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- We are not going to pursue the boson path integral in detail, but let's examine it a bit to see what is different from the σ-path integral.
  - The boson and fermion path integrals actually look the same at this point $\Rightarrow$ the differences are hidden in suppressed spin indices, c-number functions $\psi(x), \psi^\dagger(x)$ versus Grassmann functions for ψ and $\psi^\dagger$, and the boundary conditions on the fields
  - We stress again that the ψ's and $\psi^\dagger$'s in the path integral for Z_G are not operators (no ^'s), so in the boson case they commute entirely.
    - In the fermion case they are also not operators, but they anti-commute.
    - In the fermion case, we'll write

$$\psi^\dagger\psi^\dagger\psi\psi \rightarrow \psi^\dagger_\alpha(x)\psi^\dagger_\beta(x')\psi_\beta(x')\psi_\alpha(x)$$

where α and β are spin indices and there is an implied summation $\sum_{\alpha=1}^{2}$ when indices are repeated.

- Comparisons to σ-path integral
  - If we want to add a source, we'll have two types (treated independently) $\boxed{\eta^\dagger(x)\psi(x) + \psi^\dagger(x)\eta(x)}$ (the $\dagger$'s are really just complex conjugates for spinless bosons).

$$\Rightarrow \boxed{\psi(x) \rightarrow \frac{\delta}{\delta\eta^\dagger(x)} \quad \text{and} \quad \psi^\dagger(x) \rightarrow \frac{\delta}{\delta\eta(x)}}$$

  - Instead of "a" we have the differential operator

$$\boxed{a \rightarrow \frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu}$$

  - but remember that this is a shorthand for the limit of a discretized version. Then, this "term" is really a Gaussian matrix integral.

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From page (65) we have the result

$$\boxed{\int_{-\infty}^{\infty} dz_1 \cdots dz_n\, e^{-\frac{1}{2} z_i A_{ij} z_j + z_i J_i} = (2\pi)^{n/2} [\det \underline{\underline{A}}]^{-1/2} e^{1/2 J_i A^{-1}_{ik} J_k}}$$

· With a suitable generalization to two sources, we see that we get the determinant of $\left(\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu\right)$ appearing and its inverse in the place of $\mathcal{D}_0$.

The generalization is to an integral over pairs of complex conjugate variables $dz_i^* dz_i$, which is (N+O pg. 34)

$$\boxed{\int \prod_{i=1}^{n} \frac{dz_i^* dz_i}{2\pi i}\, e^{-z_i^* H_{ij} z_j + \eta_i^* z_i + \eta_i z_i^*} = [\det H]^{-1} e^{\eta_i^* H^{-1}_{ij} \eta_j}}$$

· This result is valid for any matrix H with a positive Hermitian part.

· If H is, in fact, Hermitian, we prove it the same way that the result up top was proved:

i) define a transformation $y_i = z_i - H^{-1}_{ij}\eta_j$ and its complex conjugate

ii) transform H to diagonal form

iii) evaluate $$\int \frac{dz^* dz}{2\pi i} e^{-z^* a z} = \int_{-\infty}^{\infty} \frac{du\, dv}{\pi} e^{-a(u^2+v^2)} = \frac{1}{a}$$

where $z = u + iv$. We can take $\int \frac{dz^* dz}{2i} = \int du\, dv$ as the definition of the measure $dz^* dz$. (Stone has a more highbrow discussion of pp.153-156).

· As noted, the integration is over all u and v from $-\infty$ to $+\infty$.

· Summary: We get one over the determinant appearing (this is just the multiplicative inverse) and then with the "sources" η_i and η_i^* we have the matrix inverse.

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At this point it would be opportune to present the analogous formula for Grassmann (anti-commuting) variables. [The following is taken largely from Negele and Orland, sect 1.5]

- So we'll start with a quick introduction to the essentials of Grassmann numbers, integrals, and all that.
  - We are really dealing with algebras of anticommuting numbers, but for our purposes (for now at least) we can simply regard the following definitions and manipulations as a convenient and efficient mathematical construction that builds in all of the minus signs associated with antisymmetry.
    $\Rightarrow$ There is no need to intrepret them physically.
  - Standard reference: Brézin, LeGuillou, Zinn-Justin, Phys. Rev. D15 (1977) 1544; 1588.

- An n-dimensional Grassmann algebra is defined by a set of generators $\{\xi_\alpha\}$, $\alpha = 1, \ldots, n$ that satisfy anti-commutation relations:

$$\boxed{\{\xi_\alpha, \xi_\beta\} \equiv \xi_\alpha \xi_\beta + \xi_\beta \xi_\alpha = 0} \quad \alpha, \beta = 1, 2, \ldots, n$$

  - Note that these are not like the field operator anti-commutation relation where one could get a non-zero result — these always precisely anticommute.

  - An immediate consequence is that

$$\boxed{\xi_\alpha^2 = 0 \quad \text{for } \forall \alpha}$$

so these are somewhat unusual objects!

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- We form a basis of the algebra by considering all distinct products of the generators.

$\Rightarrow$ a number in the algebra is a linear combination of

$$\boxed{\{1, \xi_{\alpha_1}, \xi_{\alpha_1}\xi_{\alpha_2}, \dots, \xi_{\alpha_1}\xi_{\alpha_2}\cdots\xi_{\alpha_n}\}}$$

with complex coefficients with a conventional ordering of the coefficients $\alpha_1 < \alpha_2 < \dots < \alpha_n$.

- The dimension with n generators of the algebra is 2^n (exercise for the reader to verify!)

- We'll need to define complex conjugates, which we do when n is even by just assigning half of the generators ξ_α to have corresponding ξ^*_α among the other half.
  - The properties of complex conjugation are predictable:

$$\boxed{(\xi_\alpha)^* = \xi^*_\alpha \quad (\xi^*_\alpha)^* = \xi_\alpha}$$

If $\lambda \in \mathbb{C}$ (a complex number), then $\boxed{(\lambda\xi_\alpha)^* = \lambda^*\xi^*_\alpha}$

and $\boxed{(\xi_{\alpha_1}\cdots\xi_{\alpha_n})^* = \xi^*_{\alpha_n}\cdots\xi^*_{\alpha_1}}$

- Let's see what happens with only two generators, ξ and ξ^*, so the basis is $\boxed{\{1, \xi, \xi^*, \xi^*\xi\}}$.
  - A function of ξ must be linear:

$$\boxed{f(\xi) = f_0 + f_1\xi}$$

because $\xi^2 = 1$. So Taylor series always truncate!

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We define derivatives by

$$\boxed{\frac{\partial}{\partial \xi}\xi = 1}$$

but we have to anticommute the ξ until it is to the right of $\frac{\partial}{\partial \xi}$:

$$\boxed{\frac{\partial}{\partial \xi}(\xi^* \xi) = \frac{\partial}{\partial \xi}(-\xi\xi^*) = -\xi^*}$$

If $\boxed{A(\xi^*, \xi) = a_0 + a_1 \xi + \bar{a}_1 \xi^* + a_{12}\xi^* \xi}$, then

$$\boxed{\begin{aligned} \frac{\partial}{\partial \xi} A(\xi^*, \xi) &= a_1 - a_{12}\xi^* \\ \frac{\partial}{\partial \xi^*} A(\xi^*, \xi) &= \bar{a}_1 + a_{12}\xi \\ \frac{\partial}{\partial \xi^*}\frac{\partial}{\partial \xi} A(\xi^* \xi) &= -a_{12} = -\frac{\partial}{\partial \xi}\frac{\partial}{\partial \xi^*} A(\xi^*, \xi) \end{aligned}}$$

which illustrates that $\frac{\partial}{\partial \xi^*}$ and $\frac{\partial}{\partial \xi}$ (as well as their generalizations to n-dimensional algebra) anti-commute.

Ok, so what about integrals?

- We can't do anything like the Riemannian sum as for ordinary variables.
- So we define integrals as anti-derivatives, which has the effect of making integrals the same as derivatives!
  - i.e. since $\frac{\partial}{\partial \xi}\xi = 1$, the definite integral of 1 is zero:

$$\Rightarrow \boxed{\begin{aligned} \int d\xi \, 1 &= 0 \\ \int d\xi \, \xi &= 1 \end{aligned}}$$

(Don't worry if this seems weird; you just need to know the rules. But don't treat "$d\xi$" as an infinitesimal. It's not!)

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As with derivatives, you need to anti-commute $d\xi_\alpha$ and ξ_α so they are adjacent.

- Integration for complex conjugate variables is what you would imagine:

$$\boxed{\int d\xi^* \, 1 = 0 \quad \text{and} \quad \int d\xi^* \, \xi^* = 1}$$

- So the integral exactly removes the corresponding variable (0 if not available)

OK, we can apply these rules to $f(\xi)$ and $A(\xi^*, \xi)$ as defined earlier:

$$\boxed{\int d\xi \, f(\xi) = \int d\xi \, (f_0 + f_1 \xi) = f_0 \cdot 0 + f_1 \cdot 1 = f_1}$$

$$\int d\xi \, A(\xi^*, \xi) = \int d\xi \, (a_0 + a_1 \xi + \bar{a}_1 \xi^* + a_{12} \xi^* \xi) = a_1 - a_{12} \xi^*$$

$$\int d\xi^* \, A(\xi^*, \xi) = \bar{a}_1 + a_{12} \xi$$

$$\int d\xi^* \, d\xi \, A(\xi^*, \xi) = -a_{12} = -\int d\xi \, d\xi^* \, A(\xi^*, \xi)$$

- We can define a δ-function as (with η another Grassmann variable)

$$\boxed{\delta(\xi, \xi') \equiv \int d\eta \, e^{-\eta(\xi - \xi')} = \int d\eta \, (1 - \eta(\xi - \xi')) = -(\xi - \xi')}$$

stops since $\eta^2 = 0$

Check: $\int d\xi' \, \delta(\xi, \xi') f(\xi') = -\int d\xi' (\xi - \xi')(f_0 + f_1 \xi') = f_0 + f_1 \xi = f(\xi)$ ✓

- Finally, we define the scalar product of Grassmann functions $\langle f | g \rangle$ by

$$\langle f | g \rangle \equiv \int d\xi^* d\xi \, e^{-\xi^* \xi} f^*(\xi) g(\xi^*)$$

note: not ξ^*

$$= \int d\xi^* d\xi \, (1 - \xi^* \xi)(f_0^* + f_1^* \xi)(g_0 + g_1 \xi^*)$$

$$= -\int d\xi^* d\xi \, \xi^* \xi \, f_0^* g_0 + \int d\xi^* d\xi \, f_1^* g_1 \xi \xi^* = f_0^* g_0 + f_1^* g_1$$

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OK, now we can generalize to $2n$ generators and think about Gaussian integrals.

· Suppose we have just ξ and ξ^*, then (with "a" a number)

$$\boxed{\int d\xi^* d\xi \, e^{-\xi^* a \xi} = \int d\xi^* d\xi \left(1 - \xi^* a \xi\right) = a}$$

$\Rightarrow$ we get "a" instead of "1/a" as in the ordinary Gaussian integral.

Now try $\xi_1, \xi_2, \xi_1^*, \xi_2^*$! [class try this!]

$$\int d\xi_1^* d\xi_1 \, d\xi_2^* d\xi_2 \, e^{-\xi_i^* H_{ij} \xi_j}$$

$$= \int d\xi_1^* d\xi_1 \, d\xi_2^* d\xi_2 \, e^{-\left[\xi_1^* H_{11} \xi_1 + \xi_2^* H_{21} \xi_1 + \xi_1^* H_{12} \xi_2 + \xi_2^* H_{22} \xi_2\right]}$$

expand to 2nd order

$$= \int d\xi_1^* d\xi_1 \, d\xi_2^* d\xi_2 \, \frac{(-1)^2}{2!} \left(\xi_1^* H_{11} \xi_1 + \xi_2^* H_{21} \xi_1 + \xi_1^* H_{12} \xi_2 + \xi_2^* H_{22} \xi_2\right)$$
$$\times \left(\xi_1^* H_{11} \xi_1 + \xi_2^* H_{21} \xi_1 + \xi_1^* H_{12} \xi_2 + \xi_2^* H_{22} \xi_2\right)$$

$$= \frac{1}{2!}\left(H_{11} H_{22} - H_{21} H_{12} - H_{12} H_{21} + H_{22} H_{11}\right)$$

$$= H_{11} H_{22} - H_{12} H_{21} = \det \underline{\underline{H}}$$

· So, again, the determinant is in the numerator rather than the denominator.

· The generalization, including Grassmann "sources" η_i and η_i^*, is

$$\boxed{\int \prod_{i=1}^{n} d\xi_i^* d\xi_i \, e^{-\xi_i^* H_{ij} \xi_j + \eta_i^* \xi_i + \xi_i \eta_i^*} = \left[\det H\right] e^{\eta_i^* H_{ij}^{-1} \eta_j}}$$

This works for H Hermitian.

· The proof (which requires showing how to change Grassmann variables), is given in Negele + Orland section 1.5.

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- So we see that the boson and fermion Gaussian integral formulas are quite similar, with the main distinction being $[\det H]^{-1}$ in the former and $[\det H]$ in the latter.
  - We have restrictions on H for bosons (positive definite) to ensure that the integral converges
  - No restrictions on H for fermions, since we can expand the exponential $e^{a\xi^*\xi}$ and it terminates at first order, so we get a finite integral no matter what "a" is.

- This distinction embodies the difference that the Pauli principle makes — 0, or 1 occupation numbers for fermions versus 0, 1, 2, ... for bosons.

- Recall the noninteracting Grand Canonical partition function evaluated in the occupation number basis:

$$Z_G^0 = \mathrm{Tr}\, e^{-\beta(\hat{H}_0 - \mu\hat{N})} = \prod_\alpha \sum_{n_\alpha} e^{-\beta(\epsilon_\alpha - \mu)n_\alpha}$$

where $n_\alpha = 0,1$ for fermions, and $n_\alpha = 0,1,2,\ldots$ for bosons.

$\Rightarrow$ fermions: $Z_F^0 = \prod_\alpha \left(1 + e^{-\beta(\epsilon_\alpha - \mu)}\right)$ $\quad\therefore$ just two terms

bosons: $Z_B^0 = \prod_\alpha \left(1 + \sum_{n=1}^{\infty} \left(e^{-\beta(\epsilon_\alpha - \mu)}\right)^n\right)$

- For fermions, $\epsilon_\alpha - \mu$ can be anything, but for bosons, $\epsilon_\alpha - \mu > 0$ for all α or the partition function is not finite. $\Rightarrow$ $\hat{H} - \mu\hat{N}$ must be positive definite for bosons but not fermions.