This document documents the conventions used for deuteron wave functions and form factors.

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To do:
1) Plenty of things :)

1 Fourier transform conventions

We use the usual definition of Fourier transform

$$\tilde{\psi}(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int d^3\mathbf{r}\, \psi(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}} \; . \tag{1}$$

Invoking the plane wave expansion

$$\begin{aligned} e^{i\mathbf{k}\cdot\mathbf{r}} &= \sum_{l=0}^{\infty} i^l (2l+1) j_l(kr) P_l(\cos\theta) \\ &= 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} i^l j_l(kr) Y_{lm}(\theta_r,\phi_r) Y^*_{lm}(\theta_k,\phi_k) \\ &= 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} i^l j_l(kr) Y^*_{lm}(\theta_r,\phi_r) Y_{lm}(\theta_k,\phi_k) \; . \end{aligned} \tag{2}$$

From Eqs. (1) and (2) and letting $\psi(\mathbf{r}) = R(r) Y_{l',m'}(\theta_r,\phi_r)$ we have

$$\tilde{\psi}(\mathbf{k}) = \frac{4\pi}{(2\pi)^{3/2}} \int r^2\, dr\, d\Omega\, R(r)\, Y_{l'm'}(\theta_r,\phi_r) \sum_{l=0}^{\infty} \sum_{m=-l}^{l} i^l j_l(kr) Y^*_{lm}(\theta_r,\phi_r) Y_{lm}(\theta_k,\phi_k) \; . \tag{3}$$

Using the property of spherical harmonics,

$$\int d\Omega\, Y^*_{lm}(\Omega) Y_{l'm'}(\Omega) = \delta_{ll'}\delta_{mm'} \tag{4}$$

Eq (3) becomes

$$\tilde{\psi}(\mathbf{k}) = \sqrt{\frac{2}{\pi}} \int r^2\, dr\, R_{l'}(r)\, i^{l'} j_{l'}(kr)\, Y_{l'm'}(\theta_k,\phi_k) \tag{5}$$

$Y_{l'm'}(\theta_k,\phi_k)$ is the angular part. Thus just the 'radial' part of the Fourier transformed wave function is

$$\tilde{R}_{l'}(k) = \sqrt{\frac{2}{\pi}} \int r^2\, dr\, R_{l'}(r)\, i^{l'} j_{l'}(kr) \tag{6}$$

For deuteron the relevant channels are $l = 0$ and $l = 2$ and the useful equations are

$$\tilde{u}(k) = \sqrt{\frac{2}{\pi}} \int r\, dr\, u(r)\, j_0(kr) \tag{7}$$

$$\tilde{w}(k) = -\sqrt{\frac{2}{\pi}} \int r\, dr\, w(r)\, j_2(kr) \tag{8}$$

which obey the normalization condition

$$\int_0^\infty \left[u^2(r) + w^2(r)\right]\, dr = \int_0^\infty \left[\tilde{u}^2(k) + \tilde{w}^2(k)\right] k^2\, dk = 1\ . \tag{9}$$

Similarly, we can write transformations from $k-$space to $r-$space.

$$\psi(\mathbf{r}) = \sqrt{\frac{2}{\pi}} \int k^2 dk\, \tilde{R}_l(k)\, (-i)^l j_l(kr) Y_{lm}(\theta_r, \phi_r) \tag{10}$$

$$R(r) = \sqrt{\frac{2}{\pi}} \int k^2 dk\, \tilde{R}_l(k)\, (-i)^l j_l(kr) \tag{11}$$

$$\frac{u(r)}{r} = \sqrt{\frac{2}{\pi}} \int k^2 dk\, \tilde{u}(k)\, j_0(kr) \tag{12}$$

$$\frac{w(r)}{r} = -\sqrt{\frac{2}{\pi}} \int k^2 dk\, \tilde{w}(k)\, j_2(kr) \tag{13}$$

The expressions for u, w, $\tilde{u}$ and $\tilde{w}$ derived here match those in [1] up to a phase (The phase of the wave function is not an observable).

2 Nucleon Form factors

Before we delve into the deuteron cross section and form factor let us review our knowledge about the nucleon form factors.

Scattering of spin-$\frac{1}{2}$ particles with charge $|Z_1| = 1$ from *spinless* target particles has been treated by Mott, and the cross section for Mott scattering is [2]

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = 4(Ze^2)^2 \frac{E^2}{(qc)^4} \left(1 - \beta^2 \sin^2 \frac{\theta}{2}\right) \tag{14}$$

where E is the energy of the incident electron and $v = \beta c$ its velocity. The Mott formula ignores the internal structure of the probe and the target.

If the target possesses an internal structure (probes are usually point-like leptons), the Mott scattering formula is modified as (let us assume that the target particle possesses a spherically symmetric density distribution)

$$\left(\frac{d\sigma}{d\Omega}\right) = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} |F(\mathbf{q})|^2\ . \tag{15}$$

The multiplicative $F(\mathbf{q}^2)$ is called the form factor and $\mathbf{q}^2 = (\mathbf{p} - \mathbf{p}')^2$ is the square of momentum transfer. It can be shown that the form factor can be written as the Fourier transform of the probability density

$$F(\mathbf{q}^2) = \int d^3r\, |\psi(\mathbf{r})|^2\, exp(i\mathbf{q} \cdot \mathbf{r}/\hbar) \tag{16}$$

Eqs. (15) and (16) connect the experimentally observed cross section to the theoretical calculations and give information about the internal structure of the target.

For the scattering from a spin-$\frac{1}{2}$ particle (let us say proton), we need to invoke two form factors. The laboratory cross section can be written as

$$\left(\frac{d\sigma}{d\Omega}\right) = \left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{Mott}} \left[\frac{G_E(q^2)^2 + \tau\, G_M^2(q^2)}{1+\tau} + 2\tau\, G_M^2(q^2)\tan^2\left(\frac{\theta}{2}\right)\right] \tag{17}$$

where $\tau \equiv -q^2/4M_p^2$ and q is the four-momentum transferred. G_E and G_M are called the electric and magnetic form factors respectively. We shall soon motivate the reason for the nomenclature. Eq. (17) is called the Rosenbluth formula. We shall now sketch the derivation of the Rosenbluth formula.

The current (say for proton) is given by

$$J_Q^\mu(x) = \bar{u}(p_2)\Gamma_Q^\mu(q)u(p_1)e^{iq\cdot x} \tag{18}$$

Using Lorentz and parity invariance we can write $\Gamma_Q^\mu(q)$ most generally as

$$\Gamma_Q^\mu(q) = \left[\gamma^\mu F_1^p(q^2) + \frac{i\sigma^{\mu\nu}}{2M_p} q_\nu F_2^p(q^2)\right] \tag{19}$$

with

$$F_1^p(0) = 1 \quad F_2^p(0) = \kappa_p \text{ [4]}. \tag{20}$$

κ_p is the anomalous magnetic moment defined as follows

$$\vec{\mu}_p = +g_p\mu_N\vec{S}_p\ , \qquad g_p = 2(1+\kappa_p)\ , \quad \mu_N = e/2M_p\ . \tag{21}$$

$\mu_p = 2.793\mu_N$. Thus $\kappa_p = 1.793$. We define the *Sachs* or the electric and magnetic form factors as

$$\begin{aligned} G_E^p(q^2) &= F_1^p(q^2) - \tau F_2^p(q^2) \\ G_M^p(q^2) &= F_1^p(q^2) + F_2^p(q^2) \end{aligned} \tag{22}$$

The nomenclature is motivated by the form of the matrix elements in the Breit (see the next section for the introduction to Breit frame) frame. In the Breit frame [3]

$$\begin{aligned} J_0 = \rho &= G_E\,\delta_{S_iS_f} \\ \vec{J} &= G_M\,\frac{i}{2M}\,\chi_{S_f}^\dagger(\vec{\sigma}\times\vec{q})\,\chi_{S_i} \end{aligned} \tag{23}$$

We can write down similar formulas for the neutron. In case of neutron

$$F_1^n(0) = 0\ , \quad F_2^n(0) = \kappa_n \sim -1.91\ , \quad g_n = 2\kappa_n\ . \tag{24}$$

The values for the Sachs form factors can be summarized as

$$\begin{aligned} G_E^p(0) = 1\ , &\quad G_E^n(0) = 0\ , \\ G_M^p(0) = 2.79\ , &\quad G_M^n(0) = -1.91 \end{aligned} \tag{25}$$

It is sometimes useful to define the isovector and isoscalar form factors as

$$F_{1,2}^V = \frac{1}{2}(F_{1,2}^p - F_{1,2}^n)\ , \quad F_{1,2}^S = \frac{1}{2}(F_{1,2}^p + F_{1,2}^n) \tag{26}$$

and similarly for $G_{E,M}^{V,S}$.

3 Breit frame

In the elastic electron-nucleon scattering $q = k - k' = p' - p$. The invariant mass of the photon is $q^2 = -4EE' \sin^2 \theta/2$, which is always negative. This means that we can always find a frame in which the energy transfer $q^0 \equiv \nu = 0$ and $q^2 = -\vec{q}^{\,2} \equiv -Q^2$. The four-vectors in the Breit frame are

$$q = (0,\, 0,\, 0,\, \sqrt{Q^2}), \quad p_1 = (E,\, 0,\, 0,\, -\sqrt{Q^2}/2), \quad p_2 = (E,\, 0,\, 0,\, \sqrt{Q^2}/2)\ . \tag{27}$$

If we choose to evaluate the form factors in the Breit frame, then the relativistic and non-relativistic momentum transfers are identical, $Q^2 = \vec{q}^{\,2}$, and the relativistic nucleon form factors can be used without corrections.

4 Deuteron: radius and form factors

4.1 Radius

The characteristic size of the deuteron r_m (matter radius) is defined as the rms-half distance between the two nucleons

$$r_m^2 = \int_0^\infty \left[u^2(r) + w^2(r)\right] \left(\frac{r}{2}\right)^2 dr = \frac{1}{4}\int_0^\infty \left[u^2(r) + w^2(r)\right] r^2\, dr \tag{28}$$

As mentioned in [5] and [6] $r_m^2 = 3.90$ fm^2 or $r_m = 1.975$ fm. (r is the distance between the proton and the neutron.)

4.2 Magnetic moments

The magnetic moments of the neutron and proton are [7]

$$\mu_p = 2.793\, \mu_N \qquad \mu_n = -1.913\, \mu_N \quad \text{where } \mu_N = \frac{e\hbar}{2M_N} \tag{29}$$

and M_N is the nucleon mass. The isoscalar nucleon magnetic moment is

$$\mu_s = \mu_p + \mu_n = 0.8798\, \mu_N\ . \tag{30}$$

Since the spins of proton and neutron are parallel in the deuteron, one might expect its magnetic moment, $\mu_d \approx \mu_s$. The measured value of the deuteron magnetic dipole however is $0.857\, \mu_N$ [9]. The positive value of the quadrupole moment suggests that deuteron is prolate (cigar-shaped).

We know that the deuteron wave function has a $D-$state component and this will have an effect on the magnetic moment. Thus

$$\vec{\mu}_d = \mu_p \vec{\sigma}_p + \mu_n \vec{\sigma}_n + \vec{L}_p\ . \tag{31}$$

$\vec{L}_p$ is the orbital angular momentum of the proton. We assume that neutron being neutral, its motion will not contribute to the magnetic moment. Thus we are neglecting the two body contribution. (See the comments in the quadrupole section below.) $\vec{L}_p = \frac{1}{2}\vec{L}$ where L is the deuteron angular momentum. To eliminate $\vec{\sigma}$'s we use

$$\vec{J} = \vec{L} + \vec{S} = \vec{L} + \frac{1}{2}(\vec{\sigma}_p + \vec{\sigma}_n)\ . \tag{32}$$

$$
\begin{aligned}
\vec{\mu}_d &= \mu_p \vec{\sigma}_p + \mu_n \vec{\sigma}_n + \vec{L}_p \\
&= \frac{1}{2}(\mu_p + \mu_n)(\vec{\sigma}_p + \vec{\sigma}_n) + \frac{1}{2}(\mu_p - \mu_n)(\vec{\sigma}_p - \vec{\sigma}_n) + \frac{\vec{L}}{2} \\
&= (\mu_p + \mu_n)(\vec{J} - \vec{L}) + \frac{\vec{L}}{2} \\
&= (\mu_p + \mu_n)\vec{J} - (\mu_p + \mu_n - \frac{1}{2})\vec{L} \ . \qquad (33)
\end{aligned}
$$

We have used the fact that $\sigma_n - \sigma_p$ vanishes for an isosinglet triplet state. Invoking the projection theorem (see [8] for details and proof) we write

$$
\langle \vec{\mu}_d \rangle = \left\langle \frac{\vec{\mu}_d \cdot \vec{J}}{\vec{J}^2} \right\rangle \langle \vec{J} \rangle \ . \qquad (34)
$$

Using $\vec{J}^2 = J(J+1) = 2$ and $\vec{J} \cdot \vec{L} = \frac{1}{2}(\vec{L}^2 + \vec{J}^2 - \vec{S}^2)$ we find

$$
\langle \vec{\mu}_d \rangle_z = \mu_n + \mu_p - \frac{3}{2}(\mu_n + \mu_p - \frac{1}{2})P_D \qquad (35)
$$

where P_D is the $D-$state probability (which is not an observable). As noted above this theoretical expression for μ_d is only an approximation in a simple potential model of the deuteron. 'The magnetic moment is very sensitive to relativistic corrections and interaction currents which can easily alter this result significantly' [1].

4.3 Quadrupole Moment

Most of the discussion below is from [9, 5].

The deuteron quadrupole moment is calculated considering contributions from the proton alone, since the neutron carries no charge. (This is an approximation. This is what we would write down if we had just a free proton. We are neglecting the two-body contribution.)

$$
Q_{\alpha\beta} = e\,(3x^p_\alpha x^p_\beta - \delta_{\alpha\beta} r_p^2) = \frac{1}{4} e\,(3\,x_\alpha x_\beta - \delta_{\alpha\beta} r^2) \ . \qquad (36)
$$

r_p is the position vector of the proton measured from the center of mass, whereas the deuteron wave function is expressed in terms of the neutron-proton distance, $r = 2r_p$.

$$
\langle Q_{\alpha\beta} \rangle = \int \left| \Psi_J^M(r) \right|^2 Q_{\alpha\beta} d\tau \ . \qquad (37)
$$

For the deuteron $J = 1$. It can be seen that $\langle Q_{\alpha\beta} \rangle = 0$ for $\alpha \neq \beta$ and

$$
\langle Q_{11} \rangle = \langle Q_{22} \rangle = -\frac{1}{2} \langle Q_{33} \rangle \ . \qquad (38)
$$

The quadrupole moment of the deuteron, Q_d is *defined* to be the expectation value of $\langle Q_{33} \rangle$ in the state of maximum quantum number $M = 1$. Thus

$$
Q_d = \frac{1}{4} e \int \left| \Psi_1^1(r) \right|^2 (3z^2 - r^2) d\tau \ . \qquad (39)
$$

Substituting the expression for Ψ_1^1 gives the familiar expression

$$Q_d = \frac{e}{\sqrt{50}} \int_0^\infty w(r) \left[u(r) - \frac{w(r)}{\sqrt{8}} \right] r^2 dr \ . \tag{40}$$

The experimental value for the deuteron quadrupole moment is 0.285 e.fm^2. Note however that the quantity measured experimentally is not exactly given by Eq. (40) which is derived on the assumption that photon couples only to the proton in the deuteron. In general the operator in Eq. (40) will be modified by exchange currents (photon could couple to intermediate pions) and other corrections. The expectation value of Q_d as given by Eq. (40) would therefore change slightly under SRG (even when both wave function and the operator are consistently evolved).

4.4 Deuteron wave function

The nonrelativistic NN wave function of the deuteron can be written as [1, 5]

$$\begin{aligned} \psi^M(\mathbf{r}) &= \sum_L \sum_{m_s} \frac{z_L(r)}{r} Y_{L,M-m_s}(\hat{\mathbf{r}}) |S=1, m_s\rangle \langle L, M-m_s; S=1, m_s | J=1, M\rangle \\ &= \frac{u(r)}{r} Y_{00}(\theta,\phi)|S=1,M\rangle + \frac{w(r)}{r} \sum_{m_s} \langle JM | L=2, M-m_s; S=1, m_s\rangle Y_{L=2,M-m_s}(\theta,\phi)|S=1,m_s\rangle \end{aligned} \tag{41}$$

It can be shown that [1] (*It would be great if I could derive it!*)

$$\sum_{m_s} \langle J=1M | L=2, M-m_s; S=1, m_s\rangle Y_{L=2,M-m_s}(\theta,\phi)|S=1,m_s\rangle = \frac{1}{\sqrt{32\pi}} (3\,\vec{\sigma}_1\cdot\hat{\mathbf{r}}\,\vec{\sigma}_2\cdot\hat{\mathbf{r}} - \vec{\sigma}_1\cdot\vec{\sigma}_2)\chi_1^M \tag{42}$$

$\chi_1^M = |S=1,M\rangle$ is the triplet spin function with projection M. Thus

$$\begin{aligned} \psi^M(\mathbf{r}) &= \frac{1}{\sqrt{4\pi}} \left[\frac{u(r)}{r} \vec{\sigma}_1\cdot\vec{\sigma}_2 + \frac{1}{\sqrt{8}} \frac{w(r)}{r} (3\,\vec{\sigma}_1\cdot\hat{\mathbf{r}}\,\vec{\sigma}_2\cdot\hat{\mathbf{r}} - \vec{\sigma}_1\cdot\vec{\sigma}_2) \right] \chi_1^M \\ &= \frac{1}{\sqrt{4\pi}} \left[\frac{u(r)}{r} + \frac{w(r)}{r} \frac{1}{\sqrt{8}} S_{12} \right] \chi_1^M \end{aligned} \tag{43}$$

where $S_{12} = (3\,\vec{\sigma}_1\cdot\hat{\mathbf{r}}\,\vec{\sigma}_2\cdot\hat{\mathbf{r}} - \vec{\sigma}_1\cdot\vec{\sigma}_2)$ is the familiar tensor operator. Also note that

$$\vec{\sigma}_1\cdot\vec{\sigma}_2 \chi_1^M = 1\,\chi_1^M \ . \tag{44}$$

In momentum space (using the convention in Eq. (1)) Eq. (43) becomes

$$\tilde{\psi}^M(\mathbf{p}) \equiv \frac{1}{\sqrt{(2\pi)^3}} \int d^3r\, e^{i\mathbf{p}\cdot\mathbf{r}} \psi^M(\mathbf{r}) = \frac{1}{\sqrt{4\pi}} \left[\tilde{u}(p)\vec{\sigma}_1\cdot\vec{\sigma}_2 + \frac{\tilde{w}(p)}{\sqrt{8}} (3\,\vec{\sigma}_1\cdot\hat{\mathbf{p}}\,\vec{\sigma}_2\cdot\hat{\mathbf{p}} - \vec{\sigma}_1\cdot\vec{\sigma}_2) \right] \chi_1^M \tag{45}$$

Note that in [1] the expression for $\tilde{\psi}^M(\mathbf{p})$ has a negative sign in front of $\tilde{w}(p)$. However, the Fourier transform $\tilde{w}(p)$ (see Eq. (8)) is defined in [1] without a minus sign and thus equations are consistent. In deriving Eq. (45) the $D-$ state contribution to the Fourier transform wave function can be can be obtained through the use of Eqs. (42), (2), and (4).

Other useful relation which comes in handy for many derivations is

$$S_{12}^2 \chi_1^M = (8 - 2S_{12})\,\chi_1^M \tag{46}$$

Eq. (46) is derived using the property of Pauli matrices $(\hat{n}\cdot\vec{\sigma})^{2n} = I$ and Eq. (44).

4.5 Deuteron Form Factors

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