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Wednesday 880.05 Class

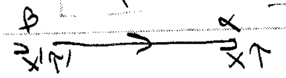
· Handout: spin sums in Mathematica.

Summary points from Monday:

- We can calculate $\ln Z/Z_0$ or $\mathcal{G}_{\alpha\beta}$ by diagrams.
  $\Rightarrow$ put Feynman rules from (136) on board.
- Do symmetry factors on (137) as warm-up.
- Recall

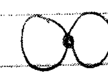
$$\boxed{\mathcal{G}^0_{\alpha\beta}(\vec{x}\tau;\vec{x}'\tau') = \delta_{\alpha\beta}\int\frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\, e^{-(\epsilon_k^0-\mu)(\tau-\tau')} \times\left[\theta(\tau-\tau'-\eta)(1-n_k^0)-\theta(\tau'-\tau+\eta)\,n_k^0\right]}$$

$$n_k^0 = \frac{1}{1+e^{\beta(\epsilon_k^0-\mu)}}$$

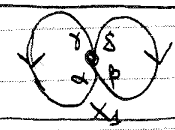


- Let's do the first orders of $\ln Z/Z_0$ in the $T\to 0$ limit
  - We need the noninteracting part. Do that on the next couple of pages.
  - We already have the order λ part from the notes, but let's do it from the Feynman rules:

Order λ^1

a. Draw all distinct diagrams with 1 vertex 

b. Assign $x_1 = (\vec{x}_1,\tau_1)$ to the vertex and $-\lambda$.

$\mathcal{G}^0_{\alpha\gamma}(x_1,x_1)$  $\mathcal{G}^0_{\beta\delta}(x_1,x_1)$ vertex $(\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma})$

· look at spin sum notebook

c. Spins $\underbrace{\delta_{\alpha\gamma}\delta_{\beta\delta}}_{\text{from } \mathcal{G}^0\text{'s}}(\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma}) = \delta_{\alpha\alpha}\delta_{\beta\beta}+\delta_{\gamma\delta}\delta_{\delta\gamma} = (-\nu)^2+\delta_{\gamma\gamma}$
$= \nu^2-\nu = \nu(\nu-1)$

d. $\boxed{\mathcal{G}^0(x_1,x_1) = \int\frac{d^3k}{(2\pi)^3}(-n_k^0) = \mathcal{G}^0(0,0)}$

$$\boxed{\int dx_1\,[\mathcal{G}^0(0,0)]^2 = [\mathcal{G}^0(0,0)]^2\int_0^\beta d\tau_1\int d^3x_1 = \beta V[\mathcal{G}^0(0,0)]^2}$$

e. symmetry factor $1\cdot\frac{1}{2}\cdot 1 \Rightarrow \frac{1}{2}$

$$\Rightarrow \ln\frac{Z}{Z_0} = \frac{1}{2}(\beta V)(-\lambda)(\nu^2-\nu)\left(\int\frac{d^3k}{(2\pi)^3}n_k^0\right)^2 = -\beta(\Omega-\Omega_0)$$

$$\text{or}\quad \Omega = \Omega_0 + V\frac{\lambda}{2}\left(1-\frac{1}{\nu}\right)\left(\nu\int\frac{d^3k}{(2\pi)^3}n_k^0\right)^2$$

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- Note that Z_0 is just the partition function for a non-interacting system of fermions.
  - We could calculate it from the path integral — our formula for Gaussian integrals tells us it is proportional to the determinant of $(\mathcal{G}^0)^{-1} \Rightarrow$ later
  - However, we could just as well take it from a 2nd quantization calculation. In particular, on the following pages are derived results for $\ln Z_0$ (which is what we need, really) for a noninteracting Fermi gas with degeneracy ν in a box of volume V.

- Let's summarize some of the basic results:

$$\boxed{\begin{aligned}\Omega_0(V,T,\mu) &= -\frac{1}{\beta}\sum_{\vec{k}\alpha}\ln\left(1+e^{-\beta(\epsilon_{\vec{k}}-\mu)}\right)\\ &\xrightarrow[V\to\infty]{} -\frac{g}{\beta}\frac{V}{(2\pi)^3}\int d^3k\,\ln\left(1+e^{-\beta(\epsilon_{\vec{k}}-\mu)}\right)\end{aligned}}$$

where $\boxed{\epsilon_k = \frac{\hbar^2k^2}{2m}}$

A change of variables to ϵ_k (which we just called ϵ) yields forms that are simpler to evaluate at finite temperature:

$$\boxed{\begin{aligned}\Rightarrow\ \Omega_0 &= -\frac{\nu}{\beta}\frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty d\epsilon\,\epsilon^{1/2}\ln\left(1+e^{\beta(\mu-\epsilon)}\right)\\ \overset{\text{partial integration}}&{=} \frac{\nu V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\frac{2}{3}\int_0^\infty d\epsilon\,\frac{\epsilon^{3/2}}{e^{\beta(\epsilon-\mu)}+1} = PV\end{aligned}}$$

We can find N from thermodynamics:

$$\boxed{\begin{aligned}N = -\frac{\partial\Omega_0}{\partial\mu} &= -\frac{\nu V}{\beta(2\pi)^3}\int d^3k\,\frac{1}{1+e^{-\beta(\epsilon_{\vec{k}}-\mu)}}\,e^{-\beta(\epsilon_{\vec{k}}-\mu)}\beta = \frac{\nu V}{(2\pi)^3}\int d^3k\,\underbrace{\frac{1}{e^{\beta(\epsilon_{\vec{k}}-\mu)}+1}}_{n_k^0}\\ &= \frac{\nu V}{4\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty d\epsilon\,\frac{\epsilon^{1/2}}{e^{\beta(\epsilon-\mu)}+1}\end{aligned}}$$

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We'll be taking the zero temperature limit in the interacting case; let's warm up with the noninteracting case.

- We'll calculate the noninteracting ground-state energy E_0 from

$$\boxed{E_0 = \lim_{T\to 0}(\Omega_0 + TS + \mu N) = \lim_{\beta\to\infty}(\Omega_0 + \mu_0 N)}$$

- Start with N. We'll use the expression with the integral over k.

$$\boxed{n_k^0 \equiv \frac{1}{e^{\beta(\epsilon_k-\mu)}+1} \longrightarrow \begin{cases} 0 & \text{if } \epsilon_k-\mu_0>0 \\ 1 & \text{if } \epsilon_k-\mu_0<0 \end{cases}}$$

$\Rightarrow$ the occupation number $n_k \to \theta(\mu_0-\epsilon_k)$.

So we fill levels until the last filled state has energy μ. Since this is the Fermi energy in the non-interacting Fermi gas, we have the result:

$$\text{at } T=0 \quad \boxed{\mu_0 = \epsilon_F = \frac{\hbar^2 k_F^2}{2m}} \Rightarrow \boxed{\rho = \frac{\nu}{6\pi^2}\left(\frac{2m\mu_0}{\hbar^2}\right)^{3/2}} = \frac{\nu}{6\pi^2}k_F^3$$

If we now consider Ω_0, then let $z \equiv \beta(\epsilon_k-\mu_0)$.

If $z\to+\infty$, then $\ln(1+e^{-z})\to 0$

If $z\to-\infty$, then $\ln(1+e^{-z})\to -z$

$$\boxed{\begin{aligned} \text{So } \Omega_0 \xrightarrow[\beta\to\infty]{} & -\frac{\nu}{\cancel{\beta}}\frac{V}{(2\pi)^3}\int d^3k\, -\cancel{\beta}(\epsilon_k-\mu_0)\,\theta(\mu_0-\epsilon_k) \\ &= +\frac{\nu V}{(2\pi)^3}\int d^3k\, \epsilon_k\,\theta(\mu_0-\epsilon_k) - \mu_0\frac{\nu V}{(2\pi)^3}\int d^3k\,\theta(\mu_0-\epsilon_k) \\ &= \sum_{k,\alpha}^{k_F}\frac{\hbar^2k^2}{2m} - \mu_0\sum_{k,\alpha}^{k_F}1 = \frac{3}{5}\epsilon_F V - \mu_0 N \end{aligned}}$$

From which we recover $\boxed{E_0 = \frac{3}{5}\epsilon_F\rho V = \frac{3}{5}\epsilon_F N}$

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As on 134, the factor $\boxed{\nu \int \frac{d^3k}{(2\pi)^3} n_k^0 = \frac{1}{V}\sum_{k,\alpha} n_k^0}$ is just the (thermal averaged) density of the system.

- The $O(\lambda)$ term has no μ dependence, so $\frac{\partial\Omega_1}{\partial\mu}=0$. In the $T=0$ limit this means

$$\boxed{N = \frac{\partial\Omega}{\partial\mu} = \frac{\partial\Omega_0}{\partial\mu}} \quad \text{and} \quad \boxed{\Omega_0 = E_0 + \mu N}$$

$$\Rightarrow \boxed{E = E_0 + E_1 = \frac{3}{5}\frac{\hbar^2 k_F^2}{2m}\cdot N + V\frac{\lambda}{2}\left(1-\frac{1}{\nu}\right)\rho^2}$$

or $$\boxed{\frac{E^{(0)}}{N}+\frac{E^{(1)}}{N} = \frac{3}{5}\frac{\hbar^2k_F^2}{2m} + \frac{\lambda}{2}\left(1-\frac{1}{\nu}\right)\rho = \frac{3}{5}\frac{\hbar^2k_F^2}{2m} + \frac{\lambda}{2}\left(1-\frac{1}{\nu}\right)\left(\frac{\nu k_F^3}{6\pi^2}\right)}$$

energy per particle

- shortly we'll find that $\lambda = \frac{4\pi a_s}{m}$ where a_s is the "scattering length"

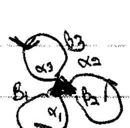
Energy density is $\frac{E}{V} = \frac{E}{N}\cdot\frac{N}{V} = \frac{E}{N}\rho$

$$\Rightarrow \boxed{\frac{E}{N} = \frac{k_F^2}{2m}\left[\frac{3}{5} + \underbrace{(\nu-1)\frac{2}{3\pi}}_{\text{order unity}}(k_F a_s) + O\left((k_F a_s)^2\right)\right]}$$

So looks like an expansion in $k_F a_s$ ⇒ return to this below.

- Note that if $\nu=1$, we get that the energy per particle is just that of the non interacting system.

- What if we had a 3-body interaction, $\propto \psi^\dagger_\gamma \psi^\dagger_\beta \psi^\dagger_\alpha \psi_\alpha \psi_\beta \psi_\gamma$

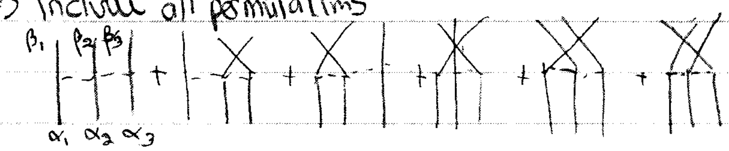


$\Rightarrow \propto \beta V$ from $\int d^3x_1$

$\propto \rho^3$ from $[\mathcal{G}^0(0,0)]^3$.

What about spin sum? Generalize the 2-body vertex rule

⇒ include all permutations



+ signs because "$-\nu$" rule takes care of "−"'s.

$$\Rightarrow \underbrace{\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2}\delta_{\alpha_3\beta_3}}_{\text{from propagator}}\big(\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2}\delta_{\alpha_3\beta_3} + \delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_3}\delta_{\alpha_3\beta_2} + \delta_{\alpha_1\beta_2}\delta_{\alpha_2\beta_1}\delta_{\alpha_3\beta_3}$$
$$+ \delta_{\alpha_1\beta_3}\delta_{\alpha_2\beta_2}\delta_{\alpha_3\beta_1} + \delta_{\alpha_1\beta_3}\delta_{\alpha_2\beta_1}\delta_{\alpha_3\beta_2} + \delta_{\alpha_1\beta_2}\delta_{\alpha_2\beta_3}\delta_{\alpha_3\beta_2}\big)$$

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If we give this to the Mathematica notebook that uses the deltasimplify package, we obtain $-(\nu-2)(\nu-1)\nu$

- Why does this make sense?
  - ⇒ now we need at least <u>3</u> different "spins".

- Comments on the deltasimplify.m package
  - A Mathematica package has definitions that can be loaded for use in a Mathematica notebook.
  - The "usage" commands give the help text returned when you do ?DeltaSimplify or ?del.
  - A Private context is used — this is like a separate namespace so there isn't interference if the user happens to use the same variable names.
    - Note that nu is in the <u>global</u> context.
  - The deltasimprules are a series of pattern matching rules: if the left side is matched, replace it with the right side.
    - The first rule ensures that all the products of sums & δ's are expanded out, so that we have $\delta_{ab}\delta_{bc}$ etc. multiplying each other.
    - Note that del[a,b] is just an object with two slots that has some rules associated with it!
    - $\delta_{ab}\delta_{bc} = \delta_{ac}$ from the first, and then all permutations.
    - finally, the key rule: $\delta_{aa} = -\nu$.
- The DeltaSimplify command, when applied to an expression, simply applies the rules over and over until the expression stops changing.
  - don't worry about the obscure syntax. Just copy it for your own rules!

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What would a spin-dependent force look like?

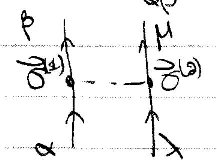
· If short-ranged:

$$\boxed{V_s(\vec{x}_1,\vec{x}_2)_{\alpha\beta,\lambda\mu} = \lambda_s\, \vec{\sigma}(1)_{\alpha\beta}\cdot\vec{\sigma}(2)_{\lambda\mu}\,\delta^3(\vec{x}_1-\vec{x}_2)}$$

where the Pauli matrices are labeled for particle 1 or particle 2.

· So the difference from the spin-independent case is

$$\delta^{(1)}_{\alpha\beta}\to\vec{\sigma}^{(1)}_{\alpha\beta} \quad\text{and}\quad \delta_{\lambda\mu}(2)\to\vec{\sigma}^{(2)}_{\lambda\mu}$$



so the σ-matrices live along the lines for particle 1 and 2

· The interaction term in the Lagrangian is generally

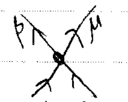
$$\tfrac{1}{2}\psi^\dagger_\alpha(x_1)\psi^\dagger_\lambda(x_2)V(\vec{x}_1,\vec{x}_2)_{\alpha\beta,\lambda\mu}\psi_\mu(x_2)\psi_\beta(x_1)$$

spin-independent $V(\vec{x}_1,\vec{x}_2)_{\alpha\beta,\lambda\mu} = \lambda\,\delta_{\alpha\beta}\delta_{\lambda\mu}\delta^3(\vec{x}_1-\vec{x}_2)$

$$\Rightarrow \tfrac{\lambda}{2}\psi^\dagger_\alpha(x)\psi^\dagger_\lambda(x)\psi_\lambda(x)\psi_\alpha(x) \quad \text{as before.}$$

Now spin-dependent $\Rightarrow \tfrac{\lambda_s}{2}\psi^\dagger_\alpha(x)\psi^\dagger_\lambda(x)\psi_\mu(x)\psi_\beta(x)\,\vec{\sigma}_{\alpha\beta}\cdot\vec{\sigma}_{\lambda\mu}$

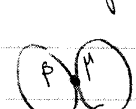
· The Feynman rule



$\Rightarrow (-\lambda)\,\delta_{\alpha\beta}\delta_{\lambda\mu}+\delta_{\alpha\mu}\delta_{\lambda\beta}$ and $-\nu$ for $\delta_{\alpha\alpha}$

now becomes (replacing δ's with σs): $(-\lambda_s)(\vec{\sigma}_{\alpha\beta}\cdot\vec{\sigma}_{\lambda\mu}+\vec{\sigma}_{\alpha\mu}\cdot\vec{\sigma}_{\lambda\beta})$

· Try the bowtie



$(-\lambda_s)\tfrac{1}{2}[\mathcal{G}^0(0,0)]^2\beta V\times\delta_{\alpha\beta}\delta_{\lambda\mu}(\vec{\sigma}_{\alpha\beta}\cdot\vec{\sigma}_{\lambda\mu}+\vec{\sigma}_{\alpha\mu}\cdot\vec{\sigma}_{\lambda\beta})$

Spin factor is $\cancel{\vec{\sigma}_{\alpha\alpha}\cdot\vec{\sigma}_{\lambda\lambda}}^{\,0}+\vec{\sigma}_{\alpha\mu}\cdot\vec{\sigma}_{\mu\alpha} = 3\,\delta_{\alpha\alpha} = -3\nu$ instead of $\nu(\nu-1)$

from dot product

$\Rightarrow$ we get 3 times the Fock term from the spin-independent case.

· The average over spin makes the Hartree term vanish.

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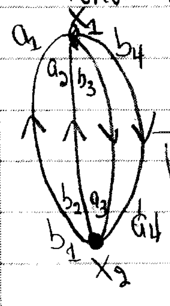
Let's do the famous beachball diagram... (With the $T\to 0$ or $\beta\to\infty$ limit in mind)

· Follow the Feynman rules for (using $\alpha\to a$, $\beta\to b$)



Convention to label lines pointing from b to a

· symmetry factor $1\times\left(\frac{1}{2}\right)^2\times\frac{1}{2}=\frac{1}{8}$

$$\left[\ln\frac{Z}{Z_0}\right]_{\text{beachball}} = -\beta\Omega^{(2)}_{\text{beachball}}$$

$$= \frac{1}{8}(-\lambda)^2\int d\tau_1 d^3x_1\int d\tau_2 d^3x_2\, \mathcal{G}^0_{k_1}(x_1,x_2)\,\mathcal{G}^0_{k_2}(x_1,x_2)\,\mathcal{G}^0_{k_3}(x_2,x_1)\,\mathcal{G}^0_{k_4}(x_2,x_1)$$

Mathematica says $2\nu(\nu-1)$ $\Rightarrow$

$$\times\underbrace{\delta_{a_1b_1}\delta_{a_2b_2}\delta_{a_3b_3}\delta_{a_4b_4}}_{\text{from } \mathcal{G}^0\text{'s}}\underbrace{\left(\delta_{a_1b_3}\delta_{a_2b_4}+\delta_{a_1b_4}\delta_{a_2b_3}\right)}_{\text{top vertex}}\underbrace{\left(\delta_{a_3b_1}\delta_{a_4b_2}+\delta_{a_3b_2}\delta_{a_4b_1}\right)}_{\text{bottom vertex}}$$

· We've peeled off the spin labels from the $\mathcal{G}^0$'s and added labels for the value of k in

$$\mathcal{G}^0_{k_i}(\vec{x}_1\tau_1,\vec{x}_2\tau_2) = \int\frac{d^3k_i}{(2\pi)^3}\, e^{i\vec{k}_i\cdot(\vec{x}_1-\vec{x}_2)}\, e^{-(\epsilon^0_{k_i}-\mu)(\tau_1-\tau_2)} \times\left[\theta(\tau_1-\tau_2-\eta)(1-n^0_k)-\theta(\tau_2-\tau_1+\eta)\,n^0_k\right]$$

to each line.

· All of the $\mathcal{G}^0$'s in the beachball depend on $\vec{x}_1-\vec{x}_2$, $\tau_1-\tau_2$ so we can do those integrals:

$$\int d^3x_1\int d^3x_2\, e^{i(\vec{k}_1+\vec{k}_2-\vec{k}_3-\vec{k}_4)\cdot(\vec{x}_1-\vec{x}_2)} \underset{\vec{y}=\vec{x}_1-\vec{x}_2}{=} \int d^3x_1\int d^3y\, e^{i(\vec{k}_1+\vec{k}_2-\vec{k}_3-\vec{k}_4)\cdot\vec{y}}$$

$$= V\,(2\pi)^3\,\delta^3(\vec{k}_1+\vec{k}_2-\vec{k}_3-\vec{k}_4)$$

$\Rightarrow$ momentum conservation! (always happens $\Rightarrow$ momentum Feynman rules)

· Note that we get a single factor of the volume V. What if there were disconnected diagrams? Eg. $\bigcirc\!\!\!\!|\;\times\infty \Rightarrow V^2$ (and so on)

· But $\ln Z/Z_0 \propto \Omega-\Omega_0$, which should be extensive.

$\Rightarrow$ the linked cluster expansion ensures extensivity!

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What about τ_1 and τ_2 integrals?

- Shift from $[0,\beta]$ to $[-\beta/2,\beta/2]$ so $\beta\to\infty$ is more convenient!

$$\boxed{\begin{matrix}\tau_1'=\tau_1-\beta/2\\ \tau_2'=\tau_2-\beta/2\end{matrix}}\qquad \boxed{\int_0^\beta d\tau_1\int_0^\beta d\tau_2 = \int_{-\beta/2}^{\beta/2}d\tau_1'\int_{-\beta/2}^{\beta/2}d\tau_2' \xrightarrow[\beta\to\infty]{} \beta\int_{-\infty}^{\infty}dy_0}\quad \text{where } y_0=\tau_1-\tau_2$$

What about Θ functions? $\Theta(\tau_1-\tau_2)\times\Theta(\tau_1-\tau_2)=\Theta(\tau_1-\tau_2)$

$\Theta(\tau_1-\tau_2)\times\Theta(\tau_2-\tau_1)=0$

$\Rightarrow$ only 2 terms survive and they are equal (just interchange $1,2\Leftrightarrow 3,4$)

so keep one of them and overall factor of 2

$$\Rightarrow\ 2\beta\int_{-\infty}^{\infty}dy_0\,\Theta(y_0)\,e^{-[(\epsilon^0_{k_1}-\mu)+(\epsilon^0_{k_2}-\mu)-(\epsilon^0_{k_3}-\mu)-(\epsilon^0_{k_4}-\mu)]y_0}\times(1-n^0_{k_1})(1-n^0_{k_2})(-n^0_{k_3})(-n^0_{k_4})$$

$$=2\beta(1-n^0_{k_1})(1-n^0_{k_2})n^0_{k_3}n^0_{k_4}\underbrace{\int_0^\infty dy_0\,e^{-(\epsilon^0_{k_1}+\epsilon^0_{k_2}-\epsilon^0_{k_3}-\epsilon^0_{k_4})y_0}}_{+\frac{1}{\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4}}$$

The μ's cancel! Only μ dependence in

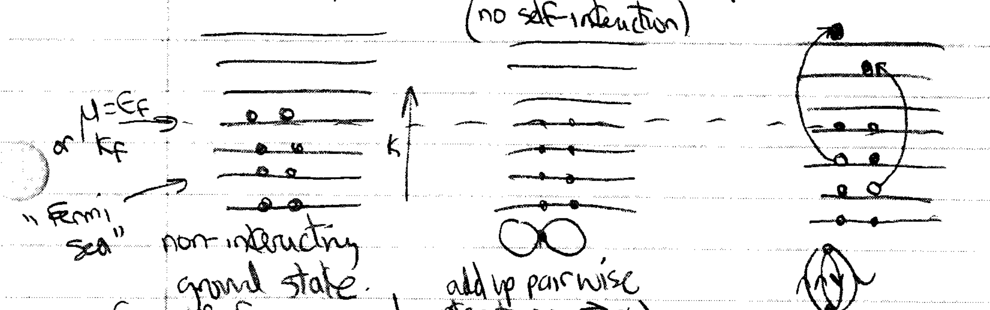
This is an energy denominator!

Put in all together! symmetry factor / spin sum / 2 sets of Θ's

$$\boxed{\begin{aligned}\left(\ln\frac{Z}{Z_0}\right)_{\text{beachball}} &= -\beta\Omega^{(2)}_{\text{beachball}}=\frac{1}{8}[2\nu(\nu-1)]\cdot 2\,\beta V\int\frac{d^3k_1\cdots d^3k_4}{(2\pi)^{12}}(2\pi)^3\delta^4(\vec{k}_1+\vec{k}_2-\vec{k}_3-\vec{k}_4)\frac{(-\lambda)^2}{\epsilon^0_{k_1}+\epsilon^0_{k_2}-\epsilon^0_{k_3}-\epsilon^0_{k_4}}\\ &=-\beta V\mathcal{E}^{(2)}_{\text{beachball}} \leftarrow T=0 \text{ energy density}\end{aligned}}$$

Consider pictures in the $T\to 0$ $(\beta\to\infty)$ limit

(no self-interaction)



non-interacting ground state.
- Sum ϵ_k for occupied

add up pairwise interactions. $\Rightarrow\alpha\lambda$
- Account for antisymmetry

"2p-2h"
excited states have 2 particles above, leaving two "holes" below
$\Rightarrow$ sum over allowed possibilities

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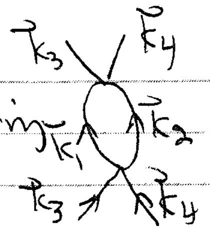
Scattering  ⇐ virtual excitations (violate energy conservation)

Momentum conservation holds! $\vec{k}_1+\vec{k}_2=\vec{k}_3+\vec{k}_4$ is required (as vectors)

• Why not 1p-1h? (can't conserve momentum)

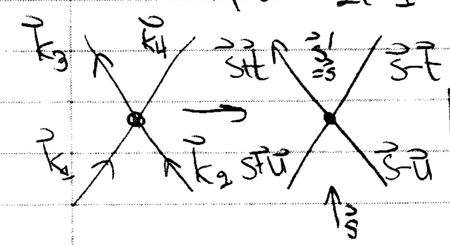
• This is 2nd-order perturbation theory for many-body system!

$$\boxed{H=H_0+H_1,\ H_0|\Phi_n\rangle=E_n|\Phi_n\rangle} \Rightarrow \boxed{\delta E^{(2)}=-\sum_{n\neq 0}\frac{\langle\Phi_0|H_1|\Phi_n\rangle\langle\Phi_n|H_1|\Phi_0\rangle}{E_n-E_0}}$$

(note order and "−" out front)

Switch variable to make clearer what is happening
⇒ identify total and relative momenta

e.g. $k_F\vec{s}=\frac{1}{2}(\vec{k}_1+\vec{k}_2)=\frac{1}{2}(\vec{k}_3+\vec{k}_4)\equiv k_F\vec{s}\,'$ momentum [could define without ½]
and $k_F\vec{u}=\frac{1}{2}(\vec{k}_1-\vec{k}_2)$, $k_F\vec{t}=\frac{1}{2}(\vec{k}_3-\vec{k}_4)$



Claim:
$$\boxed{\int d^3k_1\cdots d^3k_4\,\delta^3(\vec{k}_1+\vec{k}_2-\vec{k}_3-\vec{k}_4)=8k_F^9\int d^3s\,d^3u\,d^3t}$$

[Proof: Use $k_F\vec{s}\,'$ ⇒ $\delta(k_{1x}+k_{2x}-k_{3x}-k_{4x})=\delta(2k_F(s_x-s_x'))=\frac{1}{2k_F}\delta(s_x-s_x')$
for each Cartesian coordinate. Then $k_{1x}=k_F(s_x+u_x)$ and so on

$$\Rightarrow \int dk_{1x}dk_{2x}dk_{3x}dk_{4x}\,\delta(k_{1x}+k_{2x}-k_{3x}-k_{4x})=k_F^4\int ds_x\,ds_x'\,du_x\,dt_x\,\frac{1}{2k_F}\delta(s_x-s_x')\begin{vmatrix}\frac{\partial k_{1x}}{\partial s_x} & \frac{\partial k_{2x}}{\partial s_x}\\ \frac{\partial k_{1x}}{\partial u_x} & \frac{\partial k_{2x}}{\partial u_x}\end{vmatrix}\begin{vmatrix}\frac{\partial k_{3x}}{\partial s_x'} & \frac{\partial k_{4x}}{\partial s_x'}\\ \frac{\partial k_{3x}}{\partial t_x} & \frac{\partial k_{4x}}{\partial t_x}\end{vmatrix}$$

$$=k_F^3(2\cdot2\cdot\tfrac{1}{2})\int ds_x\,du_x\,dt_x\Big)$$

$$\left(\begin{vmatrix}+1 & +1\\ -1 & -1\end{vmatrix}=2\right)$$

$\Rightarrow 2^3=8$ overall. QED]

• note: choosing different variables might make this easier if Jacobian could be simpler!

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Now

$$\varepsilon^0_{k_1}+\varepsilon^0_{k_2}=\frac{k_f^2}{2m}\left((s+u)^2+(s-u)^2\right)=\frac{k_f^2}{m}(s^2+u^2)$$

$$-\quad \varepsilon^0_{k_3}+\varepsilon^0_{k_4}=\frac{k_f^2}{2m}\left((s+t)^2+(s-t)^2\right)=\frac{k_f^2}{m}(s^2+t^2)$$

$$\overline{\qquad\frac{k_f^2}{2m}(u^2-t^2)\qquad}$$

So finally, in the $T\to 0$ where $\boxed{n_k\to\theta(k_f-|\vec{k}|)}$, $\boxed{1-n_k\to\theta(|\vec{k}|-k_f)}$ we have

$$\boxed{\varepsilon^{(2)}_{bb}=-4\lambda^2 m\,\nu(\nu-1)\,k_f^7\int\frac{d^3s}{(2\pi)^3}\frac{d^3t}{(2\pi)^3}\frac{d^3u}{(2\pi)^3}\,\theta(1-|\vec{s}+\vec{t}|)\,\theta(1-|\vec{s}-\vec{t}|)\,\theta(|\vec{s}+\vec{u}|-1)\,\theta(|\vec{s}-\vec{u}|-1)\times\frac{1}{u^2-t^2}}$$

can this vanish?

- Now s and t integrals are limited so $|\vec{s}|\le 1$, $|\vec{t}|\le 1$ but look at $\vec{u}$ large

$$\Rightarrow\ \theta(|\vec{s}+\vec{u}|-1)=\theta(|\vec{s}-\vec{u}|-1)\longrightarrow 1$$

$$\Rightarrow\ \int_{u\ \text{large}}\frac{d^3u}{(2\pi)^3}\frac{1}{u^2-t^2}\ \propto\ \int^{\infty}\frac{u^2du}{u^2}\longrightarrow\infty!$$ (like $\int^{\Lambda}du\propto\Lambda$, so called a "linear divergence")

How do we deal with this?

Note that it came from having $V(\vec{x}_1-\vec{x}_2)=\lambda\,\delta^3(\vec{x}_1-\vec{x}_2)$.
Is it enough to take $\delta^3\to e^{-|\vec{x}_1-\vec{x}_2|^2/b^2}$?

- would be finite, but sensitive to b...

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- Nonperturbative approximations
  - We have diagrammatic expansions to calculate the ground state energy or finite temperature thermodynamic functions in perturbation theory.
  - What if pert. theory is inadequate?

- Here: Nonperturbative "conserving approximation" that sum infinite classes of diagrams. [Also called "phi-derivable"]

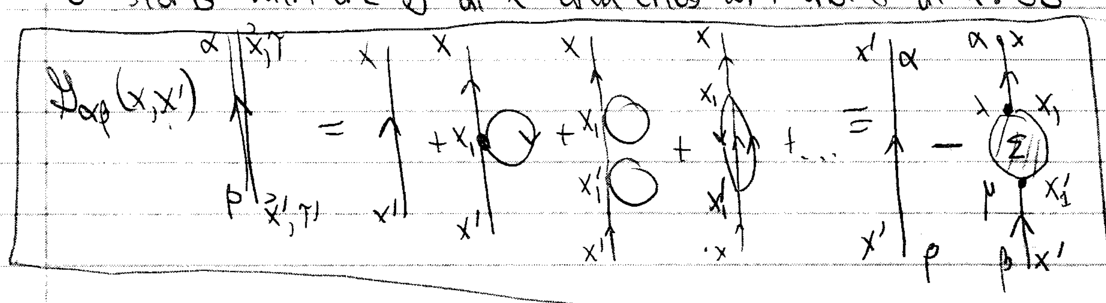
- First, go back to one body Green's function:

$$\mathcal{G}_{\alpha\beta}(\vec{x}\tau;\vec{x}'\tau') = \frac{\int \mathcal{D}(\psi^\dagger\psi)\,\psi_\alpha(\vec{x}\tau)\,\psi^\dagger_\beta(\vec{x}'\tau')\,e^{-S_E}}{\int \mathcal{D}(\psi^\dagger\psi)\,e^{-S_E}}$$

$$= \mathrm{Tr}\left[e^{-\beta(\hat{H}-\mu\hat{N})}\,T\left[\hat{\psi}_\alpha(\vec{x}\tau)\,\hat{\psi}^\dagger_\beta(\vec{x}'\tau')\right]\right] / \mathrm{Tr}\left[e^{-\beta(\hat{H}-\mu\hat{N})}\right]$$

  - see (129) – (130) for more details on field operators.

- In the diagrammatic expansion, each term is either $\mathcal{G}^0_{\alpha\beta}(x,x')$ with $x=(\vec{x},\tau)$ or starts with one $\mathcal{G}^0$ at x' and ends with another at x. So



is the general structure, where the "improper self-energy" Σ stands for all possible (connected) diagram insertions.

In equation form, this is an integral equation (defining the self energy)

$$\boxed{\mathcal{G}_{\alpha\beta}(x,x') = \mathcal{G}^0_{\alpha\beta}(x,x') - \int d\tau_1 d^3x_1 \int d\tau_1' d^3x_1'\, \mathcal{G}^0_{\alpha\lambda}(x,x_1)\,\Sigma(x_1,x_1')_{\lambda\mu}\,\mathcal{G}_{\mu\beta}(x_1',x')}$$

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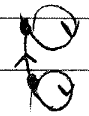
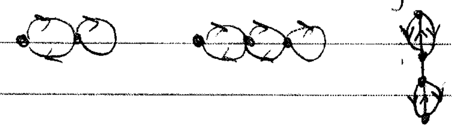
But we can go further. Compare two 2nd order contributions to Σ:

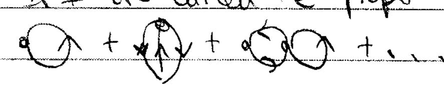
 and 

$\Rightarrow$ each part of the left diagram looks like a self-energy piece, because there is a single line joining them.

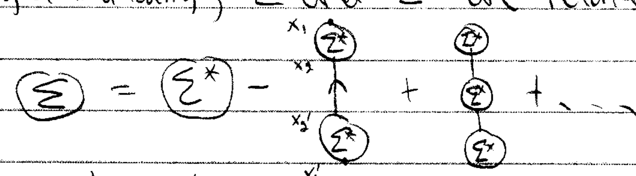
$\Rightarrow$ call it 1PR "one-particle reducible"

- The right diagram is 1PI "one-particle irreducible" because it does not fall into two pieces when a single line is cut (unlike the left diagram).
  - Check: are the following 1PR or 1PI?



- The diagrams in Σ that are 1PI are called the "proper" self-energy and designated Σ^* = 

- Diagrammatically, Σ and Σ^* are related by



or, in equations (suppressing spin indices)

$$\boxed{\Sigma(x_1,x_1') = \Sigma^*(x_1,x_1') - \int d^4x_2\, d^4x_2'\, \Sigma^*(x_1,x_2)\, G^0(x_2,x_2')\, \Sigma^*(x_2',x_1') + \dots}$$

where $\int d^4x_2$ means $\int d\tau_2\, d^3x_2$, etc.

- Now we can insert this equation back into our original equation for G to derive another integral equation:

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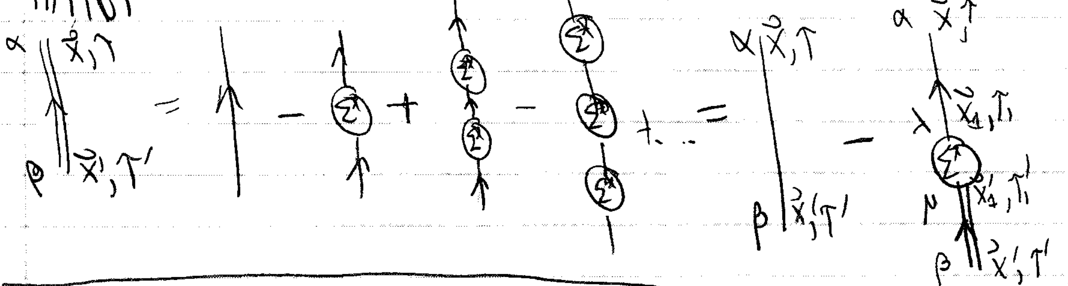

$$\boxed{\Rightarrow \mathcal{G}_{\alpha\beta}(x,x') = \mathcal{G}^0_{\alpha\beta}(x,x') - \int^\beta d\tau_1\, d^3x_1 \int^\beta d\tau_1'\, d^3x_1'\; \mathcal{G}^0_{\alpha\lambda}(x,x_1)\,\Sigma^*(x_1,x_1')_{\lambda\mu}\, \mathcal{G}_{\mu\beta}(x_1',x')}$$

- This is "Dyson's equation" for the propagator ("2-point function")
  - Note that it is like the previous equation for $\mathcal{G}$ except in the integral $\Sigma \to \Sigma^*$ and $\mathcal{G}^0 \to \mathcal{G}$.

- If we approximate Σ^* to some order in perturbation theory, we get an <u>all</u> orders approximation to $\mathcal{G}$!

- Think of the Dyson's equation as a matrix equation (in spin <u>and</u> space time indices)

$$\underline{\underline{\mathcal{G}}} = \underline{\underline{\mathcal{G}}}^0 + \underline{\underline{\mathcal{G}}}^0 \underline{\underline{\Sigma}}^* \underline{\underline{\mathcal{G}}} \Rightarrow \underline{\underline{\mathcal{G}}} - \underline{\underline{\mathcal{G}}}^0 \underline{\underline{\Sigma}}^* \underline{\underline{\mathcal{G}}} = \underline{\underline{\mathcal{G}}}^0$$

or $(\underline{\underline{1}} - \underline{\underline{\mathcal{G}}}^0 \underline{\underline{\Sigma}}^*)\underline{\underline{\mathcal{G}}} = \underline{\underline{\mathcal{G}}}^0 \Rightarrow \underline{\underline{\mathcal{G}}} = (\underline{\underline{1}} - \underline{\underline{\mathcal{G}}}^0 \underline{\underline{\Sigma}}^*)^{-1} \underline{\underline{\mathcal{G}}}^0 \Rightarrow \boxed{\underline{\underline{\mathcal{G}}}^{-1} = \underline{\underline{\mathcal{G}}}^{0\,-1}(\underline{\underline{1}} - \underline{\underline{\mathcal{G}}}^0 \underline{\underline{\Sigma}}^*) = \underline{\underline{\mathcal{G}}}^{0\,-1} - \underline{\underline{\Sigma}}^*}$

Recall that $\underline{\mathcal{G}}^{0\,-1} = \frac{\partial}{\partial\tau} + \frac{\nabla^2}{2m}$, so Σ^* is like an external potential.

- How do we get the (free) energy? We might think $\int \mathcal{G}\Sigma \Rightarrow$ Σ^* because, if expanded, it has all the diagrams for $\ln Z/Z_0$
  - But the factors are incorrect!

$\Rightarrow$ We'll come back and see how to do it correctly!