PHYSICS 880.05 PROBLEM SET #3 SOLUTIONS

1. "Directly Solving for $\mathcal{G}^0$"

Goal: solve
$$\boxed{\left(\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu\right)\mathcal{G}^0(\vec{x}\tau;\vec{x}'\tau') = \delta^3(\vec{x}-\vec{x}')\delta(\tau-\tau')} \quad (1)$$

subject to the finite temperature fermion boundary conditions:
$$\boxed{\mathcal{G}^0(\vec{x}\beta;\vec{x}'\tau') = -\mathcal{G}^0(\vec{x}0;\vec{x}'\tau')} \quad (2)$$

a) We first introduce the Fourier transforms to (three-)momentum space, which will enable us to change the spatial part from a differential equation in $\vec{x},\vec{x}'$ to an algebraic equation in $\vec{k}$ (leaving just a first-order differential equation in τ).

$$\boxed{\mathcal{G}^0(\vec{k};\tau-\tau') = \int d^3x\, e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}\mathcal{G}^0(\vec{x}\tau;\vec{x}'\tau')} \quad (3)$$

$$\boxed{\mathcal{G}^0(\vec{x}\tau;\vec{x}'\tau') = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\mathcal{G}^0(\vec{k},\tau-\tau')} \quad (4)$$

Since (4) isolates the $\vec{x}$ and $\vec{x}'$ dependence, we can directly apply $\left(\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu\right)$ to both sides and interchange with $\int\frac{d^3k}{(2\pi)^3}$:

$$\boxed{\left(\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu\right)\mathcal{G}^0(\vec{x}\tau;\vec{x}'\tau') = \int\frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\left(\frac{\partial}{\partial\tau} + \frac{k^2}{2m} - \mu\right)\mathcal{G}^0(\vec{k},\tau-\tau')} \quad (5)$$

but $\boxed{\delta^3(\vec{x}-\vec{x}') = \int\frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}}$ (6) so we can equate coefficients of each $\vec{k}$ (or else project in out):

$$\Rightarrow \boxed{\left(\frac{\partial}{\partial\tau} + \frac{k^2}{2m} - \mu\right)\mathcal{G}^0(\vec{k},\tau-\tau') = \delta(\tau-\tau')} \quad (7)$$

- There are many ways to have known that $\mathcal{G}^0$ is a function of $(\vec{x}-\vec{x}')$ and $(\tau-\tau')$: by physics, since it applies for a translationally invariant system (in space and time) so there can be no preferred $\vec{x}$ or τ; by Fourier transforming $\mathcal{G}^0$ with respect to $\vec{x}\to\vec{k}$ and $\vec{x}'\to\vec{k}'$ and observing that we get $\delta^3(\vec{k}-\vec{k}')$, which manifests $\vec{x}-\vec{x}'$; by deriving the Lehmann or spectral representation, as in class; by assumption, as we did above.

b) In the two regions, $\tau-\tau'>0$ and $\tau-\tau'<0$, $\delta(\tau-\tau')=0$, so we simply have

$$\boxed{\left(\frac{\partial}{\partial\tau}+\epsilon_k^0-\mu\right)\mathcal{G}^0(\vec{k},\tau-\tau')=0} \quad (8)$$

where $\boxed{\epsilon_k^0 \equiv \frac{k^2}{2m}}$ (9) (or $\frac{\hbar^2k^2}{2m}$ if $\hbar\neq 1$). The general solution is

$$\boxed{(\tau-\tau')>0 \quad \mathcal{G}^0(\vec{k},\tau-\tau') = A\,e^{-(\epsilon_k^0-\mu)(\tau-\tau')}} \quad (10)$$

$$\boxed{(\tau-\tau')<0 \quad \mathcal{G}^0(\vec{k},\tau-\tau') = B\,e^{-(\epsilon_k^0-\mu)(\tau-\tau')}} \quad (11)$$

c) Since $\tau=\beta \Rightarrow \tau-\tau'>0$ and $\tau=0 \Rightarrow \tau-\tau'<0$, the boundary condition relates solutions (10) and (11)

$$\Rightarrow \boxed{\mathcal{G}^0(\vec{k},\beta-\tau') = A\,e^{-(\epsilon_k^0-\mu)(\beta-\tau')} = -\mathcal{G}^0(\vec{k},0-\tau') = B\,e^{-(\epsilon_k^0-\mu)(-\tau')}} \quad (12)$$

$$\Rightarrow \boxed{A\,e^{-\beta(\epsilon_k^0-\mu)} = -B} \quad (13)$$

We could just apply (7) to the solutions (10) and (11), including the implied step functions, but instead we'll integrate (7) from $\tau=\tau'-\epsilon$ to $\tau'+\epsilon$:

$$\int_{\tau'-\epsilon}^{\tau'+\epsilon} d\tau\left(\frac{\partial}{\partial\tau}+\frac{k^2}{2m}-\mu\right)\mathcal{G}^0(\vec{k},\tau-\tau') = \int_{\tau'-\epsilon}^{\tau'+\epsilon}\delta(\tau-\tau')\,d\tau = 1$$

$$\Rightarrow \boxed{\mathcal{G}^0(\vec{k},\epsilon)-\mathcal{G}^0(\vec{k},-\epsilon) = 1} \quad (14)$$

(since $\mathcal{G}^0$ has a finite discontinuity so only the $\frac{\partial}{\partial\tau}\mathcal{G}^0$ part contributes.)

From (10) and (11):

$$A\,e^{-(\epsilon_k^0-\mu)\epsilon} - B\,e^{-(\epsilon_k^0-\mu)(-\epsilon)} = 1 \Rightarrow \boxed{A-B=1} \quad (15)$$

Solving, $\boxed{A = \frac{1}{1+e^{-\beta(\epsilon_k^0-\mu)}} = \frac{e^{\beta(\epsilon_k-\mu)}}{e^{\beta(\epsilon_k-\mu)}+1} = 1-n_k^0}$ (16) and $\boxed{B=-n_k^0}$ (17)

$$\Rightarrow \boxed{\mathcal{G}^0(\vec{x}\tau;\vec{x}'\tau') = \int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}e^{-(\epsilon_k^0-\mu)(\tau-\tau')}\left[\theta(\tau-\tau')(1-n_k^0)-\theta(\tau'-\tau)\,n_k^0\right]} \quad (18)$$

as before!

2. The Beachball Diagram

(a)  • On page (125) we derived the expression for the contribution $\mathcal{E}_2$ of the beach ball diagram to the energy density to be:

$$\boxed{\mathcal{E}_2 = -4\lambda^2 M g(g-1) k_F^7 \int\frac{d^3s}{(2\pi)^3}\int\frac{d^3t}{(2\pi)^3}\int\frac{d^3u}{(2\pi)^3}\,\Theta(1-|\vec{s}+\vec{t}|)\,\Theta(1-|\vec{s}-\vec{t}|) \times \Theta(|\vec{s}+\vec{u}|-1)\,\Theta(|\vec{s}-\vec{u}|-1)\,\frac{1}{u^2-t^2-i\eta}} \quad (1)$$

[Note: The expression is not yet regularized and renormalized, but it is this expression we need to compare to perturbation theory.]

- We recall that $k_F\vec{s} = \frac{1}{2}(\vec{k}_1+\vec{k}_2) = \frac{1}{2}(\vec{k}_3+\vec{k}_4)$, $k_F\vec{u} = \frac{1}{2}(\vec{k}_1-\vec{k}_2)$, $k_F\vec{t} = \frac{1}{2}(\vec{k}_3-\vec{k}_4)$ so that $\frac{k_F^2}{2m}(u^2-t^2) = \frac{k_F^2}{m}\left((s^2+u^2)-(s^2+t^2)\right) = \omega_{k_1}+\omega_{k_2}-\omega_{k_3}-\omega_{k_4}$.

- We recall from PS#1, problem (3) that 2nd-order perturbation theory is:

$$\boxed{E^{(2)} = \sum_{j\neq 0}\frac{|\langle 0|H_1|j\rangle|^2}{E_0-E_j}} \quad (2)$$

where $|j\rangle$ are "2p-2h" states: they differ from the ground state by having two particles above k_F $\Rightarrow [\Theta(|\vec{s}+\vec{u}|-1)\times\Theta(|\vec{s}-\vec{u}|-1)]$ and two holes (missing particles) below $k_F \Rightarrow [\Theta(1-|\vec{s}+\vec{t}|)\,\Theta(1-|\vec{s}-\vec{t}|)]$, and the matrix element is just λ^2 (the minus sign is from $E_0<E_j$). The energy denominator is just

$$\boxed{\frac{1}{E_j-E_0} = \frac{1}{\omega_{k_1}+\omega_{k_2}-\omega_{k_3}-\omega_{k_4}}} \quad (3)$$

and the integrals and $4g(g-1)$ factors are the sum over $j\neq 0$.

- The divergence is manifest in the u integration:

$$\boxed{\int\frac{d^3u}{(2\pi)^3}\frac{1}{u^2-t^2-i\eta} \to \int^{\infty}\frac{u^2\,du}{u^2} \to \infty} \quad (4)$$

a linear divergence.

b) In momentum space, on pg 130 we got the expression:

$$\mathcal{E}_2 = -\frac{i\lambda^2}{4} g(g-1) \int \frac{d^4\tilde{k}}{(2\pi)^4} \int \frac{d^4\tilde{p}}{(2\pi)^4} \int \frac{d^4\tilde{q}}{(2\pi)^4} G^0(\tilde{p}) G^0(\tilde{k}) G^0(\tilde{k}+\tilde{q}) G^0(\tilde{p}-\tilde{q})$$

with
$$\boxed{G^0(\tilde{k}) = \frac{\Theta(|\vec{k}|-k_F)}{k_0 - \omega_k^0 + i\varepsilon} + \frac{\Theta(k_F - |\vec{k}|)}{k_0 - \omega_k^0 - i\varepsilon}} \quad (5)$$

- We'll make the same change of variables eventually to $\vec{s}, \vec{t}$, and $\vec{n}$ (see page 126), but first we have to do the frequency integrals over k_0, p_0, q_0.
  - Each variable appears in two G^0's.

- Step through the integrals:

$$\int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \Bigg(\underset{ⓐ}{\frac{\Theta(|\vec{k}|-k_F)}{k_0-\omega_k^0+i\varepsilon}} + \underset{ⓑ}{\frac{\Theta(k_F-|\vec{k}|)}{k_0-\omega_k^0-i\varepsilon}} \Bigg) \Bigg(\underset{ⓒ}{\frac{\Theta(|\vec{k}+\vec{q}|-k_F)}{k_0+q_0-\omega_{\vec{k}+\vec{q}}^0+i\varepsilon}} + \underset{ⓓ}{\frac{\Theta(k_F-|\vec{k}+\vec{q}|)}{k_0+q_0-\omega_{\vec{k}+\vec{q}}^0-i\varepsilon}} \Bigg)$$

We evaluate these as contour integrals. We can close in either half plane, so we get a non zero result only for products of poles in opposite half planes $\Rightarrow$ ⓐ×ⓓ and ⓑ×ⓒ. Close in the upper half plane $\Rightarrow 2\pi i$ for each:

$$2\pi i \frac{1}{2\pi} \Bigg[\underset{ⓐ'}{\frac{\Theta(|\vec{k}|-k_F)\Theta(k_F-|\vec{k}+\vec{q}|)}{-q_0+\omega_{\vec{k}+\vec{q}}^0-\omega_k^0+i\varepsilon}} + \underset{ⓑ'}{\frac{\Theta(k_F-|\vec{k}|)\Theta(|\vec{k}+\vec{q}|-k_F)}{q_0+\omega_k^0-\omega_{\vec{k}+\vec{q}}^0+i\varepsilon}} \Bigg]$$

The p_0 integral is the same with $\vec{k}\to\vec{p}$ and $q_0 \to -q_0$, $\vec{q}\to-\vec{q}$

$$2\pi i \frac{1}{2\pi} \Bigg[\underset{ⓒ'}{\frac{\Theta(|\vec{p}|-k_F)\Theta(k_F-|\vec{p}-\vec{q}|)}{q_0+\omega_{\vec{p}-\vec{q}}^0-\omega_p^0+i\varepsilon}} + \underset{ⓓ'}{\frac{\Theta(k_F-|\vec{p}|)\Theta(|\vec{p}-\vec{q}|-k_F)}{-q_0+\omega_p^0-\omega_{\vec{p}-\vec{q}}^0+i\varepsilon}} \Bigg]$$

Now just the q_0 integral left. Once again, only the terms with poles in opposite half planes survive: ⓐ'ⓒ' and ⓑ'ⓓ'

Keeping the $i^2=-1$ from the other integrals,

$$-2\pi i \frac{1}{2\pi}\left[\frac{\theta(|\vec{k}|-k_f)\,\theta(k_f-|\vec{k}+\vec{q}|)\,\theta(|\vec{p}|-k_f)\,\theta(k_f-|\vec{p}-\vec{q}|)}{\omega^0_{\vec{k}+\vec{q}}-\omega^0_{k}+\omega^0_{\vec{p}-\vec{q}}-\omega^0_{p}+i\varepsilon} + \frac{\theta(k_f-|\vec{k}|)\,\theta(|\vec{k}+\vec{q}|-k_f)\,\theta(k_f-|\vec{p}|)\,\theta(|\vec{p}-\vec{q}|-k_f)}{\omega^0_{p}-\omega^0_{\vec{p}-\vec{q}}+\omega^0_{k}-\omega^0_{\vec{k}+\vec{q}}+i\varepsilon}\right]$$

If we take $\vec{k}\to\vec{k}'=\vec{k}+\vec{q}$, $\vec{p}\to\vec{p}'=\vec{p}-\vec{q}$, $\vec{q}\to\vec{q}'=-\vec{q}$, we see that the two terms are the same
Now let $\vec{p}=k_f(\vec{s}+\vec{t})$, $\vec{k}=k_f(\vec{s}-\vec{t})$, $\vec{q}=k_f(\vec{t}-\vec{u})$ $\overset{\Downarrow}{\text{factor of } 2}$

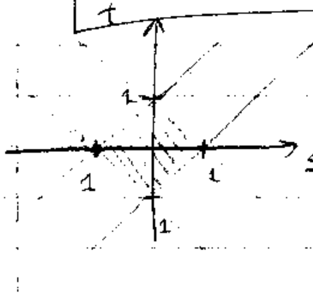
$$\boxed{\begin{aligned}\Rightarrow \omega^0_p+\omega^0_k-\omega^0_{\vec{p}-\vec{q}}-\omega^0_{\vec{k}+\vec{q}}+i\varepsilon &= \frac{k_f^2}{2m}\left(s^2+2\vec{s}\cdot\vec{t}+t^2+s^2-2\vec{s}\cdot\vec{t}+t^2-s^2-2\vec{s}\cdot\vec{u}-u^2-s^2+2\vec{s}\cdot\vec{u}-u^2\right)\\ &= \frac{k_f^2}{m}\left(t^2-u^2\right)-i\varepsilon\end{aligned}} \quad (6$$

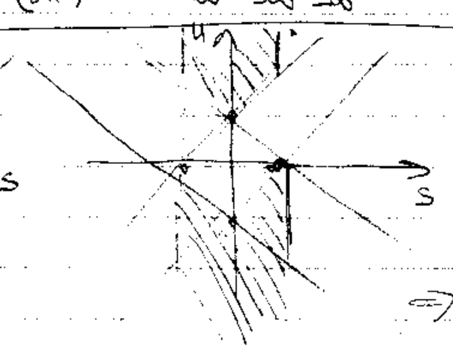
The 2 from the two terms combines with 8 from the Jacobian to obtain $16\times\frac{1}{4}=4$. The m comes from here and $k_f^9/k_f^2=k_f^7$ as expected.

$\Rightarrow$ we reproduce Equation (1).

c) In one dimension, the Feynman rules lead to essentially the same expression as (1), only without vector signs and with $\int\frac{d^3s}{(2\pi)^3}\to\int_{-\infty}^{\infty}\frac{ds}{2\pi}$ and similarly with t and u. $k_f^7\to k_f^6$ and the Jacobian is only 2 instead of 8 $\Rightarrow$ Factor of 4 smaller

$$\boxed{\Rightarrow \mathcal{E}_2 = -\frac{\lambda^2 M g(g-1)k_f}{(2\pi)^3}\int_{-\infty}^{\infty}ds\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}du\,\theta(1-|s+t|)\,\theta(1-|s-t|)\,\theta(|s+u|-1)\,\theta(|s-u|-1)\times\frac{1}{u^2-t^2-i\eta}}$$





The integration regions are shown.
We can restrict s to 0 to 1
t from 0 to $s-1$ and
u from $s+1$ to ∞
$\Rightarrow$ factor of $2\times2\times2=8$

(PS 3-6)

- So we find

$$\boxed{\begin{aligned} \mathcal{E}_2 &= -\frac{1}{\pi^3}\lambda^2 m\, g(g-1)\, k_F \underbrace{\int_0^1 ds \int_{1+s}^{\infty} du \int_0^{1-s} dt\, \frac{1}{u^2-t^2}}_{\pi^2/24 \text{ from Mathematica}} \\ &= -\frac{1}{24\pi}\lambda^2 m\, g(g-1)\, k_F \end{aligned}} \quad (8)$$

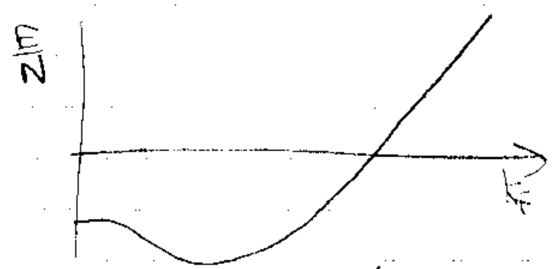
The Mathematica command to do the integral is:

```
Integrate[Integrate[1/(u^2-t^2), {u, 1+s, Infinity}], {t, 0, 1-s}],
          {s, 0, 1}]
```

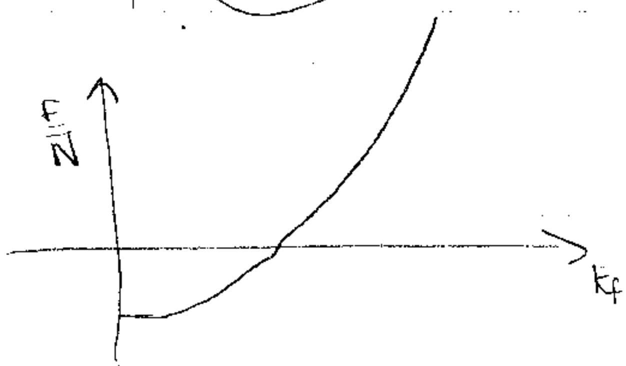
- Recall that $\rho = g k_F/\pi \Rightarrow E_2/N = \mathcal{E}_2/\rho = -\frac{1}{24}\lambda^2 m(g-1)$
- The zeroth and first order results combined with this new result:

$$\boxed{\begin{aligned} \frac{E}{N} &= \frac{1}{3}\frac{k_F^2}{2m} + (g-1)\frac{k_F\lambda}{2\pi} - \frac{1}{24}\lambda^2 m(g-1) + \dots \\ \lambda = \frac{1}{m a_\lambda} \quad &= \frac{k_F^2}{2m}\left(\frac{1}{3} + (g-1)\frac{1}{\pi k_F a_\lambda} - (g-1)\frac{1}{12(k_F a_\lambda)^2}\right) + \dots \end{aligned}} \quad (9)$$

attractive: $a_\lambda < 0$

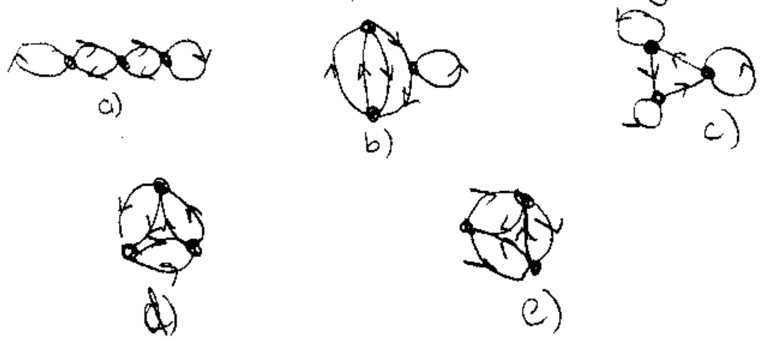


repulsive: $a_\lambda > 0$



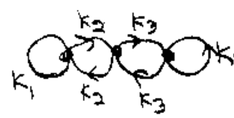
3. Vanishing Diagrams at 3rd Order

a) · The third-order diagrams with only a contact interaction are



· We can use the Feynman rules in either coordinate space or momentum space. We'll use momentum space here.

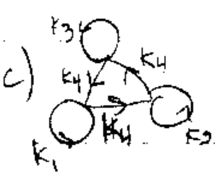
Order: (symmetry factor $*$ (spin sum) $*$ (iG^0 factors) $\times(-i\lambda)^3$ from vertices [spin sums from part b)]

a) 

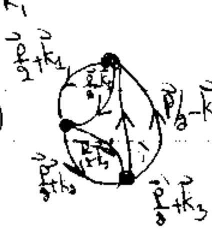
$$\left(\frac{i}{2}\right)\times\left(g(g-1)^3\right)\times\left(\int\frac{d^4k_2}{(2\pi)^4}\left(iG^0(\tilde{k}_2)\right)^2\right)^2\left(\int\frac{d^4k_1}{(2\pi)^4}\left(iG^0(\tilde{k}_1)\right)e^{ik_0\eta}\right)^2(-i\lambda)^3$$

b) 

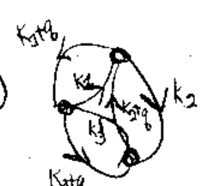
$$\left(\frac{i}{2}\right)*\left(-2g(g-1)^2\right)\times(-i\lambda)^3\left(\int\frac{d^4k_1}{(2\pi)^4}\,iG^0(\tilde{k}_1)e^{ik_0\eta}\right)\left(\int\frac{d^4k_2}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4}\frac{d^4k_4}{(2\pi)^4}\left(iG^0(\tilde{k}_2)\right)^2 iG^0(\tilde{k}_3)\,iG^0(\tilde{k}_4)\times iG^0(\tilde{k}_3+\tilde{k}_2-\tilde{k}_4)\right)$$

c) 

$$\left(\frac{i}{3}\right)*\left(g(g-1)^3\right)\left(\int\frac{d^4k_1}{(2\pi)^4}\,iG^0(\tilde{k}_1)e^{ik_0\eta}\right)^3\left(\int\frac{d^4k_4}{(2\pi)^4}\left(iG^0(\tilde{k}_4)\right)^3\right)(-i\lambda)^3$$

d) 

$$\left(\frac{i}{(2!)^3\,3}\right)*\left(4g(g-1)\right)\int\frac{d^4p}{(2\pi)^4}\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4}\left(iG^0\left(\widetilde{\tfrac{p}{2}+k_1}\right)iG^0\left(\widetilde{\tfrac{p}{2}-k_1}\right)iG^0\left(\widetilde{\tfrac{p}{2}+k_2}\right)\,iG^0\left(\widetilde{\tfrac{p}{2}-k_2}\right)iG^0\left(\widetilde{\tfrac{p}{2}+k_3}\right)iG^0\left(\widetilde{\tfrac{p}{2}-k_3}\right)\right)\times(-i\lambda)^3$$

e) 

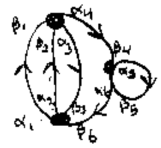
$$\left(\frac{i}{3!}\right)\times\left(-g(g-1)(g-3)\right)\times(-i\lambda)^3\int\frac{d^4q}{(2\pi)^4}\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4}\left(iG^0(\widetilde{k_1+q})\,iG^0(\tilde{k}_1)\,iG^0(\widetilde{k_2+q})\,iG^0(\tilde{k}_2)\,iG^0(\widetilde{k_3+q})\,iG^0(\tilde{k}_3)\right)$$

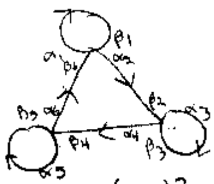
[**Note**: many other assignments of momenta are possible!]

b) Spin sums — see Mathematica notebook spinsums_ps3.nb

a) 

$g(g-1)^3$

b)  $-2g(g-1)^2$

c) 

$g(g-1)^3$

d) 

$4g(g-1)$

e) 

$-g(g-1)(g-3)$

c) We can immediately conclude that diagrams a) and c) are anomalous and therefore vanish at $T=0$ because they have two or more lines with the same $\vec{k}$ $\Rightarrow$ $\Theta(|\vec{k}|-k_F)\ast\Theta(k_F-|\vec{k}|)=0$ factors.

Diagrams d) and e) do not vanish but b) is not so clear.

In fact, diagram b) does vanish. The analysis is essentially the same as for the beachball diagram in momentum space [problem 2.b)], except that now one of the Green's functions (it doesn't matter which) is squared. (There is also an integral for the tadpole, but that factors.)

- The squared term means that the denominators in the final expression (after the 3rd momentum integration) are squared and there is a relative minus sign between the terms
- For the beachball, a change of variables shows the two terms are equal and add, giving a factor of 2.
- For diagram b), they cancel identically $\Rightarrow$ zero!

4. Three-body Forces.

a) The relevant term in the Lagrangian is $-\frac{\beta}{3!}(\psi^\dagger\psi)^3$ (where we're being somewhat schematic in the notation).

The lowest order diagram is 

b) The Feynman rules, generalized appropriately, give us the contribution to the energy density:

- $-i\beta$ for the vertex
- $1/3!$ for 1 equivalent 3-tuple of lines to the same vertex times the overall i factor
- spin sum

$$\left(\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2}\delta_{\alpha_3\beta_3} + \delta_{\alpha_1\beta_2}\delta_{\alpha_2\beta_1}\delta_{\alpha_3\beta_3} + \delta_{\alpha_1\beta_2}\delta_{\alpha_2\beta_3}\delta_{\alpha_3\beta_1} + \delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_3}\delta_{\alpha_3\beta_2} + \delta_{\alpha_1\beta_3}\delta_{\alpha_2\beta_2}\delta_{\alpha_3\beta_1} + \delta_{\alpha_1\beta_3}\delta_{\alpha_2\beta_1}\delta_{\alpha_3\beta_2}\right)$$
$$\times\left(\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2}\delta_{\alpha_3\beta_3}\right)$$

$$=(-g)^3+(-g)^2+(-g)+(-g)^2+(-g)^2+(-g) = -g^3+3g^2-2g$$

$$=-g(g-1)(g-2)$$

- $$\left(\lim_{\eta\to0^+}\int\frac{d^4k}{(2\pi)^4}e^{ik_0\eta}\,iG_0(k)\right)^3 = \left(-\int\frac{d^3k}{(2\pi)^3}\theta(k_F-k)\right)^3 = -\left(\frac{k_F^3}{6\pi^2}\right)^3$$

$$\Rightarrow \boxed{\mathcal{E}_3 = \beta g(g-1)(g-2)\frac{k_F^9}{16\cdot81\cdot\pi^6} = \frac{\beta}{3!}\left(1-\frac{1}{g}\right)\left(1-\frac{2}{g}\right)\rho^3}$$