2. Stochastic Variational Method Revisited.

The estimate for the energy is

$$E_{estimate} = \frac{\sum_{ij} c_i c_j \langle \varphi_i | H | \varphi_j \rangle}{\sum_{i'j'} c_{i'} c_{j'} \langle \varphi_{i'} | \varphi_{j'} \rangle} \equiv \frac{\sum_{ij} c_i c_j H_{ij}}{\sum_{i'j'} c_{i'} c_{j'} B_{i'j'}}$$

- For this to be stationary with respect to variations in the coefficients $\{c_i\}$, we need

$$\frac{\partial E_{estimate}}{\partial c_k} = 0 \quad \text{for all } k.$$

$$\Rightarrow \quad 0 = \frac{\sum_j c_j H_{kj} + \sum_i c_i H_{ik}}{\sum_{i'j'} c_{i'} c_{j'} B_{i'j'}} - \frac{\left(\sum_{ij} c_i c_j H_{ij}\right)\left(\sum_{j'} c_{j'} B_{kj'} + \sum_{i'} c_{i'} B_{i'k}\right)}{\left(\sum_{i'j'} c_{i'} c_{j'} B_{i'j'}\right)^2}$$

- Now $\sum_{i'j'} c_{i'} c_{j'} B_{i'j'} > 0$ and $H_{ij} = H_{ji}$, $B_{ij} = B_{ji}$ so we can cancel a denominator factor and an overall 2:

$$\Rightarrow \quad \sum_j H_{kj} c_j - \underbrace{\left(\frac{\sum_{ij} c_i c_j H_{ij}}{\sum_{i'j'} c_{i'} c_{j'} B_{i'j'}}\right)}_{E_{estimate}} \sum_j B_{kj} c_j = 0$$

or

$$\boxed{\sum_j H_{kj} c_j - E \sum_j B_{kj} c_j = 0} \quad \text{for all } k.$$

will give up $E = E_{estimate}$. This is precisely the generalized eigenvalue problem, PS#1-3

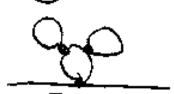
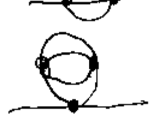
3. Model Partition Function

a) Here we use $Z = N\int d\xi\, e^{-\xi^2/2 - \lambda\xi^4/4}$ where the normalization N will drop out of $\langle \xi^2 \rangle$.

$$\boxed{\langle \xi^2 \rangle = \frac{\int d\xi\ \xi^2 e^{-(a\xi^2/2 + \lambda\xi^4/4)}}{\int d\xi\ e^{-(a\xi^2/2 + \lambda\xi^4/4)}}} \Rightarrow \quad —— \ \frac{1}{a}$$

$$\times \quad -\frac{\lambda}{4} 4! = -6\lambda$$

The Feynman rules to find the λ^3 contribution say to sum the contributions from all connected diagrams with two external lines and three vertices (one for each λ). The disconnected diagrams from the numerator cancel with those from the denominator.

i) diagrams	ii) symmetry factors	iii) contribution
	$\frac{1}{2^3} \times 1 \times 1$	$\frac{1}{8}(-6\lambda)^3\, 1/a^7$
	$\frac{1}{2^2} \times \frac{1}{2} \times 1$	$\frac{1}{8}(-6\lambda)^3\, 1/a^7$
	$\frac{1}{2^2} \times \frac{1}{2} \times 1$	$\frac{1}{8}(-6\lambda)^3\, 1/a^7$
	$\frac{1}{2} \times \frac{1}{3!} \times 1$	$\frac{1}{12}(-6\lambda)^3\, \frac{1}{a^7}$
	$\frac{1}{2} \times \frac{1}{3!} \times 1$	$\frac{1}{12}(-6\lambda)^3\, \frac{1}{a^7}$
	$\frac{1}{2^2} \times 1 \times \frac{1}{2}$	$\frac{1}{8}(-6\lambda)^3\, \frac{4}{a^7}$
	$\frac{1}{2} \times (\frac{1}{2})^2 \times 1$	$\frac{1}{8}(-6\lambda)^3\, \frac{1}{a^7}$
	$1 \times (\frac{1}{2})^2 \times 1$	$\frac{1}{4}(-6\lambda)^3\, \frac{1}{a^7}$
	$\frac{1}{2} \times \frac{1}{2} \times 1$	$\frac{1}{4}(-6\lambda)^3\, \frac{1}{a^7}$
	$1 \times \frac{1}{3!} \times \frac{1}{2}$	$\frac{1}{2}(-6\lambda)^3\, \frac{1}{a^7}$

total $\boxed{-297\lambda^3/a^7}$

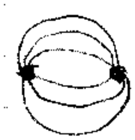
(The symmetry factors are in the usual order.)

3b) Now $\boxed{Z=\int d\xi\; e^{-(a\xi^2/2+\alpha\xi^6/6)}}$

i) The rule $\longrightarrow = \frac{1}{a}$ follows exactly as before.
Now we have a new vertex with 6 legs $\Rightarrow$ $\ast\ \left(-\frac{\alpha}{6}\right)6! = -5!\,\alpha$
(you can see why $-\frac{\alpha'}{6!}\xi^6$ would be smarter than $-\alpha\xi^6/6!$)
Otherwise, we calculate graphs as before.

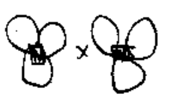
ii) Z_α/Z_0 includes both connected and disconnected closed diagrams (no external legs). Order α^1 has one α vertex and order α^2 has two vertices: [note: order $\alpha^0 = 1$ trivially.]

 $\left(\frac{1}{2}\right)^3 \times \frac{1}{3!} \times 1 \times \left(-\frac{\alpha}{6}6!\right)\frac{1}{a^3} = \boxed{-\frac{5}{2}\frac{\alpha}{2a^3}}$ $O(\alpha)$

$+$  $1 \times \frac{1}{6!} \times \frac{1}{2} \times \left(-\frac{\alpha}{6}6!\right)^2 \frac{1}{a^6} = 10\frac{\alpha^2}{a^6}$

$+$  $\left(\frac{1}{2}\right)^2 \times \frac{1}{4!} \times \frac{1}{2} \times \left(-\frac{\alpha}{6}6!\right)^2 \frac{1}{a^6} = 75\frac{\alpha^2}{a^6}$

$+$  $\left(\frac{1}{2}\right)^4 \times \left(\frac{1}{2!}\right)^3 \times \frac{1}{2} \times \left(-\frac{\alpha}{6}6!\right)^2 \frac{1}{a^6} = \frac{225}{4}\frac{\alpha^2}{a^6}$

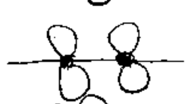
$+$  $\left(\frac{1}{2}\right)^6 \times \left(\frac{1}{3!}\right)^2 \times \frac{1}{2} \times \left(-\frac{\alpha}{6}6!\right)^2 \frac{1}{a^6} = \frac{25}{8}\frac{\alpha^2}{a^6}$

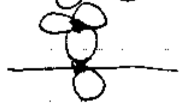
$= \boxed{\frac{1155}{8}\frac{\alpha^2}{a^6}}$ $O(\alpha^2)$

These results agree with Mathematica.

iii) $\langle\xi^2\rangle$ has two external lines and one ($O(\alpha)$) or two ($O(\alpha^2)$) vertices:

 $\left(\frac{1}{2}\right)^2 \times \frac{1}{2!} \times 1 \times \left(-\frac{\alpha}{6}6!\right)\frac{1}{a^4} = \boxed{-15\,\alpha/a^4}$ $O(\alpha)$ ✓

 $\left(\frac{1}{2}\right)^4 \times \left(\frac{1}{2!}\right)^2 \times 1 \times \left(-\frac{\alpha}{6}6!\right)^2 \frac{1}{a^7}$

 $\left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2!}\right)^2 \times 1 \times \left(-\frac{\alpha}{6}6!\right)^2 \frac{1}{a^7}$

 $\frac{1}{2} \times \frac{1}{4!} \times 1 \times \left(-\frac{\alpha}{6}6!\right)^2 \frac{1}{a^7}$

 $1 \times \frac{1}{5!} \times 1 \times \left(-\frac{\alpha}{6}6!\right)^2 \frac{1}{a^7}$

 $\left(\frac{1}{2}\right)^2 \times \frac{1}{3!} \times 1 \times \left(-\frac{\alpha}{6}6!\right)^2 \frac{1}{a^7}$

$= \left(\frac{1}{64}+\frac{1}{32}+\frac{1}{48}+\frac{1}{120}+\frac{1}{24}\right)120^2\frac{\alpha^2}{a^7}$

$\boxed{= 1695\frac{\alpha^2}{a^7}}$ ✓

PS#1-5

3c) i) If we multiply and divide O_n by $\boxed{\dfrac{\int d\xi_1\, e^{-[\ ]/T_0}}{\int d\xi_1\, e^{-[\ ]/T_0}}}$, then

$$\boxed{O_n = \frac{\left(\int d\xi_1\, \xi_1^2 e^{-[\ ]/T_0}\right)}{\left(\int d\xi_1\, e^{-[\ ]/T_0}\right)} \times \underbrace{\left(\int d\xi_1\, e^{-[\ ]/T_0}\right)\left(\int d\xi_2\, e^{-[\ ]/T_0}\right)\cdots\left(\int d\xi_n\, e^{-[\ ]/T_0}\right)}_{n \text{ copies}}}$$

so the n dependence is just in the n copies. If we set $n=0$, then we are left with the first ratio, which is the definition of $\langle \xi^2 \rangle$.

ii) The external legs come only from the ξ_1^2 term in the numerator of the ratio, so the index 1 is all that appears.

iii) As discussed in class for the partition function, each connected piece can only have the same number. If disconnected, then there will be a factor of n for each disconnected piece (since $1, 2, \ldots n$ copies) $\Rightarrow$ the $n \neq 0$ parts are exactly the connected ones.

iv) changing ξ^2 to a general $O(\xi)$ operator changes nothing in any of the previous parts (only the <u>number</u> of external legs with change). $\Rightarrow$ the argument works for other operators.