Physics 880.05: Problem Set #1

This is a set of exercises to help you build a good foundation for later developments. The MATLAB exercises are due on Monday, Oct. 12 and the other problems are due on Friday, Oct. 16. See the online lecture notes for details of any of the individual topics covered in this problem set. Check the 880.05 webpage for suggestions and hints. Please give feedback early and often (and stop by to ask about anything).

1. **MATLAB Sandbox.** MATLAB can be very helpful in building intuition and testing approximations for matrix manipulations by considering concrete examples. [Note: You can substitute any other program like Mathematica that has the same capabilities, but we will assume MATLAB. We recommend trying MATLAB even if you are familiar with another program.] The following exercises are to get you used to MATLAB; you need only turn in a printout of your session(s) with brief annotations. A summary guide to using MATLAB for playing matrix games has been posted on the website to help you.

    (a) Check by example that a complex Hermitian matrix is diagonalized by a matrix of its eigenvectors. The steps are:

    i. Create a random Hermitian matrix `A` of some manageable size (e.g., 4-by-4). [See Sections 4d, e, f and 5m of the guide.]

    ii. Diagonalize with `eig` to find the matrix `V` of eigenvectors [see 5h].

    iii. Use `V` explicitly to diagonalize `A`. What kind of matrix is `V`?

    (b) Generate three random 5×5 matrices A, B, C and compare $\mathrm{Tr}[ABC]$ to $\mathrm{Tr}[BCA]$ to $\mathrm{Tr}[BAC]$ to $\mathrm{Tr}[CAB]$. Conclusion?

    (c) Check for a couple of (random) examples that $\det A = e^{\mathrm{Tr}\ln A}$. [See Section 5. Be careful of `logm` vs. `log`.] Can A be complex? Does it have to be Hermitian?

    (d) Check that exponentiation of a matrix by using a power series works eventually, but is not efficient. Do this by generating a normally distributed 3×3 random matrix M [Section 4f] and comparing e^M to the power series expansion with 2 to 5 terms. [You will need 4a and 5i.] How accurate is the expansion?

    (e) Verify that $\mathrm{Tr}[e^{-\beta(T+V)}] = \mathrm{Tr}[(e^{-\epsilon(T+V)})^N]$ for T and V hermitian by considering random T and V (your choice of size), $\beta = 1$ and $N\epsilon = \beta$ for $N = 10, 100, 1000$. Now compare to $\mathrm{Tr}[(e^{-\epsilon T}e^{-\epsilon V})^N]$ to determine how the error scales with ϵ [use `format long` to get enough digits]. Can you explain the result?

2. **Stochastic Variational Method Revisited.** In Lecture 2, we gave an overview of the SVM. Show that the best variational estimate for a (finite) non-orthogonal basis is found by solving the generalized eigenvalue problem described on page 22 of the notes. [Hint: Require the estimated energy (see page 21) to be stationary under arbitrary variations of the expansion coefficients $\{c_i\}$.]

3. **Model Partition Function.**

   (a) Apply the Feynman rules for the model partition function discussed in class, $Z = \int d\xi \, e^{-a\xi^2/2 - \lambda\xi^4/4}$, to derive an expression for the λ^3 contribution to $\langle \xi^2 \rangle$.

      i. Draw all of the distinct diagrams that contribute. Explain how you know what type of diagram to draw (e.g., number of vertices, connected vs. disconnected, etc.).

      ii. Find the symmetry factor for each diagram.

      iii. Use the symmetry factors and the other Feynman rules to find an expression for $\langle \xi^2 \rangle$ in terms of a and λ.

      [Note: The solution to this problem is available on PS#2 from the old course website. Refer to it if you get stuck, but please try it on your own first!]

   (b) Consider a model partition function that has the analog of three-body forces: $Z = \int d\xi \, e^{-a\xi^2/2 - \alpha\xi^6/6}$, where α is a new coupling constant.

      i. What are the Feynman rules for this "theory"?

      ii. Draw the diagrams that contribute to Z_α/Z_0 through α^2.

      iii. Draw the diagrams that contribute to $\langle \xi^2 \rangle$ through α^2.

      iv. [Bonus.] Evaluate the last two contributions, including symmetry factors, in terms of a and α.

   (c) [Bonus.] Use the replica method (pages 58-59 in the lecture notes) to show that the expectation value $\langle \xi^2 \rangle$ is given by the connected diagrams only. Do this by defining O_n by:

   $$O_n \equiv \left(\int d\xi_1 \, \xi_1^2 \, e^{-a\xi_1^2/2 - \lambda\xi_1^4/4}/Z_0 \right) \left(\int d\xi_2 \, e^{-a\xi_2^2/2 - \lambda\xi_2^4/4}/Z_0 \right) \cdots \left(\int d\xi_n \, e^{-a\xi_n^2/2 - \lambda\xi_n^4/4}/Z_0 \right)$$

   where ξ^2 only appears in the first integral (with index 1).

      i. Argue that O_0 is equal to $\langle \xi^2 \rangle$, so we can calculate O_n and then set n to zero.

      ii. Consider the perturbative expansion of O_n. What indices appear on the external legs?

      iii. Argue that the $n = 0$ part is precisely the sum of the connected diagrams.

      iv. Does a similar argument work for other operators?