Observable properties of a normal Fermi liquid

We now study the predictions of Fermi liquid theory.

1.) Equilibrium properties

a) Specific heat

Consider the specific heat at constant volume

$$c_V = \frac{1}{V}\left(\frac{\partial E}{\partial T}\right)_V$$

A change in the temperature leads to a change in the quasiparticle distribution function

$$c_V = \frac{1}{V}\sum_{\vec{k},\sigma} \frac{\delta E}{\delta n_{\vec{k}\sigma}} \frac{\delta n_{\vec{k}\sigma}}{\delta T}$$

$$= \frac{1}{V}\sum_{\vec{k},\sigma}\left(\varepsilon_{qp}(\vec{k}\sigma) + \frac{1}{V}\sum_{\vec{k}'\sigma'} f_{\vec{k}\sigma,\vec{k}'\sigma'}\, \delta n_{\vec{k}'\sigma'} \right)\frac{\delta n_{\vec{k}\sigma}}{\delta T}$$

Now, the distribution is a Fermi-Dirac distribution close to Fermi surface,

$$n_{\vec{k}\sigma} = \frac{1}{e^{\frac{\varepsilon_{qp}(\vec{k}\sigma)-\mu}{T}} + 1} \qquad k_B \equiv 1$$

And therefore, we have

depends on $n_{\vec{k}\sigma}$ which depends on μ

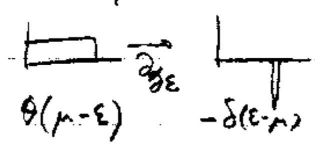
$$\frac{\delta n_{\vec{k}\sigma}}{\delta T} = \frac{\delta n_{\vec{k}\sigma}}{\delta\left(\frac{\varepsilon_{qp}(\vec{k},\sigma)-\mu}{T}\right)}\left(\frac{\delta\left(\frac{\varepsilon_{qp}(\vec{k}\sigma)-\mu}{T}\right)}{\delta T}\right) = \frac{\delta n_{\vec{k}\sigma}}{\delta \varepsilon_{qp}(\vec{k}\sigma)}\, T\left(-\frac{1}{T^2}\left(\varepsilon_{qp}(\vec{k}\sigma)-\mu\right) + \frac{1}{T}\frac{\delta\left(\varepsilon_{qp}(\vec{k}\sigma)-\mu\right)}{\delta T}\right)$$

$$\Rightarrow C_V = \frac{1}{V}\sum_{\vec{k}\sigma}\left(\varepsilon_{qp}(\vec{k}\sigma) + \frac{1}{V}\sum_{\vec{k}'\sigma'} f_{\vec{k}\sigma\vec{k}'\sigma'}\,\delta n_{\vec{k}'\sigma'}\right)\frac{\partial n}{\partial \varepsilon_{qp}}\left(-\frac{\varepsilon_{qp}(\vec{k}\sigma)-\mu}{T} + \frac{\delta(\varepsilon_{qp}(\vec{k}\sigma)-\mu)}{\delta T}\right)$$

higher order in the temperature, involve δn

Use Sommerfeld expansion for $\frac{\partial n}{\partial \varepsilon}$:

$$\frac{\partial n}{\partial \varepsilon} = \frac{\partial}{\partial \varepsilon}\frac{1}{e^{\frac{\varepsilon-\mu}{T}}+1} \underset{T\to 0}{=} -\delta(\varepsilon-\mu) - \frac{\pi^2}{6}T^2\frac{\partial^2}{\partial\varepsilon^2}\delta(\varepsilon-\mu) + \mathcal{O}(T^4)$$



The leading term is given by

$$C_V = \frac{1}{V}\sum_{\vec{k},\sigma}\varepsilon_{qp}(\vec{k},\sigma)\left(-\frac{\varepsilon_{qp}(\vec{k})-\mu}{T}\right)\left(-\delta(\varepsilon_{qp}(\vec{k},\sigma)-\mu) - \frac{\pi^2}{6}T^2\frac{\partial^2}{\partial\varepsilon^2}\delta(\varepsilon-\mu)\Big|_{\varepsilon=\varepsilon_{qp}(\vec{k},\sigma)}\right)$$

$= 0$

quadratic ε^2 integrate by parts

$$\Rightarrow C_V = \frac{1}{V}\sum_{\vec{k},\sigma}\frac{\pi^2}{6}T\cdot 2\,\delta(\varepsilon_{qp}(\vec{k})-\mu)$$

$$= 2\int\frac{d^3k}{(2\pi)^3}\frac{\pi^2}{3}T\frac{1}{\frac{d\varepsilon_{qp}(\vec{k})}{dk}\Big|_{k=k_F}}\delta(k-k_F)$$

spin sum in general: g

$$= 2\int\frac{d^3k}{(2\pi)^3}\frac{\pi^2}{3}T\frac{m^*}{k_F}\delta(k-k_F)$$

$$= 2\cdot 4\pi\frac{\pi^2}{3}\frac{1}{(2\pi)^3}k_F^2\frac{m^*}{k_F}T = \boxed{\frac{1}{3}k_F m^* T = C_V}$$

as for a free Fermi gas, $\sim T$, <u>but</u> with effective mass $m^* \neq m$

Thus, the specific heat provides a measurement for the effective mass. This is however less clear for finite systems, such as the heavy nuclei, such as ^{206}Pb and ^{205}Tl.

For liquid ^3He: $\frac{m^*}{m} \simeq 3$ at zero pressure.

More exotic are "heavy fermion materials", e.g.,

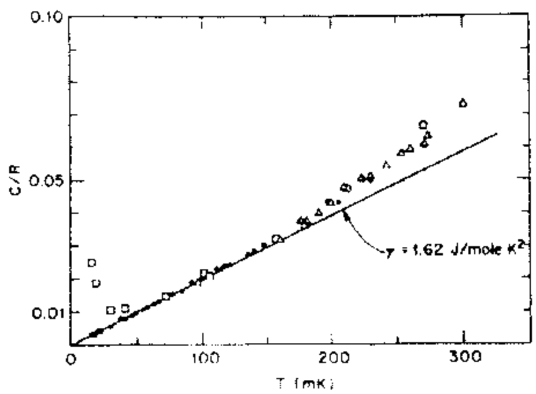


Fig. 2. Specific heat of $CeAl_3$ at low temperatures from Andres *et al.* [28]. The slope of the linear specific heat is about 3000 times that of the linear specific heat of say Cu. But the high-temperature cut-off of this linear term is smaller than that of Cu by a similar amount. The rise of the specific heat in a magnetic field at low temperatures is the nuclear contribution, irrelevant to our discussion.

b) Effective mass

Is there a relation between $\frac{m^*}{m}$ and the quasi-particle interaction? (Galilean transformations mix $\varepsilon_{p\sigma}$ and f !)

Momentum per unit volume $\vec{P} = \frac{1}{V}\sum_{\vec{k}\sigma} \vec{k}\, n_{\vec{k}\sigma}$

Due to the one-to-one correspondence between particles and quasiparticles, the number of particles = number of quasiparticles. And with the quasiparticle (group) velocity $\frac{d\varepsilon_{qp}(\vec{k}\sigma)}{d\vec{k}}$, the total momentum is also given by

$$\vec{P} = \frac{1}{V}\sum_{\vec{k},\sigma} m \frac{d\varepsilon_{qp}(\vec{k}\sigma)}{d\vec{k}} n_{\vec{k}\sigma} = \frac{1}{V}\sum_{\vec{k}\sigma} \vec{k}\, n_{\vec{k}\sigma}$$

$$\frac{\delta}{\delta n_{\vec{k}'\sigma'}} : \quad m \frac{d\varepsilon_{qp}(\vec{k}',\sigma')}{d\vec{k}'} + \underbrace{m \sum_{\sigma} \int \frac{d^3k}{(2\pi)^3} \frac{d}{d\vec{k}} \left(\frac{\delta \varepsilon_{qp}(\vec{k}\sigma)}{\delta n_{\vec{k}'\sigma'}} \right) n_{\vec{k}\sigma}}_{\circledast} = \vec{k}'$$

$$\Rightarrow \frac{\vec{k}'}{m} = \frac{d\varepsilon_{qp}(\vec{k}',\sigma')}{d\vec{k}'} + \sum_{\sigma} \int \frac{d^3k}{(2\pi)^3} \frac{d}{d\vec{k}} \left(f_{\vec{k}\sigma,\vec{k}'\sigma'} \right) n_{\vec{k}\sigma}$$

integrate by part

$$\Rightarrow \frac{\vec{k}'}{m} = \frac{d\varepsilon_{qp}(\vec{k}'\sigma')}{d\vec{k}'} - \sum_{\sigma} \int \frac{d^3k}{(2\pi)^3} f_{\vec{k}\sigma,\vec{k}'\sigma'} \underbrace{\frac{dn_{\vec{k}\sigma}}{d\vec{k}}}_{-\hat{k}\,\delta(k_F - k)}$$

for $|\vec{k}'| = k_F$

$$\Rightarrow \frac{\vec{k}'}{m} = \frac{\vec{k}'}{m^*} + \sum_{\sigma} \int \frac{k_F^2 \, d\Omega_{\vec{k}}}{(2\pi)^3} \sum_{\ell} \left(f_\ell + g_\ell\, \vec{\sigma}\cdot\vec{\sigma}' \right) P_\ell(\hat{k}\cdot\hat{k}')\, \hat{k}$$

Multiplying by $\hat{k}'$ and taking the z-axis of $\vec{k}$ along $\vec{k}'$

$$\Rightarrow \frac{k_F}{m} = \frac{k_F}{m^*} + \sum_{\sigma} \frac{k_F^2}{(2\pi)^2} \int d\cos\theta \sum_{\ell} \left(f_\ell + g_\ell\, \vec{\sigma}\cdot\vec{\sigma}' \right) P_\ell(\cos\theta) \cos\theta$$

Spin sum $\sum_\sigma$ averages out $\vec{\sigma}\cdot\vec{\sigma}'$

$$\sum_\sigma \vec{\sigma}\cdot\vec{\sigma}' = \sum_{m_s=\uparrow,\downarrow} \langle \chi_{m_s}|\sigma^i|\chi_{m_s}\rangle \sigma'^i = \sigma'^z - \sigma'^z = 0$$

$$\Rightarrow \frac{k_F}{m} = \frac{k_F}{m^*} + \underset{\uparrow \text{ spin sum}}{2}\frac{k_F^2}{(2\pi)^2}\int d\cos\theta \sum_\ell f_\ell P_\ell(\cos\theta)\underset{\overset{\|}{P_1(\cos\theta)}}{\cos\theta}$$

Use $\frac{2\ell+1}{2}\int dx\, P_\ell(x)\, P_m(x) = \delta_{\ell m}$

$$\Rightarrow \frac{\cancel{k_F}}{m} = \frac{k_F}{m^*} + \frac{2}{3}\,2\frac{k_F^{\cancel{2}}}{(2\pi)^2} f_1 \quad \Rightarrow \quad \frac{m^*}{m} = 1 + \frac{1}{3}\underbrace{\frac{k_F m^*}{\pi^2} f_1}_{F_1}$$

$$\left(\text{in general } F_\ell = \frac{k_F m^*}{\pi^2} f_\ell\right)$$

$$\Rightarrow \boxed{\frac{m^*}{m} = 1 + \frac{F_1}{3}}$$

The effective mass determines the $l=1$ moment of the quasiparticle interaction.

The second part (⊛ above) is due to the interaction with the other particles in the medium, which represents the other particles dragged along with the quasiparticle ("drag current", see also the discussion in matter). This is in addition to the current associated with the quasiparticle $\frac{\vec{k}}{m^*}$.

c) Compressibility χ (solid state notation) / Incompressibility K (nuclear) and speed of sound

↳ sometimes also compression modulus

$$\boxed{\frac{1}{\chi} = \rho \frac{\partial P}{\partial \rho} = \rho \frac{k_F^2}{3m^*}(1+F_0)}$$

$$\boxed{K = k_F^2 \frac{\partial^2}{\partial k_F^2}\left(\frac{E}{A}\right) = 9\frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial E/A}{\partial \rho}\right) = \frac{3k_F^2}{m^*}(1+F_0)} \quad \left(K = \frac{9}{\chi\rho}\right)$$

binding energy per nucleon

Speed of sound:

$$c_s^2 = \frac{1}{m\rho\chi} = \frac{1}{9m}K \sim (1+F_0)$$

due to spherical $l=0$ average

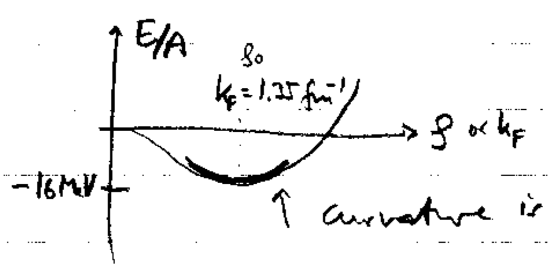
$\chi \to \infty$, $c_s^2 \to 0$ as $F_0 \to -1$: The system becomes unstable against density oscillations

In general, the requirement that the ground state energy be a minimum, and not simply stationary, restricts the possible Landau parameters to the range

$$F_l,\ G_l > -(2l+1)$$

Nuclear matter:

$K = 230 \pm 20$ MeV at nuclear matter saturation density



↑ curvature is given by incompressibility

d) Magnetic susceptibility

Magnetization m induced by magnetic field H.
$\chi_M = \frac{\partial m}{\partial H}$ will depend on the spin-dependent part of f

$$\boxed{\chi_M = \frac{\partial m}{\partial H} = \frac{\gamma^2 k_F\, m^*}{4\pi^2(1+G_0)}}$$

$\gamma = \frac{e\hbar}{mc}$ gyromagnetic ratio

e) Symmetry energy

In nuclear matter: $f_{\vec{k}\sigma\tau,\vec{k}'\sigma'\tau'} = f + f'\vec{\tau}\cdot\vec{\tau}' + g\,\vec{\sigma}\cdot\vec{\sigma}' + g'\,\vec{\tau}\cdot\vec{\tau}'\,\vec{\sigma}\cdot\vec{\sigma}'$

isospin

$$\boxed{E_{sym} = \frac{k_F^2}{6m^*}\left(1 + F_0'\right)}$$

2.) Nonequilibrium properties

The dynamics of a normal Fermi liquid close to equilibrium is governed by a Boltzmann equation for the qp distribution function

We consider only low-energy, long-wavelength (low-momentum) excitations, where we have well-defined qp's. Furthermore, we assume (this can be viewed as a first step) that $n_{\vec{k}\sigma}(\vec{r},t)$ can be treated as a classical dist. function ($n_{\vec{k}\sigma}$ for every space-time point)

The basic assumption of the qp kinetic theory is that $\varepsilon_{qp}(\vec{k},\sigma;\vec{r},t)$ $= \varepsilon_{\vec{k}\sigma}(\vec{r},t)$ plays the role of the quasiparticle Hamiltonian, i.e.,

(we drop the qp from now on)

velocity $\quad \vec{v}_{\vec{k}\sigma}(\vec{r},t) = \frac{\partial}{\partial \vec{k}}\, \varepsilon_{\vec{k}\sigma}(\vec{r},t)$

force $\quad \vec{f}_{\vec{k}\sigma}(\vec{r},t) = -\frac{\partial}{\partial \vec{r}}\, \varepsilon_{\vec{k}\sigma}(\vec{r},t)$

Suppressing the diff. indices, the Boltzmann equation is obtained by

$$\frac{d n_{\vec{k}\sigma}(\vec{r},t)}{dt} = I\left[n_{\vec{k}\sigma}(\vec{r},t)\right] = \underset{\text{no collisions}}{\text{Collision integral} = 0 \text{ for}}$$

or $$\frac{\partial n}{\partial t} + \underbrace{\frac{\partial n}{\partial \vec{r}}\cdot\frac{\partial \varepsilon}{\partial \vec{k}} - \frac{\partial n}{\partial \vec{k}}\cdot\frac{\partial \varepsilon}{\partial \vec{r}}}_{\text{Poisson bracket}} = I[n]$$

For small deviations from equilibrium, we expand n and ε around their equilibrium values:

only $\vec{r}$ dep.

$$n_{\vec{k}\sigma}(\vec{r},t) = n^{eq}_{\vec{k}\sigma} + \delta n_{\vec{k}\sigma}(\vec{r},t)$$

$$\varepsilon_{\vec{k}\sigma}(\vec{r},t) = \underset{\varepsilon_{qp}(\vec{k}\sigma)}{\overset{\shortparallel}{\varepsilon^{eq}_{\vec{k}\sigma}}} + \frac{1}{V}\sum_{\vec{k}'\sigma'} f_{\vec{k}\sigma,\vec{k}'\sigma'}\, \delta n_{\vec{k}'\sigma'}(\vec{r},t)$$

Expanding the Boltzmann equation to first order in δn, leads the linearized kinetic equation

$$\boxed{\frac{\partial}{\partial t}\delta n_{\vec{k}\sigma}(\vec{r},t) + \frac{\vec{k}}{m^*}\cdot\frac{\partial}{\partial \vec{r}}\,\delta n_{\vec{k}\sigma}(\vec{r},t) - \frac{\partial n^{eq}_{\vec{k}\sigma}}{\partial \vec{k}}\cdot\frac{1}{V}\sum_{\vec{k}'\sigma'} f_{\vec{k}\sigma,\vec{k}'\sigma'}\frac{\partial}{\partial \vec{r}}\,\delta n_{\vec{k}'\sigma'}(\vec{r},t) = I[n]}$$

Again using $\frac{\partial n^{eq}_{\vec{k}\sigma}}{\partial \vec{k}} = -\delta(\mu - \varepsilon^{eq}_{\vec{k}\sigma})\frac{\vec{k}}{m^*}$, we find

$$\boxed{\frac{\partial}{\partial t}\delta n_{\vec{k}} + \frac{\vec{k}}{m^*}\cdot\frac{\partial}{\partial \vec{r}}\delta n_{\vec{k}} + \frac{\vec{k}}{m^*}\,\delta(\mu - \varepsilon^{eq}_{\vec{k}})\,\frac{1}{V}\sum_{\vec{k}'\sigma'} f_{\vec{k}\vec{k}'}\frac{\partial}{\partial \vec{r}}\delta n_{\vec{k}'} = I[n]}$$

One can derive conservation laws from the qp kinetic equation, (see Negele + Orland p. 30c) but we continue and discuss a collective mode of the Fermi liquid predicted by Landau and subsequently observed in liquid ^{3}He.

a) Zero sound

Thermodynamic (or first) sound relies on local thermodynamic equilibrium, i.e., that the sound frequency ω is small compared to the inverse collision time $\omega \ll \frac{1}{\tau}$. The qp lifetime and the inverse collision is of order

$$\frac{1}{\tau} \sim (\varepsilon^{qp} - \mu)^2 \sim T^2$$

for excited qp's at temperature T $\varepsilon^{qp} = \mu \pm T$.

$\Rightarrow$ Ordinary sound propagates for $\omega \ll T^2$ and therefore cannot propagate for sufficiently low T (or at fixed T sufficiently high frequency).

At sufficiently low T, ordinary sound ceases to propagate, because collisions are negligible. In the collisionless regime, we can neglect the collision integral $I[n]$ and solve for the relevant collective modes, by solving the linearized Boltzmann equation.

Ansatz for $\delta n_{\vec{k}}(\vec{r},t) = e^{i(\vec{q}\cdot\vec{r}-\omega t)}\phi_{\vec{k}}$

lin. Boltzmann

$$\Rightarrow \quad \left(\vec{q}\cdot\frac{\vec{k}}{m^*}-\omega\right)\phi_{\vec{k}} + \vec{q}\cdot\frac{\vec{k}}{m^*}\,\delta(\mu-\varepsilon^{qp}_{\vec{k}})\frac{1}{V}\sum_{\vec{k}'\sigma'} f_{\vec{k}\sigma,\vec{k}'\sigma'}\,\phi_{\vec{k}'} = I[n] = 0$$

$\Rightarrow \phi_{\vec{k}} \sim \delta(\mu-\varepsilon^{qp}_{\vec{k}})$ and we define

$$\phi_{\vec{k}} = \delta(\mu-\varepsilon^{qp}_{\vec{k}})\,v_F\,u_{\vec{k}} \quad {}^{1)} \qquad \left(v_F = \frac{k_F}{m^*}\right)$$

Upon inserting this in linearized Boltzmann equation

$$\left(\vec{q}\cdot\frac{\vec{k}}{m^*}-\omega\right)u_{\vec{k}} + \vec{q}\cdot\frac{\vec{k}}{m^*}\frac{1}{V}\sum_{\vec{k}'\sigma'} f_{\vec{k}\sigma\vec{k}'\sigma'}\,\delta(\mu-\varepsilon^{qp}_{\vec{k}'})\,u_{\vec{k}'} = 0$$

and $|\vec{k}| = k_F$

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1) $u_{\vec{k}}$ is the displacement of the Fermi surface at momentum $\vec{k}$, because

spin indep. case

$$n_{\vec{k}\sigma}(\vec{r},t) = n^{eq}_{\vec{k}} + \delta n_{\vec{k}}(\vec{r},t)$$

$$= n^{eq}_{\vec{k}} + e^{i(\vec{q}\cdot\vec{r}-\omega t)}\delta(\mu-\varepsilon^{qp}_{\vec{k}})\,v_F\,u_{\vec{k}}$$

$$= n^{eq}_{\vec{k}} - e^{i(\vec{q}\cdot\vec{r}-\omega t)}\left.\frac{\partial n^{eq}_{\vec{k}}}{\partial|\vec{k}|}\right|_{k_F} u_{\vec{k}} \qquad \text{(expanded around F.S.)}$$

Since the momenta are restricted to the Fermi surface

$$u_{\vec{k}} = u(\Omega_{\vec{k}}, \sigma) = u(\underset{\uparrow \atop \hat{k}\cdot\hat{q}}{\theta}, \varphi, \sigma)$$

When the two spin species oscillate in phase, $u_{\vec{k}} = u(\theta, \varphi)$ is spin independent, which we consider here.

We define $\quad s = \dfrac{\omega}{q v_F} = \dfrac{\omega m^*}{q k_F} = \dfrac{\text{velocity of collective mode } \omega/q}{\text{qp velocity } v_F = k_F/m^*}$

$$\Rightarrow (s - \cos\theta)\, u(\theta, \varphi) = \cos\theta \sum_{\sigma'} \int \frac{d^3k'}{(2\pi)^3} \delta\left(\frac{k_F}{m^*}(k' - k_F)\right) f_{\vec{k}\sigma, \vec{k}'\sigma'}\, u(\theta', \varphi')$$

$$= \cos\theta \int \frac{k_F^2\, d\Omega'}{(2\pi)^3} \frac{m^*}{k_F} \underbrace{\sum_{\sigma'} f_{\vec{k}\sigma, \vec{k}'\sigma'}}\, u(\theta', \varphi')$$

both $|\vec{k}| = |\vec{k}'| = k_F$

averages out spin-dependent part as for $\frac{m^*}{m}$

$$\Rightarrow \boxed{(s - \cos\theta)\, u(\theta, \varphi) = \cos\theta \int \frac{d\Omega'}{4\pi} \sum_{l} F_l\, P_l(\hat{k}\cdot\hat{k}')\, u(\hat{k}'\hat{q}, \varphi')}$$

Keeping only the $l=0$ moment

$$\boxed{(s - \cos\theta)\, u(\theta, \varphi) = \cos\theta F_0 \int \frac{d\Omega'}{4\pi}\, u(\theta', \varphi')}$$

$$\Rightarrow \quad u(\Omega) = \frac{\cos\theta}{s - \cos\theta}\, C \qquad \text{where } C = F_0 \int \frac{d\Omega'}{4\pi}\, u(\Omega')$$

By inserting back we find

$$C = F_0 \int \frac{d\Omega'}{4\pi} \frac{\cos\theta}{s-\cos\theta}\, C$$

$$\Rightarrow \boxed{\frac{1}{F_0} = \int_{-1}^{1} \frac{dx}{2}\, \frac{x}{s-x} = \frac{s}{2} \ln\left(\frac{s+1}{s-1}\right)}$$ Condition for zero sound

A real solution requires $S > 1$ corresponding to $F_0 > 0$

and

$$s \xrightarrow[F_0 \to 0]{} 1 + 2e^{-2/F_0}$$

$$\xrightarrow[F_0 \to \infty]{} \sqrt{\frac{F_0}{3}}$$

$$u(\cos\theta, \varphi) = \frac{\cos\theta}{\underset{s>1}{s} - \cos\theta}\, C$$

shows that the Fermi surface for the longitudinal zero sound mode oscillates as sketched below in comparison to first sound

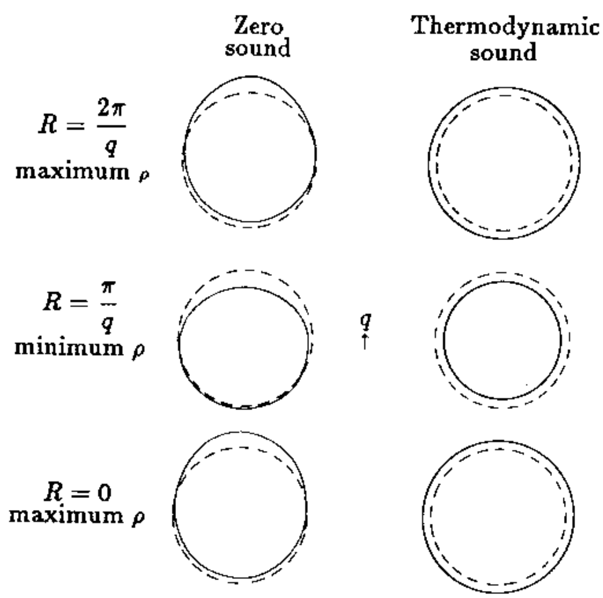

Fig. 5.16 The distribution function $f(p, R)$ for zero sound and thermodynamic sound. The equilibrium Fermi sphere is shown at each spatial position by the dashed line and the solid line denotes the momentum distribution of the propagating mode.

For F_0 and $F_1 \neq 0$ and $u(\theta,\varphi) = u(\theta)$, s has to satisfy

$$\frac{1}{2} s \ln\left(\frac{s+1}{s-1}\right) - 1 = \frac{1 + F_1/3}{F_0 + s^2 F_1 + \frac{1}{3} F_0 F_1}$$

The observed zero sound velocity in liquid ^3He at 0.32 atm pressure is found to be (compared to first sound $c_0 = c_1$)

$$\left.\frac{\overset{= \frac{\omega}{q}}{c_0} - c_1}{c_1}\right|_{0.32\text{ atm}} = 0.035 \pm 0.003$$

Compare to predictions from Landau parameters

$$\left.\begin{array}{ll} F_0 = 10.77 & \text{(from first sound)} \\ F_1 = 6.25 & \text{(from specific heat)} \end{array}\right\} \text{at } 0.28 \text{ atm}$$

$\Rightarrow$ $\frac{c_0 - c_1}{c_1}$ corresponds to $s = 3.60 \pm 0.01$

Using only $l=0$ F_0, Fermi liquid theory predicts $s = 2.05$
" $l=0$ and $l=1$, " $\underline{s = 3.60}$!