* Here we derive the SRG-evolved pair momentum distribution $n_\lambda^{\tau\tau'}(q,Q)$ starting with the following:

$$n^{\tau\tau'}(q,Q) = \frac{1}{2}\sum_{\sigma\sigma'} a^\dagger_{\frac{\vec{Q}}{2}+\vec{q},\sigma\tau}\, a^\dagger_{\frac{\vec{Q}}{2}-\vec{q},\sigma'\tau'}\, a_{\frac{\vec{Q}}{2}-\vec{q},\sigma'\tau'}\, a_{\frac{\vec{Q}}{2}+\vec{q},\sigma\tau} \quad (1)$$

τ isospin projection, σ spin projection
$\vec{Q} = \vec{k}_1 + \vec{k}_2$, $\vec{q} = \frac{1}{2}(\vec{k}_1 - \vec{k}_2)$ (total, relative momentum)

$$U_\lambda = \mathbb{1} + \frac{1}{4}\sum_{\sigma_1\sigma_2\sigma_3\sigma_4}\sum_{\tau_1\tau_2\tau_3\tau_4}\sum_{\vec{k}\vec{k}'\vec{K}'} \langle \vec{k}\,\sigma_1\tau_1\sigma_2\tau_2 | \delta\tilde{U} | \vec{k}'\sigma_3\tau_3\sigma_4\tau_4\rangle \times$$

$$a^\dagger_{\frac{\vec{K}}{2}+\vec{k},\sigma_1\tau_1}\, a^\dagger_{\frac{\vec{K}}{2}-\vec{k},\sigma_2\tau_2}\, a_{\frac{\vec{K}}{2}-\vec{k}',\sigma_4\tau_4}\, a_{\frac{\vec{K}}{2}+\vec{k}',\sigma_3\tau_3} + \text{3-body} + \ldots \quad (2)$$

where $\delta\tilde{U}_{1234} = -\delta\tilde{U}_{1243}$

Simplify the notation in the following:

$\left(\frac{\vec{Q}}{2}+\vec{q}, \sigma, \tau\right) \rightarrow \alpha$, $\left(\frac{\vec{Q}}{2}-\vec{q}, \sigma', \tau'\right) \rightarrow \alpha'$

$\sum_{\sigma\sigma'} \rightsquigarrow \sum_{\alpha,\alpha'}$ (not including momenta or τ,τ')

$\left(\frac{\vec{K}}{2}+\vec{k}, \sigma_1, \tau_1\right) \rightarrow 1, \ldots, \left(\frac{\vec{K}'}{2}-\vec{k}''', \sigma_8, \tau_8\right) \rightarrow 8$

$$U_\lambda n^{\tau\tau'}(\vec{q}, \vec{a}) U_\lambda^\dagger = \left(\mathbb{1} + \frac{1}{4}\sum_{1234} \langle 12|\delta\tilde{U}|34\rangle a_1^\dagger a_2^\dagger a_4 a_3\right) \times$$

$$\left(\frac{1}{2}\sum_{\alpha\alpha'} a_\alpha^\dagger a_{\alpha'}^\dagger a_{\alpha'} a_\alpha\right)\left(\mathbb{1} + \frac{1}{4}\sum_{5678} \langle 56|\delta\tilde{U}|78\rangle a_5^\dagger a_6^\dagger a_8 a_7\right)^\dagger$$

$$= \frac{1}{2}\sum_{\alpha\alpha'} a_\alpha^\dagger a_{\alpha'}^\dagger a_{\alpha'} a_\alpha \qquad \text{1 term}$$

$$\left.\begin{aligned} &+ \frac{1}{8}\sum_{\alpha\alpha'}\sum_{1234} \langle 12|\delta\tilde{U}|34\rangle a_1^\dagger a_2^\dagger a_4 a_3 a_\alpha^\dagger a_{\alpha'}^\dagger a_{\alpha'} a_\alpha \\ &+ \frac{1}{8}\sum_{\alpha\alpha'}\sum_{5678} \langle 78|\delta\tilde{U}^\dagger|56\rangle a_\alpha^\dagger a_{\alpha'}^\dagger a_{\alpha'} a_\alpha a_7^\dagger a_8^\dagger a_6 a_5 \end{aligned}\right\} \begin{matrix}\delta U \\ \text{term}\end{matrix}$$

$$\underbrace{+ \frac{1}{32}\sum_{\alpha\alpha'}\sum_{12345678} \langle 12|\delta\tilde{U}|34\rangle\langle 78|\delta\tilde{U}^\dagger|56\rangle a_1^\dagger a_2^\dagger a_4 a_3 a_\alpha^\dagger a_{\alpha'}^\dagger a_{\alpha'} a_\alpha a_7^\dagger a_8^\dagger a_6 a_5}_{\delta U \delta U^\dagger \text{ term}} \quad (3)$$

- Evaluate contractions with respect to $|0\rangle$ first

1 term: No possible contractions.

δU term: $a_1^\dagger a_2^\dagger a_4 a_3 a_\alpha^\dagger a_{\alpha'}^\dagger a_{\alpha'} a_\alpha$

$$\rightarrow a_1^\dagger a_2^\dagger a_4 a_3 a_\alpha^\dagger a_{\alpha'}^\dagger a_{\alpha'} a_\alpha + a_1^\dagger a_2^\dagger a_4 a_3 a_\alpha^\dagger a_{\alpha'}^\dagger a_{\alpha'} a_\alpha$$

Antisymmetrized so evaluate first one and multiply by 2

$$\frac{1}{8}\sum_{\alpha\alpha'}\sum_{1234} \langle 12|\delta\tilde{U}|34\rangle\, 2\,\delta_{4\alpha'}\delta_{3\alpha}\, a_1^\dagger a_2^\dagger a_{\alpha'} a_\alpha$$

$$\vec{K}/2+\vec{k}' = \vec{G}/2+\vec{q}$$
$$\vec{K}/2-\vec{k}' = \vec{G}/2-\vec{q}$$
$$\Rightarrow \vec{K}=\vec{G},\ \vec{k}'=\vec{q}$$

$$= \frac{1}{4}\sum_{\alpha\alpha'}\sum_{12} \langle 12|\delta\tilde{U}|\alpha\alpha'\rangle\, a_1^\dagger a_2^\dagger a_{\alpha'} a_\alpha \qquad (4)$$

Similarly $\delta U^\dagger$ gives $\frac{1}{4}\sum_{\alpha\alpha'}\sum_{56} \langle \alpha\alpha'|\delta\tilde{U}^\dagger|56\rangle\, a_\alpha^\dagger a_{\alpha'}^\dagger a_6 a_5$

$\vec{K}'=\vec{G}\quad \vec{k}'''=\vec{q}$

$\delta U\delta U^\dagger$ term: $\quad a_1^\dagger a_2^\dagger a_4 a_3\, a_\alpha^\dagger a_{\alpha'}^\dagger a_{\alpha'} a_\alpha\, a_7^\dagger a_8^\dagger a_6 a_5$

$\rightarrow \quad a_1^\dagger a_2^\dagger a_4 a_3 a_\alpha^\dagger a_{\alpha'}^\dagger a_{\alpha'} a_\alpha a_7^\dagger a_8^\dagger a_6 a_5$

$+$ 3 others

Antisymmetrized $\rightarrow$ factor of 4

$$\vec{K}/2+\vec{k}' = \vec{G}/2+\vec{q}$$
$$\vec{K}/2-\vec{k}' = \vec{G}/2-\vec{q}$$
$$\vec{K}'/2+\vec{k}''' = \vec{G}/2+\vec{q}$$
$$\vec{K}'/2-\vec{k}''' = \vec{G}/2-\vec{q}$$
$$\Rightarrow \vec{k}'=\vec{q},\ \vec{K}=\vec{G}$$
$$\vec{k}'''=\vec{q},\ \vec{K}'=\vec{G}$$

$$\frac{1}{32}\sum_{\alpha\alpha'}\sum_{12345678} \langle 12|\delta\tilde{U}|34\rangle\langle 78|\delta\tilde{U}^\dagger|56\rangle\, 4\,\delta_{4\alpha'}\delta_{3\alpha}\delta_{\alpha'8}\delta_{\alpha 7}\, a_1^\dagger a_2^\dagger a_6 a_5$$

$$= \frac{1}{8}\sum_{\alpha\alpha'}\sum_{1256} \langle 12|\delta\tilde{U}|\alpha\alpha'\rangle\langle\alpha\alpha'|\delta\tilde{U}^\dagger|56\rangle\, a_1^\dagger a_2^\dagger a_6 a_5 \qquad (5)$$

In total,

$$n_\lambda^{\tau\tau'}(\vec{q},\vec{G}) = \frac{1}{2}\sum_{\alpha\alpha'} a_\alpha^\dagger a_{\alpha'}^\dagger a_{\alpha'} a_\alpha + \frac{1}{4}\sum_{\alpha\alpha'}\sum_{12}\langle 12|\delta\tilde{U}|\alpha\alpha'\rangle a_1^\dagger a_2^\dagger a_{\alpha'} a_\alpha +$$

$$\frac{1}{4}\sum_{\alpha\alpha'}\sum_{56}\langle\alpha\alpha'|\delta \tilde U^\dagger|56\rangle a^\dagger_{\alpha} a^\dagger_{\alpha'} a_6 a_5 +$$

$$\frac{1}{8}\sum_{\alpha\alpha'}\sum_{1256}\langle 12|\delta\tilde U|\alpha\alpha'\rangle\langle\alpha\alpha'|\delta\tilde U^\dagger|56\rangle a_1^\dagger a_2^\dagger a_6 a_5 \qquad (6)$$

Relabel and take continuum limit

$$= \frac{1}{2}\sum_{\sigma\sigma'} a^\dagger_{\frac{\vec Q}{2}+\vec q,\sigma\tau}\, a^\dagger_{\frac{\vec Q}{2}-\vec q,\sigma'\tau'}\, a_{\frac{\vec Q}{2}-\vec q,\sigma'\tau'}\, a_{\frac{\vec Q}{2}+\vec q,\sigma\tau}$$

1 term

$$+\frac{1}{4}\sum_{\sigma\sigma'}\sum_{\sigma_1\sigma_2}\sum_{\tau_1\tau_2}\int\frac{d^3k}{(2\pi)^3}\langle\vec k\,\sigma_1\tau_1\sigma_2\tau_2|\delta\tilde U|\vec q\,\sigma\tau\sigma'\tau'\rangle \times$$

$$a^\dagger_{\frac{\vec Q}{2}+\vec k,\sigma_1\tau_1}\, a^\dagger_{\frac{\vec Q}{2}-\vec k,\sigma_2\tau_2}\, a_{\frac{\vec Q}{2}-\vec q,\sigma'\tau'}\, a_{\frac{\vec Q}{2}+\vec q,\sigma\tau}$$

$$+\frac{1}{4}\sum_{\sigma\sigma'}\sum_{\sigma_3\sigma_4}\sum_{\tau_3\tau_4}\int\frac{d^3k'}{(2\pi)^3}\langle\vec q\,\sigma\tau\sigma'\tau'|\delta\tilde U^\dagger|\vec k'\,\sigma_3\tau_3\sigma_4\tau_4\rangle \times$$

$$a^\dagger_{\frac{\vec Q}{2}+\vec q\,\sigma\tau}\, a^\dagger_{\frac{\vec Q}{2}-\vec q\,\sigma'\tau'}\, a_{\frac{\vec Q}{2}-\vec k'\,\sigma_4\tau_4}\, a_{\frac{\vec Q}{2}+\vec k'\,\sigma_3\tau_3}$$

δU term

$$+\frac{1}{8}\sum_{\sigma\sigma'}\sum_{\sigma_1\sigma_2\sigma_3\sigma_4}\sum_{\tau_1\tau_2\tau_3\tau_4}\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3k'}{(2\pi)^3}\langle\vec k\,\sigma_1\tau_1\sigma_2\tau_2|\delta\tilde U|\vec q\,\sigma\tau\sigma'\tau'\rangle \times$$

$$\underbrace{\langle\vec q\,\sigma\tau\sigma'\tau'|\delta\tilde U^\dagger|\vec k'\,\sigma_3\tau_3\sigma_4\tau_4\rangle\, a^\dagger_{\frac{\vec Q}{2}+\vec k\,\sigma_1\tau_1}\, a^\dagger_{\frac{\vec Q}{2}-\vec k\,\sigma_2\tau_2}\, a_{\frac{\vec Q}{2}-\vec k'\,\sigma_4\tau_4}\, a_{\frac{\vec Q}{2}+\vec k'\,\sigma_3\tau_3}}_{\delta U\delta U^\dagger \text{ term}} \qquad (7)$$

Now do contractions with respect to filled Fermi sea $|F\rangle$ where $\langle F|a^\dagger_\beta a_\beta|F\rangle = \Theta(k_F^\beta - k_\beta)$.

1 term: $\langle F | a^\dagger_{\frac{\vec{Q}}{2}+\vec{q}\sigma\tau} a^\dagger_{\frac{\vec{Q}}{2}-\vec{q}\sigma'\tau'} a_{\frac{\vec{Q}}{2}-\vec{q}\sigma'\tau'} a_{\frac{\vec{Q}}{2}+\vec{q}\sigma\tau} | F \rangle$

$a^\dagger a^\dagger a a + a^\dagger a^\dagger a a \rightarrow \frac{\vec{Q}}{2}+\vec{q} = \frac{\vec{Q}}{2}-\vec{q} \Rightarrow \vec{q}=0$ but

we only consider $q>0$!

$\sum_{\sigma\sigma'} \rightarrow$ factor of 4

$(2\pi)^3\delta^3(0)(2\pi)^3\delta^3(0) = V^2$?

Not sure how V factors work here.

$$\boxed{= V^2 2\Theta\left(k_F^\tau - \left|\frac{\vec{Q}}{2}+\vec{q}\right|\right)\Theta\left(k_F^{\tau'} - \left|\frac{\vec{Q}}{2}-\vec{q}\right|\right)} \tag{8}$$

δU term: $\frac{1}{4}\sum_{\sigma\sigma'}\sum_{\sigma_1\sigma_2}\sum_{\tau_1\tau_2}\int\frac{d^3k}{(2\pi)^3}\langle \vec{k}\,\sigma_1\tau_1\sigma_2\tau_2 | \delta\tilde{V} | \vec{q}\,\sigma\tau\sigma'\tau'\rangle \times$

$a^\dagger_{\frac{\vec{Q}}{2}+\vec{k},\sigma_1\tau_1} a^\dagger_{\frac{\vec{Q}}{2}-\vec{k},\sigma_2\tau_2} a_{\frac{\vec{Q}}{2}-\vec{q},\sigma'\tau'} a_{\frac{\vec{Q}}{2}+\vec{q},\sigma\tau}$

Two contractions: $a^\dagger a^\dagger a a + a^\dagger a^\dagger a a$

Do first and multiply by 2

$= \frac{1}{2}\sum_{\sigma\sigma'}\sum_{\sigma_1\sigma_2}\sum_{\tau_1\tau_2}\int\frac{d^3k}{(2\pi)^3}\langle \vec{k}\,\sigma_1\tau_1\sigma_2\tau_2 | \delta\tilde{V} | \vec{q}\,\sigma\tau\sigma'\tau'\rangle (2\pi)^3\delta^3(\vec{k}-\vec{q}) \times$

$\delta_{\sigma_1\sigma}\delta_{\tau_1\tau}\,(2\pi)^3\delta^3(\vec{q}-\vec{k})\,\delta_{\sigma_2\sigma'}\delta_{\tau_2\tau'}\langle F | a^\dagger a\, a^\dagger a | F\rangle$

$\rightarrow (2\pi)^3\delta^3(0) = V$

$$= \frac{V}{2} \sum_{\sigma\sigma'} \langle \vec{q}\sigma\tau\sigma'\tau' | \delta\tilde{U} | \vec{q}\sigma\tau\sigma'\tau' \rangle \Theta\left(k_F^{\tau} - \left|\frac{\vec{Q}}{2} + \vec{q}\right|\right) \Theta\left(k_F^{\tau'} - \left|\frac{\vec{Q}}{2} - \vec{q}\right|\right)$$

δU^{+} term gives identical contribution $\rightarrow$ combine!

$$\boxed{= V \sum_{\sigma\sigma'} \langle \vec{q}\sigma\tau\sigma'\tau' | \delta\tilde{U} | \vec{q}\sigma\tau\sigma'\tau' \rangle \Theta\left(k_F^{\tau} - \left|\frac{\vec{Q}}{2} + \vec{q}\right|\right) \Theta\left(k_F^{\tau'} - \left|\frac{\vec{Q}}{2} - \vec{q}\right|\right)} \quad (9)$$

$\delta U \delta U^{+}$ term:

$$\frac{1}{8} \sum_{\sigma\sigma'} \sum_{\sigma_1\sigma_2\sigma_3\sigma_4} \sum_{\tau_1\tau_2\tau_3\tau_4} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \langle \vec{k}\,\sigma_1\tau_1\sigma_2\tau_2 | \delta\tilde{U} | \vec{q}\,\sigma\tau\sigma'\tau' \rangle \times$$

$$\langle \vec{q}\,\sigma\tau\sigma'\tau' | \delta\tilde{U}^{+} | \vec{k}'\sigma_3\tau_3\sigma_4\tau_4 \rangle\, a^{+}_{\frac{\vec{Q}}{2}+\vec{k}\sigma_1\tau_1} a^{+}_{\frac{\vec{Q}}{2}-\vec{k}\sigma_2\tau_2} a_{\frac{\vec{Q}}{2}-\vec{k}'\sigma_4\tau_4} a_{\frac{\vec{Q}}{2}+\vec{k}'\sigma_3\tau_3}$$

Two contractions $a^{+}a^{+}aa$, $a^{+}a^{+}aa$

Take first and multiply by 2

$$= \frac{1}{4} \sum_{\sigma\sigma'} \sum_{\sigma_1\sigma_2\sigma_3\sigma_4} \sum_{\tau_1\tau_2\tau_3\tau_4} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \langle \vec{k}\,\sigma_1\tau_1\sigma_2\tau_2 | \delta\tilde{U} | \vec{q}\,\sigma\tau\sigma'\tau' \rangle \times$$

$$\langle \vec{q}\,\sigma\tau\sigma'\tau' | \delta\tilde{U}^{+} | \vec{k}'\sigma_3\tau_3\sigma_4\tau_4 \rangle (2\pi)^3 \delta(\vec{k} - \vec{k}')\, \delta_{\sigma_1\sigma_3} \delta_{\tau_1\tau_3} (2\pi)^3 \times$$

$$\underbrace{\delta^3(\vec{k}' - \vec{k})}_{\hookrightarrow V} \delta_{\sigma_2\sigma_4} \delta_{\tau_2\tau_4} \langle F | a^{+}a\, a^{+}a | F \rangle$$

(relabel 1,2 → ″,‴)

$$= \frac{V}{4} \sum_{\sigma\sigma'\sigma''\sigma'''} \sum_{\tau''\tau'''} \int \frac{d^3k}{(2\pi)^3} \langle \vec{k}\,\sigma''\tau''\sigma'''\tau''' | \delta\tilde{U} | \vec{q}\,\sigma\tau\sigma'\tau' \rangle \times$$

$$\langle \vec{q}\,\sigma\tau\sigma'\tau' | \delta\tilde{U}^{+} | \vec{k}\,\sigma''\tau''\sigma'''\tau''' \rangle \theta\left(k_F^{\tau''} - \left|\frac{\vec{Q}}{2} + \vec{k}\right|\right) \theta\left(k_F^{\tau'''} - \left|\frac{\vec{Q}}{2} - \vec{k}\right|\right) \quad (10)$$

Go to partial waves

$2V^2 \theta\left(k_F^{\tau} - \left|\frac{\vec{Q}}{2} + \vec{q}\right|\right) \theta\left(k_F^{\tau'} - \left|\frac{\vec{Q}}{2} - \vec{q}\right|\right)$ does not change

δU term $(\delta U + \delta U^{+})$:

$$V \sum_{\sigma\sigma'} \langle \vec{q}\,\sigma\tau\sigma'\tau' | \delta\tilde{U} | \vec{q}\,\sigma\tau\sigma'\tau' \rangle \theta\left(k_F^{\tau} - \left|\frac{\vec{Q}}{2} + \vec{q}\right|\right) \theta\left(k_F^{\tau'} - \left|\frac{\vec{Q}}{2} - \vec{q}\right|\right)$$

Insert complete set of states using

$$\sum_{S=0}^{1} \sum_{M_S=-S}^{S} |S M_S\rangle\langle S M_S| = 1 \quad (11)$$

$$\sum_{T=0}^{1} \sum_{M_T=-T}^{T} |T M_T\rangle\langle T M_T| = 1 \quad (12)$$

$$\sqrt{\frac{2}{\pi}} \sum_{L M_L} |k L M_L\rangle\langle k L M_L| = 1 \quad (13)$$

$$\sum_{J=|L-S|}^{L+S} \sum_{M_J=-J}^{J} |J M_J L S\rangle\langle J M_J L S| = 1 \quad (14)$$

$$= V \frac{2}{\pi} \sum_{\sigma\sigma'} \sum_{SM_S} \sum_{S'M_S'} \sum_{TM_T} \sum_{T'M_T'} \sum_{LM_L} \sum_{L'M_L'} \sum_{JM_J} \sum_{J'M_J'} \langle \sigma\sigma' | SM_S \rangle \times$$

$$\langle \tau\tau' | TM_T \rangle \langle \vec{q} | qLM_L \rangle \langle SM_S LM_L | JM_J LS \rangle \langle qJLST | \delta\tilde{V} | qJ'L'S'T' \rangle \times$$

$$\langle J'M_J' L'S' | S'M_S' L'M_L' \rangle \langle qL'M_L' | \vec{q} \rangle \langle T'M_T' | \tau\tau' \rangle \langle S'M_S' | \sigma\sigma' \rangle \Theta's$$

Apply $\sum_{\sigma\sigma'} \langle \sigma\sigma' | SM_S \rangle \langle S'M_S' | \sigma\sigma' \rangle = \delta_{SS'} \delta_{M_S M_S'}$

Average over $\vec{q}$-angle $\int \frac{d\Omega_{\vec{q}}}{4\pi} \langle \vec{q} | qLM_L \rangle \langle qL'M_L' | \vec{q} \rangle = \delta_{LL'} \delta_{M_L M_L'}$

$$\sum_{M_L M_S} \langle SM_S LM_L | JM_J LS \rangle \langle J'M_J' LS | SM_S LM_L \rangle = \delta_{JJ'} \delta_{M_J M_J'}$$

$$\sum_{M_J} [\dots] = (2J+1)$$

$\delta\tilde{V} \propto \delta_{TT'} \delta_{M_T M_T'}$

$$\boxed{= \frac{V}{4\pi} \frac{2}{\pi} \sum_{TM_T} \sum_{S} \sum_{L} \sum_{J} (2J+1) \langle \tau\tau' | TM_T \rangle \langle qJLST | \delta\tilde{V} | qJLST \rangle \times \langle TM_T | \tau\tau' \rangle \Theta\left(k_F^{\tau} - \left|\frac{\vec{Q}}{2} + \vec{q}\right|\right) \Theta\left(k_F^{\tau'} - \left|\frac{\vec{Q}}{2} - \vec{q}\right|\right) \quad (15)}$$

* We'll move on to $\delta U \delta U^{\dagger}$ before doing anything with $\vec{Q}$.

$\delta U \delta U^{\dagger}$ term:

$$= \frac{V}{4} \sum_{\sigma\sigma'\sigma''\sigma'''} \sum_{\tau''\tau'''} \int \frac{d^3k}{(2\pi)^3} \langle \vec{k}\sigma''\tau''\sigma'''\tau''' | \delta\tilde{U} | \vec{q}\sigma\tau\sigma'\tau' \rangle \times$$

$$\langle \vec{q}\sigma\tau\sigma'\tau' | \delta U^{\dagger} | \vec{k}\sigma''\tau''\sigma'''\tau''' \rangle \theta\left(k_F^{\tau''} - \left|\frac{\vec{Q}}{2} + \vec{k}\right|\right) \theta\left(k_F^{\tau'''} - \left|\frac{\vec{Q}}{2} - \vec{k}\right|\right)$$

$\downarrow$ Insert $SM_S, \ldots, S'''M_S''', \ldots, J'''M_J'''$ states

$$= \frac{V}{4} \frac{1}{(2\pi)^3} \left(\frac{2}{\pi}\right)^2 \sum_{\sigma\sigma'\sigma''\sigma'''} \sum_{\tau''\tau'''} \sum_{TM_T \cdots T'''M_T'''} \sum_{SM_S \cdots S'''M_S'''} \sum_{LM_L \cdots L'''M_L'''} \sum_{JM_J \cdots J'''M_J'''} \times$$

$$\int_0^{\infty} dk\, k^2 \int d\Omega_{\vec{k}} \langle \sigma''\sigma''' | SM_S \rangle \langle \tau''\tau''' | TM_T \rangle \langle \vec{k} | kLM_L \rangle \times$$

$$\langle SM_S LM_L | JM_J LS \rangle \langle kJLST | \delta\tilde{U} | qJ'L'S'T' \rangle \langle J'M_J'L'S' | S'M_S'L'M_L' \rangle \times$$

$$\langle qL'M_L' | \vec{q} \rangle \langle T'M_T' | \tau\tau' \rangle \langle S'M_S' | \sigma\sigma' \rangle \langle \sigma\sigma' | S''M_S'' \rangle \langle \tau\tau' | T''M_T'' \rangle \times$$

$$\langle \vec{q} | qL''M_L'' \rangle \langle S''M_S''L''M_L'' | J''M_J''L''S'' \rangle \langle qJ''L''S''T'' | \delta U^{\dagger} | kJ'''L'''S'''T''' \rangle \times$$

$$\langle J'''M_J'''L'''S''' | S'''M_S'''L'''M_L''' \rangle \langle kL'''M_L''' | \vec{k} \rangle \langle T'''M_T''' | \tau''\tau''' \rangle \times$$

$$\langle S'''M_S''' | \sigma''\sigma''' \rangle \ \theta\text{'s}$$

Apply $\displaystyle\sum_{\sigma''\sigma'''} \langle \sigma''\sigma''' | SM_S \rangle \langle S'''M_S''' | \sigma''\sigma''' \rangle = \delta_{SS'''} \delta_{M_S M_S'''}$

$\displaystyle\sum_{\sigma\sigma'} \langle S'M_S' | \sigma\sigma' \rangle \langle \sigma\sigma' | S''M_S'' \rangle = \delta_{S'S''} \delta_{M_S' M_S''}$

$$\int d\Omega_{\vec{k}} \langle \hat{k} | k L M_L \rangle \langle k L''' M_L''' | \hat{k} \rangle = \delta_{L L'''} \delta_{M_L M_L'''}$$

$$(\text{avg.}) \int \frac{d\Omega_{\vec{q}}}{4\pi} \langle q L' M_L' | \vec{q} \rangle \langle \vec{q} | q L'' M_L'' \rangle = \delta_{L' L''} \delta_{M_L' M_L''} \frac{1}{4\pi}$$

$$\sum_{M_L M_S} \langle S M_S L M_L | J M_J L S \rangle \langle J''' M_J''' L S | S M_S L M_L \rangle = \delta_{J J'''} \delta_{M_J M_J'''}$$

$$\sum_{M_L' M_L'} \langle J' M_J' L' S' | S' M_S' L' M_L' \rangle \langle S' M_S' L' M_L' | J'' M_J'' L' S' \rangle = \delta_{J' J''} \delta_{M_J' M_J''}$$

$\delta U, \delta U^{\dagger}$ - diagonal in S, M_S, T, M_T, J, M_J

$$\sum_{M_J} [\ldots] \rightarrow (2J+1)$$

$$= \frac{V}{4} \frac{1}{(2\pi)^3} \left(\frac{2}{\pi}\right)^2 \frac{1}{4\pi} \sum_{\tau''\tau'''} \sum_{T M_T T' M_T'} \sum_{S} \sum_{L L'} \sum_{J} (2J+1) \int_0^{\infty} dk\, k^2 \times$$

$$\langle \tau''\tau''' | T M_T \rangle \langle k J L S T | \delta U | q J L' S T \rangle \langle T M_T | \tau \tau' \rangle \langle \tau \tau' | T' M_T' \rangle \times$$

$$\langle q J L' S T' | \delta U^{\dagger} | k J L S T' \rangle \langle T' M_T' | \tau'' \tau''' \rangle \, \theta\left(k_F^{\tau''} - \left|\frac{\vec{Q}}{2} + \vec{k}\right|\right) \times$$

$$\theta\left(k_F^{\tau'''} - \left|\frac{\vec{Q}}{2} - \vec{k}\right|\right) \qquad (16)$$

Now combine all terms :

additional V?

$$\Lambda_\lambda^{\tau\tau'}(q,\vec{Q}) = V\Big[2V\,\theta\big(k_F^\tau - |\tfrac{\vec{Q}}{2}+\vec{q}|\big)\,\theta\big(k_F^{\tau'} - |\tfrac{\vec{Q}}{2}-\vec{q}|\big)\Big]$$

1 term

$$+\frac{1}{4\pi}\frac{2}{\pi}\sum_{TM_T}\sum_S\sum_L\sum_J (2J+1)\langle\tau\tau'|TM_T\rangle\langle qJLST|\delta\tilde{V}|qJLST\rangle\,\times$$

$$\langle TM_T|\tau\tau'\rangle\,\theta\big(k_F^\tau - |\tfrac{\vec{Q}}{2}+\vec{q}|\big)\,\theta\big(k_F^{\tau'} - |\tfrac{\vec{Q}}{2}-\vec{q}|\big)$$

δV term

$$+\frac{1}{4}\frac{1}{(2\pi)^3}\left(\frac{2}{\pi}\right)^2\frac{1}{4\pi}\sum_{\tau''\tau'''}\sum_{TM_TT'M_T'}\sum_S\sum_{LL'}\sum_J (2J+1)\int_0^\infty dk\,k^2\,\times$$

$$\langle\tau''\tau'''|TM_T\rangle\langle kJLST|\delta\tilde{V}|qJL'ST\rangle\langle TM_T|\tau\tau'\rangle\langle\tau\tau'|T'M_T'\rangle\,\times$$

$$\langle qJL'ST'|\delta\tilde{V}^\dagger|kJLST'\rangle\langle T'M_T'|\tau''\tau'''\rangle\,\theta\big(k_F^{\tau''} - |\tfrac{\vec{Q}}{2}+\vec{k}|\big)\,\times$$

$$\theta\big(k_F^{\tau'''} - |\tfrac{\vec{Q}}{2}-\vec{k}|\big)$$

$\delta V \delta V^\dagger$ term

(17)

* How to deal with $\vec{Q}$
    1. Take $\vec{Q}=0$
    2. Do $\int d^3Q\ \Lambda(q,\vec{Q}) = \Lambda(q)$

Let's do 1. first. Then (17) simplifies to

$$\Lambda_\lambda^{\tau\tau'}(q,\vec{Q}=0) = V\Big[2V\,\theta(k_F^\tau-q)\theta(k_F^{\tau'}-q)\Big] \quad \text{1 term}$$

$$+\frac{1}{4\pi}\frac{2}{\pi}\sum_{TM_T}\sum_S\sum_L\sum_J (2J+1)(\tau\tau'|TM_T)\langle qJLST|\delta\tilde{U}|qJLST\rangle \times$$

$$\langle TM_T|\tau\tau'\rangle\theta(k_F^\tau-q)\theta(k_F^{\tau'}-q) \quad \delta U \text{ term}$$

$$+\frac{1}{4}\frac{1}{(2\pi)^3}\left(\frac{2}{\pi}\right)^2\frac{1}{4\pi}\sum_{\tau''\tau'''}\sum_{TM_TT'M_T'}\sum_S\sum_{LL'}\sum_J(2J+1)\int_0^\infty dk\,k^2 \times$$

$$\langle\tau''\tau'''|TM_T\rangle\langle kJLST|\delta\tilde{U}|qJL'ST\rangle\langle TM_T|\tau\tau'\rangle\langle\tau\tau'|T'M_T'\rangle\times$$

$$\langle qJL'ST'|\delta\tilde{U}^\dagger|kJLST'\rangle\langle T'M_T'|\tau''\tau'''\rangle\,\theta(k_F^{\tau''}-k)\theta(k_F^{\tau'''}-k)$$

$\delta U \delta U^\dagger$ term

(18)

Evaluate through $L=0-2$. Start with $\tau=+\frac{1}{2}$ and consider cases for τ'. $\tau=-\frac{1}{2}$ will be nearly identical.

δU term:

A) $\tau'=+\frac{1}{2}$ $\quad M_T=1,\ T=1 \quad$ CG's $=1$

A1) $S=0,\ J=L \Rightarrow L$ even only: ${}^1S_0, {}^1D_2, \ldots$

A2) $S=1$, $J=|L-S|,\dots,L+S \Rightarrow L$ odd only : ${}^3P_0, {}^3P_1, {}^3P_2-{}^3F_2, \dots$

This gives $\sim \left(\delta U_{^1S_0} + \delta U_{^3P_0} + 3\,\delta U_{^3P_1} + 5\,\delta U_{^3P_2-^3P_2} + 5\,\delta U_{^1D_2} \right)$

$\times\, \Theta\left(k_F^p - \left|\frac{\vec{Q}}{2}+\vec{q}\right|\right) \Theta\left(k_F^p - \left|\frac{\vec{Q}}{2}-\vec{q}\right|\right)$ (19) (δU pp term)

B) $\tau' = -\frac{1}{2}$ $M_T = 0$, $T = 0, 1$: CG's $= \frac{1}{\sqrt{2}}$

A1) $T = 1$

i) $S=0$, $J=L \Rightarrow L$ even only : ${}^1S_0, {}^1D_2, \dots$

ii) $S=1$, $J=|L-S|,\dots,L+S \Rightarrow L$ odd : ${}^3P_0, {}^3P_1, {}^3P_2-{}^3F_2$

A2) $T=0$

i) $S=0$, $J=L \Rightarrow L$ odd only : ${}^1P_1$

ii) $S=1$, $J=|L-S|,\dots,L+S \Rightarrow L$ even : ${}^3S_1-{}^3D_1, {}^3D_2, {}^3D_3-{}^3G_3$

δU pn term $\sim \left(\frac{1}{2}\delta U_{^1S_0} + \frac{1}{2}\delta U_{^3P_0} + \frac{3}{2}\delta U_{^1P_1} + \frac{3}{2}\delta U_{^3P_1} \right.$

$+ \frac{5}{2}\delta U_{^3P_2-^3P_2} + \frac{5}{2}\delta U_{^1D_2} + \frac{3}{2}\delta U_{^3S_1-^3S_1} + \frac{3}{2}\delta U_{^3D_1-^3D_1} + \frac{5}{2}\delta U_{^3D_2}$

$\left. + \frac{7}{2}\delta U_{^3D_3-^3D_3} \right) \Theta\left(k_F^n - \left|\frac{\vec{Q}}{2}+\vec{q}\right|\right) \Theta\left(k_F^p - \left|\frac{\vec{Q}}{2}-\vec{q}\right|\right)$ (20)

$\delta U \delta U^{\dagger}$ term :

A) $\tau' = +\frac{1}{2}$ $M_T = 1 \Rightarrow T = 1$, $M_T' = 1 \Rightarrow T' = 1$ CG's $= 1$

$\tau'' = \tau''' = +\frac{1}{2}$

A1) $S = 0$ $J = L$, $L' = L$ $\rightarrow$ L even : ${}^1S_0$, ${}^1D_2$

A2) $S = 1$ $J = |L-S|, \dots, L+S$ $\rightarrow$ L odd : ${}^3P_0$, ${}^3P_1$, ${}^3P_2 - {}^3F_2$

$$\boxed{\begin{aligned} &\sim \Big[\Big(\delta U_{{}^1S_0} \delta U^{\dagger}_{{}^1S_0} + \delta U_{{}^3P_0} \delta U^{\dagger}_{{}^3P_0} + 3\, \delta U_{{}^3P_1} \delta U^{\dagger}_{{}^3P_1} + 5\, \delta U_{{}^3P_2 - {}^3P_2} \delta U^{\dagger}_{{}^3P_2 - {}^3P_2} + \\ &5\, \delta U_{{}^3P_2 - {}^3F_2} \delta U^{\dagger}_{{}^3F_2 - {}^3P_2} + 5\, \delta U_{{}^1D_2} \delta U^{\dagger}_{{}^1D_2} \Big) \Theta_p^{+} \Theta_p^{-} \qquad (21) \end{aligned}}$$

$\delta U \delta U^{\dagger}$ pp term

B) $\tau' = -\frac{1}{2}$ $M_T = 0$, $M_T' = 0$ $\Rightarrow \tau'' = \pm\frac{1}{2}$ $\tau''' = \mp\frac{1}{2}$

$\Rightarrow (\Theta_p \Theta_n + \Theta_n \Theta_p)$ CG's $= 1/\sqrt{2}$

A1) $T = 1$ $T' = 1$

i) $S = 0$, $J = L$ $\Rightarrow$ L even : ${}^1S_0$, ${}^1D_2$

ii) $S = 1$, $J = |L-S|, \dots, L+S \Rightarrow L$ odd : ${}^3P_0$, ${}^3P_1$, ${}^3P_2 - {}^3F_2$

A2) $T = 0$ $T' = 0$

i) $S = 0$, $J = L$ $\Rightarrow$ L odd : ${}^1P_1$

ii) $S = 1$, $J = |L-S|, \dots, L+S$ $\Rightarrow$ L even : ${}^3S_1 - {}^3D_1$, ${}^3D_2$, ${}^3D_3 - {}^3G_3$

$\delta U \delta U^{+}$ pn term

$$\frac{1}{4}\Big(\delta U_{^1S_0}\,\delta U^{+}_{^1S_0} + 5\,\delta U_{^1D_2}\,\delta U^{+}_{^1D_2} + \delta U_{^3P_0}\,\delta U^{+}_{^3P_0} + 3\,\delta U_{^3P_1}\,\delta U^{+}_{^3P_1} +$$

$$5\,\delta U_{^3P_2-^3P_2}\,\delta U^{+}_{^3P_2-^3P_2} + 5\,\delta U_{^3P_2-^3F_2}\,\delta U^{+}_{^3F_2-^3P_2} + 3\,\delta U_{^1P_1}\,\delta U^{+}_{^1P_1} +$$

$$3\,\delta U_{^3S_1-^3S_1}\,\delta U^{+}_{^3S_1-^3S_1} + 3\,\delta U_{^3S_1-^3D_1}\,\delta U^{+}_{^3D_1-^3S_1} + 3\,\delta U_{^3D_1-^3D_1}\,\delta U^{+}_{^3D_1-^3D_1}$$

$$+ 5\,\delta U_{^3D_2}\,\delta U^{+}_{^3D_2} + 7\,\delta U_{^3D_3-^3D_3}\,\delta U^{+}_{^3D_3-^3D_3} + 7\,\delta U_{^3D_3-^3G_3}\,\delta U^{+}_{^3G_3-^3D_3}\Big) \times$$

$$\boxed{\left(\Theta_p^{+}\Theta_n^{-} + \Theta_n^{+}\Theta_p^{-}\right) \qquad (22)}$$

where $\Theta_N^{\pm} \equiv (k_F^N - |\frac{1}{2}\vec{K} \pm \vec{q}\,|)$. This gives the pp and pn for δU and $\delta U \delta U^{+}$ terms. For nn and np, just flip $k_F^p \to k_F^n$ in every case.