$$|\Psi_\lambda\rangle = \frac{1}{2}\sum_{AB}\int\frac{d\vec{k}_A}{(2\pi)^3}\int\frac{d\vec{k}_B}{(2\pi)^3}\, a^\dagger_{\vec{k}_A,A}\, a^\dagger_{\vec{k}_B,B}\,|0\rangle\,\Psi_{AB}(\vec{k}_A,\vec{k}_B) \qquad (1)$$

where $\Psi_{AB}(\vec{k}_A,\vec{k}_B) = (2\pi)^3\delta^3(\vec{k}_A+\vec{k}_B)\langle \vec{k}_{AB}|\Psi_\lambda\rangle$ (2)

Then $\langle\Psi_\lambda|\Psi_\lambda\rangle = (2\pi)^3\delta^3(0)\sum_{AB}\int\frac{d\vec{k}_{AB}}{(2\pi)^3}\,|\Psi_{AB}(\vec{k}_{AB})|^2$ (3)

as in SB notes.

* We want to evaluate $\dfrac{\langle\Psi_\lambda|\hat{n}_\lambda(\vec{q},\vec{G})|\Psi_\lambda\rangle}{\langle\Psi_\lambda|\Psi_\lambda\rangle}$

Start with $\hat{n}_\lambda(\vec{q},\vec{G})$ where we've evaluated contractions w.r.t. $|0\rangle$ and taken the continuum limit:

Term 1

$$\hat{n}_\lambda(\vec{q},\vec{G}) = \frac{1}{2}\sum_{\sigma\sigma'}\sum_{\tau\tau'} a^\dagger_{\frac{\vec{G}}{2}+\vec{q}\sigma\tau}\, a^\dagger_{\frac{\vec{G}}{2}-\vec{q}\sigma'\tau'}\, a_{\frac{\vec{G}}{2}-\vec{q}\sigma'\tau'}\, a_{\frac{\vec{G}}{2}+\vec{q}\sigma\tau} \; +$$

Term 2

$$\frac{1}{4}\sum_{\sigma\sigma'}\sum_{\tau\tau'}\sum_{12}\int\frac{d^3k}{(2\pi)^3}\langle\vec{k}\,12|\delta\hat{U}|\vec{q}\,\sigma\tau\sigma'\tau'\rangle\, a^\dagger_{\frac{\vec{G}}{2}+\vec{k},1}\, a^\dagger_{\frac{\vec{G}}{2}-\vec{k},2} \times a_{\frac{\vec{G}}{2}-\vec{q}\sigma'\tau'}\, a_{\frac{\vec{G}}{2}+\vec{q}\sigma\tau}$$

Term 3

$$+\frac{1}{4}\sum_{\sigma\sigma'}\sum_{\tau\tau'}\sum_{34}\int\frac{d^3k'}{(2\pi)^3}\langle\vec{q}\,\sigma\tau\sigma'\tau'|\delta\hat{U}^\dagger|\vec{k}'\,34\rangle\, a^\dagger_{\frac{\vec{G}}{2}+\vec{q}\sigma\tau}\, a^\dagger_{\frac{\vec{G}}{2}-\vec{q}\sigma'\tau'} \times a_{\frac{\vec{G}}{2}-\vec{k}',4}\, a_{\frac{\vec{G}}{2}+\vec{k}',3}$$

Term 4

$$+\frac{1}{8}\sum_{\sigma\sigma'}\sum_{\tau\tau'}\sum_{1234}\int\frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\langle\vec{k}\,12|\delta\hat{V}^{\dagger}|\vec{q}\,\sigma\tau\sigma'\tau'\rangle\langle\vec{q}\,\sigma\tau\sigma'\tau'|\delta\hat{V}^{\dagger}|\vec{k}'\,34\rangle\times$$

$$a^{\dagger}_{\frac{\vec{G}}{2}+\vec{k},1}\,a^{\dagger}_{\frac{\vec{G}}{2}-\vec{k},2}\,a_{\frac{\vec{G}}{2}-\vec{k}',4}\,a_{\frac{\vec{G}}{2}+\vec{k}',3} \qquad (4)$$

* Next we will evaluate the expectation value $\langle\Phi_d^{\dagger}|\hat{n}_{\lambda}(\vec{q},\vec{\alpha})|\Phi_d^{\dagger}\rangle$ doing contractions first term-by-term

Term 1 ~ $\langle 0|\,a_{\vec{k}_B,B}\,a_{\vec{k}_A,A}\,a^{\dagger}_{\frac{\vec{G}}{2}+\vec{q}\sigma\tau}\,a^{\dagger}_{\frac{\vec{G}}{2}-\vec{q}\sigma'\tau'}\,a_{\frac{\vec{G}}{2}-\vec{q}\sigma'\tau'}\times$

$a_{\frac{\vec{G}}{2}+\vec{q}\sigma\tau}\,a^{\dagger}_{\vec{k}_C,C}\,a^{\dagger}_{\vec{k}_D,D}\,|0\rangle$

(antisymmetric ME's so just multiply by 4)

$$=4\,(2\pi)^3\delta^3\left(\vec{k}_A-\frac{\vec{G}}{2}-\vec{q}\right)\delta_{A,\sigma\tau}\,(2\pi)^3\delta^3\left(\vec{k}_B-\frac{\vec{G}}{2}+\vec{q}\right)\delta_{B,\sigma'\tau'}\times$$
$$(2\pi)^3\delta^3\left(\frac{\vec{G}}{2}-\vec{q}-\vec{k}_D\right)\delta_{\sigma'\tau',D}\,(2\pi)^3\delta^3\left(\frac{\vec{G}}{2}+\vec{q}-\vec{k}_C\right)\delta_{\sigma\tau,C}$$

So $\langle\Phi_d^{\dagger}|\hat{n}_{\lambda}(\vec{q},\vec{\alpha})|\Phi_d^{\dagger}\rangle^{(1)}=\frac{1}{8}\sum_{ABCD}\sum_{\sigma\sigma'\tau\tau'}\int\frac{d^3k_A}{(2\pi)^3}\frac{d^3k_B}{(2\pi)^3}\frac{d^3k_C}{(2\pi)^3}\frac{d^3k_D}{(2\pi)^3}\times$

$$(2\pi)^3\delta^3(\vec{k}_A+\vec{k}_B)\,\psi_d^{\dagger}\left(\frac{\vec{k}_A-\vec{k}_B}{2};AB\right)\Big[4\,(2\pi)^3\delta^3\left(\vec{k}_A-\frac{\vec{G}}{2}-\vec{q}\right)\delta_{A,\sigma\tau}\times$$

$$(2\pi)^3\delta^3\left(\vec{k}_B-\frac{\vec{G}}{2}+\vec{q}\right)\delta_{B,\sigma'\tau'}\,(2\pi)^3\delta^3\left(\frac{\vec{G}}{2}-\vec{q}-\vec{k}_D\right)\delta_{\sigma'\tau',D}\,(2\pi)^3\times$$

$$\delta^3\left(\frac{\vec{G}}{2}+\vec{q}-\vec{k}_C\right)\delta_{\sigma\tau,C}\;\psi_d\left(\frac{\vec{k}_C-\vec{k}_D}{2};CD\right)(2\pi)^3\delta^3(\vec{k}_C+\vec{k}_D)$$

$$= \frac{1}{2}(2\pi)^3\delta^3(\vec{Q})(2\pi)^3\delta^3(\vec{Q})\sum_{\sigma\sigma'\tau\tau'}\psi_d^*(\vec{q};\sigma\tau,\sigma'\tau')\times$$

$$\psi_d(\vec{q};\sigma\tau,\sigma'\tau') \qquad (5)$$

(Integrating over $\int\frac{d^3Q}{(2\pi)^3}$ gives factor

$\underbrace{\frac{1}{2}(2\pi)^3\delta^3(0)}_{\equiv V}$ in front of sum)

Term 2 $\sim$ $\langle 0|a_{\vec{k}_B B}a_{\vec{k}_A A}a^\dagger_{\frac{\vec{Q}}{2}+\vec{k},1}a^\dagger_{\frac{\vec{Q}}{2}-\vec{k},2}a_{\frac{\vec{Q}}{2}-\vec{q}\sigma\tau'}a_{\frac{\vec{Q}}{2}+\vec{q}\sigma\tau}a^\dagger_{\vec{k}_C,C}a^\dagger_{\vec{k}_D,D}|0\rangle$

$$= 4(2\pi)^3\delta^3(\vec{k}_B-\tfrac{\vec{Q}}{2}+\vec{k})\delta_{B,2}(2\pi)^3\delta^3(\vec{k}_A-\tfrac{\vec{Q}}{2}-\vec{k})\delta_{A,1}\times$$
$$(2\pi)^3\delta^3(\tfrac{\vec{Q}}{2}-\vec{q}-\vec{k}_D)\delta_{\sigma\tau',D}(2\pi)^3\delta^3(\tfrac{\vec{Q}}{2}+\vec{q}-\vec{k}_C)\delta_{\sigma\tau,C}$$

$$\langle\Psi_d^\lambda|\hat{\Lambda}_\lambda(\vec{q},\vec{Q})|\Psi_d^\lambda\rangle^{(2)} = \frac{1}{16}\sum_{ABCD}\sum_{\sigma\sigma'\tau\tau'}\sum_{12}\int\frac{d^3k_A}{(2\pi)^3}\frac{d^3k_B}{(2\pi)^3}\frac{d^3k_C}{(2\pi)^3}\frac{d^3k_D}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\times$$

$$(2\pi)^3\delta^3(\vec{k}_A+\vec{k}_B)\psi_d^*(\tfrac{\vec{k}_A-\vec{k}_B}{2};AB)\langle\vec{k}12|\delta\hat{U}|\vec{q}\sigma\tau\sigma'\tau'\rangle\Big[4(2\pi)^3\delta^3(\vec{k}_B-\tfrac{\vec{Q}}{2}+\vec{k})\times$$

$$\delta_{B,2}(2\pi)^3\delta^3(\vec{k}_A-\tfrac{\vec{Q}}{2}-\vec{k})\delta_{A,1}(2\pi)^3\delta^3(\tfrac{\vec{Q}}{2}-\vec{q}-\vec{k}_D)\delta_{\sigma\tau',D}(2\pi)^3\delta^3(\tfrac{\vec{Q}}{2}+\vec{q}-\vec{k}_C)\times$$

$$\delta_{\sigma\tau,C}\Big](2\pi)^3\delta^3(\vec{k}_C+\vec{k}_D)\psi_d(\tfrac{\vec{k}_C-\vec{k}_D}{2};CD)$$

$$= \frac{1}{4}(2\pi)^3\delta^3(\vec{Q})(2\pi)^3\delta^3(\vec{Q})\sum_{\sigma\sigma'\tau\tau'}\sum_{12}\int\frac{d^3k}{(2\pi)^3}\Psi_d^*(\vec{k};12)\times$$

$$\langle\vec{k}\,12|\delta\tilde{U}|\vec{q}\,\sigma\tau\sigma'\tau'\rangle\,\Psi_d(\vec{q};\sigma\tau,\sigma'\tau') \qquad (6)$$

Term 3 $= \frac{1}{4}(2\pi)^3\delta^3(\vec{Q})(2\pi)^3\delta^3(\vec{Q})\sum_{\sigma\sigma'\tau\tau'}\sum_{34}\int\frac{d^3k'}{(2\pi)^3}\times$

$$\Psi_d^*(\vec{q};\sigma\tau,\sigma'\tau')\langle\vec{q}\,\sigma\tau\sigma'\tau'|\delta\tilde{U}^+|\vec{k}'34\rangle\Psi_d(\vec{k}';34)$$

Term 4 $\sim \langle 0|a_{\vec{k}_B,B}\,a_{\vec{k}_A,A}\,a^+_{\frac{\vec{Q}}{2}+\vec{k},1}\,a^+_{\frac{\vec{Q}}{2}-\vec{k},2}\,a_{\frac{\vec{Q}}{2}-\vec{k}',4}\,a_{\frac{\vec{Q}}{2}+\vec{k}',3}\times$

$$a^+_{\vec{k}_C,C}\,a^+_{\vec{k}_D,D}|0\rangle$$

$$= 4(2\pi)^3\delta^3\left(\vec{k}_B-\frac{\vec{Q}}{2}+\vec{k}\right)\delta_{B,2}\,\delta^3\left(\vec{k}_A-\frac{\vec{Q}}{2}-\vec{k}\right)\delta_{A,1}\times$$
$$(2\pi)^3\delta^3\left(\frac{\vec{Q}}{2}-\vec{k}'-\vec{k}_D\right)\delta_{4,D}\,(2\pi)^3\delta^3\left(\frac{\vec{Q}}{2}+\vec{k}'-\vec{k}_C\right)\delta_{3,C}$$

$$\langle\Psi_d^1|\hat{n}_4(\vec{q},\vec{Q})|\Psi_d^1\rangle^{(4)} = \frac{1}{32}\sum_{ABCD}\sum_{\sigma\sigma'\tau\tau'}\sum_{1234}\int\frac{d^3k_A}{(2\pi)^3}\cdots\frac{d^3k_D}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\times$$

$$(2\pi)^3\delta^3(\vec{k}_A+\vec{k}_B)\,\Psi_d^*\left(\frac{\vec{k}_A-\vec{k}_B}{2};AB\right)\langle\vec{k}\,12|\delta\tilde{U}|\vec{q}\,\sigma\tau\sigma'\tau'\rangle\quad\Big[4(2\pi)^3\times$$

$$\delta^3\left(\vec{k}_B-\frac{\vec{Q}}{2}+\vec{k}\right)\delta_{B,2}\,\delta^3\left(\vec{k}_A-\frac{\vec{Q}}{2}-\vec{k}\right)\delta_{A,1}\,(2\pi)^3\delta^3\left(\frac{\vec{Q}}{2}-\vec{k}'-\vec{k}_D\right)\delta_{4,D}(2\pi)^3\times$$

$$\delta^3\left(\frac{\vec{Q}}{2}+\vec{k}'-\vec{k}_C\right)\delta_{3,C}\Big](2\pi)^3\delta^3(\vec{k}_C+\vec{k}_D)\,\Psi_d\left(\frac{\vec{k}_C-\vec{k}_D}{2};CD\right)\times$$

$$\langle\vec{q}\,\sigma\tau\sigma'\tau'|\delta\tilde{U}^+|\vec{k}'34\rangle \qquad (7)$$

$$= \frac{1}{8}(2\pi)^3\delta^3(\vec{G})(2\pi)^3\delta^3(\vec{G})\sum_{\sigma\sigma'\tau\tau'}\sum_{1234}\int\frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\times$$

$$\psi_\lambda^*(\vec{k};12)\langle\vec{k}12|\delta\tilde{U}|\vec{q}\sigma\tau\sigma'\tau'\rangle\langle\vec{q}\sigma\tau\sigma'\tau'|\delta\tilde{U}^\dagger|\vec{k}'34\rangle\psi_\lambda(\vec{k};34) \quad (8)$$

Let's now combine all four terms where we integrate over $\int\frac{d^3G}{(2\pi)^3}$ and let $(2\pi)^3\delta^3(0)\equiv V$

$$\boxed{\begin{aligned}\langle\Phi_\lambda^\lambda|\hat{\Lambda}_\lambda(\vec{q})|\Phi_\lambda^\lambda\rangle = {} & \frac{V}{2}\sum_{\sigma\sigma'\tau\tau'}|\psi_\lambda^\lambda(\vec{q};\sigma\tau,\sigma'\tau')|^2 + \\ & \frac{V}{4}\sum_{\sigma\sigma'\tau\tau'}\sum_{12}\int\frac{d^3k}{(2\pi)^3}\psi_\lambda^{\lambda*}(\vec{k};12)\langle\vec{k}12|\delta\tilde{U}|\vec{q}\sigma\tau\sigma'\tau'\rangle\psi_\lambda^\lambda(\vec{q};\sigma\tau,\sigma\tau') + \\ & \frac{V}{4}\sum_{\sigma\sigma'\tau\tau'}\sum_{34}\int\frac{d^3k'}{(2\pi)^3}\psi_\lambda^{\lambda*}(\vec{q};\sigma\tau,\sigma'\tau')\langle\vec{q}\sigma\tau,\sigma'\tau'|\delta\tilde{U}^\dagger|\vec{k}'34\rangle\psi_\lambda^\lambda(\vec{k};34) + \\ & \frac{V}{8}\sum_{\sigma\sigma'\tau\tau'}\sum_{1234}\int\frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\psi_\lambda^{\lambda*}(\vec{k};12)\langle\vec{k}12|\delta\tilde{U}|\vec{q}\sigma\tau\sigma'\tau'\rangle\times \\ & \langle\vec{q}\sigma\tau\sigma'\tau'|\delta\tilde{U}|\vec{k}'34\rangle\psi_\lambda^\lambda(\vec{k};34) \qquad (9)\end{aligned}}$$

In partial waves replace $\frac{1}{2}\sum_{\sigma\sigma'\tau\tau'} \to \sum_{^1S_0,\,^3S_1,\,^3D_1,\ldots}$

$\frac{1}{2}\sum_{12} \to \sum_{^1S_0,\,^3S_1,\,^3D_1}$. ψ_λ only has $^3S_1$ and $^3D_1$ components.

We will assume $\psi_\lambda^\lambda(^3D_1)$ is negligible.

Replace $\int\frac{d^3k}{(2\pi)^3} \to \frac{2}{\pi}\int_0^\infty dk\,k^2$ (not sure about factors of 4π)

Then $\langle \Phi_\alpha^\lambda | \hat{n}_\alpha(q) | \Phi_\alpha^\lambda \rangle = V \int |\psi_\alpha^\lambda(q; {}^3S_1)|^2 +$

$$\sum_{x={}^3S_1,{}^3D_1} \frac{2}{\pi}\int_0^\infty dk\, k^2\, \psi_\alpha^{\lambda *}(k;x)\, \delta\tilde{U}_{x-{}^3S_1}(k,q)\, \psi_\alpha^\lambda(q;{}^3S_1) +$$

$$\frac{2}{\pi}\sum_{x={}^3S_1,{}^3D_1} \int_0^\infty dk'\, k'^2\, \psi_\alpha^{\lambda *}(q;{}^3S_1)\, \delta\tilde{U}^\dagger_{{}^3S_1-x}(q,k')\, \psi_\alpha^\lambda(k';x) +$$

$$\sum_{x,y} \left(\frac{2}{\pi}\right)^2 \int_0^\infty dk\, k^2 \int_0^\infty dk'\, k'^2\, \psi_\alpha^{\lambda *}(k;x)\, \delta\tilde{U}_{x-y}(k,q)\, \delta\tilde{U}^\dagger_{y-x}(q,k') \times \psi_\alpha^\lambda(k';x) \qquad (10)$$

Restrict $x = {}^3S_1$

fm^3

$$\langle \Phi_\alpha^\lambda | \hat{n}_\alpha(q) | \Phi_\alpha^\lambda \rangle_{{}^3S_1} = V \int |\psi_\alpha^\lambda(q;{}^3S_1)|^2 + \qquad fm^3$$

$$\frac{2}{\pi}\int_0^\infty \underbrace{dk}_{fm^{-3}}\, \underbrace{k^2}_{fm^2}\, \psi_\alpha^{\lambda *}(k;{}^3S_1)\, \underbrace{\delta\tilde{U}_{{}^3S_1-{}^3S_1}(k,q)}_{fm^3}\, \underbrace{\psi_\alpha^\lambda(q;{}^3S_1)}_{fm^{3/2}} +$$

$$\frac{2}{\pi}\int_0^\infty dk'\, k'^2\, \psi_\alpha^{\lambda *}(q;{}^3S_1)\, \delta\tilde{U}^\dagger_{{}^3S_1-{}^3S_1}(q,k)\, \psi_\alpha^\lambda(k';{}^3S_1) +$$

$$\left(\frac{2}{\pi}\right)^2 \sum_{x={}^3S_1,{}^3D_1} \int_0^\infty \underbrace{dk}_{fm^{-3}}\, \underbrace{k^2}_{fm^{-3}} \int_0^\infty \underbrace{dk'}_{fm^{-3}}\, \underbrace{k'^2}_{fm^{3/2}}\, \psi_\alpha^{\lambda *}(k;{}^3S_1)\, \underbrace{\delta\tilde{U}_{{}^3S_1-x}(k,q)}_{fm^3}\, \underbrace{\delta\tilde{U}^\dagger_{x-{}^3S_1}(q,k')}_{fm^3} \times \underbrace{\psi_\alpha^\lambda(k';{}^3S_1)}_{fm^{3/2}} \qquad (11)$$