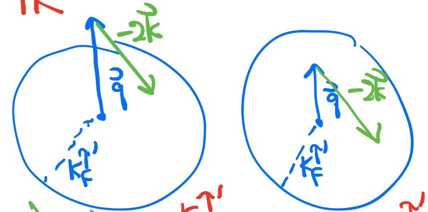
Consider the angle-averaged Θ function that appears in the SU term of the single particle momentum distribution:

$$h(q,k) \equiv \frac{1}{2}\int_{-1}^{1} dx\ \Theta\left(k_F^{\uparrow\prime} - |\vec{q} - 2\vec{k}|\right) \quad \text{with } \Theta(k_F^{\uparrow} - q)$$

x is defined as the cosine of the angle between $\vec{q}$ and $2\vec{k}$

$$\Rightarrow |\vec{q} - 2\vec{k}|^2 = q^2 - 4qkx + 4k^2$$

We'll take $\vec{q}$ to be the polar axis



As in the pictures, given q we need to identify all vectors $-2\vec{k}$ (specified by k and x) such that $\vec{q} + (-2\vec{k})$ lies within the circle of radius $k_F^{\uparrow\prime}$.

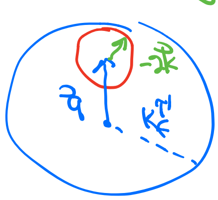
a) $q > k_F^{\uparrow\prime}$ b) $q < k_F^{\uparrow\prime}$

We'll need to consider the cases b) $0 \le q < k_F^{\uparrow\prime}$ and a) $k_F^{\uparrow\prime} < q < k_F^{\uparrow}$ separately because the constraints on allowed k are different (as will become apparent, if not already). Assumes $k_F^{\uparrow} \geq k_F^{\uparrow\prime}$

Let's do b) first: $q < k_F^{\uparrow\prime}$ (if $k_F^{\uparrow} < k_F^{\uparrow\prime}$, this holds automatically)

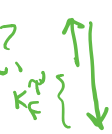
The basic idea is to determine the range of x allowed given q and k.

i) $2k < k_F^{\uparrow\prime} - q$ means $2\vec{k}$ is short enough that it fits inside the circle (which is really the cross section of a sphere!) for any $-1 \le x \le 1$



$$\Rightarrow h(q,k) = \frac{1}{2}\int_{-1}^{1} dx \cdot 1 = 1$$

ii) How large can k be?



$\Rightarrow 2k < k_F^{\uparrow\prime} + q$

So the range of k given q is $k_F^{\uparrow\prime} - q < 2k < k_F^{\uparrow\prime} + q$

For any k in this range, $x=+1$ is ok but there is a lower limit we can solve for from $|\vec{q}-2\vec{k}|^2 \leq k_F^{\pi'2}$

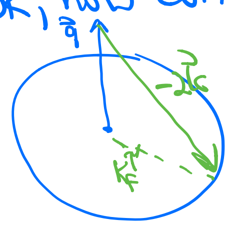
$$\Rightarrow q^2 - 4kqx + 4k^2 < k_F^{\pi'2}$$

$$\text{or} \quad x > \frac{q^2+4k^2-k_F^{\pi'2}}{4kq}$$

$$\Rightarrow h(q,k) = \frac{1}{2}\int_{\frac{q^2+k^2-k_F^{\pi'2}}{4kq}}^{1} dx \cdot 1 = \frac{1}{2}\left(1 - \frac{q^2+k^2-k_F^{\pi'2}}{4kq}\right)$$

$$= \frac{1}{2}\left(\frac{k_F^{\pi'2} - (q-2k)^2}{4kq}\right) > 0 \quad \text{(and less than 1)}$$

Ok, now consider $k_F^{\pi'} < q < k_F^{\pi}$



Here we see that there is a minimum 2k of $q - k_F^{\pi'}$ to get into the circle and then the maximum 2k is $q + k_F^{\pi'}$

$$\Rightarrow q - k_F^{\pi'} < 2k < k_F^{\pi'} + q$$

Once again we have $h(q,k)$ given by the same integral, so

$$h(q,k) = \frac{1}{2}\left(\frac{k_F^{\pi'2} - (q-2k)^2}{4kq}\right)$$

Summary!

1. If $q < k_F^{\pi'}$ and $2k < k_F^{\pi'} - q \Rightarrow h(q,k) = 1$

2. If $q < k_F^{\pi'}$ and $k_F^{\pi'} - q < 2k < k_F^{\pi'} + q$

$$\Rightarrow h(q,k) = \frac{1}{2}\left(\frac{k_F^{\pi'2} - (q-2k)^2}{4kq}\right)$$

3. If $k_F^{\pi'} < q < k_F^{\pi}$ and $q - k_F^{\pi'} < 2k < k_F^{\pi'} + q$

$$\Rightarrow h(q,k) = \frac{1}{2}\left(\frac{k_F^{\pi'2} - (q-2k)^2}{4kq}\right)$$

Lets check some limiting cases:

- $q=0$ then we expect $h \to 1$ for $k < k_F^{\uparrow\downarrow}$ and 0 otherwise.
  $\Rightarrow$ follows from ①.
- $q = k_F^{\uparrow\downarrow} \Rightarrow$ ② says $0 < 2k < 2k_F^{\uparrow\downarrow}$ and $h = \frac{1}{2}\left(\frac{k_F^{\uparrow\downarrow 2} - k_F^{\uparrow\downarrow 2} + 4kk_F^{\uparrow\downarrow} - 4k^2}{4kk_F^{\uparrow\downarrow}}\right)$
  or $k < k_F^{\uparrow\downarrow}$
  $= \frac{1}{2}\left(1 - \frac{k}{k_F}\right)$
  which works for $k=0$ $[0<x<1]$
  and $k=k_F \Rightarrow x=1$ only
- $q \gg k_F^{\uparrow\downarrow} \Rightarrow 2k = q+\varepsilon \Rightarrow h \to 0$ which makes sense.