Derivation of the RG equation

Starting from the Lippmann-Schwinger equation for the half-on-shell T-matrix with a cutoff

$$\langle k'|T(k^2)|k\rangle = \langle k'|V_{low\,k}|k\rangle + \frac{2}{\pi}\mathcal{P}\int_0^\Lambda \frac{\langle k'|V_{low\,k}|p\rangle\langle p|T(k^2)|k\rangle}{k^2-p^2}\,p^2dp$$

$$\frac{d}{d\Lambda}\langle k'|T(k^2)|k\rangle = 0$$

$$\Rightarrow \frac{d}{d\Lambda}\langle k'|V_{low\,k}|k\rangle = -\frac{2}{\pi}\frac{\Lambda^2}{k^2-\Lambda^2}\langle k'|V_{low\,k}|\Lambda\rangle\langle\Lambda|T(k^2)|k\rangle$$

$$+\frac{2}{\pi}\mathcal{P}\int_0^\Lambda \frac{\frac{d}{d\Lambda}\left(\langle k'|V_{low\,k}|p\rangle\right)\langle p|T(k^2)|k\rangle}{k^2-p^2}\,p^2dp$$

$$\Leftrightarrow \frac{2}{\pi}\mathcal{P}\int_0^\Lambda p^2dp\,\frac{d}{d\Lambda}\left(\langle k'|V_{low\,k}|p\rangle\right)\left(\frac{\pi}{2}\underbrace{\frac{\delta(p-k)}{p^2}}_{\langle p|k\rangle}+\frac{\langle p|T(k^2)|k\rangle}{k^2-p^2}\right)$$

$$= -\frac{2}{\pi}\frac{\Lambda^2}{k^2-\Lambda^2}\langle k'|V_{low\,k}|\Lambda\rangle\langle\Lambda|T(k^2)|k\rangle$$

With the normalization $\langle p|k\rangle = \frac{\pi}{2}\frac{\delta(p-k)}{p^2}$

and the scattering states expressed in terms of the T matrix

$$|\chi_k\rangle = |k\rangle + \frac{2}{\pi}\mathcal{P}\int_0^\Lambda p^2dp\,\frac{1}{k^2-p^2}\,T(p,k;k^2)\,|p\rangle$$

Scattering state with energy k^2, i.e., $H|\chi_k\rangle = k^2|\chi_k\rangle$

We have

$$\frac{2}{\pi}\mathcal{P}\int_0^\Lambda p^2 dp \frac{d}{d\Lambda}\langle k'|V_{\mathrm{low}\,k}|p\rangle\langle p|\chi_k\rangle = \frac{2}{\pi}\,\frac{\langle k'|V_{\mathrm{low}\,k}|\Lambda\rangle\langle\Lambda|T(k^2)|k\rangle}{1-(k/\Lambda)^2}$$

Multiply by $\langle\chi_k|p'\rangle$ *) and integrate over k

$$\frac{2}{\pi}\mathcal{P}\int p^2dp\,\underbrace{\frac{2}{\pi}\mathcal{P}\int k^2dk\,\frac{d}{d\Lambda}\langle k'|V_{\mathrm{low}\,k}|p\rangle\langle p|\chi_k\rangle\langle\chi_k|p'\rangle}_{=\mathbb{1}}$$

$$= \frac{2}{\pi}\,\frac{2}{\pi}\mathcal{P}\int k^2dk\,\frac{\langle k'|V_{\mathrm{low}\,k}|\Lambda\rangle}{1-(k/\Lambda)^2}\underbrace{\langle\Lambda|T(k^2)|k\rangle}_{=\langle\Lambda|V_{\mathrm{low}\,k}|\chi_k\rangle}\langle\chi_k|p'\rangle$$

$$\Longleftrightarrow\ \frac{2}{\pi}\mathcal{P}\int p^2dp\,\frac{d}{d\Lambda}\langle k'|V_{\mathrm{low}\,k}|p\rangle\langle p|p'\rangle$$

$$= \frac{2}{\pi}\,\frac{2}{\pi}\mathcal{P}\int k^2dk\,\frac{\langle k'|V_{\mathrm{low}\,k}|\Lambda\rangle}{\Lambda^2-k^2}\,\Lambda^2\langle\Lambda|V_{\mathrm{low}\,k}|\chi_k\rangle\langle\chi_k|p'\rangle$$

$$\Longleftrightarrow\ \frac{d}{d\Lambda}\langle k'|V_{\mathrm{low}\,k}|p'\rangle = \frac{2}{\pi}\Lambda^2\,\frac{2}{\pi}\mathcal{P}\int k^2dk\,\langle k'|V_{\mathrm{low}\,k}|\Lambda\rangle\langle\Lambda|V_{\mathrm{low}\,k}\,\frac{1}{\Lambda^2-k^2}|\chi_k\rangle\langle\chi_k|p'\rangle$$

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*) Strictly speaking the $\langle\chi_k|$ is the bi-orthogonal complement here. We will ignore this detail, since the derivation only makes use of completeness:

$$\frac{2}{\pi}\mathcal{P}\int_0^\Lambda k^2dk\,|\chi_k\rangle\langle\chi_k| = \mathbb{1}$$

Notice $\frac{1}{\Lambda^2 - k^2} |\chi_k\rangle = \frac{1}{\Lambda^2 - H} |\chi_k\rangle$

and with $G(E) = \frac{1}{\underline{\underline{E-H}}}$ full Green's function

$$= G(\Lambda^2)\, |\chi_k\rangle$$

$$\Rightarrow \frac{d}{d\Lambda} \langle k' | V_{lowk} | p' \rangle = \frac{2}{\pi} \Lambda^2 \, \langle k' | V_{lowk} | \Lambda \rangle \langle \Lambda | V_{lowk} \, G(\Lambda^2) | p' \rangle$$

Can we simplify $V_{lowk}\, G(E)$?

We have $\boxed{T(E) = V_{lowk} + V_{lowk} \underbrace{G_0(E)}_{\frac{1}{E-H_0}} T(E)}$

and $\boxed{G(E) = \frac{1}{E-H} = \frac{1}{E-H_0-V} = G_0(E) + G_0(E)\, T(E)\, G_0(E)}$ *)

$$\Rightarrow V_{lowk} = T(E) \left(1 + G_0(E)\, T(E) \right)^{-1}$$

$$\Rightarrow V_{lowk}\, G(E) = T(E) \left(1 + G_0(E)\, T(E) \right)^{-1} \underbrace{\left(G_0(E) + G_0(E)\, T(E)\, G_0(E) \right)}_{\left(1 + G_0(E) T(E)\right) G_0(E)}$$

$$\Rightarrow \boxed{V_{lowk}\, G(E) = T(E)\, G_0(E)} \quad \text{cancel}$$

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*) drop (E) label for derivation and lowk) label

$$G^{-1} = E - H = E - H_0 - V = G_0^{-1} - V \qquad \text{(Dyson equation)}$$

From $T = V + V G_0 T \Rightarrow T = V(1 + G_0 T) \Rightarrow V = T(1 + G_0 T)^{-1}$

$$\Rightarrow G^{-1} = G_0^{-1} - T(1 + G_0 T)^{-1} \Rightarrow G^{-1}(1 + G_0 T) = G_0^{-1}(1 + G_0 T) - T$$

$$\Rightarrow G^{-1}(1 + G_0 T) = G_0^{-1} + \cancel{T} - T \Rightarrow G_0 = (1 + G_0 T)^{-1} G \Rightarrow (1 + G_0 T) G_0 = G$$

Now, we can rewrite the RG equation as

$$\boxed{\frac{d}{d\Lambda} \langle k' | V_{low\,k} | p' \rangle = \frac{2}{\pi} \Lambda^2 \langle k' | V_{low\,k} | \Lambda \rangle \langle \Lambda | T(\Lambda^2) G_0(\Lambda^2) | p' \rangle}$$

$$\boxed{= \frac{2}{\pi} \Lambda^2 \langle k' | V_{low\,k} | \Lambda \rangle \langle \Lambda | T(\Lambda^2) | p' \rangle \frac{1}{\Lambda^2 - p'}}$$

Besides the solution to $\Lambda \sim 2.1\ fm^{-1}$, we can study how $V_{low\,k}$ changes with the cutoff. Consider the change of $V_{low\,k}(0,0) = \langle 0\ fm^{-1} | V_{low\,k} | 0\ fm^{-1} \rangle$ with cutoff shown below

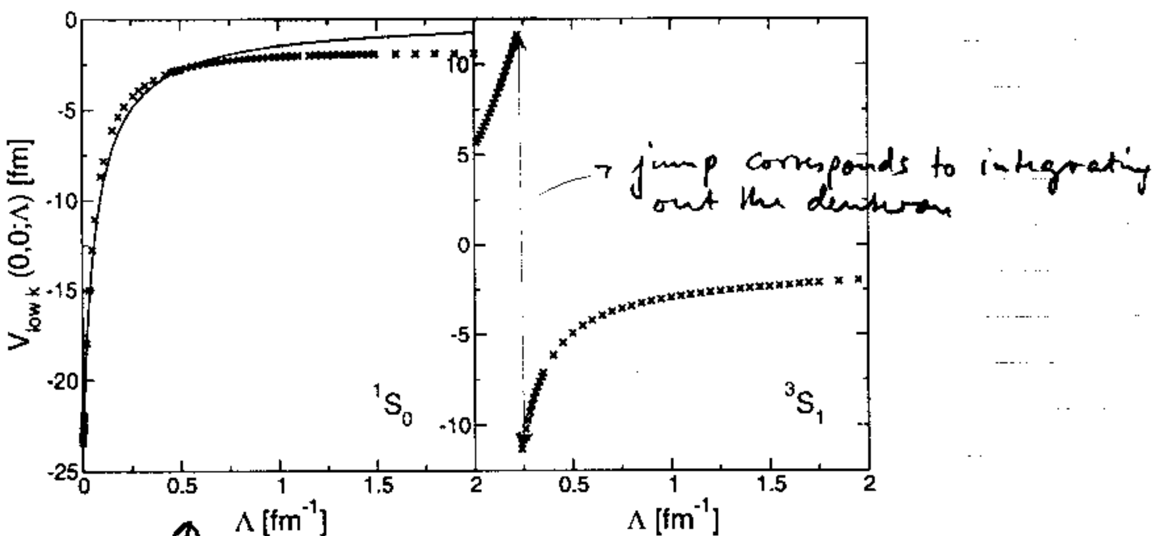

Fig. 5. RG flow of $V_{\rm low\,k}(0,0;\Lambda)$ versus cutoff Λ in the $^1S_0$ and $^3S_1$ partial waves. The solid line represents the solution of the RG equation for small Λ as discussed in the text.

Consider the $^1S_0$ partial wave. Can we understand the strong falloff at small cutoffs?

For small Λ, the T matrix $T = V_{low} + \int_0^\Lambda V_{low}\, G_0\, T$

may be approximated by $T \approx V_{low}$ $\Lambda \to 0$

and the RG equation reads

$$\frac{d}{d\Lambda} \langle 0 | V_{low} | 0 \rangle = \frac{2}{\pi} \cancel{\Lambda^2} \frac{\langle 0 | V_{low} | \Lambda \rangle \langle \Lambda | V_{low} | 0 \rangle}{\cancel{\Lambda^2 - 0^2}}$$

Further approximate the cutoff dep. in $\langle 0 | V_{low} | \Lambda \rangle$ by

$$\langle 0 | V_{low} | \Lambda \rangle \approx \langle 0 | V_{low} | 0 \rangle \quad \text{for } \Lambda \to 0$$

With $C_0 = \langle 0 | V_{low} | 0 \rangle$ (constant part of the interaction)

$$\frac{d}{d\Lambda} C_0 = \frac{2}{\pi} C_0^2 \qquad (1)$$

What are the boundary conditions?

For small Λ: $\langle 0 | V_{low} (\Lambda \to 0) | 0 \rangle \longrightarrow \langle 0 | T(0) | 0 \rangle = a_s$

scattering length

(recall $\langle k | T(k^2) | k \rangle = \dfrac{-1}{-\frac{1}{a_s} + \frac{1}{2} r_0 k^2}$ at small k)

Solve (1) by integration

$$\frac{2}{\pi} \int_0^\Lambda d\Lambda = \int_{a_s}^{C_0(\Lambda)} dc_0 \frac{1}{c_0^2} \iff \frac{2}{\pi} \Lambda = -\frac{1}{C_0(\Lambda)} + \frac{1}{a_s}$$

$$\Rightarrow \quad C_0(\Lambda) = \frac{1}{\frac{1}{a_s} - \frac{2}{\pi} \Lambda} \qquad a_s = -23.73 \text{ fm in } {}^1S_0$$

plotted as line in Fig. 5 above. It works!