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Review: Contour Integrals

- We will very often find ourselves doing integrals over frequency (rather than time).
  - At zero temperature, these are integrals from $-\infty$ to $+\infty$, which are usually evaluated as contour integrals.
  - At finite temperature, the "time" (really τ) integration is over a finite interval (e.g. 0 to β), which leads to a frequency sum rather than integral. These require additional techniques, which we won't describe here.

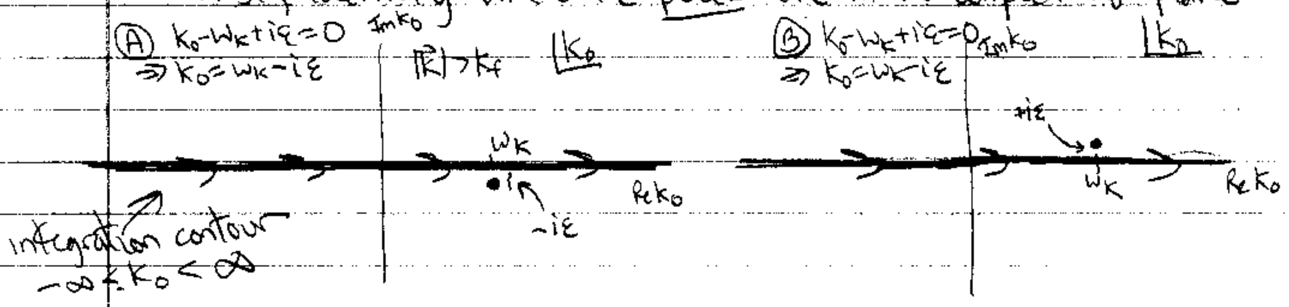
- So let's review by example the sort of contour integral we need.

- The lowest order energy:  will involve this integral:

$$\boxed{\int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \Big[\underbrace{\frac{\theta(|\vec{k}|-k_F)}{k_0-\omega_k+i\varepsilon}}_{(A)} + \underbrace{\frac{\theta(k_F-|\vec{k}|)}{k_0-\omega_k-i\varepsilon}}_{(B)} \Big] e^{ik_0\eta} = \frac{1}{2\pi}(2\pi i)\,\theta(k_F-|\vec{k}|)}$$

let's "deconstruct" this result.

- First, identify where the poles are in the complex k_0 plane



- 2nd, consider the behavior of the integrand in each half plane:

upper half plane: $\boxed{\mathrm{Im}\,k_0>0 \Rightarrow e^{ik_0\eta} = e^{i(\mathrm{Re}\,k_0)\eta}\, e^{-(\mathrm{Im}\,k_0)\eta} \xrightarrow[|k_0|\to\infty]{} 0}$

lower half plane: $\boxed{\mathrm{Im}\,k_0<0 \Rightarrow e^{ik_0\eta} = e^{i(\mathrm{Re}\,k_0)\eta}\, e^{+(\mathrm{Im}\,k_0)\eta} \xrightarrow[|k_0|\to\infty]{} \infty}$

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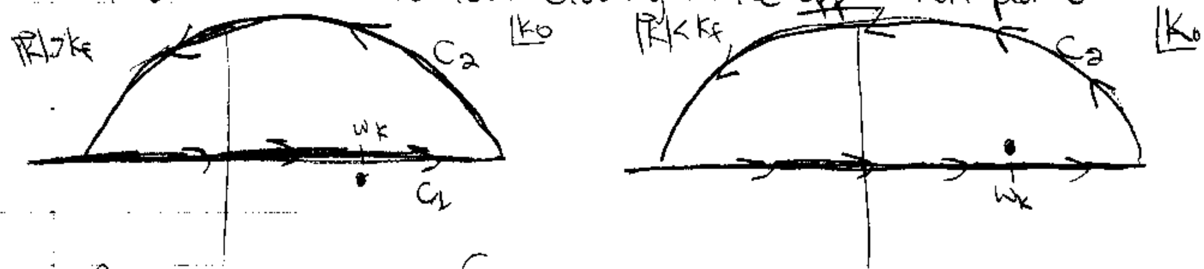
- 3rd, based on these results extends the integration to a <u>closed contour</u> (to which we can apply Cauchy's magic!)

Here: Draw a contour closing in the <u>upper</u> half plane



The contour integral $\oint_{C_1+C_2}$ is given by

$$\boxed{\oint_{C_1+C_2} \frac{dk_0}{2\pi}\left(\frac{\theta(|\vec{k}|-k_f)}{k_0-w_k+i\epsilon}+\frac{\theta(k_f-|\vec{k}|)}{k_0-w_k-i\epsilon}\right)e^{ik_0\eta} = 2\pi i \times \left(\begin{array}{c}\text{sum of residues}\\ \text{at \underline{enclosed} poles}\end{array}\right)}$$

$+2\pi i$ because counterclockwise contour ($-2\pi i$ if clockwise contour)

But this integral is also the sum of the integral we want (contour C_1) and one we can evaluate separately (C_2).

- In most case we will consider, the integral on C_2 will <u>vanish</u>. Here it does because of the $e^{ik_0\eta}$ factor.

- 4th evaluate the residues.

$\theta(|\vec{k}|-k_f) \Rightarrow$ no poles $\Rightarrow \oint(\quad) = 0.$

$\theta(k_f-|\vec{k}|) \Rightarrow$ residue is $\frac{\theta(k_f-|\vec{k}|)}{2\pi} \Rightarrow \boxed{\oint_{C_1+C_2}(\quad) = \frac{1}{2\pi}(2\pi i)\,\theta(k_f-|\vec{k}|)}$

so the final result is

$$\boxed{\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\Big[\qquad\qquad\Big]e^{ik_0\eta} = i\,\theta(k_f-|\vec{k}|)}$$ as advertised.

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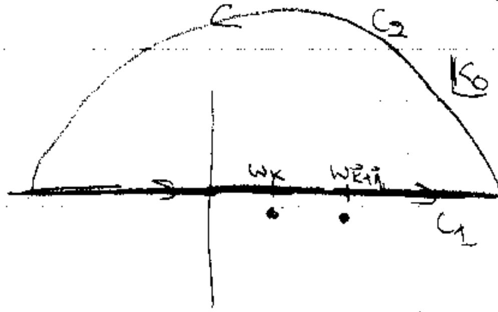
Now suppose we have

$$\boxed{\int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \frac{\theta(|\vec{k}|-k_f)}{k_0-\omega_k+i\varepsilon} \frac{\theta(|\vec{k}+\vec{q}|-k_f)}{k_0-\omega_{\vec{k}+\vec{q}}+i\varepsilon}}$$



① the poles are in the lower half plane, as shown.

② The integral converges in either half plane (by this we mean that the half-circle integral C_2 as $|k_0|\to\infty$ is finite $\Rightarrow$ vanishes here)

$$\text{for } |k_0| \text{ large: } \boxed{\oint_{C_2} \frac{dk_0}{2\pi}\frac{1}{k_0^2} \xrightarrow{k_0=r_0e^{i\theta}} \int_0^{\pi} \frac{1}{2\pi}\frac{r_0e^{i\theta}\,d\theta}{r_0^2e^{2i\theta}} = \frac{1}{2\pi r_0}\int_0^{\pi} d\theta\, e^{-i\theta} \propto \frac{(\text{constant})}{r_0} \xrightarrow[r_0\to\infty]{} 0}$$

③ Since we can close in the upper half plane, choose that one (as in the figure) because there are no poles enclosed.

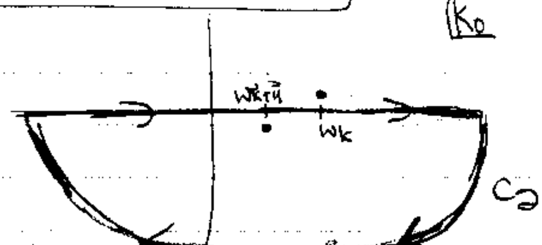
④ no residues $\Rightarrow$ $\boxed{\text{integral is zero}}$!

• If we wanted to work hard to get the same answer, we could close in the lower half plane

$$\Rightarrow \boxed{\frac{1}{2\pi}(-2\pi i)\left(\frac{1}{\omega_{|\vec{k}+\vec{q}|}-\omega_k} + \frac{1}{\omega_k-\omega_{\vec{k}+\vec{q}}}\right) = 0}$$

Finally, consider

$$\boxed{\int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \frac{\theta(k_f-|\vec{k}|)}{k_0-\omega_k-i\varepsilon} \frac{\theta(|\vec{k}+\vec{q}|-k_f)}{k_0-\omega_{\vec{k}+\vec{q}}+i\varepsilon}}$$



① poles in opposite half planes, as shown

② converges in either half plane

③ choose to close in lower

④ $$\Rightarrow \boxed{\frac{1}{2\pi}(-2\pi i)\frac{1}{\omega_{\vec{k}+\vec{q}}-\omega_k-i\varepsilon}\,\theta(k_f-|\vec{k}|)\,\theta(|\vec{k}+\vec{q}|-k_f)}$$ That's it!

note!

• You should check that the same result is obtained when we choose to close in the upper half plane.

* note that we can drop the $i\varepsilon$ for $q>0$

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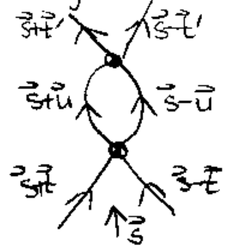
Time to return to the beachball diagram:

$$\Rightarrow \boxed{\mathcal{E}_2 = -4\lambda^2 M\, g(g-1)\, k_F^7 \times \int \frac{d^3s}{(2\pi)^3}\int\frac{d^3t}{(2\pi)^3}\int\frac{d^3u}{(2\pi)^3}\,\frac{1}{u^2-t^2-i\eta} \times \theta\left(1-|\vec{s}+\vec{t}|\right)\theta\left(1-|\vec{s}-\vec{t}|\right)\theta\left(|\vec{s}+\vec{u}|-1\right)\theta\left(|\vec{s}-\vec{u}|-1\right)}$$

- Because the θ functions restrict the sum and difference of $\vec{s}$ and $\vec{t}$ to be less than 1, both of these integrals are bounded.
- However, the $\vec{u}$ integration runs to $|\vec{u}|\to\infty$. For large u, $\theta(|\vec{s}+\vec{u}|-1)=\theta(|\vec{s}-\vec{u}|-1)=1$ and $\frac{1}{u^2-t^2}\to\frac{1}{u^2}$, so the integral goes like $\int^\infty \frac{u^2du}{u^2}\sim\int^\infty du\to\infty$.

- To analyze the divergence and see how to renormalize it, we will "unfold" the diagram into the closely related scattering diagram:



- We've labeled the final legs with t', but to generate the energy, we set $\vec{t}=\vec{t}'$ or $\vec{t}=-\vec{t}'$ and close the legs.
- The variables in the $\mathcal{E}_2$ expression (s,t,u) are dimensionless, but we can always put back the powers of k_F and make them momenta again.

- In the scattering case, the intermediate state, specified by momenta $\vec{s}+\vec{u}$ and $\vec{s}-\vec{u}$ is unrestricted. In the contribution to $\mathcal{E}_2$, these momenta must be greater than k_F. But the divergence is for large $\vec{u}$, where there is no restriction.
  - $\therefore$ If we fix the divergence for the scattering case, we'll fix in for finite density

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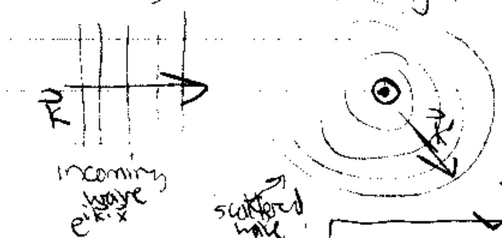
- The last conclusion is a general result about <u>ultraviolet</u> (short wavelength ⇒ large momentum) divergences. The Pauli blocking effects at finite density are low momentum (e.g. compared to k_F) effects and so the same UV divergences in free space will appear in the <u>same form</u> at finite density.
  ⇒ if we renormalize in free space, we should automatically renormalize in the many-body system. (This <u>better</u> work, because we have no choice!)

- So considering the scattering problem . . . first a one-body picture:

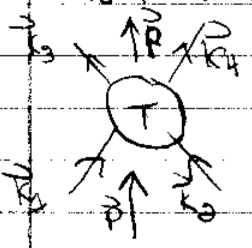
lab frame:



incoming $\vec{k}$ scatters to $\vec{k}'$
elastic: $|\vec{k}| = |\vec{k}'|$

Scattering wave function $\boxed{\psi_{\vec{k}}^{(+)}(\vec{x}) \sim e^{i\vec{k}\cdot\vec{x}} + f(\vec{k}',\vec{k})\frac{e^{ikx}}{x}}$ as $x \to \infty$

- We consider the scattering of two particles with potential V, and use k_F to scale the momenta (so we make a connection to the finite density problem — k_F doesn't have any particular meaning!).



- The total momentum is $\boxed{\vec{P} = \vec{k}_1 + \vec{k}_2 = 2k_F\vec{s}}$

- The relative momenta are

$$\boxed{\vec{k} \equiv \tfrac{1}{2}|\vec{k}_1 - \vec{k}_2|} \quad \boxed{\vec{k}' = \tfrac{1}{2}|\vec{k}_3 - \vec{k}_4|} \Rightarrow \begin{array}{l}\vec{k}_1, \vec{k}_2 = \vec{P} \pm \vec{k} \\ \vec{k}_3, \vec{k}_4 = \vec{P} \pm \vec{k}'\end{array}$$

which correspond to the variables in the equivalent one-body picture above.

- Galilean invariance requires the interaction between the particles to be independent of their center-of-mass momentum $\vec{P}$. That is, you cannot tell what reference frame you are in.
- In contrast, in a finite density system, the other particles define a preferred reference frame.

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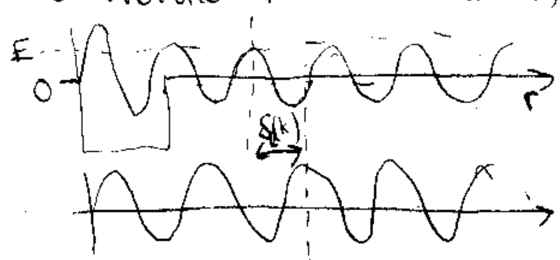
- The scattering amplitude f for elastic scattering depends only on the magnitude $k \equiv |\vec{k}| = |\vec{k}'|$ and the angle between $\vec{k}$ and $\vec{k}'$ (the scattering angle θ). The differential cross section $d\sigma/d\Omega$ is basically $|f|^2$.
  - f can be expanded in Legendre polynomials in $\cos\theta$:

$$\boxed{\begin{aligned} f(k,\theta) &= \sum_{l=0}^{\infty} \frac{2l+1}{k} e^{i\delta_l(k)} \sin\delta_l(k) P_l(\cos\theta) \\ &= \sum_{l=0}^{\infty} \frac{2l+1}{k\cot\delta_l - ik} P_l(\cos\theta) \end{aligned}}$$

where $\delta_l(k)$ is the "phase shift" for the l^{th} partial wave.

- Recall that one can determine $\delta_l(k)$ by comparing the phase of the radial wavefunction outside the range of the potential to the phase of a free wavefunction (hence the name!)



- We'll use a slightly different normalization, indicated by T:

$$\boxed{T(k, \cos\theta) = \frac{4\pi}{M} \sum_{l=0}^{\infty} \frac{2l+1}{k\cot\delta_l(k) - ik} P_l(\cos\theta)}$$

- For short-range interaction at low momentum, $k^{2l+1}\cot\delta_l$ has a power series expansion, called the "effective range expansion":

$$\boxed{k\cot\delta_0(k) = -\frac{1}{a_s} + \frac{1}{2} r_s k^2 + \ldots} \qquad \boxed{k^3 \cos\delta_1(k) = -\frac{3}{a_p^3} + \ldots}$$

which defines the s-wave ($l=0$) scattering length a_s and effective range r_s, and the p-wave ($l=1$) scattering length a_p.

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- Unless the system has a bound-state (for given l) near zero energy (either just bound, or just missing being bound), the expected size of a_s, r_s, a_p is about R, the range of the potential.
  - For example, for "hard spheres" of radius R (ie, the potential is zero for $r>R$ but is infinitely repulsive at $r=R$), $a_s = a_p = R$ and $r_s = 2R/3$.
  - We define Λ to be the momentum scale $\Lambda = 1/R$.

- If we consider momenta $k \ll \Lambda$, then we can expand $T(k, \cos\theta)$ in a perturbation series using the effective range expansion for $k\cot\delta_l$:

$$\boxed{\begin{aligned} T(k,\cos\theta) = &-\frac{4\pi a_s}{M}\left[1 - i a_s k + (a_s r_s/2 - a_s^2)k^2\right] \\ &- \frac{4\pi a_p^3}{M} k^2\cos\theta + \mathcal{O}(k^3/\Lambda^3) \end{aligned}}$$

- So at very low momentum, the scattering is described to good accuracy by specifying just the scattering length a_s.
  - The next correction requires r_s and a_p, and so on.

** • There are an infinite number of potentials V (which means $\langle \vec{k}'|V|\vec{k}\rangle$) that have the same $a_s, r_s, \ldots$ (effective range parameters). (We mean, a finite number of the same parameters.)
  $\Rightarrow$ we can reproduce this momentum expansion systematically (order-by-order in k^2/Λ^2)

- In an effective field theory (EFT), we carry out this expansion using a local Lagrangian density to define the EFT.
  - For low momentum $k \ll \Lambda = \frac{1}{R}$, all interactions in the EFT are short-ranged and we have only contact interactions (eg. delta functions) between the fermions.

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The possible terms in the Lagrangian and constrained by some symmetries: Galilean, parity, and time-reversal invariances.

- That means that we only include terms that don't change (up to total derivatives) under these transformations.
- The Lagrangian is organized by the number of $\vec{\nabla}$'s that appear in an interaction term. Each $\vec{\nabla}$ acting on a field gives a momentum, so more $\vec{\nabla}$'s means higher order in the momentum expansion.

- Here is a Lagrangian we can use:

$$\boxed{\mathcal{L}_{EFT} = \psi^\dagger\left[i\frac{\partial}{\partial t} + \frac{\vec{\nabla}^2}{2m}\right]\psi - \frac{C_0}{2}(\psi^\dagger\psi)^2 + \frac{C_2}{16}\left[(\psi\psi)^\dagger(\psi\overleftrightarrow{\nabla}^2\psi) + \text{h.c.}\right] + \frac{C_2'}{8}(\psi\overleftrightarrow{\nabla}\psi)^\dagger\cdot(\psi\overleftrightarrow{\nabla}\psi) - \frac{D_0}{6}(\psi^\dagger\psi)^3 + \ldots}$$

where "h.c." mean Hermitian conjugate (so that the combined sum in [ ]'s is Hermitian) and $\overleftrightarrow{\nabla}$ is the Galilean invariant derivative:

$$\boxed{\overleftrightarrow{\nabla} = \overleftarrow{\nabla} - \overrightarrow{\nabla}}$$

Using this derivative ensures that the Lagrangian is unchanged if all of the particle momenta are "boosted" by $\vec{v}$: $\vec{p} \to \vec{p} + m\vec{v}$.

- Our favorite δ-function Lagrangian corresponds to $C_0 = \lambda$, $C_2 = C_2' = D_0 = \ldots = 0$.
- This is a general, but not unique, form of the Lagrangian for short-range, spin-independent, interactions. One can perform "field redefinitions" which are basically changes of variable for ψ, which lead to different, but physically equivalent, forms (eg. more time derivatives.)
- We've including the leading ($\Rightarrow$ no derivatives) <u>three-body</u> interaction as well.

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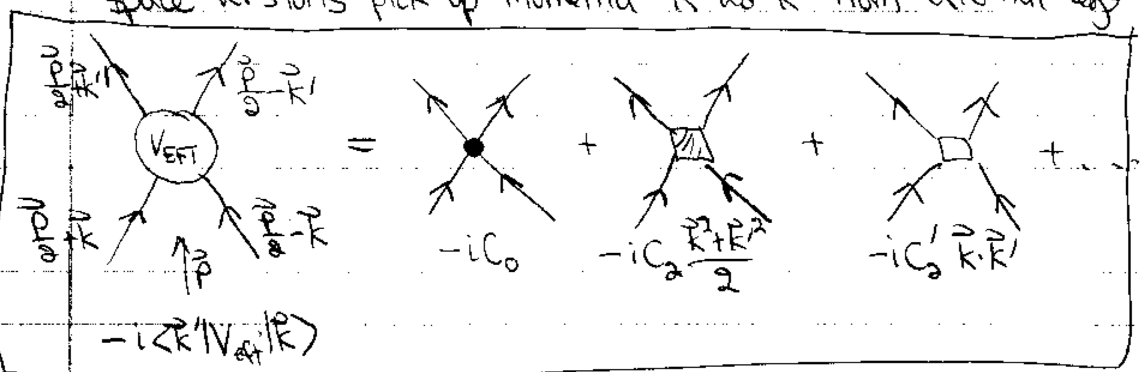
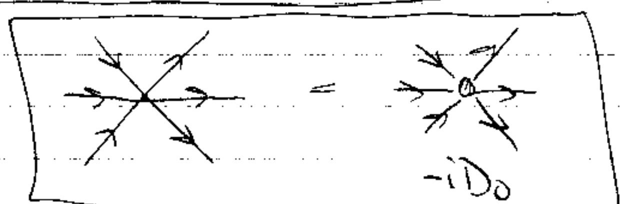
- The Feynman rules for the vertices beyond C_0 can be derived from the path integral expression. The momentum space versions pick up momenta $\vec{k}$ and $\vec{k}'$ from external legs





- Note that $\vec{p}$ drops out $\Rightarrow$ this is Galilean invariance at work!
- The terms involving only four fermion fields are simple polynomial vertices, equivalent to a momentum expansion of a potential V_{EFT} for particle-particle scattering:

$$\boxed{\langle\vec{k}'|V_{EFT}|\vec{k}\rangle = C_0 + C_2(\vec{k}'^2+\vec{k}^2)/2 + C_2'(\vec{k}'\cdot\vec{k}) + \ldots}$$

- For example, if the "true" underlying physics were due to a Yukawa potential (the tree-level exchange of a meson ϕ with mass Λ and coupling g, for example):

$$\boxed{V(\vec{x}) = -\frac{g^2}{4\pi}\frac{e^{-\Lambda x}}{x}}$$

with $\boxed{x \equiv |\vec{x}|}$, then

$$\begin{aligned}\langle\vec{k}'|V|\vec{k}\rangle &= \int d^3x\,\langle\vec{k}'|\vec{x}\rangle V(\vec{x})\langle\vec{x}|\vec{k}\rangle = \frac{-g^2}{4\pi}\frac{1}{(2\pi)^3}\int d^3x\, e^{-i(\vec{k}'-\vec{k})\cdot\vec{x}}\frac{e^{-\Lambda x}}{x} \\ &= -\frac{g^2}{4\pi}\frac{1}{(2\pi)^2}\int_0^\infty x^2dx\,\frac{e^{-\Lambda x}}{x}\int_{-1}^{1}d\chi\, e^{-iqx\chi} = \frac{-g^2}{4\pi}\frac{2}{(2\pi)^2}\int_0^\infty x^2dx\,\frac{\sin qx}{qx}\frac{e^{-\Lambda x}}{x}\end{aligned}$$

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or, finally,

$$\boxed{\langle \vec{k}' | V | \vec{k} \rangle = -\left(\frac{g^2}{4\pi}\right)\frac{2}{(2\pi)^2}\frac{1}{\Lambda^2+q^2} = -\left(\frac{g^2}{4\pi}\right)\frac{2}{(2\pi)^2}\frac{1}{\Lambda^2+(\vec{k}'-\vec{k})^2}}$$

where $\vec{q}$ is the momentum transfer.

- From the calculation of $\langle \vec{k}' | V | \vec{k} \rangle$ in this case, we can generalize that any local potential (ie., V only depends on $\vec{x}$) will have $\langle \vec{k}' | V | \vec{k} \rangle$ be a function of $\vec{q} = \vec{k}' - \vec{k}$ alone.
  - More generally, $\boxed{\langle \vec{x}' | V | \vec{x} \rangle \neq V(\vec{x})\,\delta^3(\vec{x}-\vec{x}')}$ and therefore depends separately on $\vec{k}$ and $\vec{k}'$.
  - The Schrodinger equation:

$$\boxed{(\hat{H}_0 + \hat{V})|\psi\rangle = E|\psi\rangle}$$

in coordinate space is

$$\int d^3x' \left(\underbrace{\langle \vec{x} | \hat{H}_0 | \vec{x}' \rangle}_{= -\frac{\nabla^2}{2m}\delta^3(\vec{x}-\vec{x}')} \langle x' | \psi \rangle + \langle \vec{x} | V | \vec{x}' \rangle \langle \vec{x}' | \psi \rangle\right) = E \langle \vec{x} | \psi \rangle$$

or

$$\boxed{-\frac{\nabla^2}{2m}\psi(\vec{x}') + \int d^3x' \, \langle \vec{x} | \hat{V} | \vec{x}' \rangle \psi(\vec{x}') = E\psi(\vec{x})}$$

- The Yukawa potential, for $|\vec{k}|, |\vec{k}'| \ll \Lambda$, has a simple power series expansion.
  - One might expect that we determine $C_0, C_2, C_2', \ldots$ by simply matching this expansion to the expression for $\langle \vec{k}' | V_{EFT} | \vec{k} \rangle$.
  - BUT THIS WOULD BE WRONG!
  - The complication is that the scattering amplitude is given by more than just the "tree level" diagrams (ie., more than just first-order perturbation theory).

$\Rightarrow$ include diagrams with "loops"

$\Rightarrow$ summation over intermediate states with high momentum.