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Linear Response and Correlation Functions

- When we derived a Lehmann representation for the one-particle Green's function [notes 171-179 and PS#4]

$$\boxed{iG_{\alpha\beta}(\vec{x}t,\vec{x}'t') = \langle \Psi_0^N | T[\hat{\psi}_{H\alpha}(\vec{x}t)\hat{\psi}^\dagger_{H\beta}(\vec{x}'t')] | \Psi_0^N \rangle}$$

we found

$$G_{\alpha\beta}(\vec{x},\vec{x}';\omega) = \int_{-\infty}^{\infty} d(t-t')\, e^{i\omega(t-t')} G_{\alpha\beta}(\vec{x}t,\vec{x}'t')$$

$$= \sum_n \frac{\langle \Psi_0^N | \hat{\psi}_\alpha(\vec{x}) | \Psi_n^{N+1}\rangle \langle \Psi_n^{N+1} | \hat{\psi}^\dagger_\beta(\vec{x}') | \Psi_0^N \rangle}{\omega - (E_n^{N+1} - E_0^N) + i\eta}$$

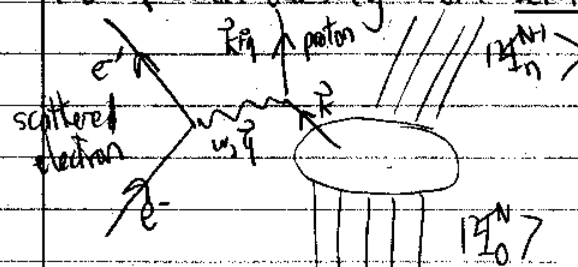
$$- \sum_n \frac{\langle \Psi_0^N | \hat{\psi}^\dagger_\beta(\vec{x}') | \Psi_n^{N-1}\rangle \langle \Psi_n^{N-1} | \hat{\psi}_\alpha(\vec{x}) | \Psi_0^N \rangle}{\omega + E_n^{N-1} - E_0^N - i\eta}$$

Why the minus sign? (time-ordered)

So this function tells us about states with one particle more or one less than the original ground state.

- the pole positions give excitation energies
- overlap matrix elements from residues.

- We claimed on 179 that, with some assumptions, we could learn about the spectral density from semi-inclusive electron scattering $\Rightarrow (e,e'p)$



← don't measure final state but measure e' (tells $\omega,\vec{q}$) and knocked out proton $\Rightarrow$ deduce $\vec{k}$

Impulse approximation implies we learn about the momentum of the proton in the nucleus

$$\boxed{\sigma = 2\pi \sum_n |\langle \Psi_n^{N-1} | a_{\vec{k}} | \Psi_0^N \rangle|^2 \,\delta(E_n^{N-1} + E_p - E_0^N - \omega) = \rho^-(\vec{k}, \omega + \mu^N - E_p)}$$

$\nwarrow \int e^{-i\vec{k}\cdot\vec{x}}\, \hat{\psi}(\vec{x})\, d^3x$

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- What if we wanted to learn about excited states with the same number of particles?
  - ⇒ bosonic collective excitations, such as phonons, spin waves, ...
  - excite (and measure) using probes that couple to particle density, charge density, spin density, or some other particle conserving operator.
    - e.g. scatter E/M waves, electrons, neutrons
  - weakly interacting probes ⇒ Born approximation is valid

- So apply a weak perturbation that is time dependent; How does a nucleus respond?
  - If we vary the frequency (and wavelength) of the probe, there will be different regions with different types of excitations.
    - low energy — discrete "particle-hole" transitions (think of → shell model picture)
    - intermediate energy ⇒ collective modes ⇒ giant resonances (think of vibrating liquid drops — e.g. isoscalar monopole resonance → "breathing mode" → which is related to the compressibility, or the giant dipole resonance p n where protons vibrate against neutrons. [Why are these called "giant" resonances?]
    - quasi-elastic scattering at still higher energy ⇒ we'll model this today!

- We learn about all of these things via linear response and correlation functions.
  - The basic idea of linear response is to "bang" on a system at some place and time (or with some frequency and wavelength) and ask how a property of the system changes ("responds") at another place and (later) time (since causal).
  - related to probability to absorb photons (or whatever) at particular wavelength and frequency (later!).

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· We'll follow chapter 5 of Fetter and Walecka in the initial discussion and then switch to Negele and Orland's discussion. Our notation is a compromise.

Start with our regular Hamiltonian $\hat{H}$ and the Schrödinger (or S-) picture state $|\Psi_s(t)\rangle$, which satisfies

$$\boxed{i\frac{\partial}{\partial t}|\Psi_s(t)\rangle = \hat{H}|\Psi_s(t)\rangle}$$

which has the (formal) solution

$$\boxed{|\Psi_s(t)\rangle = e^{-i\hat{H}t}|\Psi_s(0)\rangle}$$

(that is, $e^{-i\hat{H}t}$ is the evolution operator).

Now turn on the weak external perturbation $\hat{H}^{ext}(t)$ at $t=t_0$. We have in mind something like

$$\boxed{\hat{H}^{ext}(t) = \int d^3x\ \varphi^{ext}(\vec{x}t)\,\hat{O}_1(\vec{x})}$$

where $\hat{O}_1(\vec{x})$ is an S-picture operation like $\hat{n}(\vec{x}) = \hat{\psi}^{\dagger}(\vec{x})\hat{\psi}(\vec{x})$.

The subsequent wave function $|\tilde{\Psi}_s(t)\rangle$ satisfies

$$\boxed{i\frac{\partial}{\partial t}|\tilde{\Psi}_s(t)\rangle = (\hat{H} + \hat{H}^{ext}(t))|\tilde{\Psi}_s(t)\rangle}$$

The major time dependence is given by $\hat{H}$, so we pull that out, which defines $\hat{A}(t)$:

$$\boxed{|\tilde{\Psi}_s(t)\rangle = e^{-i\hat{H}t}\hat{A}(t)|\Psi_s(0)\rangle}$$

with $\boxed{\hat{A}(t) = 1}$ for $t \le t_0$ (causal boundary condition)

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So $\hat{A}(t)$ represents the modifications to the evolution from adding $\hat{H}^{ex}(t)$. We project its differential equation:

$$\boxed{\begin{aligned} i\frac{\partial}{\partial t}\left[e^{-i\hat{H}t}\hat{A}(t)|\Psi_s(0)\rangle\right] &= \hat{H}e^{-i\hat{H}t}A(t)|\Psi_s(0)\rangle + e^{-i\hat{H}t}\, i\frac{\partial}{\partial t}\hat{A}(t)|\Psi_s(0)\rangle \\ \text{from the right side} \quad &= \hat{H}e^{-i\hat{H}t}A(t)|\Psi_s(0)\rangle + H^{ext}(t)\,\hat{A}(t)|\Psi_s(0)\rangle \end{aligned}}$$

which then implies (since true for any $|\Psi_s(0)\rangle$) that

$$\boxed{e^{-i\hat{H}t}\, i\frac{\partial}{\partial t}\hat{A}(t) = \hat{H}^{ext}\, e^{-i\hat{H}t}A(t)}$$

or $$\boxed{i\frac{\partial\hat{A}(t)}{\partial t} = e^{i\hat{H}t}\hat{H}^{ext}(t)\,e^{-i\hat{H}t}\hat{A}(t) \equiv \hat{H}^{ext}_H(t)\,A(t)}$$

(Heisenberg picture denoted by subscript H)

We can solve this equation by iteration for $t > t_0$, by first integrating for t_0 to t

$$\boxed{i\int_{t_0}^{t}\frac{\partial\hat{A}(t')}{\partial t'}dt' = i\hat{A}(t) - i\overset{1}{\hat{A}(t_0)} = \int_{t_0}^{t}\hat{H}^{ext}_H(t')\,\hat{A}(t')\,dt'}$$

$$\Rightarrow \boxed{\hat{A}(t) = 1 - i\int_{t_0}^{t}dt'\,\hat{H}^{ext}_H(t')\,\hat{A}(t')\cdot dt'}$$

and start with $\hat{A}(t) = 1$ on the right side, then plug the result back, etc. so

$$\boxed{\hat{A}(t) = 1 - i\int_{t_0}^{t}dt'\,\hat{H}^{ext}_H(t') + (-i)^2\int_{t_0}^{t}dt'\int_{t_0}^{t'}dt''\,H^{ext}_H(t')\,H^{ext}_H(t'') + \ldots}$$

(which could be written as a time-ordered exponential, but we won't need to!)

and the corresponding state vector is

$$\boxed{|\hat{\Psi}_s(t)\rangle = e^{-i\hat{H}t}|\Psi_s(0)\rangle - i\,e^{-i\hat{H}t}\int_{t_0}^{t}dt'\,\hat{H}^{ext}_H(t')|\Psi_s(0)\rangle + \ldots}$$

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- So now suppose we have an S-picture operator $\hat{O}_{2S}(t)$ (which may have an explicit time dependence)
  - This could be the density as well $\psi^\dagger_\beta(\vec{x}')\psi_\beta(\vec{x}')$, for example

- We look for it's expectation value after the perturbation:

$$\boxed{\begin{aligned}\langle\hat{O}_2(t)\rangle_{ext} &\equiv \langle\hat{\Psi}_S(t)|\,\hat{O}_{2S}(t)\,|\hat{\Psi}_S(t)\rangle \\ &= \langle\Psi_S(0)|\left[1 + i\int_{t_0}^{t}dt'\,\hat{H}^{ext}_H(t') + \ldots\right]e^{i\hat{H}t}\,\hat{O}_{2S}(t)\,e^{-i\hat{H}t} \\ &\qquad\times\left[1 - i\int_{t_0}^{t}dt'\,\hat{H}^{ext}_H(t') + \ldots\right]|\Psi_S(0)\rangle \\ &= \langle\Psi_H(0)|\,\hat{O}_{2H}(t)\,|\Psi_H(0)\rangle + i\langle\Psi_H(0)|\int_{t_0}^{t}dt'\left[\hat{H}^{ext}_H(t'),\hat{O}_{2H}(t)\right]|\Psi_H(0)\rangle \\ &\qquad + \ldots\end{aligned}}$$

where we only keep the linear term, since the external perturbation is weak.

- From here on we'll take the initial state to be the ground state (for finite temperature we would use an ensemble of states) with N particles $\Rightarrow |\Psi_H(0)\rangle \to |\Psi_0^N\rangle$

- Then the change in $\langle\hat{O}_2(t)\rangle$ is

$$\boxed{\begin{aligned}\delta\langle\hat{O}_2(t)\rangle &\equiv \langle\hat{O}_2(t)\rangle_{ext} - \langle\hat{O}_2(t)\rangle \\ &= i\int_{t_0}^{t}dt'\,\langle\Psi_0^N|\left[H^{ext}_H(t'),\hat{O}_{2H}(t)\right]|\Psi_0^N\rangle \\ &= i\int_{-\infty}^{\infty}dt'\,\theta(t-t')\langle\Psi_0^N|\left[H^{ext}_H(t'),\hat{O}_{2H}(t)\right]|\Psi_0^N\rangle\end{aligned}}$$

← replaced by $\theta(t-t')$

↖ allows general $H^{ext}_H(t')$

- So the response of an "observable" represented by $\hat{O}_J(t)$ to a perturbation $H^{ext}_H(t)$ is given by a <u>commutator</u> times a θ-function.

- Thus we can find this response if $\hat{O}_J$ and $\hat{H}^{ext}_H$ are one body operators by considering special cases of <u>two</u>-particle Green's functions (two $\psi^\dagger$'s and two ψ's), which we call correlation or response functions:

$$\boxed{D(x_1,x_2) \equiv iG^2(x_1 x_2 | x_1^+ x_2^+) = -i\langle \Psi^N_0 | T[\hat{\psi}^\dagger(x_1)\hat{\psi}(x_1)\hat{\psi}^\dagger(x_2)\hat{\psi}(x_2)] | \Psi^N_0 \rangle}$$

$$\boxed{D_R(x_1,x_2) \equiv -i\theta(t_1-t_2)\langle \Psi^N_0 | [\hat{\psi}^\dagger(x_1)\hat{\psi}(x_1), \hat{\psi}^\dagger(x_2)\hat{\psi}(x_2)] | \Psi^N_0 \rangle}$$

Here $x_i \equiv \{\vec{x}_i, t_i\}$ and we've suppressed spin indices. In general, if

$$\boxed{\hat{H}^{ext}(t') = \int d^3x\, \hat{\psi}^\dagger_\alpha(\vec{x})\, O_{\alpha\beta}(\vec{x}t)\, \hat{\psi}_\beta(\vec{x})}$$

and similarly with $\hat{O}_J(t)$, then

$$\boxed{\theta(t_1-t_2)\langle \Psi^N_0 | [\hat{\psi}^\dagger_\alpha(x_1)\hat{\psi}_\beta(x_1), \hat{\psi}_\gamma(x_2)\hat{\psi}_\delta(x_2)] | \Psi^N_0 \rangle}$$

is all we need. So, a spin density is appropriate for an external magnetic field $\Rightarrow \psi^\dagger \vec{\sigma} \psi$!

- From now on, however, we will restrict attention to the density

$$\boxed{\hat{\rho}(x) \equiv \psi^\dagger_\alpha(x)\psi(x)}$$

- Note: We assume throughout that $|\Psi^N_n\rangle$ is normalized.

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- Now we have a slight problem: Our path integral expansion tells us how to calculate time-ordered correlators of field operators, but we need the retarded correlator.

  $\Rightarrow$ use the Lehmann representation to relate.

- The procedure to construct the Lehmann representation is the same as for the one-particle Green's functions
  1) insert the Heisenberg field time dependence
  2) insert intermediate states (non-vanishing matrix elements)
  3) Fourier transform

We obtain:

$$\boxed{\left\{\begin{array}{l} D(\vec{x}_1,\vec{x}_2;\omega) \\ D_R(\vec{x}_1,\vec{x}_2;\omega)\end{array}\right\} = \sum_n \left[\frac{\langle \Psi_0^N|\hat{\rho}(\vec{x}_1)|\Psi_n^N\rangle\langle\Psi_n^N|\hat{\rho}(\vec{x}_2)|\Psi_0^N\rangle}{\omega-(E_n^N-E_0^N)+i\eta} - \frac{\langle \Psi_0^N|\hat{\rho}(\vec{x}_2)|\Psi_n^N\rangle\langle\Psi_n^N|\hat{\rho}(\vec{x}_1)|\Psi_0^N\rangle}{\omega+(E_n^N-E_0)\mp i\eta}\right]}$$

- We can also Fourier transform to momentum space if we have a uniform system

$$\hat{\rho}(\vec{q}) = \int e^{-i\vec{q}\cdot\vec{x}}\rho(\vec{x})\,d^3x = \int e^{-i\vec{q}\cdot\vec{x}}\left[\int\frac{d^3k}{(2\pi)^3}e^{-i\vec{k}\cdot\vec{x}}a_{\vec{k}}^\dagger\right]\left[\int\frac{d^3k'}{(2\pi)^3}e^{+i\vec{k}'\cdot\vec{x}}a_{\vec{k}'}\right]d^3x$$

$$= \int\frac{d^3k}{(2\pi)^3}a_{\vec{k}}^\dagger a_{\vec{k}+\vec{q}} \quad \text{and} \quad \boxed{\hat{\rho}(\vec{q})^\dagger = \hat{\rho}(-\vec{q})}$$

$$\Rightarrow \boxed{\left\{\begin{array}{l} D(\vec{k},\omega) \\ D_R(\vec{k},\omega)\end{array}\right\} = \sum_n \left[\frac{|\langle\Psi_n^N|\hat{\rho}(-\vec{q})|\Psi_0^N\rangle|^2}{\omega-(E_n^N-E_0^N)+i\eta} - \frac{|\langle\Psi_n^N|\hat{\rho}(\vec{q})|\Psi_0^N\rangle|^2}{\omega+(E_n^N-E_0^N)\mp i\eta}\right]}$$

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So the plan is:

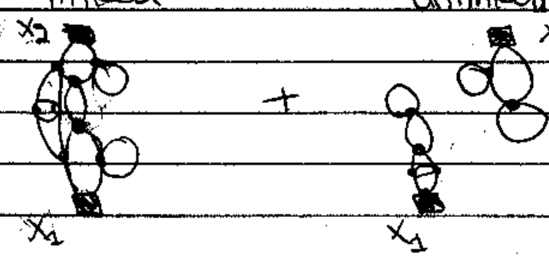
i) Calculate $\bar{D}$ in some approximation

ii) Find $\bar{D}_R$ using

$$\mathrm{Re}\,\bar{D}(\omega) = \mathrm{Re}\,\bar{D}_R(\omega)$$
$$\mathrm{Im}\,\bar{D}(\omega) = \mathrm{sgn}(\omega - E_0^N)\,\mathrm{Im}\,\bar{D}_R \quad \text{(try it!)}$$

• What does the diagrammatic expansion of $\bar{D}(x_1, x_2)$ look like? Typical diagrams fall into two classes:

linked unlinked



where ■ x_1 is the operator $\hat{\psi}^\dagger(x_1)\hat{\psi}(x_1)$.

• For excited states, only the linked diagrams contribute, so we usually introduce the density fluctuation operator

$$\boxed{\tilde{\hat{\rho}} \equiv \hat{\rho} - \langle \hat{\rho} \rangle}$$

and the corresponding correlation functions (no bars now!)

$$\boxed{\begin{aligned} iD(x_1, x_2) &= \langle \Psi_0^N | T[\tilde{\rho}(x_1), \tilde{\rho}(x_2)] | \Psi_0^N \rangle = i\bar{D}(x_1, x_2) - \langle \hat{\rho}(x_1) \rangle \langle \hat{\rho}(x_2) \rangle \\ iD_R(x_1, x_2) &= \theta(t_1 - t_2) \langle \Psi_0^N | [\tilde{\rho}(x_1), \tilde{\rho}(x_2)] | \Psi_0^N \rangle = i\bar{D}_R(x_1, x_2) \end{aligned}}$$

• The diagrammatic expansion of $iD(x_1, x_2)$ has only linked diagrams. The Feynman rules for the diagrams follow directly from our previous rules (see Negele and Orland for specifics).

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- We'll work for now with a uniform system (even though we'll apply the result to a finite nucleus!)

$$\Rightarrow \boxed{D^R(x_1, x_2) = D^R(x_1 - x_2)}$$

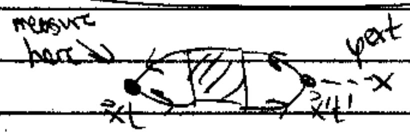
$$\Rightarrow \boxed{D^R(\vec{k}, \omega) \equiv \int d^3x \int dt\ e^{-i\vec{k}\cdot\vec{x}}\ e^{i\omega t}\ D^R(\vec{x}t)}$$

If we take

$$\boxed{\hat{H}^{ext}(t) = \int d^3x\ \hat{\rho}(\vec{x}t)\ \varphi^{ext}(\vec{x}t)}$$

then

$$\boxed{\delta\langle\hat{\rho}(\vec{x}t)\rangle = \int_{-\infty}^{\infty} dt' \int d^3x'\ D^R(\vec{x}t, \vec{x}'t')\ \varphi^{ext}(\vec{x}'t')}$$

$\Rightarrow$ 

Transforming,

$$\boxed{\varphi^{ext}(\vec{k}, \omega) \equiv \int d^3x \int dt\ e^{-i\vec{k}\cdot\vec{x}}\ e^{i\omega t}\ \varphi^{ext}(\vec{x}t)}$$

$$\boxed{\delta\langle\hat{\rho}(\vec{k}, \omega)\rangle \equiv \int d^3x \int dt\ e^{-i\vec{k}\cdot\vec{x}}\ e^{i\omega t}\ \delta\langle\hat{\rho}(\vec{x}t)\rangle}$$

which yields (substitute into last expressions the transforms of D^R and φ^{ext}):

$$\boxed{\delta\langle\hat{\rho}(\vec{k}, \omega)\rangle = D^R(\vec{k}, \omega)\ \varphi^{ext}(\vec{k}, \omega)}$$

$\Rightarrow$ The system responds at the same wave vector and frequency as the perturbation $\Rightarrow$ linear!

- we need nonlinear response to get frequency doubling and other phenomena where the response is at a different frequency.

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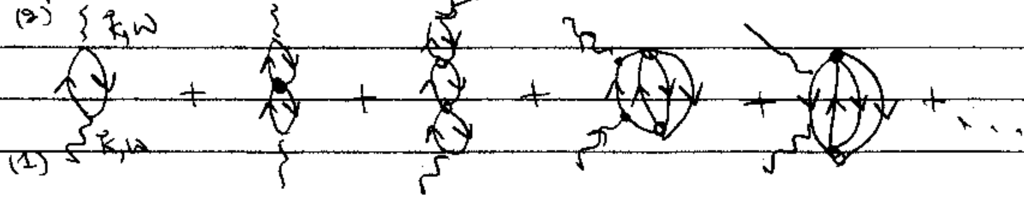
The diagrammatic expansion for $D(\vec{k},\omega)$ in perturbation theory looks like



$\Rightarrow$ "inject" $\vec{k}$ at (1) and remove it at (2).

What are the simplest approximations to the full sum?

i) a single ring $\Rightarrow D_0(\vec{q},\omega)$ 

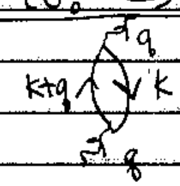
ii) sum of rings $\Rightarrow$ RPA ("random phase approximation")

- in general, replace • by anti-symmetrized interaction
- this is the propagator for σ we found in the large-N expansion!

• Do the single ring first

- gives response function for free Fermi gas (non-interacting system)
- assume spin-½ fermions
- 1 closed loop, no interactions, no equivalent lines, two propagators iG_0 $\Rightarrow (-1)^{\text{interactions}+1} \times (-1)^{\text{closed loops}}$

$$\Rightarrow D_0(\vec{q},q_0) = $$



$$= -2i\int\frac{d^4k}{(2\pi)^4}\,G_0(k+q)\,G_0(k)$$

$\boxed{n_k \equiv \Theta(k_F-|\vec{k}|)}$

$$= -2i\int\frac{d^3k\,dk_0}{(2\pi)^4}\left(\frac{1-n_{\vec{k}+\vec{q}}}{k_0+q_0-\epsilon_{\vec{k}+\vec{q}}+i\eta}+\frac{n_k}{k_0+q_0-\epsilon_{\vec{k}+\vec{q}}-i\eta}\right)\left(\frac{1-n_k}{k_0-\epsilon_k+i\eta}+\frac{n_k}{k_0-\epsilon_k-i\eta}\right)$$

$\boxed{\epsilon_k \equiv \frac{k^2}{2m}}$

$$= 2\int\frac{d^3k}{(2\pi)^3}\left[\frac{(1-n_{\vec{k}+\vec{q}})n_k}{q_0+\epsilon_{\vec{k}}-\epsilon_{\vec{k}+\vec{q}}+i\eta}-\frac{n_{\vec{k}+\vec{q}}(1-n_{\vec{k}})}{q_0+\epsilon_{\vec{k}}-\epsilon_{\vec{k}+\vec{q}}-i\eta}\right]$$

The calculation is carried out in the Fetter and Walecka handout. The function Π^0 is the same as D^0 (up to $\hbar = 1$).

- note how the individual cases for q_0 and $\tilde{q}$ are considered to find the imaginary part.

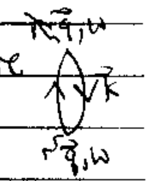
- If we define $\boxed{\tilde{q} \equiv \frac{q}{k_F}}$ and $\boxed{\tilde{\omega} \equiv \frac{q_0 m}{k_F^2}}$, then

$$\boxed{\begin{aligned} \mathrm{Re}\, D_0(\vec{q}, q_0) = \frac{m k_F}{2\pi^2}\Big\{ -1 &+ \frac{1}{2\tilde{q}}\Big[1 - \Big(\frac{\tilde{\omega}}{\tilde{q}} - \frac{\tilde{q}}{2}\Big)^2\Big] \ln\left|\frac{1 + \left(\frac{\tilde{\omega}}{\tilde{q}} - \frac{\tilde{q}}{2}\right)}{1 - \left(\frac{\tilde{\omega}}{\tilde{q}} - \frac{\tilde{q}}{2}\right)}\right| \\ &- \frac{1}{2\tilde{q}}\Big[1 - \Big(\frac{\tilde{\omega}}{\tilde{q}} + \frac{\tilde{q}}{2}\Big)^2\Big] \ln\left|\frac{1 + \left(\frac{\tilde{\omega}}{\tilde{q}} + \frac{\tilde{q}}{2}\right)}{1 - \left(\frac{\tilde{\omega}}{\tilde{q}} + \frac{\tilde{q}}{2}\right)}\right| \Big\} \end{aligned}}$$

and

$$\boxed{\mathrm{Im}\, D_0(\vec{q}, q_0) = \begin{cases} -\dfrac{m k_F}{4\pi \tilde{q}}\Big[1 - \Big(\dfrac{\tilde{\omega}}{\tilde{q}} - \dfrac{\tilde{q}}{2}\Big)^2\Big] & \left|\dfrac{\tilde{q}^2}{2} - \tilde{q}\right| \le \tilde{\omega} \le \dfrac{\tilde{q}^2}{2} + \tilde{q} \\ -\dfrac{m k_F}{4\pi \tilde{q}}\, 2\tilde{\omega} & 0 \le \tilde{\omega} \le \tilde{q} - \dfrac{\tilde{q}^2}{2} \end{cases}}$$

- The imaginary part is what we'll need for the inclusive cross section (e.g. for inelastic electron scattering).



- It counts the ways that an unoccupied state with $|\vec{k}| < k_F$ can be scattered to an unoccupied state $|\vec{k}+\vec{q}| > k_F$ by transferring momentum $\vec{q}$ and energy

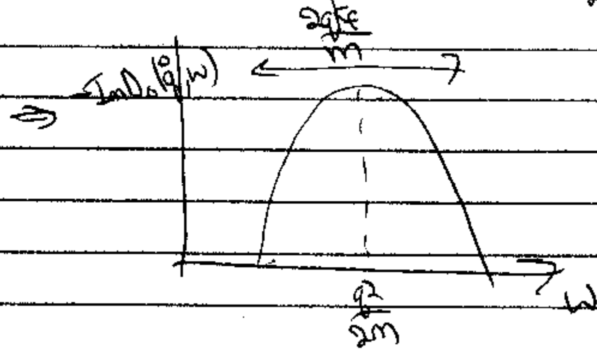
$$\boxed{\begin{aligned} q_0 \equiv \omega = \epsilon_{\vec{k}+\vec{q}} - \epsilon_{\vec{k}} &= \big((\vec{k}+\vec{q})^2 - \vec{k}^2\big)/2m \\ &= \vec{k}\cdot\vec{q}/m + \vec{q}^{\,2}/2m \end{aligned}}$$

- From the F+W result there are basically two regions.

If $q > 2k_F$, then $|\vec{k}+\vec{q}|$ is always greater than k_F.

- Then $\vec{k}\cdot\vec{q}/m$ goes from $-|kq|/m$ to $+|kq|/m$ so the extreme values of ω are when $|\vec{k}| = k_F$

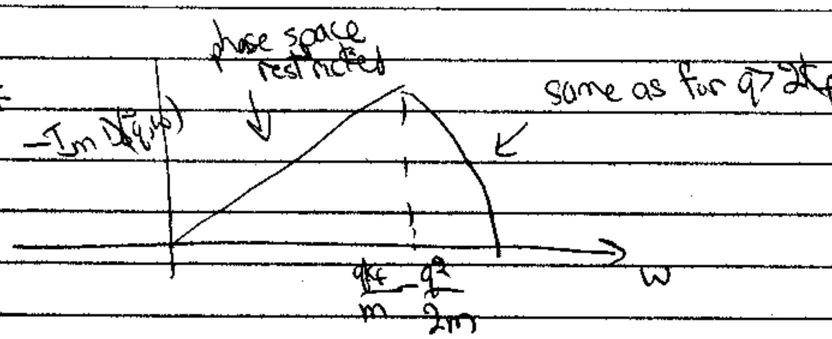
$\Rightarrow \operatorname{Im} D_0$ is nonzero for $\frac{q^2}{2m} - \frac{k_F q}{m} \le \omega \le \frac{q^2}{2m} + \frac{k_F q}{2m}$



- Fixed $\vec{q}$ here
- maximum at $\frac{q^2}{2m}$ because all transverse $|\vec{k}| < k_F$ contribute.

$\Rightarrow$ measures the "Fermi motion" $\Rightarrow k_F$!

- For $q < k_F$



- See the pictures in the FtW handout to understand the phase space restrictions

- We can see these shapes directly in inelastic electron scattering in the quasielastic region.

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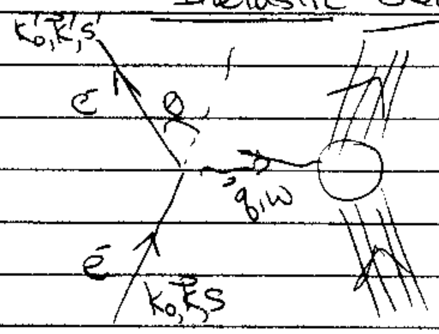
Inelastic Electron Scattering



- scatter an electron from plane wave $|\vec{k}s\rangle$ to $|\vec{k}'s'\rangle$
- interaction between lepton and nuclear four-currents

- Here just keep the charge-charge interaction

$$\boxed{\hat{H}^{ext} = -e^2 \iint \frac{\hat{\rho}_{el}(\vec{x}')\,\hat{\rho}_{N}(\vec{x})}{|\vec{x}-\vec{x}'|}\, d^3x\, d^3x'}$$

with $\hat{\rho}_{el}(\vec{x}')$ the charge density operator for the scattered electron while $\hat{\rho}_N(\vec{x})$ is the corresponding operator for the target nucleus (mostly the proton density)

- The interaction is Coulomb $1/|\vec{x}-\vec{x}'|$ but the discussion would go through similarly with another interaction (e.g. $\delta^3(\vec{x}-\vec{x}')$).

- For the electron,

$$\boxed{\langle \vec{k}'s' | \hat{\rho}_{el}(\vec{x}') | \vec{k}s\rangle = \underbrace{\frac{1}{\Omega}}_{\text{normalization of plane waves}} e^{i\vec{q}\cdot\vec{x}'}\, \underbrace{U^{\dagger}_{s'}(\vec{k}')\, U_s(\vec{k})}_{\text{Dirac 4-component spinors}}}$$

where $\vec{q} \equiv \vec{k}-\vec{k}'$ is the momentum transfer.

Ultrarelativistic energies

$\Rightarrow \boxed{k_0^2 - \vec{k}^2 = m_e^2 \doteq 0 = k_0'^2 - \vec{k}'^2}$. Since $\omega = k_0' - k_0$,

$$\boxed{\begin{aligned} q^2 = \omega^2 - \vec{q}^{\,2} &= (k_0'-k_0)^2 - (\vec{k}-\vec{k}')^2 = (k_0^2-\vec{k}^2) + (k_0'^2-\vec{k}'^2) - 2k_0k_0' - 2\vec{k}\cdot\vec{k}' \\ &= -2|\vec{k}||\vec{k}'|(1+\cos\theta) = -4|\vec{k}||\vec{k}'|\sin^2\theta/2 \leq 0 \end{aligned}}$$

$\Rightarrow$ space-like four momentum

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- So the experiment is like the resonant absorption of light but instead of $|\vec{q}|=\omega$, we can have any $|\vec{q}|>\omega$

- Fermi's Golden rule yields the cross section:

exact Heisenberg states of target

$$\boxed{d^2\sigma = \frac{2\pi}{\hbar}\underbrace{\frac{1}{2}\sum_s\sum_{s'}}_{\text{sum over final spin, average over initial}}\sum_n \delta\left[\hbar\omega - (E_n^N - E_0^N)\right]\left|\langle \Psi_n^N|\hat{\rho}_{ch}(-\vec{q})|\Psi_0^N\rangle\right|^2 \frac{\Omega\, d^3k'}{(2\pi)^3} \times \left(\frac{4\pi}{q^2}\right)^2\left(\frac{e^2}{\Omega}\right)^2 \left|u^\dagger_{s'}(\vec{k}')u_s(\vec{k})\right|^2 \left(\frac{c}{\Omega}\right)^{-1}}$$

normalization of plane waves — flux

The $\frac{\hat{\rho}_{ch}(-\vec{q})}{q^2}$ arises from

$$\boxed{\int e^{i\vec{q}\cdot\vec{x}'}\frac{1}{|\vec{x}-\vec{x}'|}\hat{\rho}_{ch}(\vec{x})\, d^3x\, d^3x' = \int d^3x\, e^{i\vec{q}\cdot\vec{x}}\Big(\underbrace{\int e^{-i\vec{q}\cdot(\vec{x}-\vec{x}')}\frac{1}{|\vec{x}-\vec{x}'|}d^3x'}_{\frac{4\pi}{q^2}}\Big)\hat{\rho}_{ch}(\vec{x}) = \frac{4\pi}{q^2}\hat{\rho}_{ch}(-\vec{q})}$$

- The spin sum yields $\boxed{\frac{1}{2}\sum_s\sum_{s'}\left|u^\dagger_{s'}(\vec{k}')u_s(\vec{k})\right|^2 = \cos^2\tfrac{1}{2}\theta}$

$$\Rightarrow \boxed{\frac{1}{\sigma_m}\frac{d^2\sigma}{d\Omega\, d\epsilon'} = \sum_n\left|\langle \Psi_n^N|\hat{\rho}_{ch}(-\vec{q})|\Psi_0^N\rangle\right|^2 \delta\left(\omega - (E_n - E_0)\right)}$$

with $\boxed{\sigma_m \equiv \left(\frac{e^2}{\hbar c}\right)^2\frac{4k'^2}{q^4}\cos^2\frac{\theta}{2} \underset{q^2\gg\omega^2}{=} \left(\frac{e^2}{\hbar c}\right)^2\frac{\cos^2\theta/2}{4k^2\sin^4\theta/2}}$

this is why large q count rates are low!

We can relate this directly to the imaginary part of the correlator:

$$\sum_n |\langle \Psi_n^N | \hat{\rho}_{ch}(-\vec{q}) | \Psi_0^N \rangle|^2 \, \delta(\omega - (E_n - E_0))$$

$$= -\frac{1}{\pi} \mathrm{Im} \sum_n \left[\frac{\langle \Psi_0^N | \hat{\rho}^\dagger_{ch}(-\vec{q}) | \Psi_n^N \rangle \langle \Psi_n^N | \hat{\rho}_{ch}(-\vec{q}) | \Psi_0^N \rangle}{\omega - (E_n - E_0) + i\eta} \right.$$

doesn't contribute for $\omega > 0$ $\rightarrow$

$$\left. - \frac{\langle \Psi_0^N | \hat{\rho}^\dagger_{ch}(\vec{q}) | \Psi_n^N \rangle \langle \Psi_n^N | \hat{\rho}_{ch}(\vec{q}) | \Psi_0^N \rangle}{\omega + (E_n - E_0) \mp i\eta} \right]$$

$$= -\frac{1}{\pi} \mathrm{Im}\, D_{ch}(\vec{q}, \vec{q}; \omega) = -\frac{1}{\pi} \mathrm{Im}\, D^R_{ch}(\vec{q}, \vec{q}; \omega)$$

where

$$i D_{ch}(x, y) = \langle \Psi_0^N | T[\hat{\rho}_{ch}(x) \hat{\rho}_{ch}(y)] | \Psi_0 \rangle$$

$$= \int \frac{d^3q}{(2\pi)^3} \int \frac{d^3q'}{(2\pi)^3} \int \frac{d\omega}{2\pi} e^{i\vec{q}\cdot\vec{x}} e^{-i\omega(t_x - t_y)} e^{-i\vec{q}'\cdot\vec{y}} \, i D_{ch}(\vec{q}, \vec{q}'; \omega)$$

In a uniform system,

$$D_{ch}(\vec{q}, \vec{q}'; \omega) = V \delta_{\vec{q}\vec{q}'} D_{ch}(\vec{q}, \omega)$$

So we can approximate the cross section for a real nucleus as

$$\frac{1}{\sigma_m} \frac{d^2\sigma}{d\Omega' d\epsilon'} \doteq -\frac{V}{\pi} \mathrm{Im}\, D_{ch}(\vec{q}, \omega) \doteq -\frac{3\pi Z}{k_F^3} \mathrm{Im}\, D_{ch}(\vec{q}, \omega)$$

- We'll try the single ring approximation $D \rightarrow D^0$
  - account for interactions with m^* or $\bar{\epsilon}$.

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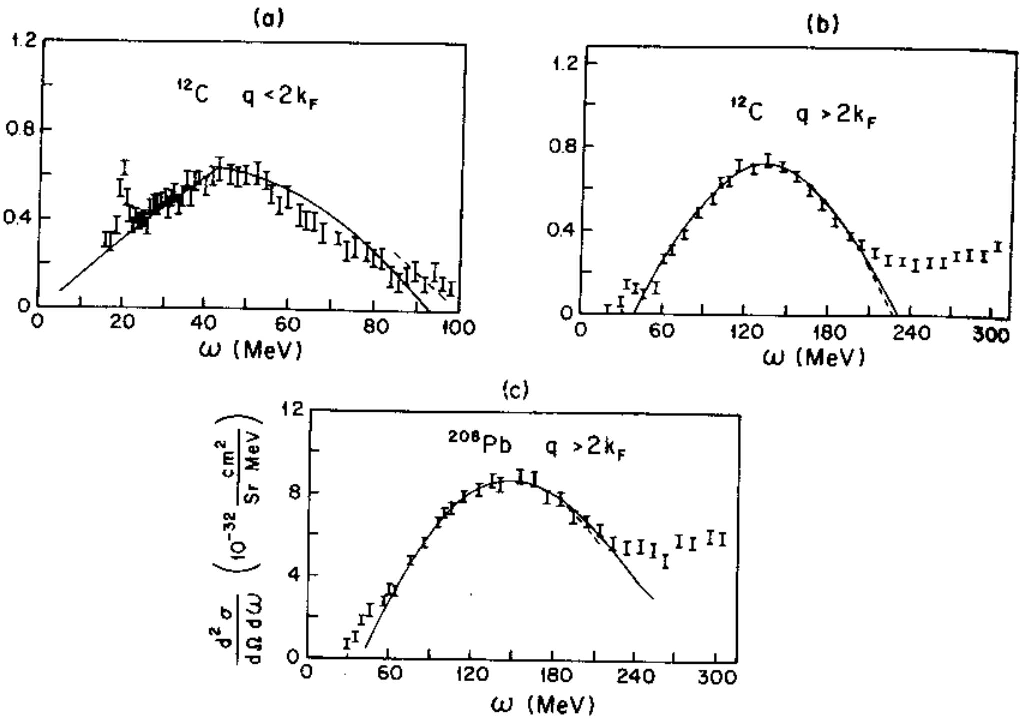

Fig. 5.19 Inclusive electron scattering from atomic nuclei. The inclusive cross sections as a function of energy loss ω for ^{12}C at incident energy $E = 198.5$ MeV and scattering angle $\theta = 135°$ in (a) correspond to momentum transfer q less than $2k_F$ whereas the $E = 500$ MeV data at $\theta = 60°$ shown for ^{12}C in (b) and ^{208}Pb in (c) correspond to momentum transfer greater than $2k_F$. The solid curves correspond to the Fermi gas response functions described in the text. The low energy data is from Leiss and Taylor (1963) and the high energy data from Moniz *et al.* (1971).

- The nucleon self-energy is strongly momentum dependent
  - ⇒ a nucleon in the Fermi sea feels strong attraction while a nucleon above the Fermi sea feels weaker attraction
  - ⇒ model with either m^* or binding correction $\bar{\epsilon} \doteq \frac{1}{2}(\Sigma_{in} - \Sigma_{out})_{avg}$.
    - shifts curves without distorting shape
- In figure
  - ^{12}C fit with $k_F = 1.19\,fm^{-1}$ and $m^*/m = 0.7$ in a), $k_F = 1.14\,fm^{-1}$, $\bar{\epsilon} = 25$ MeV in b)
  - ^{208}Pb fit with $k_F = 1.36\,fm^{-1}$ and $\bar{\epsilon} = 44$ MeV
  - $\bar{\epsilon}$'s consistent with average single particle energies, Pions or Δ's at high ω.