac{1}{2}\right)^2\frac{1}{a^2}\,4! + O(\lambda^2)\right) \\ &= \frac{1}{a} - \frac{\lambda}{a^3}\left[\frac{6\cdot 5\cdot 4}{4\cdot 8} - \frac{4\cdot 3}{4\cdot 4}\right] + O(\lambda^2) = \frac{1}{a} - \frac{3\lambda}{a^3} + O(\lambda^2)\end{aligned}}$$

$\frac{\partial}{\partial j^6}(jjjjjj) = 6!$

· To be clear: The first $\frac{\partial}{\partial j}$ in $\frac{\partial}{\partial j^6}(jjjjjj)$ has 6 choices, the next 5 choices, the next 4, until the last, which has only 1, $\Rightarrow 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1 = 6!$ terms.

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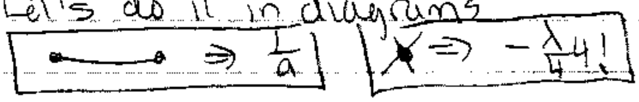
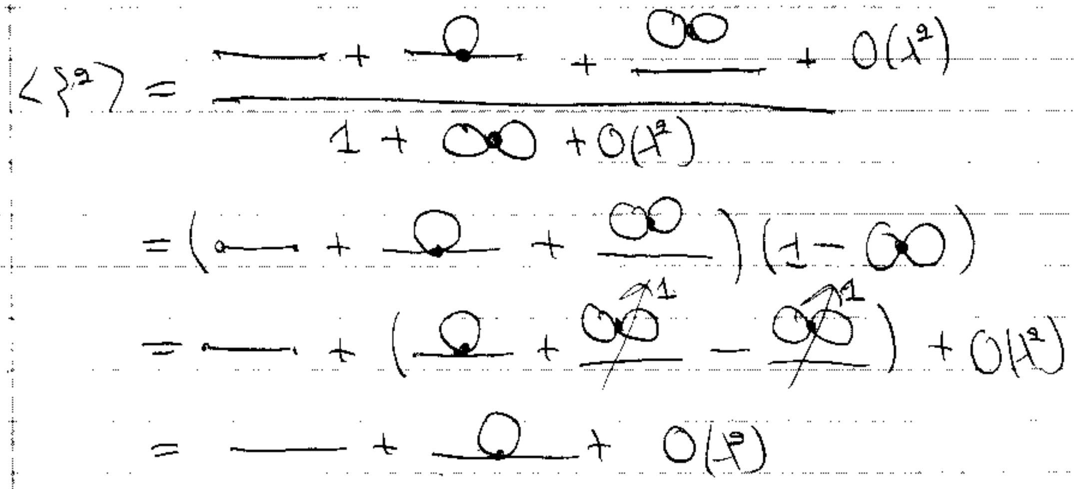
- Let's do it in diagrams



(The 4! is from $\left(\frac{\partial}{\partial j}\right)^4 j^4$)



$\Rightarrow$ The "disconnected" parts  cancel. This is a general result. [Again, not yet convincing because we don't know the factors.]

- Make sure you understand what each diagram corresponds to. For example, what is the difference between

?

- What do the Feynman rules give?
  - We're suppose to get $\frac{1}{a} - \frac{3\lambda}{a^3}$ but we get $\frac{1}{a} - \frac{6\lambda}{a^3}$.
    This happens because of an "overcounting."
  - We fix it with another Feynman rule: the "symmetry factor."

- First we'll take a brief aside to show how disconnected diagrams do not contribute,

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<u>Replica method</u>: The idea is to consider n identical copies of the partition function multiplied together: $\tilde{Z}^n \equiv \left(\frac{Z}{Z_0}\right)^n$.
Now rewrite $\tilde{Z}^n$ so it takes the form of a series expansion in n:

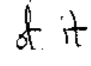
$$\boxed{\tilde{Z}^n = e^{\ln \tilde{Z}^n} = e^{n \ln \tilde{Z}} = 1 + n(\ln \tilde{Z}) + \frac{1}{2} n^2 (\ln \tilde{Z})^2 + \ldots}$$

So to find $\ln \tilde{Z}$, we calculate $\tilde{Z}^n$ and then identify the linear term in n!

But $\tilde{Z}^n$ is just a product of n $\tilde{Z}$'s, with variables $\xi_1, \xi_2, \ldots, \xi_n$ and corresponding sources:

$$\boxed{\begin{aligned} \tilde{Z}^n &= \left(\int d\xi_1\, e^{-\frac{1}{2} a_1 \xi_1^2 - \frac{\lambda_1}{4} \xi_1^4 + j_1 \xi_1}\right)\left(\int d\xi_2\, e^{-\frac{1}{2} a_2 \xi_2^2 - \frac{\lambda_2}{4} \xi_2^4 + j_2 \xi_2}\right) \cdots \left(\int d\xi_n\, e^{\cdots}\right) \\ &= \int d\xi_1 \cdots d\xi_n\, e^{-\frac{1}{2} \sum_{i=1}^n a_i \xi_i^2 - \frac{1}{4} \sum_i \lambda_i \xi_i^4 + \sum_i j_i \xi_i} \end{aligned}}$$

- Even though the copies are identical, we add indices to a and λ.

- Now use $\frac{\partial}{\partial j_i}$ as before to remove interaction terms from <u>all</u> of the integrals.
    - A vertex has the <u>same</u> index i for all lines coming out of it , so the $1/a_i$'s have the same i index at each end.

    - we also sum over i from 1 to n.

- It follows that a connected diagram can only have vertices of one i at a time. E.g.  can't happen, because the 1 vertex comes from $-\frac{\lambda_1}{4}\left(\frac{\partial}{\partial j_1}\right)^4$ and each $\frac{\partial}{\partial j_1}$ is only nonzero if acting on $\frac{1}{2} j_1 a_1^{-1} j_1$. So the lines from "1" to "2" must be "1" lines. But the same argument applied to the other vertex says they must be "2" lines, so the diagram cannot occur.

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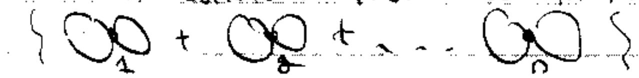
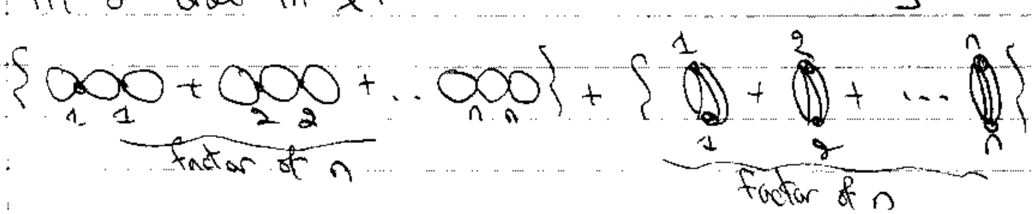
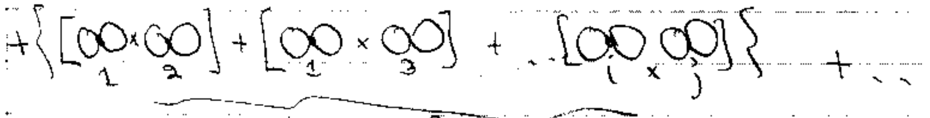
- So if we consider first order in λ:



$\Rightarrow$ overall factor of n [all diagrams have the same value]

- At 2nd order in λ:



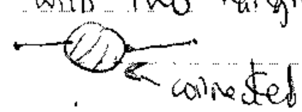


- So if we have 2 disconnected pieces, the total is $\propto n^2$, three disconnected pieces the total is $\propto n^3$, and so on.

$\Rightarrow$ The terms linear in n are precisely those that are connected. But that is also $\ln \tilde{Z}$

$\Rightarrow$ $\ln Z - \ln Z_0$ is exactly given by the sum of connected diagrams, with all of the correct factors!

- The replica argument can be used to show that $\langle \xi^2 \rangle$ is only from connected diagrams with two hanging lines. [see PS#1] QFD, That is, diagrams of the form



only contribute.

- It follows similarly that the expectation value $\langle \hat{O} \rangle$ for any operator $\hat{O}$ is given by the sum of connected diagrams.

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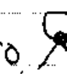
Now let's do <u>symmetry factors</u> . . .

- The $1/n!$ from the Taylor series expansion of the exponentials almost always cancels the $n!$ ways of interchanging the vertices. Further, the factor of $4!$ from $\left(\frac{\delta}{\delta j}\right)^4 j^4$ is taken into account in the Feynman rule $-\frac{\lambda}{4!} \cdot 4!$
  $\Rightarrow$ The symmetry factor is the correction when these cancellations are incomplete.

- There are three types of symmetry factors in our diagrams:

① Factor of $1/2$ from each line that starts and ends on the same vertex: like 

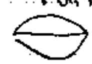
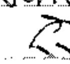
- This goes away for lines for which the ends of different (this will be case for fermions later on)
- We'll write these factors as fractions, but often they are collected first as a factor S and then the diagram is multiplied by $1/S$.
- The factor follows by comparing  to :

$$\left(\frac{\delta}{\delta j}\right)^4\left(\frac{-\lambda}{4}\right)\left(\frac{\delta}{\delta j}\right)^4 \frac{1}{4!}\left(\frac{1}{2}\right)^4\left(j a^{-1} j\right)^4 = -\frac{6\lambda}{a^4} \Rightarrow \tfrac{1}{2} \text{ factor}$$

vs.

$$\left(\frac{\delta}{\delta j}\right)^2\left(-\frac{\lambda}{4}\right)\left(\frac{\delta}{\delta j}\right)^4 \frac{1}{3!}\left(\frac{1}{2}\right)^3\left(j a^{-1} j\right)^3 = -\frac{3\lambda}{a^3}$$

② Factor of $\frac{1}{n!}$ for each set of n "equivalent lines", which are lines that run between two different vertices:

 $\frac{1}{2!}$  $\frac{1}{3!}$  1  $\frac{1}{2!}$ (if $\rightarrow$ means ends are different)

③ Factor of $1/P$ for permutations P of the vertices that leave the diagram unchanged (including any arrows)
- external points stay fixed when considering permutations.

- These factors are derived simply by expanding to low order and observing the patterns.

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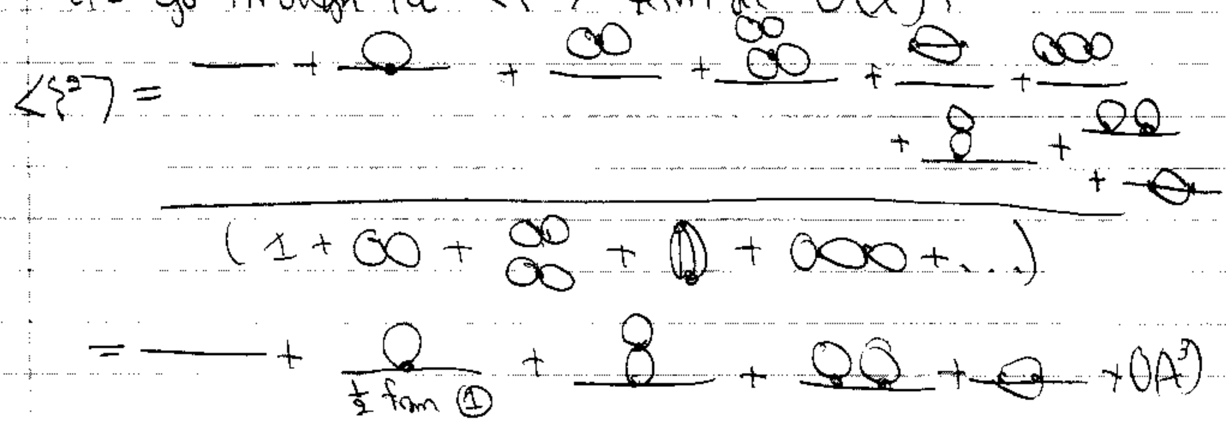
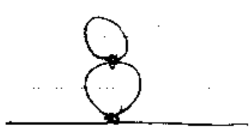
Let's go through the $\langle \xi^2 \rangle$ term at $O(\lambda^2)$:



so once again the denominator makes disconnected pieces cancel.

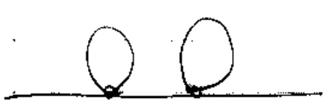
Fill in the symmetry factors



① lines to same vertex:

② equivalent lines:

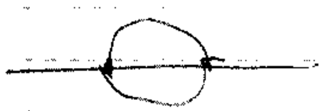
③ vertex permutations



① lines to same vertex:

② equivalent lines:

③ vertex permutations:



① lines to same vertex:

② equivalent lines:

③ vertex permutations:

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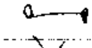
- Calculation of $\ln \frac{Z}{Z_0}$ at $O(\lambda^3)$

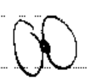
$$\boxed{\frac{Z}{Z_0} = \sum_{n=0}^{\infty} \lambda^n Z_n} \quad \text{with} \quad \boxed{Z_n = \frac{(-1)^n}{n!}\frac{(4n-1)!!}{4^n}\frac{1}{a^{2n}}}$$ the exact answer

$$\Rightarrow \boxed{\lambda Z_1 = -\frac{3}{4}\frac{\lambda}{a^2}} \quad \boxed{\lambda^2 Z_2 = \frac{105}{32}\frac{\lambda^2}{a^4}} \quad \boxed{\lambda^3 Z_3 = -\frac{3465}{128 a^6}\lambda^3}$$

Mathematica
$$\Rightarrow \boxed{\ln\frac{Z}{Z_0} = -\frac{3\lambda}{4a^2} + \frac{3\lambda^2}{a^4} - \frac{99\lambda^3}{4a^6} + O(\lambda^4)}$$

rules:  $\frac{1}{a}$

 $-\frac{\lambda}{4}4! = -6\lambda$

λ^1:  $\left(-\frac{\lambda}{4}4!\right)\frac{1}{a^2}$

① $\frac{1}{2}\cdot\frac{1}{2}$
② $1/2$ $\Rightarrow \frac{1}{a^2}(-6\lambda)\frac{1}{8} = -\frac{3\lambda}{4a^2}$ ✓
③ 1

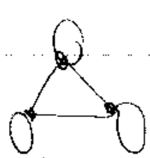
λ^2:  a) +  b)

	a)	b)
①	1	$\frac{1}{2}\cdot\frac{1}{2}$
②	$\frac{1}{4!}$	$\frac{1}{2}$
③	$\frac{1}{2}$	$1/2$

$$\Rightarrow (-6\lambda)^2\frac{1}{a^4}\left(\frac{1}{48}+\frac{1}{16}\right) = +\frac{3\lambda^2}{a^4} \checkmark$$

λ^3: a) 

b) 

c) 

d) 

	a)	b)	c)	d)
①	$\frac{1}{2}\cdot\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$	1
②	$\frac{1}{2}\cdot\frac{1}{2}$	$\frac{1}{3!}$	1	$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$
③	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3!}$	$\frac{1}{3!}$
	$1/32$ +	$1/24$ +	$1/48$ +	$1/48$

$$(-6\lambda)^3\frac{1}{a^6} \quad \Rightarrow -\frac{\lambda^3}{a^6}6^3\left(\frac{11}{96}\right) = -\frac{99\lambda^3}{4a^6} \checkmark$$

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If the lines have arrows that indicate one end is different from the other (like fermions) then the rules are modified:

① This is always 1

② only equivalent lines if in the same direction  yes  no

③ the permutations cannot change the flow of the lines. E.g.

 $\neq$  ← reversed

But we count  as $\frac{1}{2}$ under ② still.

If the potential is not contracted to a point:

where $\alpha, \gamma, \beta, \delta$ are spin indices, then we lose ② and have only ③.

- For n-point functions (as opposed to energy diagrams), the remaining symmetry factor is at most $1/2$ and for many cases there are no symmetry factors.

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Generalizing our previous result for $\langle\xi^2\rangle$:

$$\boxed{\langle\xi^4\rangle = \frac{N\int_{-\infty}^{\infty} d\xi\, \xi^4 e^{-\frac{a}{2}\xi^2 - \frac{\lambda}{4}\xi^4}}{N\int_{-\infty}^{\infty} d\xi\, e^{-\frac{a}{2}\xi^2-\frac{\lambda}{4}\xi^4}} = \frac{\left[\left(\frac{\partial}{\partial j}\frac{\partial}{\partial j}\frac{\partial}{\partial j}\frac{\partial}{\partial j}\right) e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4} e^{\frac{1}{2} j a^{-1} j}\right]\Big|_{j=0}}{\left[e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4} e^{\frac{1}{2} j a^{-1} j}\right]\Big|_{j=0}}}$$

· The leading contributions to $\langle\xi^4\rangle$ are:

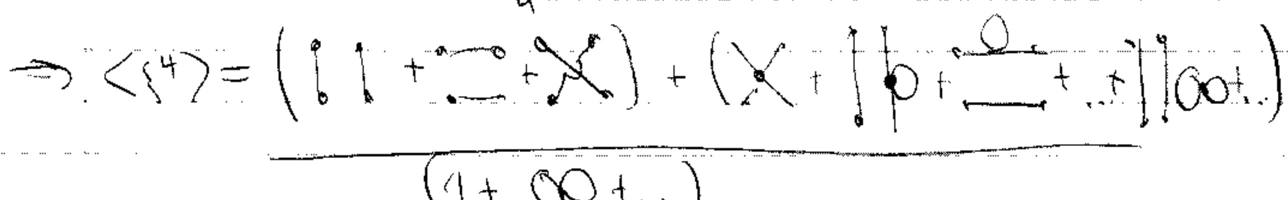
$$\langle\xi^4\rangle = \frac{\left\{\left(\frac{\partial}{\partial j}\frac{\partial}{\partial j}\frac{\partial}{\partial j}\frac{\partial}{\partial j}\right)\left[1-\frac{1}{4}\lambda\left(\frac{\partial}{\partial j}\right)^4+\ldots\right]\left[1+\left(\frac{j a^{-1} j}{2}\right)+\frac{1}{2!}\left(\frac{j a^{-1} j}{2}\right)^2+\frac{1}{3!}\left(\frac{j a^{-1} j}{2}\right)^3+\frac{1}{4!}\left(\frac{j a^{-1} j}{2}\right)^4+\ldots\right]\right\}\Big|_{j=0}}{\left\{\left[1-\frac{1}{4}\lambda\left(\frac{\partial}{\partial j}\right)^4+\ldots\right]\left[1+\left(\frac{j a^{-1} j}{2}\right)+\frac{1}{2!}\left(\frac{j a^{-1} j}{2}\right)^2+\ldots\right]\right\}\Big|_{j=0}}$$

$$= \frac{\left(\frac{1}{2!}\cdot\frac{1}{4}\cdot 4!\,\frac{1}{a^2} - \frac{1}{4}\lambda\frac{1}{a^4}\frac{1}{4!}\frac{1}{2^4}8!\right)}{1-\frac{\lambda}{4}\cdot\frac{1}{2!}\frac{1}{4}4!\,\frac{1}{a^2}} \quad \leftarrow \frac{8\cdot 7\cdot 6\cdot 5}{4\cdot 16} = \frac{105}{4}$$

$$= \left(\frac{3}{a^2} - \frac{105\lambda}{4a^4}\right)\left(1+\frac{3}{4}\frac{\lambda}{a^2}\right) = \frac{3}{a^2} - \frac{24\lambda}{a^4} + O(\lambda^2)$$

What is this in the diagrammatic expansion?

· You should imagine that ξ^4 is represented by 4 fixed points $\vdots\,\vdots$ to which we can join "propagators" $\text{—} \Rightarrow \frac{1}{a}$ and vertices $\bullet \Rightarrow \frac{\lambda}{4}\cdot 4!$



The [diagram] type pieces come from $\langle\xi^2\rangle^2$ so subtract those off and look at $\langle\xi^4\rangle - \langle\xi^2\rangle\langle\xi^2\rangle$. This removes all of these pieces, leaving the totally connected diagrams like [diagram]