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METHOD OF STEEPEST DESCENT FOR PATH INTEGRALS

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To estimate a Feynman path integral for a nonrelativistic particle with one degree of freedom in an arbitrary potential $V(x)$, it is proposed to use a functional method of steepest descent, the analog of the method for finite-dimensional integrals, without going over to the Euclidean form of the theory. The concepts of functional Cauchy—Riemann conditions and Cauchy theorem in a complex function space are introduced and used essentially. After the choice in this space of a "contour of steepest descent," the original Feynman integral is reduced to a functional integral of a decreasing exponential. In principle, the obtained result can serve as a basis for constructing the measure of Feynman path integrals.

1. INTRODUCTION

The complexity of the modern formalism of functional integration is to a large degree due to the circumstance that, as a mathematical object, the Feynman path integral is not defined [1—5]. The reason for this state of affairs is the poor behavior of the integrand exponential $\exp\{iS\}$ in the case of integration over an infinite function space. The struggle to overcome the infinities by various regularizations and the associated ambiguities are what lead to the excessive complexity of the formalism [5,6]. The number of regularizations (artificial improvements of the convergence) can be reduced by applying to the functional integral the estimation method known as the "pass" (saddle point, steepest descent, stationary phase) method from the theory of finite-dimensional integrals. It should be noted that this method has apparently already been used to estimate functional integrals. In reality, apart from the use of the terms given above, nothing on the essence of the method could be found in the literature.

The paper is arranged as follows. We first recall the basic idea of the classical methods of Laplace and steepest descent in the simplest form for estimating a simple integral [7]. We then apply this idea to estimate a functional integral.

2. LAPLACE'S METHOD

Laplace's method is used to estimate integrals of the form

$$F(\lambda) = \int_a^b e^{\lambda f(x)} dx \, , \tag{1}$$

where λ is a large real parameter, and the function $f(x)$ on the interval (a, b), which may also be infinite, has one maximum at a point $x_0 \in (a, b)$; then the function $e^{\lambda f(x)}$ has a sharp and high maximum at the point x_0. It is clear that the main contribution to the integral is made by a small neighborhood of the point x_0:

$$\int_a^b e^{\lambda f(x)} dx \approx \int_{x_0-\varepsilon}^{x_0+\varepsilon} e^{\lambda f(x)} dx \approx \int_{x_0-\varepsilon}^{x_0+\varepsilon} e^{\lambda[f(x_0)+\frac{1}{2}f''(x_0)(x-x_0)^2]} dx$$

$$= e^{\lambda f(x_0)} \int_{x_0-\varepsilon}^{x_0+\varepsilon} e^{\frac{\lambda}{2}f''(x_0)(x-x_0)^2} dx \quad ,$$

where $\varepsilon > 0$ is a small number, and small quantities have been omitted in the expansion of $f(x)$ near the point x_0. The remaining integral on the right can be readily calculated if the limits are replaced by infinite limits; this does not introduce a large error on account of the rapid decrease of the integrand with increasing distance from the point x_0. The obtained estimate is fairly accurate if the ends of the interval of integration do not make a large contribution to the integral. If there are several points of maximum of the function $f(x)$ on the interval (a, b), then the estimate of the integral is a sum of the contributions of each point.

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3. METHOD OF STEEPEST DESCENT

This is a somewhat more general method for estimating integrals of the form

$$F(\lambda) = \int_C \exp[\lambda f(z)]dz\,, \tag{2}$$

where $\lambda \to \infty$, $f(z)$ is an analytic function, C is a contour in the z plane, and the contribution of the ends of the contour C to the integral is assumed to be small. According to Laplace's idea, to estimate the integral (2) it is necessary to take into account the contribution of the points of the contour at which the modulus of the integrand, $\exp[\lambda \mathrm{Re}(f(z)]$, is large. By virtue of the analyticity of $f(z)$, we can, in accordance with Cauchy's theorem, deform the contour C without changing the positions of its ends and the values of the integral itself. Let the contour C be deformed into a contour $\tilde{C}$ such that $\tilde{C}$ passes through the stationary point z_0 of the function $f(z)$ (which may also lie far away from the contour C) in the direction of steepest descent of the function $\mathrm{Re}\, f(z)$. Then the main contribution to the integral (2) is determined by a small neighborhood $\Delta\tilde{C}$ of the point z_0 on the contour $\tilde{C}$. Near the point z_0

$$f(z) = u(x,y) + iv(x,y) \approx f(z_0) + \frac{1}{2}f''(z_0)(z-z_0)^2.$$

Let

$$f''(z_0) = \rho e^{i\theta}, \quad z - z_0 = re^{i\phi}.$$

Then

$$u(x,y) = u(x_0,y_0) + \frac{1}{2}\rho r^2\cos(\theta+2\phi), \quad v(x,y) = v(x_0,y_0) + \frac{1}{2}\rho r^2\sin(\theta+2\phi). \tag{3}$$

It can be seen from (3) that the direction of steepest descent ϕ for the function $u(x, y)$ with saddle point (x_0, y_0) is determined by the condition

$$\cos(\theta+2\phi) = -1, \quad \to \quad \phi = \frac{1}{2}(-\theta \pm \pi).$$

Now

$$\int_C e^{\lambda f(z)}dz = \int_{\tilde{C}} e^{\lambda f(z)}dz \approx \int_{\Delta\tilde{C}} \exp\left(\lambda\left[f(z_0) + \frac{1}{2}f''(z_0)(z-z_0)^2\right]\right)dz.$$

Making the substitution $z \to r$ in the integrand and integrating over r in infinite limits, which does not introduce a large error on account of the rapid decrease of the integrand, we obtain

$$F(\lambda) \approx e^{\lambda f(z_0)}e^{i\phi}\sqrt{\frac{2\pi}{\lambda\rho}}\,,$$

which is indeed the leading term of the asymptotic expansion in powers of $1/\lambda$ as $\lambda \to \infty$.

Thus, the integral (3) can be calculated approximately if the function $f(z)$ has stationary points. The contribution of one point can be found by Laplace's method for the contour of steepest descent — this is the essence of the method. It is also important to note that the phase of the exponential in the integrand remains constant during motion along $\tilde{c}$, and therefore there are no oscillations, which would complicate the estimate.

4. FUNCTIONAL METHOD OF STEEPEST DESCENT

The method of estimating the integral (2) presented above can be generalized almost literally to the case of infinite-dimensional (functional) integrals.

The simplest Feynman path integral (transition amplitude) for a particle with one degree of freedom moving in a potential $V(x)$ has the form

$$\langle x_f, t_f | x_i, t_i \rangle = \int Dx(t)\exp[iS(x(t))], \quad S[x(t)] = \int_{t_i}^{t_f}\left[\frac{\dot{x}^2}{2} - V(x(t))\right]dt. \tag{4}$$

We consider the more general expression

$$\langle x_f,\theta_f|x_i,\theta_i\rangle = \int Dx(\theta)\exp[iS(x(\theta))], \quad S(x(\theta)) = \int_C \left[\frac{\dot{x}^2}{2} - V(x(\theta))\right] d\theta, \tag{5}$$

where $\theta = t + i\tau$ is a complex time, $x(\theta)$ is a complex trajectory, C is a contour in the θ plane with ends θ_i, θ_f, and $x(\theta)$ and $V(x)$ are analytic functions. The functional $S(x(\theta))$ takes complex values, and it can be regarded as an analytic function in the sense that

$$\frac{\delta}{\delta u}\mathrm{Re}\,S(x) = \frac{\delta}{\delta v}\mathrm{Im}\,S(x), \quad \frac{\delta}{\delta v}\mathrm{Re}\,S = -\frac{\delta}{\delta u}\mathrm{Im}\,S; \quad x(\theta) = u(t,\tau) + iv(t,\tau). \tag{6}$$

These relations are functional analogs of the Cauchy—Riemann conditions and can be verified directly. Of course, the expressions (6) are a consequence of the analyticity of $x(\theta)$ and $V(x)$. The fulfillment of the relations (6) enables us to say that the functional integral (5) is independent of the "position of the contour of integration" in the function space (functional Cauchy theorem).

Let Γ be the original contour in the integral (5) [in (4), this is the real "axis"]. If in (5) the ends θ_i and θ_f of the contour C lie on the real axis, then the integral (5) is identical to (4). This is obvious from the fact that $S(x(\theta)) = S(x(t))$ by virtue of Cauchy's theorem, and from it

$$\ln Dx(\theta) = \ln\prod_\theta dx(\theta) = \int_C \ln dx(\theta)d\theta = \int_{t_i}^{t_f} \ln dx(t)dt = \ln Dx(t),$$

i.e., $D(x(\theta)) = D(x(t))$.

The integral (5) can be understood as a curvilinear integral in a complex two-dimensional function space. Using now the functional Cauchy theorem, we rewrite (5) in the form

$$\langle x_f,\theta_f|x_i,\theta_i\rangle = \int_\Gamma Dx(\theta)\exp[iS(x(\theta))] = \int_{\tilde{\Gamma}} Dx(\theta)\exp[iS(x(\theta))],$$

where $\tilde{\Gamma}$ is a new functional contour whose "ends" coincide with the "ends" of the contour Γ. It is clear that the largest contribution to the integral is made by the functions $x(\theta)$ for which the modulus of the integrand is large, i.e., $\exp[\mathrm{Re}(iS)]$ or $\mathrm{Re}(iS)$ are large. If it is assumed that the functional $S(x(\theta))$ has one (for simplicity) stationary point $x_0(\theta)$, then the contour $\tilde{\Gamma}$ must pass through the classical solution $x_0(\theta)$ in the direction of steepest descent for $\mathrm{Re}[iS(x)]$ in the function space. Then along $\tilde{\Gamma}$ the functional $\mathrm{Re}(iS)$ will be maximal for $x_0(\theta)$. Further, in accordance with Laplace's idea, integration around $\tilde{\Gamma}$ can be approximately replaced by integration along a small neighborhood $\Delta\tilde{\Gamma}$ of the stationary point $x_0(\theta)$ on the contour $\tilde{\Gamma}$:

$$\int_{\tilde{\Gamma}} Dx(\theta)\exp[iS(x(\theta))] \approx \int_{\Delta\tilde{\Gamma}} Dx(\theta)\exp[iS(x(\theta))]. \tag{7}$$

Thus, the problem reduces to finding the direction of steepest descent of $\mathrm{Re}(iS)$ at the point $x_0(\theta)$ of the function space.

Using the fact that on the right-hand side of (7) the functions $x(\theta)$ are close to $x_0(\theta)$, we expand $S(x(\theta))$, restricting ourselves to quadratic terms:

$$S(x(\theta)) = S(x_0(\theta)) + \delta^2 S, \quad x(\theta) = x_0(\theta) + \delta x(\theta), \quad \delta^2 S = \int_C [\delta\dot{x}^2(\theta) - V''(x_0)\delta x^2]d\theta. \tag{8}$$

Then

$$\langle x_f,\theta_f|x_i,\theta_i\rangle = e^{iS_0}\int_{\Delta\tilde{\Gamma}} D(\delta x(\theta))\exp[i\delta^2 S(\delta x)], \quad S_0 = S(x_0(\theta)). \tag{9}$$

It is in fact often the use of the expansion (8) and the representation of the amplitude in the form (9) that is called the method of steepest descent. However, in reality the essence of the method is associated with the calculation of the preexponential functional integral in (9).

Let $\delta x(\theta) = r(t,\tau)\exp[i\phi(t,\tau)]$, where the function $r(t,\tau)$ is small and is the deviation from the saddle point $x_0(\theta)$ in the function space in the direction $\phi(t,\tau)$. It is required to find the direction of steepest descent $\phi(t,\tau)$ for the functional $\mathrm{Re}(i\delta^2 S)$ for a small displacement from $x_0(\theta)$, i.e., for small $r(t,\tau)$. Obviously, this direction is found from the requirement of a minimum of the functional $\mathrm{Re}(i\delta^2 S(r,\phi))$ with respect to the function $\phi(t,\tau)$:

$$\frac{\delta}{\delta\phi}\mathrm{Re}\,(i\delta^2 S) = 0. \tag{10}$$

We find this minimum. The functional $\mathrm{Re}(i\delta^2 S)$ can be reduced to the form

$$\mathrm{Re}\,(i\delta^2 S) = \int_C ((Adt - Bd\tau)\cos 2\phi - (Bdt + Ad\tau)\sin 2\phi)\,, \tag{11}$$

where

$$A = 2r_t r_\tau + r^2 W(t,\tau), \qquad B = r_t^2 - r_\tau^2 - r^2 U(t,\tau), \qquad V''(x_0) = U(t,\tau) + iW(t,\tau),$$

$$r_t = \frac{\partial r}{\partial t}, \quad r_\tau = \frac{\partial r}{\partial \tau}.$$

We have here taken into account the Cauchy—Riemann conditions for $\delta x(\theta)$: $\phi_t = -r_\tau/r$, $\phi_\tau = r_t/r$, which make it possible to eliminate the derivatives ϕ_t and ϕ_τ from (11). Therefore, the condition (10) takes the trivial form

$$\frac{\delta}{\delta\phi}\mathrm{Re}\,(i\delta^2 S) = [B\tau'(\lambda) - At'(\lambda)]\sin 2\phi - [A\tau'(\lambda) + Bt'(\lambda)]\cos 2\phi = 0. \tag{12}$$

Here λ is the parameter along the contour C. Further, from (12) we find

$$\tan 2\phi = \frac{A\tau' + Bt'}{B\tau' - At'}.$$

There are two solutions of this relation:

$$\cos 2\phi = \frac{B\tau' - At'}{\sqrt{(A^2+B^2)(t'^2+\tau'^2)}}, \qquad \sin 2\phi = \frac{A\tau' + Bt'}{\sqrt{(A^2+B^2)(t'^2+\tau'^2)}}, \tag{13}$$

$$\cos 2\phi' = \frac{At' - B\tau'}{\sqrt{(A^2+B^2)(t'^2+\tau'^2)}}, \qquad \sin 2\phi' = -\frac{A\tau' + Bt'}{\sqrt{(A^2+B^2)(t'^2+\tau'^2)}}. \tag{14}$$

Substituting (13) in the functional $i\delta^2 S$, we find

$$i\delta^2 S = \mathrm{Re}\,(i\delta^2 S) = -\int_{\lambda_i}^{\lambda_f} \sqrt{(A^2+B^2)(t'^2+\tau'^2)}d\lambda, \qquad \mathrm{Im}\,(i\delta^2 S) = 0. \tag{15}$$

Thus, for small displacements from the point $x_0(\theta)$ in the direction $\phi(t, \tau)$ given by (13), the exponential in the integrand of the path integral (7) decreases rapidly. At the same time, the imaginary part of the functional $iS(x(\theta))$ remains constant, and, hence, the oscillations do not spoil the picture. Because of the good decrease of the functional in the integrand, it is possible to integrate with respect to all functions $r(t, \tau)$ without making a restriction to small $r(t, \tau)$.

The solution (14) with the functions $\phi'(t, \tau)$ gives the "direction of steepest ascent."

Further, making the substitution $\delta x(\theta) \to r(t, \tau)$ of the functional variable of integration, and using also the expressions (13), we finally obtain

$$\langle x_f, \theta_f | x_i, \theta_i \rangle = e^{iS_0} \int Dr(t,\tau) \exp\left[-\int_{\lambda_i}^{\lambda_f} \sqrt{A^2+B^2}\left(\sqrt{t'^2+\tau'^2}d\lambda\right)\right] \det\left[\frac{\delta}{\delta r}(re^{i\phi})\right], \tag{16}$$

where

$$e^{i\phi} = \left(\frac{B+iA}{B-iA}\frac{\tau'+it'}{\tau'-it'}\right)^{1/4}.$$

5. DISCUSSION

The expression (16) shows that the poorly defined Feynman path integral can be reduced by means of the method of steepest descent to a functional integral of an exponential that decreases well. Is it not possible in this way to introduce the Feynman transition amplitude as a measure in a function space?

In the case of an arbitrary potential $V(x)$, the right-hand side of (16) is a non-Gaussian integral since the argument of the exponential contains a form of the fourth degree under the root of the function $r(t, \tau)$. Only for a free particle does one obtain a Gaussian integral of the type

$$\langle x_f, \theta_f | x_i, \theta_i \rangle = e^{iS_0} \int Dr \exp\left[-\int_{\lambda_i}^{\lambda_f} (r_t^2 + r_\tau^2)(\sqrt{t'^2 + \tau'^2} d\lambda) \right] \det\left[\frac{\delta}{\delta r}(re^{i\phi}) \right], \tag{17}$$

and this is not even an ordinary Gaussian integral. It is not the presence of the determinant but the fact that the quadratic form in the argument of the exponential is a curvilinear integral. Moreover, in the given case the functional integral in (16) is, so to speak, a rectified curvilinear integral. Thus, the well-known assertion that the expansion of the action near a stationary point to quadratic terms always leads to Gaussian integrals is here incorrect.

It should be noted that in (16) we have obtained a "damping" exponential without any appeal to Euclidean methods, which are so popular today and for which one of the main motives for their introduction is to improve the convergence of functional integrals. In the light of the proposed functional method of steepest descent, Euclidization appears as a special device, and it should be replaced by complexification (or, rather, analytic continuation). For example, in the case of the double-hump potential $V(x) = \lambda(x^2 - \eta^2)^2$ the transition to the Euclidean form of the theory leads to the finding of the classical solution (instanton) [8]:

$$x_0(\tau) = \pm\eta \tanh(\frac{\omega}{2}\tau), \quad \omega^2 = 8\lambda\eta^2.$$

In reality, this is a complex solution $x_0(\theta) = \pm\eta \tan(\omega/2\theta)$, the substitution of which in (16) reproduces the well-known exponential factor of the amplitude $\exp(iS_0) = \exp(\omega^3/12\lambda)$, and it remains to calculate the preexponential, which is a well-defined but non-Gaussian integral.

In the example with the instanton there are no real solutions with real time, and therefore in that case the transition to the Euclidean form is in general justified. But there is an example of a real solution in real time with finite action (the potential is the same). Let the particle be at rest at the point $x=0$ in the state of unstable equilibrium. The solution that describes the particle leaving a hump at the time $t = -\infty$ and returning at the time $t = \infty$ has the form

$$x_0(t) = \pm \frac{\eta\sqrt{2}}{\cosh(\omega t/\sqrt{2})}.$$

To this solution, there corresponds an exponential factor in the amplitude $\exp(i\omega^3\sqrt{2}/12\lambda)$, which in modulus is equal to unity. To find the preexponential factor, it is necessary to calculate a functional integral, and going over to the Euclidean form in this case only makes matters worse, since it introduces unnecessary singularities in $V''(x_0)$. Hence, in this case it is necessary to use the method of steepest descent. Would it be possible to find a (real) solution in the theory of Yang—Mills fields analogous to the one described here? (Just as the BPShT instanton [9] is a quantum-field analog of a nonrelativistic instanton.)

The expression (16) is obtained on the basis of the approximation (7), which is a consequence of the "functional Cauchy theorem." Of course, this theorem is as yet a not too well-founded hypothesis.

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