Nonlocality of Potentials in One Dimension

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Abstract

To better understand the nature of the non-locality of $V_{\mathrm{low}\,k}$ and SRG potentials in three dimensions, in these internal notes we look at simple model potentials in one dimension.

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In the standard implementations of the SRG and $V_{\mathrm{low}\,k}$ RG, the potential becomes increasingly non-local as the evolution proceeds. At the same time, we get decoupling: matrix elements between low and high momenta are increasingly suppressed, with all of the favorable consequences. We would like to understand whether decoupling is necessarily accompanied by increased non-locality. Naively this is not the case; one could imagine a potential that becomes more diagonal in each partial wave but which stays a function only of the momentum transfer $\mathbf{q} = \mathbf{k} - \mathbf{k}'$ (here $\mathbf{k}$ and $\mathbf{k}'$ are the final and initial relative momenta). So, is this possible and can we prove it one way or another? If not possible, then can we characterize the non-locality in a simple way, such as a form factor independent of dynamics times a local potential? (Or, does non-locality arise because we evolve separately in each partial wave; is there a different choice of SRG generator that maintains locality?)

These questions get complicated in three-dimensions by working in a single partial-wave at a time. The S-wave projection of a local potential doesn't look local anymore and we can't test locality only with the S-wave. (E.g., imagine two potentials, one fully local with all the partial-wave projections and another defined to be zero except for the S-wave projection; same S-waves but one is local and the other is non-local.) Working with the full three-dimensional potential is complicated (and it's hard to represent efficiently). So we turn to one space dimension, where life is simpler but the basic principles should still be testable. As illustrated below, the clear signature of a local potential in one dimension is that it is constant along a line parallel to the main diagonal (where $k = k'$). The analogous requirement to keeping all partial waves to test locality is to keep both even and odd parts of the potential. (It is demonstrated below that the even part alone is not local.)

We start with a simple one-dimensional non-local potential,

$$\langle x|V_\beta|x'\rangle \equiv V_\beta(x,x') = V_0\, e^{-[(x+x')/2]^2/2\alpha^2} \frac{1}{\sqrt{2\pi\beta^2}} e^{-(x-x')^2/2\beta^2} \;, \tag{1}$$

which in the limit $\beta \to 0$ becomes a local potential:

$$\lim_{\beta\to 0} V_\beta(x,x') = V_0\, e^{-x^2/2\alpha^2}\, \delta(x-x') \;. \tag{2}$$

It is evident that α determines the range of the interaction and β determines the range of the non-locality. (What is not immediately clear is how to best assign factors of 2 or $\sqrt{2}$ or whatever to these ranges so that the length scale is manifested.) The analogous potential in three dimensions served as a test case for the density matrix expansion (DME).

Our conventions are:

$$1 = \int_{-\infty}^{+\infty} dx\, |x\rangle\langle x| = \int_{-\infty}^{+\infty} dk\, |k\rangle\langle k| \;, \tag{3}$$

and

$$\langle k|x\rangle = \frac{e^{-ikx}}{\sqrt{2\pi}} \;, \qquad \langle x|k\rangle = \frac{e^{ikx}}{\sqrt{2\pi}} \;, \tag{4}$$

so that

$$\langle k|V|k'\rangle \equiv V(k,k') = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx'\, e^{-ikx}\, V(x,x')\, e^{ik'x'} \tag{5}$$

and

$$\langle x|V|x'\rangle \equiv V(x,x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dk'\, e^{ikx}\, V(k,k')\, e^{-ik'x'} \;. \tag{6}$$

The variables $u = (x + x')/2$ and $v = x - x'$ are natural to use with our gaussian potential, with conjugate momentum variables

$$q \equiv k - k' \qquad p \equiv (k + k')/2 \; . \tag{7}$$

(Note: The analogous definition for p given in three dimensions elsewhere does not have the 1/2 factor.) The Jacobians from $\{x, x'\}$ to $\{u, v\}$ and from $\{k, k'\}$ to $\{q, p\}$ are unity.

[For completeness, we note that x and x' are relative coordinates, so we've used the fact that the free-space potential is translationally invariant, which means that the potential is independent of the center-of-mass coordinate (so the original four coordinates for a two-to-two interaction kernel are reduced to two). Similarly, k and k' are relative momenta, having used that Galilean invariance means there is no dependence on the total momentum.]

Transforming to momentum representation, we find a factorization in the p and q variables (we'll use V when the arguments are k and k' and $\widetilde{V}$ when the arguments are p and q),

$$\widetilde{V}_\beta(p, q) = V_0 \, \frac{\alpha}{\sqrt{2\pi}} \, e^{-\alpha^2 q^2/2} e^{-\beta^2 p^2/2} = V_\beta(p + q/2, p - q/2) \; . \tag{8}$$

It is manifest that the local limit, which means β set to zero, yields a potential independent of p. This will be the most obvious signature of locality in this representation.

We can also think about a mixed representation, where we make a "velocity expansion" of $V(x, x')$ to some order. The analog of the usual three-dimensional discussion writes:

$$V(x, x') = V_L(x, \frac{d}{dx}) \delta(x - x') \; . \tag{9}$$

The three-d structure of $V_L(x, \frac{d}{dx})$ was worked out in the non-relativistic case by Okubo and Marshak in Ann. Phys. **4**, 166 (1958).

Here are pictures for the local limit. We plot on the left the potential as a function of k and k' as a color contour plot. (We've plotted positive k and k' only, but the potential is defined over the entire $\{k, k'\}$ plane, which is used to interpolate for the plots on the right. Alternatively, we could have worked with the odd and even potentials for $k > 0$, $k' > 0$.) Independence of p means that the potential is constant along any 45° line parallel to the main diagonal, which corresponds to $k = k' +$ constant (i.e., to constant q). If we change the value of α, the fall-off in the q direction, which is perpendicular to the main diagonal, changes with larger α causing a faster fall-off (from $e^{-\alpha^2 q^2/2}$).
On the right are plots of the potential as a function of p for several fixed values of q. Independence of p is manifest. (Note: we've interpreted distances to be in fm and momenta to be in fm^{-1}.) We might want to divide by $V(0, q_0)$ to normalize the p dependence.

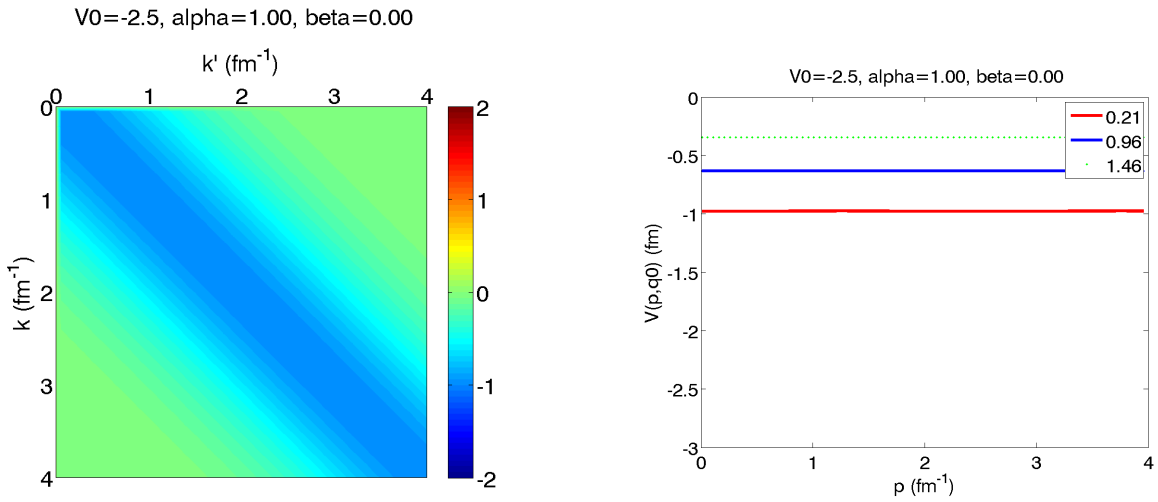

FIG. 1: Potential from Eq. (8) with $\alpha = 1$ and $\beta = 0$. On the right are curves of the potential as a function of p at several fixed q_0 values. [*Should rescale the y-axis to 1 on the right.*]

From Eq. (8) we see that a non-locality of β translates into a fall-off in p at fixed q with momentum width $1/\beta$. We can see this for $\beta = 0.5$ and $\beta = 1.0$:

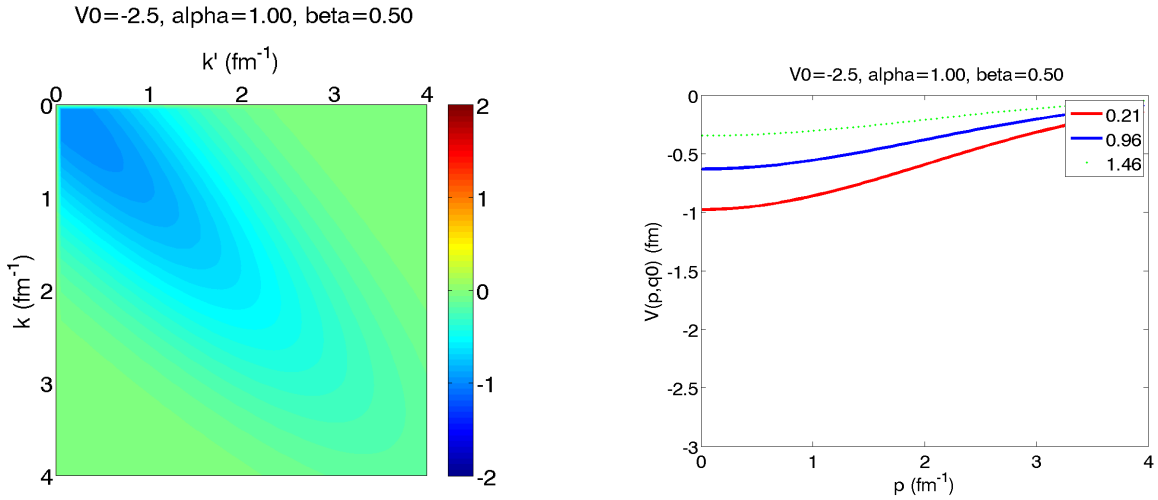

FIG. 2: Potential starting from Eq. (8) with $\alpha = 1$ and $\beta = 0.5$. On the right are curves of the potential as a function of p at several fixed q_0 values.

This potential is particularly simple since the fall-off is the same for any value of q, so looking along any diagonal in the contour plots leads to the same conclusion. This will not be true in general. (Note again that although we only plot positive k and k', one should imagine extending it to negative values.) Again, it might be more useful to normalize the curves to their value at $V(0, q_0)$.

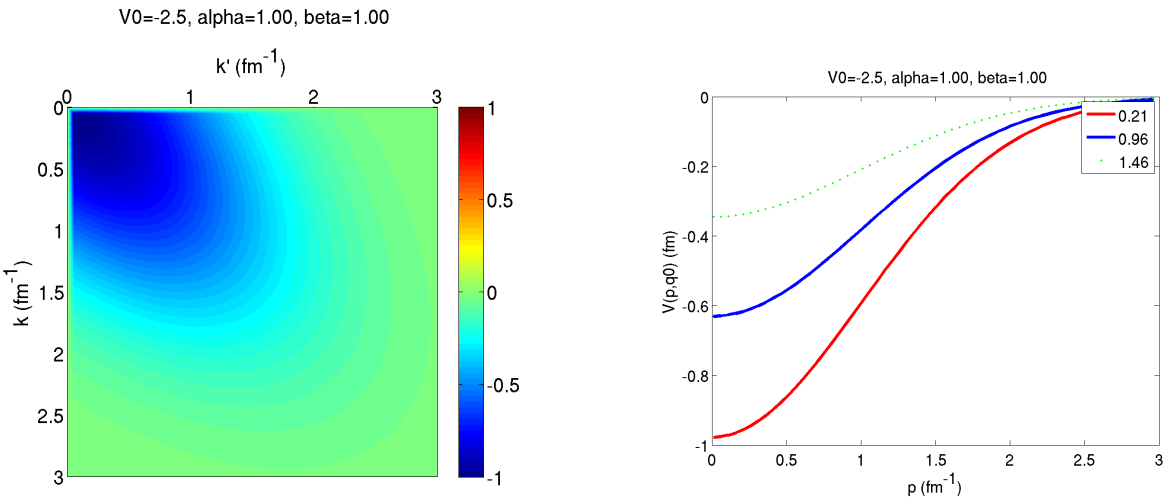

FIG. 3: Potential starting from Eq. (8) with $\alpha = 1$ and $\beta = 1$. On the right are curves of the potential as a function of p at several fixed q_0 values.

Now we can look at the some SRG-evolved results. These are for $\beta = 0$ evolved to $\lambda = 4.0$, 3.0, 2.0, 1.5, and 1.0.

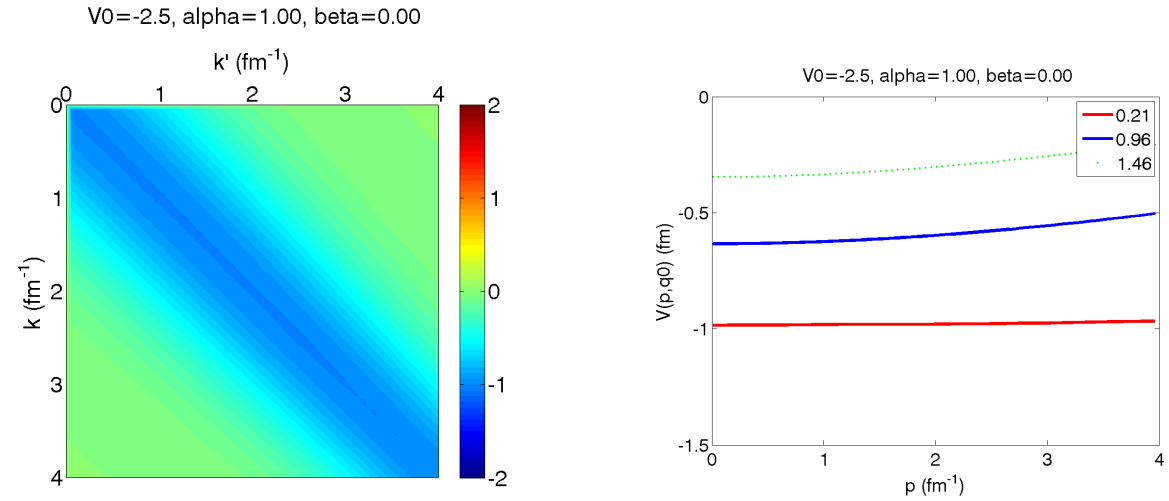

FIG. 4: SRG potential starting from Eq. (8) with $\alpha = 1$ and $\beta = 0$, evolved to $\lambda = 4$. On the right are curves of the potential as a function of p at several fixed q_0 values.

When we do the SRG evolution, we evolve separately the even and odd parts of the potential:

$$V_s^{(+)}(k,k') = \frac{1}{2}[V_s(k,k') + V_s(k,-k')] \,, \tag{10}$$

$$V_s^{(-)}(k,k') = \frac{1}{2}[V_s(k,k') - V_s(k,-k')] \,. \tag{11}$$

However, since taking $k' \to -k'$ is the same as $q \to 2p$ and $p \to q/2$, a local potential will

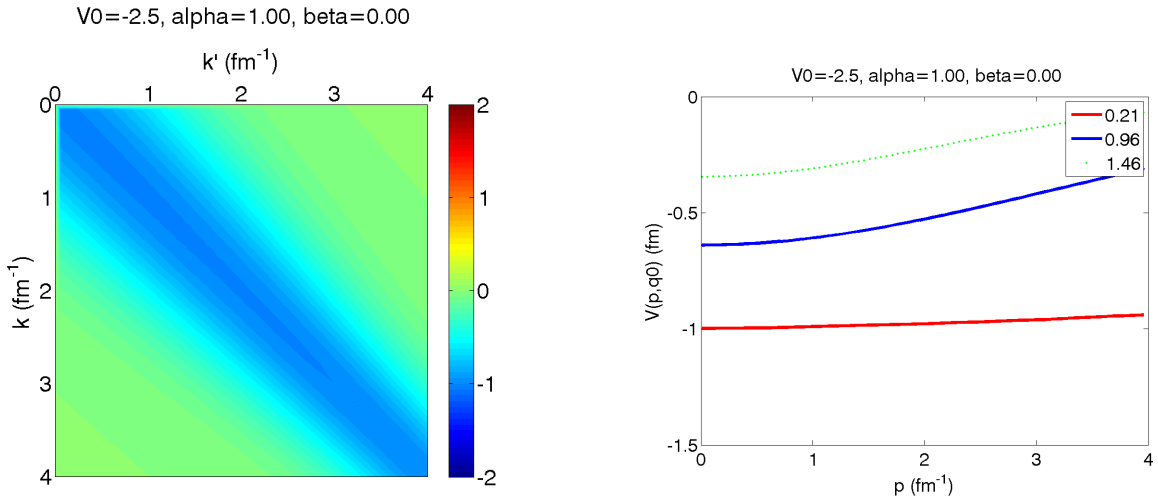

FIG. 5: SRG potential starting from Eq. (8) with $\alpha = 1$ and $\beta = 0$, evolved to $\lambda = 3$. On the right are curves of the potential as a function of p at several fixed q_0 values.

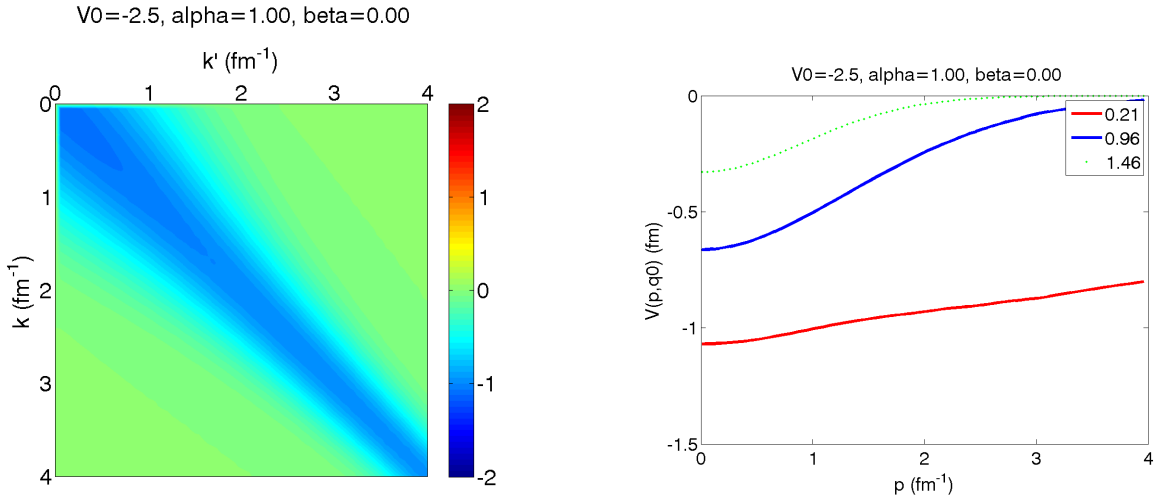

FIG. 6: SRG potential starting from Eq. (8) with $\alpha = 1$ and $\beta = 0$, evolved to $\lambda = 2$. On the right are curves of the potential as a function of p at several fixed q_0 values.

not look local if we use only $V_s^{(+)}$ or $V_s^{(-)}$. For example, in the local limit,

$$V_\beta^{(+)}(k,k') \xrightarrow{\beta\to 0} V_0 \frac{\alpha}{\sqrt{2\pi}} \left[e^{-\alpha^2(k-k')^2/2} + e^{-\alpha^2(k+k')^2/2}\right]/2 \,, \tag{12}$$

which is not independent of p.

Let's compare to what we expect from basic SRG considerations. From the decoupling discussions [9, 10], we expect a fall off of the matrix elements with s or λ to go roughly like (with $\hbar^2/m = 1$)

$$\langle k|V_s|k'\rangle \approx \langle k|V_{s=0}|k'\rangle\, e^{-s(\epsilon_k-\epsilon_{k'})^2} = \langle k|V_{s=0}|k'\rangle\, e^{-[(k^2-k'^2)/\lambda^2]^2} \,. \tag{13}$$

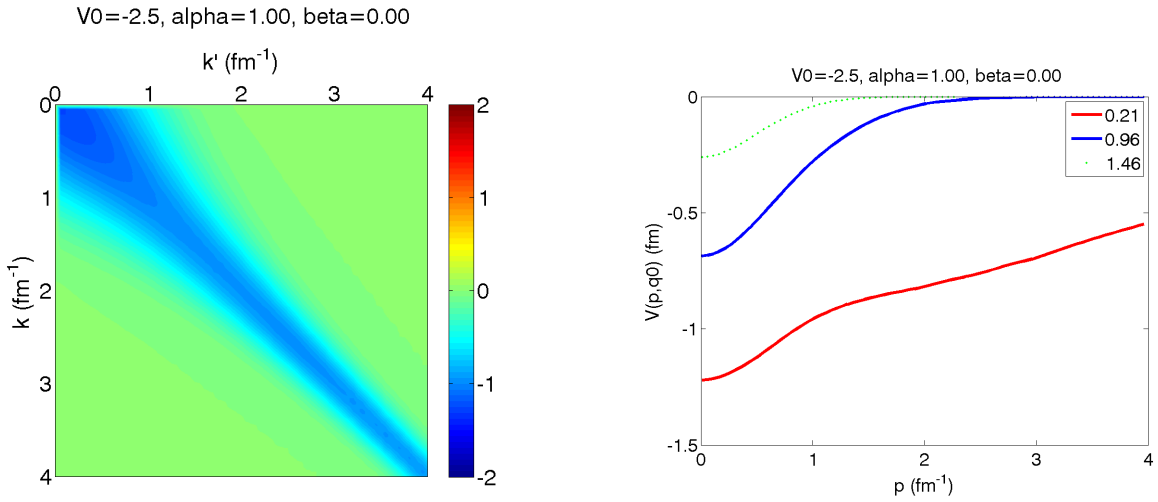

FIG. 7: SRG potential starting from Eq. (8) with $\alpha = 1$ and $\beta = 0$, evolved to $\lambda = 1.5$. On the right are curves of the potential as a function of p at several fixed q_0 values.

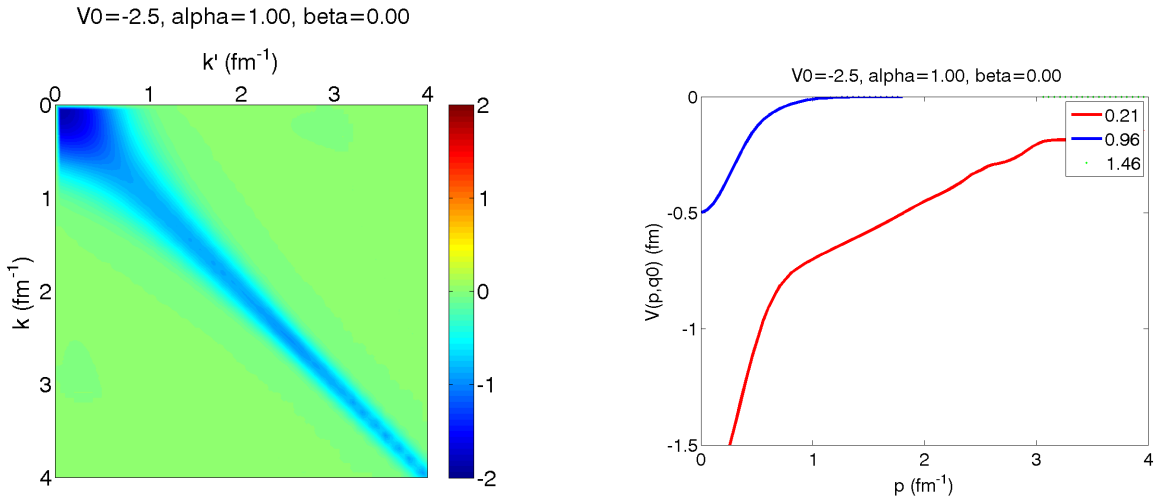

FIG. 8: SRG potential starting from Eq. (8) with $\alpha = 1$ and $\beta = 0$, evolved to $\lambda = 1$. On the right are curves of the potential as a function of p at several fixed q_0 values.

If we take the initial potential to be local and therefore independent of p, then the piece to compare to a gaussian fall-off in p with non-locality β is

$$e^{-[(k^2-k'^2)/\lambda^2]^2} = e^{-(2pq/\lambda^2)^2} = e^{-(\sqrt{8}q/\lambda^2)^2 p^2/2} \Longleftrightarrow e^{-\beta^2 p^2/2}\,, \tag{14}$$

where we've used $k^2 - k'^2 = 2pq$. So for the evolved potential there is an *effective* $\beta \sim \sqrt{8}q/\lambda^2$ that is q dependent. This dependence looks consistent with the figures above for all λ's except for small q_0 at small λ.

What are the implications?

- As q_0 increases, we have greater non-locality. There isn't a simple $\beta \sim 1/\lambda$ relation, as we have suggested in other places.

- The high q part becomes the most non-local while low q is more local. Here are some numbers (here $p_{1/2}$ is the p value for 50% fall-off):

q_0	λ	β	$p_{1/2}$
0.5	2.5	0.23	5.20
0.5	2.0	0.35	3.33
0.5	1.5	0.63	1.87
1.0	2.5	0.45	2.60
1.0	2.0	0.71	1.67
1.0	1.5	1.26	0.94
1.5	2.5	0.68	1.73
1.5	2.0	1.06	1.11
1.5	1.5	1.89	0.62
2.0	2.5	0.91	1.30
2.0	2.0	1.41	0.83
2.0	1.5	2.51	0.47

  If we convert to fm's, the non-locality ranges seem just fine for the low-q region.

- Note how rapidly the non-locality grows with decreasing λ. Is there a simple relation?

We have credited decoupling with being the key feature of the SRG evolution that simplifies (softens) the potentials. But the trend toward diagonalization is associated (at least naively) with a rapid fall-off in q, with no explicit consequence for the p dependence. Is there any reason we can't have a more diagonal potential that is independent of p? There may be constraints that make it impossible to do both, which reflect the impossibility (?) of describing the S-wave phase shifts without a fairly hard core (even with supersoft potentials!). [*Try enforcing a one-D evolution with the independence of p used to constrain the generator.*] For the standard SRG, if the kinetic energy dominates, as in all examples considered so far, then the fall-off is like Eq. (12) and the non-locality has the same form as just discussed. What about for $V_{\text{low}\,k}$?

What is the one-dimensional analog of the construction Robert proposed for S-wave three-dimensional potentials? He suggested

$$V(k,k') \stackrel{?}{=} e^{-(k^2-k'^2)^2/\lambda^4} V_{\text{local}}(k,k') \;, \tag{15}$$

where

$$V_{\text{local}}(k,k') \equiv \frac{1}{kk'}\left[\frac{(k+k')^2}{4}\,V\left(\frac{k+k'}{2},\frac{k+k'}{2}\right) - \frac{(k-k')^2}{4}\,V\left(\frac{k-k'}{2},\frac{k-k'}{2}\right)\right] \;. \tag{16}$$

The condition $V(k,k') = V_{\text{local}}(k,k')$ for all k and k' is a necessary condition for the S-wave part of a local potential. The idea is that the non-locality of an SRG-evolved potential will have the form of the prefactor in (14). Note that this is the same decomposition as the suggested approximation in Eq. (12) with V_{local} the same as the $s = 0$ potential (which we chose to be local).

An empirical test of how well this decomposition works is to see if $e^{(k^2-k'^2)^2/\lambda^4}\,V_s(k,k')$ passes the tests for locality. Here are some pictures comparing the p dependence before and after applying this factor:

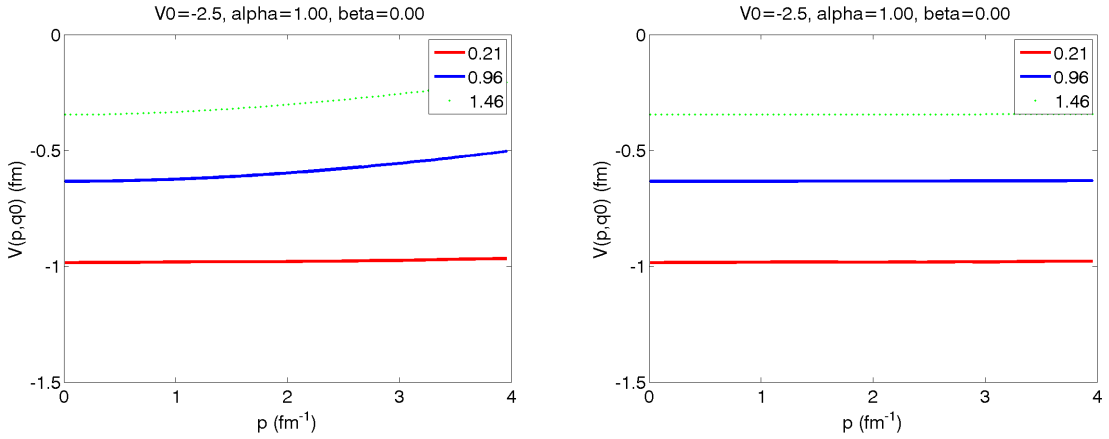

FIG. 9: SRG potential starting from Eq. (8) with $\alpha = 1$ and $\beta = 0$, evolved to $\lambda = 4$. On the left are curves of the potential as a function of p at several fixed q_0 values while on the right it has the factor $e^{(k^2-k'^2)^2/\lambda^4}$.

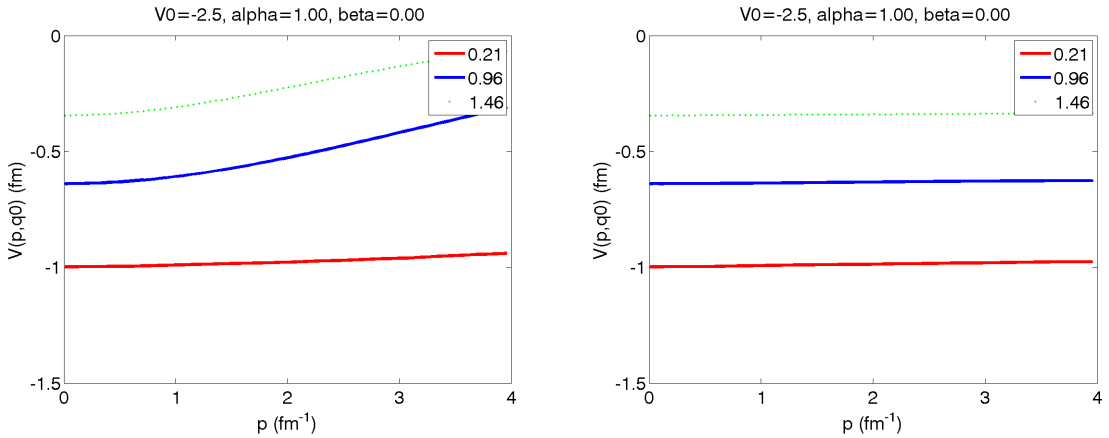

FIG. 10: SRG potential starting from Eq. (8) with $\alpha = 1$ and $\beta = 0$, evolved to $\lambda = 3$. On the left are curves of the potential as a function of p at several fixed q_0 values while on the right it has the factor $e^{(k^2-k'^2)^2/\lambda^4}$.

The correction is very effective in flattening out the p dependence for $\lambda = 4$, 3, and 2. Below this, it increasingly fails. Should this bother us, given that $\alpha = 1$?

Can the potential be approximately decomposed as:

$$\widetilde{V}(p,q) = V_{\rm local}(q) + V_{\rm local}(p) \ , \tag{17}$$

(a minus sign would lead to $V(0,0) = 0$)? Note that this is the form of the even part of a local potential (up to factors of 2 in our definitions of p and q).

These discussions are reminiscent of the "diagonal parametrization" discussed in the book by Kehrein [8]. Translating to our notation, the ansatz is:

$$V_s(k,k') = V_s(\overline{kk'})\, e^{-s(\epsilon_k - \epsilon'_k)^2} \ , \tag{18}$$

with

$$V_s(p) \equiv V_s(p,p) \tag{19}$$

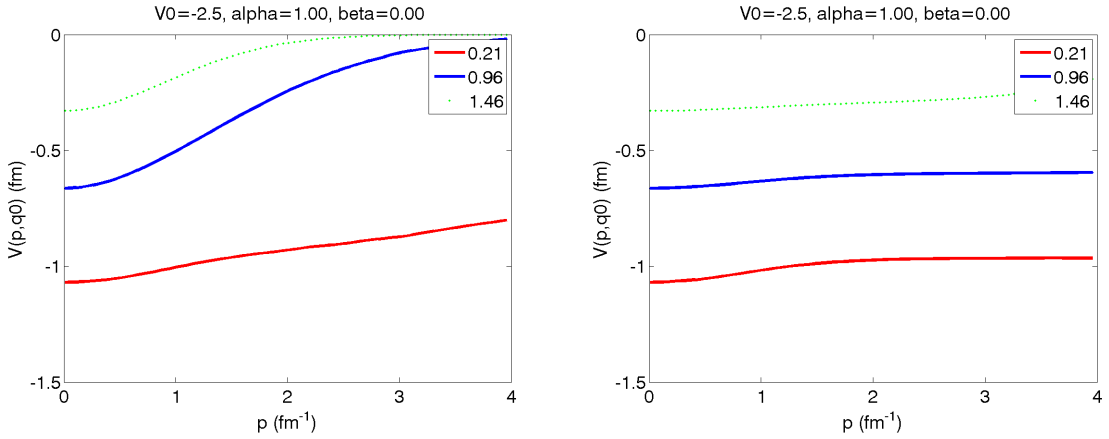

FIG. 11: SRG potential starting from Eq. (8) with $\alpha = 1$ and $\beta = 0$, evolved to $\lambda = 2$. On the left are curves of the potential as a function of p at several fixed q_0 values while on the right it has the factor $e^{(k^2-k'^2)^2/\lambda^4}$.

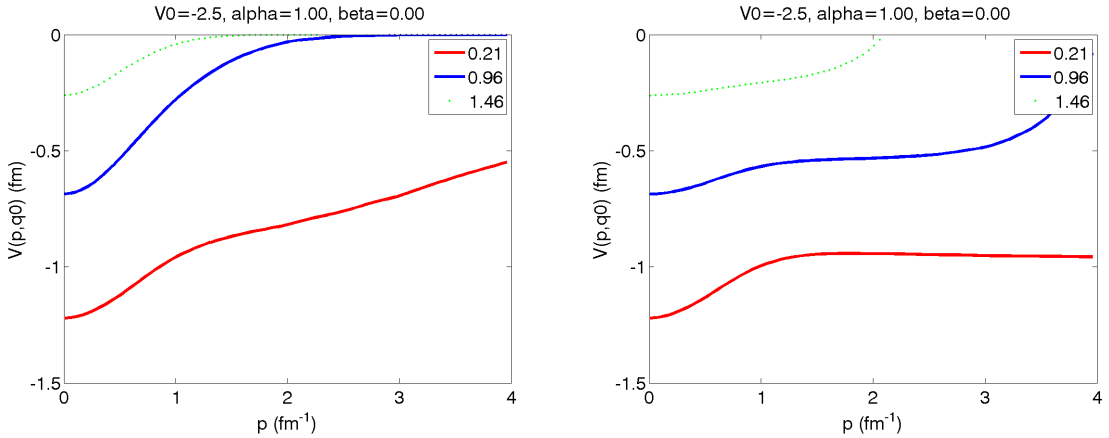

FIG. 12: SRG potential starting from Eq. (8) with $\alpha = 1$ and $\beta = 0$, evolved to $\lambda = 1.5$. On the left are curves of the potential as a function of p at several fixed q_0 values while on the right it has the factor $e^{(k^2-k'^2)^2/\lambda^4}$.

and where

$$\epsilon_{\overline{kk'}} \equiv \frac{1}{2}(\epsilon_k + \epsilon_{k'}) \tag{20}$$

is the arithmetic mean. Evidently this ansatz is meant to be particular to the special case of a constant initial potential in momentum representation. Note that in one dimension $V_s(p)$ only keeps track of changes down the main diagonal, so it has p dependence only. Be careful of jumping to three dimensions without thinking, since $k - k'$ there is *not* the magnitude of the momentum transfer $\mathbf{k} - \mathbf{k}'$.

Robert has observed that if we are trying to devise an SRG transformation that preserves locality, not only does V_s have to remain local, but the change in T has to be absorbed into a local V_s. To try: an SRG equation with the constraint on G_s such that V_s is independent

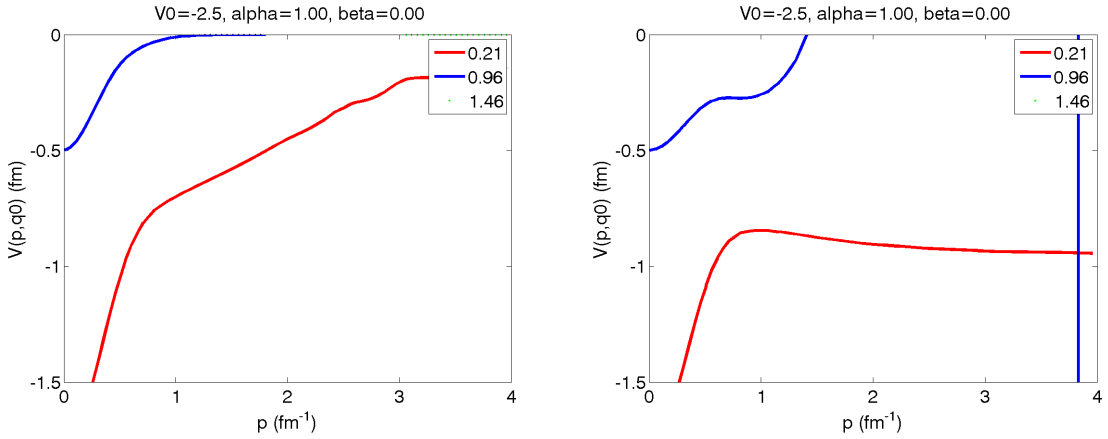

FIG. 13: SRG potential starting from Eq. (8) with $\alpha = 1$ and $\beta = 0$, evolved to $\lambda = 1.0$. On the left are curves of the potential as a function of p at several fixed q_0 values while on the right it has the factor $e^{(k^2-k'^2)^2/\lambda^4}$.

of p for *all* k,k'. Do an infinitesimal step and see what happens.

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