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- Call the bound-state energy for two particles in free space in one dimension B_2.

- Let's simply solve the Schrödinger equation for the two body wave function $\psi(x_1, x_2)$:

$$\boxed{H\psi(x_1,x_2) = E\psi(x_1,x_2)}$$

$$\Rightarrow \boxed{-\frac{1}{2m}\left(\frac{d^2}{dx_1^2}+\frac{d^2}{dx_2^2}\right)\psi(x_1,x_2) - \lambda\,\delta(x_1-x_2)\psi(x_1,x_2) = E\psi(x_1,x_2)}$$

(with $\hbar=1$ units here).

- We'll assume spin ½ ($g=2$) for now. The wave function $\psi(x_1,x_2)$ has a spin part as well. Since the interaction is independent of spin and acts at zero distance, the spin wave function will be restricted by the Pauli principle.
  - If we say that one particle is spin up and one is spin down, then we don't need to consider spin at this point.

- We can separate the S-eqn into center-of-mass (cm) and relative pieces with

$$\boxed{X_{cm} \equiv \tfrac{1}{2}(x_1+x_2)} \quad \text{and} \quad \boxed{x \equiv x_1 - x_2}$$

$$\Rightarrow \boxed{\frac{d}{dx_1} = \frac{dX_{cm}}{dx_1}\frac{d}{dX_{cm}} + \frac{dx}{dx_1}\frac{d}{dx} = \frac{1}{2}\frac{d}{dX_{cm}} + \frac{d}{dx}}$$

$$\boxed{\frac{d}{dx_2} = \frac{dX_{cm}}{dx_2}\frac{d}{dX_{cm}} + \frac{dx}{dx_2}\frac{d}{dx} = \frac{1}{2}\frac{d}{dX_{cm}} - \frac{d}{dx}}$$

$$\Rightarrow \boxed{\frac{d^2}{dx_1^2} + \frac{d^2}{dx_2^2} = 2\cdot\frac{1}{4}\frac{d^2}{dX_{cm}^2} + 2\frac{d^2}{dx^2} + \emptyset \times \text{cross terms}}$$

So that the Hamiltonian separates into a sum of two

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sub-Hamiltonians for the center of mass and relative degrees of freedom.

$$\Rightarrow \quad \boxed{H = H_{cm} + H_{rel}}$$

and the wave function $\boxed{\Psi(x_1, x_2) \rightarrow \Psi_{cm}(x_{cm})\,\varphi(x)}$

and $\boxed{E = E_{cm} + B_2}$

(where we use B_2 in anticipation of a single 2-body bound state solution).

$$\Rightarrow \quad \boxed{-\frac{1}{4m}\frac{\partial^2}{\partial x_{cm}^2}\Psi_{cm}(x_{cm}) = E_{cm}\,\Psi_{cm}(x_{cm})}$$

which simply has plane wave solutions. For now we'll work in the cm frame and focus on the relative wave function:

$$\boxed{-\frac{1}{m}\frac{d^2\varphi(x)}{dx^2} - \lambda\,\delta(x)\,\varphi(x) = B_2\,\varphi(x)}$$

We can solve this directly using similar manipulations to finding the Green's function in PS#3.

If we look for $B_2 < 0$, we can define

$$\boxed{\varkappa^2 \equiv |B_2|\,m} \qquad \boxed{\tilde{\lambda} = m\lambda}$$

and solve the eigenvalue problem

$$\boxed{\frac{d^2\varphi}{dx^2} + \tilde{\lambda}\,\delta(x)\,\varphi(x) = \varkappa^2\varphi(x)}$$

Plan: Write down general solutions for $x>0$ and $x<0$ and join them by integrating the S-equation across the delta function.

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By inspection, the solutions for $x>0$ and $x<0$ that are normalizable are

$$\boxed{\psi(x) = \begin{cases} A_+ e^{-\kappa x} & x>0 \\ A_- e^{+\kappa x} & x<0 \end{cases}}$$

· Continuity of $\psi(x)$ at $x=0$ $\Rightarrow$ $A_+ = A_- \equiv A$.

Integrating the S-equation from $-\varepsilon$ to $+\varepsilon$:

$$\boxed{\int_{-\varepsilon}^{\varepsilon} \frac{d}{dx}\left(\frac{d\psi}{dx}\right) dx + \int_{-\varepsilon}^{\varepsilon} \tilde{\lambda}\,\delta(x)\psi(x)\, dx = O(\varepsilon)} \quad \text{by continuity}$$

$$\Rightarrow \epsilon\to 0: \boxed{\left.\frac{d\psi}{dx}\right|_{0^+} - \left.\frac{d\psi}{dx}\right|_{0^-} + \tilde{\lambda}\psi(0) = 0}$$

$$\Rightarrow \boxed{-\kappa A - \kappa A + \tilde{\lambda} A = 0} \Rightarrow \boxed{\kappa = \frac{\tilde{\lambda}}{2}}$$

$$\text{or } \boxed{B_2 = -\frac{\kappa^2}{m} = -\frac{\tilde{\lambda}^2}{4m} = -\frac{m\lambda^2}{4}}$$

· So there is always exactly one bound state, with a binding energy that grows as the strength of the delta function squared.

· The extent of the wavefunction $\sim \frac{2}{\kappa} \propto \frac{1}{\lambda}$ so for strong coupling the bound state is very localized while for weak coupling it is very spread out.

· One can solve the problem for a greater degeneracy with the Bethe ansatz.

· So at very low density, we expect that the system becomes a gas of noninteracting two-fermion composite particles.

· At the other limit of high density, we found a noninteracting Fermi gas.

$\Rightarrow$ What happens in between?

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- So one question is: How does the presence of other particles (the "Fermi sea") affect the formation of bound states.
  - We'll first look at the role of "Pauli blocking", which means the impact of having already filled levels not allowing us to use those momentum components for our wave function.
  - So this means we want to look at our bound state problem formulated in momentum space (rather than coordinate/position space).

- Let's imagine we have two fermions with opposite spins ($g=2$) and zero center-of-mass momentum.
  - Call the relative wave function $\mathcal{Y}(x)$, and the bound-state energy B_2 (so $\mathcal{Y} \to \psi$ and $E_2 \to B_2$ in free space).
  - Decompose $\mathcal{Y}(x)$ in momentum space:

$$\boxed{\mathcal{Y}(x) = \sum_{k'} C_{k'} e^{ik'x}}$$

(so find the Fourier coefficients $C_{k'}$).
  - We could also use a continuous k' ($L \to \infty$ limit) and write $C_{k'} \to \tilde{\mathcal{Y}}(k')$ as the momentum-space wave function.
- Substituting in the S-equation (in free space for now)

$$\boxed{-\frac{1}{m}\sum_{k'} C_{k'}(-k'^2) e^{ik'x} - \lambda\,\delta(x)\sum_{k'} C_{k'} e^{ik'x} = E_2 \sum_{k'} C_{k'} e^{ik'x}}$$

- We project out the C_k's by multiplying by e^{-ikx}, and integrating over $\int_{-\infty}^{\infty} dx$ (or $\int_{-L}^{L} dx$) using

$$\boxed{\int_{-\infty}^{\infty} dx\, e^{i(k'-k)x} = L\,\delta_{kk'}}$$

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to obtain

$$\boxed{2L\left(\frac{k^2}{2m}\right)C_k - \lambda\sum_{k'} C_{k'} = LE_2 C_k}$$

or
$$\boxed{\left(2\epsilon_k^0 - E_2\right)C_k = \frac{\lambda}{L}\sum_{k'} C_{k'}}$$

with $\boxed{\epsilon_k^0 \equiv \frac{k^2}{2m}}$

[If we have two particles with momentum k_1 and k_2, then

$$\boxed{k_{cm} \equiv k_1 + k_2 = 0} \text{ here and } \boxed{k = \tfrac{1}{2}(k_1 - k_2)}$$

$\Rightarrow |k_1| = |k_2| = k$

Their non-interacting energy is $\frac{k_1^2}{2m} + \frac{k_2^2}{2m} = \frac{k^2}{m} = 2\epsilon_k^0$. We will usually consider particles at or near the Fermi surface, in which case $k \to k_F$.]

Alternatively, we could have started with

$$\boxed{(H_0 + V)|\psi\rangle = E_2|\psi\rangle}$$

$$\Rightarrow \boxed{\langle k|H_0|\psi\rangle + \langle k|V|\psi\rangle = E_2\langle k|\psi\rangle} \quad \text{and} \quad \boxed{\langle k|\psi\rangle \equiv C_k}$$

$$\Rightarrow \boxed{\frac{k^2}{m}C_k + \sum_{k'} \underbrace{\langle k|V|k'\rangle}_{\text{just } \lambda \text{ for delta function}}\langle k'|\psi\rangle = E_2 C_k}$$

or $\boxed{2\epsilon_k^0 C_k + \lambda\sum_{k'} C_{k'} = E_2 C_k}$

as before.

$$\boxed{\langle k|V|k'\rangle = \int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dx'\langle k|x\rangle\langle x|V|x'\rangle\langle x'|k'\rangle = \lambda\int_{-\infty}^{\infty}dx\,dx'\,e^{ikx}e^{-ik'x'}\delta(x)\delta(x-x') = \lambda}$$

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How do we find solutions to this equation, given K?

- Here's a standard approach...

let $\sum_{k'} C_{k'} \equiv A$, then solve the equation for C_k, and sum over K to form A again:

$$\boxed{C_k = \frac{1}{\varepsilon_k^0 - E_2}\frac{\lambda}{L}A}$$

$$\Rightarrow \boxed{\sum_k C_k = A = \frac{\lambda}{L}A\sum_k \frac{1}{\varepsilon_k^0 - E_2}}$$

or, if $A \neq 0$, the eigenvalue condition is

$$\boxed{1 = \frac{\lambda}{L}\sum_k \frac{1}{\varepsilon_k^0 - E_2}}$$

- In free space, $\boxed{\sum_k \xrightarrow[L\to\infty]{} \frac{L}{2\pi}\int_{-\infty}^{\infty} dk}$

$$\Rightarrow \boxed{1 = \frac{\lambda}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{k^2/m + |E_2|}} \quad \text{for } E_2 < 0$$

$$= \frac{\lambda}{2}\sqrt{\frac{m}{|E_2|}}$$ from Mathematica (with Assumptions → {$|E_2|>0$, $m>0$}

or $\boxed{|E_2| = -B_2 = \frac{m\lambda^2}{4}}$ as before.

So we reproduce the bound state solution from before (and there is only one with $E_2 < 0$).

- The states with $E > 0$ are scattering states which have energy equal to the asymptotic kinetic energy.

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· Now what about in the medium? Pauli-blocking means that the states with $-k_f < k < k_f$ are excluded from the integral in the eigenvalue equation.

$$\Rightarrow \int_{-\infty}^{\infty} dk \longrightarrow \int_{-\infty}^{-k_f} dk + \int_{k_f}^{\infty} dk = 2\int_{k_f}^{\infty} dk \quad \text{if the integrand is even.}$$

It will be most useful to integrate over $\epsilon_k^0 \equiv \epsilon$ instead of k. We change variables using $\epsilon = \frac{k^2}{2m} \Rightarrow d\epsilon = \frac{1}{m} k\,dk$ with $k = \sqrt{2m\epsilon}$

$$\Rightarrow \boxed{1 = \frac{\lambda}{2\pi} 2\int_{k_f}^{\infty} \frac{dk}{k^2/m + |E_2|} \longrightarrow 1 = \frac{\lambda}{\pi}\sqrt{\frac{m}{2}} \int_{\epsilon_f}^{\infty} \frac{d\epsilon}{\sqrt{\epsilon}} \frac{1}{2\epsilon - E_2}}$$

[Note the 1-d density of states with length L is $g(\epsilon) = \sqrt{\frac{m}{2}} \frac{L}{\pi} \frac{1}{\sqrt{\epsilon}}$.]

· Let's look at this integral for $E_2 < 0$ and $0 < E_2 < 2\epsilon_f$.

· First: $\boxed{E_2 < 0}$ $\rightarrow$

$$\boxed{\int_{\epsilon_f}^{\infty} \frac{d\epsilon}{\sqrt{\epsilon}} \frac{1}{2\epsilon + |E_2|} = \sqrt{\frac{2}{|E|}} \tan^{-1}\sqrt{\frac{|E|}{2\epsilon_f}} = \frac{1}{\sqrt{\epsilon_f}\sqrt{|z|}} \tan^{-1}(|z|)} \quad \boxed{z < 0}$$

where $\boxed{z \equiv E_2/2\epsilon_f}$ (so we'll consider $z < 0$ and $0 < z < 1$).

· If $\boxed{0 < E_2 < 2\epsilon_f}$ then

$$\boxed{\int_{\epsilon_f}^{\infty} \frac{d\epsilon}{\sqrt{\epsilon}} \frac{1}{2\epsilon - E_2} = \frac{1}{2\sqrt{\epsilon_f}} \frac{1}{\sqrt{z}} \ln\left|\frac{1+\sqrt{z}}{1-\sqrt{z}}\right|} \quad \boxed{0 < z < 1}$$

[These results are directly obtained from Mathematica.]

(Note: we can relate the results by analytic continuation of z.)

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So we have transcendental equations for nontrivial ($z \neq 0$) solutions:

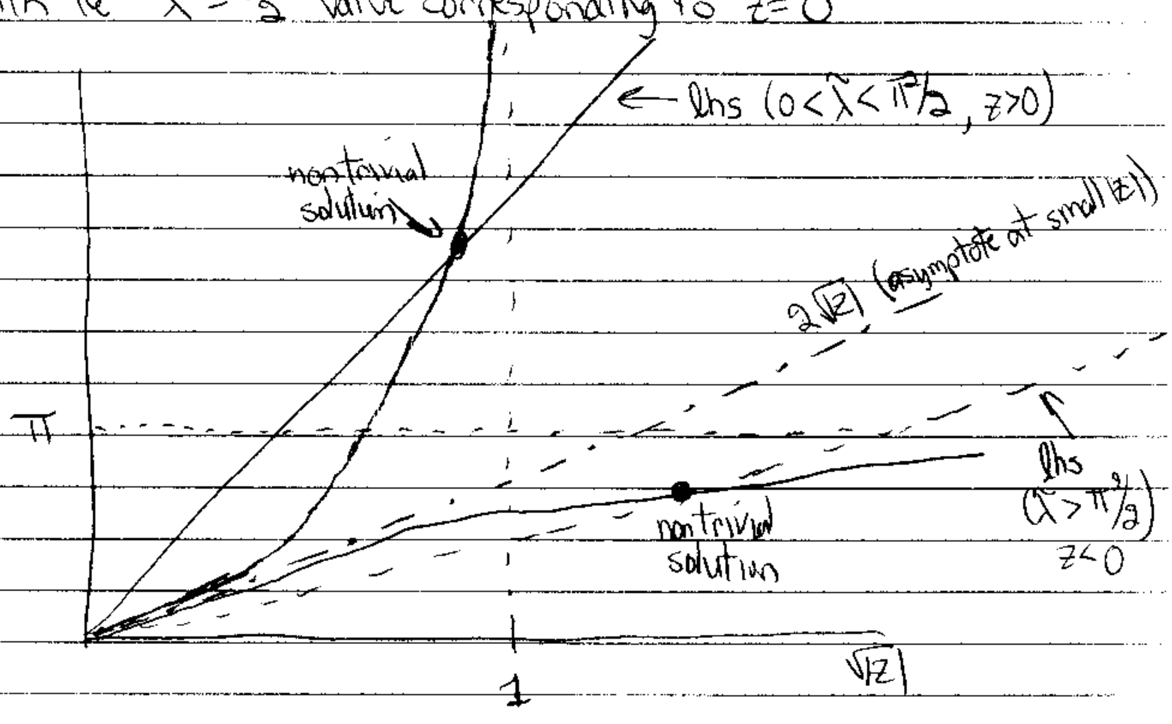
$$\boxed{0<z<1: \quad \pi^2 \sqrt{z}/\hat{\lambda} = \ln\left[\frac{1+\sqrt{z}}{1-\sqrt{z}}\right]}$$

$$\boxed{z<0: \quad \pi^2 \sqrt{|z|}/\hat{\lambda} = 2\tan^{-1}\sqrt{|z|}}$$

- We can look for solutions graphically. Both right-hand sides (rhs) behave like $2\sqrt{|z|} + O(|z|^{3/2})$ for small $|z|$, but one is concave up in $|z|$ and the other is concave down.
  $\Rightarrow$ there are nontrivial ($|z|>0$) solutions when

$$\boxed{\begin{array}{l} \text{for } z>0 \text{ if } 0<\hat{\lambda}<\pi^2/2 \\ \text{for } z<0 \text{ if } \hat{\lambda}>\pi^2/2 \end{array}}$$

with the $\hat{\lambda} = \frac{\pi^2}{2}$ value corresponding to $z=0$



- So what does this mean and how do we find the full energy density? Stay tuned!