

AST notes (3/24/21)

①

- * Here we derive the SRG-evolved single-nucleon momentum distribution $\Lambda_{\lambda}^{\tau}(\vec{q})$.

- * Start by defining the operators in second quantization.

$$\Lambda^{\tau}(\vec{q}) = \sum_{\sigma} a_{\vec{q}\sigma\sigma}^+ a_{\vec{q}\sigma\sigma} \quad \begin{matrix} \text{spin projection} \\ \tau \text{ isospin projection} \end{matrix} \quad (1)$$

↑ NOT relative momentum

$$U_{\lambda} = \mathbb{1} + \frac{1}{4} \sum_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} \sum_{\tau_1 \tau_2 \tau_3 \tau_4} \sum_{\vec{k} \vec{k}' \vec{k}''} \langle \vec{k} \sigma_1 \tau_1 \sigma_2 \tau_2 | \delta U | \vec{k}' \sigma_3 \tau_3 \sigma_4 \tau_4 \rangle \times$$

$$a_{\frac{\vec{K}}{2} + \vec{k}, \sigma_1 \tau_1}^+ a_{\frac{\vec{K}}{2} - \vec{k}, \sigma_2 \tau_2}^+ a_{\frac{\vec{K}}{2} - \vec{k}', \sigma_3 \tau_3} a_{\frac{\vec{K}}{2} + \vec{k}', \sigma_4 \tau_4} + \text{3-body} \dots \quad (2)$$

where δU is antisymmetrized ($\delta U_{1234} = -\delta U_{1243}$).

Our task is to evaluate $U_{\lambda} \Lambda^{\tau}(\vec{q}) U_{\lambda}^+$ truncating up through 2-body. We will use simplified notation for momentum, spin, isospin basis.

$$\sum_{\alpha} \equiv \sum_{\sigma} \quad \text{NOT} \quad \sum_{\vec{q} \sigma \tau}$$

$$(\vec{q}, \sigma, \tau) \rightarrow \alpha, (\frac{\vec{K}}{2} + \vec{k}, \sigma_1, \tau_1) \rightarrow 1, \dots$$

$$(\frac{\vec{K}}{2} - \vec{k}', \sigma_4, \tau_4) \rightarrow 4, (\frac{\vec{K}}{2} + \vec{k}'', \sigma_5, \tau_5) \rightarrow 5, \dots$$

$$(\frac{\vec{K}}{2} - \vec{k}''', \sigma_8, \tau_8) \rightarrow 8$$

So we have

(2)

$$\left(1 + \frac{1}{q} \sum_{1234} \langle 12 | \tilde{sU} | 34 \rangle a_1^+ a_2^+ a_4 a_3 \right) \left(\sum_{\alpha} a_{\alpha}^+ a_{\alpha} \right) \times \\ \left(1 + \frac{1}{q} \sum_{5678} \langle 56 | \tilde{sU} | 78 \rangle a_5^+ a_6^+ a_8 a_7 \right)^+$$

$$= \sum_{\alpha} a_{\alpha}^+ a_{\alpha} \quad \text{1 term}$$

$$+ \frac{1}{q} \sum_{\alpha} \sum_{1234} \langle 12 | \tilde{sU} | 34 \rangle a_1^+ a_2^+ a_4 a_3 a_{\alpha}^+ a_{\alpha} \quad] \quad \text{SU term}$$

$$+ \frac{1}{q} \sum_{\alpha} \sum_{5678} \langle 78 | \tilde{sU} | 56 \rangle a_{\alpha}^+ a_{\alpha} a_7^+ a_8^+ a_6 a_5$$

$$+ \frac{1}{16} \sum_{\alpha} \sum_{1234} \sum_{5678} \langle 12 | \tilde{sU} | 34 \rangle \langle 78 | \tilde{sU}^+ | 56 \rangle a_1^+ a_2^+ a_4 a_3 a_{\alpha}^+ a_{\alpha} a_7^+ a_8^+ a_6 a_5$$

L (3)

SUSU⁺ term

- Evaluate contractions with respect to |0> first

1 term : No possible contractions

SU term : $a_1^+ a_2^+ a_4 a_3 a_{\alpha}^+ a_{\alpha}$

$$\rightarrow a_1^+ a_2^+ a_4 \overbrace{a_3 a_{\alpha}^+ a_{\alpha}} + a_1^+ a_2^+ \overbrace{a_4 a_3} a_{\alpha}^+ a_{\alpha}$$

Antisymmetrized so evaluate first one and multiply by

(3)

$$\frac{1}{4} \sum_{\alpha} \sum_{1234} \langle 12 | \delta U | 34 \rangle 2 \delta_{3,\alpha} a_1^+ a_2^+ a_4 a_{\alpha} \quad \rightarrow \quad \vec{k}' + \vec{k}'' = \vec{q}$$

$$= \frac{1}{2} \sum_{\alpha} \sum_{124} \langle 12 | \delta U | \alpha 4 \rangle a_1^+ a_2^+ a_4 a_{\alpha} \quad \vec{k}' \rightarrow \vec{q} - \frac{\vec{k}}{2}$$

δU^+ term gives $\frac{1}{2} \sum_{\alpha} \sum_{568} \langle \alpha 8 | \delta U^+ | 56 \rangle a_{\alpha}^+ a_8^+ a_6 a_5 \quad (4)$

$$\vec{k}' + \vec{k}''' = \vec{q} \quad \vec{k}''' = \vec{q} - \frac{\vec{k}}{2}$$

$\delta U \delta U^+$ term: $a_1^+ a_2^+ a_4 a_3 a_6 a_{\alpha} a_7^+ a_8^+ a_6 a_5$

$$\rightarrow a_1^+ a_2^+ \underbrace{a_4 a_3}_{a_6 a_{\alpha}} \underbrace{a_7^+ a_8^+}_{a_6 a_5} a_6 a_5$$

$$+ a_1^+ a_2^+ \underbrace{a_4 a_3}_{a_6 a_{\alpha}} \underbrace{a_7^+ a_8^+}_{a_6 a_5} a_6 a_5$$

$$+ a_1^+ a_2^+ \underbrace{a_4 a_3}_{a_7 a_{\alpha}} \underbrace{a_7^+ a_8^+}_{a_6 a_5} a_6 a_5$$

$$+ a_1^+ a_2^+ a_4 a_3 a_6 a_{\alpha} a_7^+ a_8^+ a_6 a_5$$

$$\vec{k}_1' + \vec{k}' = \vec{q} \rightarrow \frac{\vec{k}}{2} = \vec{q} - \vec{k}'$$

$$\frac{\vec{k}}{2} + \vec{k}''' = \vec{q} \rightarrow \frac{\vec{k}}{2} = \vec{q} - \vec{k}'''$$

$$\vec{k}_1' - \vec{k}' = \frac{\vec{k}}{2} - \vec{k}'''$$

$$\vec{q} - 2\vec{k}' = \vec{q} - 2\vec{k}''' \rightarrow \vec{k}' = \vec{k}'''$$

$$\Rightarrow \vec{k}' = \vec{k} \quad \vec{k}' = \vec{q} - \frac{\vec{k}}{2}$$

Antisymmetrized \rightarrow Factor of 4 (do last one only)

$$= \frac{1}{16} \sum_{\alpha} \sum_{1234} \sum_{5678} \langle 12 | \delta U | 34 \rangle \langle 78 | \delta U^+ | 56 \rangle 4 \delta_{3,\alpha} \delta_{2,7} \delta_{4,8} a_1^+ a_2^+ a_6 a_5$$

$$= \frac{1}{4} \sum_{\alpha} \sum_{12456} \langle 12 | \delta U | \alpha 4 \rangle \langle \alpha 4 | \delta U^+ | 56 \rangle a_1^+ a_2^+ a_6 a_5 \quad (5)$$

So in total, we have

$$N_x^U(\vec{q}) = \sum_{\alpha} a_{\alpha}^+ a_{\alpha} + \frac{1}{2} \sum_{\alpha} \sum_{124} \langle 12 | \delta U | \alpha 4 \rangle a_1^+ a_2^+ a_4 a_{\alpha} +$$

(4)

$$\frac{1}{2} \sum_{\alpha} \sum_{S68} \langle \alpha_8 | \delta U | 56 \rangle a_{\alpha}^+ a_8^+ a_6 a_5 +$$

$$\frac{1}{4} \sum_{\alpha} \sum_{12456} \langle 12 | \delta U | 14 \rangle \langle 14 | \delta U | 56 \rangle a_1^+ a_2^+ a_6 a_5 \quad (6)$$

Rewrite then take continuum limit

$$= \sum_{\sigma} a_{\vec{\sigma} \tau_2}^+ a_{\vec{\sigma} \tau_2} + \frac{1}{2} \sum_{\sigma} \sum_{\sigma_1 \sigma_2 \sigma_4} \sum_{\tau_1 \tau_2 \tau_4} \int \frac{d^3 k}{(2\pi)^3} \int \frac{d^3 K}{(2\pi)^3} \times$$

$$\langle \vec{k} \sigma_1 \tau_1 \sigma_2 \tau_2 | \delta U | \vec{q} - \vec{k} \sigma_2 \sigma_4 \tau_4 \rangle a_{\vec{k} + \vec{t}, \sigma_1 \tau_1}^+ a_{\vec{k} - \vec{t}, \sigma_2 \tau_2}^+ a_{\vec{k} - \vec{q}, \sigma_4 \tau_4} a_{\vec{q} \sigma_2}$$

$$+ \frac{1}{2} \sum_{\sigma} \sum_{\sigma_3 \sigma_8} \sum_{\tau_5 \tau_6 \tau_8} \int \frac{d^3 k'}{(2\pi)^3} \int \frac{d^3 K'}{(2\pi)^3} \langle \vec{q} - \vec{k}' \sigma_2 \sigma_8 \tau_8 | \delta U | \vec{k}' \sigma_1 \tau_1 \sigma_2 \tau_2 \rangle \times$$

$$a_{\vec{k}' - \vec{q} \sigma_8 \tau_8}^+ a_{\vec{q} \sigma_2}^+ a_{\vec{k}' - \vec{k}' \sigma_6 \tau_6}^+ a_{\vec{k}' + \vec{t}' \sigma_5 \tau_5}^+ + \frac{1}{4} \sum_{\sigma} \sum_{\sigma_1 \sigma_4 \sigma_8} \times$$

$$\sum_{\tau_1 \tau_2 \tau_4 \tau_6} \int \frac{d^3 k}{(2\pi)^3} \int \frac{d^3 k'}{(2\pi)^3} \int \frac{d^3 K}{(2\pi)^3} \langle \vec{k} \sigma_1 \tau_1 \sigma_2 \tau_2 | \delta U | \vec{q} - \vec{k} \sigma_4 \tau_4 \rangle \times$$

$$\langle \vec{q} - \vec{k} \sigma_2 \sigma_4 \tau_4 | \delta U | \vec{k}' \sigma_5 \tau_5 \sigma_6 \tau_6 \rangle a_{\vec{k} + \vec{t}, \sigma_1 \tau_1}^+ a_{\vec{k} - \vec{t}, \sigma_2 \tau_2}^+ \times$$

$$a_{\vec{k}' - \vec{k}', \sigma_6 \tau_6}^+ a_{\vec{k}' + \vec{t}', \sigma_5 \tau_5}^+ \quad (7)$$

Now do contractions with respect to a filled Fermi sea $|F\rangle$ where $\langle F | a_{\beta}^+ a_{\beta} | F \rangle = \delta(k_{\beta}^0 - k_{\beta})$.

(5)

$$1 \text{ term: } \sum_{\sigma} \langle F | a_{q\sigma\sigma}^+ a_{\bar{q}\sigma\sigma}^- | F \rangle \quad (1r)^3 \delta^3(\vec{q} - \vec{q})$$

\downarrow

$$\text{One contraction: } \overbrace{a_{q\sigma\sigma}^+ a_{\bar{q}\sigma\sigma}^-} = V \theta(k_F^z - q)$$

$$\times \sum_{\sigma} \rightarrow 2$$

$$= V 2 \theta(k_F^z - q) \quad (8)$$

$$SU \text{ term: } \frac{1}{2} \sum_{\sigma} \sum_{\sigma_1 \sigma_2 \sigma_3} \sum_{\tau_1 \tau_2 \tau_3} \int \frac{d^3 k}{(2\pi)^3} \int \frac{d^3 K}{(2\pi)^3} \langle \vec{k} \sigma_1 \tau_1 \sigma_2 \tau_2 | S \vec{v} | \vec{q} - \vec{K} \sigma_3 \tau_3 \sigma_4 \tau_4 \rangle \times$$

$$a_{\frac{\vec{K}}{2} + \vec{k}, \sigma_1 \tau_1}^+ a_{\frac{\vec{K}}{2} - \vec{k}, \sigma_2 \tau_2}^+ a_{\vec{K} - \vec{q}, \sigma_3 \tau_3}^- a_{\vec{q} \sigma_4 \tau_4}^-$$

$$\text{Two contractions: } \overbrace{a^+ a^+ a a} \text{ and } \overbrace{a^+ a^+ a a}$$

Do the first and multiply by 2

$$= \sum_{\sigma} \sum_{\sigma_1 \sigma_2 \sigma_3} \sum_{\tau_1 \tau_2 \tau_3} \int \frac{d^3 k}{(2\pi)^3} \int \frac{d^3 K}{(2\pi)^3} \langle \vec{k} \sigma_1 \tau_1 \sigma_2 \tau_2 | S \vec{v} | \vec{q} - \vec{K} \sigma_3 \tau_3 \sigma_4 \tau_4 \rangle \times$$

$$\delta_{\sigma_2, \sigma_4} \delta_{\tau_2, \tau_4} (2\pi)^3 \delta^3(\vec{K} - \vec{k} - \vec{K} + \vec{q}) \delta_{\sigma_1 \sigma} \delta_{\tau_1 \tau} \delta^3(\vec{K} + \vec{k} - \vec{q}) (2\pi)^3$$

$$\langle F | a_{\frac{\vec{K}}{2} + \vec{k}, \sigma_1 \tau_1}^+ a_{\vec{q} \sigma_2}^- a_{\frac{\vec{K}}{2} - \vec{k}, \sigma_3 \tau_3}^+ a_{\vec{K} - \vec{q}, \sigma_4 \tau_4}^- | F \rangle$$

$$V_0 \int \frac{d^3 K}{(2\pi)^3} (2\pi)^3 \delta^3(\vec{K} + \vec{k} - \vec{q}) [\dots]$$

$$\rightarrow \vec{\frac{K}{2}} + \vec{k} - \vec{q} = 0 \Rightarrow \vec{\frac{K}{2}} = \vec{q} - \vec{k}$$

(6)

$$= \sum_{\sigma_2, \sigma_4} \sum_{\tau_2, \tau_4} \int \frac{d^3 k}{(2\pi)^3} \langle \vec{k} \sigma_2 \sigma_2 \tau_2 | \delta \tilde{U} | \vec{k} \sigma_2 \sigma_4 \tau_4 \rangle \times$$

$$\underbrace{\delta_{\sigma_2, \sigma_4} \delta_{\tau_2, \tau_4} (2\pi)^3 \delta^3(\vec{q} - \vec{k} - \vec{q} + \vec{k})}_{= (2\pi)^3 \delta^3(0)} \times$$

$$\equiv V$$

$$\langle F | \hat{n}_{\vec{q}\sigma_2} \hat{a}_{\vec{q}-2\vec{k}\sigma_2\tau_2}^\dagger a_{\vec{q}-2\vec{k}\sigma_4\tau_4} | F \rangle$$

Rerelabel $\sigma_2, \tau_2 \rightarrow \sigma'_1 \tau'_1$ and evaluate

$$\langle F | \hat{n}_{\vec{q}\sigma_2} \hat{n}_{\vec{q}-2\vec{k}\sigma'_1\tau'_1} | F \rangle = \Theta(k_F^2 - q) \Theta(k_F^2 - |\vec{q} - 2\vec{k}|)$$

$$= V \sum_{\sigma_2} \sum_{\tau'_1} \int \frac{d^3 k}{(2\pi)^3} \langle \vec{k} \sigma_2 \sigma'_1 \tau'_1 | \delta \tilde{U} | \vec{k} \sigma_2 \sigma'_1 \tau'_1 \rangle \times$$

$$\Theta(k_F^2 - q) \Theta(k_F^2 - |\vec{q} - 2\vec{k}|)$$

SU^+ term gives identical contribution since

$$\langle \vec{k} | \delta \tilde{U} | \vec{k} \rangle = \langle \vec{k} | \delta U^+ | \vec{k} \rangle. \text{ Factor of 2}$$

$$= V 2 \sum_{\sigma_2} \sum_{\tau'_1} \int \frac{d^3 k}{(2\pi)^3} \langle \vec{k} \sigma_2 \sigma'_1 \tau'_1 | \delta \tilde{U} | \vec{k} \sigma_2 \sigma'_1 \tau'_1 \rangle \times$$

$$\Theta(k_F^2 - q) \Theta(k_F^2 - |\vec{q} - 2\vec{k}|) \quad (9)$$



(7)

$$\delta U \delta U^+ \text{ term} : \frac{1}{4} \sum_{\sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_6} \sum_{\tau_1 \tau_2 \tau_3 \tau_4} \int \frac{d^3 k}{(2\pi)^3} \int \frac{d^3 k'}{(2\pi)^3} \int \frac{d^3 K}{(2\pi)^3} \times$$

$$\left\langle \vec{k} \sigma_1 \tau_1 \sigma_2 \tau_2 | \delta \tilde{U} | \vec{q} - \frac{\vec{K}}{2} \sigma_3 \tau_3 \sigma_4 \tau_4 \right\rangle \left\langle \vec{q} - \frac{\vec{K}}{2} \sigma_3 \sigma_4 \tau_3 \tau_4 | \delta \tilde{U}^+ | \vec{k}' \sigma_5 \tau_5 \sigma_6 \tau_6 \right\rangle \times$$

$$a_{\vec{k} + \vec{L}}^\dagger a_{\sigma_1 \tau_1} a_{\vec{k}' - \vec{L}}^\dagger a_{\sigma_2 \tau_2} a_{\vec{k} - \vec{L}' - \vec{L}}^\dagger a_{\sigma_5 \tau_5} a_{\vec{k}' + \vec{L}' - \vec{L}}^\dagger a_{\sigma_6 \tau_6}$$

Two contractions : $\overbrace{a^\dagger a^\dagger a a}^m$ and $\overbrace{a^\dagger a^\dagger a a}^n$

Do first one and multiply by 2.

$$= \frac{1}{2} \sum_{\sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_6} \sum_{\tau_1 \tau_2 \tau_3 \tau_4} \int \frac{d^3 k}{(2\pi)^3} \int \frac{d^3 k'}{(2\pi)^3} \int \frac{d^3 K}{(2\pi)^3} \left\langle \vec{k} \sigma_1 \tau_1 \sigma_2 \tau_2 | \delta \tilde{U} | \vec{q} - \frac{\vec{K}}{2} \sigma_3 \tau_3 \sigma_4 \tau_4 \right\rangle \times$$

$$\left\langle \vec{q} - \frac{\vec{K}}{2} \sigma_3 \tau_3 \sigma_4 \tau_4 | \delta \tilde{U}^+ | \vec{k}' \sigma_5 \tau_5 \sigma_6 \tau_6 \right\rangle \delta_{\sigma_1 \sigma_5} \delta_{\tau_1 \tau_5} (2\pi)^3 \delta^3(\vec{k}' + \vec{L} - \frac{\vec{K}}{2} - \vec{L}')$$

$$\times \delta_{\sigma_2 \sigma_6} \delta_{\tau_2 \tau_6} (2\pi)^3 \delta^3(\vec{k}' - \vec{L} - \frac{\vec{K}}{2} + \vec{L}') \langle F | a^\dagger a^\dagger a a | F \rangle$$

(Similar to evaluation in δU term)

$$= \sqrt{\frac{1}{2}} \sum_{\sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5 \sigma_6} \sum_{\tau_1 \tau_2 \tau_3 \tau_4 \tau_5 \tau_6} \int \frac{d^3 k}{(2\pi)^3} \int \frac{d^3 K}{(2\pi)^3} \left\langle \vec{k} \sigma_1 \tau_1 \sigma_2 \tau_2 | \delta \tilde{U} | \vec{q} - \frac{\vec{K}}{2} \sigma_3 \tau_3 \sigma_4 \tau_4 \right\rangle$$

$$\times \left\langle \vec{q} - \frac{\vec{K}}{2} \sigma_3 \tau_3 \sigma_4 \tau_4 | \delta \tilde{U}^+ | \vec{k}' \sigma_5 \tau_5 \sigma_6 \tau_6 \right\rangle \Theta(k_f'' - |\vec{k}' + \vec{L}|) \times$$

$$\Theta(k_f'' - |\vec{k}' + \vec{L}|)$$

(10)

(8)

Go to partial waves for each term separately

$2V\Theta(k_F^\tau - q)$ does not change.

δU term (includes $\delta U + \delta U^\dagger$):

$$= V 2 \sum_{\sigma\sigma'} \sum_{\tau\tau'} \int \frac{d^3 k}{(2\pi)^3} \langle \hat{k}\sigma\tau\sigma'\tau' | \delta U | \hat{k}\sigma\tau\sigma'\tau' \rangle \Theta(k_F^\tau - q) \times$$

$$\Theta(k_F^{\tau'} - |\vec{q} - \vec{k}|)$$

Insert complete set of states using

$$\sum_{S=0}^1 \sum_{M_S=-S}^S |SM_S\rangle \langle SM_S| = 1 \quad (9)$$

$$\sum_{T=0}^1 \sum_{M_T=-T}^T |TM_T\rangle \langle TM_T| = 1 \quad (10)$$

$$\sqrt{\frac{2}{\pi}} \sum_{LM_L} |hLM_L\rangle \langle hLM_L| = 1 \quad (11)$$

$$\sum_{J=|L-S|}^{L+S} \sum_{M_J=-J}^J |JM_JLS\rangle \langle JM_JLS| = 1 \quad (12)$$

$$= V 2 \frac{1}{(2\pi)^3} \frac{3}{\pi} \sum_{\sigma\sigma'} \sum_{\tau\tau'} \sum_{SM_S} \sum_{S'M'_S} \sum_{TM_T} \sum_{T'M'_T} \sum_{LM_L} \sum_{L'M'_L} \sum_{JM_J} \sum_{J'M'_J} \int_0^\infty dk k^2 \times$$

$$\int d\Omega_{\vec{k}} \langle \sigma\sigma' | SM_S \rangle \langle \tau\tau' | TM_T \rangle \langle \hat{k} | hLM_L \rangle \langle SM_S LM_L | JM_J LS \rangle \times$$

(9)

$$\langle kJLS\tau | \delta\tilde{U} | k'J'L'S'T' \rangle \langle kL'M_L | \tilde{k} \rangle \langle T'M_T | \tau\tau' \rangle \langle S'M_S | \sigma\sigma' \rangle \Theta's$$

Apply $\sum_{\sigma\sigma'} \langle \sigma\sigma' | S_M \rangle \langle S'M'_S | \sigma\sigma' \rangle = \delta_{SS'} \delta_{MM'_S}$

$$\int dR_E \langle \tilde{k} | kLM_L \rangle \langle k'L'M'_L | \tilde{k} \rangle = \delta_{LL'} \delta_{MM'_L}$$

$$\sum_{M_L M_S} \langle S_M S_L M_L | JM_S \rangle \langle J'M'_J | S'M_S L M_S \rangle = \delta_{JJ'} \delta_{MM'_S}$$

$$\sum_{M_J} [\dots] = (2J+1)$$

$$\delta\tilde{U} \propto \delta_{TT'} \delta_{MM'_T}$$

$$= V 2 \frac{1}{(2\pi)^3} \frac{2}{\pi} \sum_{\tau} \sum_{TM_T} \sum_{S} \sum_{L} \sum_{J} \int_0^{\infty} dk h^2 \langle \tau\tau' | TM_T \rangle \times$$

$$\langle kJLS\tau | \delta\tilde{U} | k'J'L'S'T' \rangle \langle T'M_T | \tau\tau' \rangle \Theta(k_f^2 - q) \Theta(L_f^2 - |\vec{q} - \vec{k}|)$$

(13)

But we still have dependence on $|\vec{q} - \vec{k}| = \sqrt{q^2 + k^2 - 2qkx}$

where $x = \vec{q} \cdot \vec{k}$. Average by evaluating

$$\int_{-1}^1 \frac{dx}{2} [\dots]$$

$$= V 2 \frac{1}{(2\pi)^3} \frac{2}{\pi} \sum_{\tau} \sum_{TM_T} \sum_{S} \sum_{L} \sum_{J} \int_0^{\infty} dk h^2 \int_{-1}^1 \frac{dx}{2} \langle \tau\tau' | TM_T \rangle \times$$

$$\langle kJLS\tau | \delta\tilde{U} | k'J'L'S'T' \rangle \langle T'M_T | \tau\tau' \rangle \Theta(k_f^2 - q) \Theta(L_f^2 - |\vec{q} - \vec{k}|) \quad (14)$$

(10)

$SUSU^\dagger$ term

$$= V \frac{1}{2} \sum_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} \sum_{\sigma_1' \sigma_2' \sigma_3' \sigma_4'} \int \frac{d^3 k}{(2\pi)^3} \int \frac{d^3 K}{(2\pi)^3} \langle \vec{k} | \sigma_1 \sigma_2 \sigma_3 \sigma_4 | \vec{\sigma}'_1 \vec{\sigma}'_2 \vec{\sigma}'_3 \vec{\sigma}'_4 | \vec{k}' | \vec{K} | \vec{\sigma}_1 \vec{\sigma}_2 \vec{\sigma}_3 \vec{\sigma}_4 \rangle$$

$$\times \langle \vec{q} - \frac{\vec{K}}{2} | \sigma_2 \sigma_3 \sigma_4 | \vec{\sigma}'_1 | \vec{\sigma}'_2 \vec{\sigma}'_3 \vec{\sigma}'_4 \rangle \Theta(|\vec{k}'| - |\vec{K} + \vec{k}|) \times$$

$$\Theta(|\vec{k}'| - |\vec{K} - \vec{k}|)$$

Insert complete set of states for $S M_S T M_T$ etc.

$$= V \left(\frac{2}{\pi}\right)^2 \frac{1}{(2\pi)^6} \sum_{S M_S T M_T} \sum_{T M_T T' M_T' T' M_T'} \sum_{S M_S S' M_S' S'' M_S''} \sum_{L M_L L' M_L' L'' M_L''} \sum_{J M_J J' M_J' J'' M_J''} \sum_{J M_J J' M_J' J'' M_J''}$$

$$\int_0^\infty dk k^2 \int_0^\infty dK K^2 \int dR_{k'} dR_K \langle \sigma' \sigma'' | S M_S \rangle \langle \tau' \tau'' | T M_T \rangle \times$$

$$\langle \vec{k} | k L M_L \rangle \langle S M_S L M_L | J M_J L S \rangle \langle k J L S T | \vec{S} \vec{U} | \vec{q} - \frac{\vec{K}}{2} | J' L' S' T' \rangle \times$$

$$\langle J' M_J' L' S' | S' M_S' L' M_S' \rangle \langle \vec{q} - \frac{\vec{K}}{2} | L' M_S' | \vec{q} - \frac{\vec{K}}{2} \rangle \langle T' M_T' | \tau' \tau'' \rangle \times$$

$$\langle S' M_S' | \sigma \sigma'' \rangle \langle \sigma \sigma''' | S'' M_S'' \rangle \langle \tau \tau''' | T'' M_T'' \rangle \langle \vec{q} - \frac{\vec{K}}{2} | L'' M_S'' \rangle \times$$

$$\langle S'' M_S'' L'' M_S'' | S'' M_S'' L'' M_S'' \rangle \langle \vec{q} - \frac{\vec{K}}{2} | J'' L'' S'' T'' | \vec{S} \vec{U}^\dagger | h J'' L'' S'' T'' \rangle \times$$

$$\langle J''' M_S''' L''' S''' | S''' M_S''' L''' M_S''' \rangle \langle k L''' M_S''' | \vec{k} \rangle \langle T''' M_T''' | \tau''' \rangle \langle S''' M_S''' | \vec{\sigma} \vec{\phi}'' \rangle \times$$

$\times G's$

$$\text{Apply } \sum_{\sigma' \sigma''} \langle \sigma' \sigma'' | S M_S \rangle \langle S'' M_S'' | \sigma \sigma'' \rangle = \delta_{S S''} \delta_{M_S M_S''} \quad (11)$$

$$\sum_{\sigma \sigma''} \langle S' M_S' | \sigma \sigma'' \rangle \langle \sigma \sigma'' | S'' M_S'' \rangle = \delta_{S' S''} \delta_{M_S' M_S''}$$

$$\int dR_{\vec{k}} \langle \vec{k} | k L M_L \rangle \langle k L'' M_L'' | \vec{k} \rangle = \delta_{L L''} \delta_{M_L M_L''}$$

$$(\text{aug-}) \quad \int \frac{dR_{\vec{q}-\frac{1}{2}\vec{k}}}{4\pi} \langle \vec{q}-\frac{1}{2}\vec{k} | L' M_L' | \vec{q}-\frac{1}{2}\vec{k} \rangle \langle \vec{q}-\frac{1}{2}\vec{k} | \vec{q}-\frac{1}{2}\vec{k} | L'' M_L'' \rangle \\ = \frac{1}{4\pi} \delta_{L L''} \delta_{M_L' M_L''}$$

$$\sum_{M_S M_S'} \langle S M_S L M_L | T M_J L S \rangle \langle T'' M_J'' L S | S M_S L M_L \rangle = \delta_{T T''} \delta_{M_J M_J''}$$

$$\sum_{M_L' M_S'} \langle J' M_J' L' S' | S' M_S' L' M_L' \rangle \langle S' M_S' L' M_L' | J'' M_J'' L' S' \rangle = \delta_{J J''} \delta_{M_J' M_J''}$$

$$\int dR_{\vec{k}} = 4\pi$$

$\delta V, \delta V^+$ - diagonal in S, M_S, T, M_T, J, M_J

$$\sum_{M_J} [...] = (2J+1) [...]$$

$$D_0 \int_{-1}^1 \frac{dy}{2} [...] \text{ where } |\frac{1}{2}\vec{k} \pm \vec{k}| = \sqrt{\frac{K^2}{4} + k^2 \pm K k_y}$$

$$\boxed{= V \frac{1}{2} \left(\frac{2}{\pi}\right)^2 \frac{1}{(2\pi)^6} \sum_{T T'' L'' L''} \sum_{T M_T T' M'_T} \sum_S \sum_{L L'} \sum_J \int_0^\infty dk k^2 \int_0^\infty dk' k'^2 \times} \\ \boxed{\int_{-1}^1 \frac{dy}{2} \langle T' L'' | T M_T \rangle \langle L J L' S T | \delta O | \vec{q} - \frac{1}{2}\vec{k} | J L' S T \rangle \langle T M_T | T T'' \rangle_x}$$

(12)

$$\langle \tau\tau'''|T'M_T' \rangle \langle \vec{q}-\frac{1}{2}\vec{k} | JL'ST' | S\tilde{U}^+ | kJLST \rangle \langle T'M_T' | \tau\tau'' \rangle \times$$

$$\Theta(k_F'' - |\frac{1}{2}\vec{k} + \vec{\ell}|) \Theta(k_F'' - |\frac{1}{2}\vec{k} - \vec{\ell}|) \quad (15)$$

Overall we have :

$$N_1(\eta) = V \left[2 \Theta(k_F - \eta) \right] \quad \text{1 term}$$

$$+ 2 \frac{1}{(2\pi)^3} \frac{3}{\pi} \sum_s \sum_{T' M_T} \sum_L \sum_J \int_0^\infty dk h^2 \int_{-1}^1 \frac{dx}{2} \langle \tau\tau' | T M_T \rangle \times \quad \text{SU term}$$

$$\langle kJLST | S\tilde{U} | kJLST \rangle \langle T M_T | \tau\tau' \rangle \Theta(k_F - \eta) \Theta(k_F' - |\vec{q} - 2\vec{\ell}|)$$

$$+ \frac{1}{2} \frac{1}{(2\pi)^6} \left(\frac{2}{\pi}\right)^2 \sum_{\tau\tau''\tau'''} \sum_{T' M_T T' M_T'} \sum_s \sum_{LL'} \sum_I \int_0^\infty dk h^2 \int_0^\infty dK K^2 \times$$

$$\int_{-1}^1 \frac{dx}{2} \underbrace{\langle \tau\tau''' | T M_T \rangle \langle kJLST | S\tilde{U} | \vec{q} - \frac{1}{2}\vec{k} | JL'ST \rangle \langle T M_T' | \tau\tau' \rangle}_{(2J+1)} \times$$

$$\langle \tau\tau' | T M_T' \rangle \langle \vec{q} - \frac{1}{2}\vec{k} | JL'ST' | S\tilde{U}^+ | kJLST' \rangle \langle T M_T' | \tau\tau''' \rangle \times$$

$$\Theta(k_F'' - |\frac{1}{2}\vec{k} + \vec{\ell}|) \Theta(k_F'' - |\frac{1}{2}\vec{k} - \vec{\ell}|) \quad (16)$$

$$\text{Where } |\vec{q} - 2\vec{\ell}| = \sqrt{q^2 + 4k - 4q\vec{k} \cdot \vec{x}},$$

SUSU[†]

$$|\vec{q} - \frac{1}{2}\vec{k}| \approx q^2 + \frac{k^2}{4}, \text{ and}$$

(13)

$$\left| \frac{1}{2}\vec{k} \pm \vec{h} \right| = \sqrt{\frac{k^2}{4} + h^2 \pm kh_y}$$

Let's evaluate this up through $L=2$.

Strategy : start with $\tau = +\frac{1}{2}$ then consider cases for τ' .

δV term :

(A) $\tau' = +\frac{1}{2} \quad M_T = 1, T = 1 : \quad CG\xi = 1$

A1) $S = 0, J = L \quad \begin{cases} L \text{ even } ^1S_0, ^1D_2, \dots \\ L \text{ odd } \text{not allowed} \end{cases}$

A2) $S = 1, J = |L-S|, \dots, L+S \quad \begin{cases} L \text{ even } \text{not allowed} \\ L \text{ odd } ^3P_0, ^3P_1, ^3P_2, \dots \end{cases}$

(B) $\tau' = -\frac{1}{2} \quad M_T = 0, T = 0, 1 : \quad CG\xi = \frac{1}{\sqrt{2}}$

A1) $T = 1$

i) $S = 0, J = L \quad \begin{cases} L \text{ even } ^1S_0, ^1D_2, \dots \\ L \text{ odd } \text{not allowed} \end{cases}$

(14)

ii) $S = 1, J = |L-S|, \dots, L+S$ $\left\{ \begin{array}{l} L \text{ even not allowed} \\ L \text{ odd } {}^3P_0, {}^3P_1, {}^3P_2 - {}^3P_1 \end{array} \right.$

AL) $T=0$

i) $S = 0, J = L$ $\left\{ \begin{array}{l} L \text{ even not allowed} \\ L \text{ odd } {}^1P_1, \dots \end{array} \right.$

ii) $S = 1, J = |L-S| \dots L+S$ $\left\{ \begin{array}{l} L \text{ even } {}^3S_1, {}^3P_1, {}^3D_2, {}^3D_3 \\ L \text{ odd not allowed} \dots \end{array} \right.$

$$\left[\left(\underbrace{\delta U_{1S_0} + \delta U_{3P_0} + 3 \delta U_{3P_1} + 5 \delta U_{3P_2-3P_1} + 5 \delta U_{1D_2}}_{\Theta(k_F^\rho - |\vec{q} - \vec{k}|)} \right) + \left(\underbrace{\frac{1}{2} \delta U_{1S_0} + \frac{1}{2} \delta U_{3P_0} + \frac{3}{2} \delta U_{3P_1}}_{\frac{3}{2} \delta U_{3P_1} + \frac{5}{2} \delta U_{3P_2-3P_1} + \frac{5}{2} \delta U_{1D_2}} \right. \right. \\ \left. \left. + \underbrace{\frac{3}{2} \delta U_{3P_1} + \frac{5}{2} \delta U_{3P_2-3P_1} + \frac{5}{2} \delta U_{1D_2} + \frac{3}{2} \delta U_{3S_1-3S_1} + \frac{3}{2} \delta U_{3P_1-3P_1}}_{\frac{5}{2} \delta U_{3P_1} + \frac{7}{2} \delta U_{3D_3-3D_3}} \right) \Theta(k_F^\rho - (\vec{q} - \vec{k})) \right] \times \\ \Theta(k_F^\rho - q) \quad (17)$$

$\rho\rho$ underlined in green

$\rho\eta$ underlined in blue

(15)

 $SU\delta U^\dagger$ term:

$$\textcircled{A} \quad \tau' = +\frac{1}{2} \quad M_T = 1, T = 1, M_T' = 1, T' = 1 \quad CGS = 1$$

$$\tau'' = \tau''' = +\frac{1}{2}$$

$$A1) \quad S = 0, J = L, L' = L$$

$$\begin{cases} L \text{ even } ^1S_0, ^1D_2, \dots \\ L \text{ odd } \text{not allowed} \end{cases}$$

$$A2) \quad S = 1, J = |L-S|, \dots, L+S$$

$$\begin{cases} L \text{ even } \text{not possible} \\ L \text{ odd } ^3P_0, ^3P_1, ^3P_2 - ^3P_2, ^3P_2 - ^3F_2 \end{cases}$$

$$\textcircled{B} \quad \tau' = -\frac{1}{2} \quad M_T = 0 \quad M_T' = 0 \quad \tau'' = \pm \frac{1}{2} \quad \tau''' = \mp \frac{1}{2}$$

$$\Rightarrow (G_p G_n + G_n G_p)$$

$$A1) \quad T = 1, T' = 1 \quad CGS = \frac{1}{\sqrt{2}}$$

$$i) \quad S = 0, J = L \quad \begin{cases} L \text{ even } ^1S_0, ^1D_2 \\ \text{not possible} \end{cases}$$

$$ii) \quad S = 1, J = |L-S|, \dots, L+S$$

$$\begin{cases} L \text{ even } \text{not possible} \\ L \text{ odd } ^3P_0, ^3P_1, ^3P_2 - ^3P_2, ^3P_2 - ^3F_2 \end{cases}$$

(16)

$$A2) \quad T=0, \quad T'=0 \quad GGS = \frac{1}{\sqrt{2}}$$

$$i) \quad S=0, \quad J=L \quad \begin{cases} L \text{ even not poss.} \\ L \text{ odd } lP_1 \end{cases}$$

$$ii) \quad S=1, \quad J=|L-S| \dots L+S \quad 3D_1 - 3D_1$$

$$\begin{cases} L \text{ even } 3S_1 - 3S_1, 3S_1 - 3D_1, 3D_2, 3D_3 - 3D_3, 3D_3 - 3G_3 \\ L \text{ odd not possible} \end{cases}$$

$$\left[\underbrace{(\delta U_{1S_0} \delta U_{1S_0}^+ + \delta U_{3P_0} \delta U_{3P_0}^+ + 3 \delta U_{3P_1} \delta U_{3P_1}^+ + 5 \delta U_{3P_2} \delta U_{3P_2}^+ + \delta U_{3P_2-3P_2}^+ + 5 \delta U_{3P_2-3F_2} \delta U_{3F_2-3P_2}^+ + 5 \delta U_{3D_2} \delta U_{3D_2}^+)}_{\Theta_p^+ \Theta_p^-} + \frac{1}{4} \left(\underbrace{\delta U_{1S_0} \delta U_{1S_0}^+ + 5 \delta U_{1P_1} \delta U_{1P_1}^+ + \delta U_{3P_0} \delta U_{3P_0}^+ + 3 \delta U_{3P_1} \delta U_{3P_1}^+ + 5 \delta U_{3P_2} \delta U_{3P_2}^+ + 5 \delta U_{3P_2-3F_2} \delta U_{3F_2-3P_2}^+ + 3 \delta U_{3P_2-3P_2}^+ + 3 \delta U_{3F_2-3P_2} \delta U_{3F_2-3P_2}^+ + 3 \delta U_{1P_1} \delta U_{1P_1}^+ + 3 \delta U_{3S_1-3S_1} \delta U_{3S_1-3S_1}^+ + 3 \delta U_{3S_1-3P_1} \delta U_{3S_1-3P_1}^+ + 3 \delta U_{3P_1-3D_1} \delta U_{3P_1-3D_1}^+ + 5 \delta U_{3D_2} \delta U_{3D_2}^+ + 7 \delta U_{3P_2-3V_3} \delta U_{3P_2-3V_3}^+ + 7 \delta U_{3V_3-3G_3} \delta U_{3V_3-3G_3}^+)}_{\Theta_p^+ \Theta_n^- + \Theta_n^+ \Theta_p^-} \right) \times \right] \quad (18)$$

$$\text{where } G_x^{\pm} \equiv G(k_F^x - |\vec{k}^{\pm}|) \quad (17)$$

This gives $p\bar{p}$, $p\bar{n}$, and $n\bar{p}$ contributions to

$$n_x^p(q) \text{ from } S U S U^+ \text{ term.}$$

In LDA, we simply average over $k_F^p(r)$ and $k_F^n(r)$ by evaluating

$$\langle n_x^p(q) \rangle_A = 4\pi \int_0^\infty dr r^2 n_x^p(q; k_F^p(r), k_F^n(r))$$

$$\text{where } k_F^N(r) = \left[3\pi^2 \rho_A^N(r) \right]^{1/3}$$

(swap $p \rightarrow n$, $n \rightarrow p$ for neutron distribution)