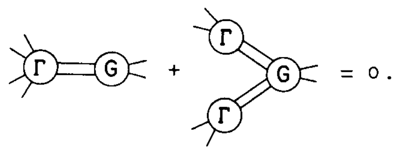


(2.196b)

The first equation shows that the matrix of four-point vertex functions is the inverse of the matrix of two-particle Green's functions. In the absence of symmetry-breaking, all $\mathcal{G}$'s and Γ's must have equal numbers of incoming and outgoing lines and the pairs of lines in these equations represent all combinations of aligned or anti-aligned propagators consistent with this restriction. By the same arguments used for Eqs. (2.186), Eqs. (2.196) show that the Γ's are composed of two-particle irreducible amputated connected diagrams and that the n-particle Green's functions is obtained from tree graphs involving fewer particle Green's functions and $2n$-point and fewer point vertex functions. It is an instructive exercise to derive these equations in detail (see Problem 2.12).

2.5 STATIONARY-PHASE APPROXIMATION AND LOOP EXPANSION

Whereas perturbation theory is valuable for the formal developments of Section 2.3 and is directly applicable to a limited class of physical problems with weak interactions characterized by a small expansion parameter, many problems of physical interest involve strong many-body interactions for which a perturbation expansion in the interaction strength is inappropriate. A natural approach for such problems is to reorganize the perturbation expansion into a series in powers of a new small parameter. We have already seen examples of such infinite resummations of perturbation theory in Section 2.4, where solution of Dyson's equations with some finite set of self energy diagrams or, in general, calculation of Green's functions with any finite set of diagrams for n-particle irreducible vertex functions Γ resums an infinite number of terms of the original perturbation series. Other physically motivated resummations will be presented subsequently for specific systems. In view of the general lack of mathematical control on the convergence properties of the original series and the obvious ambiguity associated with regrouping divergent series, any such resummations must be understood ultimately on physical grounds.

In this section, we consider a specific systematic regrouping of terms obtained by applying the stationary-phase approximation to the functional integral for the partition function. This regrouping will be seen to be ordered in the number of loops occuring in the Feynman diagrams. For certain systems, this approximation generates an asymptotic expansion in a small parameter, such as $\hbar$ or the number of degrees of freedom associated with an internal symmetry.

We first review the stationary-phase approximation in the case of a one-dimensional integral, and then generalize to the cases of the Feynman path integral and the partition function for many-particle systems. As usual, the primary emphasis will be on the essence of the method and its physical interpretation, rather than mathematical rigor.

ONE-DIMENSIONAL INTEGRAL

The stationary-phase approximation, also referred to as the saddle point approximation or method of steepest descent, is a method for developing an asymptotic expansion in powers of $\frac{1}{\ell}$ for an integral of the form

$$I(\ell) = \int_{-\infty}^{\infty} dt e^{-\ell f(t)} \tag{2.197}$$

where ℓ is a real parameter and in general $f(t)$ is an analytic function in the complex t-plane.

For simplicity, we will first consider the special case of a real function $f(t)$ with an absolute minimum at $t = t_0$. As ℓ increases, the integral becomes sharply peaked around the point t_0, and the dominant contribution to the integral arises from the vicinity of t_0. Expanding $f(t)$ around t_0, recognizing the fact that $f'(t_0) = 0$ and $f''(t_0) > 0$ since t_0 is the minimum, the integral may be rewritten:

$$\begin{aligned} I(\ell) &= e^{-\ell f_0} \int_{-\infty}^{\infty} dt\, e^{-\frac{1}{2} f_0''(t-t_0)^2 - \ell \sum_{n=3}^{\infty} \frac{(t-t_0)^n}{n!} f_0^{(n)}} \\ &= \sqrt{\frac{2\pi}{\ell f_0''}}\, e^{-\ell f_0} \int_{-\infty}^{\infty} \frac{d\tau}{\sqrt{2\pi}} e^{-\frac{\tau^2}{2} - \sum_{n=3}^{\infty} \frac{\tau^n}{n!} \frac{f_0^{(n)}}{\ell^{(\frac{n}{2}-1)} (f_0'')^{\frac{n}{2}}}} \end{aligned} \tag{2.198}$$

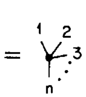
where derivatives evaluated at t_0 are denoted f_0'' and $f_0^{(n)}$ and the change of variables $\tau = (t - t_0)\sqrt{\ell f_0''}$ has been introduced to rescale the Gaussian to unit width. As $\ell \rightarrow \infty$, the terms with $n \geq 3$ go to zero and we may expand $I(\ell)$ in powers of $\frac{1}{\ell}$. For the problems of physical interest, we will usually be interested in expanding the logarithm of $I(\ell)$, and in the present example, it is straightforward to expand the exponential in Eq. (2.149), perform the Gaussian integrals, and exponentiate the result to the desired power of $\frac{1}{\ell}$ (see Problem 2.13).

A more economical derivation, however, may be obtained by utilizing our knowledge of Wick's theorem and the linked cluster expansion. Just as we defined contractions in connection with Eq. (2.84), we may define the contraction of τ as

$$\overbrace{\tau \cdot \tau} = \int_{-\infty}^{\infty} \frac{d\tau}{\sqrt{2\pi}} \tau \cdot \tau e^{-\frac{\tau^2}{2}} = 1 \ . \tag{2.199}$$

The coefficients of τ^n for $n \geq 3$ in Eq. (2.198) are regarded as vertices with n lines

$$V_n = \frac{1}{n!} \frac{f_0^{(n)}}{\ell^{(\frac{n}{2}-1)} (f_0'')^{\frac{n}{2}}} =$$



. (2.200)

Diagrams representing all possible contractions contributing to Eq. (2.198) are obtained by drawing any number of vertices $V_{n_1} V_{n_2} \dots V_{n_N}$ and connecting them with propagators equal to 1. In addition to the vertices, V_n, a diagram of order N has the overall factor $\frac{(-1)^N}{N!}$. By the linked cluster theorem,

$$I(\ell) = e^{-\ell f_0} \sqrt{\frac{2\pi}{\ell f_0''}}\, e^{(\text{sum of all linked diagrams})} \ . \tag{2.201}$$

The following diagrams contribute to lowest order in $\frac{1}{\ell}$

(a) (b) (c) (2.202)

Counting the 3×2 ways of contracting the three lines of each vertex in diagram (a) and the 3×3 ways of picking one line from each vertex to contract in diagram (b), the contribution of these two diagrams is

$$\frac{(-1)^2}{2!}(3 \times 2 + 3 \times 3)\left(\frac{f_0^{(3)}}{3!}\right)^2 \frac{1}{\ell (f_0'')^3} \quad .$$

Similarly, counting the three distinct contractions, diagram (c) contributes $(-1)\frac{3}{4!}\frac{f_0^{(4)}}{\ell (f_0'')^2}$, so that the asymptotic expansion of $I(\ell)$ is thus

$$I(\ell) = e^{-\ell f_0 + \frac{1}{2} ln\left(\frac{2\pi}{\ell f_0''}\right) + \frac{1}{\ell}\left(\frac{5}{24}\frac{(f_0^{(3)})^2}{(f_0'')^3} - \frac{1}{8}\frac{f_0^{(4)}}{(f_0'')^2}\right) + O\left(\frac{1}{\ell^2}\right)} \quad . \tag{2.203}$$

The contribution of order $\frac{1}{\ell^2}$ involves diagrams containing $V_6, V_5V_3, (V_4)^2, V_4(V_3)^2$, and $(V_3)^4$ and is treated in Problem 2.13.

The previous discussion generalizes straighforwardly to a complex function $f(t)$ which is analytic in some region of the complex t plane. Consider the case in which $f(t)$ has a single stationary point t_0 such that $f'(t_0) = 0$. Since by the Cauchy integral formula $\mathrm{Re}\, f(t_0)$ is equal to the average of $\mathrm{Re}\, f(t)$ on a circle centered on t_0, t_0 must be a saddle point for the function $\mathrm{Re}\, f(t)$. For an arbitrary contour passing through t_0, $\mathrm{Im}\, f(t)$ will vary along the contour giving rise to arbitrarily rapid oscillations in the integrand as $\ell \to \infty$. However, by selecting a contour such that $\mathrm{Im}\, f(t)$ has the constant value $\mathrm{Im}\, f(t_0)$ in the vicinity of t_0, the integral assumes the form

$$I(\ell) = e^{-\ell \mathrm{Im}\, f(t_0)} \int_c e^{-\ell \mathrm{Re}\, f(t)} dt \tag{2.204}$$

in the region of t_0. Furthermore, writing $f''(t_0) = \rho_0 e^{i\phi_0}$ in polar form and expanding to second order around t_0

$$f(t_0 + \rho e^{i\phi}) = f(t_0) + \frac{1}{2}\rho_0 e^{i\phi_0} \rho^2 e^{2i\phi} \tag{2.205}$$

it is evident that the two directions $\phi = -\frac{\phi_0}{2}$ and $\phi = -\frac{\phi_0}{2} + \frac{\pi}{2}$ which keep $\mathrm{Im}\, f$ constant correspond to the directions of maximum positive and negative curvature for $\mathrm{Re}\, f(t)$. Hence, the countour c is deformed from the real axis such that in the vicinity of t_0 it coincides with the curve having $\mathrm{Im}\, f(t) = \mathrm{Im}\, f(t_0)$ which passes through the saddle point of the function $e^{\ell \mathrm{Re}\, f(t)}$ in the direction of steepest descent, and the resulting real integral, Eq. (2.204) is evaluated as in the previous case.

In the case of multiple stationary points, the analysis is complicated in two respects. The first, essentially technical, complication, is the necessity of globally analyzing the surface of $e^{-\ell \mathrm{Re}\, f(t)}$ in order to connect the positive and negative t axes at infinity with a contour traversing a sequence of intermediate saddle points. The more substantive problem is the fact that integration of an infinite expansion around each of two separate stationary points has the potential for double counting contributions to the integral. For well-separated stationary points, it is often assumed (without justification) that low-order contributions from each stationary point may simply be added. When two stationary points come sufficiently close together, the combined contribution of both stationary points is treated by an appropriate form of uniform approximation (Berry (1966), Miller (1970), Connor and Marcus, (1971)).

FEYNMAN PATH INTEGRAL

The Feynman path integral for the evolution operator of a particle in a potential $V(x)$ is the limit of a product of integrals over the variables x_k at each time slice k, Eq. (2.39), so that the stationary-phase approximation may be applied straightforwardly to each integral in the product. For convenience, we will use here the continuum notation of Eq. (2.41)

$$\mathcal{U}(x_f t_f; x_i t_i) = \int_{(x_i,t_i)}^{(x_f,t_f)} \mathcal{D}[x(t)] e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt\left[\frac{1}{2}m\left(\frac{\partial x}{\partial t}\right)^2 - V(x(t))\right]} = \int_{(x_i,t_i)}^{(x_f,t_f)} \mathcal{D}[x(t)] e^{\frac{i}{\hbar}S[x(t)]} \tag{2.206}$$

with the understanding that $\int dt$ and $\left(\frac{\partial x}{\partial t}\right)$ are defined by the discrete expressions (2.40). Due to the multiplicative factor $\frac{i}{\hbar}$ in the exponent, it is evident that the stationary-phase approximation will generate the semi-classical expansion of the evolution operator in powers of $\hbar$.

Since $S[x(t)]$ is the classical action, stationarity of $S[x(t)]$ yields the Euler-Lagrange equation of motion for the classical trajectory $x_c(t)$

$$m\frac{d^2 x_c}{dt^2} = -\nabla V(x_c) \tag{2.207a}$$

with the boundary conditions

$$x_c(t_i) = x_i \qquad x_c(t_f) = x_f \quad . \tag{2.207b}$$

Expanding the action around the classical trajectory $x_c(t)$ and introducing the change of variables

$$\eta(t) = \frac{1}{\sqrt{\hbar}}(x(t) - x_c(t)) \tag{2.208}$$

we obtain

$$\mathcal{U}(x_f t_f, x_i t_i) = e^{\frac{i}{\hbar}S(x_c(t))} \int_{(0,t_i)}^{(0,t_f)} \mathcal{D}[\eta(t)] \\ \times e^{i\int_{t_i}^{t_f} dt\left\{\frac{1}{2}\eta(t)\left[-m\frac{d^2}{dt^2} - V''(x_c(t))\right]\eta(t) + \sum_{n=3}^{\infty} \frac{\hbar^{\left(\frac{n}{2}-1\right)}}{n!} V^{(n)}(x_c(t))\eta^n\right\}} \tag{2.209}$$

where $\eta(t)$ is required to vanish at the end points because $x(t)$ and $x_c(t)$ satisfy the same boundary conditions. Introducing an infinitesimal real term,

ϵ, to render the integral well-defined and noting that the quadratic form $\int dt \frac{1}{2}\eta(t)\left[-\epsilon - im\frac{d^2}{dt^2} - iV''(x_c(t))\right]\eta(t)$ actually represents a discrete sum $\eta_k A_{k\ell}\eta_\ell$, the Gaussian integral may be performed using Eq. (1.179) with the result

$$\mathcal{U}(x_f t_f, x_i t_i) = e^{\frac{i}{\hbar}S(x_c(t)) - \frac{1}{2}\ln\det\left(m\frac{d^2}{dt^2} + V''(x_c(t))\right) + \mathcal{O}(\hbar)} \tag{2.210}$$

where the determinant may either be calculated directly from the discrete expression for the quadratic form and the measure Eq. (2.42) or from the appropriately normalized product of eigenvalues E_n of the equation

$$(m\frac{d^2}{dt^2} + V''(x_c(t)) - E_n)\psi_n(t) = 0 \tag{2.211a}$$

with

$$\psi_n(t_i) = \psi_n(t_f) = 0 \ . \tag{2.211b}$$

It is an instructive exercise to explicitly evaluate the determinant for the case of the harmonic oscillator to obtain the exact propagator (see Problem 2.14).

Physically, the leading term in Eq. (2.210) of order $\frac{1}{\hbar}$ is given by the classical trajectory, with the next term of order unity corresponding to the sum of all possible quadratic fluctuations around the classical trajectory. Higher order terms in $\hbar$ may be obtained by summing linked diagrams in which the vertices $V^{(n)}(x_c)$ are connected by propagators $[m\frac{d^2}{dt^2} + V''(x_c)]^{-1}$ as in the previous section.

It is interesting to note at this point that application of the stationary-phase approximation to the imaginary-time path integral, Eq. (2.54) corresponds to an analogous expansion around the stationary solution

$$m\frac{d^2x_c}{dt^2} = -\nabla(-V(x_c)) \ . \tag{2.212}$$

Here the minus sign associated with the transformation (2.53) has been grouped with the potential to indicate that the stationary trajectory corresponds to the classical solution in the inverted potential. Thus, in tunneling and barrier penetration problems for which the classically forbidden region does not support appropriate classical solutions, these stationary solutions in the inverted potential will serve as the starting point for a quantum mechanical expansion.

MANY-PARTICLE PARTITION FUNCTION

The coherent state functional integral for the many-particle evolution operator, Eq. (2.62), may be written in coordinate representation as

$$\begin{aligned}\mathcal{U}(\phi_f^*(x), t_f; \phi_i(x), t_i) = &\int_{\phi(x,t_i)=\phi_i(x)}^{\phi^*(x,t_f)=\phi_f^*(x)} \mathcal{D}[\phi^*(x,t),\phi(x,t)]e^{\int dx\phi^*(x,t_f)\phi(x,t_f)} \\ &\times e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt\left[\int dx\phi^*(x,t)(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\nabla^2)\phi(x,t) - \frac{1}{2}\int dxdy\phi^*(x,t)\phi^*(y,t)v(x-y)\phi(y,t)\phi(x,t)\right]}\end{aligned} \tag{2.213}$$

Whereas an explicit factor $\frac{i}{\hbar}$ multiplies the integral $\int dt$ in the exponent, in contrast to the Feynman path integral in the previous section, $\hbar$ also appears in the integrand of the exponent. Thus application of the stationary-phase approximation to Eq. (2.213) does not strictly yield a semi-classical expansion in powers of $\hbar$. A similar situation arises in field theory where, for example, the action associated with a scalar field has the form

$$\frac{1}{\hbar}S(\phi) = \frac{1}{\hbar}\int dtd^3x\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}(\frac{m}{\hbar})^2\phi^2 + \lambda\phi^4\right] \tag{2.214}$$

and $\hbar$ appears explicitly in the mass term. In either case, to interpret the stationary-phase approximation as an expansion in $\hbar$, one must imagine two separate $\hbar$'s, with the $\hbar$ appearing within the integrand as a fixed constant and only the multiplicative factor as the expansion parameter.

In the absence of a strict expansion in an explicit small parameter, our present treatment of the stationary-phase approximation will be analogous to that of perturbation theory. We will introduce a parameter ℓ multiplying the action, which for the sake of the derivation will be assumed to be large, just as the formal parameter λ which is often introduced in the potential in perturbation theory is assumed to be small. In this way, it is straightforward to develop the stationary-phase expansion and demonstrate its correspondence to perturbation theory. From subsequent application to specific examples and the treatment of the closely related $\frac{1}{N}$ expansion, the physical conditions will become apparent under which a suitable expansion parameter arises and the method becomes preferable to perturbation theory.

With the introduction of the parameter ℓ and suppressing factors of $\hbar$, the partition function Eq. (2.66) may be written

$$\begin{aligned}Z(\ell) = &\int_{\phi(x,\beta)=\varsigma\phi(x,0)} \mathcal{D}(\phi^*(x,\tau)\phi(x,\tau))e^{-\ell\int_0^\beta d\tau\int dx\phi^*(x,\tau)\left(\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu\right)\phi(x,\tau)} \\ &\times e^{-\frac{\ell}{2}\int_0^\beta d\tau\int dxdy\phi^*(x,\tau)\phi^*(y,\tau)v(x-y)\phi(y,\tau)\phi(x,\tau)} \ .\end{aligned} \tag{2.215a}$$

Variation of the exponent yields the following equations for the stationary solutions ϕ_c^* and ϕ_c:

$$\frac{\delta S}{\delta\phi_c^*(x,\tau)} = \left[\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu + \int dy\, v(x-y)\phi_c^*(y,\tau)\phi_c(y,\tau)\right]\phi_c(x,\tau) = 0 \tag{2.215b}$$

$$\frac{\delta S}{\delta\phi_c(x,\tau)} = \left[-\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu + \int dy\, v(x-y)\phi_c^*(y,\tau)\phi_c(y,\tau)\right]\phi_c^*(x,t) = 0 \tag{2.215c}$$

with the boundary conditions

$$\phi_c(x,\beta) = \varsigma\phi_c(x,0) \ . \tag{2.215d}$$

One trivial solution about which to expand for either Fermions or Bosons is the solution $\phi_c(x,\tau) = \phi_c^*(x,\tau) = 0$. In this case, $Z(\ell)$ is evaluated in the usual way using perturbation theory with propagators $\frac{1}{\ell}(\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu)^{-1}$ and interaction $\ell v(x-y)$. The ℓ-dependence of each linked diagram for the grand potential is then $\ell^{n_V - n_I}$ where n_V denotes the number of vertices and n_I indicates the number of internal propagator lines. Since each vertex is connected to four lines and each line is connected to two vertices, $n_I = 2n_V$. Finally for a translationally invariant system, let n_M denote the number of momentum loops, defined as the number of independent momentum integrals performed in the diagram. Note that the number of momentum loops differs from n_L, the number of closed propagator loops defined earlier to determine the sign of a Feynman diagram. Recalling from Section 2.3 that there are $n_V - 1$ momentum conserving δ-functions, the number of independent momenta is $n_M = n_I - n_V + 1$. The ℓ-dependence of a general diagram may therefore be expressed as ℓ^{-n_V} or ℓ^{-n_M+1}. The former result may also be obtained trivially by rescaling the fields ϕ in Eq. (2.215) by a factor $\frac{1}{\sqrt{\ell}}$ and observing that only the ratio $\frac{v}{\ell}$ occurs in the rescaled action.

Thus ordinary perturbation theory may be regarded as an expansion in $\frac{1}{\ell}$ and is equivalent to a loop expansion in the number of independent momentum integrals. Note, however, that when expanding around $\phi_c = \phi_c^* = 0$, the leading contribution to Ω is of order $\ln \ell$.

In addition to the trivial solution $\phi_c = \phi_c^* = 0$, Eqs. (2.215) admit non-trivial static and time-dependent solutions. In the case of Fermions, the stationary solutions in terms of Grassman variables lie outside the space of physical observables and the functional integrals over the shifted variables must be performed to obtain a physical result. The most efficient way to proceed for Fermions is thus to introduce an auxiliary field to enable the integrations over Grassman variables to be done exactly, and then apply the stationary-phase approximation to the integral over the auxiliary field. We defer this treatment of Fermions until Chapter 7, and address the simpler case of Bosons here.

For Bosons, application of the stationary-phase approximation to $\int \mathcal{D}(\phi^*\phi)\, e^{F(\phi^*\phi)}$ may be regarded as approximating a double real integral $\int du dv\, e^{F(u,v)}$ where u and v are real variables representing the real and imaginary parts of ϕ. Since the general method of steepest descent requires consideration of all complex stationary points of u and v, in principle we should consider solutions for which ϕ_c^* is not necessarily the complex conjugate of ϕ_c. Again, the most general case will be deferred and we consider here the special case that ϕ_c^* is the complex conjugate of ϕ_c. Because of the opposite signs of the time derivatives in (2.215b) and (2.215c), such solutions must necessarily be time-independent and satisfy the static Hartree equation:

$$\left(-\frac{\nabla^2}{2m} - \mu + \int dy\, v(x-y)|\phi_c(y)|^2\right)\phi_c(x) = 0 \ . \tag{2.216}$$

The general solutions to Eq. (2.216) for translationally invariant systems are plane waves. We will assume for the present treatment that $\tilde{v}(0)$, which is the zero-momentum Fourier transform or volume integral of $V(x-y)$, is positive. In this case the solution which minimizes the action is the zero-momentum Bose condensate

$$\psi_c = \sqrt{\frac{\mu}{\tilde{v}(0)}} \tag{2.217}$$

with action

$$\ell S_c = -\frac{\ell\beta}{2}\mathcal{V}\frac{\mu^2}{\tilde{v}(0)}$$

where $\mathcal{V}$ is the volume of the system, and we have chosen the arbitrary phase to make ψ_c real. The significance of the phase of ψ_c will be addressed in Chapter 4 in the general discussion of order parameters. With the change of variables

$$\begin{aligned} \psi(x,\tau) &= \sqrt{\ell}(\phi(x,\tau) - \psi_c) \\ \psi^*(x,\tau) &= \sqrt{\ell}(\phi^*(x,\tau) - \psi_c) \end{aligned} \tag{2.218}$$

the action becomes

$$\ell S\,(\psi^*(x,\tau), \psi(x,\tau)) = \ell S_c + S^{(2)} + \frac{1}{\sqrt{\ell}}S^{(3)} + \frac{1}{\ell}S^{(4)} \tag{2.219a}$$

where

$$S^{(2)} = \frac{1}{2}\int_0^\beta d\tau \int dxdy [\psi^*(x,\tau)\psi(x,\tau)] D \begin{bmatrix} \psi(y,\tau) \\ \psi^*(y,\tau) \end{bmatrix} \tag{2.219b}$$

$$D = \begin{bmatrix} \delta(x-y)\left(\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m}\right) + \psi_c^2 v(x-y) & \psi_c^2 v(x-y) \\ \psi_c^2 v(x-y) & \delta(x-y)\left(-\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m}\right) + \psi_c^2 v(x-y) \end{bmatrix} \tag{2.219c}$$

$$S^{(3)} = \psi_c \int_0^\beta d\tau \int dxdy\, v(x-y)|\psi(x,\tau)|^2(\psi^*(y,\tau) + \psi(y,\tau)) \tag{2.219d}$$

and

$$S^{(4)} = \int_0^\beta d\tau \int dxdy\, v(x-y)|\psi(x,\tau)|^2|\psi(y,\tau)|^2 \ . \tag{2.219e}$$

It is convenient to simplify the notation by suppressing (x,τ) arguments, using the matrix D, Eq. (2.219c), and abbreviating the vertices in $S^{(3)}$ and $S^{(4)}$ as g_3 and g_4, so that the partition function is written as

$$Z(\ell) = e^{-\ell S_c}\int \mathcal{D}(\psi^*\psi) e^{-\frac{1}{2}(\psi^*\psi)D\binom{\psi}{\psi^*} - \frac{1}{\sqrt{\ell}}|\psi|^2 g_3(\psi+\psi^*) - \frac{1}{\ell}|\psi|^2 g_4 |\psi|^2} \ . \tag{2.220}$$

Although the quadratic form appearing in this functional integral differs slightly from that in Eq. (2.84) used in establishing Wick's theorem due to the presence of the $\psi^*\psi^*$ and $\psi\psi$ terms, it is shown in Problem 2.15 that a straightforward generalization of the usual linked diagram expansion is obtained. Writing the inverse of the matrix D in the block form

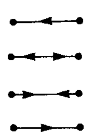
$$(D)\begin{pmatrix} G_{\psi\psi^*} & G_{\psi\psi} \\ G_{\psi^*\psi^*} & G_{\psi^*\psi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.221}$$

the contractions of the fields $\{\psi^*, \psi\}$ are given by

$$\begin{aligned} \int \mathcal{D}(\psi^*\psi)e^{-\frac{1}{2}(\psi^*\psi)D\binom{\psi}{\psi^*}} \begin{Bmatrix} \psi(1)\psi^*(2) & \psi(1)\psi(2) \\ \psi^*(1)\psi^*(2) & \psi^*(1)\psi(2) \end{Bmatrix} \\ = \begin{Bmatrix} G_{\psi\psi^*}(1,2) & G_{\psi\psi}(1,2) \\ G_{\psi^*\psi^*}(1,2) & G_{\psi^*\psi}(1,2) \end{Bmatrix} \ . \end{aligned} \tag{2.222}$$

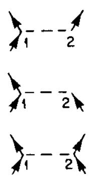
Note that the positivity of D requires that ψ_c correspond to a minimum of the action.

The propagators corresponding to these contractions are written as follows:



$$
\begin{aligned}
&= G_{\psi\psi^*}(1,2) = \langle \psi(1)\psi^*(2)\rangle \\
&= G_{\psi\psi}(1,2) = \langle \psi(1)\psi(2)\rangle \\
&= G_{\psi^*\psi^*}(1,2) = \langle \psi^*(1)\psi^*(2)\rangle \\
&= G_{\psi^*\psi}(1,2) = \langle \psi^*(1)\psi(2)\rangle \ .
\end{aligned} \tag{2.223}
$$

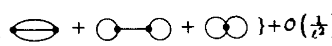
As usual, arrows entering or leaving a dot indicate a ψ or ψ^* respectively and the new feature arising from the shifted fields ψ is the introduction of lines with two arrows pointing in opposite directions. The vertices corresponding to the cubic and quartic terms in Eq. (2.219) are denoted



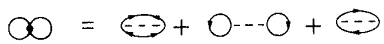
$$
\begin{aligned}
&= \frac{1}{\sqrt{\ell}}\psi_c v(1,2) \\
&= \frac{1}{\sqrt{\ell}}\psi_c v(1,2) \\
&= \frac{1}{\ell} v(1,2) \ .
\end{aligned} \tag{2.224}
$$

By calculating the linked diagrams composed of these vertices connected by the propagators (2.223), the partition function may be written

$$Z(\ell) = e^{-\ell S_c - \frac{1}{2}\ln(\det D) + \frac{1}{\ell}\{\ \cdots\ \} + O(\frac{1}{\ell^2})} \tag{2.225a}$$



where the diagrams denote the sum of all contractions of the indicated topology, for example



(2.225b)

and similarly for the remaining diagrams.

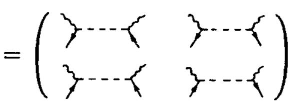
Note that in contrast to the expansion of $Z(\ell)$ about $\phi = 0$, which corresponded to ordinary perturbation theory in powers of v or $\frac{1}{\ell}$, the expansion about ψ_c in Eq. (2.205a) yields contributions to Ω of order ℓ and order unity. Such terms cannot be obtained by expansion in powers of ℓ, so the stationary-phase expansion corresponds to an infinite resummation of perturbation theory.

This resummation may be understood physically in terms of Bose condensation. The zero-momentum mode is macroscopically occupied, with amplitude $\psi_c = \sqrt{\frac{\mu}{\tilde{v}(0)}}$ controlled by the chemical potential. The shift in variables (2.218) then gives rise to vertices in which one or more of the extremities has a zero-momentum condensate factor ψ_c which we will denote by the symbol rather than a field variable ψ or ψ^* indicated in the usual way by or . We will write the quadratic martix D from Eq. (2.219b) and its inverse G from Eq. (2.221) in the following schematic form:

$$D = G_0^{-1} + V = G^{-1} \tag{2.226a}$$

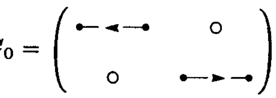
where

$$G_0^{-1} = \begin{pmatrix} \frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} & 0 \\ 0 & -\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} \end{pmatrix} \qquad V = \tag{2.226b}$$

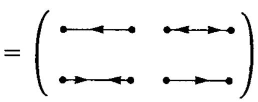


and will denote G_0 and G by broken and solid propagators:

$G_0 =$

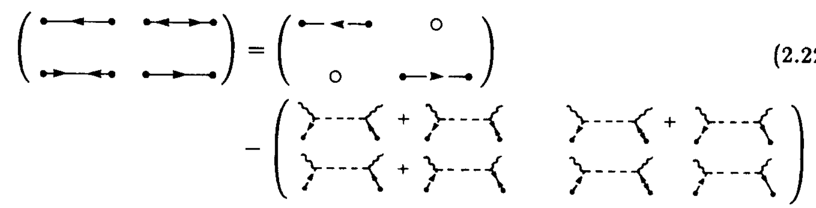


$G =$



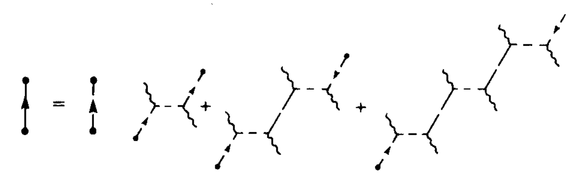
(2.226c)

The Dyson equation $G = G_0 - G_0 V G$ following from (2.226a) then has the diagrammatic form:



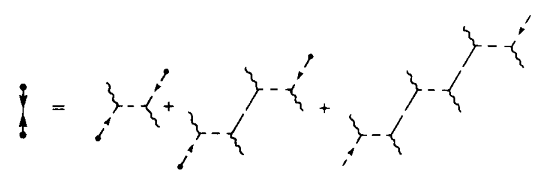
(2.227a)

Iteration yields the series:

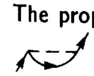


(2.227b)

and



(2.227c)

where internal lines are summed over G_0 propagators with arrows in each direction. The propagator G thus sums self-energy insertions analogous to the exchange term  of Eq. (2.180b) where the factor ψ_c^2 plays the role of the occupation number n_γ for the zero momentum state. Note that in deriving Eq. (2.199), direct terms analogous to the Hartree self-energy of the form $\tilde{v}(0)\psi_c^2$ exactly cancelled the chemical potential, so in fact both direct and exchange terms have been resummed to all orders.

The term of order $\ell^{(0)}$ in Eq. (2.225a), $\frac{1}{2}\ln(\det D)$, also has a simple diagrammatic interpretation. We first write

$$\det D = det\,(G_0^{-1} + V) = \det G_0^{-1} \det\,(1 + G_0 V) \ . \tag{2.228}$$

Since from Eq. (2.68), $\det\left(\frac{\partial}{\partial\tau}-\frac{\nabla^2}{2m}\right)^{-1}$ is the partition function $Z_0 = e^{-\beta\Omega_0}$ for the non-interacting system and because the block form for G_0^{-1}, Eq. (2.226b) contains two such matrices with equal determinants, it follows that

$$[\det G_0^{-1}]^{-\frac{1}{2}} = e^{-\beta\Omega_0} \quad . \tag{2.229}$$

Using the relation

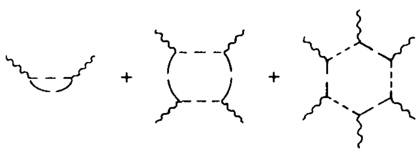
$$\det M = e^{\mathrm{tr}\,\ln M} \tag{2.230}$$

we obtain the desired result

$$\begin{aligned} -\frac{1}{2}\ln\det D &= -\beta\Omega_0 - \frac{1}{2}\mathrm{tr}\,\ln(1+G_0V) \\ &= -\beta\Omega_0 - \frac{1}{2}\mathrm{tr}\sum_{n=1}^{\infty}\frac{1}{n}(-G_0V)^n \quad . \end{aligned} \tag{2.231a}$$

The factor $(G_0V)^n$ corresponds to a series of propagators and interactions of the form in Eq. (2.227), which are then connected into a closed loop by the trace and weighted by the symmetry factor $\frac{1}{2n}$ accounting for the number of rotations and reflections of the resulting diagram. Thus, the term of order ℓ^0 sums the one-loop diagrams

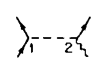
$$-\frac{1}{2}\ln\det D = -\beta\Omega_0 + \quad + \quad + \quad + \cdots \tag{2.231b}$$



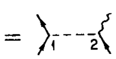
where again the line without an arrow denotes a sum over G_0 propagators in each direction.

The first few orders of the expansion of the grand potential in Eq. (2.225a) thus are clearly ordered by the number of momentum loops, n_M, with ℓ dependence $\ell^{-(n_M-1)}$. The leading term ℓS_c is just the classical action Eq. (2.217) with no loops. The second term, of order $\ell^{(0)}$, sums the one-loop contributions shown in Eq. (2.231b) and the third term, of order ℓ^{-1}, sums the two loop diagrams shown in Eq. (2.225).

The fact that the stationary-phase expansion in powers of $\frac{1}{\ell}$ systematically orders diagrams by the number of loops is easily seen as follows. We will explicitly draw the condensate factor on the vertices in (2.224) as an external line so that the vertices



$$= \frac{1}{\sqrt{\ell}}\psi_c v(1,2) = \tag{2.232a}$$



connect to three internal lines and one external line and the vertex

$$= \frac{1}{\ell}v(1,2) \tag{2.232b}$$

connects to four internal lines. Now, consider a general diagram obtained by contracting n_V vertices with propagators. Of these n_V vertices, assume n_{VE} are of the form of (2.232a) containing an external condensate line and the remaining $n_V - n_{VE}$ are of the form (2.232b) with no external lines. Since the propagators are independent of ℓ, the overall ℓ-dependence of the diagram from (2.232) will be $\ell^{-n_V+\frac{1}{2}n_{VE}}$ If one counts the lines coming out of all vertices, the internal lines connecting to two vertices will be counted twice and the external condensate lines will be counted once, giving the topological constraint

$$2n_I + n_{VE} = 4n_V \tag{2.233a}$$

where n_I denotes the number of internal lines. As in the analysis with no condensate, the number of independent momentum loops, n_M, after accounting for the $n_V - 1$ momentum conservation constraints is

$$n_M = n_I - n_V + 1 \tag{2.233b}$$

or using (2.233a)

$$n_M = n_V - \frac{1}{2}n_{VE} + 1 \quad . \tag{2.233c}$$

Thus, for an arbitrary diagram, the overall ℓ-dependence may be written $\ell^{-(n_M-1)}$ establishing the loop expansion. An analogous analysis shows that the series for the effective action is also a loop expansion. In contrast to the expansion around $\psi = 0$, for which the loop expansion coincided with a perturbation expansion, in the presence of a condensate diagrams with n_M loops contain from $n_M - 1$ to $2(n_M - 1)$ vertices of the form of Eq. (2.232) as well as the infinite resummation of interactions Eq. (2.227) contained in the propagators.

PROBLEMS FOR CHAPTER 2

The first two problems review elementary aspects of quantum statistical mechanics used in this chapter. Problem 3 indicates the derivation of the time-ordered exponential used in the text, for those unfamiliar with it. The two most crucial problems for understanding path integrals are Problems 4 and 5, which are emphasized by an *. The next two problems treat discrete and continuum derivations of the propagators. For those who wish to broaden their understanding of fundamental derivations in the text and establish contact with traditional derivations in the original literature, alternative derivations of Wick's theorem, the linked-cluster theorem, the evaluation of expectation values, and the stationary phase expansion are presented in Problems 8, 10, 11, and 13. Generalizations of topics introduced in the text are treated in Problems 9, 12, and 15 and a specific example of the evaluation of the determinant arising from quadratic fluctuations is given in Problem 14.

PROBLEM 2.1 Show that Ω defined in quantum statistical mechanics by Eq. (2.40) is equal to the grand canonical potential defined in thermodynamics by the Legendre transformation

$$\Omega_T(T,V,\mu) = U - TS - \mu N$$

where $U(S,V,N)$ is the state function specifying the internal energy. First, define $\phi(T,V,\mu) = \Omega - \Omega_T$. Derive Eqs. (2.4) and use these results to show that ϕ must be a constant times T, introducing at most an additive constant in the entropy. Then, take the zero temperature limit of $\frac{\partial\Omega}{\partial T}$ to show that this additive constant is zero.