

1.4.† Show that the second-order contribution to the ground-state energy of an electron gas is given by $E^{(2)} = (Ne^2/2a_0)(\epsilon_2^s + \epsilon_2^b)$, where

$$\epsilon_2^s = -\frac{3}{8\pi^5} \int \frac{d^3q}{q^4} \int_{|k+q|>1} d^3k \int_{|p+q|>1} d^3p \frac{\theta(1-k)\theta(1-p)}{q^2 + \mathbf{q} \cdot (\mathbf{k} + \mathbf{p})}$$

$$\epsilon_2^b = \frac{3}{16\pi^5} \int \frac{d^3q}{q^2} \int_{|k+q|>1} d^3k \int_{|p+q|>1} d^3p \frac{\theta(1-k)\theta(1-p)}{(\mathbf{q} + \mathbf{k} + \mathbf{p})^2 [q^2 + \mathbf{q} \cdot (\mathbf{k} + \mathbf{p})]}$$

1.5. The exchange term ϵ_2^b in Prob. 1.4 is finite, while the direct term ϵ_2^s diverges.

(a) Consider the function $f(q)$ defined as

$$f(q) = \int_{|k+q|>1} d^3k \int_{|p+q|>1} d^3p \frac{\theta(1-k)\theta(1-p)}{q^2 + \mathbf{q} \cdot (\mathbf{k} + \mathbf{p})}$$

Show that $f(q) \approx (4\pi/3)^2 q^{-2}$ as $q \rightarrow \infty$ and $f(q) \approx \frac{2}{3}(2\pi)^2(1 - \ln 2)q$ as $q \rightarrow 0$. Hence conclude that $\epsilon_2^s = -(3/8\pi^5) \int d^3q f(q)q^{-4}$ diverges logarithmically for small q .

(b) The polarizability of the intervening medium modifies the effective interaction between two electrons at long wavelength, where it behaves as

$$V(q)_{\text{eff}} \approx 4\pi e^2 [q^2 + (4r_s/\pi)k_F^2(4/9\pi)^{\frac{1}{2}}]^{-1}$$

for $q \rightarrow 0$. [See Eq. (12.65).] In the limit $r_s \rightarrow 0$, use this result to demonstrate that

$$\epsilon_2^s = 2\pi^{-2}(1 - \ln 2) \ln r_s + \text{const} = 0.0622 \ln r_s + \text{const}$$

(c) How does ϵ_2^s of part b affect the equation of state? Find the density at which the compressibility becomes negative.

1.6. Consider a polarized electron gas in which N_{\pm} denotes the number of electrons with spin-up (-down).¹

(a) Find the ground-state energy to first order in the interaction potential as a function of $N = N_+ + N_-$ and the polarization $\zeta = (N_+ - N_-)/N$.

(b) Prove that the ferromagnetic state ($\zeta = 1$) represents a lower energy than the unmagnetized state ($\zeta = 0$) if $r_s > (2\pi/5)(9\pi/4)^{\frac{1}{2}}(2^{\frac{1}{2}} + 1) = 5.45$. Explain why this is so.

(c) Show that $\partial^2(E/N)/\partial\zeta^2|_{\zeta=0}$ becomes negative for $r_s > (3\pi^2/2)^{\frac{2}{3}} = 6.03$.

(d) Discuss the physical significance of the two critical densities. What happens for $5.45 < r_s < 6.03$?

1.7. Repeat Prob. 1.6 for a potential $V(|\mathbf{x} - \mathbf{y}|) = g\delta^{(3)}(\mathbf{x} - \mathbf{y})$. Show that the system is partially magnetized for $20/9 < gN/V\bar{T} < (5/3)2^{\frac{2}{3}} = 2.64$, where \bar{T} is the mean kinetic energy per particle in the unmagnetized state, and N/V is the corresponding particle density. What happens outside of these limits?

† See footnote on p. 31.

¹ F. Bloch, *Z. Physik*, **57**:545 (1929).

2 Statistical Mechanics

Before formulating the quantum-mechanical description of many-particle assemblies, it is useful to review some thermodynamic relations. The elementary discussions usually consider assemblies containing a fixed number of particles, but such a description is too restricted for the present purposes. We must therefore generalize the treatment to include the possibility of variable number of particles N . This approach is most simply expressed in the grand canonical ensemble, which is generally more tractable than the canonical ensemble (N fixed). In addition, there are physical systems where the variable number of particles is an essential feature, rather than a mathematical convenience; for example, the macroscopic condensate in superfluid helium and in superconductors acts as a particle bath that can exchange particles with the remainder of the system. Indeed, these systems are best described with model hamiltonians that do not even conserve N , and the more general description must be used.

4. REVIEW OF THERMODYNAMICS AND STATISTICAL MECHANICS

Although it is possible to treat systems containing several different kinds of particles, the added generality is not needed for most physical applications, and we shall consider only single-component systems. The fundamental thermodynamic identity

$$dE = T dS - P dV + \mu dN \quad (4.1)$$

specifies the change in the internal energy E arising from small independent changes in the entropy S , the volume V , and the number of particles N . Equation (4.1) shows that the internal energy is a thermodynamic function of these three variables, $E = E(S, V, N)$, and that the temperature T , the pressure P , and the chemical potential μ are related to the partial derivatives of E :

$$T = \left(\frac{\partial E}{\partial S} \right)_{VN} \quad -P = \left(\frac{\partial E}{\partial V} \right)_{SN} \quad \mu = \left(\frac{\partial E}{\partial N} \right)_{SV} \quad (4.2)$$

In the particular case of a quantum-mechanical system in its ground state, the entropy vanishes, and the chemical potential reduces to

$$\mu = \left(\frac{\partial E}{\partial N} \right)_V \quad S = 0 \quad (4.3)$$

where E is the ground-state energy. More generally, Eqs. (4.1) and (4.2) may be interpreted as defining the chemical potential.

The internal energy is useful for studying isentropic processes; in practice, however, experiments are usually performed at fixed T , and it is convenient to make a Legendre transformation to the variables (T, V, N) or (T, P, N) . The resulting functions are known as the *Helmholtz free energy* $F(T, V, N)$ and the *Gibbs free energy* $G(T, P, N)$, defined by

$$F = E - TS \quad G = E - TS + PV \quad (4.4)$$

The differential of these two equations may be combined with Eq. (4.1) to yield

$$dF = -S dT - P dV + \mu dN \quad dG = -S dT + V dP + \mu dN \quad (4.5)$$

which demonstrates that F and G are indeed thermodynamic functions of the specified variables. In particular, the chemical potential may be defined as

$$\mu = \left(\frac{\partial F}{\partial N} \right)_{TV} = \left(\frac{\partial G}{\partial N} \right)_{TP} \quad (4.6)$$

Furthermore, it is often important to consider the set of independent variables (T, V, μ) , which is appropriate for variable N . A further Legendre transformation leads to the *thermodynamic potential*

$$\Omega(T, V, \mu) = F - \mu N = E - TS - \mu N \quad (4.7)$$

with the corresponding differential

$$d\Omega = -S dT - P dV - N d\mu \quad (4.8)$$

The coefficients are immediately given by

$$S = - \left(\frac{\partial \Omega}{\partial T} \right)_{V\mu} \quad P = - \left(\frac{\partial \Omega}{\partial V} \right)_{T\mu} \quad N = - \left(\frac{\partial \Omega}{\partial \mu} \right)_{TV} \quad (4.9)$$

which will be particularly useful in subsequent applications.

Although E , F , G , and Ω represent formally equivalent ways of describing the same system, their natural independent variables differ in one important way. In particular, the set (S, V, N) consists entirely of extensive variables, proportional to the actual amount of matter present. The transformation to F and then to G or Ω may be interpreted as reducing the number of extensive variables in favor of intensive ones that are independent of the total amount of matter. This distinction between extensive and intensive variables leads to an important result. Consider a scale change in which all extensive quantities (including E , F , G , and Ω) are multiplied by a factor λ . For definiteness, we shall study the internal energy, which becomes

$$\lambda E = E(\lambda S, \lambda V, \lambda N)$$

Differentiate with respect to λ and set $\lambda = 1$:

$$E = S \left(\frac{\partial E}{\partial S} \right)_{VN} + V \left(\frac{\partial E}{\partial V} \right)_{SN} + N \left(\frac{\partial E}{\partial N} \right)_{SV} = TS - PV + \mu N \quad (4.10)$$

where Eq. (4.2) has been used. Equation (4.10), which here arises from physical arguments, is a special case of Euler's theorem on homogeneous functions. The remaining thermodynamic functions are immediately found as

$$F = -PV + \mu N \quad G = \mu N \quad \Omega = -PV \quad (4.11)$$

which shows that the chemical potential in a one-component system is the Gibbs free energy per particle $\mu = N^{-1} G(T, P, N)$ and that $P = -V^{-1} \Omega(T, V, \mu)$. This last result is also an obvious consequence of Eq. (4.9), because Ω and V are extensive, whereas T and μ are intensive.

To this point, we have used only macroscopic thermodynamics, which merely correlates bulk properties of the system. The microscopic content of the theory must be added separately through statistical mechanics, which relates the thermodynamic functions to the hamiltonian of the many-particle assembly. In the grand canonical ensemble at chemical potential μ and temperature

$$T \equiv \frac{1}{k_B \beta} \quad (4.12)$$

where k_B is Boltzmann's constant, $k_B = 1.381 \times 10^{-16}$ erg/degree, the grand partition function Z_G is defined as

$$\begin{aligned} Z_G &\equiv \sum_N \sum_j e^{-\beta(E_j - \mu N)} \\ &= \sum_N \sum_j \langle Nj | e^{-\beta(H - \mu \hat{N})} | Nj \rangle \\ &= \text{Tr}(e^{-\beta(H - \mu \hat{N})}) \end{aligned} \quad (4.13)$$

where j denotes the set of all states for a fixed number of particles N , and the sum implied in the trace is over both j and N . A fundamental result from statistical mechanics then asserts that

$$\Omega(T, V, \mu) = -k_B T \ln Z_G \quad (4.14)$$

which allows us to compute all the macroscopic equilibrium thermodynamics from the grand partition function.

The statistical operator $\hat{\rho}_G$ corresponding to Eq. (4.13) is given by

$$\hat{\rho}_G = Z_G^{-1} e^{-\beta(H - \mu \hat{N})} \quad (4.15)$$

With the aid of Eq. (4.14), $\hat{\rho}_G$ may be rewritten compactly as

$$\hat{\rho}_G = e^{\beta(\Omega - H + \mu \hat{N})} \quad (4.16)$$

For any operator \hat{O} , the ensemble average $\langle \hat{O} \rangle$ is obtained with the prescription

$$\begin{aligned} \langle \hat{O} \rangle &= \text{Tr}(\hat{\rho}_G \hat{O}) \\ &= \text{Tr}(e^{\beta(\Omega - H + \mu \hat{N})} \hat{O}) \\ &= \frac{\text{Tr}(e^{-\beta(H - \mu \hat{N})} \hat{O})}{\text{Tr} e^{-\beta(H - \mu \hat{N})}} \end{aligned} \quad (4.17)$$

The utility of these expressions will be illustrated in Sec. 5, which reviews the thermodynamic behavior of ideal Bose and Fermi gases.

5. IDEAL GAS¹

We now apply these results by reviewing the properties of noninteracting Bose and Fermi gases. Throughout our discussion we use the notational simplification

$$\beta = (k_B T)^{-1} \quad (5.1)$$

¹ The arguments in this section are contained in any good book on statistical mechanics, for example, L. D. Landau and E. M. Lifshitz, "Statistical Physics," chap. V, Pergamon Press, London, 1958.

If Eq. (4.13) is written out in detail with the complete set of states in the abstract occupation-number Hilbert space, we have

$$\begin{aligned} Z_G &= \text{Tr}(e^{-\beta(H_0 - \mu \hat{N})}) \\ &= \sum_{n_1 \dots n_\infty} \langle n_1 \dots n_\infty | e^{\beta(\mu \hat{N} - H_0)} | n_1 \dots n_\infty \rangle \\ &= e^{-\beta \Omega_0(T, V, \mu)} \end{aligned} \quad (5.2)$$

Since these states are eigenstates of the hamiltonian \hat{H}_0 and the number operator \hat{N} , both operators can be replaced by their eigenvalues

$$Z_G = \sum_{n_1 \dots n_\infty} \langle n_1 \dots n_\infty | \exp \left[\beta \left(\mu \sum_i n_i - \sum_i \epsilon_i n_i \right) \right] | n_1 \dots n_\infty \rangle \quad (5.3)$$

The exponential is now a c number and is equivalent to a product of exponentials; hence the sum over expectation values factors into a product of traces, one referring to each mode,

$$Z_G = \sum_{n_1} \langle n_1 | e^{\beta(\mu n_1 - \epsilon_1 n_1)} | n_1 \rangle \dots \sum_{n_\infty} \langle n_\infty | e^{\beta(\mu n_\infty - \epsilon_\infty n_\infty)} | n_\infty \rangle \quad (5.4)$$

which may be written compactly as

$$Z_G = \prod_{i=1}^{\infty} \text{Tr}_i e^{-\beta(\epsilon_i - \mu) \hat{n}_i} \quad (5.5)$$

For bosons the occupation numbers are unrestricted so that we must sum n_i over all integers in Eq. (5.5)

$$Z_G = \prod_{i=1}^{\infty} \sum_{n=0}^{\infty} (e^{\beta(\mu - \epsilon_i)})^n = \prod_{i=1}^{\infty} (1 - e^{\beta(\mu - \epsilon_i)})^{-1} \quad (5.6)$$

The logarithm of Eq. (5.6) yields the thermodynamic potential

$$\Omega_0(T, V, \mu) = -k_B T \ln \prod_{i=1}^{\infty} (1 - e^{\beta(\mu - \epsilon_i)})^{-1} \quad (5.7)$$

$$\Omega_0(T, V, \mu) = k_B T \sum_{i=1}^{\infty} \ln(1 - e^{\beta(\mu - \epsilon_i)}) \quad \text{Bose} \quad (5.8)$$

The mean number of particles is obtained from Ω_0 by differentiating with respect to the chemical potential, as in Eq. (4.9), keeping T and V (equivalently the ϵ_i) fixed:

$$\langle N \rangle \equiv \sum_{i=1}^{\infty} n_i^0 = \sum_{i=1}^{\infty} \frac{1}{e^{\beta(\epsilon_i - \mu)} - 1} \quad \text{Bose} \quad (5.9)$$

where n_i^0 is the mean occupation number in the i th state.

For fermions, the occupation numbers are either 0 or 1, and the sum in Eq. (5.5) is restricted to these values

$$Z_G = \prod_{i=1}^{\infty} \sum_{n=0}^1 (e^{\beta(\mu-\epsilon_i)})^n = \prod_{i=1}^{\infty} (1 + e^{\beta(\mu-\epsilon_i)}) \quad (5.10)$$

Taking the logarithm of both sides, we have

$$\Omega_0(T, V, \mu) = -k_B T \sum_{i=1}^{\infty} \ln(1 + e^{\beta(\mu-\epsilon_i)}) \quad \text{Fermi} \quad (5.11)$$

while the number of particles becomes

$$\langle N \rangle \equiv \sum_{i=1}^{\infty} n_i^0 = \sum_{i=1}^{\infty} \frac{1}{e^{\beta(\epsilon_i-\mu)} + 1} \quad \text{Fermi} \quad (5.12)$$

Although bosons and fermions differ only by the sign in the denominator in Eqs. (5.9) and (5.12), this sign leads to rather remarkable differences in the behavior of these assemblies.

BOSONS

We shall first consider a collection of noninteracting bosons, where the energy spectrum is given by

$$\epsilon_p = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m} \quad (5.13)$$

We assume that the assembly is contained in a large volume V and apply periodic boundary conditions on the single-particle wave functions. Just as in Eq. (3.26), sums over single-particle levels can be replaced by an integral over wavenumbers according to

$$\sum_i \rightarrow g \int d^3n = gV(2\pi)^{-3} \int d^3k \quad (5.14)$$

where g is the degeneracy of each single-particle momentum state. For example, $g = 1$ for spinless particles. With Eq. (5.13), the density of states in Eq. (5.14) can be rewritten as

$$\frac{gV}{(2\pi)^3} 4\pi k^2 dk = \frac{gV}{2\pi^2} \left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \frac{\epsilon d\epsilon}{2\epsilon^{\frac{1}{2}}} = \frac{gV}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \epsilon^{\frac{1}{2}} d\epsilon \quad (5.15)$$

and the thermodynamic potential Eq. (5.8) for an ideal Bose gas becomes

$$-\frac{\Omega_0}{k_B T} = \frac{PV}{k_B T} = -\frac{gV}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \int_0^{\infty} d\epsilon \epsilon^{\frac{1}{2}} \ln(1 - e^{\beta(\mu-\epsilon)}) \quad (5.16)$$

A simple partial integration then yields

$$PV = \frac{gV}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \frac{2}{3} \int_0^{\infty} d\epsilon \frac{\epsilon^{\frac{3}{2}}}{e^{\beta(\epsilon-\mu)} - 1} \quad (5.17)$$

Alternatively, a combination of Eqs. (4.9), (4.10), and (5.8) allows us to write

$$E = \sum_i n_i^0 \epsilon_i = \frac{gV}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \int_0^{\infty} d\epsilon \frac{\epsilon^{\frac{3}{2}}}{e^{\beta(\epsilon-\mu)} - 1} \quad (5.18)$$

showing that the equation of state of an ideal Bose gas is given by

$$PV = \frac{2}{3} E \quad (5.19)$$

In a similar way, the number of particles becomes

$$\frac{N}{V} = \frac{g}{4\pi^2} \left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}} \int_0^{\infty} d\epsilon \frac{\epsilon^{\frac{1}{2}}}{e^{\beta(\epsilon-\mu)} - 1} \quad (5.20)$$

Although Eqs. (5.17) and (5.20) determine the thermodynamic variables of an ideal Bose gas as functions of T , V , and μ , Eq. (5.20) can in principle be inverted to obtain the chemical potential as a function of the number of particles. Substitution into Eq. (5.17) then yields the thermodynamic variables as a function of T , V , and N .

Equation (5.20) is meaningful only if

$$\epsilon - \mu \geq 0$$

Otherwise the mean occupation number n^0 would be less than zero for some values of ϵ . In particular, ϵ can vanish so that the chemical potential of an ideal Bose gas must satisfy the condition

$$\mu \leq 0 \quad (5.21)$$

To understand this relation, we recall the classical limit of the chemical potential for fixed N :[†]

$$\frac{\mu}{k_B T} \rightarrow -\infty \quad T \rightarrow \infty \quad (5.22)$$

In this limit we see that Bose and Fermi gases give the same expression

$$N \rightarrow \sum_i e^{\beta(\mu-\epsilon_i)} \quad T \rightarrow \infty \quad (5.23)$$

which is just the familiar Boltzmann distribution, and

$$\Omega_0 = -PV = -k_B T \sum_i e^{\beta(\mu-\epsilon_i)} = -Nk_B T \quad T \rightarrow \infty \quad (5.24)$$

which is the equation of state of an ideal classical gas. The sum may be evaluated approximately as an integral

$$\begin{aligned} \sum_i e^{-\beta\epsilon_i} &= g(2\pi)^{-3} V \int d^3k e^{-\hbar^2 k^2 / 2mk_B T} \\ &= gV \left(\frac{mk_B T}{2\pi\hbar^2}\right)^{\frac{3}{2}} \end{aligned} \quad (5.25)$$

[†] L. D. Landau and E. M. Lifshitz, *op. cit.*, sec. 45.

and Eq. (5.23) then yields the classical expression μ_c for the chemical potential

$$\frac{\mu_c}{k_B T} = \ln \left[\frac{N}{gV} \left(\frac{2\pi\hbar^2}{mk_B T} \right)^{\frac{3}{2}} \right] \quad (5.26)$$

in terms of T , V , N . Note that Eq. (5.22) is indeed satisfied as $T \rightarrow \infty$. Furthermore, it is clear that Fermi, Bose, and Boltzmann statistics now coincide, since the particles are distributed over many states. Thus the mean occupation number of any one state is much less than one, and quantum restrictions play no role.

The classical chemical potential of Eq. (5.26) is sketched in Fig. 5.1. As the temperature is reduced at fixed density, $\mu_c/k_B T$ passes through zero and

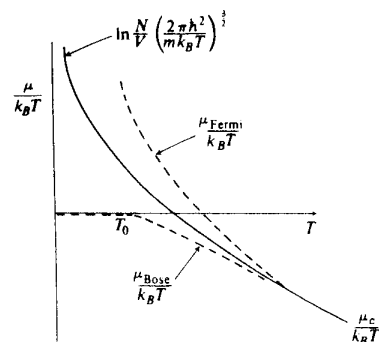


Fig. 5.1 The chemical potential of ideal classical, Fermi, and Bose gases for fixed N and V .

becomes positive, diverging to $+\infty$ at $T=0$. Since this behavior violates Eq. (5.21), the chemical potential for an ideal Bose gas must lie below the classical value, staying negative or zero. Let T_0 be the temperature where the chemical potential of an ideal Bose gas vanishes. This critical temperature is readily determined with Eq. (5.20)

$$\frac{N}{V} = \frac{g}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{\frac{3}{2}} \int_0^\infty d\epsilon \frac{\epsilon^{\frac{1}{2}}}{e^{\epsilon/k_B T_0} - 1} \quad (5.27)$$

which may be rewritten with the new variable $x \equiv \epsilon/k_B T_0$ as

$$\frac{N}{V} = \frac{g}{4\pi^2} \left(\frac{2mk_B T_0}{\hbar^2} \right)^{\frac{3}{2}} \int_0^\infty dx \frac{x^{\frac{1}{2}}}{e^x - 1} \quad (5.28)$$

The integral is evaluated in Appendix A

$$\frac{N}{V} = \frac{g}{4\pi^2} \left(\frac{2mk_B T_0}{\hbar^2} \right)^{\frac{3}{2}} \zeta\left(\frac{3}{2}\right) \Gamma\left(\frac{3}{2}\right) \quad (5.29)$$

and Eq. (5.29) may be inverted to give

$$T_0 = \frac{\hbar^2}{2mk_B} \left(\frac{4\pi^2}{g\Gamma\left(\frac{3}{2}\right)\zeta\left(\frac{3}{2}\right)} \right)^{\frac{2}{3}} \left(\frac{N}{V} \right)^{\frac{2}{3}} = \frac{3.31}{g^{\frac{2}{3}}} \frac{\hbar^2}{mk_B} \left(\frac{N}{V} \right)^{\frac{2}{3}} \quad (5.30)$$

as the temperature at which the chemical potential of an ideal Bose gas reaches zero. This value has the simple physical interpretation that the thermal energy $k_B T_0$ is comparable with the only other intensive energy for a perfect gas, the zero-point energy $(\hbar^2/m)(N/V)^{\frac{2}{3}}$ associated with localizing a particle in a volume V/N .

What happens as we lower the temperature below T_0 ? It is clear physically that many bosons will start to occupy the lowest available single-particle state, namely the ground state. For $\mu=0$ and $T < T_0$, however, the integral in Eq. (5.20) is less than N/V because these conditions increase the denominator relative to its value at T_0 . Thus the theory appears to break down because Eq. (5.20) will not reproduce the full density N/V . This difficulty can be traced to the replacement of the sum by an integral in Eq. (5.14), and we therefore examine the original sum

$$N = \sum_i (e^{\beta(\epsilon_i - \mu)} - 1)^{-1}$$

As $\mu \rightarrow 0$, all of the terms except the first approach a finite limit; the sum of these finite terms is just that given by the integral evaluated above.¹ In contrast, the first term has been lost in passing to the integral because the $\epsilon^{\frac{1}{2}}$ in the density of states vanishes at $\epsilon=0$. We see, however, that this first term becomes arbitrarily large as $\mu \rightarrow 0$ and can therefore make up the rest of the particles. This behavior reflects the macroscopic occupation of the single quantum state $\epsilon=0$.

For temperatures $T < T_0$, we conclude that the chemical potential μ must be infinitesimally small and negative

$$\mu = 0^- \quad \text{for } T \leq T_0 \quad (5.31)$$

In this temperature range, the density of particles with energies $\epsilon > 0$ becomes

$$\frac{dN_\epsilon}{V} = \frac{g}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{\frac{3}{2}} \frac{\epsilon^{\frac{1}{2}} d\epsilon}{e^{\beta\epsilon} - 1} \quad (5.32)$$

with the integrated value

$$\frac{N_{\epsilon>0}}{V} = \frac{g}{4\pi^2} \left(\frac{2mk_B T}{\hbar^2} \right)^{\frac{3}{2}} \int_0^\infty dx \frac{x^{\frac{1}{2}}}{e^x - 1} = \frac{N}{V} \left(\frac{T}{T_0} \right)^{\frac{3}{2}} \quad (5.33)$$

The remaining particles are then in the lowest energy state with $\epsilon=0$

$$\frac{N_{\epsilon=0}}{V} = \frac{N}{V} \left[1 - \left(\frac{T}{T_0} \right)^{\frac{3}{2}} \right] \quad (5.34)$$

¹ Strictly speaking, the occupation number of the low-lying excited states is of order $N^{\frac{1}{3}}$, which becomes negligible only in the thermodynamic limit. A rigorous discussion of the Bose-Einstein condensation may be found in R. H. Fowler and H. Jones, *Proc. Cambridge Phil. Soc.*, **34**:573 (1938).

In the degenerate region ($T < T_0$) where the chemical potential is given by Eq. (5.31), the energy of the Bose gas arises entirely from those particles not in the condensate

$$\frac{E}{V} = \frac{g}{4\pi^2} \left(\frac{2mk_B T}{\hbar^2} \right)^{\frac{3}{2}} k_B T \int_0^\infty dx \frac{x^{\frac{3}{2}}}{e^x - 1}$$

This integral is again treated in Appendix A, and we find

$$\frac{E}{V} = \frac{g}{4\pi^2} \left(\frac{2mk_B T}{\hbar^2} \right)^{\frac{3}{2}} k_B T \zeta\left(\frac{5}{2}\right) \Gamma\left(\frac{5}{2}\right) \quad (5.35)$$

which may be rewritten in terms of T_0 from Eq. (5.29) as

$$E = \frac{\zeta\left(\frac{5}{2}\right) \Gamma\left(\frac{5}{2}\right)}{\zeta\left(\frac{3}{2}\right) \Gamma\left(\frac{3}{2}\right)} N k_B T \left(\frac{T}{T_0} \right)^{\frac{5}{2}} = 0.770 N k_B T \left(\frac{T}{T_0} \right)^{\frac{5}{2}} \quad T \leq T_0 \quad (5.36)$$

The constant-volume heat capacity then becomes

$$C_V = \frac{5}{2} \left[0.770 N k_B \left(\frac{T}{T_0} \right)^{\frac{3}{2}} \right] \quad T \leq T_0 \quad (5.37)$$

which varies as $T^{\frac{3}{2}}$ and vanishes at $T = 0$. Equation (5.35) also can be used to rewrite the equation of state:

$$\begin{aligned} P &= \frac{2}{3} \frac{E}{V} = \frac{2}{3} \frac{g}{4\pi^2} \left(\frac{2mk_B T}{\hbar^2} \right)^{\frac{3}{2}} \zeta\left(\frac{5}{2}\right) \Gamma\left(\frac{5}{2}\right) \frac{m^{\frac{3}{2}} (k_B T)^{\frac{5}{2}}}{\hbar^3} \\ &= 0.0851 m^{\frac{3}{2}} (k_B T)^{\frac{5}{2}} \hbar^{-3} g \quad T \leq T_0 \end{aligned} \quad (5.38)$$

The pressure vanishes at zero temperature because all of the particles are in the zero-momentum state and therefore exert no force on the walls of the container. Furthermore, the pressure is independent of the density N/V , depending only on the temperature $T \leq T_0$.

We have seen that the ideal Bose gas has a critical temperature T_0 where the chemical potential changes its analytic form. Since $\mu(T, V, N)$ is related to the free energy by Eq. (4.6), it is natural to expect similar discontinuities in other thermodynamic functions, and we now show that the heat capacity at constant volume C_V has a discontinuous slope at T_0 . The behavior for $T < T_0$ is given in Eq. (5.37); the corresponding quantity for $T > T_0$ can be found as follows: Define the (fictitious) number of particles computed for $\mu = 0$ and $T > T_0$ by

$$N_0(T) \equiv \frac{gV}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{\frac{3}{2}} \int_0^\infty d\epsilon \frac{\epsilon^{\frac{3}{2}}}{e^{\epsilon/k_B T} - 1}$$

This expression clearly implies

$$\frac{N_0(T)}{N_0(T_0)} = \frac{N_0(T)}{N} = \left(\frac{T}{T_0} \right)^{\frac{3}{2}} \quad T > T_0$$

Equation (5.20), which determines the actual $\mu(T, V, N)$ for $T > T_0$, can now be rewritten

$$N - N_0(T) = \frac{gV}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{\frac{3}{2}} \int_0^\infty d\epsilon \epsilon^{\frac{3}{2}} \left\{ \frac{1}{e^{(\epsilon - \mu)/k_B T} - 1} - \frac{1}{e^{\epsilon/k_B T} - 1} \right\}$$

The dominant contribution to this integral arises from small values of ϵ because $\mu/k_B T$ is small and negative for $0 < T - T_0 \ll T_0$. Thus we shall expand the integrand to give

$$\begin{aligned} N - N_0(T) &\approx \frac{gV}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{\frac{3}{2}} \mu k_B T \int_0^\infty d\epsilon \frac{1}{\epsilon^{\frac{3}{2}} (\epsilon + |\mu|)} \\ &\approx - \frac{gV}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{\frac{3}{2}} \pi k_B T_0 |\mu|^{\frac{1}{2}} \end{aligned}$$

where we have set $T = T_0$ to leading order in $T - T_0$. A combination with Eq. (5.29) leads to

$$\begin{aligned} \mu &\approx - \left[\frac{\zeta\left(\frac{3}{2}\right) \Gamma\left(\frac{3}{2}\right)}{\pi} \right]^2 k_B T_0 \left[\frac{N_0(T)}{N} - 1 \right]^2 \\ &= - \left[\frac{\zeta\left(\frac{3}{2}\right) \Gamma\left(\frac{3}{2}\right)}{\pi} \right]^2 k_B T_0 \left[\left(\frac{T}{T_0} \right)^{\frac{3}{2}} - 1 \right]^2 \quad T \gtrsim T_0 \end{aligned}$$

Note that μ vanishes quadratically as $T \rightarrow T_0^+$ so that $\mu(T, V, N)$ has a discontinuous second derivative at T_0 (see Fig. 5.1).

The remaining calculation can be carried out by differentiating the equation of state (5.19)

$$\left(\frac{\partial E}{\partial \mu} \right)_{TV} = \frac{3}{2} \left[\frac{\partial(PV)}{\partial \mu} \right]_{TV} = \frac{3}{2} N$$

where the last equality follows from Eqs. (4.9) and (4.11). This result allows us to find the change in energy arising from a small change in μ at constant T and V . If $E(T, V)$ is the energy for zero chemical potential [Eq. (5.35)], then the actual energy is given approximately as

$$E = \begin{cases} E(T, V) & T < T_0 \\ E(T, V) + \frac{3}{2} N \mu & T > T_0 \end{cases}$$

We now change variables to T , V , and N using the expression obtained above for $\mu(T, V, N)$. The jump in the slope of C_V is then given by¹

$$\begin{aligned} \Delta \left[\frac{\partial C_V}{\partial T} \right]_{T_0} &= - \frac{3}{2} N k_B T_0 \left[\frac{\zeta\left(\frac{3}{2}\right) \Gamma\left(\frac{3}{2}\right)}{\pi} \right]^2 \left\{ \frac{\partial^2}{\partial T^2} \left[\left(\frac{T}{T_0} \right)^{\frac{3}{2}} - 1 \right]^2 \right\}_{T_0} \\ &= - \frac{27}{4} \left[\frac{\zeta\left(\frac{3}{2}\right) \Gamma\left(\frac{3}{2}\right)}{\pi} \right]^2 \frac{N k_B}{T_0} = -3.66 \frac{N k_B}{T_0} \end{aligned} \quad (5.39)$$

¹ F. London, "Superfluids," vol. II, sec. 7, Dover, New York, 1964; we here follow the approach of L. D. Landau and E. M. Lifshitz, *op. cit.*, p. 170.

The full curve is sketched in Fig. 5.2. Such discontinuities imply that an ideal Bose gas exhibits a *phase transition* at a temperature T_0 . This temperature has a physical interpretation as the point where a finite fraction of all the particles begins to occupy the zero-momentum state. Below T_0 the occupation number n_0^0 of the lowest single-particle state is of order N , rather than of order 1. As emphasized by F. London,¹ the assembly is ordered in *momentum* space and not in coordinate space; this phenomenon is called *Bose-Einstein condensation*.

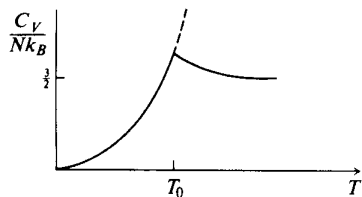


Fig. 5.2 Constant-volume heat capacity C_V of an ideal Bose gas.

To estimate the magnitude of the quantities involved, we recall that the density of liquid He^4 at low temperature is

$$\rho_4 = 0.145 \text{ g cm}^{-3}$$

Inserting this quantity into Eq. (5.30), we find the value

$$T_0 = 3.14^\circ\text{K} \quad (5.40)$$

as the transition temperature of an ideal Bose gas with the parameters appropriate to liquid helium. Below this temperature, the foregoing discussion indicates that the assembly consists of two different components, one corresponding to the particles that occupy the zero-momentum state and therefore have no energy, and the other corresponding to the particles in the excited states. Indeed, it is an experimental fact that liquid He^4 has a transition at 2.2°K (the λ point) between the two phases He I and He II. Below this temperature He^4 acts like a mixture of a superfluid and a normal fluid, and the superfluid has no heat capacity or viscosity. It is also true that the fraction of normal component vanishes as the temperature goes to zero. The Bose-Einstein condensation of the ideal Bose gas therefore provides a qualitative description of actual He^4 . In detail, however, the ideal Bose gas is an oversimplified model. For example, the actual specific heat varies as T^3 at low temperature and becomes logarithmically infinite at the λ point for liquid He^4 . In addition, it is incorrect to identify the superfluid component of He II with the particles in the zero-momentum state. Indeed, the excitation spectrum of the ideal Bose gas precludes superfluidity at any finite velocity. These questions are discussed in detail in Chaps. 6, 10, and 14, where we show that the interparticle interactions play a crucial role in understanding the properties of quantum fluids such as liquid He^4 .

¹ F. London, *op. cit.*, pp. 39, 143.

FERMIONS

We now discuss Eqs. (5.11) and (5.12) referring to an assembly of fermions, which serves as a model for many physical systems. The basic equation is the mean occupation number

$$n_i^0 = (e^{\beta(\epsilon_i - \mu)} + 1)^{-1} \quad (5.41)$$

With the nonrelativistic energy spectrum [Eq. (5.13)], the same analysis as for bosons gives

$$PV = \frac{2}{3}E = \frac{2}{3} \frac{gV}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{\frac{3}{2}} \int_0^\infty d\epsilon \frac{\epsilon^{\frac{3}{2}}}{e^{\beta(\epsilon - \mu)} + 1} \quad (5.42)$$

$$\frac{N}{V} = \frac{g}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{\frac{3}{2}} \int_0^\infty d\epsilon \frac{\epsilon^{\frac{1}{2}}}{e^{\beta(\epsilon - \mu)} + 1} \quad (5.43)$$

where g is the degeneracy factor ($g = 2$ for a spin- $\frac{1}{2}$ Fermi gas). As noted previously, the only difference between bosons and fermions is the minus or plus sign in the denominators of Eqs. (5.42) and (5.43).

Consider the distribution function n^0 . Equation (5.41) shows that the condition

$$n^0 \leq 1$$

is guaranteed for all values of μ and T . It is interesting to invert Eq. (5.43) and determine the chemical potential for fixed N ; this function is sketched in Fig. 5.1. In the high-temperature or classical limit, we again find

$$n^0 = e^{\beta(\mu - \epsilon)} \quad T \rightarrow \infty$$

which is just the familiar Boltzmann distribution. Unlike the situation for bosons, however, there is nothing to prevent the chemical potential from becoming positive as the temperature is reduced; in particular, we have

$$n^0 = \frac{1}{2} \quad \text{when } \epsilon = \mu$$

In the zero-temperature limit, the Fermi distribution reduces to a step function

$$\frac{1}{e^{(\epsilon - \mu)/k_B T} + 1} \xrightarrow{T \rightarrow 0} \begin{cases} 0 & \epsilon > \mu \\ 1 & \epsilon < \mu \end{cases} = \theta(\mu - \epsilon) \quad (5.44)$$

This behavior is readily understood, because the lowest energy state of the system is obtained by filling the energy levels up to

$$\mu = \epsilon_F \quad \text{at } T = 0$$

Hence the chemical potential of an ideal Fermi gas at zero temperature is a finite positive number, equal to the Fermi energy. The equilibrium distribution numbers in three representative cases are sketched in Fig. 5.3.

We shall first evaluate the properties of an ideal Fermi gas at $T = 0$. From Eqs. (5.43) and (5.44), the density is given by

$$\frac{N}{V} = \frac{g}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{\frac{3}{2}} \int_0^\mu d\epsilon \epsilon^{\frac{1}{2}}$$

because the distribution number is then a step function. This integral is easily evaluated as

$$\frac{N}{V} = \frac{g}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{\frac{3}{2}} \frac{2}{3} \mu^{\frac{3}{2}} \quad (5.45)$$

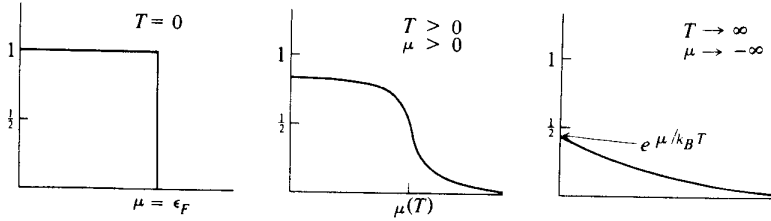


Fig. 5.3 Schematic distribution functions $n(\epsilon)$ for an ideal Fermi gas at various temperatures.

which may be inverted to find the Fermi energy

$$\epsilon_F \equiv \mu(T=0) = \left(\frac{6\pi^2}{g} \right)^{\frac{2}{3}} \frac{\hbar^2}{2m} \left(\frac{N}{V} \right)^{\frac{2}{3}} \equiv \frac{\hbar^2 k_F^2}{2m} \quad (5.46)$$

or the Fermi wavenumber

$$k_F = \left(\frac{6\pi^2 N}{gV} \right)^{\frac{1}{3}} \quad (5.47)$$

Similarly, the energy is obtained from

$$\frac{E}{V} = \frac{g}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{\frac{3}{2}} \int_0^\mu d\epsilon \epsilon^{\frac{3}{2}} = \frac{g}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{\frac{3}{2}} \frac{2}{5} \mu^{\frac{5}{2}}$$

A combination with Eq. (5.45) yields

$$\frac{E}{N} = \frac{3}{5} \mu = \frac{3}{5} \epsilon_F \quad (5.48)$$

Finally, the equation of state (5.42) becomes

$$PV = \frac{3}{2} E = \frac{3}{2} N \epsilon_F \quad (5.49a)$$

$$P = \frac{2}{5} \left(\frac{6\pi^2}{g} \right)^{\frac{2}{3}} \frac{\hbar^2}{2m} \left(\frac{N}{V} \right)^{\frac{5}{3}} \quad (5.49b)$$

which shows that a Fermi gas exerts a finite pressure at zero temperature. This result arises because the Pauli principle requires that the momentum states be filled up to the Fermi momentum, and these higher momentum states exert a pressure on the walls of any container.

We now turn to small but finite temperature, where the difficult part is the inversion of Eq. (5.43) to determine the chemical potential in terms of the total number of particles. Define the variable $x \equiv (\epsilon - \mu)/k_B T$. Equation (5.42) may then be rewritten as

$$PV = \frac{2}{3} \frac{gV}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{\frac{3}{2}} (k_B T)^{\frac{5}{2}} \int_{-\mu/k_B T}^{\infty} dx \frac{(x + \mu/k_B T)^{\frac{3}{2}}}{e^x + 1} \quad (5.50)$$

It is also convenient to introduce $\alpha \equiv \mu/k_B T$; since μ is finite as $T \rightarrow 0$, we are interested in the limit $\alpha \rightarrow \infty$. Consider the integral

$$I(\alpha) \equiv \int_{-\alpha}^{\infty} dx \frac{(x + \alpha)^{\frac{3}{2}}}{e^x + 1} = \int_{-\alpha}^0 dx \frac{(\alpha + x)^{\frac{3}{2}}}{e^x + 1} + \int_0^{\infty} dx \frac{(\alpha + x)^{\frac{3}{2}}}{e^x + 1} \quad (5.51)$$

The change of variable $x \rightarrow -x$ in the first integral and use of the identity $(e^{-x} + 1)^{-1} \equiv 1 - (e^x + 1)^{-1}$ yield

$$I(\alpha) = \int_0^{\alpha} dx (\alpha - x)^{\frac{3}{2}} + \int_0^{\infty} dx \frac{(\alpha + x)^{\frac{3}{2}} - (\alpha - x)^{\frac{3}{2}}}{e^x + 1} + \int_{\alpha}^{\infty} dx \frac{(\alpha - x)^{\frac{3}{2}}}{e^x + 1}$$

The last term is exponentially small in the limit of large α , and we can approximate the numerator in the second integral as

$$(\alpha + x)^{\frac{3}{2}} - (\alpha - x)^{\frac{3}{2}} = 3\alpha x^{\frac{1}{2}} + O(\alpha^{-\frac{1}{2}}) \quad \alpha \rightarrow \infty$$

A straightforward calculation (see Appendix A) therefore gives the asymptotic expansion

$$I(\alpha) = \frac{2}{5} \alpha^{\frac{5}{2}} + \frac{\pi^2}{4} \alpha^{\frac{3}{2}} + \dots = \left(\frac{1}{k_B T} \right)^{\frac{5}{2}} \left[\frac{2}{5} \mu^{\frac{5}{2}} + (k_B T)^2 \frac{\pi^2}{4} \mu^{\frac{3}{2}} + \dots \right] \quad (5.52)$$

Thus Eq. (5.50) can be written to order T^2 or $1/\alpha^2$ as

$$PV \underset{T \rightarrow 0}{=} \frac{gV}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{\frac{3}{2}} \frac{2}{3} \left[\frac{2}{5} \mu^{\frac{5}{2}} + (k_B T)^2 \frac{\pi^2}{4} \mu^{\frac{3}{2}} + \dots \right] \quad (5.53)$$

Note that this result gives $PV(T, V, \mu)$, which are the proper thermodynamic variables for the thermodynamic potential. The correction terms in this equation (indicated by dots) are of higher order in T^2 and thus negligible to the present order.

The number of particles can be determined immediately from this expression:

$$N = \left[\frac{\partial(PV)}{\partial \mu} \right]_{TV} = \frac{gV}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{\frac{3}{2}} \frac{2}{3} \left[\mu^{\frac{3}{2}} + (k_B T)^2 \frac{\pi^2}{8\mu^{\frac{1}{2}}} + \dots \right] \quad (5.54)$$

If N/V is rewritten in terms of the Fermi energy ϵ_F using Eq. (5.46) we have

$$\mu = \epsilon_F \left[1 + \frac{\pi^2}{8} \left(\frac{k_B T}{\mu} \right)^2 + \dots \right]^{-\frac{1}{3}} \quad (5.55)$$

which may be solved for μ as a power series in T^2

$$\mu = \epsilon_F \left[1 - \frac{\pi^2}{12} \left(\frac{k_B T}{\epsilon_F} \right)^2 + \dots \right] \quad (5.56)$$

The entropy can be determined from Eq. (5.53) by differentiating at fixed V and μ

$$S(T, V, \mu) = \left[\frac{\partial(PV)}{\partial T} \right]_{V, \mu} = \frac{gV}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{\frac{3}{2}} \frac{2}{3} \left[\frac{2\pi^2}{4} k_B^2 T \mu^{\frac{1}{2}} + \dots \right] \quad (5.57)$$

Since S is a thermodynamic function, it may be expressed in any variables; in particular, substitution of Eq. (5.56) yields

$$S(T, V, N) = Nk_B \frac{k_B T}{\epsilon_F} \frac{\pi^2}{2} \quad (5.58)$$

to lowest order in T . We can thus compute the heat capacity from the relation

$$C_V = T \left(\frac{\partial S}{\partial T} \right)_{V, N} = \frac{\pi^2}{2} Nk_B \frac{k_B T}{\epsilon_F} \quad (5.59)$$

which gives

$$C_V = S = \frac{mk_B^2}{\hbar^2} \left(\frac{g\pi}{6} \right)^{\frac{3}{2}} NT \left(\frac{V}{N} \right)^{\frac{1}{3}} \quad T \rightarrow 0 \quad (5.60)$$

Note that the heat capacity for a Fermi gas at low temperature is linear in the temperature. In contrast, at high temperature, where Boltzmann statistics apply, the heat capacity of a perfect (Bose or Fermi) gas is

$$C_V \rightarrow \frac{3}{2} Nk_B \quad T \rightarrow \infty \quad (5.61)$$

and the heat capacity of an ideal Fermi gas at all temperatures is indicated in Fig. 5.4. Note that a Fermi gas has no discontinuities in the thermodynamic variables at any temperature.

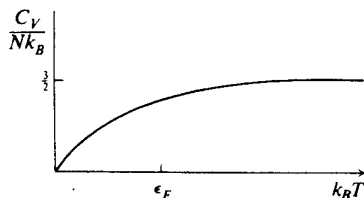


Fig. 5.4 Constant-volume heat capacity of an ideal Fermi gas.

The noninteracting Fermi gas forms a useful first approximation in the theory of metals, in the theory of liquid He^3 , in studies of nuclear structure, and even for understanding such diverse phenomena as the structure of white-dwarf and neutron stars. For a detailed understanding of the behavior of these many-body assemblies, however, we must include the interactions between the particles, which forms the central problem of this book.

PROBLEMS

2.1. Prove that the entropy of an ideal quantum gas is given by

$$S = -k_B \sum_i [n_i^0 \ln n_i^0 \mp (1 \pm n_i^0) \ln (1 \pm n_i^0)]$$

where the upper (lower) signs refer to bosons (fermions). Find the corresponding expression for Boltzmann statistics. Prove that the internal energy is given by $E = \sum_i \epsilon_i n_i^0$ for all three cases.

2.2. Given the energy spectrum $\epsilon_p = [(pc)^2 + m_0^2 c^4]^{\frac{1}{2}} \rightarrow pc$ ($p \rightarrow \infty$), prove that an ultrarelativistic ideal gas satisfies the equation of state $PV = E/3$ where E is the total energy. [Compare with Eqs. (5.19) and (5.42).]

2.3. Show that there is no Bose-Einstein condensation at any finite temperature for a two-dimensional ideal Bose gas.

2.4. Consider an ideal gas in a cubical box ($V = L^3$) with the boundary condition that the single-particle wave function vanish at the walls.

(a) Find the density of states. In the thermodynamic limit, show that the thermodynamic functions for both bosons and fermions reduce to those obtained in Sec. 5.

(b) Discuss the onset of Bose condensation and compute the properties for $T < T_0$.

2.5. When a metal is heated to a sufficiently high temperature, electrons are emitted from the metal surface and can be collected as thermionic current. Assuming the electrons form a noninteracting Fermi gas, derive the Richardson-Dushman equation¹ for the current $i = (4\pi emk_B^2 T^2 / h^3) e^{-W/k_B T}$, where W is the work function for the metal (i.e., the energy necessary to remove an electron).

2.6. Prove that the paramagnetic spin susceptibility of a free Fermi gas of spin- $\frac{1}{2}$ particles at $T = 0$ is given by $\chi(T = 0) = \frac{3}{2} (2m/\hbar^2 k_F^2) \mu_0^2 N/V$ where μ_0 is the magnetic moment of one of the particles. Derive the corresponding high-temperature result $\chi(T \rightarrow \infty) = \mu_0^2 N/k_B TV$.

¹ S. Dushman, *Rev. Mod. Phys.*, 2:381 (1930).

2.7. For a first approximation to atomic nuclei, consider the nucleus as a degenerate noninteracting Fermi gas of neutrons and protons.

- What is the degeneracy factor for each level?
- If the radius of a nucleus with A nucleons is given by $R = r_0 A^{1/3}$ with $r_0 \approx 1.2 \times 10^{-13}$ cm, what are k_F and ϵ_F ? How do they vary with A ?
- What is the pressure exerted by this Fermi gas?
- If each nucleon is considered to be moving in a constant potential of depth V_0 , how large must V_0 be?
- At what temperature will the nucleus act like a collection of particles described by Boltzmann statistics?

2.8. As a model of a white-dwarf star, consider an electrically neutral gas composed of fully ionized He (α particles) and degenerate electrons.

- Write the equation of *local hydrostatic equilibrium* in the low-density (non-relativistic electron gas) and high-density (relativistic electron gas) limits assuming an ideal Fermi system.
- Hence find expressions for the density $\rho(r)$ and the relation $M = M(R)$ between the total mass M and the radius R of the star.
- Show there exists a maximum mass M_{\max} comparable with the solar mass M_\odot . Explain the physics of why this is so.
- Check the initial model using the typical parameters of a white dwarf $\rho \approx 10^7$ g/cm³ $\approx 10^7 \rho_\odot$, $M \approx 10^{33}$ g $\approx M_\odot$, central temperature $\approx 10^7$ K $\approx T_\odot$. Note the following results obtained by numerical integration:¹

- $$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{df}{d\xi} \right) = -f^2 \quad \left. \begin{array}{l} f(0) = 178.2 \\ f'(0) = 0; f(1) = 0 \end{array} \right\} \text{implies } f'(1) = -132.4$$
- $$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{df}{d\xi} \right) = -f^3 \quad \left. \begin{array}{l} f(0) = 6.897 \\ f'(0) = 0; f(1) = 0 \end{array} \right\} \text{implies } f'(1) = -2.018$$

¹ L. D. Landau and E. M. Lifshitz, *op. cit.*, sec. 106.

part two

Ground-state (Zero-temperature) Formalism