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Monday 880.05

- Comments on PS#4!
  - It will be based on both formal and numerical questions for a one-dimensional (1+1 space+time) system of fermions.
  - Any codes and supplementary notes you need will be provided.
  - We'll also complete the Bose limit discussion but in 1d!
- Take-away points on effective action:
  - Follows from Legendre transform of $\ln Z[J]$ from J to field(s) coupled to J
  - When minimized, Γ is the energy up to a constant when minimized with respect to the conjugate field (expectation value). Expand about non-zero minimum $\Rightarrow$ symmetry breaking
- Follow-up to effective action for large ν, but now don't assume uniform system. (Drop μ here.)

$$\boxed{Z[J] = e^{iW[J]} = \int \mathcal{D}(\psi^\dagger\psi)\,\mathcal{D}\sigma\, e^{i\left[\int d^4x\, \psi^\dagger_\alpha \underbrace{\left(i\frac{\partial}{\partial t} - \frac{\nabla^2}{2m} - C_0\sigma(x)\right)}_{G^{-1}}\psi_\alpha(x) + \frac{1}{2}C_0(\sigma(x))^2 + J(x)\sigma(x)\right]} = \int \mathcal{D}\sigma\, e^{\nu \operatorname{Tr}\ln G^{-1}(x,y)}\, e^{\frac{i}{2}C_0\int d^4x (\sigma(x))^2}\, e^{i\int d^4x\, J(x)\sigma(x)}}$$

Effective action was:

$$\boxed{\Gamma[\sigma_c] \equiv W[J] - \int d^4x\, J(x)\sigma(x) = \underbrace{\frac{\nu}{i}\operatorname{Tr}\ln\left[G_H^{-1}(x,y)\right] + \frac{C_0}{2}\int d^4x (\sigma_c(x))^2}_{\Gamma_{LO}[\sigma_c]} + \text{higher order}}$$

Here $\boxed{G_H^{-1}(x,y) = \left[i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m} - C_0\sigma_c(x)\right]\delta^4(x-y)}$

Energy $\boxed{E_{LO} = -\Gamma_{LO}[\sigma_c^0]/T}$ where $\boxed{\left.\frac{\delta\Gamma_{LO}}{\delta\sigma_c}\right|_{\sigma_c^0} = 0}$

- So what if $\sigma_c \to \sigma_c(\vec{x})$, time independent in the ground state?
  - Then $\frac{\partial}{\partial t}$ still diagonalized by p_0.

- Compare to uniform system $\to$ frequency and momentum space

$$\boxed{g\operatorname{Tr}\ln G_H^{-1} = \nu\sum_{\vec{p}}\sum_{p_0}\ln\left(p_0 - e_{\vec{p}} + i\eta\,\mathrm{sgn}(|\vec{p}| - k_F)\right)} \quad \boxed{e_{\vec{p}} = p^2/2m + C_0\sigma_c} \leftarrow \text{const}$$

$$\boxed{\left.\frac{\delta\Gamma_{LO}}{\delta\sigma_c}\right|_{\sigma_c^0} = 0 \Rightarrow \sigma_c^0 = \rho} \quad \text{and} \quad \boxed{\mathcal{E}_{LO} = -\Gamma_{LO}[\sigma_c^0]/VT = \rho\left(\frac{3}{5}\frac{k_F^2}{2m} + \frac{C_0}{2}\rho\right)}$$

- Consider the plot of $\mathcal{E}_{LO}$ $\Rightarrow$ positive pressure.

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Now we find ϵ_α instead of $\epsilon_{\vec{p}}$ with $\sigma_c \to \sigma_c(\vec{x})$ time independent in the ground state (which means $\frac{\partial}{\partial t}$ is still diagonalized by going to p_0).

$$\Rightarrow \text{solve} \quad \boxed{\left(-\frac{\nabla^2}{2m} + C_0\sigma_c(\vec{x})\right)\varphi_\alpha(\vec{x}) = \epsilon_\alpha \varphi_\alpha(\vec{x})}$$

for all $\epsilon_\alpha < \epsilon_F$ (just count them until N states!)

$$\text{Then} \quad \boxed{-\frac{\Gamma_{LO}[\sigma_c^0]}{T} = \sum_\alpha^{\epsilon_F} \epsilon_\alpha - \frac{1}{2}C_0\int d^3x\,(\sigma_c(\vec{x}))^2}$$

example of Feynman-Hellman type result

$$\text{We need} \quad \frac{\delta \epsilon_\alpha}{\delta \sigma_c(\vec{x})} = \frac{\delta}{\delta \sigma_c(\vec{x})}\int d^3y\, \varphi_\alpha^\dagger(\vec{y})\left(-\frac{\nabla_y^2}{2m} + C_0\sigma(\vec{y})\right)\varphi_\alpha(\vec{y})$$

$$= C_0\varphi_\alpha^\dagger(\vec{x})\varphi_\alpha(\vec{x}) \quad \left[\text{Why not } \frac{\delta\varphi_\alpha(\vec{y})}{\delta\sigma_c(\vec{x})} \text{ terms?}\right]$$

$$\Rightarrow -\frac{\delta\Gamma_{LO}[\sigma_c^0]/T}{\delta\sigma_c^0(\vec{x})} = 0 = C_0\sum_\alpha^{\epsilon_F}\varphi_\alpha^\dagger(\vec{x})\varphi_\alpha(\vec{x}) - C_0\sigma_c(\vec{x})$$

$$\text{or} \quad \boxed{\sigma_c(\vec{x}) = \sum_\alpha^{\epsilon_F}\varphi_\alpha^\dagger(\vec{x})\varphi_\alpha(\vec{x}) \equiv \rho(\vec{x})}$$

Iteration plan:

1. guess $\sigma_c(\vec{x}) = \rho_{old}(\vec{x})$
2. plug into S-eqn and solve for $\{\epsilon_\alpha, \varphi_\alpha(\vec{x})\}$
3. find $\rho_{new}(\vec{x}) = \sum_\alpha \varphi_\alpha^\dagger(\vec{x})\varphi_\alpha(\vec{x})$
4. repeat 2, 3 until $\rho(\vec{x})$ stops changing
5. evaluate E.

Claim: We can make this procedure general with density functional theory (DFT)!

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Effective Action and Pairing

- So far we have been restricted to repulsive interactions, with scattering length $a_s > 0$, so that the ground state was stable (pressure is positive, so we could confine it).
  - When we have attraction, then our non interacting ground state is not a good starting point (at least near the Fermi surface) $\Rightarrow$ pairing happens!

- We'd like to extend our effective field theory approach to include pairing.
  - Systematic power counting is the goal.
  - Should be present even in the dilute, natural system we have considered.
  - Particularly important in large scattering length $k_F a_s \gtrsim 1$ problem.

- So how do we find pairing in a field theory / path integral framework? Answer: In the effective action formalism.
  - Here: Show the basic idea and highlight some questions (of which there are many!).

- Recall analogy of effective action and spin systems
  - A lattice of spins $s_i = \pm 1$ with Hamiltonian in external magnetic field H (we'll sum the interaction over all pairs here):

$$\boxed{\mathcal{H} = -\frac{1}{2N} J \sum_{\{ij\}} s_i s_j - H \sum_i s_i}$$

with partition function

$$\boxed{Z(\beta, H, N) = \sum_{\{s_i\}} e^{\beta\left(\frac{1}{2}\frac{J}{N}\sum_{\{ij\}} s_i s_j + H \sum_i s_i\right)} \longrightarrow \int \mathcal{D}s \, e^{-\beta \int dx \left[\mathcal{H}(s) - H s(x)\right]}}$$

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One relevant question is whether the ground state at zero external magnetic field ($H=0$) has a non-zero magnetization.

- The magnetization m is the expectation value of $\sum_i s_i$:

$$\boxed{m = \frac{1}{Z} \mathrm{Tr}\left\{ \left(\sum_i s_i\right) e^{-\beta\left(\frac{1}{2}\frac{I}{N}\sum_{i,j} s_i s_j + H \sum_i s_i\right)} \right\} = -\frac{\partial F(H)}{\partial H}}$$

where F is the Helmholtz free energy:

$$\boxed{F = -\frac{1}{\beta}\ln Z} \quad \text{or} \quad \boxed{Z = e^{-\beta F(H)}}$$

- We could calculate $F(H)$ in a perturbative expansion, but it will always predict $m=0$ if $H=0$.
  - But, we can do a Legendre transformation to the Gibb's free energy: invert $m = -\frac{\partial F(H)}{\partial H}$ to find $H(m)$, then

$$\boxed{G(m) = F(H(m)) + m H(m)}$$

  - Note that

$$\boxed{\frac{\partial G}{\partial m} = \overbrace{\frac{\partial F}{\partial H}}^{-m}\frac{\partial H}{\partial m} + H + m\frac{\partial H}{\partial m} = H}$$

so G as a function of m is minimized $\frac{\partial G}{\partial m}=0$ when there is no external field.

** A perturbative approximation to $G(m)$ <u>can</u> have nontrivial (that is $m \neq 0$) solutions to $\partial G/\partial m = 0$.

- Similarly, our expansion of the dilute Fermi gas in EFT counting (powers of k_F/Λ) did not reveal pairing.
  $\Rightarrow$ do analog of magnetic Legendre transformation.

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- The magnetization example is one of spontaneously broken symmetry: in particular, the global symmetry of rotational invariance is broken by the magnetization in the ground state picking out a direction.
  - (Put back the vectors — $m \to \vec{m}$ and $\vec{H}\cdot\sum_i \vec{S}_i$ to see this)
  - The external field H acts as a source term to break the symmetry. Then we see from $G(m)$ whether it survives as $|\vec{H}| \to 0$.

- What is the corresponding symmetry in the fermion case with delta-function interaction?

$$\boxed{\mathcal{L} = \psi_\alpha^\dagger \left[i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m} + \mu \right]\psi_\alpha - \frac{1}{2}C_0\left(\psi_\alpha^\dagger\psi_\alpha\right)^2}$$

(equation of motion for ψ from $\frac{\partial \mathcal{L}}{\partial \psi_\alpha^\dagger} = 0$)

has the usual Galilean invariance and parity and time reversal symmetries.

- But there is also a global (perform the same transformation at every space-time point) $U(1)$ symmetry:

$$\boxed{\psi_\alpha \to e^{-i\theta}\psi_\alpha} \quad \boxed{\psi_\alpha^\dagger \to e^{+i\theta}\psi_\alpha^\dagger}$$

since ψ_α always appears with $\psi_\alpha^\dagger$. Infinitesimal $\Rightarrow \psi_\alpha \to (1-i\theta)\psi_\alpha$

$\Delta\psi_\alpha = -i\psi_\alpha$.

- Noether's theorem says that, to each symmetry of a local Lagrangian, there corresponds a conserved current. (This is the classical version; see Peskin + Schroeder: $j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_\alpha)}\Delta\varphi_\alpha$ where $\varphi_\alpha \to \varphi_\alpha + \epsilon\Delta\varphi_\alpha$ up to total divergence)
  - We can find the current (use $\psi_\alpha^\dagger \frac{\nabla^2}{2m}\psi_\alpha \to -\frac{1}{2m}(\nabla\psi^\dagger)\cdot(\nabla\psi)$)

$$\Rightarrow \boxed{j_0(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0\psi^\dagger)}\Delta\psi^\dagger + \frac{\partial \mathcal{L}}{\partial(\partial_0\psi)}\Delta\psi = \psi^\dagger\psi}$$

$$\boxed{\vec{j}(x) = -\frac{i}{2m}\left(\psi^\dagger\vec{\nabla}\psi - (\vec{\nabla}\psi^\dagger)\psi\right)}$$

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Symmetry Transformations and Noether's Theorem.

- We want to consider the idea of symmetries from the point of view of path integrals.
  - A symmetry is manifested by a transformation of the fields that does not change the action.
  - But this should just correspond to a change of variables in the path integral, $\Rightarrow$ in general we can do this without changing the integral. This is called a field redefinition and while the Lagrangian might change, S-matrix elements and thermodynamic observables don't.
  - If the action doesn't change, then we have additional consequences if the Jacobian is unity. If not we have quantum anomalies. Here, the U(1) symmetry transformation leaves the Lagrangian invariant and has Jacobian unity in the path integral. (Because it is a unitary transformation among the fields)

Careful classical vs. QM

- A main consequence of a continuous symmetry transformation is that there is a conserved (time-independent) charge Q associated with a conserved current $\partial_\mu J^\mu = 0$.
  - Consider $\mathcal{L} = \mathcal{L}(\varphi_i, \partial_\mu \varphi_i)$ where $\partial_0 \equiv \frac{\partial}{\partial t}$, $\partial_i = \vec{\nabla}_i$
  - It is sufficient to consider an infinitesimal transformation. (Why? And what do you gain?)

$$\varphi_i'(x) = \varphi_i(x) + \epsilon f_i(\varphi)$$

  - For internal symmetry, let $\epsilon \to \epsilon(x)$:

$$\hat{\varphi}_i(x) = \varphi_i(x) + \epsilon(x) f_i(\varphi)$$

then

$$J^\mu(x) \equiv \frac{\partial}{\partial(\partial_\mu \epsilon(x))} \mathcal{L}(\hat{\varphi}, \partial\hat{\varphi}) \Rightarrow \partial_\mu J^\mu = \frac{\partial \mathcal{L}}{\partial \epsilon(x)} = 0 \quad \text{from } \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0$$

and

$$Q = \int d^3x \, J_0 \Rightarrow \frac{dQ}{dt} = \int d^3x \, \partial_0 J_0 = -\int d^3x \, \vec{\nabla} \cdot \vec{J} = 0.$$

Try $\theta(\vec{x}, t)$: $\int d^4x \, \mathcal{L}' = \int d^4x \left[\psi_\alpha^\dagger (1 + \theta(x)) \left(i \frac{\partial}{\partial t} + \mu\right)(1 - i\theta) \psi_\alpha - \vec{\nabla}(\psi_\alpha^\dagger (1 + i\theta(x))) \frac{1}{2m} \vec{\nabla}(1 - i\theta(x)) \psi_\alpha \right.$

$= \int d^4x \, \mathcal{L} + \int d^4x \left[\psi_\alpha^\dagger \psi_\alpha \left(\frac{\partial}{\partial t} \theta\right) - \frac{i}{2m} \left(\psi_\alpha^\dagger (\nabla\theta) \vec{\nabla} \psi_\alpha - (\nabla \psi_\alpha^\dagger)(\nabla\theta) \psi_\alpha \right) \right] = \int d^4 \mathcal{L} - \int d^4x \, \theta(x) \left[\frac{\partial}{\partial t} (\psi_\alpha^\dagger \psi_\alpha) - \vec{\nabla} \cdot \right.$
$\left. (\psi_\alpha^\dagger \nabla \psi_\alpha - (\nabla \psi_\alpha^\dagger) \psi_\alpha) \right]$

arbitrary $\theta(x) \Rightarrow [\ ]$ vanishes.

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so the conserved charge is the fermion number $N = \int \psi^\dagger\psi\, dx$.

- This will be a broken symmetry if we have a non-zero expectation value for $\langle \psi_\uparrow \psi_\downarrow \rangle$. Why?
- So we'll need a chemical potential.

- Let's show one way to proceed, following M. Stone's "The Physics of Quantum Fields" discussion.
  - We'll work in Euclidean space at temperature $1/\beta$ (with $\hbar = 1$) and take $\beta \to \infty$ early on.
  - We'll consider spin-1/2 only, and attractive $C_0 = -|\lambda| < 0$. (so $\nu = 2$)

- The partition function is (in 1-D, the 3-D generalization is immediate):

$$\boxed{\begin{aligned} Z &= \mathrm{Tr}\left(e^{-\beta(\hat{H} - \mu\hat{N})}\right) \\ &= \int \mathcal{D}(\psi^\dagger\psi)\, e^{-\int_0^\beta dx\, d\tau \left[\sum_{\alpha=1}^{2} \psi_\alpha^\dagger \left(\frac{\partial}{\partial\tau} - \frac{1}{2m}\frac{d^2}{dx^2} - \mu\right)\psi_\alpha - |\lambda| \psi_1^\dagger \psi_2^\dagger \psi_2 \psi_1\right]} \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{or } -\tfrac{1}{2}|\lambda|(\psi_\alpha^\dagger\psi_\alpha)(\psi_\beta^\dagger\psi_\beta) \end{aligned}}$$

where $\alpha = 1, 2$ correspond to $\uparrow, \downarrow$.

- If we think about Grassmann anticommuting fields, we don't need to worry about anticommutators so we can see we have simply rearranged the usual $\frac{1}{2}(\psi_\alpha^\dagger\psi_\alpha)^2$ term: (all at x,t)

$$\boxed{\begin{aligned} \tfrac{1}{2}\psi_\alpha^\dagger\psi_\alpha\psi_\beta^\dagger\psi_\beta &= \tfrac{1}{2}(\psi_1^\dagger\psi_1 + \psi_2^\dagger\psi_2)(\psi_1^\dagger\psi_1 + \psi_2^\dagger\psi_2) \\ &= \tfrac{1}{2}(\psi_2^\dagger\psi_2\psi_1^\dagger\psi_1 + 1 \leftrightarrow 2) \\ &= \psi_1^\dagger\psi_2^\dagger\psi_2\psi_1 \end{aligned}}$$

(2 interchanges $\Rightarrow$ same sign)

- Note that the Minkowski $e^{iS} = e^{i\int dx\,dt\, \mathcal{L}(x,t)}$ becomes the Euclidean $Z = e^{-S_E} = e^{-\int dx\, d\tau\, \mathcal{L}_E(x,t)}$ where

$$\boxed{t \to -i\tau} \quad \text{and} \quad \boxed{\mathcal{L}_E(x,\tau) = -\mathcal{L}(x, -i\tau)} \qquad \left(\text{so } i\frac{\partial}{\partial t} \to -\frac{\partial}{\partial\tau}\right)$$

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The basic plan is to eliminate the $\psi^\dagger_\uparrow \psi^\dagger_\downarrow \psi_\downarrow \psi_\uparrow$ term by introducing an auxiliary field, just as we did for the $(\psi^\dagger\psi)^2$ term in the large N discussion.

- What can we say about the field?
- $\psi^\dagger\psi$ is Hermitian, but $\psi_\downarrow\psi_\uparrow$ is not so we can expect to need a <u>charged</u> scalar. We'll call it Δ and Δ^*.

- We note the Gaussian integral over Δ and Δ^* can be shifted, so we can write

$$\boxed{1 = \frac{\int \mathcal{D}(\Delta,\Delta^*)\, e^{-\frac{1}{|\lambda|}\int dx\, d\tau\, (\Delta^* - |\lambda|\psi_1^\dagger\psi_2^\dagger)(\Delta - |\lambda|\psi_2\psi_1)}}{\int \mathcal{D}(\Delta,\Delta^*)\, e^{-\frac{1}{|\lambda|}\int dx\, d\tau\, |\Delta|^2}}}$$

or $|\lambda| \int (\Delta^* - \psi_1^\dagger\psi_2^\dagger)(\Delta - \psi_2\psi_1)$

where we have a nice convergent integral at each x, τ.

- The sign of the quartic term in the exponent is just right to kill the corresponding term in Z.

So we can write (absorbing the constant denominator in the equation above into the measure):

$$\boxed{Z = \int \mathcal{D}(\psi^\dagger,\psi)\,\mathcal{D}(\Delta,\Delta^*)\, e^{-\int_0^\beta dx\, d\tau \left\{ \psi^\dagger_\alpha \left(\frac{\partial}{\partial\tau} - \frac{1}{2m}\frac{\partial^2}{\partial x^2} - \mu\right)\psi_\alpha - \Delta^*\psi_2\psi_1 - \Delta\psi_1^\dagger\psi_2^\dagger + \frac{1}{|\lambda|}|\Delta|^2 \right\}}}$$

Note that the equation of motion for Δ is $\boxed{\Delta = |\lambda|\psi_2\psi_1}$

$\left(\frac{\delta S_E}{\delta\Delta^*} = 0 \Rightarrow \frac{1}{|\lambda|}\Delta - \psi_2\psi_1 = 0\right)$

vs.

We can verify that the new "interaction term" is Hermitian:

$$\boxed{\left(\Delta^*\psi_2\psi_1 + \Delta\psi_1^\dagger\psi_2^\dagger\right)^\dagger = +\Delta^*\psi_2\psi_1 + \Delta\psi_1^\dagger\psi_2^\dagger}$$

noting that Δ and Δ^* commute with the Grassman variables.

- Also the sign of the interaction is not relevant since $\Delta \leftrightarrow -\Delta$ and $\Delta^* \leftrightarrow -\Delta^*$ leaves Z invariant.

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- To consider the same global U(1) symmetry when Δ and Δ^* have been added, we need to take

$$\boxed{\Delta \to e^{+2i\theta}\Delta} \quad \text{and} \quad \boxed{\Delta^* \to e^{-2i\theta}\Delta^*}$$

to keep each term individually invariant.
  - $|\Delta|^2$ doesn't change since it involves the magnitude.

- So now if Δ gets an expectation value, that picks out a phase and we no longer have the invariance.
  - Just as in the magnetization case, we can add a source to single out a direction (an external magnetic field in that case) and then take it to zero at the end to see if the symmetry breaking persists.

- This implies we should add sources coupled to Δ and Δ^*

$$\Rightarrow \boxed{Z \to Z[j,j^*] = \int \mathcal{D}(\psi^\dagger\psi)\mathcal{D}(\Delta\Delta^*)\, e^{-\tilde{S}_E + \int dx\,d\tau\,(j^*(x)\Delta(x) + j(x)\Delta^*(x))}}$$

where $\tilde{S}_E$ includes the Δ, Δ^* fields.

- At this stage Stone shifts to $\beta \to \infty$, which we can accomplish by simply taking the τ limits to be from $-\infty$ to $+\infty$.

- The usual plan at this point is to identify a classical expectation value and then to expand about it.
  - What is the Legendre transform?

If $\boxed{\Gamma[\Delta_c] \equiv W[j,j^*] - \int dx\,d\tau\,(\Delta_c^* j + j^*\Delta_c)}$ then $\boxed{\Delta_c^*(x) = \frac{\delta W}{\delta j(x)}}$ and $\boxed{\Delta_c(x) = \frac{\delta W}{\delta j^*(x)}}$.

So find $\Delta_c[j(x)]$ and $\Delta_c^*[j^*(x)]$. Note that $\frac{\delta \Gamma}{\delta \Delta_c} = -j \to 0$ when $j \to 0$.

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- In our previous effective action example, $\sigma \propto \psi^\dagger_\alpha \psi_\alpha$ and we expanded $\sigma(x,t) = \sigma_c(x) + \eta(x,t)$.
  - Here we write

$$\boxed{\Delta(x,\tau) = \Delta_c(x) + \eta(x,\tau)}$$

and expand in fluctuations η about Δ_c.

- For our initial discussion we'll just need the "classical" piece. So we take $\Delta \to \Delta_c$ and $\Delta^* \to \Delta_c^*$ in $Z[j,j^*]$ and do the Gaussian $\psi^\dagger_\alpha, \psi_\alpha$ integrations.

- Previously, when we introduced σ, we had (for $\nu=2$)

$$\boxed{\begin{pmatrix} \psi_1^\dagger & \psi_2^\dagger \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu - |\lambda|\sigma_c(x) & 0 \\ 0 & \frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu - |\lambda|\sigma_c(x) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}}$$

and so the determinant of the matrix (from the Gaussian integral) simply picked up a factor of 2 (or ν in general) from the 2×2 submatrix. ↖ ln det gets factor of 2, det get the square

- The present matrix mixes ψ_1 and ψ_2, so the above form doesn't work. But we can swap ψ_2 and $\psi_2^\dagger$ (remembering they are Grassmann!):

$$\boxed{\begin{pmatrix} \psi_1^\dagger & \psi_2 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu & -\Delta_c \\ -\Delta_c^* & \frac{\partial}{\partial\tau} + \frac{\nabla^2}{2m} + \mu \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2^\dagger \end{pmatrix}}$$

- We can just redefine $\psi_2 \to \tilde{\psi}_2^\dagger$ and $\psi_2^\dagger \to \tilde{\psi}_2$ and we have a more ordinary looking Gaussian integral, with a non diagonal submatrix.

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Plan: Work in a basis that diagonalizes the $\frac{\partial}{\partial \tau} \pm \frac{\nabla^2}{2m}$ parts and then just calculate the determinant of the 2×2 matrix.

for experts! [· You might imagine introducing quasi particle operators instead to diagonalize it!

· So we work in the Fourier basis

· Then we need

minus sign

$$\boxed{\begin{aligned}\Gamma_{\text{fermion}}[\Delta_c] &= -\ln \prod_{\omega}\prod_{k} \det -\begin{pmatrix} i\omega + \frac{k^2}{2m} - \mu & \Delta_c \\ \Delta_c^* & i\omega - \frac{k^2}{2m} + \mu \end{pmatrix} \\ &= -\ln \prod_{\omega}\prod_{k}\left[\omega^2 + \left(\frac{k^2}{2m} - \mu\right)^2 + |\Delta_c|^2\right] \\ &= -\sum_{\omega}\sum_{k} \ln\left[\omega^2 + \left(\frac{k^2}{2m}-\mu\right)^2 + |\Delta_c|^2\right] \\ &\longrightarrow -LT \int \frac{d\omega}{2\pi}\int \frac{dk}{2\pi} \ln\left[(\omega^2 + \xi_k^2) + |\Delta_c|^2\right]\end{aligned}}$$

with $\boxed{\xi_k \equiv \frac{k^2}{2m} - \mu}$ [note: no Hartree-Fock part!]

· Now we can use our derivative trick again:

· let $|\Delta_c|^2 \to \gamma|\Delta_c|^2$ and take $\int_0^1 d\gamma \frac{d}{d\gamma}[\quad]$ which will give us $\Gamma[\Delta_c] - \Gamma[0]$.

$$\Rightarrow \boxed{\begin{aligned}&\int_0^1 d\gamma \int \frac{d\omega}{2\pi} \frac{d}{d\gamma}\left(\ln\left[(\omega^2 + \xi_k^2) + \gamma|\Delta_c|^2\right]\right) \\ &= \int_0^1 d\gamma \int \frac{d\omega}{2\pi} \frac{|\Delta_c|^2}{\omega^2 + \xi_k^2 + \lambda|\Delta_c|^2} \\ &= \int_0^1 d\gamma\, \frac{1}{2}\frac{|\Delta_c|}{\sqrt{\xi_k^2 + \lambda|\Delta_c|^2}} \\ &= \sqrt{\xi_k^2 + \Delta_c^2} - |\xi_k|\end{aligned}}$$

alternative: $\ln(\omega^2 + \xi_k^2) + \ln\left(1 + \frac{|\Delta_c|^2}{(\omega^2 + \xi_k^2)}\right)$

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So the leading order (LO) effective action is

$$\boxed{\Gamma_{LO}[\Delta_c] - \Gamma_{LO}[0] = -LT\left(\frac{|\Delta_c|^2}{|\lambda|} - \int \frac{dk}{2\pi}\left(\sqrt{\xi_k^2 + |\Delta_c|^2} - |\xi_k|\right)\right)}$$

• can plot this

Minimize with respect to Δ_c to get the ground state:

$$\frac{\partial \Gamma_{LO}}{\partial \Delta_c} = -LT\left(\frac{2\Delta_c^*}{|\lambda|} - 2\Delta_c^* \int \frac{dk}{2\pi} \frac{1}{\sqrt{\xi_k^2 + |\Delta_c|^2}} \cdot \frac{1}{2}\right)$$

$$= -LT\Delta_c^*\left(\frac{2}{|\lambda|} - \int \frac{dk}{2\pi} \frac{1}{\sqrt{\xi_k^2 + |\Delta_c|^2}}\right)$$

note that $\Delta_c^* = 0$ is a solution, but is it the lowest?

Setting $\partial\Gamma_{LO}/\partial \Delta_c = 0$ and looking for a non-zero Δ_c^* yields the gap equation:

$$\frac{2}{|\lambda|} - \int \frac{dk}{2\pi} \frac{1}{\sqrt{\xi_k^2 + |\Delta_c|^2}} = 0$$

or $$\boxed{1 = \frac{|\lambda|}{4\pi} \int_{-\infty}^{\infty} dk \frac{1}{\sqrt{\xi_k^2 + |\Delta_c|^2}}}$$

The thermodynamic potential follows from

$$\boxed{\Omega[\Delta_c] = -\Gamma[\Delta_c]/T} \quad \text{and} \quad \boxed{E = \Omega + \mu N = \Omega - \mu\left(\frac{\partial \Omega}{\partial \mu}\right)_L}$$

So $$\boxed{\frac{N}{L} = \frac{1}{LT}\frac{\partial \Gamma[\Delta_c]}{\partial \mu} = -\int \frac{dk}{2\pi}\left(\frac{\xi_k}{\sqrt{\xi_k^2 + |\Delta_c|^2}} - 1\right)}$$

and $$\boxed{\frac{E}{L} = \frac{-\Gamma[\Delta_c]}{LT} + \mu \int dk \left(1 - \frac{\xi_k}{\sqrt{\xi_k^2 + |\Delta_c|^2}}\right)}$$

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Let's return to the quadratic term before doing the fermion integration

$$\boxed{\int d\tau\, dx \begin{pmatrix} \psi_1^+ & \psi_2 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu & -\Delta_c \\ -\Delta_c^* & \frac{\partial}{\partial\tau} + \frac{\nabla^2}{2m} + \mu \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2^+ \end{pmatrix} + \frac{1}{|\lambda|}|\Delta_c|^2}$$

and go to a uniform system $\Delta_c(x) \to \Delta_c$ and Fourier transform: (also take $1 \to \uparrow$ and $2 \to \downarrow$)

$$\boxed{\sum_\omega \sum_k \begin{pmatrix} \psi_\uparrow^+(k) & \psi_\downarrow(-k) \end{pmatrix} \begin{pmatrix} -i\omega + \xi_k & -\Delta_c \\ -\Delta_c^* & -i\omega - \xi_k \end{pmatrix} \begin{pmatrix} \psi_\uparrow(k) \\ \psi_\downarrow^+(-k) \end{pmatrix} + LT \frac{|\Delta_c|^2}{|\lambda|}}$$

Now think about diagonalizing the matrix $\Rightarrow$ introduce a rotated basis of Grassman fields:

$$\boxed{\begin{pmatrix} \psi_\uparrow(k) \\ \psi_\downarrow^+(-k) \end{pmatrix} = \begin{pmatrix} \cos\chi_k & +\sin\chi_k \\ -\sin\chi_k & \cos\chi_k \end{pmatrix} \begin{pmatrix} \alpha(k) \\ \beta^+(-k) \end{pmatrix}}$$

and choose χ_k so that we get a diagonal matrix.

This is accomplished (surprise, surprise!) by

$$\boxed{\cos 2\chi_k = \frac{\xi_k}{\sqrt{\xi_k^2 + |\Delta_c|^2}}} \qquad \boxed{\sin 2\chi_k = \frac{|\Delta_c|}{\sqrt{\xi_k^2 + |\Delta_c|^2}}}$$

(checked with Mathematica)

Inverting the transformation,

$$\boxed{\begin{pmatrix} \alpha(k) \\ \beta^+(-k) \end{pmatrix} = \begin{pmatrix} \cos\chi_k & -\sin\chi_k \\ +\sin\chi_k & \cos\chi_k \end{pmatrix} \begin{pmatrix} \psi_\uparrow(k) \\ \psi_\downarrow^+(-k) \end{pmatrix}}$$

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which is precisely the Bogolyubov Transformation when we make the identifications of u_k and v_k:

$$\boxed{u_k \equiv \cos\chi_k} \quad \text{and} \quad \boxed{v_k \equiv \sin\chi_k}$$

Mathematica check:

```
In[1]:= r1 = {{Cos[χ], Sin[χ]}, {-Sin[χ], Cos[χ]}};

In[2]:= MatrixForm[r1]

Out[2]//MatrixForm=
 ( Cos[χ]   Sin[χ] )
 ( -Sin[χ]  Cos[χ] )

In[3]:= r1t = Transpose[r1];

In[4]:= MatrixForm[r1t]

Out[4]//MatrixForm=
 ( Cos[χ]  -Sin[χ] )
 ( Sin[χ]   Cos[χ] )

In[5]:= mid = {{-I ω + ξ, -Δ}, {-Δ, -I ω - ξ}};

In[6]:= MatrixForm[mid]

Out[6]//MatrixForm=
 ( ξ - i ω    -Δ     )
 (  -Δ     -ξ - i ω  )

In[7]:= Ek = Sqrt[ξ^2 + Δ^2]

Out[7]= Sqrt[Δ^2 + ξ^2]

In[8]:= MatrixForm[Simplify[r1t.mid.r1] /. {Cos[2 χ] → ξ / Ek, Sin[2 χ] → Δ / Ek}]

Out[8]//MatrixForm=
```

$$\begin{pmatrix} \frac{\Delta^2}{\sqrt{\Delta^2+\xi^2}} + \frac{\xi^2}{\sqrt{\Delta^2+\xi^2}} - i\,\omega & 0 \\ 0 & -\frac{\Delta^2}{\sqrt{\Delta^2+\xi^2}} - \frac{\xi^2}{\sqrt{\Delta^2+\xi^2}} - i\,\omega \end{pmatrix}$$

We can simplify that last result a bit more to get

$$\boxed{\begin{aligned} S &= TL\frac{|\Delta_c|^2}{|\lambda|} + \sum_\omega\sum_k \begin{bmatrix} \alpha^\dagger(k) \\ \beta(-k) \end{bmatrix}^T \begin{pmatrix} -i\omega + \sqrt{\xi_k^2+|\Delta_c|^2} & 0 \\ 0 & -i\omega - \sqrt{\xi_k^2+|\Delta_c|^2} \end{pmatrix} \begin{pmatrix} \alpha(k) \\ \beta^\dagger(-k) \end{pmatrix} \\ &= TL\frac{|\Delta_c|^2}{|\lambda|} + \sum_\omega\sum_k \left(-i\omega + \sqrt{\xi_k^2+\Delta_0^2}\right)\left(\alpha^\dagger(k)\alpha(k) + \beta^\dagger(k)\beta(k)\right) \end{aligned}}$$

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Which is the action corresponding to the Hamiltonian:

$$\boxed{\mathcal{H}_1 = \sum_k E_k \left(\alpha^\dagger(k)\alpha(k) + \beta^\dagger(k)\beta(k) \right)}$$

What about the Hartree-Fock piece we had before?

- In our two effective action expansions, we made two different choices for getting rid of the $|\lambda| \psi_1^\dagger \psi_2^\dagger \psi_2 \psi_1$ term

$$\boxed{\begin{array}{ll} \text{i)} & \sigma \to \psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2 \quad \left[\tfrac{1}{2}|\lambda| (\psi_\alpha^\dagger \psi_\alpha)^2\right] \\ \text{ii)} & \Delta \to |\lambda| \psi_2 \psi_1 \end{array}}$$

- The leading term in the σ-expansion included the Hartree-Fock contribution $\boxed{\mathcal{E}_{HF} = -\frac{|\lambda|}{2}\rho^2}$ or $\boxed{E_{HF}/N = -\frac{|\lambda|}{2}\rho}$

- If we solved either expansion exactly (i.e. to all orders), we should get the same answer whether we choose i), ii) or $\frac{1}{2}[\text{i)} + \text{ii)}]$.
- However, if we truncate the expansion, we are incorporating two different types of physics. Nagaosa in his "QFT in Condensed Matter Physics" book says: "Here, a physical picture or intuition is necessary, because no general method exists."
  - We would like effective field theory (EFT) to provide such a method, but we haven't figured it out yet.

- Nagaosa argues that the most physically reasonable approximation is i) + ii) (rather than $\frac{1}{2}[\text{i)}+\text{ii)}]$) $\Rightarrow$ if restricted in momentum space, actually non-overlapping $\Rightarrow$ do both.
  - In practice this is what is done in nuclear models.