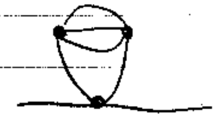
1/29/03

Clarifications from the model partition function discussion...

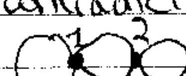
(a) We note on page 72 how the symmetry factor rules are changed when the propagator lines have "arrows" on them. This is misleading because of the convention of using arrows to indicate momentum flow in momentum-space Feynman diagrams.

- The change in rules occurs when the arrow means that one end of the propagator is different from the other end (so it is <u>not</u> a symmetry operation to exchange the ends).
- In general, fermion propagators have a direction (and therefore follow the modified rules for factors) while some bosonic propagators do not (so they follow the rules as originally given). We'll see examples of both.

(b) Here's a warm-up diagram for the problem in the problem set. What is the symmetry factor?



(c) To clarify the replica method proofs, consider some special cases and argue by contradiction.

- E.g., could we ever get 

where the 1 and 2 label the vertices?
  - Well, the 1 vertex comes from $\left(\frac{\partial}{\partial j_1}\right)^4$, and each $\frac{\partial}{\partial j_1}$ is nonzero only when acting on $\frac{1}{2} j_1 \sigma_1' j_1$, so <u>each</u> of the lines connecting "1" and "2" must be "<u>1</u>" lines.
  - But by the same argument, they must be "2" lines, so this diagram cannot occur.
- Now suppose we have $\langle x^2 \rangle$ and we start to build a diagram knowing the external lines are 1's $\underline{\ 1\ }$? $\underline{\ 1\ }$ By the same argument, the 1 propagators here can only connect to <u>1</u> vertices, which connect to 1 propagators $\Rightarrow$ the entire diagram <u>must</u> be 1's.

1/29/03

- As an intermediate step between our model partition function and the case of fermions interacting with a short-range force, let's consider a path integral over a field σ:

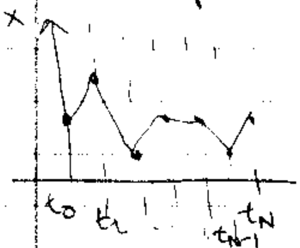
$$\boxed{Z_\lambda \equiv \int \mathcal{D}\sigma \, e^{-\int d^4x \left[\frac{1}{2} a\,\sigma(x)^2 + \frac{\lambda}{4}[\sigma(x)]^4\right]}}$$

- A path integral over such a field $\sigma(x)$ will arise naturally when we consider the "large N" expansion of the fermionic path integral we're working up to. (However, we'll see problems with this one!)
- The $\int d^4x$ is shorthand for an integral over space $\int d^3x$ and over Euclidean time $d\tau$. The "x" stands for $(\tau, \vec{x})$.
  - The real time version replaces $-\int d^4x \to i\int d^4x$ with an integral over dt.
  - At finite temperature, the τ integration will only run over a finite interval of size $\beta\hbar$ (with $\beta \equiv 1/kT$).
  - In the zero temperature ($\beta \to \infty$) limit, which we'll assume here, the integration over τ is from $-\infty$ to $+\infty$, just like the dx_i integrations.
- The path integral measure $\mathcal{D}\sigma$ will, as usual, hide an awkward constant, which we won't need to write explicitly.
- More generally, we expect to have time and space derivatives acting on the $\sigma(x)$ (in the quadratic term, for example). But in the example we're mimicking here, they don't appear. However, the "interaction" term is also different from σ^4.

- The path integral is defined as the limit of a discretized version in a finite time interval and a finite box at which we have lattice points: a mesh in both τ and the x-directions.
  $\Rightarrow$ label them $x_i \to (\tau_i, x_i, y_i, z_i)$, for example.

1/29/03

- Previously, when considering a path integral for single-particle quantum mechanics, a "path" or "trajectory" was specified by the value of x at each discrete time t_i or τ_i:



The path integral "summed" over all paths (with a specified weighting factor for a given path) by integrating over the value of x at each t_i. The measure is $\prod_i dx_i$ (along with some normalization factors).

- In the present case, we sum over "configurations." A given configuration is specified by the value of σ at each discrete time τ_i and each discrete spatial lattice point $\vec{x}_i \equiv (x_i, y_i, z_i)$.
  - The configurations are "summed" over by integrating over σ at each lattice point. $\Rightarrow \mathcal{D}\sigma \propto \prod_i d\sigma_i$ where i runs over the lattice
  - In the "continuum limit," where the discretization spacing is taken to zero, we are supposed to get the partition function (or other function) we seek.

- Let's mimic the model partition function exercise: We seek expressions for

i) $\boxed{\ln(Z_\lambda/Z_0)}$, where $\boxed{Z_0 = \int \mathcal{D}\sigma \, e^{-\int d^4x \, \frac{1}{2}a[\sigma(x)]^2}}$ and

ii) $$\boxed{D_\sigma(y,z) \equiv \frac{\int \mathcal{D}\sigma \, \sigma(y)\sigma(z) \, e^{-\int d^4x \left[\frac{1}{2}a[\sigma(x)]^2 + \lambda[\sigma(x)]^4\right]}}{\int \mathcal{D}\sigma \, e^{-\int d^4x \left(\frac{1}{2}a\sigma^2 + \lambda\sigma^4\right)}}}$$

- We introduce a "source term" $j(\tau, \vec{x}) \equiv j(x)$, which we will use to develop a perturbative expansion for i) and ii).
  - To do so, we need to review (or introduce, if you haven't seen them) the idea and properties of functional derivatives.

  $\Rightarrow$ pause to consider them ...

1/29/03

Aside: Functionals and Functional Derivatives

Ref.: Appendix B in the text by N. Nagaosa is the source for much of this presentation. Most field theory books have an introduction to functionals in their discussions of path integrals.

- First let's contrast Functions and Functionals:
  - i) function f: you give me a real number $x \in \mathbb{R}$ and I'll give you another real number:
  $$\boxed{f: x \in \mathbb{R} \to f(x) \in \mathbb{R}}$$
    - generalizations include different domains and ranges (e.g. complex numbers)
  - ii) functional F: rather than a single x, give me an entire function $f(x)$ in an interval $\alpha \le x \le \beta$ and I'll give you a single real number back:
  $$\boxed{F: f(x) \in \mathcal{H} \to F(\{f(x)\}) \equiv F[f] \equiv F[f(x)] \in \mathbb{R}}$$
    - we've indicated a variety of notations you might find in the literature
    - $\mathcal{H}$ signifies a Hilbert space
    - Note that when we use the notation $F[f(x)]$, there is no particular value of x involved; the entire function (in a relevant interval) is the input.

- It is useful (as usual) to imagine putting a functional on a computer.
  - We can represent a function $f(x)$ as breaking up the interval $\alpha < x < \beta$ into a discrete mesh (set) of points $\{x_i\}$ with $\boxed{\alpha \equiv x_1 < x_2 < \cdots < x_{N-1} < x_N \equiv \beta}$

1/29/03

- These numbers might be stored in an array, x[i], for example, with $i = 1$ to N.
- The function $f(x)$ is the set of N numbers $f_i \equiv f(x_i)$, (which we can also store as f[i].

$\Rightarrow$ A functional F is a <u>function</u> of the N numbers $\{f_i\}$ (<u>not</u> of the N numbers $\{x_i\}$!)

- A <u>functional</u> <u>derivative</u> tells us about what happens to the value of a functional when we let $f(x)$ become $f(x) + \delta f(x) \Rightarrow$ a slightly different function.
  - In the discrete version, we take $f_i \to f_i + \delta f_i$ (just like one take $x_i \to x_i + \Delta x_i$ in an ordinary derivate).

- Suppose
$$\boxed{F[f(x)] = \int_a^b f(x)\, g(x)\, dx}$$

where $g(x)$ is some other <u>fixed</u> function of x.
  - Note that we integrate over x, so there is no free $x \Rightarrow F$ is <u>not</u> a function of x!
  - Let's make the simplest discrete version:

$$\boxed{F[f(x)] \to \sum_i f(x_i)\, g(x_i)\, \Delta x_i \equiv \sum_i f_i g_i \Delta x_i}$$

Now let $f_i \to f_i + \delta f_i$:

$$\boxed{\begin{aligned} \delta F &= F(f_1 + \delta f_1, \ldots, f_N + \delta f_N) - F(f_1, \ldots, f_N) \\ &\doteq \sum_{i=1}^{N} \frac{\partial F(f_1, \ldots, f_N)}{\partial f_i}\, \delta f_i \; + O(\delta f_i^2) \end{aligned}}$$

- Notice the δ's instead of ∂'s.

1/29/03

We define the functional derivative $\frac{\delta F}{\delta f(x)}$ as

$$\boxed{\frac{\delta F[f]}{\delta f(x)} = \lim_{\Delta x_i \to 0} \frac{1}{\Delta x_i} \frac{\partial F(f_1, \ldots, f_N)}{\partial f_i}} \quad \text{where } f_i = f(x_i)$$

this x matters!

- The x on the left corresponds to x_i on the right; in the limit that we make the mesh very fine.

$\Rightarrow$ We look how much the functional changes when we change the function at the i^{th} mesh point.

- Apply this definition to the example:

$$\frac{\delta F[f(x)]}{\delta f(x)} = \lim_{\Delta x_i \to 0} \frac{1}{\Delta x_i} \frac{\partial}{\partial f_i}\left(\sum_j f_j g_j \Delta x_j\right)$$

$$= \lim_{\Delta x_i \to 0} \frac{1}{\Delta x_i} \sum_j \delta_{ij} g_j \Delta x_j$$

$$= \lim_{\Delta x_i \to 0} \frac{1}{\cancel{\Delta x_i}} g_i \cancel{\Delta x_i} = \lim_{\Delta x_i \to 0} g_i = g(x)$$

- Rewriting this:

$$\boxed{\frac{\delta}{\delta f(x)} \int f(x') g(x') \, dx' = g(x)}$$

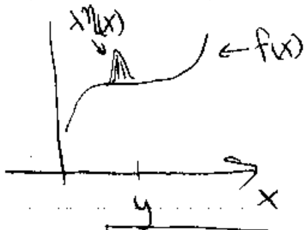
- note again that the "x" argument on the left is not a dummy argument.

- Special case: $F[f(x)] = \int f(x) \, dx$ (or $g(x) \equiv 1$)

$$\Rightarrow \boxed{\frac{\delta}{\delta f(x)} \int f(x') \, dx' = 1}$$

1/29/03

- For completeness, we present an alternative definition of the functional derivative that is more common in the literature.



As in the picture at the left we add a "bump" at y, which we take to be a delta function with strength λ, and see how much the functional changes:

$$\boxed{\frac{\delta F[f]}{\delta f(y)} = \lim_{\lambda\to 0} \frac{F[f(x)+\lambda\delta(x-y)] - F[f(x)]}{\lambda}}$$

Try it out on the example:

$$\frac{\delta}{\delta f(y)} \int f(x)\, g(x)\, dx = \lim_{\lambda\to 0} \frac{1}{\lambda}\left[\int [f(x)+\lambda\delta(x-y)]\, g(x)\, dx - \int f(x)\, g(x)\, dx\right]$$

$$= \lim_{\lambda\to 0} \frac{1}{\lambda}\lambda \int \delta(x-y)\, g(x)\, dx = g(y)$$ as before.

- Another special case, which can be used as the definition of the functional derivative as well: $g(x) = \delta(x-z)$

$$\Rightarrow \boxed{\frac{\delta}{\delta f(x)} \int f(x')\, \delta(x'-z)\, dx' = \frac{\delta f(z)}{\delta f(x)} = \delta(x-z)}$$

where the last equality is what we want.

- Note that this is the analog of $\frac{\partial}{\partial j} j = 1$, which we used repeatedly in the model problem.
  - Here, if we let $f(x) \Rightarrow f_i \to j_i$, then the equivalent formula is

$$\boxed{\frac{\partial j_i}{\partial j_k} = \delta_{jk} \Rightarrow \frac{\delta j(x)}{\delta j(y)} = \delta(x-y)}$$

1/29/13

Let's try another example we'll come across

$$\boxed{\begin{aligned}&\frac{\delta}{\delta f(x)}\int f(x_1)C(x_1,x_2)f(x_2)\,dx_1 dx_2 \quad \text{with } C(x_2,x_1)=C(x_1,x_2)\\ &= \lim_{\Delta x_i\to 0}\frac{1}{\Delta x_i}\frac{\partial}{\partial f_i}\sum_{jk} f_j C_{jk} f_k \Delta x_j \Delta x_k\\ &= \lim_{\Delta x_i\to 0}\frac{1}{\Delta x_i}\sum_{jk}\left(\delta_{ij}C_{jk}f_k + f_j C_{jk}\delta_{ik}\right)\Delta x_j\Delta x_k\\ &= \lim_{\Delta x_i\to 0}\frac{2}{\cancel{\Delta x_i}}\sum_k C_{ik}f_k \cancel{\Delta x_i}\Delta x_k = 2\int C(x,x_2)f(x_2)\,dx_2\end{aligned}}$$

- More generally, if $C_n(x_1, x_2, \cdots, x_n)$ is totally symmetric under interchange of any two x_i, then

$$\boxed{\frac{\delta^n F[f]}{\delta f(x_1)\cdots\delta f(x_n)} = n!\, C_n(x_1,\dots,x_n)}$$

- We get the special case of $n=2$ by taking another functional derivative of the example up top.

- Exercise for the diligent reader: rederive these results using the alternative definition of a functional derivative.

- The familiar properties of ordinary (partial) derivatives, such as the product and chain rules, carry over to functional derivatives in a natural way.

1/29/03

So consider

$$\frac{\delta}{\delta f(x)}\int (f(y))^3 g(y)\,dy = \lim_{\Delta x_i \to 0} \frac{1}{\Delta x_i}\frac{\partial}{\partial f_i}\sum_j (f_j)^3 g_j \Delta x_j$$

$$= \lim_{\Delta x_i \to 0} \frac{1}{\Delta x_i}\sum_j 3(f_j)^2 \delta_{ij}\, g_j \Delta x_j$$

$$= \lim_{\Delta x_i \to 0} 3(f_i)^3 g_i = 3(f(x))^2 g(x)$$

and so we have the chain rule:

$$\boxed{\frac{\delta}{\delta f(x)}\int g[f(y)]\,dy = g'[f(x)]}$$

where g' means to take the derivative of g with respect to f as if it were a partial derivative.

- Another form of the chain rule we'll use a lot is:

$$\boxed{\frac{\delta}{\delta j(y)} e^{-\int dx\,(a\varphi(x)^2 + j(x)\varphi(x))} = -\varphi(y)\,e^{-\int dx\,(a\varphi(x)^2 + j(x)\varphi(x))}}$$

- What if we have a function of $\vec{x}$ or $\vec{x}$ and t (or τ)?
  - We can combine $\vec{x}$ and t into a "four-vector" x^μ, so it is sufficient to consider an n-vector $\vec{x} = (x_1, \ldots, x_n)$
  - Consider

$$F[f(\vec{x})] = \int_0^b f(\vec{x})\, g(\vec{x})\, dx_1 \cdots dx_n$$

$$\Rightarrow F[f(\vec{x})] \to \sum_{i_1, i_2, \ldots i_n} f_{i_1,\ldots,i_n}\, g_{i_1,\ldots,i_n}\, \Delta x_{i_1} \cdots \Delta x_{i_n}$$

where we now have n-dimensional arrays $x[i_1,\ldots,i_n]$ and $f[i_1,\ldots,i_n]$, where i_1 runs over the mesh for x_1, i_2 runs over the mesh for i_2, and so on.

1/29/03

Let's take $n=2$ for clarity. Then our mesh is a two-dimensional grid and the function value is defined at each grid point (i,j) to be f_{ij}.

$$\Rightarrow F[f(\vec{x})] = \sum_{i,j} f_{ij}\, g_{ij}\, \Delta x_i \Delta y_j \quad \text{with } \vec{x} = (x,y)$$

and the functional derivative

$$\boxed{\begin{aligned} \frac{\delta F[f]}{\delta f(\vec{x})} &= \lim_{\substack{\Delta x_i \to 0 \\ \Delta y_j \to 0}} \frac{1}{\Delta x_i \Delta y_j} \frac{\partial}{\partial f_{ij}} \sum_{i',j'} f_{i'j'}\, g_{i'j'}\, \Delta x_{i'} \Delta y_{j'} \\ &= \lim_{\substack{\Delta x_i \to 0 \\ \Delta y_j \to 0}} \frac{1}{\Delta x_i \Delta y_j} \sum_{i',j'} \delta_{ii'} \delta_{jj'}\, g_{i'j'}\, \Delta x_{i'} \Delta y_{j'} \\ &= \lim_{\substack{\Delta x_i \to 0 \\ \Delta y_j \to 0}} g_{ij} = g(\vec{x}) \end{aligned}}$$

Similarly,

$$\boxed{\frac{\delta f(\vec{x})}{\delta f(\vec{x}')} = \delta^n(\vec{x}-\vec{x}')}$$

and so on.

1/29/03

· What if there is a derivative in the functional?

$$\frac{\delta}{\delta f(x)}\int \frac{df(x')}{dx'}\, g(x')\,dx' \;?$$

· We can do this several ways. Here are two:

i) partially integrate

$$\boxed{\begin{aligned}\frac{\delta}{\delta f(x)}\int \frac{df(x')}{dx'}\, g(x')\,dx' &= \frac{\delta}{\delta f(x)}\left[-\int f(x')\frac{dg(x')}{dx'}\,dx' + \text{surface term}\right] \\ &= -\frac{dg}{dx} \quad \text{(if the surface term doesn't contribute)}\end{aligned}}$$

or

ii) discrete version:

$$\boxed{\begin{aligned}\frac{\delta}{\delta f(x)}\int \frac{df(x')}{dx'}\, g(x')\,dx' &= \lim_{\Delta x_i \to 0} \frac{1}{\Delta x_i}\frac{\partial}{\partial f_i}\sum_j \frac{f_{j+1}-f_j}{\cancel{\Delta x_j}}\, g_j \cancel{\Delta x_j} \\ &= \lim_{\Delta x_i \to 0} \frac{1}{\Delta x_i}\sum_j (\delta_{i,j+1}-\delta_{ij})\, g_j \\ &= \frac{1}{\Delta x_i}(g_{i-1}-g_i) = -\frac{dg}{dx}\end{aligned}}$$

· note that the possible surface terms in this case are left over from taking $\sum_j f_{j+1} g_j - \sum_j f_j g_j$ and changing dummy indices in the first.

· Ok, that's it for now ⇒ more later!

1/29/03

Ok, let's derive perturbative expansions, Introduce

$$\boxed{Z[J(x)] = \int \mathcal{D}\sigma \, e^{-\int d^4x \left[\frac{1}{2}a(\sigma(x))^2 + \frac{\lambda}{4}(\sigma(x))^4 - J(x)\sigma(x)\right]}}$$

- functional derivatives bring down $\sigma(x)$ at the x-point of the derivative. I.e., we can write D_σ as

$$\boxed{D_\sigma(y,z) = \frac{\left(\frac{\delta}{\delta J(y)}\frac{\delta}{\delta J(z)} Z[J(x)]\right)\Big|_{J(x)=0}}{Z[J(x)]\Big|_{J(x)=0}}}$$

- We set the source $J(x)$ identically to zero after taking any functional derivatives.

- Now pull $e^{-\int d^4x \frac{\lambda}{4}(\sigma(x))^4}$ out of the path integral by replacing $\sigma(x)$ by $\frac{\delta}{\delta J(x)}$:

$$\boxed{Z[J(x)] = e^{-\int d^4x' \frac{\lambda}{4}\left(\frac{\delta}{\delta J(x')}\right)^4} \int \mathcal{D}\sigma \, e^{-\int d^4x \left[\frac{1}{2}a(\sigma(x))^2 - J(x)\sigma(x)\right]}}$$

- remember that the $\sigma(x)$'s and $J(x)$'s are just functions, there are no operators (in the occupation number space) in these expressions.
- As always, the exponential out front is defined by its Taylor series.
- Note that we've used x' in the integral in the exponent out front to emphasize that it is a different label than the x in the integral in the exponent inside the integral.

- Next we "complete the square" in the exponent in the path integral (it's convenient to drop the x's while doing this):

$$\boxed{\frac{1}{2}a\left(\sigma^2 - \frac{2}{a}J\sigma + \frac{J^2}{a^2}\right) - \frac{1}{2}Ja^{-1}J = \frac{1}{2}a\left(\sigma(x) - \frac{1}{a}J(x)\right)^2 - \frac{1}{2}J(x)a^{-1}J(x)}$$

- we'll see below that "a" is usually a differential operator, not just a number.

1/29/03

Now we can change the path integral "variable" from $\sigma(x) \to \sigma'(x) = \sigma(x) - \frac{1}{a}J(x)$.

- Recall that x is just a label for a lattice point. So for each x_i, when we integrate over $\sigma_i \equiv \sigma(x_i)$, we can change variables to $\sigma_i' = \sigma_i - \frac{1}{a}J_i$ and $d\sigma_i' = d\sigma_i$ (and the integration limits are $-\infty$ to ∞, so no problem there).

$$\Rightarrow \mathcal{D}\sigma = \mathcal{D}\sigma'$$

$$\Rightarrow \boxed{Z[J(x)] = e^{-\int d^4x' \frac{\lambda}{4}\left(\frac{\delta}{\delta J(x')}\right)^4} e^{+\int d^4x \frac{1}{2}J(x)a^{-1}J(x)} \underbrace{\int \mathcal{D}\sigma' e^{-\int d^4x'' \frac{1}{2}a(\sigma'(x''))^2}}_{Z_0!}}$$

or

$$\boxed{\frac{Z[J(x)]}{Z_0} = e^{-\int d^4x' \frac{\lambda}{4}\left(\frac{\delta}{\delta J(x')}\right)^4} e^{\int d^4x \frac{1}{2}J(x)a^{-1}J(x)}}$$

and

$$\boxed{D_\sigma(y,z) = \frac{\frac{\delta}{\delta J(y)}\frac{\delta}{\delta J(z)} e^{-\int d^4x' \frac{\lambda}{4}\left(\frac{\delta}{\delta J(x')}\right)^4} e^{\int d^4x \frac{1}{2}J(x)a^{-1}J(x)}\Big|_{J=0}}{e^{-\int d^4x' \frac{\lambda}{4}\left(\frac{\delta}{\delta J(x')}\right)^4} e^{\int d^4x \frac{1}{2}J(x)a^{-1}J(x)}\Big|_{J=0}}}$$

$\Rightarrow$ we see a very close analogy to the model partition function!

- The diagramatic expansion follows as in that case: by expanding exponentials to a desired order in λ and so that the J's are exactly wiped out by the same number of $\frac{\delta}{\delta J}$'s.
  - Things are a bit more involved because the J's now carry labels $\Rightarrow J(x) \to J(x_i) = J_i$.

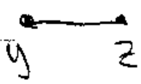
- First do $D_\sigma(y,z)$ to zeroth order $\Rightarrow$ call it $D_\sigma^0(y,z)$:

$$\boxed{D_\sigma^0(y,z) = \frac{\delta}{\delta J(y)}\frac{\delta}{\delta J(z)} \underbrace{\int d^4x \frac{1}{2}J(x)a^{-1}J(x)}_{1}\Big|_{J=0} = \frac{\delta}{\delta J(y)} a^{-1}J(z) = a^{-1}\delta(y-z)}$$

$$(\delta(y-z) \equiv \delta(\tau_y - \tau_z)\delta^3(\vec{y}-\vec{z}))$$

1/29/03

- So now we can use the endpoints of —— to mark the points y and z (these are "spacetime points")

$\Rightarrow$  $\Rightarrow \quad \boxed{\sigma^{-1}\delta(y-z) = D_0^0(y,z)}$

is our Feynman rule for •—.

- Now we can rewrite $\int \frac{1}{2} J\sigma^{-1} J$ in terms of D_0^0 as:

$$\boxed{\int d^4x \, \tfrac{1}{2} J(x)\, \sigma^{-1} J(x) = \tfrac{1}{2}\int d^4x\, d^4x' \, J(x)\, D_0^0(x,x')\, J(x')}$$

which is the form we'll see more often. Note that $D_0^0(x,x') = D_0^0(x',x)$.

- Ok, now expand Z_λ/Z_0 to leading order in λ:

$$\boxed{\left.\frac{Z[J(x)]}{Z_0}\right|_{J=0} = 1 - \left[\int d^4x' \, \frac{\lambda}{4}\left(\frac{\delta}{\delta J(x')}\right)^4\right] \int d^4x_1 d^4x_2 \, \tfrac{1}{2} J(x_1) D_0^0(x_1,x_2) J(x_2) \times \int d^4x_3 d^4x_4 \, \tfrac{1}{2} J(x_3) D_0^0(x_3,x_4) J(x_4)}$$

- We see that the $\frac{\delta}{\delta J(x')}$ can wipe out any of $J(x_1), J(x_2), J(x_3), J(x_4)$
  $\Rightarrow$ there is a 4! factor, just as in the model.

- Note that $\frac{\delta}{\delta J(x')}\frac{\delta}{\delta J(x')} \int d^4x_1 \int d^4x_2 \, \frac{1}{2} J(x_1) D_0^0(x_1,x_2) J(x_2) = \frac{\delta}{\delta J(x')} \int d^4x_1 \, J(x_1) D_0^0(x_1,x')$

$$= D_0^0(x',x')$$

- So if we mark the vertex position x' with a •, then this is 
  - The other term is similar so we get 
  - We'll get the same symmetry factor as in the model by applying our rules ① : ½ × ½ ② ½ ③ 1 ⇒ 1/8.
  - We'll get $-\frac{\lambda}{4} \cdot 4! = -6\lambda$ for the vertex.
  - We still have an overall integral over x'

$\Rightarrow \boxed{Z_\lambda/Z_0 = 1 - 6\lambda \int d^4x' \, D_0^0(x',x') D_0^0(x',x')}$

1/29/03

If we substitute $D_\sigma(x,y) = a^{-1}\delta(x-y)$, this expression looks very badly behaved!

- There is a factor of the space-time volume, which is ok, because this is $\int d^4x \to \beta\Omega$ (where Ω is the volume) at temperature β. When we take $\frac{1}{\beta}(\ln Z/Z_0)$ we get $\Omega - \Omega_0$, which should be extensive $\Rightarrow$ proportional to Ω.
- We get a hint by thinking about the delta function in momentum space

$$D^0_\sigma(x,y) = \frac{1}{a}\delta(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-y)}\frac{1}{a} \equiv \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot(x-y)} \tilde{D}^0_\sigma(k)$$

$\Rightarrow$ the "momentum-space propagator" $\tilde{D}^0_\sigma(k) = \frac{1}{a}$, which is a constant here, independent of momentum $k \equiv (k_0, \vec{k})$

$\Rightarrow$ the high-energy, high-momentum modes are completely undamped.

- This is an extreme type of divergence (cf. PS #1).

$\Rightarrow$ we'll return to such divergences soon.

- If we neglect all of the infinities for the moment ...

The diagrammatic expansion of $\ln Z/Z_0$ and $D_\sigma(y,z)$ parallel the model problem exactly.

- The replica method proofs go through as well.

- Feynman rules for λ^n contribution to $D_\sigma(y,z)$:

1. Draw all connected diagrams with n vertices at x_i, $i = 1,\dots,n$, and two external points y and z, connected by propagator lines.
2. Each vertex gets a factor of -6λ.
3. Each line between x_i and x_j gets a factor $D_\sigma(x_i,x_j)$.
4. Multiply by a symmetry factor (as described elsewhere for lines without arrows).
5. Integrate $\int d^4x_1 \cdots d^4x_n$.

Try some! e.g.

1/29/03

- Now let's return to the case of greater immediate interest: fermions with a short-range interaction.

- In second-quantized form, written in terms of field operators $\boxed{\hat{\psi}(\vec{x}) \equiv \sum_k \psi_k(\vec{x}) c_k}$ and $\boxed{\hat{\psi}^\dagger(\vec{x}) \equiv \sum_k \psi_k^\dagger(\vec{x}) c_k^\dagger}$

  for either bosons or fermions (recall that "k" stands for a complete set of quantum numbers specifying a single-particle state, such as $k = \{\vec{k}, s_z\}$ for spin-1/2 fermions),

$$\boxed{\begin{aligned}\hat{H} &= \int d^3x\, \hat{\psi}^\dagger(\vec{x}) \left(-\frac{\hbar^2\vec{\nabla}^2}{2m}\right) \hat{\psi}(\vec{x}) + \frac{1}{2}\iint d^3x\, d^3x'\, \hat{\psi}^\dagger(\vec{x})\hat{\psi}^\dagger(\vec{x}')V(\vec{x},\vec{x}')\hat{\psi}(\vec{x}')\hat{\psi}(\vec{x}) \\ &\longrightarrow \int d^3x\, \hat{\psi}^\dagger(\vec{x})\left(-\frac{\hbar^2\nabla^2}{2m}\right)\hat{\psi}(\vec{x}) + \frac{1}{2}\lambda \int d^3x\, \hat{\psi}^\dagger(\vec{x})\hat{\psi}^\dagger(\vec{x})\hat{\psi}(\vec{x})\hat{\psi}(\vec{x})\end{aligned}}$$

for $\boxed{V(\vec{x},\vec{x}') = \lambda\delta^3(\vec{x}-\vec{x}')}$

- In the boson case,

$$\boxed{\begin{aligned}&[\hat{\psi}(\vec{x}), \hat{\psi}^\dagger(\vec{x}')] = \delta(\vec{x}-\vec{x}') \\ \text{and}\quad &[\hat{\psi}(\vec{x}), \hat{\psi}(\vec{x}')] = [\hat{\psi}^\dagger(\vec{x}), \hat{\psi}^\dagger(\vec{x}')] = 0\end{aligned}}$$

- In the fermion case, we add a spin index $\hat{\psi}(\vec{x}) \to \hat{\psi}_\alpha(\vec{x})$ where α runs from 1 to g and we have anti-commutation relations

$$\boxed{\begin{aligned}\{\hat{\psi}^\dagger_\alpha(\vec{x}), \hat{\psi}_\beta(\vec{x}')\} &= \delta(\vec{x}-\vec{x}')\delta_{\alpha\beta} \\ \{\hat{\psi}_\alpha(\vec{x}), \hat{\psi}_\beta(\vec{x}')\} &= \{\hat{\psi}^\dagger_\alpha(\vec{x}), \hat{\psi}^\dagger_\beta(\vec{x}')\} = 0\end{aligned}}$$

1/29/03

In deriving path integral expressions for matrix elements of the Euclidean evolution operator, we inserted complete sets of position and momentum states at the discretized times (see pg 48 and thereabouts):

$$\boxed{\begin{aligned} U_E(x_f\tau_f, x_i\tau_i) &= \langle x_f | e^{-(\tau_f-\tau_i)\hat{H}/\hbar} | x_i\rangle \\ &= \lim_{m\to\infty} \int \left(\prod_{k=1}^{m-1} dx_k\right) \prod_{k=1}^{m} \frac{dp_k}{(2\pi\hbar)^3}\, e^{\sum_{k=1}^{m}\left(\frac{i p_k}{\hbar}(x_k - x_{k-1}) - \frac{\epsilon}{\hbar}\left(\frac{p_k^2}{2m} + V(x_{k-1})\right)\right)} \\ &\to \int_{(x_i\tau_i)}^{(x_f\tau_f)} \mathcal{D}[x(\tau)]\, e^{-\frac{1}{\hbar}S_E[x(\tau)]} \end{aligned}}$$

For bosons, we have the correspondence

$$\hat{x} \leftrightarrow \hat{\psi}(\vec{x}) \qquad \hat{p} \leftrightarrow i\hbar\hat{\psi}^\dagger(\vec{x})$$

which leads to (details elsewhere) the partition function expression

$$\boxed{\begin{aligned} Z_G &= \mathrm{Tr}\left(e^{-\beta(\hat{H}-\mu\hat{N})}\right) \\ &= \int_{\psi(\vec{x},\beta)=\pm\psi(\vec{x},0)} \mathcal{D}[\psi^\dagger(\vec{x},\tau),\psi(\vec{x},\tau)]\, e^{-\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau \int d^3x\, \psi^\dagger(\vec{x}\tau)\left(\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu\right)\psi(\vec{x},\tau)} \\ &\qquad \times e^{-\frac{1}{2}\int_0^{\beta\hbar} d\tau \int d^3x\, d^3x'\, \psi^\dagger(\vec{x}\tau)\psi^\dagger(\vec{x}'\tau)V(\vec{x}-\vec{x}')\psi(\vec{x}'\tau)\psi} \\ &\to \int \mathcal{D}[\psi^\dagger(x)\psi(x)]\, e^{-1/\hbar\int_0^{\beta\hbar} d\tau\int d^3x\, \psi^\dagger(x)\left(\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu\right)\psi(x)} \\ &\qquad \times e^{-\frac{\lambda}{2}\int_0^{\beta\hbar} d\tau \int d^3x\, \psi^\dagger(x)\psi^\dagger(x)\psi(x)\psi(x)} \end{aligned}}$$

where we've used the short hand $x \equiv \{\vec{x}, \tau\}$ (or $\{\tau, \vec{x}\}$) and the + sign on the first path integral indicates the boson boundary conditions.