Physics 880.05: PS#1 Solutions

1. MATLAB Sandbox.
   - Starting on the next page is a printout of a MATLAB session with sample results for parts (a) through (e).
   - Most of these are self-explanatory, but let's look at (e) more closely.

Wikipedia says that the Zassenhauss formula is

$$e^{t(X+Y)} = e^{tX} e^{tY} e^{-\frac{t^2}{2}[X,Y]} e^{\frac{t^3}{6}(2[Y,[X,Y]]+[X,[X,Y]])} \cdots$$

so we expect naively that approximating $e^{-\epsilon(T+V)}$ by $e^{-\epsilon T} e^{-\epsilon V}$ should make an error of order $t^2 \to \epsilon^2$ and multiplying N of them together could have an error $N\epsilon^2 \sim O(\epsilon)$. In fact, the trace $\mathrm{Tr}[(e^{-\epsilon(T+V)})^N]$ differs from $\mathrm{Tr}[((e^{-\epsilon T})(e^{-\epsilon V}))^N]$ by $O(\epsilon^2)$:

Check the formula:

$$e^{t(X+Y)} = 1 + t(X+Y) + \tfrac{1}{2}t^2 (X+Y)(X+Y) + \ldots = 1 + t(X+Y) + \tfrac{t^2}{2}(X^2+XY+YX+Y^2) + \ldots$$

$$e^{tX} e^{tY} = (1 + tX + \tfrac{t^2}{2}X^2 + \ldots)(1 + tY + \tfrac{t^2}{2}Y^2 + \ldots) = 1 + tX + tY + \tfrac{1}{2}t^2X^2 + t^2XY + \tfrac{1}{2}t^2Y^2 + \ldots$$

$$\Rightarrow e^{t(X+Y)} - e^{tX}e^{tY} = \tfrac{1}{2}t^2[XY+YX-2XY] = -\tfrac{1}{2}t^2[X,Y] \checkmark$$

But note that

$$e^{-\epsilon T} e^{-\epsilon V} e^{-\epsilon T} e^{-\epsilon V} = e^{-\epsilon T/2}\left[e^{-\epsilon T/2} e^{-\epsilon V} e^{-\epsilon T/2}\right] e^{-\epsilon T/2}$$

and the error is given by the terms in [ ]'s:

$$e^{+\frac{t}{2}X} e^{+tY} e^{+\frac{t}{2}X} = (1 + \tfrac{t}{2}X + \tfrac{1}{8}t^2X^2)(1 + tY + \tfrac{t^2}{2}Y^2)(1 + \tfrac{t}{2}X + \tfrac{t^2}{8}X^2) \ldots$$

$$= 1 + \tfrac{t}{2}X + tY + \tfrac{t}{2}X + \tfrac{t^2}{2}XY + \tfrac{1}{8}t^2X^2 + \tfrac{t^2}{2}Y^2 + \tfrac{t^2}{2}YX + \tfrac{t^2}{4}X^2 + \tfrac{t^2}{8}X^2$$

$$= 1 + \tfrac{t}{2}(X+Y) + \tfrac{t^2}{2}(X^2 + XY + YX + Y^2) + O(t^3)$$

so to this order it agrees with the exact result
$\Rightarrow$ the error is ϵ^3 rather than ϵ^2! [Note that the trace takes care of the terms at the end of the product].

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***
  The following is an output from a MATLAB session that steps through
  the tasks for problem 1 MATLAB Sandbox on 880.05 Problem Set #1.
***


                                          < M A T L A B >
                              Copyright 1984-2006 The MathWorks, Inc.
                                     Version 7.3.0.298 (R2006b)
                                          August 03, 2006


  To get started, select MATLAB Help or Demos from the Help menu.

>> % Use ";" to not print output and % for comments.

>> %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
>> % Part (a)
>> Atemp = rand(4) + i*rand(4)  % random complex

Atemp =

   0.9501 + 0.9355i   0.8913 + 0.0579i   0.8214 + 0.1389i   0.9218 + 0.2722i
   0.2311 + 0.9169i   0.7621 + 0.3529i   0.4447 + 0.2028i   0.7382 + 0.1988i
   0.6068 + 0.4103i   0.4565 + 0.8132i   0.6154 + 0.1987i   0.1763 + 0.0153i
   0.4860 + 0.8936i   0.0185 + 0.0099i   0.7919 + 0.6038i   0.4057 + 0.7468i

>> A_hermitian = (Atemp + Atemp')/2  % random hermitian

A_hermitian =

   0.9501             0.5612 - 0.4295i   0.7141 - 0.1357i   0.7039 - 0.3107i
   0.5612 + 0.4295i   0.7621             0.4506 - 0.3052i   0.3784 + 0.0945i
   0.7141 + 0.1357i   0.4506 + 0.3052i   0.6154             0.4841 - 0.2943i
   0.7039 + 0.3107i   0.3784 - 0.0945i   0.4841 + 0.2943i   0.4057

>> [V,D] = eig(A_hermitian)

V =

  -0.0290 + 0.3110i  -0.5163 + 0.4033i   0.2761 + 0.0211i   0.5531 - 0.3011i
   0.4158 + 0.2175i   0.3050 - 0.0352i  -0.5579 + 0.4103i   0.4523 - 0.0376i
   0.1932 - 0.6398i  -0.0678 - 0.1269i  -0.1576 - 0.5417i   0.4566 - 0.0770i
  -0.4853             0.6752             0.3543             0.4279


D =

   -0.3002         0         0         0
         0   -0.1455         0         0
         0         0    0.6839         0
         0         0         0    2.4952

>> V*A_hermitian*V'   % try this first --- it fails!

ans =

   0.4566 + 0.0000i   0.2258 - 0.2226i  -0.5267 + 0.3768i   0.1685 + 0.4034i
   0.2258 + 0.2226i   0.6172 - 0.0000i  -0.3776 + 0.8149i   0.3545 - 0.1078i
  -0.5267 - 0.3768i  -0.3776 - 0.8149i   1.2615 - 0.0000i  -0.2263 - 0.7257i
   0.1685 - 0.4034i   0.3545 + 0.1078i  -0.2263 + 0.7257i   0.3980 - 0.0000i

>> V'*A_hermitian*V  % diagonal ==> this order works!

ans =

  -0.3002 + 0.0000i   0.0000 - 0.0000i   0.0000 - 0.0000i  -0.0000 + 0.0000i
   0.0000 + 0.0000i  -0.1455 + 0.0000i   0.0000 - 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.6839 - 0.0000i  -0.0000 - 0.0000i
```



```
   0.0000 - 0.0000i   0.0000 - 0.0000i  -0.0000 + 0.0000i   2.4952

>> V*V'    % check: unitary matrix

ans =

   1.0000            -0.0000 + 0.0000i  -0.0000 - 0.0000i  -0.0000 + 0.0000i
  -0.0000 - 0.0000i   1.0000                  0 - 0.0000i   0.0000 + 0.0000i
  -0.0000 + 0.0000i        0 + 0.0000i   1.0000                  0 - 0.0000i
  -0.0000 - 0.0000i   0.0000 - 0.0000i        0 + 0.0000i   1.0000

>> V'*V   % check: unitary matrix

ans =

   1.0000             0.0000 + 0.0000i  -0.0000 + 0.0000i  -0.0000 - 0.0000i
   0.0000 - 0.0000i   1.0000            -0.0000 + 0.0000i   0.0000 + 0.0000i
  -0.0000 - 0.0000i  -0.0000 - 0.0000i   1.0000            -0.0000 - 0.0000i
  -0.0000 + 0.0000i   0.0000 - 0.0000i  -0.0000 + 0.0000i   1.0000

>> %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
>> % Part (b)
>> A = rand(5); B = rand(5); C = rand(5);   % three random matrices.

>> trace(A*B*C)   % original order

ans =

   17.8475

>> trace(B*C*A)   % a cyclic permutation of the original order

ans =

   17.8475

>> trace(B*A*C)  % not a cyclic permutation of the original order

ans =

   18.7847

>> trace(C*A*B)   % a cyclic permutation of the original order

ans =

   17.8475

>> %  Only traces of cyclic permutations are equal in general.
>>

>> %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
>> % Part (c)

>> exp(trace(logm(A)))  % note exp (not expm) but logm

ans =

    0.0176

>> det(B) - exp(trace(logm(B)))
Warning: Principal matrix logarithm is not defined for A with
         nonpositive real eigenvalues. A non-principal matrix
         logarithm is returned.
> In funm at 157
  In logm at 27

ans =
```

```
  -3.4694e-18 - 6.6127e-18i

>> % machine precision.  Note the warning if eigenvalues are not
>> %  positive definite since the branch cut for logm is on the negative
>> %  real axis, which is completely excluded here in the definition of
>> %  the principal logarithm.  The answer it gives could be from a
>> %  different branch from what you want (although it is likely to
>> %  be ok).

>> det(A_hermitian) - exp(trace(logm(A_hermitian)))
Warning: Principal matrix logarithm is not defined for A with
         nonpositive real eigenvalues. A non-principal matrix
         logarithm is returned.
> In funm at 157
  In logm at 27

ans =

        0 - 1.1417e-16i

>> % again, zero to machine precision and same warning.
>> %  Works for Hermitian matrices with warning if negative real eigenvalues.
>> eig(A_hermitian)

ans =

   -0.3002
   -0.1455
    0.6839
    2.4952

>> det(Atemp) - exp(trace(logm(Atemp)))

ans =

   4.0593e-16 - 1.3878e-16i

>> eig(Atemp)

ans =

   2.4638 + 1.8435i
   0.1831 - 0.2539i
   0.2946 + 0.0264i
  -0.2082 + 0.6178i

>> % No problem here!
>> det(A)

ans =

    0.0176

>> %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
>> % Part (d)
>> M = randn(3)

M =

   -0.4326    0.2877    1.1892
   -1.6656   -1.1465   -0.0376
    0.1253    1.1909    0.3273

>> exact = expm(M)

exact =

    0.3368    0.5690    1.0211
   -0.6421   -0.0005   -0.6648
```



```
   -0.5341    0.7626    1.1321

>> % In the following, we could also initialize sum = 0 and loop from 0 to 5.
>> sum = eye(3)

sum =

     1     0     0
     0     1     0
     0     0     1

>> for i = 1:5
     sum = sum + (M^i)/factorial(i)
   end

sum =

    0.5674    0.2877    1.1892
   -1.6656   -0.1465   -0.0376
    0.1253    1.1909    1.3273


sum =

    0.4959    0.7686    1.1212
   -0.3529    0.2487   -1.0125
   -0.8730    0.7212    1.4330


sum =

    0.2364    0.5510    1.0794
   -0.8024   -0.1634   -0.6035
   -0.4639    0.8469    1.0546


sum =

    0.3538    0.5823    1.0008
   -0.5693    0.0442   -0.6998
   -0.5723    0.7276    1.1441


sum =

    0.3312    0.5631    1.0234
   -0.6610   -0.0130   -0.6522
   -0.5210    0.7701    1.1251

>> expm(M)

ans =

    0.3368    0.5690    1.0211
   -0.6421   -0.0005   -0.6648
   -0.5341    0.7626    1.1321

>> % About two significant digits by 5 terms; none at 2.
>> % Try another:
>> M = randn(3)

M =

    0.1746   -0.5883    0.1139
   -0.1867    2.1832    1.0668
    0.7258   -0.1364    0.0593

>> sum = eye(3)
```

```
sum =

     1     0     0
     0     1     0
     0     0     1

>> for i = 1:5
     sum = sum + (M^i)/factorial(i)
   end

sum =

    1.1746   -0.5883    0.1139
   -0.1867    3.1832    1.0668
    0.7258   -0.1364    1.0593


sum =

    1.2862   -1.2897   -0.1865
   -0.0197    5.5485    2.2522
    0.8234   -0.5028    1.0296


sum =

    1.2636   -1.8083   -0.4376
    0.1296    7.1832    3.1231
    0.8447   -0.7873    0.9025


sum =

    1.2413   -2.0794   -0.5803
    0.2178    8.0237    3.5762
    0.8359   -0.9413    0.8253


sum =

    1.2299   -2.1913   -0.6404
    0.2553    8.3680    3.7629
    0.8301   -1.0055    0.7913

>> expm(M)  % exact

ans =

    1.2242   -2.2444   -0.6691
    0.2733    8.5308    3.8514
    0.8269   -1.0360    0.7748

>> % Some elements have 1-2 digits at 2 terms, 2-3 digits at 5
>> %  Of course, the number of terms needed to get a given accuracy
>> %  will be highly dependent on the matrix.

>> %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
>> % Part (e)
>> T = rand(5) + i*rand(5);  T = (T + T')/2

T =

   0.3759             0.4649 + 0.1725i   0.3069 + 0.1377i   0.6512 - 0.2400i   0
.7106 + 0.1286i
   0.4649 - 0.1725i   0.8447             0.6363 + 0.1032i   0.7762 + 0.0878i   0
.7152 + 0.2440i
   0.3069 - 0.1377i   0.6363 - 0.1032i   0.5692             0.4835 - 0.1691i   0
.3158 + 0.2572i
   0.6512 + 0.2400i   0.7762 - 0.0878i   0.4835 + 0.1691i   0.6555             0
```



```
.4028 + 0.1974i
   0.7106 - 0.1286i   0.7152 - 0.2440i   0.3158 - 0.2572i   0.4028 - 0.1974i   0
.6552

>> V = rand(5) + i*rand(5);  V = (V + V')/2

V =

   0.3603             0.5598 - 0.2507i   0.3468 - 0.1112i   0.7566 - 0.0051i   0
.3910 + 0.0030i
   0.5598 + 0.2507i   0.7009             0.7983 + 0.2319i   0.6763 - 0.1808i   0
.5811 - 0.2638i
   0.3468 + 0.1112i   0.7983 - 0.2319i   0.8030             0.1687 - 0.2590i   0
.9534 - 0.2718i
   0.7566 + 0.0051i   0.6763 + 0.1808i   0.1687 + 0.2590i   0.8735             0
.2927 - 0.0639i
   0.3910 - 0.0030i   0.5811 + 0.2638i   0.9534 + 0.2718i   0.2927 + 0.0639i   0
.5534

>>  % calculate the exact answer
>>  beta = 1; format long;
>>  exact = trace(expm(-beta*(T+V)))

exact =

  4.191605016071577 + 0.000000000000000i

>> % Generate a series of approximations.  (We could do this with a loop.)
>> N = 10; eps = beta/N; approx1 = trace((expm(-eps*T)*expm(-eps*V))^N)

approx1 =

  4.192843103435818 + 0.000000000000000i

>> N = 100; eps = beta/N; approx2 = trace((expm(-eps*T)*expm(-eps*V))^N)

approx2 =

  4.191617489855238 - 0.000000000000000i

>> N = 1000; eps = beta/N; approx3 = trace((expm(-eps*T)*expm(-eps*V))^N)

approx3 =

  4.191605140820158 + 0.000000000000000i

>> % Compute the relative error for each approximation.
>> (exact - approx1)/exact

ans =

     -2.953730992051764e-04 - 1.173449075275326e-17i

>> (exact - approx2)/exact

ans =

     -2.975896730074984e-06 + 4.372088718803882e-17i

>> (exact - approx3)/exact

ans =

     -2.976153067010590e-08 - 1.048141094721094e-17i

>> % So error scales like eps^2, because a factor of 10 reduction in
>> %  eps yields .01 improvement
>> %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

2. Stochastic Variational Method Revisited.

The estimate for the energy is

$$E_{estimate} = \frac{\sum_{ij} c_i c_j \langle \varphi_i | H | \varphi_j \rangle}{\sum_{i'j'} c_{i'} c_{j'} \langle \varphi_{i'} | \varphi_{j'} \rangle} \equiv \frac{\sum_{ij} c_i c_j H_{ij}}{\sum_{i'j'} c_{i'} c_{j'} B_{i'j'}}$$

· For this to be stationary with respect to variations in the coefficients $\{c_i\}$, we need

$$\frac{\partial E_{estimate}}{\partial c_k} = 0 \quad \text{for all } k.$$

$$\Rightarrow \quad 0 = \frac{\sum_j c_j H_{kj} + \sum_i c_i H_{ik}}{\sum_{i'j'} c_{i'} c_{j'} B_{i'j'}} - \frac{\left(\sum_{ij} c_i c_j H_{ij}\right)\left(\sum_{j'} c_{j'} B_{kj'} + \sum_{i'} c_{i'} B_{i'k}\right)}{\left(\sum_{i'j'} c_{i'} c_{j'} B_{i'j'}\right)^2}$$

• Now $\sum_{i'j'} c_{i'} c_{j'} B_{i'j'} > 0$ and $H_{ij} = H_{ji}$, $B_{ij} = B_{ji}$ so we can cancel a denominator factor and an overall 2:

$$\Rightarrow \sum_j H_{kj} c_j - \underbrace{\left(\frac{\sum_{ij} c_i c_j H_{ij}}{\sum_{i'j'} c_{i'} c_{j'} B_{i'j'}}\right)}_{E_{estimate}} \sum_j B_{kj} c_j = 0$$

or

$$\boxed{\sum_j H_{kj} c_j - E \sum_j B_{kj} c_j = 0} \quad \text{for all } k.$$

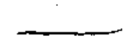
will give up $E = E_{estimate}$. This is precisely the generalized eigenvalue problem,

PS#1-3

3. Model Partition Function

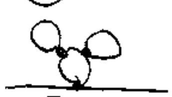
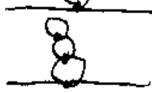
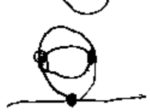
a) Here we use $Z = N\int d\xi\, e^{-\xi^2/2 - \lambda\xi^4/4}$ where the normalization N will drop out of $\langle \xi^2 \rangle$.

$$\boxed{\langle \xi^2 \rangle = \frac{\int d\xi\ \xi^2 e^{-(a\xi^2/2 + \lambda\xi^4/4)}}{\int d\xi\ e^{-(a\xi^2/2 + \lambda\xi^4/4)}}}$$

$\Rightarrow$  $\frac{1}{a}$

 $-\frac{\lambda}{4}\,4! = -6\lambda$

The Feynman rules to find the λ^3 contribution say to sum the contributions from all connected diagrams with two external lines and three vertices (one for each λ). The disconnected diagrams from the numerator cancel with those from the denominator.

i) diagrams	ii) symmetry factors	iii) contribution
	$\frac{1}{2^3}\times 1\times 1$	$\frac{1}{8}(-6\lambda)^3\, 1/a^7$
	$\frac{1}{2^2}\times\frac{1}{2}\times 1$	$\frac{1}{8}(-6\lambda)^3\, 1/a^7$
	$\frac{1}{2^2}\times\frac{1}{2}\times 1$	$\frac{1}{8}(-6\lambda)^3\, 1/a^7$
	$\frac{1}{2}\times\frac{1}{3!}\times 1$	$\frac{1}{12}(-6\lambda)^3\,\frac{1}{a^7}$
	$\frac{1}{2}\times\frac{1}{3!}\times 1$	$\frac{1}{12}(-6\lambda)^3\,\frac{1}{a^7}$
	$\frac{1}{2^2}\times 1\times\frac{1}{2}$	$\frac{1}{8}(-6\lambda)^3\,\frac{4}{a^7}$
	$\frac{1}{2}\times(\frac{1}{2})^2\times 1$	$\frac{1}{8}(-6\lambda)^3\,\frac{1}{a^7}$
	$1\times(\frac{1}{2})^2\times 1$	$\frac{1}{4}(-6\lambda)^3\,\frac{1}{a^7}$
	$\frac{1}{2}\times\frac{1}{2}\times 1$	$\frac{1}{4}(-6\lambda)^3\,\frac{1}{a^7}$
	$1\times\frac{1}{3!}\times\frac{1}{2}$	$\frac{1}{2}(-6\lambda)^3\,\frac{1}{a^7}$

total $\boxed{-297\lambda^3/a^7}$

(The symmetry factors are in the usual order.)

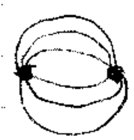
PS#1-4

3b) Now $\boxed{Z=\int d\xi\ e^{(a\xi^2/2+\alpha\xi^6/6)}}$

i) The rule —— $=\frac{1}{a}$ follows exactly as before.
New we have a new vertex with 6 legs $\Rightarrow$  $\left(-\frac{\alpha}{6}\right)6! = -5!\alpha$
(you can see why $-\frac{\alpha'}{6!}\xi^6$ would be smarter than $-\alpha\xi^6/6!$)
Otherwise, we calculate graphs as before.

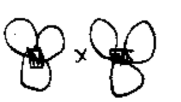
ii) Z_α/Z_0 includes both connected and disconnected closed diagrams (no external legs). Order α has one α vertex and order α^2 has two vertices: [note: order $\alpha^0 = 1$ trivially.]

 $\left(\frac{1}{2}\right)^3\times\frac{1}{3!}\times 1\times\left(-\frac{\alpha}{6}6!\right)\frac{1}{a^3} = \boxed{-\frac{5}{2}\frac{\alpha}{a^3}}$ $O(\alpha)$

$+$  $1\times\frac{1}{6!}\times\frac{1}{2}\times\left(-\frac{\alpha}{6}6!\right)^2\frac{1}{a^6} = 10\frac{\alpha^2}{a^6}$

$+$  $\left(\frac{1}{2}\right)^2\times\frac{1}{4!}\times\frac{1}{2}\times\left(-\frac{\alpha}{6}6!\right)^2\frac{1}{a^6} = 75\frac{\alpha^2}{a^6}$

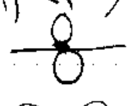
$+$  $\left(\frac{1}{2}\right)^4\times\left(\frac{1}{2!}\right)^3\times\frac{1}{2}\times\left(-\frac{\alpha}{6}6!\right)^2\frac{1}{a^6} = \frac{225}{4}\frac{\alpha^2}{a^6}$

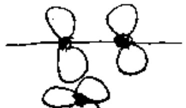
$+$  $\left(\frac{1}{2}\right)^6\times\left(\frac{1}{3!}\right)^2\times\frac{1}{2}\times\left(-\frac{\alpha}{6}6!\right)^2\frac{1}{a^6} = \frac{25}{8}\frac{\alpha^2}{a^6}$

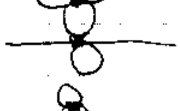
$= \boxed{\frac{1155}{8}\frac{\alpha^2}{a^6}}$ $O(\alpha^2)$

These results agree with Mathematica.

iii) $\langle\xi^2\rangle$ has two external lines and one ($O(\alpha)$) or two ($O(\alpha^2)$) vertices:

 $\left(\frac{1}{2}\right)^2\times\frac{1}{2!}\times 1\times\left(-\frac{\alpha}{6}6!\right)\frac{1}{a^4} = \boxed{-15\,\alpha/a^4}$ $O(\alpha)$

 $\left(\frac{1}{2}\right)^4\times\left(\frac{1}{2!}\right)^2\times 1\times\left(-\frac{\alpha}{6}6!\right)^2\frac{1}{a^7}$

 $\left(\frac{1}{2}\right)^3\times\left(\frac{1}{2!}\right)^2\times 1\times\left(-\frac{\alpha}{6}6!\right)^2\frac{1}{a^7}$

 $\frac{1}{2}\times\frac{1}{4!}\times 1\times\left(-\frac{\alpha}{6}6!\right)^2\frac{1}{a^7}$

 $1\times\frac{1}{5!}\times 1\times\left(-\frac{\alpha}{6}6!\right)^2\frac{1}{a^7}$

 $\left(\frac{1}{2}\right)^2\times\frac{1}{3!}\times 1\times\left(-\frac{\alpha}{6}6!\right)^2\frac{1}{a^7}$

$= \left(\frac{1}{64}+\frac{1}{32}+\frac{1}{48}+\frac{1}{120}+\frac{1}{24}\right)120^2\frac{\alpha^2}{a^7}$

$\boxed{= 1695\frac{\alpha^2}{a^7}}$

PS#1-5

3c) i) If we multiply and divide O_n by $\boxed{\dfrac{\int d\xi_1 \, e^{-[\,]/z_0}}{\int d\xi_1 \, e^{-[\,]/z_0}}}$, then

$$\boxed{O_n = \frac{\left(\int d\xi_1 \, \xi_1^2 e^{-[\;]/z_0}\right)}{\left(\int d\xi_1 \, e^{-[\;]/z_0}\right)} \times \underbrace{\left(\int d\xi_1 \, e^{-[\;]/z_0}\right)\left(\int d\xi_2 \, e^{-[\;]/z_0}\right) \cdots \left(\int d\xi_n \, e^{-[\;]/z_0}\right)}_{n \text{ copies}}}$$

so the n dependence is just in the n copies. If we set $n=0$, then we are left with the first ratio, which is the definition of $\langle \xi^2 \rangle$.

ii) The external legs come only from the ξ_1^2 term in the numerator of the ratio, so the index 1 is all that appears.

iii) As discussed in class for the partition function, each connected piece can only have the same number. If disconnected, then there will be a factor of n for each disconnected piece (since $1, 2, \ldots n$ copies) $\Rightarrow$ the n^0 parts are exactly the connected ones.

iv) changing ξ^2 to a general $O(\xi)$ operator changes nothing in any of the previous parts (only the <u>number</u> of external legs with change). $\Rightarrow$ the argument works for other operators.