3/10/03

Ok, let's back up and see where the expression we wrote on pg. 230 comes from.

Returning to $Z[J]$ after the $\psi^\dagger, \psi$ integration: (from 228)

$$\boxed{\begin{aligned} Z[J] = e^{iW[J]} &= \int \mathcal{D}\sigma\, e^{g \operatorname{Tr} \ln G^{-1}(x,y) + \frac{i}{2} C_0 \int d^4x\, (\sigma(x))^2 + i\int d^4x\, J(x)\sigma(x)} \\ &\equiv \int \mathcal{D}\sigma\, e^{i(I[\sigma] + \int J\sigma)} \end{aligned}}$$

which defines $I[\sigma]$.

- We'll use the schematic notation $\int J\sigma$ to mean $\int d^4x\, J(x)\sigma(x)$.
- Note that $I[\sigma]$ shows up in the path integral as the action for the σ field.

- With the rescaling $C_0 = c_0/g$, $\sigma = g\sigma'$, we see again there is an overall factor of g in the exponent $\Rightarrow$ saddle point evaluation.

- So we're going to expand around the stationary point. At lowest order, this is $\sigma(x) = \sigma_0(x)$ but this can change as we go higher in the expansion, unless we make sure it works. So we will by introducing a counterterm for J.

- We write
$$\boxed{\begin{aligned} I[\sigma] &= I_1[\sigma] + \delta I[\sigma] \\ J(x) &= J_1(x) + \delta J(x) \end{aligned}}$$

- $\delta I[\sigma]$ contains the "usual" counterterms we have in quantum field theory, which fix up the short-distance behavior. [see 155 - 161]
  - For our short-range effective field theory, using dimensional regularization and minimal subtraction, we will not need to write it explicitly [see 162 - 170]

3/10/03

- What about δJ? Define J_1 so that

$$\boxed{\frac{\delta I_1}{\delta \sigma}\Big|_{\sigma=\sigma_c} + J_1(x) = 0}$$

(i.e. choose $J_1(x) = -\frac{\delta I_1}{\delta \sigma}\Big|_{\sigma=\sigma_c}$).

- Then determine $\delta J(x)$ order-by-order in the expansion such that $\boxed{\langle \sigma(x)\rangle_J = \sigma_c(x)}$ as before. (see below).

So then

$$\boxed{e^{iW[J]} = \int \mathcal{D}\sigma\, e^{i(I_1[\sigma] + \int J_1\sigma)}\, e^{i(\delta I[\sigma] + \int \delta J\sigma)}}$$

- focus on the first term — do our $\sigma = \sigma_c + \eta$ expansion:

$$\boxed{\begin{aligned} I_1[\sigma] + \int J_1\sigma &= \left(I_1[\sigma_c] + \int J_1\sigma_c\right) + \int d^4x\, \eta(x)\left(\cancel{\frac{\delta I_1[\sigma]}{\delta\sigma(x)} + J_1(x)}^{\,0}\right)\Big|_{\sigma=\sigma_c} \\ &+ \frac{1}{2}\int d^4x\, d^4y\, \eta(x)\frac{\delta^2 I_1}{\delta\sigma(x)\delta\sigma(y)}\Big|_{\sigma=\sigma_c}\eta(y) + \frac{1}{3!}\int d^4x\, d^4y\, d^4z\, \eta(x)\eta(y)\eta(z)\frac{\delta^3 I[\sigma]}{\delta\sigma(x)\delta\sigma(y)\delta\sigma(z)} \\ &+ \int J_1\eta \qquad + \cdots \end{aligned}}$$

- The linear term in η vanishes by construction (recall the expansion of $f(t)$ in $\tau = t - t_0$:

$$\boxed{I(\ell) = e^{-\ell f_0}\sqrt{\frac{2\pi}{\ell}}\int_{-\infty}^{\infty} d\tau\, e^{-\frac{f_0''\tau^2}{2}}\, e^{\sum_{n=3}^{\infty}\frac{\tau^n}{n!}\frac{f_0^{(n)}}{\ell^{(n/2-1)}}}}$$

- $\ln I(\ell)$ had an expansion with the "classical" term $-\ell f_0$ (here $I_1[\sigma_c] + \int J_1\sigma_c$), a quadratic term that gave the propagator $\longrightarrow \Rightarrow \frac{1}{f_0''}$ (here the inverse of $\delta^2 I_1/\delta\sigma\delta\sigma$) and then a series of vertices $\to -f_0^{(n)}/\ell^{n/2-1}$ (here the $\frac{\delta^3 I}{\delta\sigma\delta\sigma\delta\sigma}$ and higher terms).

3/10/03

· So identify $W[J]$ neglecting the $\frac{\delta^3 I}{\delta\sigma\delta\sigma\delta\sigma}$ and higher terms at first

$$\boxed{e^{iW[J_1]} = e^{i\left[\frac{1}{i} g\,\mathrm{Tr}\ln G^{-1}(x,y)\big|_{\sigma=\sigma_c(x)} + \frac{C_0}{2}\int d^4x\,(\sigma_c(x))^2 + \int d^4x\, J_1(x)\sigma_c(x)\right]} \times \int \mathcal{D}\eta\; e^{\frac{i}{2}\int d^4x\, d^4y\, \eta(x)\left\{\left[\frac{\delta^2\left[\frac{g}{i}\mathrm{Tr}\ln G^{-1}\right]}{\delta\sigma(x)\delta\sigma(y)}\right]\Big|_{\sigma=\sigma_c} + C_0\delta^4(x-y)\right\}\eta(y)}}$$

$(\mathcal{D}\sigma = \mathcal{D}\eta)$

where we've used $\boxed{\frac{\delta}{\delta\sigma(x)}\frac{\delta}{\delta\sigma(y)}\left[\frac{1}{2}C_0\int d^4z\,(\sigma(z))^2\right] = C_0\delta^4(x-y)}$

Now $\boxed{G^{-1}(x,y)\Big|_{\sigma=\sigma_c} = \left[i\frac{\partial}{\partial t} + \frac{\nabla_x^2}{2m} + \mu - C_0\sigma_c(x)\right]\delta^4(x-y) \equiv G_H^{-1}(x,y)}$

(suppressing spin here), which we call the "Hartree" propagator.

- It is the single particle Green's function in the presence of the "background field" $\sigma_c(x)$.
  $\Rightarrow \sigma_c(x)$ is just like an external potential,
- We can solve for $G_H(x,y)$ by methods analogous to the problem set problem.

· The η integral is just a Gaussian integral, with a scary looking operator between the η's, which we define as the inverse σ propagator D_0^{-1}:

$$\boxed{\begin{aligned} iD_0^{-1}(x,y) &= C_0\delta^4(x-y) + \frac{\delta^2[g\,\mathrm{Tr}\ln G^{-1}]}{\delta\sigma\delta\sigma}\Big|_{\sigma=\sigma_c} \\ &= C_0\delta^4(x-y) + igC_0^2\, G_H(y,x)\, G_H(x,y) \end{aligned}}$$

· Here we've used (in schematic notation). Note $D_0 = $ - - - + - -O- - + - -OO- - + . . .

$\boxed{\frac{\delta\,\mathrm{Tr}\ln G^{-1}}{\delta\sigma} = G\frac{\delta G}{\delta\sigma} = -C_0 G}$ and $\boxed{\frac{\delta(GG^{-1})}{\delta\sigma} = 0 = \frac{\delta G}{\delta\sigma}G^{-1} + G\frac{\delta G^{-1}}{\delta\sigma} \text{ or } \frac{\delta G}{\delta\sigma} = -C_0 GG}$

(fill in indices for problem)

3/10/03

- If we put back the higher order η terms and treated them in perturbation theory (by taking derivatives with respect to a source)
  $\Rightarrow$ diagrams with these vertices and lines (propagators) given by $D_0(x,y)$.

- So now we can find $\boxed{\Gamma[\sigma_c] = W[J] - \int d^4x\, J(x)\sigma_c(x)}$ to quadratic order because we can just take the ln of $Z[J]$ to find $W[J]$.

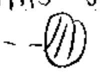
$$\Rightarrow \boxed{W[J] = \frac{g}{i}\operatorname{Tr}\ln\left[G_H^{-1}(x,y)\right] + \frac{c_0}{2}\int d^4x\,(\sigma_c(x))^2 + \int J_1(x)\sigma_c(x) + \frac{i}{2}\operatorname{Tr}\ln\left[D_0^{-1}(x,y)\right]}$$

from Gaussian boson path integral

- Now we still have $\delta I[\sigma]$ and δJ that we left behind. We expand these around σ_c as well:

$$\boxed{\underbrace{\left(\delta I[\sigma_c] + \int \delta J\,\sigma_c\right)}_{\text{just added to } W[J] \text{ above}} + \underbrace{\left(\delta I[\sigma_c+\eta] - \delta I[\sigma_c] + \int \delta J\,\eta\right)}_{\text{counterterm vertices}}}$$

- Now we can do the Legendre transformation, since we have $W[J] = \dots + \int (J_1 + \delta J)\sigma_c$
  $\Rightarrow$ J dependence in $\Gamma[\sigma_c]$ goes away, as expected!

- The counterterms $\delta I[\sigma_c]$ and $\delta I[\sigma_c+\eta] - \delta I[\sigma_c]$ do their usual job.
- The term $\int \delta J\,\eta$ takes care of ensuring $\langle\sigma\rangle = \sigma_c$ because we choose δJ (order by order) so that $\langle\eta\rangle = 0$.
  - In practice this simply means we ignore any "tadpole" diagrams  .

3/10/03

Next quarter we'll look at the higher order diagrams in the expansion.

- For now now we'll stop at:

$$\boxed{\Gamma[\sigma_c] = \frac{g}{i}\,\mathrm{Tr}\ln[G_H^{-1}(x,y)] + \frac{C_0}{2}\int d^4x\,(\sigma_c(x))^2 + \frac{i}{2}\mathrm{Tr}\ln[D_G^{-1}(x,y)]}$$

- Let's turn to our uniform dilute system.
  - Uniform $\Rightarrow$ $\sigma_c(x) = \sigma_c$, a constant
  - Later we'll need to test whether the assumption that the ground state is uniform is valid. (true for $C_0 > 0$ repulsive, not true for $C_0 < 0$ attractive!)

- We can evaluate $\mathrm{Tr}\ln[G_H^{-1}]$ most easily by diagonalizing G_H^{-1} $\Rightarrow$ Tr is simple.
  - For a uniform system, G_H^{-1} is diagonal in momentum space. Drop μ and put in boundary conditions

$$\Rightarrow \boxed{\mathrm{Tr}\ln G_H^{-1} = VT\int\frac{d^3p}{(2\pi)^3}\int\frac{dp_0}{2\pi}\ln(p_0 - e_{\vec p} + i\eta\,\mathrm{sgn}(|\vec p| - k_F))}$$

(VT is the space time volume).

- Here $\boxed{e_{\vec p} = \frac{p^2}{2m} + C_0\sigma_c}$

- As it stands, this expression is divergent, so there must be counterterm subtractions we've left out.
  - It's easiest to proceed by noticing that the same divergence is present when $C_0 = 0$, when we know the answer is $\boxed{-VT\,\mathcal{E}_{\text{noninteracting}} = -VT\,\frac{3}{5}\frac{k_F^2}{2m}\rho}$

  $\Rightarrow$ calculate the difference between $\mathrm{Tr}\ln G_H^{-1}|_{C_0}$ and $\mathrm{Tr}\ln G_H^{-1}|_{C_0=0}$.

3/10/03

So take $C_0 \to \lambda C_0$ and use $\boxed{\int_0^1 d\lambda \frac{d}{d\lambda} f(\lambda) = f(1) - f(0)}$

$$\boxed{\begin{aligned}\Rightarrow \Delta g \mathrm{Tr} \ln G_H^{-1} &= \int_0^1 d\lambda \frac{d}{d\lambda} VT \int \frac{d^3p}{(2\pi)^3} \int \frac{dp_0}{2\pi} \ln\left(p_0 - \frac{\vec{p}^2}{2m} - \lambda C_0 \sigma_c + i\eta\, \mathrm{sgn}(|\vec{p}| - k_f)\right) \\ &= gVT \int \frac{d^3p}{(2\pi)^3} \int_0^1 d\lambda \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \left(\frac{\theta(|\vec{p}| - k_f)}{p_0 - e_p + i\eta} + \frac{\theta(k_f - |\vec{p}|)}{p_0 - e_p - i\eta}\right) e^{ip_0\eta} (-\lambda C_0 \sigma_c) \\ &= -VT C_0 \sigma_c \left(\cancel{\int_0^1 d\lambda\, \lambda}^{\,1}\right) g \int \frac{d^3p}{(2\pi)^3} \left[i\theta(k_f - |\vec{p}|)\right] \\ &= -iVT C_0 \sigma_c \rho \end{aligned}}$$

convergence factor

So we get the leading order (LO) effective action as

$$\boxed{\Gamma_{LO}[\sigma_c] = VT\left(-\frac{3}{5}\frac{k_f^2}{2m}\rho - C_0 \sigma_c \rho + \frac{1}{2} C_0 \sigma_c^2\right)}$$

from $\frac{C_0}{2}\int d^4x [\sigma(x)]^2$

We require it to be stationary:

$$\boxed{\left.\frac{\delta \Gamma_{LO}}{\delta \sigma_c}\right|_{\sigma_c = \sigma_c^0} = 0} \Rightarrow \boxed{\sigma_c^0 = \rho}$$

Plugging back in to Γ_{LO},

$$\boxed{\mathcal{E}_{LO} = -\frac{1}{VT}\Gamma_{LO}[\sigma_c^0] = \rho\left(\frac{3}{5}\frac{k_f^2}{2m} + \frac{C_0}{2}\rho\right) = \rho \frac{k_f^2}{2m}\left(\frac{3}{5} + g\frac{2}{3\pi} k_f a_s\right)}$$

where we've used $\boxed{C_0 = \frac{4\pi a_s}{m}}$.

- Note that this includes Hartree but not Fock (g vs. $g-1$), consistent with the large g limit.

3/10/03

- Ok, now for Γ_{NLO}, where we use a similar evaluation of $1/2\,\mathrm{Tr}\ln\left[D_0^{-1}(x,y)\right]$.
  - We'll use in the uniform system

$$\boxed{G_H(x_1,x_2)=\int\frac{d^4p}{(2\pi)^4}\,G_H(p)\,e^{-ip\cdot(x_1-x_2)}}$$

with the $G_H(p)$ we've already used:

$$\boxed{G_H(p)=\frac{\Theta(p-k_f)}{p_0-\epsilon_p+i\varepsilon}+\frac{\Theta(k_f-p)}{p_0-e_p-i\varepsilon}}$$

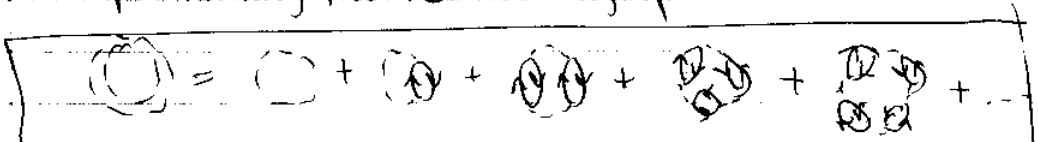
$$\Rightarrow \boxed{\begin{aligned}\Gamma_{NLO}[\sigma_c] &= \frac{i}{2}\mathrm{Tr}\ln\left[D_0^{-1}\right]\\ &= \frac{i}{2}\mathrm{Tr}_{x_1,x_2}\ln\left[\int\frac{d^4q}{(2\pi)^4}e^{-iq\cdot(x_1-x_2)}\left[-iC_0+gC_0^2\int\frac{d^4p}{(2\pi)^4}G_H(p+q)G_H(p)\right]\right]\\ &= \frac{i}{2}VT\int\frac{d^4q}{(2\pi)^4}\ln\left(-iC_0+gC_0^2\int\frac{d^4p}{(2\pi)^4}G_H(p+q)G_H(p)\right)\end{aligned}}$$

we define

$$\boxed{\Pi_0(q)\equiv -ig\int\frac{d^4p}{(2\pi)^4}G_0(p+q)G_0(p) = -ig\int\frac{d^4p}{(2\pi)^4}G_H(p+q)G_H(p)}$$

where the 2nd equality is valid for a uniform system.

- Diagramatically, we're summing up



which are exactly the diagrams (after the first few) that we want for the Bose limit!

3/10/03

We can evaluate the $\ln$ with the same trick as before:

$G_0 \Rightarrow \lambda C_0$ and take derivatives

- There will be apparent divergences when $\lambda = 0$ and elsewhere, but they're all taken care of with dimensional regularization (DR) for which

$$\boxed{\int \frac{d^D k}{(2\pi)^D} k^n = 0}$$

and while we're at it (from Peskin + Schroeder QFT text)

$$\boxed{\int \frac{d^D k}{(2\pi)^D} \frac{k^2}{\sqrt{k^2+\Lambda^2}} = \frac{\Gamma(-D/2)\Gamma(\frac{1}{2}+D/2)}{4\pi^{D/2}\Gamma(D/2)\Gamma(1/2)}(\Lambda^2)^{D/2}}$$

$$\Rightarrow \Gamma_{NLO}[G_c] = \frac{i}{2} VT \int \frac{d^4 q}{(2\pi)^4} \int_0^1 d\lambda \frac{d}{d\lambda}\left(\ln\left(-i\lambda C_0 + i\lambda^2 C_0^2 \Pi_0(q)\right)\right)$$

$$= \frac{i}{2} VT \int_0^1 d\lambda \int \frac{d^4 q}{(2\pi)^4} \frac{1}{-i\lambda C_0 + i\lambda^2 C_0^2 \Pi_0(q)}\left(-iC_0 + 2i\lambda C_0^2 \Pi_0\right)$$

$$= -\frac{i}{2} VT \int_0^1 \frac{d\lambda}{\lambda} \int \frac{d^4 q}{(2\pi)^4} \frac{\lambda C_0 \Pi_0(q)}{1-\lambda C_0 \Pi_0(q)}$$

where we've dropped a $\int \frac{d^4 q}{(2\pi)^4} 1$ term by DR.

This we can evaluate using (exercise for reader to derive!)

$$\boxed{\Pi_0(q_0,\vec{q}) = g\int \frac{d^3 p}{(2\pi)^3}\Theta(k_f - p)\left(\frac{\Theta(|\vec{p}+\vec{q}|-k_f)}{q_0 + w_p^0 - w_{p+q}^0 + i\varepsilon} - \frac{\Theta(|\vec{p}-\vec{q}|-k_f)}{q_0 - w_p^0 + w_{p-q}^0 - i\varepsilon}\right)}$$

- This is ugly and has to be done numerically.
- Here $w_p^0 \equiv \frac{p^2}{2m}$

3/10/03

- But if we apply the Bose limit $g \to \infty$, $k_f \to 0$, $\rho = \frac{g k_f^3}{6\pi^2}$ fixed at this stage, it's easy!

$$\boxed{\begin{aligned}\Pi_0(q_0,\vec{q}) &\xrightarrow[k_f \to 0]{} g\int \frac{d^3 p}{(2\pi)^3}\,\Theta(k_f - p)\left(\frac{1}{q_0 - \omega_q + i\varepsilon} - \frac{1}{q_0 + \omega_q - i\varepsilon}\right) \\ &= \frac{2\omega_q^0 \rho}{(q_0 - \omega_q^0 + i\varepsilon)(q_0 + \omega_q^0 - i\varepsilon)}\end{aligned}}$$

using $|\vec{q}| \gg k_f, p$ in the integrals.

So now

$$\boxed{\begin{aligned}\frac{\lambda C_0 \Pi_0}{1 - \lambda C_0 \Pi_0} = \frac{\lambda C_0}{(\Pi_0)^{-1} - \lambda C_0} &= \frac{2\omega_q \rho \lambda C_0}{q_0^2 - \omega_q^2 - 2\omega_q \rho \lambda C_0 + i\varepsilon} \\ &= \frac{2\omega_q \rho \lambda C_0}{q_0^2 - \epsilon_q^2 + i\varepsilon}\end{aligned}}$$

defining the Bogoliubov quasiparticle energies

$$\boxed{\epsilon_{\vec{q}} = \sqrt{\omega_q^2 + 2\omega_q \rho C_0 \lambda}}$$

$\Rightarrow$ The q_0 integral picks up a simple pole and we get

$$\boxed{\mathcal{E}_1^{Bose} = -\frac{\Gamma_{NLO}}{VT} = \frac{\rho}{2}\int_0^1 d\lambda\, C_0 \int \frac{d^3 q}{(2\pi)^3}\frac{\omega_q}{\epsilon_q} = \frac{\rho}{2} C_0 \int_0^1 d\lambda \int \frac{d^3 q}{(2\pi)^3}\frac{q^2}{q\sqrt{q^2 + 4m\rho\lambda C_0}}}$$

applying the DR formula, we get (for $D=3$)

$$\boxed{\mathcal{E}_1^{Bose} = \frac{\rho}{15\pi^2}(4m\rho)^{3/2} C_0^{5/2} = \frac{2\pi a_s \rho^2}{m}\frac{128}{15\sqrt{\pi}}\sqrt{\rho a_s^3}}$$ as desired!