

THE LANDAU THEORY OF FERMIL LIQUIDS

The general methods presented in previous chapters to calculate observables in Fermi or Bose systems have been based on perturbation theory. In the case of strongly interacting systems, even when physically motivated resummations have been made, it is difficult to assess the reliability of the resulting approximations. Hence, in this chapter we will present a completely different approach developed by Landau (1956, 1957, 1958) for Fermi liquids which provides exact relations between certain observable quantities.

The domain of validity of the Landau theory is restricted to phenomena which involve excitations very close to the Fermi surface. In this domain, the fundamental degrees of freedom of the system are quasiparticles which interact with a quasiparticle interaction $f(k, k')$. Although the theory does not specify any properties of the strongly interacting ground state, it describes small departures from the ground state and response functions. It thus allows one to express a number of observables such as the specific heat, magnetic susceptibility, sound velocity, zero sound velocity, and transport coefficients in terms of the interaction $f(k, k')$. This function can either be parameterized phenomenologically by fitting several parameters to experiment, or else it can be calculated microscopically, in which case the theory becomes exact.

The chapter is organized as follows. In the first section we present the basic assumptions of the theory in a heuristic and phenomenological manner and in the next section we show how to calculate thermodynamical observables with this set of assumptions. We emphasize that the arguments in these sections will not be rigorously deduced from general principles; rather, at key points assumptions or postulates will simply be introduced following Landau's original development. For those who wish to pursue this approach to Landau theory in further detail, the review of Baym and Pethick (1976) is particularly clear and pedagogical. The last section presents the microscopic justification of the theory. This microscopic understanding both provides a clear foundation for the postulates of the earlier sections and establishes the relation of the phenomenological interaction $f(k, k')$ to the two-body vertex function.

6.1 QUASIPARTICLES AND THEIR INTERACTIONS

Let us consider a uniform gas of spin- $\frac{1}{2}$ Fermions, containing N particles in a volume \mathcal{V} . If the Fermions are not interacting, the ground state of the system consists of a Fermi sea of plane waves of momenta $|\vec{k}| < k_F$. The Fermi momentum k_F is related to the density by:

$$\rho = \frac{k_F^3}{3\pi^2} \quad (6.1)$$

The total energy of the system is given by

$$E = \sum_{\vec{k}} \frac{\vec{k}^2}{2m} \cdot n(\vec{k}) \quad (6.2)$$

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where $n(\vec{k})$ is the occupation number of the plane wave state $|\vec{k}\rangle$.

$$n(\vec{k}) = 2\theta(k_F - |\vec{k}|) \quad (6.3)$$

If an arbitrary weak external field is coupled to the system, the net effect will be a variation of the occupation numbers, and the corresponding variation of the total energy can be written:

$$\delta E = \sum_{\vec{k}} \frac{\vec{k}^2}{2m} \cdot \delta n(\vec{k}) \quad (6.4)$$

Further, the field is very weak, it can only excite states close to the Fermi surface, so that $\delta n(\vec{k})$ will be sharply peaked around k_F .

Let us now adiabatically turn on the interaction between the particles. A normal Fermi liquid is defined as a system in which the non-interacting ground state evolves into the interacting ground state and there is a one-to-one correspondence between the bare particle states of the original system and the dressed or quasiparticle states of the interacting system. A quasiparticle state with $|k_p| > k_F$ is defined as the state obtained from a non-interacting Fermi sea plus a plane wave $|k_p\rangle$ by switching on the interaction, and similarly, a quasihole state is obtained by starting with a Fermi sea with a hole in state $|\vec{k}_h\rangle$.

Note that in order for a system to be a normal Fermi liquid, switching on the interaction must not produce bound states. For example, in a superconductor, the non-interacting ground state does not evolve into the BCS ground state and the formation of Cooper pairs destroys the one-to-one correspondence between non-interacting states and quasiparticle states. Throughout this chapter, we will tacitly assume a normal state, either by virtue of a totally repulsive interaction or by treating temperatures above the superfluid or superconducting transition temperatures. Normal Fermi liquids to which the theory is commonly applied include liquid ^4He , the electron gas in metals, and nuclear matter.

We have seen in Eq. (5.72) that the quasiparticle lifetime is proportional to $(\epsilon - \epsilon_F)^2$, so that even in a normal Fermi system, quasiparticles far away from the Fermi surface are ill-defined, unstable states. If one tried to prepare them by turning on the interaction slowly enough to be adiabatic, they would decay before the process was complete. Hence, the Landau theory is only applicable to low-lying excited states of the system, which are made of superpositions of quasiparticle excitations close to the Fermi surface. In particular, it can describe neither the ground state energy, which would require summation over all occupied states in the sea, nor highly excited states.

Let us temporarily suppress spin and denote the quasiparticle energy by $\epsilon_{\vec{k}}$ and the interaction energy of quasiparticles of momentum \vec{k} and \vec{k}' by $f(\vec{k}, \vec{k}')$. If we apply a weak perturbation to the system which takes it away from its ground state, it induces a change $\delta n(\vec{k})$ in the occupation number of quasiparticle \vec{k} , and Landau postulated that the change in the total energy of the system is given by

$$\delta E = \sum_{\vec{k}} \epsilon_{\vec{k}} \delta n(\vec{k}) + \frac{1}{2\mathcal{V}} \sum_{\vec{k}, \vec{k}'} f(\vec{k}, \vec{k}') \delta n(\vec{k}) \delta n(\vec{k}') \quad (6.5)$$

The energy of quasiparticle \vec{k} , when it is surrounded by other quasiparticles, is given by:

$$\epsilon_{\vec{k}} = \frac{\delta E}{\delta n(\vec{k})} = \epsilon_{\vec{k}}^0 + \frac{1}{V} \sum_{\vec{k}'} f(\vec{k}, \vec{k}') \delta n(\vec{k}') \quad (6.6)$$

The interaction $f(\vec{k}, \vec{k}')$ of quasiparticles \vec{k} and \vec{k}' is given by

$$\frac{1}{V} f(\vec{k}, \vec{k}') = \frac{\delta^2 E}{\delta n(\vec{k}) \delta n(\vec{k}')} \quad (6.7a)$$

and it is thus symmetric in \vec{k} and \vec{k}'

$$f(\vec{k}, \vec{k}') = f(\vec{k}', \vec{k}) \quad (6.7b)$$

Although the Landau theory is much more general than the Hartree-Fock approximation, it is instructive to note that Eq. (6.5) is of the same form as the total Hartree Fock energy where $\epsilon_{\vec{k}}^0$ is the kinetic energy, $\epsilon_{\vec{k}}$ is the self-consistent single-particle energy and $\frac{1}{V} f(\vec{k}, \vec{k}')$ is the antisymmetrized matrix element of the bare two-body interaction.

Since quasiparticles are adiabatically evolved from Fermions, they obey Fermi-Dirac statistics and the distribution functions at finite temperature is given by the usual Fermi distribution

$$n(\vec{k}) = \frac{1}{e^{\beta(\epsilon(\vec{k}) - \mu)} + 1} \quad (6.8)$$

where μ is the chemical potential.

At zero temperature, we recover a step function distribution, and at non-zero temperature, Eq. (6.8) becomes a set of self-consistent equations since $\epsilon(\vec{k})$ depends on $n(\vec{k})$ through (6.6).

We now extend the notation to include spin. Formulas (6.5) to (6.8) are unchanged, except that \vec{k} should be replaced by (\vec{k}, σ) where σ is the spin projection on a quantization axis and the sums over \vec{k} are replaced by $\sum_{\sigma=\pm 1/2} \sum_{\vec{k}}$. In the absence of a magnetic field, by symmetry $\epsilon(\vec{k}, \sigma)$ is independent of σ , so that $\epsilon(\vec{k}, \sigma) = \epsilon(\vec{k})$. Similarly, $f(\vec{k}\sigma, \vec{k}'\sigma')$ can depend only on the product $\sigma \cdot \sigma'$, and may be parametrized as

$$f(\vec{k}\sigma, \vec{k}'\sigma') = f(\vec{k}, \vec{k}') + 4\vec{\sigma} \cdot \vec{\sigma}' \varphi(\vec{k}, \vec{k}') \quad (6.9a)$$

or

$$f(\vec{k}\sigma, \vec{k}'\sigma') = f_o(\vec{k}, \vec{k}') + \delta_{\sigma\sigma'} f_e(\vec{k}, \vec{k}') \quad (6.9b)$$

with the relation

$$\begin{aligned} f_o &= f - \varphi \\ f_e &= 2\varphi \end{aligned} \quad (6.9c)$$

Note that here σ denotes a spin $\frac{1}{2}$ matrix so that a Pauli matrix would be 2σ .

A crucial simplification occurs for the translationally invariant systems. Since Landau theory is restricted to phenomena involving only quasiparticles close to the Fermi surface, $f(k, k')$ only enters observables with the two vectors k and k' on the

Fermi surface. Thus, f will depend only on the angle θ between \vec{k} and \vec{k}' , and it is convenient to expand it on the basis of Legendre polynomials as:

$$\begin{aligned} f(\vec{k}\sigma, \vec{k}'\sigma') \Big|_{|\vec{k}|=|\vec{k}'|=k_f} &= f(\theta, \sigma, \sigma') \\ &= f(\theta) + 4\vec{\sigma} \cdot \vec{\sigma}' \varphi(\theta) \\ &= \sum_{L=0}^{\infty} (f_L + 4\vec{\sigma} \cdot \vec{\sigma}' \varphi_L) P_L(\cos \theta) \end{aligned} \quad (6.10)$$

The orthogonality relation for the Legendre polynomials:

$$\frac{2L+1}{2} \int_{-1}^1 P_L(\cos \theta) P_{L'}(\cos \theta) d(\cos \theta) = \delta_{LL'} \quad (6.11)$$

implies that

$$\left\{ \begin{matrix} f_L \\ \varphi_L \end{matrix} \right\} = \frac{2L+1}{4\pi} \int d\Omega P_L(\cos \theta) \left\{ \begin{matrix} f(\vec{k}, \vec{k}') \\ \phi(\vec{k}, \vec{k}') \end{matrix} \right\} \Big|_{|\vec{k}|=|\vec{k}'|=k_f} \quad (6.12)$$

In practical applications, the Legendre coefficients f_L and ϕ_L decrease sufficiently rapidly with L that truncation after several terms yields an adequate phenomenology. In this case, a small number of empirical coefficients describes a much larger number of experimental observables and the theory has non-trivial physical content and predictive power. For metals with non-spherical Fermi surfaces or finite systems, $f(k, k')$ depends on many more parameters and the theory is correspondingly less powerful.

6.2 OBSERVABLE PROPERTIES OF A NORMAL FERMI LIQUID

We shall now calculate a number of observable properties of a normal Fermi liquid in terms of the effective interaction $f(k, k')$, assuming the validity of Eqs. (6.5–6.6). The microscopic justification of the theory and the relation of $f(k, k')$ to the bare interaction will be deferred to the following section.

EQUILIBRIUM PROPERTIES

A fundamental parameter of the theory is the effective mass. As in Eq. (5.74), m^* is defined in terms of the density of states or group velocity at the Fermi surface

$$\frac{k_F}{m^*} \equiv \frac{d\epsilon_k^0}{dk} \Big|_{k_F} = v_F \quad (6.13a)$$

By the definition of a quasiparticle as the eigenstate obtained from a non-interacting Fermi sea with a particle in state $|k\rangle$, $\epsilon_{k_F}^0$ must be the chemical potential μ and expanding in the neighborhood of the Fermi surface

$$\begin{aligned} \epsilon_k^0 &= \mu + (k - k_F) \frac{\partial \epsilon^0}{\partial k} \Big|_{k_F} \\ &= \mu + (k - k_F) v_F \\ &= \mu + (k - k_F) \frac{k_F}{m^*} \end{aligned} \quad (6.13b)$$

Specific Heat

The specific heat at constant volume is defined

$$c_V = \frac{1}{V} \left. \frac{\partial E}{\partial T} \right|_V. \quad (6.14)$$

A change in temperature induces a change in occupation numbers, so that c_V is given by

$$\begin{aligned} c_V &= \frac{1}{V} \sum_{\vec{k}, \sigma} \frac{\delta E}{\delta n(\vec{k}, \sigma)} \frac{\delta n(\vec{k}, \sigma)}{\delta T} \\ &= \frac{1}{V} \sum_{\vec{k}, \sigma} \frac{\delta n(\vec{k}, \sigma)}{\epsilon_k} \frac{\delta \epsilon_k}{\delta \epsilon_k} \left(-\frac{\epsilon_k - \mu}{T} + \frac{\partial}{\partial T} (\epsilon_k - \mu) \right). \end{aligned} \quad (6.15)$$

At low temperature, the sum $\sum_{\vec{k}} f(\vec{k}, \vec{k}') \delta n(\vec{k}')$ in Eq. (6.6) is of order T^2 so that the leading contribution to c_V may be obtained by replacing ϵ_k by ϵ_k^0 and converting the integral over k to an integral over ϵ_k^0 using Eq. (6.13). The standard method for calculating the resulting integral using the low temperature expansion for the derivative of the Fermi function

$$\frac{\partial}{\partial \epsilon} \left[\frac{1}{e^{\beta(\epsilon - \mu)} + 1} \right] \sim_{\beta \rightarrow \infty} -\delta(\epsilon - \mu) - \frac{\pi^2}{6\beta^2} \frac{\partial^2}{\partial \epsilon^2} \delta(\epsilon - \mu) + \mathcal{O}(\beta^{-4}) \quad (6.16)$$

is reviewed in Problem 6.1. The result to lowest order in T is

$$c_V = \frac{1}{3} m^* k_F k_B^2 T. \quad (6.17)$$

Physically, since the effective mass is proportional to the density of state at the Fermi surface, we expect the specific heat to depend on m^* . The result, Eq. (6.17), in fact shows that it depends on no other parameter of the theory. That is, to leading order in T the specific heat is equal to that of non-interacting Fermions of mass m^* and thus provides a direct experimental measurement of m^* . For liquid ^3He , the effective mass ranges from roughly 3m at zero pressure to over 5m at 27 atm, where the primary experimental uncertainty in measuring the specific heat arises from uncertainties in the temperature scale. These results as well as subsequent experimental Landau parameters for liquid ^3He are tabulated in Table 6.1 at the end of the chapter. So-called "heavy Fermion systems" such as CeCu_2Si_2 , UPt_3 , U_2Zn_{17} , and CeAl_3 have effective masses m^*/m of the order of $10^2 - 10^3$ (see Stewart, 1984).

Effective Mass

Having seen that m^* may be measured experimentally, we now relate it to the quasiparticle interaction $f(k, k')$. It is clear that if one postulates that Eqs. (6.5–6.6) hold in any Galilean frame, the different relative contributions of ϵ_k^0 and $f(k, k')$ in different frames will give constraints relating ϵ_k^0 and $f(k, k')$. Problem 6.2 explicitly uses Galilean invariance to relate m^* to f . Here, we present Landau's original argument which equates the momentum of a unit volume of the liquid to the mass flow.

The momentum per unit volume is given by

$$\vec{P} = \frac{1}{V} \sum_{\vec{k}, \sigma} \vec{k} n(\vec{k}, \sigma). \quad (6.18a)$$

On the other hand, since the velocity of a quasiparticle is $\frac{\partial \epsilon_k}{\partial \vec{k}}$, and the number of quasiparticles equals the number of particles, the total momentum can also be expressed as:

$$\vec{P} = \frac{1}{V} \sum_{\vec{k}, \sigma} m \frac{\partial \epsilon_k}{\partial \vec{k}} n(\vec{k}, \sigma) \quad (6.18b)$$

where m is the bare mass of a particle.

Equating (6.18a) and (6.18b), we have:

$$\sum_{\vec{k}, \sigma} \vec{k} n(\vec{k}) = \sum_{\vec{k}, \sigma} m \frac{\partial \epsilon_k}{\partial \vec{k}} n(\vec{k}, \sigma). \quad (6.19)$$

Since (6.19) is an identity, we may take a functional derivative with respect to $n(\vec{k}, \sigma)$

$$\vec{k} = m \frac{\partial \epsilon_k}{\partial \vec{k}} + m \sum_{\sigma'} \int \frac{d^3 k'}{(2\pi)^3} \frac{\partial}{\partial \vec{k}'} \left(\frac{\delta \epsilon_{k'}}{\delta n(\vec{k}, \sigma)} \right) n(\vec{k}', \sigma') \quad (6.20a)$$

which we rewrite:

$$\begin{aligned} \frac{\vec{k}}{m} &= \frac{\partial \epsilon_k}{\partial \vec{k}} + \sum_{\sigma'} \int \frac{d^3 k'}{(2\pi)^3} \frac{\partial f(\vec{k}\sigma, \vec{k}\sigma')}{\partial \vec{k}'} n(\vec{k}', \sigma') \\ &= \frac{\partial \epsilon_k}{\partial \vec{k}} - \sum_{\sigma'} \int \frac{d^3 k'}{(2\pi)^3} f(\vec{k}\sigma, \vec{k}\sigma') \frac{\partial n(\vec{k}', \sigma')}{\partial \vec{k}'} \end{aligned} \quad (6.20b)$$

In (6.20), we have used the usual replacement of the sum $\sum_{\vec{k}}$ by an integral $V \int \frac{d^3 k}{(2\pi)^3}$. At zero temperature

$$\frac{\partial n(\vec{k}, \sigma)}{\partial \vec{k}} = -\hat{k} \delta(k_F - k) \quad (6.21)$$

where \hat{k} is the unit vector along \vec{k} . Thus, using (6.13) and (6.20) and taking \vec{k} on the Fermi surface, we obtain:

$$\frac{\vec{k}}{m} = \frac{\vec{k}}{m^*} + \sum_{\sigma'} \int \frac{d^3 k'}{(2\pi)^3} f(\vec{k}\sigma, \vec{k}\sigma') \hat{k}' \delta(k_F - k'). \quad (6.22a)$$

Finally, denoting by θ the angle between \vec{k} and \vec{k}' , and using Eqs. (6.10) – (6.12) we obtain

$$\begin{aligned} \frac{1}{m} &= \frac{1}{m^*} + \frac{k_F}{(2\pi)^3} \sum_{\sigma'} 4\pi \int d(\cos \theta) \sum_L (f_L + 4\sigma \cdot \sigma' \phi_L) P_L(\cos \theta) \cos \theta \\ &= \frac{1}{m^*} + \frac{k_F}{3\pi^2} f_1. \end{aligned} \quad (6.22b)$$

Because this and all subsequent integrals over $f(k, k')$ are evaluated at the Fermi surface and thus are ultimately multiplied by the density of states at the Fermi surface, it is conventional to define new Legendre expansion coefficients which include the density of states. Using Eq. (6.13b), the density of states at the Fermi surface is

$$\begin{aligned} N(0) &= \frac{1}{V} \sum_{k, \sigma} \delta(\epsilon^0(k) - \mu) \\ &= \frac{2}{(2\pi)^3} 4\pi \int k^2 dk \delta(\epsilon^0(k) - \mu) \\ &= \frac{m^* k_F}{\pi^2} \end{aligned} \quad (6.23)$$

and we define the normalized expansion coefficients*

$$F_L \equiv \frac{k_F m^*}{\pi^2} f_L \quad (6.24a)$$

$$Z_L \equiv \frac{k_F m^*}{\pi^2} \phi_L. \quad (6.24b)$$

With these definitions, we see from Eq. (6.22) that the effective mass directly specifies the $L = 1$ component of the spin independent quasiparticle interaction

$$\frac{m^*}{m} = \left(1 + \frac{F_1}{3}\right). \quad (6.25)$$

As is also evident in Problem 6.2, we observe from Eq. (6.22) that the current associated with a quasiparticle is not simply $\frac{k}{m^*}$. Because of the interaction with all the other particles in the medium, there is another piece coming from the interaction f which represents the other particles dragged along with the quasiparticle, or the backflow of other particles around the quasiparticle.

Compressibility and Sound Velocity

The compressibility χ of a liquid characterizes the change of pressure with volume according to the standard definition†

$$\frac{1}{\chi} = -V \frac{\partial P}{\partial V} = \rho \frac{\partial P}{\partial \rho} \quad (6.26)$$

where V is the volume of the system and the density is $\rho = \frac{N}{V}$. The velocity of thermodynamic sound, c_1 , provides a convenient experimental means of measuring χ , since by Eq. (5.124)

$$c_1^2 = \frac{1}{m} \frac{\partial P}{\partial \rho} = \frac{1}{m\rho\chi}. \quad (6.27)$$

* A more modern notation for F_L is F_L^s and for Z_L is F_L^a , where the s and a denote symmetric and antisymmetric, spin combinations. Note also that our definition of Z differs by a factor of 4 from some of the early literature.

† Frequently, the compressibility of nuclear matter is expressed in terms of the compression modulus, $K \equiv k_F^2 \frac{\partial^2}{\partial k_F^2} \left(\frac{E}{A}\right)$. At equilibrium, $\frac{1}{\chi} = \frac{\rho K}{9}$.

From the observation in Fig. 5.16 that thermodynamic sound corresponds to a spherically symmetric change in the occupation of states at the Fermi surface, it is clear physically that χ and c_1 will depend on a spherically symmetric average of $f(k, k')$ and thus on the Landau parameter F_0 . We now derive the precise relation.

Because the free energy is extensive, we may write it in terms of a free energy per unit volume as follows:

$$F(T, V, N) \equiv V f\left(T, \frac{N}{V}\right). \quad (6.28)$$

Hence,

$$P = -\frac{\partial F}{\partial V} = f - \rho \frac{\partial f}{\partial \rho} \quad (6.29)$$

and thus

$$\frac{1}{\chi} = \rho^2 \frac{\partial^2}{\partial \rho^2} f(T, \rho). \quad (6.30)$$

Since it is convenient to express the final result in terms of the chemical potential, we use Eq. (6.28) to obtain

$$\mu = \frac{\partial F}{\partial N} = \frac{\partial}{\partial \rho} f(T, \rho) \quad (6.31)$$

so that

$$\frac{1}{\chi} = \rho^2 \frac{\partial \mu}{\partial \rho}. \quad (6.32)$$

In order to calculate $\frac{\partial \mu}{\partial \rho}$, we note that

$$\mu = \epsilon(k_F, n(\vec{k}, \sigma)) \quad (6.33)$$

and thus, using Eq. (6.6),

$$\frac{\partial \mu}{\partial \rho} = \frac{\partial \epsilon^0}{\partial k_F} \frac{\partial k_F}{\partial \rho} + \sum_{\sigma'} \int \frac{d^3 k'}{(2\pi)^3} f(k_F, \vec{k}') \frac{\delta n(k')}{\delta k_F} \frac{\partial k_F}{\partial \rho}. \quad (6.34a)$$

Using the relations $\rho = \frac{k_F^3}{3\pi^2}$ from Eq. (6.1), $\frac{\partial \epsilon^0}{\partial k_F} = \frac{k_F}{m^*}$ from Eq. (6.13), $\frac{\partial n(k')}{\partial k_F} = \delta(k' - k_F)$ from Eq. (6.3) and $\int \frac{d\Omega}{4\pi} f(\theta, \sigma, \sigma') = f_0 = \frac{\pi^2}{m^* k_F} F_0$ from Eqs. (6.12) and (6.24), we obtain

$$\begin{aligned} \rho \frac{\partial \mu}{\partial \rho} &= \rho \frac{\partial k_F}{\partial \rho} \left[\frac{k_F}{m^*} + \frac{k_F^2}{(2\pi)^3} \sum_{\sigma'} \int d\Omega f(\theta, \sigma, \sigma') \right] \\ &= \frac{k_F^2}{3m^*} [1 + F_0] \end{aligned} \quad (6.34b)$$

from which it follows from Eqs. (6.32) and (6.27) that

$$\frac{1}{\chi} = \frac{\rho k_F^2}{3m^*} (1 + F_0) = \frac{\rho k_F^2}{m} \frac{(1 + F_0)}{(3 + F_1)}$$

and

$$c_1^2 = \frac{k_F^2}{3mm^*}(1 + F_0) = \frac{k_F^2}{m^2} \frac{(1 + F_0)}{(3 + F_1)} \quad (6.35)$$

As anticipated, the compressibility depends upon F_0 , coming from the spherical average of $f(\theta)$, as well as F_1 , arising from the effective mass. Note that the compressibility becomes infinite, that is, the system becomes unstable against density oscillations, when $F_0 = -1$. This is a specific example of the general stability criteria $F_L > -(2L + 1)$, and $Z_L > -(2L + 1)$ derived in Problem 6.3.

Magnetic Susceptibility

In contrast to the previous properties which were independent of the Fermion spin, we now consider the magnetization which is induced by an external magnetic field. The magnetic interaction energy of a Fermion with an external field H is $-\sigma\gamma H$, where for a point particle with no anomalous moment, the gyromagnetic ratio is $\gamma = \frac{e\hbar}{mc}$ and $\sigma = \pm \frac{1}{2}$ is the spin projection along H . Hence, in the presence of an external field H , the energies of the spin $+\frac{1}{2}$ particles are lowered and the energies of the spin $-\frac{1}{2}$ particles are raised, causing the equilibrium occupation of the spin $+\frac{1}{2}$ states to be larger than the spin $-\frac{1}{2}$ states and thus producing net magnetization.

To calculate the induced magnetization, we need to calculate how $n(\sigma)$, the density of particles with spin projection σ , changes in the presence of a weak external field H . Since the total number of particles is fixed, we may write

$$\rho(\sigma) = \rho_0 + \sigma\Delta\rho \quad (6.36)$$

Since the chemical potential of spin state σ , $\mu(\sigma)$, depends on both $n(\sigma)$ and $n(-\sigma)$, the equilibrium condition $\mu(\sigma) = \mu(-\sigma)$ poses a constraint which we will use to specify Δn . The chemical potential for each spin is equal to the quasiparticle energy at the corresponding Fermi surface

$$\mu(\sigma) = \epsilon(k_F(\sigma), \sigma) \quad (6.37)$$

where, in the presence of an external field, the quasiparticle energy is

$$\epsilon(\vec{k}, \sigma) = \epsilon^0(\vec{k}, \sigma) - \sigma\gamma H + \frac{1}{V} \sum_{\vec{k}', \sigma'} f(\vec{k}\sigma, \vec{k}'\sigma') \delta n(\vec{k}'\sigma') \quad (6.38)$$

Since both $k_F(\sigma)$ and the distribution function $n(k, \sigma)$ are specified by the density $\rho(\sigma)$, $\mu(\sigma)$ is a function of $\rho(\sigma)$ and $\rho(-\sigma)$. For a weak external field, we may expand the chemical potential to first order around the result μ_0 at $H = 0$.

$$\mu(\sigma) \approx \mu_0 - \sigma\gamma H + \sum_{\sigma'} \frac{\partial \mu(\sigma)}{\partial \rho(\sigma')} \sigma' \Delta\rho \quad (6.39)$$

The derivatives of the chemical potential $\mu(\sigma)$ with respect to the densities for each spin state are evaluated as in Eq. (6.34), where we now distinguish the individual contributions of each spin population

$$\begin{aligned} \frac{\partial \mu(\sigma)}{\partial \rho(\sigma')} &= \frac{\partial k_F(\sigma')}{\partial \rho(\sigma')} \left[\frac{\partial \epsilon^0(\sigma)}{\partial k_F(\sigma)} \delta(\sigma, \sigma') + \int \frac{d^3 k'}{(2\pi)^3} f(k_F\sigma, k'\sigma') \frac{\partial n(k', \sigma')}{\partial k_F(\sigma')} \right] \\ &= \frac{2\pi^2}{k_F^2(\sigma')} \left[\frac{k_F}{m^*} \delta(\sigma, \sigma') + \frac{k_F}{2m^*} (F_0 + 4\sigma\sigma'Z_0) \right] \end{aligned} \quad (6.40)$$

Using Eqs. (6.39–6.40), the equilibrium condition $\mu(\sigma) = \mu(-\sigma)$ provides the desired constraint on $\Delta\rho$

$$\gamma H = \frac{2\pi^2}{k_F m^*} (1 + Z_0) \Delta\rho \quad (6.41)$$

Using Eqs. (6.36) and (6.41) the magnetization density is

$$\begin{aligned} m &= \sum_{\sigma=\pm\frac{1}{2}} \left(\rho_0 + \sigma\Delta\rho \right) \sigma\gamma = \frac{\gamma}{2} \Delta\rho \\ &= \frac{\gamma^2 H k_F m^*}{4\pi^2 (1 + Z_0)} \end{aligned} \quad (6.42)$$

From which it follows that the magnetic susceptibility is

$$\chi_M = \frac{\partial m}{\partial H} = \frac{\gamma^2 k_F m^*}{4\pi^2 (1 + Z_0)} \quad (6.43)$$

As in the case of the compressibility, the $L = 0$ Legendre coefficient of the quasiparticle interaction enters because of the spherically symmetric deformation of the Fermi sphere. However, since the spin $+\frac{1}{2}$ sphere expands while the spin $-\frac{1}{2}$ sphere decreases, the difference, Z_0 , between the interactions of like and unlike spins contributes rather than the sum F_0 . Again, we see a specific example of the general stability criterion $Z_L > (2L + 1)$ of Problem 6.3. In liquid ^3He , as tabulated in Table 6.1, Z_0 ranges between -0.67 at 0 atm and -0.76 at 27 atm indicating that the strong spin dependence of the effective interaction makes the system nearly unstable with respect to ferromagnetic ordering. The physical origin of this dependence is simple. By antisymmetry, the spatial wave function of a spin singlet is S -wave and appreciably samples the short-range repulsive core of the He-He potential, whereas the spatial wave function of a spin triplet is P -wave and experiences much less repulsion.

NONEQUILIBRIUM PROPERTIES AND COLLECTIVE MODES

The dynamics of a Fermi liquid close to equilibrium is governed by a Boltzmann equation for the quasiparticle distribution. Two fundamental assumptions are required. The first is that we consider only long wavelength, low-energy excitations. Then, instead of considering the quantum Wigner distribution function introduced in Section 5.4, we may treat the non-equilibrium distribution function at position \vec{r} as a classical distribution function $n(\vec{k}, \vec{r}, t)$. The second major assumption is that the local quasiparticle energy at the position \vec{r} , $\epsilon(\vec{k}, \vec{r})$, plays the role of the quasiparticle Hamiltonian, so that

$$\begin{aligned} \dot{r} &= \frac{\partial \epsilon(k, r)}{\partial k} \\ \dot{k} &= - \frac{\partial \epsilon(k, r)}{\partial r} \end{aligned} \quad (6.44)$$

Then, the Boltzmann equation is obtained by equating the total time derivative of $n(\vec{k}, \vec{r}, t)$ to a quasiparticle collision integral $I(n)$

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial \vec{r}} \cdot \frac{\partial \epsilon}{\partial \vec{k}} - \frac{\partial n}{\partial \vec{k}} \cdot \frac{\partial \epsilon}{\partial \vec{r}} = I(n) \quad (6.45)$$