AJT notes (1/11/22)

- Derivation for SRG evolved overlap

$$\langle \Psi_\alpha^{A-1} | \hat{U}_\lambda^\dagger \hat{U}_\lambda a_{\vec{p}} \hat{U}_\lambda^\dagger \hat{U}_\lambda | \Psi_0^A \rangle \quad (1)$$

where α is specified by $\{n, j, l, s, \tau\}$.

i) $a_{\vec{p}}$ must have some spin and isospin projection:

$$a_{\vec{p}} \to a_{\vec{p}\sigma\tau}$$

ii) $$\hat{U}_\lambda |\Psi_0^A\rangle = |\Psi_0^A(\lambda)\rangle \equiv |\Phi\rangle \quad (2)$$

iii) $$\langle \Psi_\alpha^{A-1} | \hat{U}_\lambda^\dagger = \langle \Psi_\alpha^{A-1}(\lambda) |$$

$$= \langle \Psi_0^A(\lambda) | a_\alpha^\dagger$$

$$= \langle \Phi | a_\alpha^\dagger \quad (3)$$

Then Eq. (1) reads

$$\langle \Phi | a_\alpha^\dagger \hat{U}_\lambda a_{\vec{p}\sigma\tau} \hat{U}_\lambda^\dagger | \Phi \rangle \quad (4)$$

The strategy is to evaluate $a_\alpha^\dagger \hat{U}_\lambda a_{\vec{p}\sigma\tau} \hat{U}_\lambda^\dagger$ at the 2-body level using the expansion of $\hat{U}_\lambda$:

$$\hat{U}_\lambda = \hat{I} + \frac{1}{4} \sum_{\sigma_1\sigma_2\sigma_3\sigma_4} \sum_{\tau_1\tau_2\tau_3\tau_4} \sum_{\vec{K}\vec{k}\vec{k}'} \langle \vec{k}\sigma_1\tau_1\sigma_2\tau_2 | \delta\tilde{U} | \vec{k}'\sigma_3\tau_3\sigma_4\tau_4 \rangle$$

$$\times\, a^\dagger_{\frac{\vec{K}}{2}+\vec{k}\,\sigma_1\tau_1}\, a^\dagger_{\frac{\vec{K}}{2}-\vec{k}\,\sigma_2\tau_2}\, a_{\frac{\vec{K}}{2}-\vec{k}'\sigma_4\tau_4}\, a_{\frac{\vec{K}}{2}+\vec{k}'\sigma_3\tau_3} + \ldots \qquad (5)$$

where $\quad \delta\tilde{U} = (1 - \hat{P}_{12})\,\delta U$

The only terms that are less than 3-body are given below

I) $\quad a_\alpha^\dagger \hat{I} a_{\vec{p}\sigma\tau} \hat{I}$

II) $\quad a_\alpha^\dagger \hat{I} a_{\vec{p}\sigma\tau} \delta\hat{\tilde{U}}^\dagger$

Start with I) :

Using $\quad a_i = \sum_j \langle i | j \rangle a_j \quad$ we can write

$a_{\vec{p}\sigma\tau}$ in terms of single-particle (s.p.) states β (these will be indicated by Greek letters):

$$a_{\vec{p}\sigma\tau} = \sum_\beta \langle \vec{p}\sigma\tau | \beta \rangle a_\beta \qquad (6)$$

where $\langle \vec{p}\sigma\tau | \beta \rangle = \Phi_\beta(|\vec{p}|)\, Y_{\ell'}^{m_\ell'}(\Omega_{\vec{p}})\, \langle \ell' m_\ell' s \sigma | j' m_j' \rangle$

and $m_\ell' = m_j' - \sigma$ and $s = \frac{1}{2}$. Then Eq. (6) reads

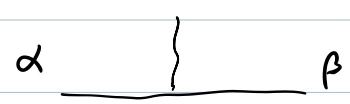
$$a_{\vec{p}\sigma\tau} = \sum_\beta \Phi_\beta(p)\, Y_{\ell'}^{m_j'-\sigma}(\Omega_{\vec{p}}) \langle \ell' m_j' - \sigma\ \tfrac{1}{2}\, \sigma | j' m_j' \rangle a_\beta \tag{7}$$

We will use Eq. (7) to also transform the creation and annihilation operators of $\hat{U}_\lambda^+$ later.

Now for the $a_\alpha^+ \hat{I}\, a_{\vec{p}\sigma\tau} \hat{I}$ term we have

$$\sum_\beta \Phi_\beta(p)\, Y_{\ell'}^{m_j'-\sigma}(\Omega_{\vec{p}}) \langle \ell' m_j' - \sigma\ \tfrac{1}{2}\, \sigma | j' m_j' \rangle\, a_\alpha^+ a_\beta \tag{8}$$

This is a 1-body operator. Diagramatically



For II), we start by transforming the creation and annihilation operators to the s.p. basis

$$\Rightarrow \frac{1}{4} \sum_{\sigma_1\sigma_2\sigma_3\sigma_4} \sum_{\tau_1\tau_2\tau_3\tau_4} \sum_{\vec{K}\vec{k}\vec{k}'} \langle \vec{k}\, \sigma_1 \tau_1 \sigma_2 \tau_2 | \delta\tilde{U}^+ | \vec{k}'\, \sigma_3 \tau_3 \sigma_4 \tau_4 \rangle$$

$$\times \sum_{\rho\sigma\mu\nu} \Phi_\rho^*(|\tfrac{\vec{K}}{2} + \vec{k}|)\, \Phi_\sigma^*(|\tfrac{\vec{K}}{2} - \vec{k}|)\, \Phi_\mu(|\tfrac{\vec{K}}{2} + \vec{k}'|)\, \Phi_\nu(|\tfrac{\vec{K}}{2} - \vec{k}'|)$$

$$\times\ Y_{l_1}^{m_{j_1}-\sigma_1 *}(\Omega_{\frac{\vec{K}}{2}+\vec{k}})\, Y_{l_2}^{m_{j_2}-\sigma_2 *}(\Omega_{\frac{\vec{K}}{2}-\vec{k}})\, Y_{l_3}^{m_{j_3}-\sigma_3}(\Omega_{\frac{\vec{K}}{2}+\vec{k}'})\, Y_{l_4}^{m_{j_4}-\sigma_4}(\Omega_{\frac{\vec{K}}{2}-\vec{k}'})$$

$$\times \langle j_1 m_{j_1} | l_1\, m_{j_1}-\sigma_1\, \tfrac{1}{2}\sigma_1 \rangle \langle j_2 m_{j_2} | l_2\, m_{j_2}-\sigma_2\, \tfrac{1}{2}\sigma_2 \rangle \langle l_3\, m_{j_3}-\sigma_3\, \tfrac{1}{2}\sigma_3 | j_3 m_{j_3} \rangle$$

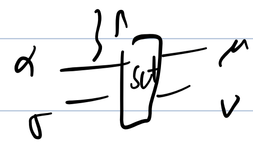
$$\times \langle l_4\, m_{j_4}-\sigma_4\, \tfrac{1}{2}\sigma_4 | j_4 m_{j_4} \rangle\ a_\alpha^\dagger a_\beta a_\rho^\dagger a_\sigma^\dagger a_\nu a_\mu \qquad (9)$$

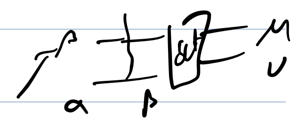
Applying Wick's theorem :

$$a_\alpha^\dagger a_\beta a_\rho^\dagger a_\sigma^\dagger a_\nu a_\mu = :a_\alpha^\dagger a_\beta \overset{3\text{-body}}{\cancel{a_\rho^\dagger a_\sigma^\dagger}} a_\nu a_\mu:$$

$$+ :a_\alpha^\dagger \overbrace{a_\beta a_\rho^\dagger} a_\sigma^\dagger a_\nu a_\mu:$$

$$- :a_\alpha^\dagger \overbrace{a_\beta a_\rho^\dagger a_\sigma^\dagger} a_\nu a_\mu:$$





$$= \delta_{\beta\rho}\, a_\alpha^\dagger a_\sigma^\dagger a_\nu a_\mu - \delta_{\beta\sigma}\, a_\alpha^\dagger a_\rho^\dagger a_\nu a_\mu \qquad (10)$$

Next we evaluate matrix elements with respect to $|\Phi\rangle$ assuming $|\Phi\rangle$ is given by independent-particle model :

$$|\Phi\rangle = \prod_{\alpha\ \text{occupied}} a_\alpha^\dagger |0\rangle \qquad (11)$$

Focusing only on the creation and annihilation operators

I) $$\langle \Phi | a_\alpha^\dagger a_\beta | \Phi \rangle = \delta_{\alpha\beta} \qquad (12)$$

and II) $\langle \Phi | \left[\delta_{\beta\rho}\, a_\alpha^\dagger a_\sigma^\dagger a_\nu a_\mu - \delta_{\beta\sigma}\, a_\alpha^\dagger a_\rho^\dagger a_\nu a_\mu \right] | \Phi \rangle$

$$= \delta_{\beta\rho}\left(\delta_{\alpha\mu}\delta_{\sigma\nu} - \delta_{\alpha\nu}\delta_{\sigma\mu}\right) - \delta_{\beta\sigma}\left(\delta_{\alpha\mu}\delta_{\rho\nu} - \delta_{\alpha\nu}\delta_{\rho\mu}\right) \qquad (13)$$

Combine all equations to find

$$\langle \Phi | a_\alpha^\dagger\, \hat{U}_\lambda\, a_{\vec{p}\sigma\tau}\, \hat{U}_\lambda^\dagger | \Phi \rangle$$

$$= \sum_\beta \Phi_\beta(p)\, Y_{l'}^{m_j'-\sigma}(\Omega_{\vec{p}}) \langle l' m_j'-\sigma\, \tfrac{1}{2}\sigma | j' m_j' \rangle\, \delta_{\alpha\beta}$$

$$+ \frac{1}{4} \sum_{\sigma_1\sigma_2\sigma_3\sigma_4} \sum_{\tau_1\tau_2\tau_3\tau_4} \sum_{\vec{K}\vec{k}\vec{k}'} \langle \vec{k}\, \sigma_1\tau_1\, \sigma_2\tau_2 | \delta\tilde{U}^\dagger | \vec{k}'\, \sigma_3\tau_3\, \sigma_4\tau_4 \rangle$$

$$\times \sum_\beta \Phi_\beta(p)\, Y_{l'}^{m_j'-\sigma}(\Omega_{\vec{p}}) \langle l' m_j'-\sigma\, \tfrac{1}{2}\sigma | j' m_j' \rangle$$

$$\times \sum_{\rho\sigma\mu\nu} \Phi_\rho^*\left(\left|\tfrac{\vec{K}}{2}+\vec{k}\right|\right) \Phi_\sigma^*\left(\left|\tfrac{\vec{K}}{2}-\vec{k}\right|\right) \Phi_\mu\left(\left|\tfrac{\vec{K}}{2}+\vec{k}'\right|\right) \Phi_\nu\left(\left|\tfrac{\vec{K}}{2}-\vec{k}'\right|\right)$$

$$\times Y_{l_1}^{m_{j_1}-\sigma_1 *}\left(\Omega_{\frac{\vec{K}}{2}+\vec{k}}\right) Y_{l_2}^{m_{j_2}-\sigma_2 *}\left(\Omega_{\frac{\vec{K}}{2}-\vec{k}}\right) Y_{l_3}^{m_{j_3}-\sigma_3}\left(\Omega_{\frac{\vec{K}}{2}+\vec{k}'}\right) Y_{l_4}^{m_{j_4}-\sigma_4}\left(\Omega_{\frac{\vec{K}}{2}-\vec{k}'}\right)$$

$$\times \langle j_1 m_{j_1} | l_1\, m_{j_1}-\sigma_1\, \tfrac{1}{2}\sigma_1 \rangle \langle j_2 m_{j_2} | l_2\, m_{j_2}-\sigma_2\, \tfrac{1}{2}\sigma_2 \rangle \langle l_3\, m_{j_3}-\sigma_3\, \tfrac{1}{2}\sigma_3 | j_3 m_{j_3} \rangle \langle l_4\, m_{j_4}-\sigma_4\, \tfrac{1}{2}\sigma_4 | j_4 m_{j_4} \rangle$$

$$\times \left[\delta_{\beta\rho}\left(\delta_{\alpha\mu}\delta_{\sigma\nu} - \delta_{\alpha\nu}\delta_{\sigma\mu}\right) - \delta_{\beta\sigma}\left(\delta_{\alpha\mu}\delta_{\rho\nu} - \delta_{\alpha\nu}\delta_{\rho\mu}\right) \right] \qquad (14)$$

$$= \Phi_\alpha(p)\, Y_\ell^{m_j-\sigma}(\Omega_{\hat p})\,(\ell\, m_j-\sigma\ \tfrac12\sigma \,|\, j\, m_j)$$

$$+ \frac14 \sum_{\sigma_1\sigma_2\sigma_3\sigma_4} \sum_{\tau_1\tau_2\tau_3\tau_4} \sum_{\vec K \vec k \vec k'} \langle \vec k\, \sigma_1\tau_1\, \sigma_2\tau_2 \,|\, \delta\tilde U^{+} \,|\, \vec k'\, \sigma_3\tau_3\, \sigma_4\tau_4 \rangle$$

$$\times \Big[\sum_{\beta\sigma} \Phi_\beta(p)\, Y_{\ell'}^{m_j'-\sigma}(\Omega_{\hat p})\,(\ell' m_j'-\sigma\ \tfrac12\sigma \,|\, j' m_j')$$

$$\times\, \Phi_\beta^*(|\tfrac{\vec K}{2}+\vec k|)\, \Phi_\sigma^*(|\tfrac{\vec K}{2}-\vec k|)\, \Phi_\alpha(|\tfrac{\vec K}{2}+\vec k'|)\, \Phi_\sigma(|\tfrac{\vec K}{2}-\vec k'|)$$

$$\times\, Y_{\ell'}^{m_j'-\sigma_1 *}(\Omega_{\frac{\vec K}{2}+\vec k})\, Y_{\ell_2}^{m_{j_2}-\sigma_2 *}(\Omega_{\frac{\vec K}{2}-\vec k})\, Y_\ell^{m_j-\sigma_3}(\Omega_{\frac{\vec K}{2}+\vec k'})\, Y_{\ell_2}^{m_{j_2}-\sigma_4}(\Omega_{\frac{\vec K}{2}-\vec k'})$$

$$\times\, (j' m_j' \,|\, \ell' m_j'-\sigma_1\ \tfrac12\sigma_1)(j_2 m_{j_2} \,|\, \ell_2\, m_{j_2}-\sigma_2\ \tfrac12\sigma_2)(\ell\, m_j-\sigma_3\ \tfrac12\sigma_3 \,|\, j\, m_j)$$

$$\times\, \langle \ell_2\, m_{j_2}-\sigma_4\ \tfrac12\sigma_4 \,|\, j_2 m_{j_2} \rangle \ -\ 3 \text{ other terms} \Big] \qquad (15)$$

* Should $\delta_{\beta p} \Rightarrow m_\ell' = m_{\ell_1}$?

Let $\overset{(s.p.)}{\sigma}, j_2, m_{j_2}, \ell_2 \rightarrow \gamma, j'', m_j'', \ell''$

Then Eq. (15) reads

$$= \Phi_\alpha(p)\, Y_\ell^{m_j-\sigma}(\Omega_{\hat p})\,(\ell\, m_j-\sigma\ \tfrac12\sigma \,|\, j\, m_j)$$

$$+ \frac14 \sum_{\sigma_1\sigma_2\sigma_3\sigma_4} \sum_{\tau_1\tau_2\tau_3\tau_4} \sum_{\vec K \vec k \vec k'} \langle \vec k\, \sigma_1\tau_1\, \sigma_2\tau_2 \,|\, \delta\tilde U^{+} \,|\, \vec k'\, \sigma_3\tau_3\, \sigma_4\tau_4 \rangle$$

$$\times \sum_{\beta\gamma} \Big[\Phi_\beta(p)\, \Phi_\beta^*(|\tfrac{\vec K}{2}+\vec k|)\, \Phi_\gamma^*(|\tfrac{\vec K}{2}-\vec k|)\, \Phi_\gamma(|\tfrac{\vec K}{2}-\vec k'|)\, \Phi_\alpha(|\tfrac{\vec K}{2}+\vec k'|)$$

$$\times\ Y_{l'}^{m_j'-\sigma}(\Omega_{\vec{p}})\langle l' m_j'-\sigma \tfrac{1}{2}\sigma | j' m_j'\rangle Y_{l'}^{m_j'-\sigma_1 *}(\Omega_{\frac{\vec{k}}{2}+\vec{k}})\langle j' m_j' | l' m_j'-\sigma_1 \tfrac{1}{2}\sigma_1\rangle$$

$$\times\ Y_{l''}^{m_j''-\sigma_2 *}(\Omega_{\frac{\vec{k}}{2}-\vec{k}})\langle j'' m_j'' | l'' m_j''-\sigma_2 \tfrac{1}{2}\sigma_2\rangle Y_{l''}^{m_j''-\sigma_4}(\Omega_{\frac{\vec{k}}{2}-\vec{k}'})\langle l'' m_j''-\sigma_4 \tfrac{1}{2}\sigma_4 | j'' m_j''\rangle$$

$$\times\ Y_{l}^{m_j-\sigma_3}(\Omega_{\frac{\vec{k}}{2}+\vec{k}'})\langle l\, m_j-\sigma_3 \tfrac{1}{2}\sigma_3 | j\, m_j\rangle - 3 \text{ other terms} \Big] \quad (16)$$

$$[\phi] = fm^{3/2} \qquad [\delta\tilde{U}^{+}] = fm^{3} \qquad \Big[\sum_{\vec{k}\vec{k}\vec{k}'}\Big] = fm^{-9}$$

$$\Rightarrow \Big[\sum_{\vec{k}\vec{k}\vec{k}'} \phi\, \phi^{*} \phi^{*} \phi\, \phi\, \delta U^{+}\Big] = fm^{3/2} \quad ✓$$

Questions :

i) How do we reduce $Y_l^{m_l}$ dependencies?

$Y_l^{m_l}$ have different Ω dependencies and l, m_l dependencies.

ii) Does the $\sum_{\sigma_1\sigma_2\sigma_3\sigma_4}$ simplify at all?

iii) First term is simple:

$$S_\alpha = \frac{1}{2j+1}\sum_{m_j}\int d^3p\ \left|\phi_\alpha(p)\, Y_l^{m_j-\sigma}(\Omega_{\vec{p}})\langle l\, m_j-\sigma\, \tfrac{1}{2}\sigma | j\, m_j\rangle\right|^2$$

$$\sum_{m_j} \int d\Omega_{\vec{p}} \left| \Phi_\alpha(p)\, Y_\ell^{m_j-\sigma}(\Omega_{\vec{p}}) \langle \ell\, m_j-\sigma\, \tfrac{1}{2}\sigma | j\, m_j \rangle \right|^2$$

$$= (2j+1) \left| \Phi_\alpha(p) \right|^2$$

$$\Rightarrow \quad S_\alpha = \frac{1}{2j+1} (2j+1) \int dp\, p^2 \left| \Phi_\alpha(p) \right|^2$$

$$= 1 \qquad \checkmark$$