(3/12/21)

- Starting point is $\hat{n}^{\tau}_{\lambda}(\vec{q})$ after evaluation of contractions. ($\vec{q}$ is not a relative momentum here!)

$$\hat{n}^{\tau}_{\lambda}(\vec{q}) = V\Big\{\sum_{\sigma}\theta(k_F^{\tau}-q) \quad \text{Term 1}$$

$$+\sum_{\sigma\sigma'\tau'}\int\frac{d^3k}{(2\pi)^3}\Big[\underbrace{\langle\vec{k}\,\sigma\tau\sigma'\tau'|\delta U|\vec{k}\,\sigma\tau\sigma'\tau'\rangle}_{\text{Term 2}}$$

$$+\underbrace{\langle\vec{k}\,\sigma\tau\sigma'\tau'|\delta U^{\dagger}|\vec{k}\,\sigma\tau\sigma'\tau'\rangle}_{\text{Term 3}}\Big]\theta(k_F^{\tau}-q)\,\theta(k_F^{\tau'}-|\vec{q}-2\vec{k}|)$$

$$+\frac{1}{2}\sum_{\sigma\sigma'\sigma''\sigma'''}\sum_{\tau'\tau''\tau'''}\int\frac{d^3k}{(2\pi)^3}\frac{d^3K}{(2\pi)^3}\langle\vec{k}\,\sigma'\tau'\sigma''\tau''|\delta U|\vec{q}-\tfrac{1}{2}\vec{K}\,\sigma\tau\sigma'''\tau'''\rangle\times$$

$$\langle\vec{q}-\tfrac{1}{2}\vec{K}\,\sigma\tau\sigma'''\tau'''|\delta U^{\dagger}|\vec{k}\,\sigma'\tau'\sigma''\tau''\rangle\,\theta(k_F^{\tau'}-|\tfrac{1}{2}\vec{K}+\vec{k}|)\,\theta(k_F^{\tau''}-|\tfrac{1}{2}\vec{K}-\vec{k}|)\Big\} \tag{1}$$

Term 1 is simple. Contribution is $2\,\theta(k_F^{\tau}-q)$.

Do 2nd + 3rd term (multiply 2nd term by 2 for full 2+3 contribution).

$$2\sum_{\sigma\sigma'\tau'}\int\frac{d^3k}{(2\pi)^3}\langle\vec{k}\,\sigma\tau\sigma'\tau'|\delta U|\vec{k}\,\sigma\tau\sigma'\tau'\rangle\,\theta(k_F^{\tau}-q)\,\theta(k_F^{\tau'}-|\vec{q}-2\vec{k}|)$$

$$=2\sum_{\sigma\sigma'\tau'}\sum_{SM_S}\sum_{S'M_S'}\sum_{TM_T}\sum_{T'M_T'}\sum_{LM_L}\sum_{L'M_L'}\sum_{JM_J}\sum_{J'M_J'}\int_0^{\infty}dk\,k^2\,d\Omega_{\vec{k}}$$

$$\langle \sigma\sigma' | S M_S \rangle \langle \tau\tau' | T M_T \rangle \langle \hat{k} | k L M_L \rangle \langle M_S M_L | J M_J \rangle \times$$
$$\langle k J L S T | \delta \tilde{U} | k J' L' S' T' \rangle \langle J' M_J' | M_S' M_L' \rangle \langle k L' M_L' | \hat{k} \rangle \times$$
$$\langle T' M_T' | \tau\tau' \rangle \langle S' M_S' | \sigma\sigma' \rangle \, \Theta(k_F^{\tau} - q) \, \Theta(k_F^{\tau'} - |\vec{q} - 2\vec{k}|)$$

Apply $\sum_{\sigma\sigma'} \langle \sigma\sigma' | S M_S \rangle \langle S' M_S' | \sigma\sigma' \rangle = \delta_{S,S'} \delta_{M_S, M_S'}$ $\quad \cancel{\sum_{S' M_S'}}$

Apply $\int d\Omega_{\hat{k}} \langle \hat{k} | k L M_L \rangle \langle k L' M_L' | \hat{k} \rangle = \frac{2}{\pi} \delta_{L,L'} \delta_{M_L M_L'}$ $\quad \cancel{\sum_{L' M_L'}}$

Apply $\sum_{M_S M_L} \langle M_S M_L | J M_J \rangle \langle J' M_J' | M_S M_L \rangle = \delta_{JJ'} \delta_{M_J M_J'}$ $\quad \cancel{\sum_{J' M_J}}$

$$= 2 \frac{2}{\pi} \sum_{\tau'} \sum_{S} \sum_{T M_T} \sum_{T' M_T'} \sum_{L} \sum_{J M_J} \int_0^\infty dk \, k^2 \langle \tau\tau' | T M_T \rangle \times$$
$$\langle k J L S T | \delta \tilde{U} | k J L S T' \rangle \langle T' M_T' | \tau\tau' \rangle \, \Theta(k_F^{\tau} - q) \, \Theta(k_F^{\tau'} - |\vec{q} - 2\vec{k}|)$$

Θ depends on $|\vec{q} - 2\vec{k}| = \sqrt{q^2 + 4k^2 - 4\vec{q}\cdot\vec{k}}$

Average over angles $\int_{-1}^{1} d(\hat{q}\cdot\hat{k}) [\ldots] / 2$

diagonal!

$$= \frac{2}{\pi} (2J+1) \sum_{\tau'} \sum_{T M_T} \cancel{\sum_{T' M_T'}} \sum_{S L J} \int_0^\infty dk \, k^2 \int_{-1}^{1} d(\hat{q}\cdot\hat{k}) \times$$
$$\langle \tau\tau' | T M_T \rangle \langle k J L S T | \delta \tilde{U} | k J L S T \rangle \langle T M_T | \tau\tau' \rangle \times$$
$$\Theta(k_F^{\tau} - q) \, \Theta(k_F^{\tau'} - |\vec{q} - 2\vec{k}|) \qquad (2)$$

$\tau = +\frac{1}{2}$

$\tau' = +\frac{1}{2} \rightarrow T = 1 \quad M_T = 1 \quad CG's = 1$

$\Rightarrow {}^1S_0 \rightarrow S=0, L=0, J=0$

$\tau' = -\frac{1}{2} \rightarrow T=1 \quad M_T = 0 \quad CG's = \frac{1}{\sqrt{2}} \qquad \langle \frac{1}{2}\, {-\frac{1}{2}} | 1 0 \rangle = \frac{1}{\sqrt{2}}$

$\Rightarrow {}^1S_0 \rightarrow S=0 \quad L=0 \quad J=0$

$\rightarrow T=0 \quad M_T = 0 \quad CG's = \frac{1}{\sqrt{2}}$

$\Rightarrow {}^3S_1 \rightarrow S=1 \quad L=0 \quad J=1$

Do we want $L > 0$?

$$\hat{\Lambda}^p_\lambda(q) \approx \frac{2}{\pi}\int_0^\infty dk\, k^2 \int_{-1}^{1} d(\hat{q}\cdot\hat{k}) \Big\{ \delta U_{^1S_0}(k,k)\Big[\theta\big(k_F^p - |\vec{q} - 2\vec{k}|\big) + \frac{1}{2}\theta\big(k_F^n - |\vec{q} - 2\vec{k}|\big)\Big] + \frac{1}{2}\delta U_{^3S_1-^3S_1}(k,k)\,\theta\big(k_F^n - |\vec{q}-2\vec{k}|\big)\Big\} \times \theta\big(k_F^p - q\big) \qquad (3)$$

(S-waves only)

Likewise

$$\hat{\Lambda}^n_\lambda(q) \approx \frac{2}{\pi}\int_0^\infty dk\, k^2 \int_{-1}^{1} d(\hat{q}\cdot\hat{k}) \Big\{ \delta U_{^1S_0}(k,k)\Big[\theta\big(k_F^n - |\vec{q} - 2\vec{k}|\big) + \frac{1}{2}\theta\big(k_F^p - |\vec{q} - 2\vec{k}|\big)\Big] + \frac{1}{2}\delta U_{^3S_1-^3S_1}(k,k)\,\theta\big(k_F^p - |\vec{q}-2\vec{k}|\big)\Big\} \times \theta\big(k_F^n - q\big) \qquad (4)$$

To do LDA $\quad 4\pi\int_0^\infty dr\, r^2\, \Lambda^p_\lambda(q; k_F^p, k_F^n) / Z$

↳ Determined by ρ functions!

Term 4 :

$$\frac{1}{2}\sum_{\sigma\sigma'\sigma''\sigma'''}\sum_{\tau\tau'\tau''\tau'''}\int\frac{d^3k}{(2\pi)^3}\frac{d^3K}{(2\pi)^3}\langle\vec{k}\,\sigma'\tau'\sigma''\tau''|\delta\tilde{U}|\vec{q}-\tfrac{1}{2}\vec{K}\,\sigma\tau\sigma'''\tau'''\rangle\times$$

$$\langle\vec{q}-\tfrac{1}{2}\vec{K}\,\sigma\tau\sigma'''\tau'''|\delta\tilde{U}^\dagger|\vec{k}\,\sigma'\tau'\sigma''\tau''\rangle\,\Theta(k_F^{\tau'}-|\tfrac{1}{2}\vec{K}+\vec{k}|)\times$$
$$\Theta(k_F^{\tau''}-|\tfrac{1}{2}\vec{K}-\vec{k}|)$$

$\delta\tilde{U}$ diagonal in J, S, T, M_J, M_S, M_T

$$=\frac{1}{2}\sum_{\sigma\sigma'\sigma''\sigma'''}\sum_{\tau'\tau''\tau'''}\sum_{SM_SS'M_S'}\sum_{TM_TT'M_T'}\sum_{LM_LL'M_L'}\sum_{L''M_L''}\sum_{L'''M_L'''}\sum_{JM_J}\sum_{J'M_J'}\times$$

$$\int_0^\infty dk\,k^2\int_0^\infty dK\,K^2\int d\Omega_k\int d\Omega_K\,\langle\sigma'\sigma''|SM_S\rangle\langle\tau'\tau''|TM_T\rangle\times$$

$$\langle\hat{k}|kLM_L\rangle\langle M_SM_L|JM_J\rangle\langle kJLST|\delta\tilde{U}||\vec{q}-\tfrac{1}{2}\vec{K}|\,JL'ST\rangle\times$$
$$\langle JM_J|M_SM_L\rangle\langle|\vec{q}-\tfrac{1}{2}\vec{K}|L'M_L'|\vec{q}-\tfrac{1}{2}\vec{K}\rangle\langle TM_T|\tau\tau'''\rangle\langle SM_S|\sigma\sigma'''\rangle\times$$
$$\langle\sigma\sigma'''|S'M_S'\rangle\langle\tau\tau'''|T'M_T'\rangle\langle\vec{q}-\tfrac{1}{2}\vec{K}\,||\vec{q}-\tfrac{1}{2}\vec{K}|L''M_L''\rangle\langle M_S'M_L''|J'M_J'\rangle\times$$
$$\langle|\vec{q}-\tfrac{1}{2}\vec{K}|J'L''S'T'|\delta\tilde{U}^\dagger|k\,J'L'''S'T'\rangle\langle J'M_J'|M_S'M_L'''\rangle\times$$
$$\langle kL'''M_L'''|\hat{k}\rangle\langle T'M_T'|\tau'\tau''\rangle\langle S'M_S'|\sigma'\sigma''\rangle\,\Theta(k_F^{\tau'}-|\tfrac{1}{2}\vec{K}+\vec{k}|)\times$$
$$\Theta(k_F^{\tau''}-|\tfrac{1}{2}\vec{K}-\vec{k}|)\qquad(5)$$

- Apply $\displaystyle\sum_{\sigma'\sigma''}\langle\sigma'\sigma''|SM_S\rangle\langle S'M_S'|\sigma'\sigma''\rangle=\delta_{SS'}\delta_{M_SM_S'}\cancel{\sum_{SM_S'}}$

$$\sum_{\sigma\sigma'''}\langle\sigma\sigma'''|SM_S\rangle\langle SM_S|\sigma\sigma'''\rangle=1$$

- Apply $\int d\Omega_{\vec{k}} \langle \hat{k} | k L M_L \rangle \langle k L''' M_L''' | \hat{k} \rangle = \frac{2}{\pi} \delta_{LL'''} \delta_{M_L M_L'''}$

$\cancel{\sum}_{L''' M_L'''}$

$$= \frac{1}{2}\frac{2}{\pi} \sum_{\tau'\tau''\tau''' T M_T T' M_T'} \sum_{S M_S} \sum_{L M_L} \sum_{L' M_L'} \sum_{L'' M_L''} \sum_{J M_J} \sum_{J' M_J'} \int_0^\infty dk\, k^2 \int_0^\infty dK\, K^2 \int d\Omega_{\vec{K}} \times$$
$$\langle \tau'\tau'' | T M_T \rangle \langle M_S M_L | J M_J \rangle \langle k J L S T | \delta\tilde{U} | |\vec{q}-\tfrac{1}{2}\vec{K}| J L' S T \rangle \langle J M_J | M_S M_L' \rangle \times$$
$$\langle |\vec{q}-\tfrac{1}{2}\vec{K}| L' M_L' | \vec{q}-\tfrac{1}{2}\vec{K} \rangle \langle T M_T | \tau\tau''' \rangle \langle \tau\tau''' | T' M_T' \rangle \langle \vec{q}-\tfrac{1}{2}\vec{K} | |\vec{q}-\tfrac{1}{2}\vec{K}| L'' M_L'' \rangle \times$$
$$\langle M_S M_L'' | J' M_J' \rangle \langle |\vec{q}-\tfrac{1}{2}\vec{K}| J' L'' S T' | \delta\tilde{U}^\dagger | k J' L S T' \rangle \times$$
$$\langle J' M_J' | M_S M_L \rangle \langle T' M_T' | \tau'\tau'' \rangle \theta(k_F^{\tau'} - |\tfrac{1}{2}\vec{K}+\vec{k}|) \theta(k_F^{\tau''} - |\tfrac{1}{2}\vec{K}-\vec{k}|)$$

- No angular dependence on $\vec{K}$ $\rightarrow$ 4π
- Average over $d(\hat{q}\cdot\hat{K})$ and $d(\hat{K}\cdot\hat{k})$
  where $|\vec{q}-\tfrac{1}{2}\vec{K}| = \sqrt{q^2 + \frac{K^2}{4} - \vec{q}\cdot\vec{K}}$,

$|\tfrac{1}{2}\vec{K} \pm \vec{k}| = \sqrt{\frac{K^2}{4} + k^2 \pm \vec{K}\cdot\vec{k}}$, $\quad \langle |\vec{q}-\tfrac{1}{2}\vec{K}| L' M_L' | \vec{q}-\tfrac{1}{2}\vec{K} \rangle$
$= \sqrt{\frac{2}{\pi}}\, Y^*_{L' M_L'}(\Omega_{\hat{q}\cdot\hat{K}})$

$\sum_{J' M_J'}$

$$= \frac{1}{2}\left(\frac{2}{\pi}\right)^2 4\pi \sum_{\tau'\tau''\tau'''} \sum_{T M_T} \sum_{T' M_T'} \sum_{S M_S} \sum_{L M_L} \sum_{L' M_L'} \sum_{L'' M_L''} \sum_{J M_J} \int_0^\infty dk\, k^2 \int_0^\infty dK\, K^2 \int \frac{d\Omega_{\hat{q}\cdot\hat{K}}}{4\pi} \times$$
$$\int_{-1}^{1} \frac{d(\hat{K}\cdot\hat{k})}{2} \langle \tau'\tau'' | T M_T \rangle \langle M_S M_L | J M_J \rangle \langle k J L S T | \delta\tilde{U} | |\vec{q}-\tfrac{1}{2}\vec{K}| J L' S T \rangle \times$$
$$\langle J M_J | M_S M_L' \rangle Y^*_{L' M_L'}(\Omega_{\hat{q}\cdot\hat{K}}) \langle T M_T | \tau\tau''' \rangle \langle \tau\tau''' | T' M_T' \rangle Y_{L'' M_L''}(\Omega_{\hat{q}\cdot\hat{K}}) \times$$

$$\langle M_S M_L'' | J' M_J' \rangle \langle |\vec{q} - \tfrac{1}{2}\vec{K}|\, J' L'' S T' | \delta \tilde{U}^{\dagger} | \hat{k}\, J' L\, S\, T' \rangle \langle J' M_J' | M_S M_L \rangle \times$$

$$\langle T' M_T' | \tau' \tau'' \rangle\, \Theta\left(k_F^{\tau'} - |\tfrac{1}{2}\vec{K} + \vec{k}|\right) \Theta\left(k_F^{\tau''} - |\tfrac{1}{2}\vec{K} - \vec{k}|\right) \qquad (6)$$