* We want to transform something like
$\langle \vec{k}\, \sigma\tau\, \sigma'\tau' | \delta\tilde{U} | \vec{k}'\, \sigma''\tau''\, \sigma'''\tau''' \rangle$ to
$\langle k\, J L S T | \delta\tilde{U} | k\, L' S T \rangle$ (or the product of $\delta\tilde{U}\,\delta\tilde{U}^{\dagger}$.)

- Start with simple stuff first

Spin and isospin completeness :

$$\sum_{S=|s_1-s_2|}^{s_1+s_2} \sum_{M_S=-S}^{S} |S M_S\rangle\langle S M_S| = 1 \qquad (1)$$

S : total spin M_S : total spin projection
$s_1 = s_2 = \frac{1}{2}$ (nucleons) M_{S_1} and M_{S_2} are the nucleon spin projections

* Note $\langle M_{S_1} M_{S_2} | S M_S \rangle = 0 \;\; \forall M_{S_1}, M_{S_2}$ where $M_{S_1} + M_{S_2} \neq M_S$.

$$\sum_{T=|t_1-t_2|}^{t_1+t_2} \sum_{M_T=-T}^{T} |T M_T\rangle\langle T M_T| = 1 \qquad (2)$$

Example : do $\delta\tilde{U}\,\delta\tilde{U}^{\dagger}$ matrix elements for pair momentum distribution ignoring L, J.

$$n_{\lambda}^{NN'}(q, Q) \approx \sum_{M_S M_S' M_S'' M_S'''} \sum_{M_T'' M_T'''} \langle \vec{k}\, M_S'' M_T''\, M_S''' M_T''' | \delta\tilde{U} | \vec{q}\, M_S M_T\, M_S' M_T' \rangle \times$$

$$\langle \vec{q}\, m_s m_t m_s' m_t' | \delta \tilde{U}^{+} | \vec{R}\, m_s'' m_t'' m_s''' m_t''' \rangle \times \ldots \qquad (3)$$

Insert completeness relations four times

$\to$ $\delta\tilde{U}$, $\delta\tilde{U}^{+}$ diagonal in S, M_S $\Rightarrow$ 2 sums!

The small m_s dependence is only in the CG's
so we can do those immediately:

$$\sum_{m_s m_s'} \langle m_s m_s' | S M_S \rangle \langle S' M_S' | m_s m_s' \rangle = \delta_{S,S'} \delta_{M_S, M_S'}$$

$\uparrow$ $\delta\tilde{U}^{+}$ term $\qquad$ $\uparrow$ $\delta\tilde{U}$ term

and $$\sum_{m_s'' m_s'''} \langle m_s'' m_s''' | S M_S \rangle \langle S M_S | m_s'' m_s''' \rangle = \delta_{S,S'} \delta_{M_S M_S'}$$

$\uparrow$ $\delta\hat{U}$ term $\qquad$ $\uparrow$ $\delta\tilde{U}^{+}$ term

Leaves just $\sum_{S M_S}$

Now for isospin.

Including $\sum_{m_t m_t' m_t'' m_t'''}$ $\to$ $\sum_{T M_T}$ (same argument as above)

$$= \sum_{T} (2T+1)$$

Not including $\sum_{m_t m_t'}$: $\to$ $\sum_{T} [\ldots] \Big|_{M_T = m_t + m_t'}$

Decomposition of $\vec{q}$, $\vec{k}$

Average over $\Omega_{\vec{q}}$ and $\Omega_{\vec{k}}$

$$\int d\Omega_{\vec{q}} \langle q L' M_L' | \vec{q} \rangle \langle \vec{q} | q L'' M_L'' \rangle \sim \delta_{L'L''} \delta_{M_L' M_L''}$$

$$\int d\Omega_{\vec{k}} \langle \vec{k} | k L M_L \rangle \langle k L''' M_L''' | \vec{k} \rangle \sim \delta_{L L'''} \delta_{M_L M_L'''}$$

Put it all together and Eq. (3) reads

$$\begin{aligned}
\Lambda_\lambda^{\nu\nu'}(\vec{q},\vec{k}) \sim & \sum_{T} \sum_{S M_S} \sum_{L M_L} \sum_{L' M_L'} \sum_{J M_J} \sum_{J' M_J'} \langle L M_L S M_S | J M_J L S \rangle \times \\
& \langle k J L S T | \delta \tilde{U} | q J L' S T \rangle \langle J M_J L' S | L' M_L' S M_S \rangle \times \\
& \langle T M_T | m_t m_t' \rangle \langle m_t m_t' | T M_T \rangle \langle L' M_L' S M_S | J' M_J' L' S \rangle \times \\
& \langle q J' L' S T | \delta \tilde{U}^\dagger | k J' L S T \rangle \langle J' M_J' L S | L M_L S M_S \rangle \times \\
& \ldots \qquad (4)
\end{aligned}$$

Doing just the $\delta\tilde{U}$ or $\delta\tilde{U}^\dagger$ terms give

$$\frac{1}{2}\sum_{m_s m_s'} \langle \vec{q}\, m_s m_t m_s' m_t' | \delta\tilde{U} | \vec{q}\, m_s m_t m_s' m_t' \rangle \ldots$$

$$\rightarrow \frac{1}{2}\sum_{m_s m_s'} \sum_{S M_S} \sum_{T M_T} \sum_{L M_L} \sum_{L' M_L'} \sum_{J M_J} \langle \vec{q} | q L M_L \rangle \langle m_s m_s' | S M_S \rangle \langle m_t m_t' | T M_T \rangle \times$$

$$\langle M_L M_S | J M_J \rangle \langle q J L S T | \delta\tilde{U} | q J L' S T \rangle \langle J M_J | M_L' M_S \rangle \times$$

$$\langle T M_T | m_t m_t' \rangle \langle S M_S | m_s m_s' \rangle \langle q L' M_L' | \vec{q} \rangle \ldots$$

where we've imposed ME $\delta_{JJ'}, \delta_{SS'}, \ldots$

- Take S-waves only: $L = L' = 0$ $(M_L = M_L' = 0)$

$m_s m_s'$ dependence only in CG's.

$$\sum_{m_s m_s'} \langle m_s m_s' | S M_S \rangle \langle S M_S | m_s m_s' \rangle = 1.$$

Lastly, $M_T = m_t + m_t'$ (remove $\sum_{M_T}$)

$$\boxed{\begin{aligned}&\rightarrow \frac{1}{2}\sum_{S M_S} \sum_{T} \sum_{J M_J} \langle \vec{q} | q 0 0 \rangle \langle m_t m_t' | T M_T \rangle \langle 0 M_S | J M_J \rangle \times \\ &\langle q J 0 S T | \delta\tilde{U} | q J 0 S T \rangle \langle J M_J | 0 M_S \rangle \langle T M_T | m_t m_t' \rangle \langle q 0 0 | \vec{q} \rangle \ldots\end{aligned}}$$