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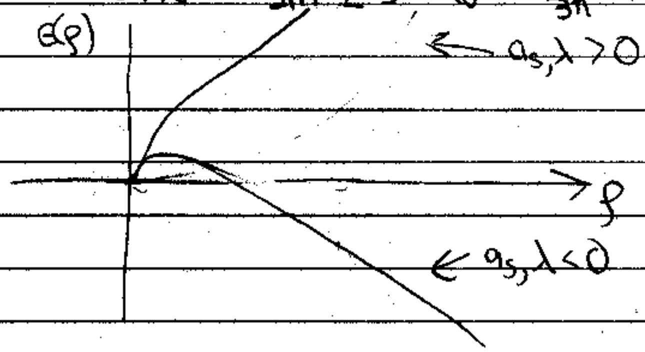
- So how does the dilute Fermi system with δ-function interaction behave? (In 3 and also 1 dimension.)

- For a fixed number of fermions N, it is relevant to consider the energy per particle $\boxed{\epsilon(\rho) \equiv E/N}$ as a function of density $\boxed{\rho = N/V}$.
  - We've switched to V for volume instead of Ω to avoid confusion, below! (Sorry!)
  - $\epsilon(\rho)$ is an intensive quantity: if we double N and V, $\epsilon(\rho)$ stays the same.
  - The other possibility would have been the energy density: $\boxed{\mathcal{E}(\rho) \equiv E/V}$, which is also intensive but not as useful here.
  - Note that $\boxed{\epsilon(\rho) = \mathcal{E}(\rho)/\rho}$

- Plot $\epsilon(\rho)$ vs. ρ. Recall that $\rho \propto k_F^3$

$$E/N = \frac{\hbar^2 k_F^2}{2m}\left[\frac{3}{5} + (g-1)\frac{2k_F a_s}{3\pi} + \ldots\right]$$



- What is the pressure? From thermodynamics,

$$P = -\left(\frac{\partial E}{\partial V}\right)_N = -\left(\frac{\partial (E/N)}{\partial (V/N)}\right)_N = -\frac{\partial \epsilon}{\partial (1/\rho)} = -\frac{\partial \epsilon}{-\frac{1}{\rho^2}\partial\rho}$$

or $\boxed{P = \rho^2 \frac{\partial \epsilon(\rho)}{\partial \rho}}$ so positive or negative slope corresponds to the sign of the pressure.

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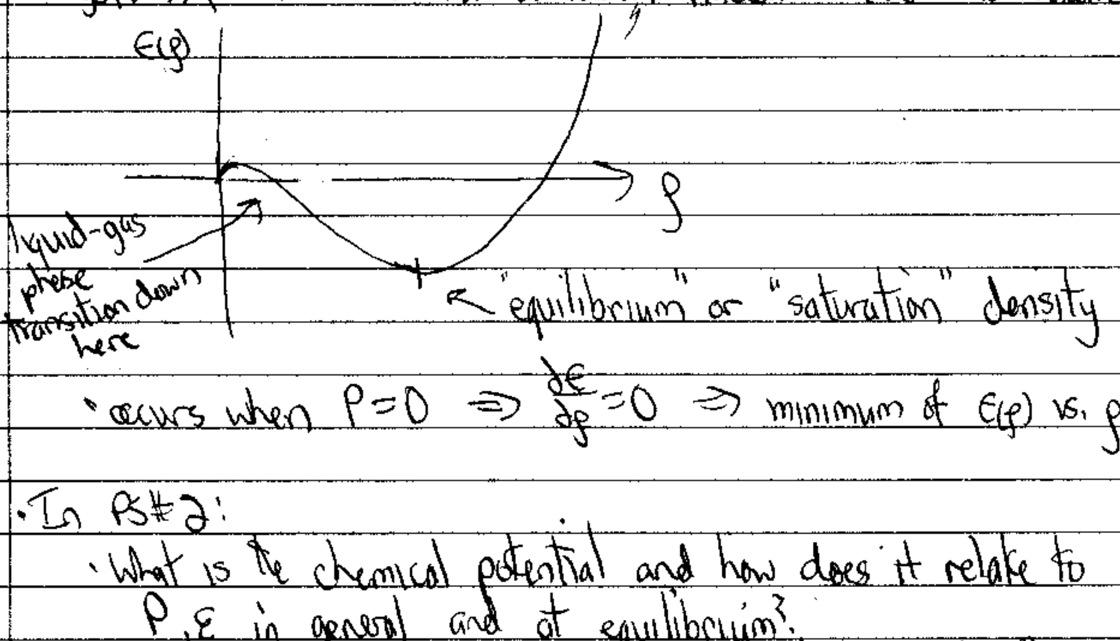
- So the pressure is always positive for the repulsive ($\lambda > 0$) case. What if $\lambda = 0$? Why is there pressure in that case? $P > 0 \Rightarrow$ system expands if no external pressure.
  - You could keep such a system stable by confining it in a trap (magnetic or optical) as done with cold atoms, or use gravity, as in neutron stars (neutron matter). $\Rightarrow$ balance pressure.

- For the attractive case, the pressure is negative except at small densities $\Rightarrow$ collapses!
  - You'll prove this is a general result (PS#1)

- What is the pressure for a nucleus (or any other self-bound system)? $P = 0$. For uniform nuclear matter (no Coulomb):



  - occurs when $P = 0 \Rightarrow \frac{\partial \mathcal{E}}{\partial \rho} = 0 \Rightarrow$ minimum of $\mathcal{E}(\rho)$ vs. ρ

- In PS#2:
  - What is the chemical potential and how does it relate to $P, \mathcal{E}$ in general and at equilibrium?
  - How do we test for stability against long-wavelength fluctuations. (eg. Can we lower the energy for a given N and V with a non-uniform system?)

- Aside: If $g=1$, the interaction term vanishes. Why?
  - For atoms in magnetic traps, $g=1$ because only one hyperfine state trapped. $\Uparrow$ essentially noninteracting

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Review of Thermodynamics/Statistical Mechanics

- To go to even the next order of perturbation theory is awkward and, in general, we will want to develop non-perturbative approximations.
  $\Rightarrow$ need a new framework.

- We will use path integral methods.
  - Later we'll show the connection to operator-based approaches.
  - The most convenient function to calculate with many-body path integrals is the grand canonical partition function.
    $\Rightarrow$ review some thermo and stat. mech.

- NOTE: Volume is V in this section, not Ω.

Recall: three ensembles

i) microcanonical: fixed E and N

ii) canonical: system exchanges energy with heat bath
$\Rightarrow$ fixed N and fixed average E

Probability of system alone with E is $\boxed{e^{-E/k_BT} = e^{-\beta E}}$ $\beta \equiv \frac{1}{k_BT}$

iii) grand canonical: system exchanges energy and particles with heat/particle bath. Neither N nor E fixed.
Probability of system alone having E and N is

$$\boxed{e^{-(E-\mu N)/k_BT} = e^{-\beta(E-\mu N)}}$$

- Suppose we have the complete set of energy and particle number eigenstates: $\hat{H}|N j_N\rangle = E_{j_N}|N j_N\rangle$; $\hat{N}|N j_N\rangle = N|N j_N\rangle$
Then the grand partition function

$$\boxed{Z_G \equiv \sum_N \sum_{j_N} e^{-\beta(E_{j_N}-\mu N)}} \qquad \beta \equiv \frac{1}{k_BT}$$

is related to the thermodynamic potential $\Omega(T,V,\mu)$ by

$$\boxed{\Omega(T,V,\mu) = -\frac{1}{\beta}\ln Z_G}$$

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- Let's recall the laundry list of thermodynamic functions
  - intensive variables: $T \quad P \quad \mu$
  - extensive variables: $S \quad V \quad N$

energy $E = E(S,V,N)$

Helmholtz free energy $F = F(T,V,N)$ · minimized for a mechanically isolated system at constant T

Gibbs free energy $G = G(T,P,N)$ · equilibrium at constant T and P when minimized

* Thermodynamic potential $\Omega(T,V,\mu)$

- First law of thermodynamics specifies how the internal energy changes with changes in independent variables:

$$\boxed{dE = T\,dS - P\,dV + \mu\,dN}$$

which implies
$$\boxed{T = \left(\frac{\partial E}{\partial S}\right)_{V,N} \quad P = -\left(\frac{\partial E}{\partial V}\right)_{S,N} \quad \mu = \left(\frac{\partial E}{\partial N}\right)_{S,V}}$$

- At $T=0$, in the ground state, $S=0 \Rightarrow \boxed{\mu = \left(\frac{\partial E}{\partial N}\right)_V}$

- To obtain the other thermodynamic functions from E, we perform Legendre transformations
  $\Rightarrow$ change the independent variables.
  - it's important to understand these, because we'll be doing the functional generalization later

$$\boxed{F = E - \left(\frac{\partial E}{\partial S}\right)_{V,N} S = E - TS \Rightarrow dF = -S\,dT - P\,dV + \mu\,dN}$$

$$\boxed{G = F + PV = E - TS + PV \Rightarrow dG = -S\,dT + V\,dP + \mu\,dN}$$

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Finally, $\boxed{\Omega = F - \mu N = E - TS - \mu N}$

$$\Rightarrow \boxed{d\Omega = -S\,dT - P\,dV - N\,d\mu}$$

so that $\boxed{S = \left(-\frac{\partial \Omega}{\partial T}\right)_{V\mu}}$ $\boxed{P = -\left(\frac{\partial \Omega}{\partial V}\right)_{T\mu}}$ $\boxed{N = -\left(\frac{\partial \Omega}{\partial \mu}\right)_{TV}}$

- Now consider $P = -\left(\frac{\partial \Omega}{\partial V}\right)_{T\mu}$
  - T and μ are both intensive but V and Ω are extensive. So if we double the size of the system, T and μ are unchanged but Ω and V both double. More generally,
    $$\Omega = (\text{const.})V \quad \text{with (const.) independent of } V.$$
  - Then the equation implies $\boxed{\Omega = -PV}$
  - Since
    $$\Omega = E - TS - \mu N = -PV \Rightarrow \boxed{E = TS - PV + \mu N}$$
    and then
    $$\boxed{F = -PV + \mu N} \quad \text{and} \quad \boxed{G = \mu N}$$

- Another way to see this is to make a scale change $\lambda = 1 + \eta$ with η infinitesimal. All extensive $\rightarrow \lambda *$ extensive.
  - Then $\lambda E = E(\lambda S, \lambda V, \lambda N)$

    or
    $$(1+\eta)E = E + \eta S\left(\frac{\partial E}{\partial S}\right)_{V,N} + \eta V\left(\frac{\partial E}{\partial V}\right)_{S,N} + \eta N\left(\frac{\partial E}{\partial N}\right)_{S,V}$$

    $$\Rightarrow E = ST + V(-P) + N\mu = TS - PV + \mu N \quad \text{as before}$$

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So we see that we can generate all the equilibrium thermodynamics if we knew $\Omega(T,V,\mu)$.

· Return to the partition function:

$$\boxed{\begin{aligned} Z_G &= \sum_N \sum_{j_N} e^{-\beta(E_{j_N}-\mu N)} \\ &= \sum_N \sum_{j_N} \langle N j_N | e^{-\beta(\hat{H}-\mu\hat{N})} | N j_N \rangle \quad \text{since diagonal in this basis} \\ &= \mathrm{Tr}\left(e^{-\beta(\hat{H}-\mu\hat{N})}\right) \end{aligned}}$$

Here Tr is the trace over any complete basis $\Rightarrow$ much more general than first expression.

$$\Omega(T,V,\mu) = -\frac{1}{\beta}\ln Z_G \Rightarrow \boxed{Z_G = e^{-\beta\Omega(T,V,\mu)} = \mathrm{Tr}\left(e^{-\beta(\hat{H}-\mu\hat{N})}\right)}$$

The ensemble average or expectation value of an operator is

$$\boxed{\langle\hat{O}\rangle = \frac{\mathrm{Tr}\left(\hat{O}e^{-\beta(\hat{H}-\mu\hat{N})}\right)}{\mathrm{Tr}\, e^{-\beta(\hat{H}-\mu\hat{N})}} = \mathrm{Tr}\left(\hat{\rho}_G\hat{O}\right)} \text{ where } \boxed{\hat{\rho}_G = \frac{1}{Z_G}e^{\beta(\Omega-\hat{H}-\mu\hat{N})}}$$

· Consider an example: $\hat{O}=\hat{N}$

$$\boxed{\langle\hat{N}\rangle = \frac{\mathrm{Tr}\left(\hat{N}e^{-\beta(\hat{H}-\mu\hat{N})}\right)}{\mathrm{Tr}\, e^{-\beta(\hat{H}-\mu\hat{N})}} = \frac{\frac{1}{\beta}\frac{\partial}{\partial\mu}\mathrm{Tr}\left(e^{-\beta(\hat{H}-\mu\hat{N})}\right)}{\mathrm{Tr}\, e^{-\beta(\hat{H}-\mu\hat{N})}} = \frac{1}{\beta}\frac{1}{Z_G}\frac{\partial Z_G}{\partial\mu} = \frac{1}{\beta}\frac{\partial}{\partial\mu}\ln Z_G}$$

check: $\ln Z_G = -\beta\Omega \Rightarrow \langle\hat{N}\rangle = \frac{-1}{\beta}\frac{\partial}{\partial\mu}\beta\Omega = -\frac{\partial\Omega}{\partial\mu} = N$ ✓

· This is a prototype of things we will do with path integrals to find expectation values.

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- Let's evaluate the trace in Z_G using the occupation number basis. In general, these are not eigenstates of $\hat{H}$, but let's start with

$$\boxed{\hat{H} \to \hat{H}_0 = \sum_i \epsilon_i a_i^\dagger a_i \qquad \hat{N} = \sum_i a_i^\dagger a_i}$$

where i runs over the single-particle quantum numbers (e.g. $\vec{k}, \alpha$ for fermions in a box).

- We'll see that the trace is easy to evaluate in this case. This, in turn, will lead to a strategy for evaluating the more general case.

- In the occupation number basis, we don't sum over E_{jN} and N, but over $n_1, n_2, \ldots$ for "modes" (single-particle states) $1, 2, 3, \ldots$

$$\boxed{\begin{aligned} Z_G = \mathrm{Tr}\, e^{-\beta(\hat{H}_0 - \mu\hat{N})} &= \sum_{n_1 \cdots n_\infty} \langle n_1 \cdots n_\infty | e^{\beta(\mu\hat{N} - \hat{H}_0)} | n_1 \cdots n_\infty \rangle \\ &= e^{-\beta\Omega_0(T,V,\mu)} \end{aligned}}$$

- Recall that $|n_1 \cdots n_\infty\rangle = |n_1\rangle |n_2\rangle \cdots$
- For bosons, each $n_i = 0, 1, 2, \ldots \infty$ ("i" labels the mode)
  for fermions, each $n_i = 0$ or 1

$$\Rightarrow \boxed{\hat{H}_0 |n_1 \cdots n_\infty\rangle = \sum_i \epsilon_i a_i^\dagger a_i |n_1 \cdots n_\infty\rangle = \sum_i n_i \epsilon_i |n_1 \cdots n_\infty\rangle}$$

$$\boxed{\hat{N} |n_1 \cdots n_\infty\rangle = \sum_i n_i |n_1 \cdots n_\infty\rangle}$$

- Recall that $[\hat{n}_i, \hat{n}_j] = 0$ for all i, j, which means we can exponentiate these results.

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Plugging in,

$$Z_G = \sum_{n_1 \cdots n_\infty} e^{-\beta \sum_i (\epsilon_i - \mu) n_i} = \left(\sum_{n_1=0}^{\infty \text{ or } 1} e^{-\beta(\epsilon_1-\mu)n_1} \right) \left(\sum_{n_2=0}^{\infty \text{ or } 1} e^{-\beta(\epsilon_2-\mu)n_2} \right) \cdots \left(\cdots \right)$$

boson or fermion; only 1 μ!

bosons: $n_i = 0, 1, 2, \ldots$

$$\boxed{\sum_{n_i=0}^{\infty} e^{-\beta(\epsilon_i-\mu)n_i} = 1 + e^{-\beta(\epsilon_i-\mu)} + e^{-2\beta(\epsilon_i-\mu)} + \cdots \quad (\text{bosons}) = \frac{1}{1-e^{-\beta(\epsilon_i-\mu)}} \quad \text{geometric series!}}$$

fermions: $n_i = 0, 1$

$$\boxed{\sum_{n_i=0}^{1} e^{-\beta(\epsilon_i-\mu)n_i} = 1 + e^{-\beta(\epsilon_i-\mu)} \quad (\text{fermions})}$$

$$\boxed{\Rightarrow \Omega_0(T,V,\mu) = -\frac{1}{\beta}\ln Z_G = \begin{cases} +\dfrac{1}{\beta}\displaystyle\sum_{i=1}^{\infty} \ln\left(1-e^{-\beta(\epsilon_i-\mu)}\right) & \text{bosons} \\ -\dfrac{1}{\beta}\displaystyle\sum_{i=1}^{\infty} \ln\left(1+e^{-\beta(\epsilon_i-\mu)}\right) & \text{fermions} \end{cases}}$$

$\Rightarrow$ just two little sign differences!

bosons:

$$\boxed{\langle \hat{N} \rangle = -\frac{\partial}{\partial\mu}\Omega_0 = -\frac{1}{\cancel{\beta}}\sum_{i=1}^{\infty} \frac{1}{1-e^{-\beta(\epsilon_i-\mu)}}\left(-e^{-\beta(\epsilon_i-\mu)}\right)(+\cancel{\beta}) = \sum_{i=1}^{\infty} \frac{1}{e^{\beta(\epsilon_i-\mu)}-1} \equiv \sum_{i=1}^{\infty} n_i^0 \quad (\text{bosons})}$$

where n_i^0 is the mean occupation number in the i^{th} state.
Can μ take on any value?

fermions:

$$\boxed{\langle \hat{N} \rangle = -\frac{\partial}{\partial\mu}\Omega_0 = -\left(-\frac{1}{\beta}\sum_{i=1}^{\infty} \frac{1}{1+e^{-\beta(\epsilon_i-\mu)}} e^{-\beta(\epsilon_i-\mu)} \beta\right) = +\sum_{i=1}^{\infty} \frac{1}{e^{\beta(\epsilon_i-\mu)}+1} \equiv \sum_{i=1}^{\infty} n_i^0 \quad (\text{fermions})}$$

$$\Rightarrow n_i^0 = \left(e^{\beta(\epsilon_i-\mu)}+1\right)^{-1} \quad \text{for fermions.}$$