(3/11/21)

$$\hat{U} = 1 + \frac{1}{4}\sum_{1234}\int \frac{d^3\vec{K}}{(2\pi)^3}\frac{d^3\vec{k}}{(2\pi)^3}\frac{d^3\vec{k}'}{(2\pi)^3}\langle \vec{k}\,12|\delta\tilde{U}|\vec{k}'\,34\rangle \times$$

$$a^{\dagger}_{\frac{1}{2}\vec{K}+\vec{k},1}\, a^{\dagger}_{\frac{1}{2}\vec{K}-\vec{k},2}\, a_{\frac{1}{2}\vec{K}-\vec{k}',4}\, a_{\frac{1}{2}\vec{K}+\vec{k}',3} + \dots \qquad (1)$$

$$\hat{n}^{\tau}_{\lambda}(\vec{q}) = \hat{U}\sum_{\sigma} a^{\dagger}_{\vec{q}\sigma\tau} a_{\vec{q}\sigma\tau}\,\hat{U}^{\dagger} \qquad (2)$$

$$|\Psi^{\lambda}_{d}\rangle = \frac{1}{2}\sum_{AB}\int \frac{d^3k_A}{(2\pi)^3}\frac{d^3k_B}{(2\pi)^3}\, a^{\dagger}_{\vec{k}_A,A}\, a^{\dagger}_{\vec{k}_B,B}\,|0\rangle\, \psi^{\lambda}_{AB}(\vec{k}_A,\vec{k}_B) \qquad (3)$$

$$\psi^{\lambda}_{AB}(\vec{k}_A,\vec{k}_B) = \underbrace{(2\pi)^3\delta^3(\vec{k}_A+\vec{k}_B)}_{\text{CoM}}\Big\langle \underbrace{\frac{\vec{k}_A-\vec{k}_B}{2}}_{\text{Relative}}\Big|\psi^{\lambda}_{d}\Big\rangle \qquad (4)$$

Split contributions into 4 terms:

Term 1

$$\langle 0|a_{\vec{k}_B B}\, a_{\vec{k}_A A}\, a^{\dagger}_{\vec{q}\sigma\tau}\, a_{\vec{q}\sigma\tau}\, a^{\dagger}_{\vec{k}_C C}\, a^{\dagger}_{\vec{k}_D D}|0\rangle$$

$$= 4\,\delta_{B,D}(2\pi)^3\delta^3(\vec{k}_B-\vec{k}_D)\,\delta_{A,\sigma\tau}(2\pi)^3\delta^3(\vec{k}_A-\vec{q}) \times$$
$$\delta_{\sigma\tau,C}(2\pi)^3\delta^3(\vec{q}-\vec{k}_C)$$

$$\text{Term 1} = \sum_{B}\sum_{\sigma}\int\frac{d^3k_B}{(2\pi)^3}\,\psi^{\lambda *}_{\sigma\tau,B}(\vec{q},\vec{k}_B)\,\psi^{\lambda}_{\sigma\tau,B}(\vec{q},\vec{k}_B)$$

$$= \sum_{B} \sum_{\sigma} \underbrace{\int \frac{d^3 k_B}{(2\pi)^3} (2\pi)^3 \delta^3(\vec{q} + \vec{k}_B)}_{\to \vec{k}_B = -\vec{q}} (2\pi)^3 \delta^3(\vec{q} + \vec{k}_B) \left| \psi^{\lambda}_{\sigma\tau, B}\left(\frac{\vec{q} - \vec{k}_B}{2}\right) \right|^2$$

$$= (2\pi)^3 \delta^3(0) \sum_{\sigma\sigma'\tau'} \left| \psi^{\lambda}_{\sigma\tau, \sigma'\tau'}(\vec{q}) \right|^2$$

(5)

$$\boxed{= V \sum_{\sigma\sigma'\tau'} \left| \psi^{\lambda}_{\sigma\tau, \sigma'\tau'}(\vec{q}) \right|^2} \quad \to \tau' \text{ is constrained by } \tau$$

Term 2 $\langle 0 | a_{\vec{k}_B B}\, a_{\vec{k}_A A}\, a^{\dagger}_{\frac{1}{2}\vec{K}+\vec{k}, 1}\, a^{\dagger}_{\frac{1}{2}\vec{K}-\vec{k}, 2}\, a_{\frac{1}{2}\vec{K}-\vec{k}', 4}\, a_{\frac{1}{2}\vec{K}+\vec{k}', 3} \times$

$a^{\dagger}_{\vec{q}\sigma\tau}\, a_{\vec{q}\sigma\tau}\, a^{\dagger}_{\vec{k}_C C}\, a^{\dagger}_{\vec{k}_D D} | 0 \rangle$

$$= 8\, \delta_{B,2} (2\pi)^3 \delta^3(\vec{k}_B - \tfrac{1}{2}\vec{K} + \vec{k})\, \delta_{A,1} (2\pi)^3 \delta^3(\vec{k}_A - \tfrac{1}{2}\vec{K} - \vec{k}) \times$$
$$\delta_{4,D} (2\pi)^3 \delta^3(\tfrac{1}{2}\vec{K} - \vec{k}' - \vec{k}_D)\, \delta_{3,\sigma\tau} (2\pi)^3 \delta^3(\tfrac{1}{2}\vec{K} + \vec{k}' - \vec{q}) \times$$
$$\delta_{\sigma\tau, C} (2\pi)^3 \delta^3(\vec{q} - \vec{k}_C)$$

$$= \frac{1}{2} \sum_{\sigma} \sum_{124} \int \frac{d^3 k}{(2\pi)^3} \frac{d^3 k'}{(2\pi)^3} \frac{d^3 K}{(2\pi)^3} (2\pi)^3 \delta^3(\tfrac{1}{2}\vec{K} + \vec{k}' - \vec{q})\, \psi^{\lambda *}_{12}(\tfrac{1}{2}\vec{K} + \vec{k}, \tfrac{1}{2}\vec{K} - \vec{k})$$

$$\langle \vec{k}\, 12 | \delta\tilde{U} | \vec{k}'\, \sigma\tau\, 4 \rangle\, \psi^{\lambda}_{\sigma\tau, 4}(\vec{q}, \tfrac{1}{2}\vec{K} - \vec{k}')$$

Do $d^3 k'$ $\quad k' \to \vec{q} - \frac{1}{2}\vec{K}$ $\quad \frac{1}{2}\vec{K} - \vec{k}' = \vec{K} - \vec{q}$

$$= \frac{1}{2} \sum_{\sigma} \sum_{124} \int \frac{d^3 k}{(2\pi)^3} \frac{d^3 K}{(2\pi)^3} (2\pi)^3 \delta^3(\vec{K}) (2\pi)^3 \delta^3(\vec{K})\, \psi^{\lambda *}_{12}(\vec{k}) \times$$

$$\langle \vec{k}\,12 | \delta\tilde{U} | \vec{q} - \tfrac{1}{2}\vec{K}\ \sigma\tau\ 4 \rangle \psi^{\lambda}_{\sigma\tau,4}(\vec{q} - \tfrac{1}{2}\vec{K})$$

$$= \frac{V}{2} \sum_{\sigma} \sum_{124} \int \frac{d^3k}{(2\pi)^3} \psi^{\lambda *}_{12}(\vec{k}) \langle \vec{k}\,12 | \delta\tilde{U} | \vec{q}\ \sigma\tau\ 4 \rangle \psi^{\lambda}_{\sigma\tau,4}(\vec{q})$$

$$\boxed{= \frac{V}{2} \sum_{\sigma\sigma'\sigma''\sigma'''} \sum_{\tau'\tau''\tau'''} \int \frac{d^3k}{(2\pi)^3} \psi^{\lambda *}_{\sigma'\tau',\sigma''\tau''}(\vec{k}) \langle \vec{k}\ \sigma'\tau'\sigma''\tau'' | \delta\tilde{U} | \vec{q}\ \sigma\tau\sigma'''\tau''' \rangle \times \psi^{\lambda}_{\sigma\tau,\sigma'''\tau'''}(\vec{q})} \tag{6}$$

Term 4 : $\langle 0 | a_B a_A a_1^\dagger a_2^\dagger a_4 a_3 a_9^\dagger a_9 a_7^\dagger a_8^\dagger a_6 a_5 a_C^\dagger a_D^\dagger | 0 \rangle$

$$= 16\, \delta_{B2} (2\pi)^3 \delta^3(\vec{k}_B - \tfrac{1}{2}\vec{K} + \vec{k})\, \delta_{A,1} (2\pi)^3 \delta^3(\vec{k}_A - \tfrac{1}{2}\vec{K} - \vec{k}) \times$$
$$\delta_{4,8} (2\pi)^3 \delta^3(\tfrac{1}{2}\vec{K} - \vec{k}' - \tfrac{1}{2}\vec{K}' + \vec{k}''')\, \delta_{3,\sigma\tau} (2\pi)^3 \delta^3(\tfrac{1}{2}\vec{K} + \vec{k}' - \vec{q}) \times$$
$$\delta_{\sigma\tau,7} (2\pi)^3 \delta^3(\vec{q} - \tfrac{1}{2}\vec{K}' - \vec{k}''')\, \delta_{6,D} (2\pi)^3 \delta^3(\tfrac{1}{2}\vec{K}' - \vec{k}'' - \vec{k}_D) \times$$
$$\delta_{5,C} (2\pi)^3 \delta^3(\tfrac{1}{2}\vec{K}' + \vec{k}'' - \vec{k}_C)$$

Let's do $A \to D$ integrations first

$$= \frac{1}{4} \sum_{\sigma} \sum_{12345678} \int \frac{d^3k}{(2\pi)^3} \cdots \frac{d^3k'''}{(2\pi)^3} \frac{d^3K}{(2\pi)^3} \frac{d^3K'}{(2\pi)^3} \psi^{\lambda *}_{12}(\tfrac{1}{2}\vec{K} + \vec{k}, \tfrac{1}{2}\vec{K} - \tfrac{1}{2}\vec{k}) \times$$

$$\langle \vec{k}\,12 | \delta\tilde{U} | \vec{k}'\,34 \rangle \langle \vec{k}'''\,78 | \delta\tilde{U}^\dagger | \vec{k}''\,56 \rangle \psi^{\lambda}_{56}(\tfrac{1}{2}\vec{K}' + \vec{k}'', \tfrac{1}{2}\vec{K}' - \vec{k}'') \times$$

$$\delta_{4,8} (2\pi)^3 \delta^3(\tfrac{1}{2}\vec{K} - \vec{k}' - \tfrac{1}{2}\vec{K}' + \vec{k}''')\, \delta_{3,\sigma\tau} (2\pi)^3 \delta^3(\tfrac{1}{2}\vec{K} + \vec{k}' - \vec{q}) \times$$

$$\delta_{\sigma\tau,7}(2\pi)^3\delta^3(\vec{q}-\tfrac{1}{2}\vec{K}'-\vec{k}''')$$

Do d^3k''', d^3k', d^3K' $\sum_{8\,3\,7}$

$$\tfrac{1}{2}\vec{K}-\vec{k}' = \tfrac{1}{2}\vec{K}'-\vec{k}''' \qquad (4,8) \qquad \vec{k}''' = \tfrac{1}{2}\vec{K}'-\tfrac{1}{2}\vec{K}+\vec{k}'$$

$$\tfrac{1}{2}\vec{K}+\vec{k}' = \vec{q} \qquad (3,\vec{q}) \qquad \vec{k}' = \vec{q}-\tfrac{1}{2}\vec{K}$$

$$\vec{q} = \tfrac{1}{2}\vec{K}'+\vec{k}''' \qquad (\vec{q},7) \qquad K' = 2(\vec{q}-\vec{k}''')$$

$$\vec{K}' = 2\vec{q} - 2\left(\tfrac{1}{2}\vec{K}'-\tfrac{1}{2}\vec{K}+\vec{k}'\right)$$

$$= 2\vec{q} - \vec{K}' + \vec{K} - 2\left(\vec{q}-\tfrac{1}{2}\vec{K}\right)$$

$$= \cancel{2\vec{q}} - \vec{K}' + \vec{K} - \cancel{2\vec{q}} + \vec{K}$$

$$\rightarrow \quad \vec{K}' = \vec{K}, \quad \vec{k}' = \vec{q}-\tfrac{1}{2}\vec{K}, \quad \vec{k}''' = \vec{k}' = \vec{q}-\tfrac{1}{2}\vec{K}$$

$$= \frac{1}{4}\sum_{\sigma}\sum_{12456}\int \frac{d^3k}{(2\pi)^3}\frac{\overset{d^3k'}{\vee}\, d^3K}{(2\pi)^3}\,\Psi^{\lambda *}_{12}\left(\tfrac{1}{2}\vec{K}+\vec{k}, \tfrac{1}{2}\vec{K}-\vec{k}\right)\times$$

$$\langle \vec{k}\,12|\delta\tilde{U}|\vec{q}-\tfrac{1}{2}\vec{K}\ \sigma\tau\ 4\rangle\langle\vec{q}-\tfrac{1}{2}\vec{K}\ \sigma\tau\ 4|\delta\tilde{U}^{+}|\vec{k}'56\rangle\times$$

$$\Psi^{\lambda}_{56}\left(\tfrac{1}{2}\vec{K}+\vec{k}', \tfrac{1}{2}\vec{K}-\vec{k}'\right) \qquad \int d_{\sigma} \quad \delta^3(\vec{k}_A+\vec{k}_B) = \delta^3(\vec{K})$$

$$= \frac{V}{4}\sum_{\substack{\sigma\sigma'\sigma''\sigma'''\tau'\tau''\tau'''\\ \sigma^{(4)}\sigma^{(5)}\tau^{(4)}\tau^{(5)}}}\int\frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\,\Psi^{\lambda *}_{\sigma'\tau'\sigma''\tau''}(\vec{k})\times$$

$$\langle\vec{k}\,\sigma'\tau'\sigma''\tau''|\delta\tilde{U}|\vec{q}\,\sigma\tau\sigma'''\tau'''\rangle\langle\vec{q}\,\sigma\tau\sigma'''\tau'''|\delta\tilde{U}^{+}|\vec{k}'\,\sigma^{(4)}\tau^{(4)}\sigma^{(5)}\tau^{(5)}\rangle$$

$$\times\ \Psi^{\lambda}_{\sigma^{(4)}\tau^{(4)}\sigma^{(5)}\tau^{(5)}}(\vec{k}') \qquad (7)$$