1.5 COHERENT STATES

Thus far we have used permanents or Slater determinants as a natural basis for the Fock space $\mathcal{B}$ or $\mathcal{F}$. Another extremely useful basis of the Fock space is the basis of coherent states which is analogous to the basis of position eigenstates in quantum mechanics. Although this is not an orthonormal basis, it spans the whole Fock space. Just as the position states $|\vec{r}\rangle$ are defined as eigenstates of $\hat{\vec{r}}$, the coherent states are defined as eigenstates of the annihilation operators.

To see why annihilation operators are selected rather than creation operators, it is useful to consider the existence of eigenstates of the creation and annihilation operators. If we denote by $|\phi\rangle$ a general vector of the Fock space, we can expand $|\phi\rangle$ as:

$$|\phi\rangle = \sum_{n=0}^{\infty} \sum_{\alpha_1\ldots\alpha_n} \phi_{\alpha_1\ldots\alpha_n} |\alpha_1\ldots\alpha_n\rangle \quad . \tag{1.111}$$

Thus, $|\phi\rangle$ necessarily has a component with a minimum number of particles and if we apply any creation operator to $|\phi\rangle$, we see that the minimum number of particles in $|\phi\rangle$ is increased by one. Thus, the resulting state cannot be a multiple of the original state and therefore a creation operator cannot have an eigenstate. On the other hand, if we apply an annihilation operator to $|\phi\rangle$, this decreases the maximum number of particles in $|\phi\rangle$ by one, since $|\phi\rangle$ may contain components with all particle numbers, nothing *a priori* forbids $|\phi\rangle$ to have eigenstates.

Assuming we have succeeded in constructing an eigenstate $|\phi\rangle$ of the annihilation operators, then

$$a_\alpha|\phi\rangle = \phi_\alpha|\phi\rangle \quad . \tag{1.112}$$

When more than one annihilation operator acts upon such a coherent state, a significant difference arises between Bosons and Fermions. The commutation or anticommutation relations (1.74) imply corresponding relations for the eigenvalues

$$[\phi_\alpha, \phi_\beta]_{-\varsigma} = 0 \quad . \tag{1.113}$$

For Fermions, the eigenvalues anticommute, and in order to accommodate this unusual feature we subsequently will need to introduce anticommuting variables called Grassmann numbers. For Bosons, however, the eigenvalues commute and we will be able to proceed straightforwardly using ordinary numbers. Hence, we shall begin by considering Bosons.

BOSON COHERENT STATES

For Bosons, the eigenvalues ϕ_α of the annihilation operators may be real or complex numbers. It is convenient to expand a Boson coherent state in occupation number representation

$$|\phi\rangle = \sum_{n_{\alpha_1}, n_{\alpha_2}\ldots n_{\alpha_p}} \phi_{n_{\alpha_1}n_{\alpha_2}\ldots n_{\alpha_p}\ldots} |n_{\alpha_1}n_{\alpha_2}\ldots n_{\alpha_p}\ldots\rangle \tag{1.114}$$

where as usual $|n_{\alpha_1}n_{\alpha_2}\ldots n_{\alpha_p}\ldots\rangle$ denotes a normalized symmetrized state with n_{α_1} particles in state $|\alpha_1\rangle$, n_{α_2} particles in state $|\alpha_2\rangle,\ldots$ and $\{|\alpha_i\rangle\}$ is an orthonormal basis.

The eigenvalue condition, Eq. (1.112) for an annihilation operator a_{α_i} acting on $|\phi\rangle$ implies the following conditions on the coefficients for all $\{n_a\}$

$$\phi_{\alpha_i}\phi_{n_{\alpha_1}n_{\alpha_2}\ldots(n_{\alpha_i}-1)\ldots} = \sqrt{n_{\alpha_i}}\,\phi_{n_{\alpha_1}n_{\alpha_2}\ldots n_{\alpha_i}\ldots} \quad . \tag{1.115}$$

Relating each coefficient by induction to the coefficient for the vacuum state which we arbitrarily set equal to 1, we obtain

$$\phi_{n_{\alpha_1}n_{\alpha_2}\ldots n_{\alpha_i}} = \frac{\phi_{\alpha_1}^{n_{\alpha_1}}\phi_{\alpha_2}^{n_{\alpha_2}}\ldots\phi_{\alpha_i}^{n_{\alpha_i}}\ldots}{\sqrt{n_{\alpha_1}!}\sqrt{n_{\alpha_2}!}\ldots\sqrt{n_{\alpha_i}!}\ldots} \quad . \tag{1.116}$$

Substituting (1.116) in (1.114) and using the fact that

$$|n_{\alpha_1}n_{\alpha_2}\ldots n_{\alpha_p}\ldots\rangle = \frac{(a^\dagger_{\alpha_1})^{n_{\alpha_1}}}{\sqrt{n_{\alpha_1}!}}\frac{(a^\dagger_{\alpha_2})^{n_{\alpha_2}}}{\sqrt{n_{\alpha_2}!}}\ldots\frac{(a^\dagger_{\alpha_p})^{n_{\alpha_p}}}{\sqrt{n_{\alpha_p}!}}\ldots|0\rangle \tag{1.117}$$

we finally obtain

$$|\phi\rangle = \sum_{n_{\alpha_1}, n_{\alpha_2}\ldots n_{\alpha_p}} \frac{(\phi_{\alpha_1}a^\dagger_{\alpha_1})^{n_{\alpha_1}}}{n_{\alpha_1}!}\frac{(\phi_{\alpha_2}a^\dagger_{\alpha_2})^{n_{\alpha_2}}}{n_{\alpha_2}!}\ldots\frac{(\phi_{\alpha_p}a^\dagger_{\alpha_p})^{n_{\alpha_p}}}{n_{\alpha_p}!}|0\rangle$$

$$= e^{\sum_\alpha \phi_\alpha a^\dagger_\alpha}|0\rangle \tag{1.118a}$$

$$\langle\phi| = \langle 0|e^{\sum_\alpha \phi^*_\alpha a_\alpha} \quad . \tag{1.118b}$$

Note by taking the adjoint of (1.112), that the adjoint of a coherent state is a left eigenstate of the creation operators:

$$\langle\phi|a^\dagger_\alpha = \langle\phi|\phi^*_\alpha \quad . \tag{1.119}$$

The action of a creation operator $a^\dagger_\alpha$ on a coherent state is given by

$$a^\dagger_\alpha|\phi\rangle = a^\dagger_\alpha e^{\sum_{\alpha'}\phi_{\alpha'}a^\dagger_{\alpha'}}|0\rangle$$

$$= \frac{\partial}{\partial\phi_\alpha}|\phi\rangle \tag{1.120a}$$

with the adjoint relation

$$\langle\phi|a_\alpha = \frac{\partial}{\partial\phi^*_\alpha}\langle\phi| \quad . \tag{1.120b}$$

The overlap of two coherent states is given by:

$$\langle\phi|\phi'\rangle = \sum_{n_{\alpha_1}\ldots n_{\alpha_p}\ldots}\;\sum_{n'_{\alpha_1}\ldots n'_{\alpha_p}\ldots} \frac{\phi^{*n_{\alpha_1}}_{\alpha_1}}{\sqrt{n_{\alpha_1}!}}\ldots\frac{\phi^{*n_{\alpha_p}}_{\alpha_p}}{\sqrt{n_{\alpha_p}!}}\ldots\frac{\phi'^{n'_{\alpha_1}}_{\alpha_1}}{\sqrt{n'_{\alpha_1}!}}\ldots\frac{\phi'^{n'_{\alpha_p}}_{\alpha_p}}{\sqrt{n'_{\alpha_p}!}}\ldots$$

$$\times\langle n_{\alpha_1}\ldots n_{\alpha_p}\ldots|n'_{\alpha_1}\ldots n'_{\alpha_p}\rangle \quad . \tag{1.121}$$

Since the basis $|\alpha\rangle$ is orthonormal, the scalar product $\langle n_{\alpha_1} \dots n_{\alpha_p} \dots | n'_{\alpha_1} \dots n'_{\alpha_p} \dots \rangle$ is equal to $\delta_{n_{\alpha_1} n'_{\alpha_1}} \dots \delta_{n_{\alpha_p} n'_{\alpha_p}} \dots$ which leads to:

$$\langle \phi | \phi' \rangle = e^{\sum_\alpha \phi^*_\alpha \phi'_\alpha} \quad . \tag{1.122}$$

A crucial property of the coherent states is their overcompleteness in the Fock space, that is, the fact that any vector of the Fock space can be expanded in terms of coherent states. This is expressed by the closure relation

$$\int \prod_\alpha \frac{d\phi^*_\alpha d\phi_\alpha}{2i\pi} e^{-\sum_\alpha \phi^*_\alpha \phi_\alpha} |\phi\rangle\langle\phi| = 1 \tag{1.123}$$

where 1 is the unit operator in the Fock space, the measure is given by:

$$\frac{d\phi^*_\alpha d\phi_\alpha}{2i\pi} = \frac{d(\mathrm{Re}\,\phi_\alpha) d(\mathrm{Im}\,\phi_\alpha)}{\pi} \tag{1.124}$$

and the integration extends over all values of $\mathrm{Re}\phi_\alpha$ and $\mathrm{Im}\phi_\alpha$.

As explained in Problem 1.2, one may verify Eq. (1.123) straightforwardly by integrating the left hand side to obtain the familiar completeness relation Eq. (1.51). A more economical proof is provided by Schur's lemma, which in the present context states that if an operator commutes with all creation and annihilation operators, it is proportional to the unit operator in the Fock space. Using equations (1.120),

$$[a_\alpha, |\phi\rangle\langle\phi|] = (\phi_\alpha - \frac{\partial}{\partial \phi^*_\alpha}) |\phi\rangle\langle\phi| \tag{1.125}$$

so that evaluating the commutator of Eq. (1.123) and integrating by parts,

$$\begin{aligned} &\left[a_\alpha, \int \prod_{\alpha'} \frac{d\phi^*_{\alpha'} d\phi_{\alpha'}}{2\pi i} e^{-\sum_{\alpha'} \phi^*_{\alpha'} \phi_{\alpha'}} |\phi\rangle\langle\phi| \right] \\ &= \int \prod_{\alpha'} \frac{d\phi^*_{\alpha'} d\phi_{\alpha'}}{2\pi i} e^{-\sum_{\alpha'} \phi^*_{\alpha'} \phi_{\alpha'}} (\phi_\alpha - \frac{\partial}{\partial \phi^*_\alpha}) |\phi\rangle\langle\phi| \\ &= 0 \quad . \end{aligned} \tag{1.126}$$

By taking the adjoint of Eq. (1.126), we observe that the left hand side of Eq. (1.123) commutes with all the creation operators as well as the annihilation operators so it must be proportional to the unit operator. The proportionality factor is calculated by taking the expectation value of the left hand side of (1.123) in the vacuum:

$$\begin{aligned} \int \prod_\alpha \frac{d\phi^*_\alpha d\phi_\alpha}{2i\pi} e^{-\sum_\alpha \phi^*_\alpha \phi_\alpha} \langle 0|\phi\rangle\langle\phi|0\rangle &= \int \prod_\alpha \frac{d\phi^*_\alpha d\phi_\alpha}{2i\pi} e^{-\sum_\alpha \phi^*_\alpha \phi_\alpha} \\ &= 1 \quad . \end{aligned} \tag{1.127}$$

This proves Eq. (1.123).

This completeness relation provides a useful expression for the trace of an operator. Let A be any operator and let $\{|n>\}$ denote a complete set of states. Then

$$\begin{aligned} \mathrm{Tr}\, A &= \sum_n \langle n|A|n\rangle \\ &= \int \prod_\alpha \frac{d\phi^*_\alpha d\phi_\alpha}{2\pi i} e^{-\sum_\alpha \phi^*_\alpha \phi_\alpha} \sum_n \langle n|\phi\rangle\langle\phi|A|n\rangle \\ &= \int \prod_\alpha \frac{d\phi^*_\alpha d\phi_\alpha}{2\pi i} e^{-\sum_\alpha \phi^*_\alpha \phi_\alpha} \langle\phi|A \sum_n |n\rangle\langle n||\phi\rangle \\ &= \int \prod_\alpha \frac{d\phi^*_\alpha d\phi_\alpha}{2\pi i} e^{-\sum_\alpha \phi^*_\alpha \phi_\alpha} \langle\phi|A|\phi\rangle \quad . \end{aligned} \tag{1.128}$$

In quantum mechanics, the completeness of the position eigenstates allows us to represent a state $|\psi\rangle = \int dx \psi(x) |x\rangle$ where $\psi(x) = \langle x|\psi\rangle$ is the coordinate representation of the state $|\psi\rangle$. Analogously, Equation (1.124) implies that any state $|\psi\rangle$ of the Fock space can be represented as:

$$|\psi\rangle = \int \prod_\alpha \frac{d\phi^*_\alpha d\phi_\alpha}{2\pi i} e^{-\sum_\alpha \phi^*_\alpha \phi_\alpha} \psi(\phi^*) \cdot |\phi\rangle \tag{1.129a}$$

where by definition:

$$\psi(\phi^*) = \langle\phi|\psi\rangle \tag{1.129b}$$

is the coherent state representation of the state $|\psi\rangle$, and ϕ denotes the set $\{\phi^*_\alpha\}$. The coherent state representation for Bosons is often referred to as the holomorphic representation, which arises from the fact that ψ is an analytic function of the variables ϕ^*_α. Physically, $\psi(\phi^*)$ is simply the wavefunction of the state $|\psi\rangle$ in the coherent state representation; that is, the probability amplitude to find the system in the coherent state $|\phi\rangle$.

Just as it is useful to know how the operators $\hat{x}$ and $\hat{p}$ act in coordinate representation, it is useful to exhibit how the operators $a^\dagger_\alpha$ and a_α act in the coherent state representation. Using Eqs. (1.119) and (1.120) we find:

$$\langle\phi|a_\alpha|f\rangle = \frac{\partial}{\partial \phi^*_\alpha} f(\phi^*) \tag{1.130a}$$

and

$$\langle\phi|a^\dagger_\alpha|f\rangle = \phi^*_\alpha f(\phi^*) \quad . \tag{1.130b}$$

Thus we can write symbolically

$$a_\alpha = \frac{\partial}{\partial \phi^*_\alpha} \tag{1.131a}$$

and

$$a^\dagger_\alpha = \phi^*_\alpha \tag{1.131b}$$

which is consistent with the Boson commutation rules:

$$\left[\phi^*_\alpha, \phi^*_\beta\right] = \left[\frac{\partial}{\partial\phi^*_\alpha}, \frac{\partial}{\partial\phi^*_\beta}\right] = 0 \tag{1.132a}$$

$$\left[\frac{\partial}{\partial\phi^*_\alpha}, \phi^*_\beta\right] = \delta_{\alpha\beta} \quad . \tag{1.132b}$$

Aside from factors of i, the behavior of a and $a^\dagger$ in the coherent state representation is thus analogous to that of $\hat{x}$ and $\hat{p}$ in coordinate representation.

The result, Eq. (1.130), yields a simple expression for the Schrödinger equation in the coherent state representation. If $H(a^\dagger_\alpha, a_\alpha)$ is the Hamiltonian in normal form, then projection of the Schrödinger equation

$$H(a^\dagger_\alpha, a_\alpha)|\psi\rangle = E|\psi\rangle \tag{1.133}$$

on the left by $\langle\phi|$ yields

$$H\left(\phi^*_\alpha, \frac{\partial}{\partial\phi^*_\alpha}\right)\psi(\phi^*) = E\psi(\phi^*) \quad . \tag{1.134a}$$

For a standard Hamiltonian with one- and two-body operators, it reads:

$$\left(\sum_{\alpha,\beta} T_{\alpha\beta}\phi^*_\alpha\frac{\partial}{\partial\phi^*_\beta} + \frac{1}{2}\sum_{\alpha\beta\gamma\delta}(\alpha\beta|v|\gamma\delta)\phi^*_\alpha\phi^*_\beta\frac{\partial}{\partial\phi^*_\delta}\frac{\partial}{\partial\phi^*_\gamma}\right)\psi(\phi^*) = E\psi(\phi^*) \quad . \tag{1.134b}$$

In the space of holomorphic functions $\psi(\phi^*)$, the unit operator is obtained by using (1.123):

$$\langle\phi|\psi\rangle = \int\prod_\alpha \frac{d\phi'^*_\alpha d\phi'_\alpha}{2\pi i}\cdot e^{-\sum_\alpha \phi'^*_\alpha\phi'_\alpha}\langle\phi|\phi'\rangle\langle\phi'|\psi\rangle \tag{1.135}$$

which implies:

$$\psi(\phi^*) = \int\prod_\alpha \frac{d\phi'^*_\alpha d\phi'_\alpha}{2\pi i} e^{-\sum_\alpha(\phi'^*_\alpha - \phi^*_\alpha)\phi'_\alpha}\cdot\psi(\phi'^*) \quad . \tag{1.136}$$

Note that this is just a general form in the complex plane for the familiar representation of a δ-function $\delta(x-x') = \int\frac{dy}{2\pi}e^{iy(x-x')}$.

Another useful property of coherent states is the simple form of matrix elements of normal-ordered operators between coherent states. If we denote by $A(a^\dagger_\alpha, a_\alpha)$ an operator in normal form, the action of the a_α to the right and $a^\dagger_\alpha$ to the left on coherent states immediately yields

$$\langle\phi|A(a^\dagger_\alpha, a_\alpha)|\phi'\rangle = A(\phi^*_\alpha, \phi'_\alpha)\, e^{\sum_\alpha \phi^*_\alpha\phi'_\alpha} \tag{1.137}$$

where $A(\psi^*_\alpha, \psi'_\alpha)$ is the normal form of the operator where the creation operators $a^\dagger_\alpha$ have been replaced by ψ^*_α and the annihilation operators a_α have been replaced by ψ'_α. For example, a two-body potential is written:

$$\begin{aligned}\langle\phi|V|\phi'\rangle &= \frac{1}{2}\sum_{\lambda\mu\nu\rho}(\lambda\mu|v|\nu\rho)\langle\phi|a^\dagger_\lambda a^\dagger_\mu a_\rho a_\nu|\phi'\rangle \\ &= \frac{1}{2}\sum_{\lambda\mu\nu\rho}(\lambda\mu|v|\nu\rho)\phi^*_\lambda\phi^*_\mu\phi'_\rho\phi'_\nu\, e^{\sum_\alpha \phi^*_\alpha\phi'_\alpha} \quad .\end{aligned} \tag{1.138}$$

From their definition, it is clear that coherent states do not have a fixed number of particles. Rather, the occupation number n_α for each state α is Poisson distributed with mean value $|\phi_\alpha|^2$

$$|\langle n_{\alpha_1} n_{\alpha_2}\ldots|\phi\rangle|^2 = \prod_\alpha\frac{|\phi_\alpha|^{2n_\alpha}}{n_\alpha!} \quad . \tag{1.139}$$

Thus the distribution of particle numbers has the average value

$$\bar{N} = \frac{\langle\phi|\hat{N}|\phi\rangle}{\langle\phi|\phi\rangle} = \frac{\sum_\alpha\langle\phi|a^\dagger_\alpha a_\alpha|\phi\rangle}{\langle\phi|\phi\rangle} = \sum_\alpha \phi^*_\alpha\phi_\alpha \tag{1.140a}$$

and variance

$$\sigma^2 = \frac{\langle\phi|\hat{N}^2|\phi\rangle}{\langle\phi|\phi\rangle} - \bar{N}^2 = \sum_\alpha \phi^*_\alpha\phi_\alpha = \bar{N} \quad . \tag{1.140b}$$

In the thermodynamic limit, where $\bar{N}\to+\infty$, the relative width $\frac{\sigma}{\bar{N}} = \frac{1}{\sqrt{\bar{N}}}$ goes to zero, and the coherent states become sharply peaked around $\bar{N}$, reflecting the fact that the product of Poisson distributions approaches a normal distribution.

GRASSMANN ALGEBRA

In order to construct coherent states for fermions which are eigenstates of annihilation operators, we have seen in Eq. (1.113) that it will be necessary to use anticommuting numbers. Algebras of anticommuting numbers are called Grassmann algebras, and in this section we briefly summarize their essential properties. For our present purposes, it is sufficient to view Grassmann algebra and the definitions of integration and differentiation as a clever mathematical construct which takes care of all the minus signs associated with antisymmetry without attempting to attach any physical significance to it. A complete treatment of Grassmann algebra is given in the treatise by Berezin (1965).

A Grassmann algebra is defined by a set of generators, which we denote by $\{\xi_\alpha\}, \alpha = 1,\ldots,n$. These generators anticommute:

$$\xi_\alpha\xi_\beta + \xi_\beta\xi_\alpha = 0 \tag{1.141a}$$

so that, in particular:

$$\xi^2_\alpha = 0 \quad . \tag{1.141b}$$

The basis of the Grassmann algebra is made of all distinct products of the generators. Thus, a number in the Grassmann algebra is a linear combination with complex coefficients of the numbers $\{1, \xi_{\alpha_1}, \xi_{\alpha_1}\xi_{\alpha_2}, \ldots \xi_{\alpha_1}\xi_{\alpha_2}\ldots\xi_{\alpha_n}\}$ where by convention the indices α_i are ordered $\alpha_1 < \alpha_2 < \ldots \alpha_n$. The dimension of a Grassmann algebra with n generators is 2^n since distinct basis elements are produced by the two possibilities of including a generator 0 or 1 times for each of n generators. Hence, a matrix representation of Grassmann numbers requires matrices of dimension at least $2^n \times 2^n$.

In an algebra with an even number $n = 2p$ of generators, one can define a conjugation operation (called involution in some texts) in the following way. We select a set of p generators ξ_α, and to each generator ξ_α, we associate a generator which we denote ξ^*_α.

The following properties define conjugation in a Grassmann algebra:

$$(\xi_\alpha)^* = \xi^*_\alpha \tag{1.142a}$$
$$(\xi^*_\alpha)^* = \xi_\alpha \quad . \tag{1.142b}$$

If λ is a complex number,

$$(\lambda\xi_\alpha)^* = \lambda^*\xi^*_\alpha \tag{1.143}$$

and for any product of generators:

$$(\xi_{\alpha_1}\ldots\xi_{\alpha_n})^* = \xi^*_{\alpha_n}\xi^*_{\alpha_{n-1}}\ldots\xi^*_{\alpha_1} \quad . \tag{1.144}$$

To simplify notation, we now consider a Grassmann algebra with two generators. We can denote the generators by ξ and ξ^*, and the algebra is generated by the four numbers $\{1, \xi, \xi^*, \xi^*\xi\}$.

Because of property (1.141b), any analytic function f defined on this algebra is a linear function:

$$f(\xi) = f_0 + f_1\xi \tag{1.145}$$

and this is the form we will obtain for the coherent state representation of a wave function. Similarly, the coherent state representation of an operator in the Grassmann algebra will be a function of ξ^* and ξ and must have the form

$$A(\xi^*, \xi) = a_0 + a_1\xi + \overline{a}_1\xi^* + a_{12}\xi^*\xi \quad . \tag{1.146}$$

As for ordinary complex functions, a derivative can be defined for Grassmann variable functions. It is defined to be identical to the complex derivative, except that in order for the derivative operator $\frac{\partial}{\partial\xi}$ to act on ξ, the variable ξ has to be anticommuted through until it is adjacent to $\frac{\partial}{\partial\xi}$. For instance:

$$\begin{aligned}\frac{\partial}{\partial\xi}(\xi^*\xi) &= \frac{\partial}{\partial\xi}(-\xi\xi^*) \\ &= -\xi^* \quad .\end{aligned} \tag{1.147}$$

With these definitions:

$$\frac{\partial}{\partial\xi}A(\xi^*, \xi) = a_1 - a_{12}\xi^* \tag{1.148a}$$
$$\frac{\partial}{\partial\xi^*}A(\xi^*, \xi) = \overline{a}_1 + a_{12}\xi . \tag{1.148b}$$
$$\begin{aligned}\frac{\partial}{\partial\xi^*}\frac{\partial}{\partial\xi}A(\xi^*, \xi) &= -a_{12} \\ &= -\frac{\partial}{\partial\xi}\frac{\partial}{\partial\xi^*}A(\xi^*\xi) \quad .\end{aligned} \tag{1.148c}$$

Note from Eq. (1.148c) that the operators $\frac{\partial}{\partial\xi^*}$ and $\frac{\partial}{\partial\xi}$ anticommute.

In defining a definite integral, there is no analog of the familiar sum motivating the Riemann integral for ordinary variables. Hence, we define integration over Grassmann variables as a linear mapping which has the fundamental property of ordinary integrals over functions vanishing at infinity that the integral of an exact differential form is zero. This requirement implies that the integral of 1 is zero, since 1 is the derivative of ξ. The only non-vanishing integral is that of ξ, since ξ is not a derivative. Hence, the definite integral is defined as follows:

$$\begin{aligned}\int d\xi\, 1 &= 0 \\ \int d\xi\, \xi &= 1\end{aligned} \tag{1.149}$$

and as in the case of a derivative, in order to apply (1.149b), one must first anticommute the variable ξ as required to bring it next to $d\xi$. A simple mnemonic for this definition is the fact that Grassmann integration is identical to Grassmann differentiation. Since half of the generators ξ^*_α have been defined arbitrarily to be conjugate variables but are otherwise equivalent to the generators ξ_α, it is natural to define integration for conjugate variables in the same way:

$$\begin{aligned}\int d\xi^*\, 1 &= 0 \\ \int d\xi^*\, \xi^* &= 1 \quad .\end{aligned} \tag{1.150}$$

Note, however, that in contrast to a Riemann integral in which dx is an infinitesimal real variable, $d\xi^*$ is not a Grassmann number and it makes no sense to apply Eq. (1.144) to the quantity $(\int d\xi\xi^*)$ to try to relate Eq. (1.149) to Eq. (1.150).

The following examples illustrate the application of these integration rules. Using (1.145), we obtain:

$$\int d\xi\, f(\xi) = f_1 \tag{1.151}$$

and using (1.146) we get:

$$\int d\xi A(\xi^*, \xi) = \int d\xi(a_0 + a_1\xi + \overline{a}_1\xi^* + a_{12}\xi^*\xi)$$

$$= a_1 - a_{12}\xi^* \tag{1.152a}$$

$$\int d\xi^* A(\xi^*, \xi) = \bar{a}_1 + a_{12}\xi \tag{1.152b}$$

and

$$\begin{aligned} \int d\xi^* d\xi A(\xi^* \xi) &= -a_{12} \\ &= -\int d\xi d\xi^* A(\xi^* \xi) \quad . \end{aligned} \tag{1.152c}$$

The motivation for this definition of integration is that with these conventions, many results look similar to those of complex integration. For instance, consider the definition of a Grassmann δ-function by:

$$\begin{aligned} \delta(\xi, \xi') &\equiv \int d\eta e^{-\eta(\xi - \xi')} \\ &= \int d\eta (1 - \eta(\xi - \xi')) \\ &= -(\xi - \xi') \quad . \end{aligned} \tag{1.153}$$

To verify that this definition has the desired behavior, we use Eq. (1.145) to obtain

$$\begin{aligned} \int d\xi' \delta(\xi, \xi') f(\xi') &= -\int d\xi' (\xi - \xi')(f_0 + f_1 \xi') \\ &= f_0 + f_1 \xi \\ &= f(\xi) \end{aligned} \tag{1.154}$$

for any function $f(\xi)$.

Motivated by Eq. (1.135) for Boson coherent states, it will be useful to define a scalar product of Grassmann functions by:

$$\langle f | g \rangle = \int d\xi^* d\xi e^{-\xi^* \xi} f^*(\xi) g(\xi^*) \tag{1.155}$$

where $f(\xi)$ is defined by (1.145) and

$$g(\xi) = g_0 + g_1 \xi \quad . \tag{1.156}$$

With definition (1.155), we see that:

$$\begin{aligned} \langle f | g \rangle &= \int d\xi^* d\xi (1 - \xi^* \xi)(f_0^* + f_1^* \xi)(g_0 + g_1 \xi^*) \\ &= -\int d\xi^* d\xi \, \xi^* \xi f_0^* g_0 + \int d\xi^* d\xi \, \xi \xi^* f_1^* g_1 \\ &= f_0^* g_0 + f_1^* g_1 \end{aligned} \tag{1.157}$$

and it can be shown that Grassmann functions have the structure of a Hilbert space. The results we have presented for the case of two generators ξ and ξ^* generalize straightforwardly to 2p generators $\xi_1 \ldots \xi_p, \xi_1^* \ldots \xi_p^*$ as shown in Problem 1.3.

FERMION COHERENT STATES

If we try to construct coherent states for Fermions, we immediately encounter the difficulty that if we expand them according to (1.114), the coefficients must be Grassmann numbers. Therefore, in order to construct coherent states, we must enlarge the Fermion Fock space.

We first define a Grassmann algebra $\mathcal{G}$ by associating a generator ξ_α with each annihilation operator a_α, and a generator ξ_α^* with each creation operator $a_\alpha^\dagger$. We then construct the generalized Fock space as the set of linear combinations of states of the Fock space $\mathcal{F}$ with coefficients in the Grassmann algebra $\mathcal{G}$. Any vector $|\psi\rangle$ in the generalized Fock space can be expanded as:

$$|\psi\rangle = \sum_\alpha \chi_\alpha |\phi_\alpha\rangle \tag{1.158}$$

where the χ_α are Grassmann numbers and the $|\phi_\alpha\rangle$ vectors of the Fock space.

In order to treat expressions containing combinations of Grassmann variables and creation and annihilation operators, it is necessary to augment the definition of the Grassmann variables to specify the commutation relations between ξ's and a's and the adjoints of mixed expressions. To obtain results analogous to those obtained previously for Bosons, it is natural and convenient to require that

$$[\tilde{\xi}, \tilde{a}]_+ = 0 \tag{1.159a}$$

and

$$(\tilde{\xi}\tilde{a})^\dagger = \tilde{a}^\dagger \tilde{\xi}^* \tag{1.159b}$$

where $\tilde{\xi}$ denotes any Grassmann variable in $\{\xi_\alpha, \xi_\alpha^*\}$ and $\tilde{a}$ is any operator in $\{a_\alpha^\dagger, a_\alpha\}$.

We now define a Fermion coherent state $|\xi\rangle$ analogous to Boson coherent states by

$$\begin{aligned} |\xi\rangle &= e^{-\sum_\alpha \xi_\alpha a_\alpha^\dagger} |0\rangle \\ &= \prod_\alpha (1 - \xi_\alpha a_\alpha^\dagger) |0\rangle \quad . \end{aligned} \tag{1.160}$$

Note that the combination $\xi_\alpha a_\alpha^\dagger$ commutes with $\xi_\beta a_\beta^\dagger$ so that the second line in Eq. (1.159) reproduces each non-vanishing term of the expansion of the exponential in the first line. Although the coherent state belongs to the generalized Fock space and not to $\mathcal{F}$, as we shall see, the crucial point is that any physical Fermion state of $\mathcal{F}$ can be expanded in terms of these coherent states.

We now verify that coherent states as defined in (1.159) are eigenstates of the annihilation operators. For a single state α, the anticommutation relations of $a_\alpha, a_\alpha^\dagger$, and ξ_α yield the relation.

$$\begin{aligned} a_\alpha (1 - \xi_\alpha a_\alpha^\dagger)|0\rangle &= +\xi_\alpha |0\rangle \\ &= \xi_\alpha (1 - \xi_\alpha a_\alpha^\dagger)|0\rangle \quad . \end{aligned} \tag{1.161}$$

Using Eqs. (1.154) and (1.160) and the fact that a_α and ξ_α both commute with the combination $\xi_\beta a^\dagger_\beta$ for $\beta \neq \alpha$, we obtain the desired eigenvalue conditions.

$$\begin{aligned}
a_\alpha|\xi\rangle &= a_\alpha \prod_\beta (1-\xi_\beta a^\dagger_\beta)|0\rangle \\
&= \prod_{\beta\neq\alpha}(1-\xi_\beta a^\dagger_\beta) a_\alpha (1-\xi_\alpha a^\dagger_\alpha)|0\rangle \\
&= \prod_{\beta\neq\alpha}(1-\xi_\beta a^\dagger_\beta) \xi_\alpha (1-\xi_\alpha a^\dagger_\alpha)|0\rangle \\
&= \xi_\alpha \prod_\beta (1-\xi_\beta a^\dagger_\beta)|0\rangle \\
&= \xi_\alpha|\xi\rangle \ .
\end{aligned} \tag{1.162}$$

Similarly, the adjoint of the coherent state is

$$\langle\xi| = \langle 0|e^{-\sum_\alpha a_\alpha \xi^*_\alpha} = \langle 0|e^{\sum \xi^*_\alpha a_\alpha} \tag{1.163a}$$

and is a left-eigenfunction of $a^\dagger_\alpha$

$$\langle\xi|a^\dagger_\alpha = \langle\xi|\xi^*_\alpha \ . \tag{1.163b}$$

The action of $a^\dagger_\alpha$ on a coherent state is analogous to the Boson result, Eq.(1.120), and differs only in sign:

$$\begin{aligned}
a^\dagger_\alpha|\xi\rangle &= a^\dagger_\alpha(1-\xi_\alpha a^\dagger_\alpha)\prod_{\beta\neq\alpha}(1-\xi_\beta a^\dagger_\beta)|0\rangle \\
&= a^\dagger_\alpha \prod_{\beta\neq\alpha}(1-\xi_\beta a^\dagger_\beta)|0\rangle \\
&= -\frac{\partial}{\partial\xi_\alpha}(1-\xi_\alpha a^\dagger_\alpha)\prod_{\beta\neq\alpha}(1-\xi_\beta a^\dagger_\beta)|0\rangle \\
&= -\frac{\partial}{\partial\xi_\alpha}|\xi\rangle
\end{aligned} \tag{1.164a}$$

and similarly one may verify

$$\langle\xi|a_\alpha = +\frac{\partial}{\partial\xi^*}\langle\xi| \ . \tag{1.164b}$$

The overlap of two coherent states is easily calculated:

$$\begin{aligned}
\langle\xi|\xi'\rangle &= \langle 0|\prod_\alpha(1+\xi^*_\alpha a_\alpha)(1-\xi_\alpha a^\dagger_\alpha)|0\rangle \\
&= \prod_\alpha(1+\xi^*_\alpha\xi_\alpha) \\
&= e^{\sum_\alpha \xi^*_\alpha\xi_\alpha} \ .
\end{aligned} \tag{1.165}$$

As in the Boson case, the closure relation may be written

$$\int \prod_\alpha d\xi^*_\alpha d\xi_\alpha e^{-\sum_\alpha \xi^*_\alpha\xi_\alpha}|\xi\rangle\langle\xi| = 1 \tag{1.166}$$

where the 1 denotes the unit operator in the physical Fermion Fock space $\mathcal{F}$. This closure relation may be proved using Schur's lemma as we proved Eq. (1.123) once integration by parts has been derived for Grassmann variables (See Problem 1.4). Here, we present an alternative proof. We define the operator A to be the left hand side of Eq. (1.166).

$$A = \int \prod_\alpha d\xi^*_\alpha d\xi_\alpha e^{-\sum_\alpha \xi^*_\alpha\xi_\alpha}|\xi\rangle\langle\xi| \ . \tag{1.167}$$

To prove (1.167), it is sufficient to prove that for any vectors of the basis of the Fock space:

$$\langle\alpha_1\ldots\alpha_n|A|\beta_1\ldots\beta_m\rangle = \langle\alpha_1\ldots\alpha_n|\beta_1\ldots\beta_m\rangle \ . \tag{1.168}$$

Using the eigenvalue property of the coherent states (1.161) we obtain

$$\begin{aligned}
\langle\alpha_1\ldots\alpha_n|\xi\rangle &= \langle 0|a_{\alpha_n}\ldots a_{\alpha_1}|\xi\rangle \\
&= \xi_{\alpha_n}\ldots\xi_{\alpha_1}
\end{aligned} \tag{1.169}$$

and the analogous adjoint equations. Thus

$$\begin{aligned}
\langle\alpha_1\ldots\alpha_n|A|\beta_1\ldots\beta_n\rangle &= \int\prod_\alpha d\xi^*_\alpha d\xi_\alpha e^{-\sum_\alpha \xi^*_\alpha\xi_\alpha}\langle\alpha_1\ldots\alpha_n|\xi\rangle\langle\xi|\beta_1\ldots\beta_n\rangle \\
&= \int\prod_\alpha d\xi^*_\alpha d\xi_\alpha \prod_\alpha(1-\xi^*_\alpha\xi_\alpha)\xi_{\alpha_n}\ldots\xi_{\alpha_1}\xi^*_{\beta_1}\ldots\xi^*_{\beta_n} \ .
\end{aligned} \tag{1.170}$$

Now consider the integrals which may arise in Eq. (1.170) for a particular state γ:

$$\int d\xi^*_\gamma d\xi_\gamma(1-\xi^*_\gamma\xi_\gamma)\begin{Bmatrix}\xi_\gamma\xi^*_\gamma\\ \xi^*_\gamma \\ \xi_\gamma \\ 1\end{Bmatrix} = \begin{Bmatrix}1\\0\\0\\1\end{Bmatrix} \ . \tag{1.171}$$

Thus, the integral in Eq.(1.170) is non-vanishing only if each state γ is either occupied in both $\langle\alpha_1\ldots\alpha_n|$ and $|\beta_1\ldots\beta_n\rangle$ or unoccupied in both states, which requires that m=n and $\{\alpha_1\ldots\alpha_n\}$ is some permutation P of $\{\beta_1\ldots\beta_n\}$. In this case, the integral is easily evaluated by writing $\xi_{\alpha_n}\ldots\xi_{\alpha_1}\xi^*_{\beta_1}\ldots\xi^*_{\beta_n} = (-1)^P\xi_{\alpha_n}\ldots\xi_{\alpha_1}\xi^*_{\alpha_1}\ldots\xi^*_{\alpha_n}$ and noting that an even number of anticommutations is required to bring the integral over each state into the form of Eq. (1.171), so that the value is just $(-1)^P$. The left hand side of Eq. (1.168) thus yields the result previously derived in Eq. (1.50) for the right hand side, so the equality is established for any vectors in the Fock space.

As in the case of Bosons, this completeness relation provides a useful expression for the trace of an operator. Because the matrix elements $\langle\psi_i|\xi\rangle$ and $\langle\xi|\psi_i\rangle$ between states $|\psi_i\rangle$ in the Fock space and coherent states contain Grassmann numbers, it follows from the anticommutation relations that

$$\langle\psi_i|\xi\rangle\langle\xi|\psi_j\rangle = \langle-\xi|\psi_j\rangle\langle\psi_i|\xi\rangle \quad . \tag{1.172}$$

Hence, if we define a complete set of states $\{|n\rangle\}$ in the Fock space, the trace of an operator A may be written

$$\begin{aligned}\mathrm{Tr}\, A &= \sum_n \langle n|A|n\rangle \\ &= \int \prod_\alpha d\xi^*_\alpha d\xi_\alpha e^{-\sum_\alpha \xi^*_\alpha\xi_\alpha} \sum_n \langle n|\xi\rangle\langle\xi|A|n\rangle \\ &= \int \prod_\alpha d\xi^*_\alpha d\xi_\alpha e^{-\sum_\alpha \xi^*_\alpha\xi_\alpha} \langle-\xi|A\sum_n |n\rangle\langle n||\xi\rangle \\ &= \int \prod_\alpha d\xi^*_\alpha d\xi_\alpha e^{-\sum_\alpha \xi^*_\alpha\xi_\alpha} \langle-\xi|A|\xi\rangle \quad .\end{aligned} \tag{1.173}$$

The overcompleteness of the Fermion coherent states allows us to define a Grassmann coherent state representation analogous to the coherent state representation for Bosons in Eq. (1.129a)

$$|\psi\rangle = \int \prod_\alpha d\xi^*_\alpha d\xi_\alpha e^{-\sum_\alpha \xi^*_\alpha\xi_\alpha} \psi(\xi^*)|\xi\rangle \tag{1.174a}$$

where

$$\langle\xi|\psi\rangle = \psi(\xi^*) \quad . \tag{1.174b}$$

Within this representation, it follows from Eqs. (1.161) and (1.164) that the creation and annihilation operators satisfy:

$$\langle\xi|a_\alpha|\psi\rangle = \frac{\partial}{\partial\xi^*_\alpha}\psi(\xi^*) \tag{1.175a}$$

$$\langle\xi|a^\dagger_\alpha|\psi\rangle = \xi^*_\alpha\psi(\xi^*) \quad . \tag{1.175b}$$

Thus, as in the Boson case, the operators a_α and $a^\dagger_\alpha$ are represented by the operators $\frac{\partial}{\partial\xi^*_\alpha}$ and ξ^*_α respectively. The anticommutation relation (1.83) is represented by

$$\left[\frac{\partial}{\partial\xi^*_\alpha}, \xi^*_\beta\right]_+ = \delta_{\alpha\beta} \quad . \tag{1.176}$$

As in the Boson case, the matrix element of a normal-ordered operator $A(a^\dagger_\alpha a_\alpha)$ between two coherent states is very simple:

$$\langle\xi|A(a^\dagger_\alpha, a_\alpha)|\xi'\rangle = e^{\sum_\alpha \xi^*_\alpha\xi'_\alpha} A(\xi^*_\alpha, \xi'_\alpha) \quad . \tag{1.177}$$

However, in contrast to the Boson case, the expectation value of the number operator is not a real number:

$$\frac{\langle\xi|N|\xi\rangle}{\langle\xi|\xi\rangle} = \sum_\alpha \xi^*_\alpha\xi_\alpha \tag{1.178}$$

and it is meaningless to speak about the average number of particles in a Fermion coherent state.

Finally, we conclude this section by contrasting the physical significance of Boson and Fermion coherent states. Boson coherent states are the physical states which emerge naturally when taking the classical limit of quantum mechanics or of a quantum field theory (See Problem 1.5). In the classical limit, when the field operators are assumed to commute, the definition of a classical field $\phi(x)$ at each point of space is identical to saying that the system is in the coherent state $|\phi\rangle = e^{\int d^3x\phi(x)\psi^\dagger(x)}|0\rangle$. For example, a classical electromagnetic field can be viewed as a coherent state of photons.

In contrast, Fermion coherent states are not contained in the Fermion Fock space, they are not physically observable, and there are no classical fields of Fermions. Nevertheless, Fermion coherent states are very useful in formally unifying many-Fermion and many-Boson problems, and we shall use this property extensively in the following chapters.

In the subsequent development, this physical difference will give rise to some significant differences in the treatment of Bosons and Fermions. For example, application of the stationary-phase approximation to an expression formulated in terms of Boson coherent states yields a useful expansion around a physical classical field configuration. For Fermions, no corresponding physical solution exists, and the Fermion degrees of freedom will have to be integrated out explicitly.

GAUSSIAN INTEGRALS

In the ensuing formal development, we will frequently evaluate matrix elements of the evolution operator in coherent states, leading to integrals of exponential functions which are polynomials in complex variables or Grassmann variables. In the case of quadratic forms, these are straightforward generalizations of the familiar Gaussian integral, and we present several useful integrals in this section for future reference. For brevity, we derive the identities for the special case of symmetric and Hermitian matrices, and refer the reader to the standard references for the general case.

We begin by proving the following identity for multi-dimensional integrals over real variables:

$$\int \frac{dx_1\ldots dx_n}{(2\pi)^{\frac{n}{2}}} e^{-\frac{1}{2}x_iA_{ij}x_j+x_iJ_i} = [\det A]^{-\frac{1}{2}} e^{\frac{1}{2}J_iA^{-1}_{ij}J_j} \tag{1.179}$$

where A is a real symmetric positive definite matrix and summation over repeated Latin indices is understood throughout this section. This identity is established straightforwardly by changing variables to reduce it to diagonal form and using the familiar Gaussian integral

$$\int_{-\infty}^{\infty} dx e^{-ax^2} = \sqrt{\frac{\pi}{a}} \quad . \tag{1.180}$$

Performing the transformations $y_i = x_i - A_{ij}^{-1}J_j$ and $z_k = O_{ki}^{-1}y_i$, where O is the orthogonal transformation which diagonalizes A, we obtain

$$
\begin{aligned}
\int dx_1 \ldots dx_n e^{-\frac{1}{2}x_i A_{ik} x_k + J_k x_k - \frac{1}{2}J_i A_{ik}^{-1} J_k}
&= \int dy_1 \ldots dy_n e^{-\frac{1}{2}y_i A_{ik} y_k} \\
&= \int dz_1 \ldots dz_n e^{-\sum_m \frac{1}{2}a_m z_m^2} \\
&= \prod_{m=1}^{n} \sqrt{\frac{2\pi}{a_m}} \\
&= \frac{(2\pi)^{\frac{n}{2}}}{[\det A]^{\frac{1}{2}}}
\end{aligned}
\tag{1.181}
$$

which proves Eq. (1.179). Note that the positivity of A is essential for convergence of the Gaussian integral.

A similar identity for integrals over pairs of conjugate complex variables is

$$
\int \prod_{i=1}^{n} \frac{dx_i^* dx_i}{2\pi i} e^{-x_i^* H_{ij} x_j + J_i^* x_i + J_i x_i^*} = [\det H]^{-1} e^{J_i^* H_{ij}^{-1} J_j}
\tag{1.182}
$$

which is valid for any matrix H with a positive Hermitian part. For the special case of a Hermitian matrix, it is proved in the same way as Eq. (1.179) by defining the transformation $y_i = x_i - H_{ij}^{-1}J_j$ and its complex conjugate, transforming H to diagonal form and evaluating the diagonal integral

$$
\begin{aligned}
\int \frac{dz^* dz}{2\pi i} e^{-z^* a z} &= \int \frac{du dv}{\pi} e^{-a(u^2+v^2)} \\
&= \frac{1}{a} \ .
\end{aligned}
\tag{1.183}
$$

Finally, we wish to establish the analogous identity for Grassmann variables

$$
\int \prod_{i=1}^{n} d\eta_i^* d\eta_i e^{-\eta_i^* H_{ij}\eta_j + \zeta_i^*\eta_i + \zeta_i \eta_i^*} = [\det H] e^{\zeta_i^* H_{ij}^{-1} \zeta_j}
\tag{1.184}
$$

where, for simplicity, H is again assumed to be Hermitian but not necessarily positive definite and $\{\eta_i, \eta_i^*, \zeta_i, \zeta_i^*\}$ are Grassmann variables. Note that Eq. (1.184) differs from Eq. (1.182) by the appearance of the determinant in the numerator instead of the denominator. To prove this identity, we need to derive two additional results for Grassmann variables: the transformation law for an integral under a change of variables and the formula for a Gaussian integral.

A Gaussian integral involving a single pair of conjugate Grassmann variables is easily evaluated as follows.

$$
\int d\xi^* d\xi e^{-\xi^* a \xi} = \int d\xi^* d\xi (1 - \xi^* a \xi) = a \ .
\tag{1.185}
$$

Note that for a single variable, the Grassmann Gaussian integral yields a in contrast to $\frac{1}{a}$ in Eq. (1.183) for the ordinary Gaussian integral. Hence, if we can bring the multivariable Grassmann integral (1.184) into diagonal form, we expect to obtain the product of eigenvalues, and thus the determinant of H, in the numerator instead of in the denominator as for complex variables.

In order to transform Eq. (1.184) into diagonal form, we need to derive the law for linear transformations of Grassmann variables

$$
\begin{aligned}
\int d\zeta_1^* d\zeta_1 \ldots d\zeta_n^* d\zeta_n P(\zeta^*, \zeta) = \left| \frac{\partial(\eta^*, \eta)}{\partial(\zeta^*, \zeta)} \right| \int d\eta_1^* d\eta_1 \ldots d\eta_n^* d\eta_n \\
\times P(\zeta^*(\eta^*, \eta), \zeta(\eta^*, \eta))
\end{aligned}
\tag{1.186}
$$

which differs from the transformation law for complex variables by the appearance of the inverse of the Jacobian instead of the Jacobian. The derivation is facilitated by relabelling the variables as follows:

$$
\begin{aligned}
(\zeta_1^* \zeta_2^* \ldots \zeta_n^* \zeta_n \zeta_{n-1} \ldots \zeta_1) &\equiv (\tilde{\zeta}_1 \tilde{\zeta}_2 \ldots \tilde{\zeta}_{2n}) \\
(\eta_1^* \eta_2^* \ldots \eta_n^* \eta_n \eta_{n-1} \ldots \eta_1) &\equiv (\tilde{\eta}_1 \tilde{\eta}_2 \ldots \tilde{\eta}_{2n})
\end{aligned}
\tag{1.187}
$$

and writing

$$
\tilde{\zeta}_i = M_{ij} \tilde{\eta}_j \ .
\tag{1.188}
$$

The only non-vanishing contributions to Eq. (1.186) come from the term in the polynomial containing each $\tilde{\zeta}_i$ as a factor, which we write as $p\prod_{i=1}^{2n}\tilde{\zeta}_i$. Thus, we must evaluate J in the equation

$$
\int d\zeta_1^* d\zeta_1 \ldots d\zeta_n^* d\zeta_n p \prod_{i=1}^{2n} \tilde{\zeta}_i = J \int d\eta_1^* d\eta_1 \ldots d\eta_n^* d\eta_n p \prod_{i=1}^{2n} \left(\sum_j M_{ij} \tilde{\eta}_j \right) \ .
\tag{1.189}
$$

The left hand side yields $p(-1)^n$ and the right hand side is evaluated by noting that the only non-vanishing contributions arise from the $(2n)!$ distinct permutations P of the variables $\{\tilde{\eta}\}$ generated by the product. Thus

$$
\begin{aligned}
p(-1)^n &= Jp \int d\eta_1^* d\eta_1 \ldots d\eta_n^* d\eta_n \prod_i \left(\sum_j M_{ij} \tilde{\eta}_j \right) \\
&= Jp \int d\eta_1^* d\eta_1 \ldots d\eta_n^* d\eta_n \sum_P \prod_i M_{iPi} \tilde{\eta}_{Pi} \\
&= Jp \sum_P \prod_i M_{iPi} (-1)^P \int d\eta_1^* d\eta_1 \ldots d\eta_n^* d\eta_n \tilde{\eta}_1 \tilde{\eta}_2 \ldots \tilde{\eta}_{2n} \\
&= Jp \det M (-1)^n
\end{aligned}
$$

so that

$$
J = (\det M)^{-1} = \left| \frac{\partial(\tilde{\eta})}{\partial(\tilde{\xi})} \right| = \left| \frac{\partial(\eta^*, \eta)}{\partial(\zeta^*, \zeta)} \right|
$$

which proves Eq. (1.186) for a general linear transformation.

Finally, Eq. (1.184) may now be proved by defining the transformations $\rho_i = \eta_i - H_{ij}^{-1}\varsigma_j, \rho_i^* = \eta_i^* - H_{ij}^{-1*}\varsigma_j^*$, diagonalizing H with a unitary transformation U, defining $\xi_i = U_{ij}^{-1}\rho_j$ and $\xi_i^* = U_{ij}^{-1*}\rho_j^*$, noting that all the Jacobians are unity, and using the Gaussian integral Eq. (1.185). Thus

$$\begin{aligned}\int \prod_{i=1}^{n} d\eta_i^* d\eta_i e^{-\eta_i^* H_{ij}\eta_j + \varsigma_i^*\eta_i + \varsigma_i\eta_i^* - \varsigma_i^* H_{ij}^{-1}\varsigma_j} &= \int \prod_{i=1}^{n} d\rho_i^* d\rho_i e^{-\rho_i^* H_{ij}\rho_j} \\ &= \int \prod_{i=1}^{n} d\xi_i^* d\xi_i e^{-\sum_i h_i \xi_i^* \xi_i} \\ &= \prod_{m=1}^{n} h_m = \det H \qquad (1.190)\end{aligned}$$

which proves Eq. (1.184). Note, as in the previous case, that the derivation may be generalized to a non-Hermitian matrix H.

It may appear curious that the Gaussian integral for Grassmann variables requires no restrictions on the matrix H whereas for ordinary variables H must be positive definite for the integral to converge. Formally, the distinction arises because the expansion of the exponential $e^{a\eta^*\cdot\eta}$ terminates at first order, yielding a finite integral irrespective of the sign of a. This formal property, however, reflects a fundamental difference between Fermions and Bosons, and it is the Pauli principle restriction that occupation numbers be either 0 or 1 which guarantees finite results for Fermions irrespective of the eigenvalues of H.

This point is illustrated by the simple case of the partition function of non-interacting particles, which will be shown in the next chapter to yield Gaussian integrals of the form of Eq. (1.184). For non-interacting particles in the Grand Canonical ensemble, the partition function may be written

$$Z = \prod_{\alpha}\sum_{n_\alpha} e^{-\beta(\epsilon_\alpha - \mu)n_\alpha}$$

where n_α denotes the occupation number of the state α. For Fermions, the series for each α terminates after two terms

$$Z_F = \prod_{\alpha}\left(1 + e^{-\beta(\epsilon_\alpha - \mu)}\right)$$

so no restriction is imposed on$(\epsilon_\alpha - \mu)$. However, for Bosons,

$$Z_B = \prod_{\alpha}\left(1 + \sum_{n=1}^{\infty}\left(e^{-\beta(\epsilon_\alpha - \mu)}\right)^n\right)$$

so that the partition function is finite only if $e^{-\beta(\epsilon_\alpha-\mu)} < 1$ for all α which requires that $(\epsilon_\alpha - \mu)$ be positive for all α. Thus the operator $(H - \mu N)$ must be positive definite for Bosons, but no such requirement arises for Fermions.

The salient results from this introductory chapter are collected for subsequent reference in Table 1.1.

$$\begin{aligned}
[a_\alpha, a_\beta^\dagger]_{-\varsigma} &= \delta_{\alpha\beta} \\
|\xi\rangle &= e^{\varsigma\sum_\alpha \xi_\alpha a_\alpha^\dagger}|0\rangle \\
a_\alpha|\xi\rangle &= \xi_\alpha|\xi\rangle \\
\langle\xi|a_\alpha^\dagger &= \langle\xi|\xi^* \\
a_\alpha^\dagger|\xi\rangle &= \varsigma\frac{\partial}{\partial\xi_\alpha}|\xi\rangle \\
\langle\xi|a_\alpha &= \frac{\partial}{\partial\xi_\alpha^*}\langle\xi| \\
\langle\xi|A(a_\alpha^\dagger, a_\alpha)|\xi'\rangle &= e^{\sum_\alpha \xi_\alpha^*\xi'_\alpha}A(\xi_\alpha^*, \xi_\alpha) \\
1 &= \int d\mu(\xi)e^{-\sum_\alpha \xi_\alpha^*\xi_\alpha}|\xi\rangle\langle\xi| \\
trA &= \int d\mu(\xi)e^{-\sum_\alpha \xi_\alpha^*\xi_\alpha}\langle\varsigma\xi|A|\xi\rangle \\
|\psi\rangle &= \int d\mu(\xi)e^{-\sum_\alpha \xi_\alpha^*\xi_\alpha}\psi(\xi_\alpha^*)|\xi\rangle \\
\psi(\xi^*) &= \langle\xi|\psi\rangle \\
\langle\xi|a_\alpha^\dagger|\psi\rangle &= \xi_\alpha^*\psi(\xi^*) \\
\langle\xi|a_\alpha|\psi\rangle &= \frac{\partial}{\partial\xi^*}\psi(\xi^*) \\
[\det H]^{-\varsigma}e^{\sum_{\alpha\beta}\eta_\alpha^* H_{\alpha\beta}^{-1}\eta_\beta} &= \int d\mu(\xi)e^{-\sum_{\alpha\beta}\xi_\alpha^* H_{\alpha\beta}\xi_\beta + \sum_\alpha(\eta_\alpha^*\xi_\alpha + \eta_\alpha\xi_\alpha^*)} \\
d\mu(\xi) &= \frac{1}{\mathcal{N}}\prod_\alpha d\xi_\alpha^* d\xi_\alpha \\
\mathcal{N} &= \begin{cases} 2\pi i & \text{Bosons} \\ 1 & \text{Fermions}\end{cases} \\
\varsigma &= \begin{cases} 1 & \text{Bosons} \\ -1 & \text{Fermions}\end{cases}
\end{aligned}$$

Table 1.1 SUMMARY OF PRINCIPAL RESULTS OF CHAPTER 1. Formulas are written in a unified form for Fermions and Bosons using the conventions that for Bosons ξ and η denote complex variables and $\varsigma = +1$ whereas for Fermions ξ and η denote Grassmann variables and $\varsigma = -1$.

PROBLEMS FOR CHAPTER 1

The subject of quantum many-body theory is too vast to treat all the formalism and illustrative examples adequately in the text. Thus the problems are intended to be an integral part of the course, and in contrast to most texts, entirely new and separate topics will often be introduced. Readers are strongly encouraged to read through all the problems and solve those which appear appropriate. As a guide,