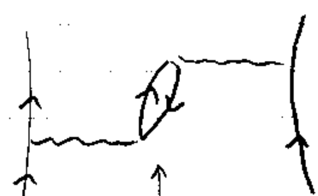
May 20, 2003

(1)

Particle-hole effects

In addition to the Pauli-blocking effect in the particle-particle ladders discussed in the last lecture, we can also have particle-hole excited states in the intermediate state. This can be thought of as a particle exciting a particle from below the Fermi surface to above, and the excited particle subsequently gives its excitation energy to another particle in form of a momentum kick. In this way, the original particle interacts with the last one. Diagrammaticly, such a process corresponds to



particle 1 particle-hole excitation particle 2

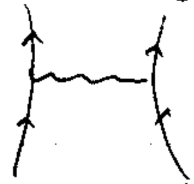
This is also referred to as particle-hole screening of the scattering amplitude or polarization of the many-body medium.

The particle-hole intermediate states are lower in energy compared to exciting two-particles, in the particle-particle ladder. For a system at low temperature, these are the dominant low-lying states of the many-body system.

Before, we move on to discuss the particle-hole (ph) channels in detail, we have to chose a convention for the momentum labels of the scattering particles. Moreover, some comments on antisymmetry / spin-statistics are appropriate.

For Fermi systems at low temperatures, the relevant degrees of freedom are particles / holes in the vicinity of the Fermi surface. We therefore take the interacting particles to lie on the Fermi surface. Our momentum conventions are as follows. (see earlier)

$$\vec{p}_1{}' = \frac{\vec{P}}{2} + \frac{\vec{q}'}{2} - \frac{\vec{q}}{2} \qquad \frac{\vec{P}}{2} + \frac{\vec{q}}{2} - \frac{\vec{q}'}{2} = \vec{p}_2{}'$$



$$\vec{p}_1 = \frac{\vec{P}}{2} + \frac{\vec{q}}{2} + \frac{\vec{q}'}{2} \qquad \frac{\vec{P}}{2} - \frac{\vec{q}}{2} - \frac{\vec{q}'}{2} = \vec{p}_2$$

Therefore, we demand $|\vec{p}_1| = |\vec{p}_2| = |\vec{p}_1{}'| = |\vec{p}_2{}'| = k_F$

$$\Rightarrow \quad \frac{P^2}{4} + \frac{q^2}{4} + \frac{q'}{4} + \frac{\vec{P}\cdot\vec{q}}{2} + \frac{\vec{P}\cdot\vec{q}'}{2} + \frac{\vec{q}\cdot\vec{q}'}{2} = k_F^2$$

$$〃 \qquad + \frac{\vec{P}\cdot\vec{q}'}{2} - \frac{\vec{P}\cdot\vec{q}}{2} - \frac{\vec{q}\cdot\vec{q}'}{2} = k_F^2$$

$$〃 \qquad - \frac{\vec{P}\cdot\vec{q}}{2} - \frac{\vec{P}\cdot\vec{q}'}{2} + \frac{\vec{q}\cdot\vec{q}'}{2} = k_F^2$$

$$〃 \qquad + \frac{\vec{P}\cdot\vec{q}}{2} - \frac{\vec{P}\cdot\vec{q}'}{2} - \frac{\vec{q}\cdot\vec{q}'}{2} = k_F^2$$

$$\Rightarrow \quad \underbrace{\vec{q}\cdot\vec{q}' = \vec{q}\cdot\vec{P} = \vec{q}'\cdot\vec{P}}_{\text{orthogonal momenta}} = 0$$

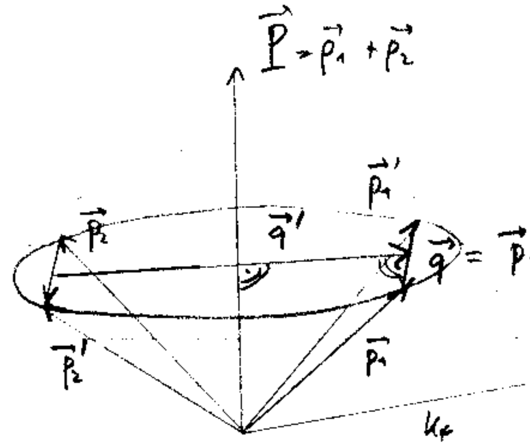


Figure above: 4 particles on the Fermi surface (momentum conserved $\vec{P} = \vec{p}_1 + \vec{p}_2 = \vec{p}_1' + \vec{p}_2'$)

For two-body scattering, the matrix elements are of the form (say at lowest order)

$$\langle 1' 2' | V | 1 2 \rangle ,$$

where $|12\rangle$ are antisymmetrized two-body states. Ignoring spin, we have

$$|12\rangle = \frac{1}{\sqrt{2}} \left(|\vec{p}_1 \vec{p}_2\rangle - |\vec{p}_2 \vec{p}_1\rangle \right)$$

$$= \frac{1}{\sqrt{2}} \left(1 - P_k \right) |\vec{p}_1 \vec{p}_2\rangle ,$$

where $P_{\vec{k}}$ is the momentum exchange operator (just as P_σ). In general

$$|12\rangle = \frac{1}{\sqrt{2}} \left(1 - P_k P_\sigma P_\tau \right) | \vec{p}_1 m_{s_1} m_{t_1} \vec{p}_2 m_{s_2} m_{t_2} \rangle .$$

Thus $\langle 1'2' | V | 12 \rangle = \frac{1}{2} \langle \vec{p}_1' \vec{p}_2' | (1 - P_{12}) V (1 - P_{12}) | \vec{p}_1 \vec{p}_2 \rangle$

and we can work in a not-antisymmetrized two-body wave function space, if we use an antisymmetric interaction

$$\frac{1}{2}(1 - P_{12}) V (1 - P_{12}) \quad \text{instead of } V$$

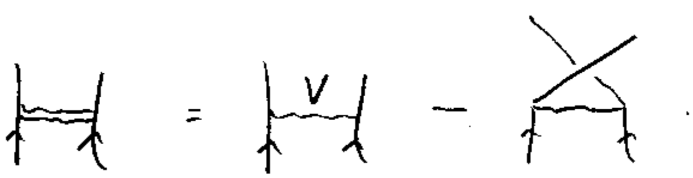
For our momentum convention

P_{12} exchanges $\vec{q}$ and $\vec{q}\,'$

$$\Rightarrow \quad V_{antisym} = \frac{1}{2} V - \frac{1}{2}\left(P_{12} V - V P_{12} \right) + \frac{1}{2} \underbrace{P_{12} V P_{12}}_{\text{twice } \vec{q} \leftrightarrow \vec{q}\,'}$$

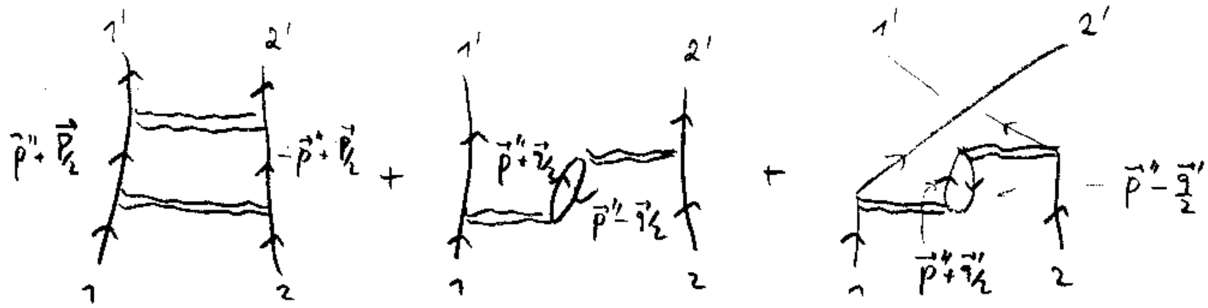
$$= V(\vec{q}, \vec{q}\,') - V(\vec{q}\,', \vec{q})$$

Diagrammatically, this amounts to introducing the antisymmetrized vertex



$V_{antisym.}$

At second order in perturbation theory, the contributions to the scattering amplitude are given by



Above we also label the momenta of the intermediate state. Thus, the momenta $\vec P, \vec q, \vec q\,'$ label the momentum transfers in the particle-particle (pp) and ph channels.

Note: that there are two ph channels (distinct!) and one pp channel.

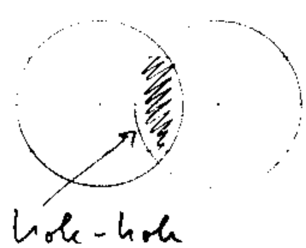
The diagrams correspond to

pp channel $\quad \underset{\text{symmetry factor}}{\frac{1}{2}} \int \frac{d^3p'}{(2\pi)^3} V_{\text{antisym}}\left(\frac{\vec q}{2}+\frac{\vec q\,'}{2}-\vec p\,', \frac{\vec q}{2}+\frac{\vec q\,'}{2}+\vec p\,'\right) \frac{(1-n_{\vec p\,''+\vec P/2})(1-n_{-\vec p\,''+\vec P/2})}{\omega_1+\omega_2-\varepsilon_{\vec p\,''+\vec P/2}-\varepsilon_{-\vec p\,''+\vec P/2}}$

$$* V_{\text{antisym}}\left(\vec p\,''-\frac{\vec q\,'}{2}+\frac{\vec q}{2}, \vec p\,''-\frac{\vec q}{2}+\frac{\vec q\,'}{2}\right)$$

pp channel inclusive hole-hole contributions

replace $(1-n)(1-n) \to (1-n)(1-n) - nn = 1 - n_{\vec p\,''+\vec P/2} - n_{-\vec p\,''+\vec P/2}$



ph channels: direct (momentum transfer $\vec{q}$) = "zero-sound" channel (ZS)

$$\int \frac{d^3\vec{p}''}{(2\pi)^3} \; V_{\text{antisym}}\left(\vec{q}, \frac{\vec{P}}{2} + \frac{\vec{q}'}{2} - \vec{p}''\right) \frac{n_{\vec{p}''+\vec{q}/2} - n_{\vec{p}''-\vec{q}/2}}{\omega_1 - \omega_1' - \varepsilon_{\vec{p}''-\vec{q}/2} + \varepsilon_{\vec{p}''+\vec{q}/2}}$$

$$* \; V_{\text{antisym}}\left(\vec{q}, \vec{p}'' - \frac{\vec{q}'}{2} + \frac{\vec{P}}{2}\right)$$

exchange (moment transfer $\vec{q}'$) = ZS'

$$- \int \frac{d^3\vec{p}''}{(2\pi)^3} \; V_{\text{antisym}}\left(\vec{q}', \frac{\vec{P}}{2} + \frac{\vec{q}}{2} - \vec{p}''\right) \frac{n_{\vec{p}''+\vec{q}'/2} - n_{\vec{p}''-\vec{q}'/2}}{\omega_1 - \omega_2' - \varepsilon_{\vec{p}''-\vec{q}'/2} + \varepsilon_{\vec{p}''+\vec{q}'/2}}$$

$$V_{\text{antisym}}\left(\vec{q}', \vec{p}'' - \vec{q}/2 + \vec{P}/2\right)$$

$$= - \int \frac{d^3\vec{p}''}{(2\pi)^3} \quad \vec{q} \leftrightarrow \vec{q}' \quad \text{and} \quad \omega_2 \to \omega_2'$$

Thus, the pp channel has poles in the $\omega_1 + \omega_2$ plane, whereas the ph channels in the $\omega_1 - \omega_2^{(1)}$ plane.

We note that in the forward scattering limit, $\vec{q} \to 0$, $1' = 1$ and $2' = 2$, the two propagators in the direct (ZS) ph channel

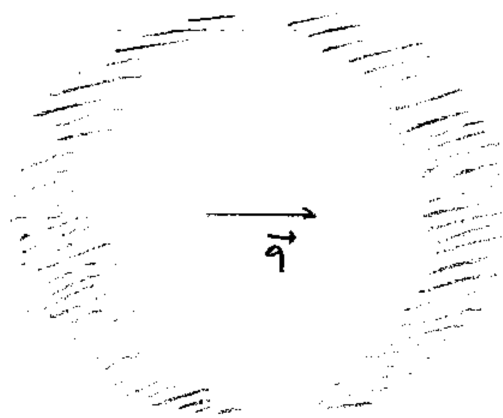
$$G(\vec{p}'' + \vec{q}/2) \;\; G(\vec{p}'' - \vec{q}/2)$$

have the same argument: $G(\vec{p}'' + 0) \;\; G(\vec{p}'' - 0)$.
This leads to a singularity, also referred to as "pinching poles" which gives rise to the propagation of the collective zero-sound mode

It is intuitive that forward scattering gives rise to collective effects, since in this limit all particles only move by a small momentum, i.e. $\vec{q} \to 0$.

The phase space in the particle-hole channel is given by the "complement" of the pp channel, but the displacement of the Fermi seas is by the momentum transfer vector $\vec{q}$ (or $\vec{q}\,'$ in the exchange ZS' channel)

$$n_{\vec{p}\,'+\vec{q}/2} - n_{\vec{p}+\vec{q}/2} \neq 0$$



Let us finally make the connection to the Fermi Liquid quasiparticle interaction f

$$\frac{1}{V} f_{\vec{k}_1 \vec{k}_2} = \frac{\delta^2 E}{\delta n_{\vec{k}_1} \delta n_{\vec{k}_2}}$$

Thus, f can be obtained from the energy diagrams by opening two lines corresponding to $n_{\vec{k}_1}$ and $n_{\vec{k}_2}$.

We go from the diagrams contributing to the scattering amplitude to the energy diagrams by closing the external lines (summing over occupied states in the Fermi sea)

Thus, we have

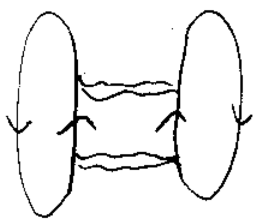
1st order

Hartree Fock

2nd order

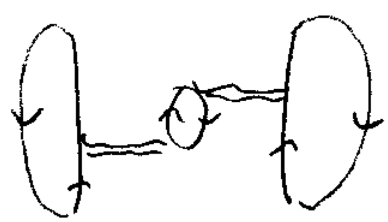
for contact interactions, we contract the lines ~~≈~~, i.e.

pp channel



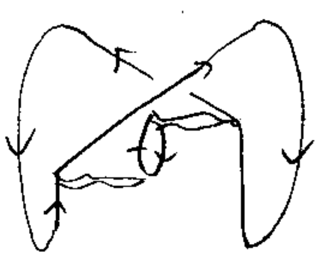
(pp) beach ball

ZS channel



anomalous

ZS' channel

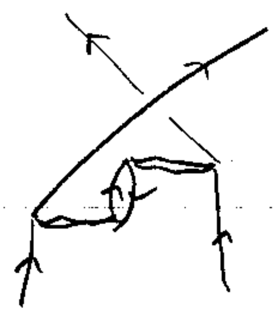


(ph) beach ball

or

The contribution in the ZS channel is anomalous, i.e. it does not contribute to the effective interaction.

Thus, at second order, the ph contribution to the effective interaction is simply given by the exchange diagram



In nuclear physics, this leads to a reasonable pairing force as well as quadrupole-quadrupole interaction. It is called the Kuo-Brown effective interaction.