where C is, again, Euler's constant and Ci(x) is the cosine integral. Numerical evaluation gives

$$\frac{g(r_0)}{r_0} = \frac{g^{(0)}(r_0)}{r_0} + \frac{1}{R}(0.824) \qquad \text{IV(28)}$$

Thus, the proton-proton scattering length a_p of (25) is

$$\frac{1}{a_p} \cong \frac{1}{a_n} + \frac{1}{R}\left(\ln\frac{r_0}{R} + 2C - 0.824\right) \cdot$$

$$\cong \frac{1}{a_n} + \frac{1}{R}\left(\ln\frac{r_0}{R} + 0.330\right), \qquad \text{IV(29)}$$

where a_n is the corresponding neutron-proton scattering length for the same nuclear potential and r_0 is the range of the nuclear force. Note that (29) involves this range only logarithmically.

Estimates show that (29) is valid to within a few percent for the potentials which fit the proton-proton scattering data. Since both a_p and a_n are large compared with the range of the nuclear forces – that is, the nuclear force such as to produce a resonance near to zero energy – large changes in a_p or a_n produce only small changes in the nuclear force. Therefore, an approximation for a_p or a_n valid to a few percent is generally quite adequate for a description of the nuclear force.

Present values* of the low-energy parameters in proton-proton scattering are:

$$\begin{aligned} a_p &= -7.822 \pm 0.004 \text{ fm}, \\ r_{0p} &= 2.830 \pm 0.017 \text{ fm}, \\ P &= 0.051 \pm 0.014, \\ Q &= 0.028 \pm 0.013. \end{aligned} \qquad \text{IV(30)}$$

Careful calculations, more accurate than the type of approximations leading to (29), by Sher, Signell and Heller show that the neutron-neutron scattering length from the same nuclear interaction as would lead to a_p of eq. (30) is

$$a_{nn} = -17.06 \text{ fm}. \qquad \text{IV(31)}$$

The neutron-neutron scattering length deduced from three-body problems** is, to within an accuracy of about 1 fm, found to be equal to this.

* M. Sher, P. Signell and L. Heller, Ann. of Phys. 58 (1970) 1.

** *Few-particle problems in the nuclear interaction*, ed. by I. Slaus, S.A. Moszkowski, R.P. Haddock and W.T.H. van Oers (North-Holland, Amsterdam, 1972).

CHAPTER V

SOME RESULTS FROM SCATTERING THEORY

V.A. Introduction

For many calculations associated with the NN interaction it is now convenient to re-express the Schrödinger equation as an integral equation in momentum space. In this section we shall sketch the derivation of the Lippman-Schwinger equation and the Low equation and shall summarize some of the properties of the T-matrix which we shall find to be particularly useful later.

We start by considering the Schrödinger equation for scattering from a potential V which falls off more rapidly than $1/r$ as r becomes large.

$$(H_0 + V)\psi_k = E\psi_k, \qquad \text{V(1)}$$

where H_0 is the kinetic energy operator, $E = k^2$, and ψ_k is the scattering eigenfunction with as yet unspecified boundary conditions. Let us break ψ_k into two orthogonal parts

$$\psi_k = \phi_k + \chi_k, \qquad \text{V(2)}$$

where ϕ_k is a solution to the free Schrödinger equation with energy E,

$$H_0\phi_k = E\phi_k \,. \qquad \text{V(3)}$$

We adopt the normalization condition

$$\langle\psi_k|\psi_k\rangle = \langle\phi_k|\phi_k\rangle. \qquad \text{V(4)}$$

Inserting eq. (2) in eq. (1) and exploiting eq. (3), we may rewrite the Schrödinger equation as

$$(E - H_0)\chi_k = V\psi_k. \qquad \text{V(5)}$$

This equation may be solved formally to yield the Lippmann-Schwinger equation for the wave function

$$\psi_k^{\pm} = \phi_k + \frac{1}{E - H_0 \pm i\epsilon} V\psi_k^{\pm}, \qquad \text{V(6)}$$

where the addition of the $\pm i\epsilon$, like the separation of eq. (2), is carried out to ensure that the operator $(E - H_0)$ is never zero for E real and positive and thus that the

inverse of this operator exists. The particular prescription used for avoiding the zero in $(E - H_0)$ is reflected in the boundary conditions satisfied by the eigenfunction, $\psi_k^{\pm}$. In particular, the choice $+i\epsilon$ (with $\epsilon > 0$) yields an outgoing scattered wave while the choice $-i\epsilon$ yields an ingoing scattered wave. It is clear that by writing the Green function $1/(E - H_0 \pm i\epsilon)$ in its spectral representation eq. (6) becomes an integral equation.

It is useful to introduce a new operator, $T^{\pm}(E)$, such that

$$T^{\pm}(E)\,\phi_k = V\psi_k^{\pm} \qquad \text{V(7)}$$

for $E = k^2$. Using the definition and the spectral representation of the Green function

$$\frac{1}{E - H_0 \pm i\epsilon} = \frac{1}{(2\pi)^3}\int \mathrm{d}^3q\,\frac{|\mathrm{q}\rangle\langle\mathrm{q}|}{E - q^2 \pm i\epsilon} \qquad \text{V(8)}$$

with the plane wave normalization

$$\langle \mathrm{q}|\mathrm{q}'\rangle = (2\pi)^3\,\delta(\mathrm{q} - \mathrm{q}') \qquad \text{V(9)}$$

we may rewrite eq. (6) as an equation for the T-matrix:

$$\langle \mathrm{k}'|T^{\pm}(E)|\mathrm{k}\rangle = \langle \mathrm{k}'|V|\mathrm{k}\rangle + \frac{1}{(2\pi)^3}\int \mathrm{d}^3q\,\frac{\langle \mathrm{k}'|V|\mathrm{q}\rangle\langle \mathrm{q}|T^{\pm}(E)|\mathrm{k}\rangle}{E - q^2 \pm i\epsilon}. \qquad \text{V(10)}$$

For many applications it is convenient to work in a single partial wave channel. In this case we may perform the angular integrals in eq. (10) explicitly to yield

$$\langle k'|T_l^{\pm}(E)|k\rangle = \langle k'|V_l|k\rangle + \frac{2}{\pi}\int_0^{\infty} q^2\mathrm{d}q\,\frac{\langle k'|V_l|q\rangle\langle q|T_l^{\pm}(E)|k\rangle}{E - q^2 \pm i\epsilon}. \qquad \text{V(11)}$$

In eq. (11) the states $|k\rangle$ are unnormalized spherical Bessel functions, $j_l(kr)$, which satisfy the relation

$$\langle k|k'\rangle = \frac{\pi}{2}\frac{1}{k^2}\,\delta(k - k'). \qquad \text{V(12)}$$

To be definite, let us see how things work out for a local potential. Then,

$$\begin{aligned}\langle \mathrm{k}'|V|\mathrm{k}\rangle &= \int \mathrm{e}^{-i\mathrm{k}'\cdot\mathrm{r}}V(r)\,\mathrm{e}^{i\mathrm{k}\cdot\mathrm{r}}\,\mathrm{d}^3r \\ &= \int \sum_{l'} i^{-l'}(2l' + 1)P_{l'}(\hat{\mathrm{k}}'\cdot\hat{\mathrm{r}})j_{l'}(k'r)V(r)\sum_{l'} i^{l}(2l + 1)P_l(\hat{\mathrm{k}}\cdot\hat{\mathrm{r}})j_l(kr)\,\mathrm{d}^3r,\end{aligned} \qquad \text{V(12.1)}$$

where we have written things out in great detail. Using the addition theorem for spherical harmonics,

$$P_{l'}(\hat{\mathrm{k}}'\cdot\hat{\mathrm{r}}) = \frac{4\pi}{2l' + 1}\sum_m Y^*_{l'm}(\hat{\mathrm{k}}\cdot\hat{\mathrm{r}})Y_{l'm}(\hat{\mathrm{k}}'\cdot\hat{\mathrm{k}}), \qquad \text{V(12.2)}$$

we carry out the angular integral to give

$$\begin{aligned}\langle \mathrm{k}'|V|\mathrm{k}\rangle &= \sum_l 4\pi(2l + 1)P_l(\hat{\mathrm{k}}'\cdot\hat{\mathrm{k}})\int j_l(k'r)V(r)j_l(kr)r^2\,\mathrm{d}r \\ &= 4\pi\sum_l (2l + 1)P_l(\hat{\mathrm{k}}'\cdot\hat{\mathrm{k}})\,\langle k'|V_l|k\rangle.\end{aligned} \qquad \text{V(12.3)}$$

Let us specialize to a Yukawa potential

$$V = \frac{V_0}{r}\mathrm{e}^{-\mu r}, \qquad \text{V(12.4)}$$

which will be used in § E. Here we easily find

$$\langle \mathrm{k}'|V|\mathrm{k}\rangle = \frac{4\pi V_0}{(\mathrm{k}' - \mathrm{k})^2 + \mu^2}. \qquad \text{V(12.5)}$$

The expansion (12.3) is easily carried out. Let us, for example, find the lowest term in this expansion. Since $P_0(\hat{\mathrm{k}}'\cdot\hat{\mathrm{k}}) = 1$,

$$\langle \mathrm{k}'|V_{l=0}|\mathrm{k}\rangle = \frac{4\pi V_0}{4\pi}\,\frac{1}{2}\int_{-1}^{1}\frac{1}{k'^2 - 2k'k\,z + k^2 + \mu^2}\,\mathrm{d}z \qquad \text{V(12.6)}$$

$$= \frac{V_0}{4k'k}\ln\frac{(k' + k)^2 + \mu^2}{(k' - k)^2 + \mu^2}. \qquad \text{V(12.7)}$$

$$= \frac{V_0}{2k'k}\,Q_0\left(\frac{k'^2 + k^2 + \mu^2}{2k'k}\right).$$

The quantity on the right-hand side is the Legendre function of the second kind; we shall return in eq. V(75) to the case of general l. We remark, though, that the analytic properties to be discussed there are easily read off of our expression, eq. (12.7).

The T-matrix as defined by eqs. (10) or (11) is complex for real, positive E. We may obtain an explicitly real form of the T-matrix by choosing the Green function as the average of the ingoing and outgoing Green functions. The use of this Green function corresponds to standing wave boundary conditions on the scattered wave function. Noting that the ingoing and outgoing Green functions may be written as

$$\frac{1}{E - q^2 \pm i\epsilon} = \frac{\mathrm{P}}{E - q^2} \mp i\pi\delta(E - q^2), \qquad \text{V(13)}$$

we may express the T-matrix in the lth partial wave as

$$\langle k'|T_l(E)|k\rangle = \langle k'|V_l|k\rangle + \frac{2}{\pi}\mathrm{P}\int_0^{\infty} q^2\mathrm{d}q\,\frac{\langle k'|V_l|q\rangle\langle q|T_l(E)|k\rangle}{E - q^2}, \qquad \text{V(14)}$$

where the symbol P denotes a Cauchy principal value. The T-matrix has only been

defined in the case when $E = k^2$. It is possible to extend its definition by allowing E to assume all possible values in eqs. (10), (11), and (14). It remains to be seen if the "off-energy-shell" T-matrices so defined have any physical application. Note, however, that we have nowhere imposed the condition that $k'^2 = k^2$.

The fully on-energy-shell T-matrices* in the partial wave representations of eqs. (11) and (14) are closely related to phase shifts. In order to obtain this relation, we start from eq. (1) and the Hermitian adjoint of eq. (3). Multiply eq. (1) from the left by $(r^2\,\phi_k^*)$, multiply the adjoint of eq. (3) from the right by $(r^2\,\psi_k)$, and subtract the two equations. Integrate the resulting equation from $r = 0$ to $r = R$. The result of these operations, following an integration by parts, is

$$\left[-(r\phi_k^*)\frac{\mathrm{d}}{\mathrm{d}r}(r\psi_k)+(r\psi_k)\frac{\mathrm{d}}{\mathrm{d}r}r\phi_k^*)\right]_{r=R} = -\int_0^R r^2\,\mathrm{d}r\,\phi_k^* V\psi_k. \qquad \text{V(15)}$$

If R is chosen to be sufficiently large that $V(r) = 0$ for $r > R$, we may evaluate eq. (15) by replacing $(r\psi_k)$ and $(r\phi_k^*)$ by their asymptotic forms and by extending the upper limit of integration on the r.h.s. of eq. (15) trivially to ∞. For standing wave boundary conditions the asymptotic forms of $(r\phi_k^*)$ and $(r\psi_k)$ in the lth partial wave are

$$\lim_{r\to\infty}[r\phi_k^*] = \sin(kr + \tfrac{1}{2}l\pi)/k, \qquad \text{V(16)}$$

$$\lim_{r\to\infty}[r\psi_k] = \sin(kr + \tfrac{1}{2}l\pi + \delta_l)/k\cos\delta_l, \qquad \text{V(16.1)}$$

where the factor of $\cos\delta_l$ in eq. (16.1) is a consequence of the normalization condition, analogous to eq. (4), that $\langle\psi_k\,|\,\phi_k\rangle$ equals $\langle\phi_k\,|\phi_k\rangle$. Using eqs. (16) and (16.1) the l.h.s. of eq. (15) is seen to be independent of R and the r.h.s. is seen to be the fully on-shell T-matrix of eq. (14):

$$\langle k|T_l(E{=}k^2)|k\rangle = -(\tan\delta_l)/k. \qquad \text{V(17)}$$

Using the appropriate asymptotic forms for the case of ingoing and outgoing boundary conditions in place of eqs. (16), we find

$$\langle k|T_l^{\pm}(E{=}k^2)|k\rangle = -\,\mathrm{e}^{\pm i\delta_l}(\sin\delta_l)/k. \qquad \text{V(18)}$$

Although definitions of the T-matrix through integral equations of the Lippmann-Schwinger type are generally most convenient for actual calculations, a slightly different formal solution to the Schrödinger equation will enable us to see certain important properties of the T-matrix in a more direct fashion. Starting again with eqs. (1) through (4), we may write the Schrödinger equation as

$$(E - H)\chi_k = V\phi_k, \qquad \text{V(19)}$$

where the full Hamiltonian H is $H_0 + V$. This equation may be solved formally to

* The T-matrix is fully on shell if $k^2 = k'^2 = E$.

yield

$$\psi_k^{\pm} = \phi_k + \frac{1}{E - H \pm i\epsilon}V\phi_k, \qquad \text{V(20)}$$

where the $\pm i\epsilon$ prescription and $\psi_k^{\pm}$ are defined as before. Eq. (20) leads to alternate expressions for the T-matrix which we may write as

$$T^{\pm}(E) = V + V\frac{1}{E - H \pm i\epsilon}V. \qquad \text{V(21)}$$

By adding eqs. (21) for $T^+(E)$ and $T^-(E)$ we obtain

$$\tilde{T}(E) = V + V\frac{\mathrm{P}}{E - H}V. \qquad \text{V(21.1)}$$

Following partial wave projection we note that $\tilde{T}_l(E)$ is identical to $T_l(E)$ defined by eq. (14) for $E < 0$. The two quantities are not identical for $E > 0$ where it can be shown* that

$$\langle p|T_l(k^2)|q\rangle = \langle p|\tilde{T}_l(k^2)|q\rangle - \frac{k\tan\delta_l(k)}{\cos^2\delta_l(k)}\langle p|\tilde{T}_l(k^2)|k\rangle\langle k|\tilde{T}_l(k^2)|q\rangle. \qquad \text{V(21.2)}$$

From eq. (21) we note that, if V is Hermitean, $(T^+(E))^{\dagger}$ is equal to $T^-(E)$ for real E and from eq. (21.1) we note that $T(E)$ is Hermitean for real E. Eqs. (21) and (21.1) may be written in more explicit form by writing the full Green function, $1/(E - H \pm i\epsilon)$, in its spectral representation. To do this we need the complete set of eigenfunctions of H; both the scattering wave functions ψ_k^+ with eigenvalues k^2 and, if they exist, the bound state wave functions ψ_n with eigenvalues $E_n < 0$. The full Green function may then be written as

$$\frac{1}{E - H \pm i\epsilon} = \sum_n \frac{|\psi_n\rangle\langle\psi_n|}{E - E_n} + \frac{1}{(2\pi)^3}\int \mathrm{d}^3q\,\frac{|\psi_q^+\rangle\langle\psi_q^+|}{E - q^2 \pm i\epsilon}, \qquad \text{V(22)}$$

where we have assumed normalizations

$$\langle\psi_n|\psi_n'\rangle = \delta_{nn}, \text{ and } \langle\psi_q|\psi_q'\rangle = (2\pi)^3\delta(\mathrm{q} - \mathrm{q}'). \qquad \text{V(22.1)}$$

Using the representation eq. (22) of the Green function we may rewrite eq. (21) as

$$\langle \mathrm{k}|T^{\pm}(E)|\mathrm{k}'\rangle = \langle \mathrm{k}|V|\mathrm{k}'\rangle + \sum_n \frac{\langle \mathrm{k}|V|\psi_n\rangle\langle\psi_n|V|\mathrm{k}'\rangle}{E - E_n} + \frac{1}{(2\pi)^3}\int \mathrm{d}^3q\,\frac{\langle \mathrm{k}|V|\psi_{\mathrm{q}}^+\rangle\langle\psi_{\mathrm{q}}^+|V|\mathrm{k}'\rangle}{E - q^2 \pm i\epsilon}. \qquad \text{V(23)}$$

* D.J. Kouri and F.S. Levin, Phys. Lett. 48B (1974) 203.

Now using the definition of the T-matrix of eq. (7) we obtain the Low equation

$$\langle \mathbf{k}|T^{\pm}(E)|\mathbf{k}'\rangle = \langle \mathbf{k}|V|\mathbf{k}'\rangle + \sum_n \frac{\langle \mathbf{k}|V|\psi_n\rangle\langle\psi_n|V|\mathbf{k}'\rangle}{E-E_n} + \frac{1}{(2\pi)^3}\int d^3q\, \frac{\langle \mathbf{k}|T^{+}(q^2)|\mathbf{q}\rangle\langle\mathbf{q}|T^{-}(q^2)|\mathbf{k}\rangle}{E-q^2\pm i\epsilon}. \quad \text{V(24)}$$

We emphasize that eqs. (10) and (24) represent alternate expressions for the same quantity.

The study of eq. (24) reveals several useful properties of the T-matrix. First, we note that the T-matrix considered as a function of E has simple poles at the energies of the bound states of H. The residues of these poles are separable (i.e., the residue of $\langle \mathbf{k}|T(E_n)|\mathbf{k}'\rangle$ may be written as the product of a function of k and a function of k′) and are related to the corresponding bound state wave functions. This relation may be made even more direct by noting that

$$\langle \mathbf{k}|V|\psi_n\rangle = (E_n - k^2)\,\langle \mathbf{k}|\psi_n\rangle. \quad \text{V(24.1)}$$

For E in the vicinity of E_n the T-matrix should be dominated by a single pole term and, in subsequent sections, this fact will be exploited in developing simple approximations to the T-matrix. If the fully on-shell T-matrix (i.e., $\langle i\kappa_n|T(E_n)|i\kappa_n\rangle$ with $E_n = -\kappa_n^2$) is to display such a pole, we see from eq. (24.1) that $\langle k|\psi_n\rangle$ must also have a pole at $k = i\kappa_n$. The bound state wave function satisfies a homogeneous LS equation of the form of eq. (6), and the presence of the free Green function in this equation guarantees such a pole in $\langle k|\psi_n\rangle$. In the case of a weakly bound state with $L = 0$ we may use the approximate form of the wave function eq. III(17) to obtain

$$\langle k|\psi_n\rangle \sim \left[\frac{2\kappa_B}{1-\kappa_B r_0}\right]^{1/2} \frac{1}{k^2+\kappa_B^2}, \quad \text{V(24.2)}$$

where the terms neglected in eq. (24.2) are due to deviations from the asymptotic form eq. III(17) at short distances and are all regular at $k = i\kappa_B$*. We find that

$$(E_n - k^2)\,\langle k|\psi_n\rangle = -\left[\frac{2\kappa_B}{1-\kappa_B r_0}\right]^{1/2} \quad \text{for } k = i\kappa_B. \quad \text{V(24.3)}$$

The finite terms neglected in eq. (24.2) make no contribution to eq. (24.3) due to the zero in $(E_n - k^2)$. Thus we see that, in the case of a weakly bound state, the residue of the fully on-shell T-matrix at the pole is simply related to the square of the normalization constant in the tail of the wave function. Using eq. II(21) this residue may also be expressed in terms of the scattering length and effective range.

Let us now look at the difference between $\langle k|T^{+}(E)|k'\rangle$ and its Hermitean adjoint $\langle k|(T^{+}(E))^{\dagger}|k'\rangle$. Constructing this difference from eq. (24) (for real V and ψ_n) we

* In obtaining eq. (24.2) we assume that $\rho(-\kappa_n^2, -\kappa_n^2)$, given by eq. II(17) is approximately r_0.

find

$$\langle \mathbf{k}|T^{+}(E) - T^{-}(E)|\mathbf{k}'\rangle = \frac{1}{(2\pi)^3}\int d^3q\, \langle \mathbf{k}|T^{+}(q^2)|\mathbf{q}\rangle \left[\frac{1}{E+i\epsilon-q^2} - \frac{1}{E-i\epsilon-q^2}\right]\langle \mathbf{q}|T^{-}(q^2)|\mathbf{k}'\rangle. \quad \text{V(25)}$$

Using eq. (13) this may be written as

$$\langle \mathbf{k}|T^{+}(E) - T^{-}(E)|\mathbf{k}'\rangle = -\frac{i}{(2\pi)^2}\int d^3q\, \delta(E-q^2)\,\langle \mathbf{k}|T^{+}(q^2)|\mathbf{q}\rangle\langle\mathbf{q}|T^{-}(q^2)|\mathbf{k}'\rangle. \quad \text{V(25.1)}$$

We may perform the integral over the magnitude of q by recalling that

$$\delta(f(x)) = \frac{1}{\left|\frac{df}{dx}\right|_{x=x_0}}\delta(x_0), \quad \text{V(25.2)}$$

where $f(x_0) = 0$. Thus eq. (25.1) becomes

$$\langle \mathbf{k}|T^{+}(E) - T^{-}(E)|\mathbf{k}'\rangle = -\frac{i}{(8\pi^2)}\sqrt{E}\int d\Omega_q\, \langle \mathbf{k}|T^{+}(E)|\mathbf{q}\rangle\langle\mathbf{q}|T^{-}(E)|\mathbf{k}'\rangle, \quad \text{V(26)}$$

where $q^2 = E$. This equation indicates that $\langle \mathbf{k}|T^{+}(E)|\mathbf{k}'\rangle$ is cut along the positive E axis with a discontinuity given by the r.h.s. of eq. (26). In the special case when $\mathbf{k} = \mathbf{k}'$ and $E = k^2$ eq. (26) reduces to the familiar optical theorem which relates the imaginary part of the forward scattering amplitude to the total cross section at energy E. A result similar to eq. (26) is obtained for the T-matrix in the lth partial wave where we find

$$\langle k|T_l^{+}(E) - T_l^{-}(E)|k'\rangle = -2i\sqrt{E}\,\langle k|T_l^{+}(E)|\sqrt{E}\rangle\langle\sqrt{E}|T_l^{-}(E)|k'\rangle, \quad \text{V(27)}$$

which is clearly consistent with eq. (18) for the fully on shell case.

We note that if V is independent of energy, it is possible to cast the Low equation into a form in which V does not appear explicitly. To do this we may write eq. (24) for $\langle \mathbf{k}|T^{\pm}(E)|\mathbf{k}'\rangle$ and subtract the corresponding equation for the half-shell* T-matrix $\langle \mathbf{k}|T^{\pm}(\mathbf{k}^2)|\mathbf{k}'\rangle$. Using eq. (24.1) we find as a result of these operations that the fully off-shell T-matrix can be expressed in terms of half-shell T-matrix elements and the bound state eigenfunctions and eigenvalues. Such a relation is frequently convenient numerically when it is necessary to calculate many off-shell T-matrix elements.

* By "half-shell" we mean that $k^2 \neq k'^2$.

V.B. Separable potentials

Solutions to the Lippmann-Schwinger equation for arbitrary potentials $\langle k|V|k'\rangle$ must be obtained numerically. This is not a difficult task and appropriate numerical methods will be discussed below. It may be useful to study first some simple potentials which allow analytic solution to the Lippmann-Schwinger equation; particularly if these simple potentials also have some physical significance. (Before embarking, however, the reader is reminded of the well-known complementarity relation between truth and clarity.) In this section we shall consider the special case of separable potentials.

First, consider the momentum space matrix element in the lth partial wave of a general, non-local potential $V(r,r')$:

$$\langle k|V_l|k'\rangle = \int r^2 \mathrm{d}r \int r'^2\ \mathrm{d}r'\ j_l(kr)\ V(r,r')\ j_l(k'r'). \qquad \text{V(28)}$$

The potential $V(r,r')$ is separable if it has the form

$$V(r,r') = \lambda\, v(r)\, v(r') \qquad \text{V(29)}$$

or alternatively

$$\langle k|V_l|k'\rangle = \lambda\, v_l(k)\, v_l(k'), \qquad \text{V(29.1)}$$

where we define

$$v_l(k) = \int r^2\ \mathrm{d}r\, j_l(kr)\, v(r). \qquad \text{V(29.2)}$$

Near threshold we see that the behaviour of $j_l(kr)$ demands that $v_l(k)$ be proportional to k^l. We also note that for large k we may exploit the asymptotic behaviour of the spherical Bessel function to see that $v_l(k)$ must vanish more rapidly than k^{-1}.

Potentials of the form eqs. (29) or (29.1) do allow analytic solution of the Lippmann-Schwinger equation and lead to T-matrices which are also separable. We recall that in the vicinity of a bound state pole the T-matrix has the form

$$\langle k|T(E)|k'\rangle \sim \frac{\langle k|V|\psi_B\rangle\langle\psi_B|V|k'\rangle}{E - E_B}, \qquad \text{V(30)}$$

which is also separable. Thus, we can expect that a separable interaction may be useful in providing simple analytic approximations to T-matrices near bound-state poles. Since both the $^1S_0$ and $^3S_1$ nucleon-nucleon interaction have such poles near $E = 0$, we may expect that the true nucleon-nucleon T-matrix near zero energy is approximately separable in S-waves. No other partial waves have either bound states or low energy resonances and separable approximations to the nucleon-nucleon interaction must be regarded with caution.

Under certain conditions even familiar local potentials may appear to be approximately separable. Consider the particularly well chosen example of a square well of radius a and depth V_0. From the relation

$$\frac{\mathrm{d}}{\mathrm{d}r}\left[r j_{l+1}(kr) j_{l+1}(k'r)\right] = \frac{r^2 kk'}{2l+3}\left[j_l(kr) j_l(k'r) - j_{l+2}(kr) j_{l+2}(k'r)\right], \qquad \text{V(31)}$$

we may immediately obtain the momentum space matrix elements

$$\langle k|V_l|k'\rangle = V_0 \int_0^a j_{l+2}(kr) j_{l+2}(k'r) r^2 \mathrm{d}r + V_0 \frac{2l+3}{kk'} a j_{l+1}(ka) j_{l+1}(k'a). \qquad \text{V(31.1)}$$

If a is sufficiently small and k and/or k' are sufficiently low, we may neglect the first term on the r.h.s. of eq. (31) to arrive at the following separable approximation to a square well potential

$$\langle k|V_l|k'\rangle \sim a^3 V_0 (2l+3) \frac{j_{l+1}(ka)}{ka} \frac{j_{l+1}(k'a)}{k'a}. \qquad \text{V(31.2)}$$

Such an approximation can be useful in describing the square well and momenta arising in the NN interaction. Also we note that the repeated use of eqs. (31) and (31.1) yields a representation of the square well as an infinite sum of separable terms

$$\langle k|V_l|k'\rangle = V_0 a^3 \sum_{n=0}^{\infty} (2l+2n+3) \frac{j_{l+2n+1}(ka)}{ka} \frac{j_{l+2n+1}(k'a)}{k'a}. \qquad \text{V(31.3)}$$

Truncation of this series after N terms yields a rank-N separable approximation to the potential for which the Lippmann-Schwinger equation may again be solved analytically. In fact, there exist systematic techniques for expressing any potential as an infinite sum of separable terms.

With some confidence that we are studying a physically interesting problem, we turn to the solution of the Lippmann-Schwinger equation for potentials of the form eq. (29.1). Starting from eq. (14) we write

$$\langle k|T(E)|k'\rangle = \lambda v(k) v(k') - \frac{2}{\pi} \mathrm{P} \int q^2 \mathrm{d}q\, \frac{\lambda v(k) v(q) \langle q|T(E)|k'\rangle}{q^2 - E}. \qquad \text{V(32)}$$

From this equation we find immediately that the T-matrix may be written as

$$\langle k|T(E)|k'\rangle = \lambda v(k) \tau(E,k') \qquad \text{V(32.1)}$$

for all k,E,k'. Inserting eq. (32.1) into eq. (32) we find

$$\tau(E,k') = v(k') - \frac{2}{\pi} \lambda\, \mathrm{P} \int \frac{q^2 v^2(q)}{q^2 - E} \tau(E,k')\, \mathrm{d}q. \qquad \text{V(32.2)}$$

The term $\tau(E,k')$ may be removed from the integrand and eq. (32.2) solved algebrai-

cally to yield

$$\langle k|T(E)|k'\rangle = \frac{\lambda\, v(k)\, v(k')}{1+\frac{2}{\pi}\lambda\, \mathrm{P}\int \frac{q^2\,\mathrm{d}q\, v^2(q)}{q^2-E}}. \qquad \text{V(33)}$$

As mentioned above, the T-matrix obtained from a separable potential is itself separable. Phase shifts may be obtained from eq. (33) by setting $k = k'$ and $E = k^2$ and using eq. (17).

As suggested by eq. (30), the T-matrix will have a simple pole in E at the energy of a bound state. This can happen in eq. (33) only when the denominator vanishes:

$$\frac{2}{\pi}\lambda \int_0^\infty \frac{q^2\,\mathrm{d}q\, v^2(q)}{q^2+\kappa_\mathrm{B}^2} = -1 \qquad \text{V(33.1)}$$

where we have written $E_\mathrm{B} = -\kappa_\mathrm{B}^2$ to emphasize that the integrand of eq. (33.1) is non-negative. The l.h.s. of eq. (33.1) is easily seen to be a monotonically decreasing function of κ_B^2. Thus the T-matrix defined by eq. (33) can have no more than one bound state. The criterion that a bound state should exist is

$$\frac{2}{\pi}\lambda \int v^2(q)\,\mathrm{d}q < -1, \qquad \text{V(33.2)}$$

which can only be satisfied for $\lambda < 0$.

It is possible to write eq. (33) in a useful alternate form in the case where eq. (33.2) is satisfied and the potential has a single bound state. Making use of eq. (33.1) we may express the T-matrix in a form which explicitly displays the bound state pole

$$\langle k|T(E)|k'\rangle = \frac{\lambda\, v(k) v(k')}{\frac{2}{\pi}\lambda\,(E-E_\mathrm{B})\,\mathrm{P}\int \frac{q^2 v^2(q)\,\mathrm{d}q}{(q^2-E_\mathrm{B})(q^2-E)}}. \qquad \text{V(34)}$$

Comparing eq. (34) with eq. (30) in the vicinity of E_B and making use of the fact that

$$\langle k|V|\psi_\mathrm{B}\rangle = (E_\mathrm{B}-k^2)\,\langle k|\psi_\mathrm{B}\rangle, \qquad \text{V(34.1)}$$

we can immediately write the bound state wave function for the potential defined by eq. (29.1) as

$$\langle k|\psi_\mathrm{B}\rangle = N^{1/2}\frac{v(k)}{k^2-E_\mathrm{B}}, \qquad \text{V(34.2)}$$

with the somewhat unusual normalization condition

$$\int_0^\infty k^2\,|\langle k|\psi_\mathrm{B}\rangle|^2\,\mathrm{d}k = \tfrac{1}{2}\pi \qquad \text{V(34.3)}$$

which is a consequence of our use of unnormalized plane waves. The wave function may be obtained in coordinate space as

$$\psi_\mathrm{B}(r) = N^{1/2}\int_0^\infty j_l(kr)\,k^2\mathrm{d}k\,\frac{v_l(k)}{k^2-E_\mathrm{B}}. \qquad \text{V(34.4)}$$

This Fourier-Bessel transform may be performed by replacing $j_l(kr)$ with $h_l^{\frac{1}{2}}(kr)$ and extending the limits of integration to $-\infty \leqslant k \leqslant +\infty$. The integral is then generally suitable for evaluation by closing the contour in the upper half-plane. In the case of a weakly bound state we may evaluate the integral in eq. (34.4) by retaining only the residue of the pole at $k = i\kappa_\mathrm{B}$ while neglecting the singularities of $v(k)$ which are assumed to be farther from $k = 0$. Thus we find

$$\psi_\mathrm{B}(r) \sim C h_l^2(i\kappa_\mathrm{B} r), \qquad \text{V(34.5)}$$

which yields the familiar result for a weakly bound state at large r,

$$\psi_\mathrm{B}(r) \sim C[\exp -\kappa_\mathrm{B} r]/r. \qquad \text{V(34.6)}$$

Using eqs. (34.1) and (34.2) we may reexpress the T-matrix only in terms of the bound state wave function and the binding energy:

$$\langle k|T(E)|k'\rangle = \frac{(E_\mathrm{B}-k^2)(E_\mathrm{B}-k'^2)}{E-E_\mathrm{B}}\,\frac{\langle k|\psi_\mathrm{B}\rangle\langle\psi_\mathrm{B}|k'\rangle}{1+\frac{2}{\pi}(E-E_\mathrm{B})\,\mathrm{P}\int \frac{q^2\langle q|\psi_\mathrm{B}\rangle^2\,\mathrm{d}q}{q^2-E}}. \qquad \text{V(35)}$$

In the vicinity of $E \sim E_\mathrm{B}$, eq. (35) reduces to the expected pole-dominated form of eq. (30). Eq. (35) has a somewhat wider use. Given the bound state wave function and E_B for any potential (not necessarily separable) we may employ eq. (35) as a convenient approximation to the T-matrix in the vicinity of E_B. This approximation has the additional virtue of being unitary (since it arises formally from solving the LS equation) and may thus have a wider useful range than the manifestly non-unitary approximation of eq. (30). We note that this approximation may be of more than formal interest. The experimentally measured charge form factor is related to the bound state wave function by

$$F_\mathrm{C}(q) = \int |\psi_\mathrm{B}(r)|^2 j_0(qr)\,r^2\mathrm{d}r. \qquad \text{V(36)}$$

Measurement of the charge form factor should thus enable us to use eq. (35) directly to approximate the T-matrix. As mentioned earlier it is not easy to measure the charge form factor of the deuteron cleanly. The difficulty comes from the fact that the electric quadrupole moment of the deuteron can interact with the electric field gradient. This quadrupole contribution is found to dominate the deuteron electric form factor for $q > 3\mathrm{fm}^{-1}$ making it essentially impossible to extract the deuteron charge form factor and thus the deuteron wave function.

As a simple example of the use of eq. (35) and in order to gain a little more experience with momentum-space calculations, we indicate how the usual expression for the scattering length may be obtained for a potential with a weakly bound state. We define the scattering length and effective range in the usual way

$$k \cot \delta \sim -1/a + \tfrac{1}{2} r_0 k^2. \qquad \text{V(37)}$$

If the binding energy is very small, we may exploit the form of eq. (35) for calculating the on-shell T-matrix for small positive energy and we may employ the asymptotic form of the wave function eq. (34.5) everywhere. Considering only S-waves we find that the properly normalized wave function is

$$\psi_B(r) = \left[\frac{2\kappa_B}{1-\kappa_B r_0}\right]^{1/2} \frac{\exp -\kappa_B r}{r} \quad \text{or} \quad \langle k|\psi_B\rangle = \frac{1}{k^2+\kappa_B^2}\left[\frac{2\kappa_B}{1-\kappa_B r_0}\right]^{1/2} \qquad \text{V(38)}$$

where the effective range r_0 has been introduced to describe the effects of deviations from the asymptotic form of the wave function at short distance on the normalization of the wave function at large distances. This use of r_0 is completely consistent with its definition in eq. II(18). Let us write the fully on-shell T-matrix using eq. (35) and the wave function defined by eq. (38):

$$\langle k|T(k^2)|k\rangle = \frac{2\kappa_B}{(k^2+\kappa_B^2)\left[(1-\kappa_B r_0) + \frac{2}{\pi}(k^2+\kappa_B^2)\,\mathrm{P}\int \frac{2\kappa_B\, q^2\, dq}{(q^2-k^2)(q^2+\kappa_B^2)^2}\right]}. \qquad \text{V(39)}$$

The integral in the denominator of eq. (39) may be performed by closing the contour in either half-plane and neglecting the poles at $q = \pm k$ as demanded by the principal value prescription to yield

$$\frac{2}{\pi} 2\kappa_B (k^2+\kappa_B^2)\,\mathrm{P}\int \frac{q^2\, dq}{(q^2-k^2)(q^2+\kappa_B^2)^2} = \frac{\kappa_B^2 - k^2}{\kappa_B^2 + k^2} \qquad \text{V(39.1)}$$

and thus

$$\langle k|T(k^2)|k\rangle = \frac{1}{\kappa_B - \frac{1}{2}\kappa_B^2 r_0 - \frac{1}{2} r_0 k^2}. \qquad \text{V(39.2)}$$

Invoking the usual relation between the on-shell T-matrix and phase shifts (given by eq. (17)) and definition eq. (37) of the scattering length, we find

$$1/a = \kappa_B - \tfrac{1}{2}\kappa_B^2 r_0, \qquad \text{V(39.3)}$$

which is familiar result obtained in eq. II(21) by studying the Schrödinger equation directly. We remark that no approximation was made other than eq. (38). We also note that the unitarity corrections which distinguish eq. (35) from eq. (30) are in no sense small even when κ_B is small. In particular, replacement of eq. (35) by the non-unitary pole form eq. (30) in the above derivation would have led to an error of a factor of two in the resulting scattering length.

The results of eqs. (34) and (35) are readily extended to the case where the interaction contains tensor components which can mix partial waves. This generalization can yield a simple approximate relation between the quadrupole moment of the deuteron and ϵ, the mixing parameter between S- and D-waves, at low energies. Such a relation is of considerable value since not even the sign of ϵ has been determined reliably at low energies. The definition of the mixing parameter is

$$\tan 2\epsilon = \frac{\langle k|T_{02}(k^2)|k\rangle + \langle k|T_{20}(k^2)|k\rangle}{\langle k|T_{00}(k^2)|k\rangle - \langle k|T_{22}(k^2)|k\rangle}. \qquad \text{V(40)}$$

Comparison with the explicit asymptotic forms of the coupled channel wave functions given in sect. II.F shows that $\tan\epsilon$ is simply the quantity η_α of eq. II(43). Using the coupled channel generalization of eq. (35), which should be approximately valid for a weakly bound state like the deuteron, we obtain

$$\tan 2\epsilon = \frac{\langle k|\psi_S\rangle\langle\psi_D|k\rangle + \langle k|\psi_D\rangle\langle\psi_S|k\rangle}{\langle k|\psi_S\rangle\langle\psi_S|k\rangle - \langle k|\psi_D\rangle\langle\psi_D|k\rangle}. \qquad \text{V(40.1)}$$

In the limit $k^2 \to 0$ we obtain

$$\epsilon \sim \frac{\langle k|\psi_D\rangle}{\langle k|\psi_S\rangle} \equiv \frac{\int_0^\infty j_2(kr)\psi_D(r) r^2\, dr}{\int_0^\infty j_0(kr)\psi_S(r) r^2\, dr}. \qquad \text{V(40.2)}$$

Keeping only the leading order terms in the spherical Bessel functions in these integrands yields

$$\epsilon \sim \tfrac{1}{15} k^2 \int_0^\infty \psi_D(r) r^4 dr \Big/ \int_0^\infty \psi_S(r) r^2 dr. \qquad \text{V(40.3)}$$

We now assume that both integrals are dominated by the contributions from outside the range of the force where we may approximate the wave functions as

$$\psi_D(r) \sim N_D \frac{\exp -\kappa_B r}{\kappa_B r}\left[1 + \frac{3}{\kappa_B r} + \frac{3}{(\kappa_B r)^2}\right],$$

$$\psi_S(r) \sim N_S \frac{\exp -\kappa_B r}{\kappa_B r} \qquad \text{V(40.4)}$$

Continuing in the spirit of neglecting terms of high-order in the D-state amplitude, we assume that the correct normalization condition for the bound state wave function is

$$\int_0^\infty \psi_S^2\, r^2\, dr = 1 \qquad \text{V(40.5)}$$

and that it may be obtained using the asymptotic form of the ψ_S

$$N_S = \sqrt{2}\kappa_B^{3/2}. \qquad \text{V(40.6)}$$

Performing the integrals in eq. (40.3) yields

$$\epsilon = \frac{N_D k^2}{\sqrt{2}\kappa_B^{7/2}}. \qquad \text{V(40.7)}$$

From eq. II(36.5) the quadrupole moment may be written as

$$Q = \sqrt{\tfrac{1}{50}} \int_0^\infty \psi_S(r)\psi_D(r) r^4 \mathrm{d}r - \tfrac{1}{20} \int_0^\infty \psi_D(r)\psi_D(r) r^4 \mathrm{d}r.$$

Neglecting the second term entirely and evaluating the first using the asymptotic wave functions yields

$$Q = \frac{N_D}{2\kappa_B^{7/2}}. \qquad \text{V(40.8)}$$

Thus, we expect that

$$\epsilon(k^2) = \sqrt{2} Q k^2. \qquad \text{V(40.9)}$$

For the Reid potential $Q = 0.282$ fm^2 which suggests that at a laboratory energy of 1 MeV $\epsilon = 0.276°$. Direct calculation yields $\epsilon = 0.201°$. The discrepancy is consistent with the neglect of terms down by order $\sqrt{P_D} \sim 0.26$. The success of eq. (40.9) in this model calculation and the positive value of the experimental deuteron quadrupole moment strongly suggest that ϵ should be positive at low energies.

The T-matrix from two potentials may be constructed directly through the LS equation. There are cases, however, when it may be desirable to take a somewhat less direct approach. Specifically, let us start from the ordinary LS equation

$$T = (V_1 + V_2) + (V_1 + V_2) GT \qquad \text{V(41)}$$

and artificially break T up into T_1 and T_2 which satisfy the equations

$$T_1 = V_1 + V_1 G T_1 + V_1 G T_2, \qquad \text{V(41.1)}$$

$$T_2 = V_2 + V_2 G T_1 + V_2 G T, \qquad \text{V(41.2)}$$

and where $T = T_1 + T_2$. Eqs. (41.1) and (41.2) may be rewritten formally as

$$T_1 = (1 - V_1 G)^{-1} V_1 [1 + G T_2], \qquad \text{V(42)}$$

$$T_2 = (1 - V_2 G)^{-1} V_2 [1 + G T_1]. \qquad \text{V(42.1)}$$

However, the quantity $(1 - V_1 G)^{-1} V_1$ is simply the formal solution to the LS equation for potential V_1 acting alone. Thus we may obtain the full T-matrix from

coupled integral equations involving the T-matrices, t_1 (t_2) obtained for potential V_1 (V_2) acting alone

$$T_1 = t_1 + t_1 G T_2, \qquad \text{V(43)}$$

$$T_2 = t_2 + t_2 G T_1. \qquad \text{V(43.1)}$$

It may be useful to note that these equations may be decoupled by the simple expedient of iterating once to obtain

$$T_1 = t_1 + t_1 G t_2 + t_1 G t_2 G T_1, \qquad \text{V(44)}$$

$$T_2 = t_2 + t_2 G t_1 + t_2 G t_1 G T_2. \qquad \text{V(44.1)}$$

A few remarks are in order before proceeding. From eqs. (43) and (43.1) or (44) and (44.1) we can see that if t_1 and t_2 are separable, T_1 and T_2 will also be separable. Thus we see that a rank-two separable potential (i.e., both V_1 and V_2 separable) leads to a rank-two separable T-matrix. In this case eqs. (44) and (44.1) may be solved algebraically following techniques completely analogous to those leading to eq. (33).

One special case of some interest is when one of the potentials is very strong (e.g., the ordinary NN interaction) while the other potential is sufficiently weak that it need be dealt with only in first order (e.g., the NN weak interaction). In this case we wish to retain only first order in V_w but all orders in V_s. From eqs. (35) and (35.1) this corresponds to

$$T = t_s + (1 + G t_s)^\dagger V_w (1 + G t_s) + O(V_w^2). \qquad \text{V(45)}$$

It may be convenient to define an effective weak interaction

$$\widetilde{V}_w = (1 + G t_s)^\dagger V_w (1 + G t_s). \qquad \text{V(45.1)}$$

Comparison with eq. (6) shows that the operator $(1 + G t_s)$ operating on solutions to the free Schrödinger equation yields solutions to the Schrödinger equation with potential V_s. In particular, if V_s contains strong short-range repulsion, the effective operator $\widetilde{V}_w$ will be appropriately cut off at short distances. The form eq. (45.1) is cumbersome to work with since $(1 + G t_s)$ is, in general, both energy dependent and non-local. Simplifications may be found in the case when t_s is separable (in which case $(1 + G t_s)$ is also separable and in the case when we merely wish to incorporate the effects of very strong short range repulsion (in which case it may be possible to approximate $(1 + G t_s)$ by a local and energy-independent function).

The results of eqs. (41) may be further generalized to include the case of n arbitrary potentials. We again find that the equations

$$T = \sum_{i=1}^{n} T_i \qquad \text{V(46)}$$

with

$$T_i = V_i + V_i G \sum_{j=1}^{n} T_j \qquad \text{V(46.1)}$$

are completely equivalent to the LS equation for T with the potential $\Sigma_{i=1}^{n} V_i$. The set of eqs. (46.1) again admits solution by algebraic means in the special case when all potentials V_i are separable. In this case we can again write T_i in the form

$$\langle k|T_i(E)|k'\rangle = \lambda_i v_i(k)\, \tau_i(E,k'). \qquad \text{V(47)}$$

The n equations defined by eq. (46.1) thus reduce to a set of n linear equations in the unknowns $\tau_i(E,k')$

$$\sum_{j=1}^{n} [\delta_{ij} + \lambda_j f_{ij}(E)]\, \tau_j(E,k') = v_i(k'), \qquad \text{V(47.1)}$$

where we have introduced the notation

$$f_{ij}(E) = \frac{2}{\pi} \mathrm{P} \int_0^\infty \frac{q^2}{q^2 - E}\, v_i(q) v_j(q)\, \mathrm{d}q. \qquad \text{V(47.2)}$$

These linear equations may be solved by any convenient method to yield $\tau_i(E,k')$ and hence T. The matrix on the l.h.s. of eq. (47.1) is independent of k' and all k' dependence appears in the inhomogeneous term. Thus we may obtain $\langle k|T(E)|k'\rangle$ for all k and k' for fixed E by a single matrix inversion. Should it occur that $f_{ij}(E) = \delta_{ij}$, eqs. (47.1) and (47.2) may be solved by inspection. We shall see in the following section that this apparently artificial situation arises naturally when we discuss the separable expansion of arbitrary potentials.

V.C. Separable expansions of arbitrary potentials

Having seen the particularly simple forms of the T-matrix which arise in the case of finite rank separable potentials, it is natural to wonder if arbitrary potentials can be approximated in a systematic way by finite rank separable potentials. We saw in eq. (31.3) that it was possible to provide such an approximation by exploiting special properties of the spherical Bessel functions. In this section we shall describe a more general method proposed by Weinberg*. For convenience we shall confine the discussion to negative E.

* S. Weinberg, Phys. Rev. 131 (1963) 440.

In deriving the LS equation from the Schrödinger equation we wrote the operator $(H_0 - E)^{-1}$ in its (separable) eigenfunction expansion. We can also cast the problem into a slightly different form by looking for an eigenfunction expansion of the slightly more complicated operator $(H_0 - E)^{-1}\, V$ where V is an arbitrary potential. Thus we wish to find solutions of the equation

$$-(H_0 - E)^{-1}\, V\psi_i = \eta_i(E)\,\psi_i, \qquad \text{V(48)}$$

which is simply the homogeneous form of the LS equation. For positive E we would expect the eigenvalues $\eta_i(E)$ to be complex. For the case of negative E, eq. (48) is simply the familiar bound state Schrödinger equation where ψ_i is a bound state eigenfunction for the potential $V/\eta_i(E)$ with energy E. In this case the eigenvalues $\eta_i(E)$ are all real. If the potential V is everywhere positive we expect all $\eta_i(E) < 0$. If V is everywhere negative, all $\eta_i(E) > 0$. In general, we expect both signs for $\eta_i(E)$.

Since the various ψ_i are solutions to *different* Schrödinger equations with the *same* energy, it should come as no surprise that they satisfy a somewhat unusual orthogonality relation. Writing the Schrödinger equation for ψ_i and the adjoint equation for ψ_j,

$$(E - H_0)|\psi_i\rangle = \frac{1}{\eta_i(E)}\, V|\psi_i\rangle, \qquad \text{V(49)}$$

$$\langle\psi_j|(E - H_0) = \langle\psi_j|V\frac{1}{\eta_j(E)}, \qquad \text{V(49.1)}$$

we obtain the orthogonality relation by multiplying eq. (49) from the left by $\langle\psi_j|$, multiplying eq. (49.1) from the right by $|\psi_i\rangle$ and subtracting to yield

$$\left[\frac{1}{\eta_i(E)} - \frac{1}{\eta_j(E)}\right]\langle\psi_j(E)|V|\psi_i(E)\rangle = 0. \qquad \text{V(50)}$$

Since we are at liberty to determine the normalization of the wave functions, we find the following choice to be particularly convenient

$$\langle\psi_i(E)|V|\psi_j(E)\rangle = \sigma_i\, \delta_{ij}, \qquad \text{V(51)}$$

where

$$\sigma_i = -\eta_i(E)/|\eta_i(E)|. \qquad \text{V(51.1)}$$

The functions ψ_i form the basis for a separable representation of V of the form

$$V = \sum_i \sigma_i V|\psi_i(E)\rangle\langle\psi_i(E)|V, \qquad \text{V(52)}$$

where the sum is, in principle, infinite. To see that this representation of V is valid we substitute eq. (52) in eq. (48) to find

$$-\sum_i \sigma_i (H - E_0)^{-1}\, V|\psi_i\rangle\langle\psi_i|V|\psi_j\rangle = \eta_j(E)|\psi_j\rangle. \qquad \text{V(53)}$$

Using eq. (48) and the orthonormality condition eq. (51) we find

$$\sigma_j^2\, \eta_j(E)|\psi_j(E)\rangle = \eta_j(E)|\psi_j(E)\rangle. \qquad \text{V(54)}$$

Since $\sigma_j^2 = 1$ for all j, this justifies the representation eq. (52). We may thus find a rank-n separable approximation to V, and hence to T, by retaining n terms in this expansion. In order to decide which terms to retain let us construct the T-matrix at energy E.

Comparison with eqs. (47.1) and (47.2) yields $\lambda_i = \sigma_i$ and $f_{ij} = -\eta_j \sigma_j \delta_{ij}$, thus verifying the claim that this particularly simple form f_{ij} can arise in physically interesting situations. The linear equations (47.1) may now be solved to yield

$$T(E) \cong \sum_{i=1}^{n} \sigma_i \frac{V|\psi_i(E)\rangle\langle\psi_i(E)|V}{1-\eta_i(E)}. \qquad \text{V(55)}$$

If it should happen that the potential V actually has a bound state in the vicinity of E, some $\eta_i(E) \sim 1$ and eq. (55) reduces to the familiar pole dominated form. The particularly simple form of eq. (55) obtains only because we have constructed the T-matrix at the same energy as the separable expansion of V. In constructing the T-matrix at the same energy from the separable representation eq. (52) we are not allowed to exploit eq. (51).

If such an approximation scheme is to be of practical use, it is necessary to introduce some quantitative measure of the difference between the exact $T(E)$ and some rank-n approximation $T^{(n)}(E)$. We shall make the following convenient choice

$$||T^2|| = \mathrm{Tr}\,[(H_0 - E)^{-1}\, T(H_0 - E)^{-1}\, T], \qquad \text{V(56)}$$

which may be written explicitly in terms of the unnormalized plane wave states $|k\rangle$ as

$$||T^2|| = \frac{2}{\pi}\int_0^\infty k^2 \mathrm{d}k\, \langle k|(H_0 - E)^{-1} T (H_0 - E)^{-1} T\, |k\rangle \qquad \text{V(56.1)}$$

Working from eq. (56) directly and using eq. (48) and the normalization conditions of eqs. (51) and (51.1), we find that

$$||T^2|| = \sum_{i=1}^{\infty} \frac{1}{(1-1/\eta_i)^2}. \qquad \text{V(56.2)}$$

It is also easy to show, by direct calculation, that the *best* rank one separable approximation to T is simply

$$T^{(1)} = \sigma_1 \frac{V|\psi_1\rangle\langle\psi_1|V}{1-\eta_1(E)} \qquad \text{V(57)}$$

where $\eta_1(E)$ is that eigenvalue which minimizes $(1 - 1/\eta_i)^2$. The obvious inductive

argument lets us find the best rank-n approximation as being precisely eq. V(55) where we have arranged the terms in order of ascending $(1 - 1/\eta_i)^2$. By *best* we clearly mean that rank-n separable approximation which minimizes $||(T - T^{(n)})^2||$ where the norm is again defined by eq. (56). It is easy to see that for this best choice of $T^{(n)}$,

$$||(T - T^{(n)})^2|| = \sum_{i=n+1}^{\infty} \frac{1}{(1-1/\eta_i)^2}. \qquad \text{V(58)}$$

Eq. (58) provides a practical aid in deciding how quickly we may truncate the expansion of T and of estimating the effect of the neglected terms.

Since it is inconvenient to perform the sum indicated in eq. (58) for infinite i, it is useful to study the asymptotic structure of the η_i for some simple, and hopefully representative potential. The simplest such example is a totally attractive square well of radius a and depth V_0. For sufficiently large i we find that

$$\eta_i \cong \frac{a^2 V_0 m}{\pi^2 \hbar^2} \frac{1}{i^2} \qquad \text{V(59)}$$

where m is the particle mass. In this case the norm eq. (58) should converge asymptotically like n^{-1}. In the table we list the first few attractive and repulsive eigenvalues for the more realistic Reid $^1S_0$ potential which provides a quantitative fit to $^1S_0$ NN phase shifts for laboratory energies below ~350 MeV. This potential has strong short-range repulsion as indicated by the small initial values of $1/\eta_i^R$. Even in this case the spectra η_i^A and η_i^R display roughly the behaviour indicated in the much simpler square well calculation. The ith attractive (or repulsive) eigenfunction ψ_i represents the ith bound state of the potential V/η_i and its coordinate space representation should thus have $(i-1)$ nodes over roughly the range of the attractive (or repulsive) part of the force. We might expect that, for $1/k$ on the order of the range of the force, the states $|k\rangle$ and $V|\psi_i\rangle$ would grow more nearly orthogonal for increasing i. Thus the convergence of $\langle k|T(E)|k'\rangle$ for small k and k' may be more rapid than the convergence suggested by the spectrum η_i and eq. (58).

Table V.1
Eigenvalues of Reid $^1S_0$ potential constructed at $E = -20$ MeV.

i	$(\eta_i^A)^{-1}$	$(\eta_i^R)^{-1}$
1	2.30	−0.071
2	12.30	−0.260
3	30.22	−0.512
4	42.66	−0.743
5	71.77	−0.992
6	156.46	−1.067
7	328.1	−1.882

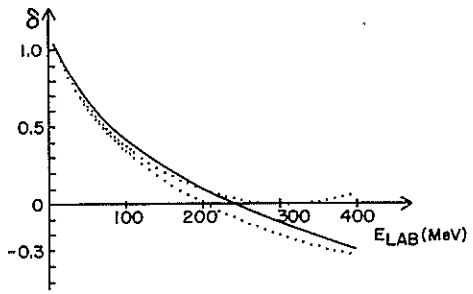

Fig. 15. Phase shifts for the Reid $^1S_0$ potential. Solid curve represents exact results, the upper and lower dotted curves represent one- and two-term separable expansions respectively.

One drawback of the Weinberg approach in practical calculations is that the eigenvalues and eigenfunctions must be constructed separately for each energy E. Levinger and co-workers have noted that a rank-n potential of the form of eq. (52) constructed at energy E will lead to a separable representation of $T(E')$ for all E'. Of course, it is only in the special case of $E = E'$ that the linear equations of eq. (47.1) may be inverted trivially to yield the particularly simple form eq. (55) and only in this case that the resulting separable approximation to $T(E')$ can be guaranteed to minimize the norm eq. (58). It is hoped that these losses will be compensated by the need to calculate the η_i and ψ_i once only at some representative energy. In practice such a scheme is frequently of use. For example, the phase shifts of the Reid $^1S_0$ potential have been calculated using the rank-one and -two approximations to V constructed at zero energy and are compared with the exact results in fig. 15. The one-term approximation is extremely good at low energies due to the fact that $1/\eta_i^A = 1.082$*. This one-term approximation, like all other rank-one separable potentials, has phase shifts of one sign. Thus the rank-one approximation breaks down for laboratory energies greater than 250 MeV. The two-term approximation does provide a reasonable qualitative description of the phase shifts at all energies. Additional results at other energies are tabulated by Harms and Newton**.

The calculations and techniques here can be extended without difficulty to the case of coupled channels. Study of the spectrum of the Reid $^3S_1 - {}^3D_1$ potential reveals a spectrum of attractive eigenvalues η_i^A similar to those given in the table. The repulsive spectrum, however, has many more states with $|\eta_i^R| < 1$ than that of the $^1S_0$ potential (i.e., 13 such states for the $^3S_1 - {}^3D_1$ potential). The convergence of the norm eq. (58) is significantly slower in this case and suggests that this approach may be of limited practical use in the case of coupled channels.

* In this case the eigenvalues and eigenfunctions were constructed at $E = 0$.

** E. Harms and V. Newton, Phys. Rev. C2 (1970) 1214.

V.D. The inverse scattering problem for rank-one separable potentials

In general, it is our intention to start with a well-defined interaction and construct the two-body observables (i.e., the two-body phases shifts and the bound state wave functions, if any). In this section we shall try to turn the problem around by investigating a simple system in which the two-body phase shifts uniquely determine the potential and thus the off-shell T-matrix. We shall see that if the unknown potential is rank-one separable and if the known phase shifts are consistent with this statement about the potential (see sect. B), the phase shifts uniquely determine the potential*. (Such a result is not unreasonable since we have already seen in eq. (35) that a bound state wave function and binding energy uniquely determine the potential if it is separable.) While this particular problem may be of restricted physical interest, both the results and the general approach to the problem have strong similarities to the more difficult inverse scattering problem for local potentials.

We recall from eq. (33) that the T-matrix for a rank-one separable potential (which is, in this case, unknown) must have the form

$$\langle k|T(E)|k'\rangle = \frac{\lambda\, v(k)\, v(k')}{D(\sqrt{E})}\,, \qquad \text{V(60)}$$

where

$$D(x) = 1 + \frac{2}{\pi}\lambda\, \mathrm{P}\int \frac{q^2 \mathrm{d}q}{q^2 - x^2}\, v^2(q). \qquad \text{V(60.1)}$$

The denominator $D(x)$ is real for all real x^2. Let us also define the quantity $D^+(x)$ as

$$D^+(x) = \lim_{\epsilon \to 0} [D(x + i\epsilon)]\,. \qquad \text{V(60.2)}$$

By direct evaluation of eq. (60.2) we find that

$$D^+(x) = D(x) + i\lambda\, v^2(x)x. \qquad \text{V(60.3)}$$

This relation has two important consequences. First, we see that

$$\mathrm{Im}\, D^+(x) = \lambda\, v^2(x)x, \qquad \text{V(60.4)}$$

which is precisely the quantity which we wish to determine in our consideration of the inverse scattering problem. Second, by using eqs. (60), (60.1) and (18) we see that

$$\mathrm{Im}\, D^+(k)/\mathrm{Re}\, D^+(k) = -\tan\delta\,(k), \qquad \text{V(60.5)}$$

which enables us to write

$$D^+(k) = |D^+(k)|\mathrm{e}^{-i\delta(k)}. \qquad \text{V(60.6)}$$

Finally, we make the apparently trivial modification of eq. (60.6) to note that

$$\ln D^+(k) = \ln|D^+(k)| - i\delta(k), \qquad \text{V(60.7)}$$

* This section follows closely the work of F. Tabakin. Phys. Rev. 177 (1969) 1443.

which shows that the imaginary part of $\ln D^+(k)$ is uniquely determined by the (known) phase shifts. These equations allow us to determine our strategy in solving the inverse scattering problem. We shall write $\ln D^+(k)$ as an appropriate Cauchy integral and thus relate the real part of $\ln D^+(k)$ to an appropriate principal value integral over the imaginary part. In other words, we shall express $\ln|D^+(k)|$ as a principal value integral over the given phase shifts. This result may be used in eqs. (60.7) and (60.5) to complete the determination of the potential.

Consider a function $f(z)$ analytic in some closed contour C in the z-plane. Cauchy's theorem allows us to write $f(z)$ as a contour integral along C

$$f(z) = \frac{1}{2\pi i} \oint \frac{f(x)\mathrm{d}x}{x - z} \qquad \text{V(61)}$$

where z is inside C. We shall consider C to run from $-\infty \leqslant z \leqslant \infty$ and close in the upper half z-plane by a semicircle. If $f(z)$ vanishes more rapidly than $1/|z|$ everywhere in the UHP, we may discard the contribution to eq. (61) from the semi-circle and write

$$f(z + i\epsilon) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(x)}{x - z - i\epsilon}\,\mathrm{d}x, \qquad \text{V(61.1)}$$

where the $+i\epsilon$ has been inserted to guarantee that we remain in the UHP. It follows from eq. (61.1) that the real part of $f(z + i\epsilon)$ is given by

$$\mathrm{Re}\,(f(z + i\epsilon)) = \frac{1}{\pi} \mathrm{P} \int_{-\infty}^{\infty} \frac{\mathrm{Im}\, f(x)}{x - z}\,\mathrm{d}x. \qquad \text{V(61.2)}$$

Unfortunately, $\ln D^+(k)$ may not have adequate analytic properties to enable us to exploit eq. (61.2). From eqs. (60.1) and (60.3) and the general requirements on $v(k)$, we see that

$$\lim_{|k|\to\infty} (D^+(k)) = 1 + \mathrm{O}\,(1/|k|^2) \qquad \text{V(62)}$$

and thus $\lim_{|k|\to\infty} (\ln D^+(k))$ is of order $1/|k|^2$ so that we may ignore the portion of the integral along the semi-circle. However, for λ sufficiently negative that the inequality of eq. (33.2) is satisfied, the T-matrix has a simple pole in E (as displayed by eq. (34)). This pole is caused by a zero in $D^+(i\kappa_\mathrm{B})$. Thus, $\ln D^+(k)$ has a branch point at $k = i\kappa_\mathrm{B}$ and we are not allowed to apply Cauchy's theorem. In order to avoid this difficulty we introduce a new function

$$\tilde{D}^+(k) = \frac{k + i\kappa_\mathrm{B}}{k - i\kappa_\mathrm{B}} D^+(k), \qquad \text{V(63)}$$

which is constructed to be finite at $k = i\kappa_\mathrm{B}$ and is free from zeros in the UHP. Also note that the additional factor in $D^+(k)$ is of magnitude 1 so that the asymptotic behaviour of eq. (62) is preserved: The asymptotic behaviour of $\ln \tilde{D}^+(k)$ still allows

us to ignore the portion of C along the semi-circle. We note that

$$\mathrm{Re}\,(\ln \tilde{D}^+(k)) = \ln|D^+(k)|, \qquad \text{V(63.1)}$$

$$i\,\mathrm{Im}\,(\ln \tilde{D}^+(k)) = -i\delta(k) + \ln \frac{k + i\kappa_\mathrm{B}}{k - i\kappa_\mathrm{B}}. \qquad \text{V(63.2)}$$

(Since the argument of the logarithm in eq. (63.2) is of magnitude 1, this quantity is explicitly imaginary.)

We may now apply eq. (61.1) to $\ln \tilde{D}^+(k)$ to obtain

$$\ln|\tilde{D}^+(k)| = -\Delta(k) + \frac{1}{\pi i} \mathrm{P} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k'}{k' - k} \ln \frac{k' + i\kappa_\mathrm{B}}{k' - i\kappa_\mathrm{B}}, \qquad \text{V(64)}$$

where we have introduced the definition

$$\Delta(k) = \frac{1}{\pi} \mathrm{P} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k'\,\delta(k')}{k' - k}. \qquad \text{V(64.1)}$$

Eq. (64.1) demands knowledge of the phase shifts for both positive and negative k. Inspection of eqs. (60), (60.1), and (18) indicates that $\delta(-k) = -\delta(k)$. The second integral in eq. (64) may be evaluated directly as

$$\frac{1}{\pi i} \mathrm{P} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k'}{k' - k} \ln \frac{k' + i\kappa_\mathrm{B}}{k' - i\kappa_\mathrm{B}} = \ln \frac{k^2 + \kappa_\mathrm{B}^2}{k^2}. \qquad \text{V(64.2)}$$

Using eqs. (64) and (60.6) we find

$$D^+(k) = \frac{k^2 + \kappa_\mathrm{B}^2}{k^2} \exp\,[-\Delta(k) - i\delta(k)], \qquad \text{V(65)}$$

where we may drop the first factor on the r.h.s. of eq. (65) in cases where there is no bound state. The solution of the inverse scattering problem is completed by using eq. (60.4) to obtain

$$\lambda v^2(k) = -\sin\delta\,(k)\,\mathrm{e}^{-\Delta(k)}/k \qquad \text{V(66)}$$

in the case of repulsive potentials or attractive potentials with no bound state. For attractive potentials with a bound state at $E = -\kappa_\mathrm{B}^2$,

$$\lambda v^2(k) = -\frac{k^2 + \kappa_\mathrm{B}^2}{k^2} \frac{\sin\delta\,(k)}{k}\,\mathrm{e}^{-\Delta(k)}. \qquad \text{V(66.1)}$$

Eqs. (66) and (66.1) indicate that the phase shifts at all energies plus the bound state energy (if any) uniquely determine the potential in those cases where it is known to be rank-one separable. We immediately recognize that the need to know phase shifts at all energies limits the practical usefulness of eqs. (66). As one would expect, lack of knowledge of high-energy phase shifts impairs our ability to determine $v(k)$ for large k or its Fourier-Bessel transform at short distances.

The inverse scattering problem has been solved by Gelfand and Levitan for local

potentials following similar but rather more complicated lines*. They arrive at the result that the phase shifts (at all energies), bound state energies and a constant describing the asymptotic normalization of each bound state wave function (if any) uniquely define the potential. This result is perhaps more important in view of the numerous local potentials used for providing phenomenological descriptions of NN phase shifts. The results of Gelfand and Levitan indicate that any discrepancies between off-shell T-matrix elements of such local phenomenological NN potentials must be due to inadequate descriptions of the fitted phase shifts, different assumptions regarding the high energy phase shifts not fitted, or different deuteron wave functions.

V.E. The N/D equations

In some situations it may be more desirable to solve the LS equation approximately by a finite number of iterations than to solve the integral equations directly. While such an approach may have certain numerical advantages, it certainly suffers the disadvantage that the T-matrix obtained from any finite number of iterations is not unitary. In particular, the fully on-shell T-matrix does not satisfy the elastic unitarity relation.

$$\text{Im}\ \langle k|T^+(E)|k\rangle = k|\langle k|T^+(E)|k\rangle|^2 \qquad \text{V(67)}$$

for $E = k^2$ and $E > 0$. In this section we shall discuss the N/D equations for potential scattering. These equations offer a technique for constructing essentially iterative solutions to the LS equation which are explicitly unitary. In order to obtain these equations we shall need several general analytic properties of the on-shell T-matrix as a function of the complex variable E. (In this section *only* $E = k^2$ except where stated otherwise.) First, we note that, independent of the choice of the potential, $T^+(E)$ has a cut from $0 \leqslant E \leqslant +\infty$ with a discontinuity given by eq. (67). In general, the on-shell amplitude also has a cut on the negative E-axis from $-\infty \leqslant E \leqslant E_0$ with a discontinuity

$$2i\ \text{Im}\ \langle k|T^+(E)|k\rangle = 2i\ \Delta(E), \qquad \text{V(68)}$$

where both $E_0 < 0$ and $\Delta(E)$ are dependent on the details of the potential. As mentioned in sect. A, $T^+(E)$ may also have simple poles in E corresponding to bound states and resonances. Finally, we note that for a wide variety of physically interesting potentials $\langle k|T^+(E)|k\rangle$ vanishes at least as fast as $1/k$ as $|E|$ approaches infinity.

We can write the fully on-shell amplitude in some partial wave channel as

$$\langle k|T^+(E)|k\rangle = N(E)/D(E), \qquad \text{V(69)}$$

where we have suppressed the angular momentum index. The properties of the individual functions $N(E)$ and $D(E)$ are not determined uniquely and we are at liberty to choose them in any convenient manner so long as the ratio N/D has the properties of the on-shell amplitude discussed above. In particular, we shall choose the numerator function such that its only singularity is a left cut from $-\infty \leqslant E \leqslant E_0$. For $E > E_0$, $N(E)$ is chosen real and, in the limit of large $|E|$, $N(E)$ is made to vanish at least as fast as $1/k$. The denominator function $D(E)$ has the unitarity cut from $0 \leqslant E \leqslant +\infty$ and is real for $E < 0$. In the limit of large $|E|$, we required that $D(E) - 1$ vanishes at least as fast as $1/k$. Finally, we require that poles in $\langle k|T^+(E)|k\rangle$, if they exist, are due to zeros in $D(E)$. Using these properties of $N(E)$ and $D(E)$ and eq. (69) we may restate the discontinuities across the cuts as given by eqs. (67) and (68). For the left cut we find that for $E < E_0$

$$2i\ \Delta(E) = 2i\ \frac{\text{Im}\ N(E)}{D(E)}. \qquad \text{V(70)}$$

For the right cut we find that for $E > 0$

$$\text{Im}\ \langle k|T^+(E)|k\rangle = -\frac{N(E)}{|D(E)|^2}\ \text{Im}\ D(E). \qquad \text{V(71)}$$

Exploiting elastic unitarity, as expressed by eq. (67), we may also write

$$\text{Im}\ \langle k|T^+(E)|k\rangle = k\,\frac{N^2(E)}{|D(E)|^2}. \qquad \text{V(71.1)}$$

Combining eqs. (71) and (71.1) allows us to express the imaginary part of $D(E)$ in the unitarity cut as

$$\text{Im}\ D(E) = -k\,N(E). \qquad \text{V(72)}$$

The conditions imposed on the numerator and denominator functions enable us to use Cauchy's theorem to write $N(E)$ and $D(E) - 1$ as integrals over the left and right cuts respectively,

$$N(E) = \frac{1}{\pi}\int_{-\infty}^{E_0} \frac{\text{Im}\ N(E')}{E' - E - i\epsilon}\,dE', \qquad \text{V(73)}$$

$$D(E) - 1 = \frac{1}{\pi}\int_{0}^{\infty} \frac{\text{Im}\ D(E')}{E' - E - i\epsilon}\,dE'. \qquad \text{V(73.1)}$$

We can obtain the usual N/D equations by combining eqs. (71) and (73) and eqs. (72) and (73.1) to yield the following coupled integral equations:

$$N(E) = \frac{1}{\pi}\int_{-\infty}^{E_0} \frac{\Delta(E')\,D(E')}{E' - E}\,dE', \qquad \text{V(74)}$$

$$D(E) - 1 = \frac{1}{\pi}\int_{0}^{\infty} \frac{N(E')\,dE'\sqrt{E'}}{E' - E}. \qquad \text{V(74.1)}$$

* See, for example, R.G. Newton, Scattering theory of waves and particles (McGraw-Hill, New York, 1966).

These equations may be decoupled by a single iteration. Information regarding the interaction is introduced in the N/D equations indirectly via the left discontinuity, $\Delta(E)$, rather than through the potential itself. Eqs. (74) may be solved using, for example, the techniques of Section V.F to yield $N(E)$ for $E > 0$ and $D(E)$ for $E < E_0$. Values of $N(E)$ and $D(E)$ for other values of E may be obtained from eqs. (74) and (74.1) respectively. Given $\Delta(E)$, solutions to the N/D equations yield the on-shell amplitude $\langle k|T^+(E)|k\rangle$ for all E. This is in contrast to the LS equation where the solutions yield all half-on-shell T-matrix elements, $\langle p|T^+(E = k^2)|k\rangle$, at a fixed energy. The solutions to the N/D equations may also be used to determine the zeros of $D(E)$ and thus the location of bound state poles and resonances.

In practice, of course, we do not know the exact form of $\Delta(E)$ for a given potential and the N/D equations do not represent a practical alternative to the exact solution of the LS equation for potential scattering. However, given some approximation to $\Delta(E)$, the N/D equations present a way to obtain an approximation to the on-shell amplitude which is guaranteed to satisfy the elastic unitarity condition eq. (67) independent of the choice of $\Delta(E)$. The exact left discontinuity may be approximated by the value obtained from an iterative solution to the LS equation of any order. If the potential is of the Yukawa form

$$V = V_0 \frac{e^{-\mu r}}{r}, \tag{V(75)}$$

the T-matrix in first Born approximation is approximated by

$$T^{(1)} = \langle k|V|p\rangle = \frac{V_0}{2pk} Q_l\left(\frac{p^2 + k^2 + \mu^2}{2pk}\right), \tag{V(75.1)}$$

where $Q_l(x)$ is the Legendre function of the second kind with branch points at $x = \pm 1$. For the fully on-shell amplitude this corresponds to a left cut from $-\infty \leqslant E \leqslant -\frac{1}{4}\mu^2$ and a discontinuity across this cut (for the case $l = 0$ only) of

$$2i\,\Delta(E) = \frac{i\pi V_0}{2E}. \tag{V(75.2)}$$

The locations of the contributions to $\Delta(E)$ from higher iterations of the LS equation may be obtained in this case without difficulty. We may define the solution to the LS equation in an iterative fashion as

$$T^+(E) = \sum_{n=1}^{\infty} T^{+(n)}(E), \tag{V(76)}$$

where

$$\langle k|T^{+(n+1)}(p^2)|p\rangle = \frac{2}{\pi}\int_0^\infty q^2 dq \frac{\langle k|V|q\rangle\langle q|T^{+(n)}(p^2)|p\rangle}{p^2 + i\epsilon - q^2} \tag{V(76.1)}$$

with the definition $T^{+(1)}(E) = V$. It can be seen that the cuts in V and $T^{+(1)}(E)$ can

pinch the contour of integration giving rise to branch points in $\langle k|T^{+(2)}(p^2)|p\rangle$ at $(p^2 + k^2 + 4\mu^2)/2pk = \pm 1$ and a contribution to $\Delta(E)$ from $-\infty \leqslant E \leqslant -\mu^2$. Repetition of such an analysis indicates that the term $\langle p|T^{+(n)}(p^2)|p\rangle$ has a branch point at $E = -\frac{1}{4}(n\mu)^2$ and thus contributes to the discontinuity $\Delta(E)$ over the region $-\infty \leqslant E \leqslant -\frac{1}{4}(n\mu)^2$. (We remark that the location of these branch points is independent of l for the potential eq. (75) although the corresponding contributions to $\Delta(E)$ are l-dependent.) If the on-shell amplitude for $E > 0$ is most strongly influenced by that part of $\Delta(E)$ closest to the physical region, it may be satisfactory to approximate $\Delta(E)$ by a low order iteration. It is important to recognize, however, that even when $\Delta(E)$ is given by eq. (75.2), the N/D equations are integral equations which must be solved numerically.

We consider one of the few cases where the N/D equations can be solved analytically. If the left discontinuity is given as or may be approximated by a δ-function

$$\Delta(E) = R\,\delta(E - E_C) \tag{V(77)}$$

with $E_C < E_0$, we may solve eq. (74) by inspection to yield

$$N(E) = \frac{R}{\pi} D(E_C) \frac{1}{E_C - E}. \tag{V(78)}$$

$D(E + i\epsilon)$ may be obtained simply from eqs. (74.1) and (78) as

$$D(E + i\epsilon) = 1 + \frac{RD(E_C)}{\pi^2}\int_0^\infty \frac{\sqrt{E'}\,dE'}{(E' - E_C)(E' - E - i\epsilon)}, \tag{V(78.1)}$$

where the constant $D(E_C)$ can be obtained directly by setting $E = E_C$ in the argument of D on the left-hand side.

$$D(E_C) = \left[1 - \frac{R}{\pi^2}\int_0^\infty \frac{\sqrt{E'}\,dE'}{(E' - E_C)^2}\right]^{-1}. \tag{V(78.2)}$$

From eq. (78.1) we note immediately that $D(E) - 1$ is a monotonic function of E for $E < 0$. Thus if $RD(E_C)$ is sufficiently negative, $D(E)$ will have a single zero corresponding to a bound state pole in $T^+(E)$. For $E > 0$ the integrals in eqs. (78.1) and (78.2) may be evaluated to yield the fully on-shell scattering amplitude and phase shifts via eq. (17)

$$k \cot\delta = -\frac{1}{a} + \tfrac{1}{2} r_0 k^2, \tag{V(79)}$$

where

$$a = \frac{2R}{\kappa(2\pi\kappa + R)}, \tag{V(79.1)}$$

$$r_0 = \frac{R - 2\pi\kappa}{R\kappa}, \tag{V(79.2)}$$

where we have introduced the notation $E_C = -\kappa^2$ with $\kappa > 0$. Eqs. (79) are exact given $\Delta(E)$ of the form of eq. (77). Comparison with eq. II(19) indicates that the quantities a and r_0 may be identified with the usual scattering length and effective range respectively. We may thus understand the effective range approximation is equivalent to replacing the left cut with a pole (i.e., replacing the left discontinuity with a δ-function)*.

Since the N/D equations remain integral equations for most physically interesting approximations to $\Delta(E)$, they present no special numerical advantage over the exact LS equation. The N/D equations may be useful, however, when it is easier to approximate the left discontinuity of the on-shell amplitude directly than to define the potential. Such a situation can arise, for example, in attempting to construct the NN interaction from boson exchanges or using dispersion theory.

V.F. Numerical solutions to the LS equation

Unfortunately, we are frequently forced to deal with potentials which are not finite rank separable. In such cases it is necessary to solve the LS equation numerically. In this section we shall give some attention to this problem. First, let us consider the LS equation

$$\langle k|T(E)|k'\rangle = \langle k|V|k'\rangle - \frac{2}{\pi}\,\mathrm{P}\int_0^\infty \frac{q^2}{q^2 - E}\,\mathrm{d}q\,\langle k|V|q\rangle\langle q|T(E)|k'\rangle \qquad \text{V(80)}$$

for $E < 0$ so that we may ignore the principal value prescription on the integral. If the integral on the r.h.s. of eq. (80) contained *known* functions, we would know how to evaluate eq. (80) numerically. We would simply perform the integral numerically by making the replacement

$$\int_0^\infty f(q)\,\mathrm{d}q \to \sum_{i=1}^{N} f(q_i)\,w_i, \qquad \text{V(81)}$$

where the choice of points and weights $\{f_i, w_i\}$ is specified by the structure of the integrand. In the present case, the integrand contains the unknown function $\langle q|T(E)|k'\rangle$ and the choice of points and weights is generally justified *a posteriori* by demonstrating that the calculated values of $\langle q|T(E)|k'\rangle$ are insensitive to increasing

* This pole is not the bound state pole which was used to obtain the effective range approximation in sect. 2. This pole is an approximation to the cut in the on-shell T-matrix and can be seen to generate a bound state pole in the case where $R < 0$ and $a > 0$. In this case the T-matrix, eq. (39.2), in the effective-range approximation will have both poles for negative E corresponding to two zeros in the denominator, the one being the approximation to the cut from the potential, the other, to the bound state.

the number of points. (Such a procedure may be uneconomical. As shown by Chao and Jackson*, it is often possible to determine the analytic structure of $\langle q|T(E)|k'\rangle$ as a function of q without solving the entire problem and to exploit this information using conformal mapping techniques to find significantly more efficient choices of points and weights.) Making a replacement of the form eq. (81) we may write

$$\langle k|T(E)|k'\rangle = \langle k|V|k'\rangle - \frac{2}{\pi}\sum_{i=1}^{N} \frac{q_i^2}{q_i^2 - E}\,\langle k|V|q_i\rangle\langle q_i|T(E)|k'\rangle\,. \qquad \text{V(82)}$$

Writing eq. (82) N times for $k = q_j$, we obtain N linear equations in the unknowns $\langle q_i|T(E)|k'\rangle$ which may be solved by any convenient technique†. If it is desired to construct $\langle k|T(E)|k'\rangle$ for arbitrary k', it is perhaps more convenient to construct the inverse of the matrix M

$$(M)_{ij} = \delta_{ij} + \frac{2}{\pi}\frac{q_j^2}{q_j^2 - E}\,w_j\,\langle q_i|V|q_j\rangle. \qquad \text{V(82.1)}$$

The quantity $\langle q_i|T(E)|k'\rangle$ for arbitrary k' is now found as a simple matrix product

$$\langle q_i|T(E)|k'\rangle = \sum_j (M^{-1})_{ij}\,\langle q_j|V|k'\rangle. \qquad \text{V(82.2)}$$

Since the number of operations in inverting a matrix is of order N^3, it is important

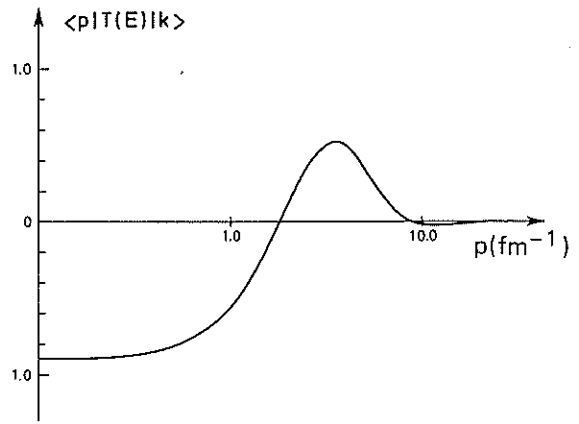

Fig. 16. Plot of $\langle p|T(E)|k\rangle$ for Reid $^1S_0$ state; $k = 1.0$ fm^{-1}, $E = -20$ MeV.

* Y. Chao and A. Jackson, Nucl. Phys. A215 (1973) 157.

† See, e.g. J.H. Wilkinson, The algebraic eigenvalue problem (Oxford, 1965) ch. 4 for a discussion of suitable techniques for the solution of linear equations.

to keep the size of eqs. (82) small if this approach is to be practical. Thus, we would prefer to work with potentials such that the integrand of eq. (82) is smooth and relatively free of oscillations (i.e., we would prefer to work with superpositions of Yukawa potentials which are relatively smooth rather than square wells whose discontinuities lead to persistent oscillations in the integrand of eq. (82)). In fig. 16 we show $\langle p|T(E)|k\rangle$ for the Reid $^1S_0$ potential as a function of p. The relative smoothness of this function provides intuitive justification for the fact that eqs. (82) can usually be solved with $N \leqslant 20$ for the potentials and values of E and k' of interest in nuclear physics. Convenient choices of $\{q_i, w_i\}$ are Gauss-Laguerre or Gauss-Hermite points and weights for which extensive tabulations are readily available*.

The eqs. (82) may also serve, without further modification, for the determination of bound state energies and wave functions. Recalling that the T-matrix has a simple pole at $E = E_B$, we note that the determinant of the linear equations (82) must vanish at $E = E_B$. Eq. (30) indicates that the solutions $\langle q_i|T(E)|k\rangle$ for $E \sim E_B$ are proportional to $\langle q_i|V|\psi_B\rangle$. Using eq. (34.1) we may obtain the unnormalized bound state wave function $\langle q_i|\psi_B\rangle$ which, due to our consistent use of unnormalized plane waves, may be normalized as

$$N\frac{2}{\pi}\sum_{i=1}^{N} w_i |\langle q_i|\psi_B\rangle|^2 = 1. \qquad \text{V(83)}$$

In the case when $E > 0$ we are no longer able to neglect the principal value prescription for avoiding the singularity in the Green function at $k^2 = E$ in eq. (80) and we must modify eqs. (82). Such modifications may be performed in a variety of ways of varying mathematical rigour. Let us first consider a particularly simple, but formally unacceptable, way of removing the difficulties of this principal value prescription. Consider, without loss of generality, the case when $E = k'^2$. We note that

$$\mathrm{P}\int_0^\infty \frac{dq}{q^2 - k'^2} = 0, \qquad \text{V(84)}$$

so that we may rewrite eq. (80) as

$$\langle k|T(E)|k\rangle = \langle k|V|k\rangle - \frac{2}{\pi}\int_0^\infty \frac{dq}{q^2 - k^2}\left[q^2\langle k|V|q\rangle\langle q|T(E)|k\rangle - k^2\langle k|V|\rangle\langle k|T(E)|k\rangle\right], \qquad \text{V(85)}$$

where we have again dropped the principal value prescription since the integrand is

* See, e.g. M. Abramowitz and I.A. Stegun, Handbook of mathematical functions (Dover Publications, 1965).

no longer singular. Again making the replacement of eq. (81) and defining $q_{N+1} = k$, we obtain the $(N+1)$ linear equations

$$\langle q_j|T(E)|k\rangle = \langle q_j|V|k\rangle - \frac{2}{\pi}\sum_{i=1}^{N}\frac{w_i q_i^2}{q_i^2 - E}\langle q_j|V|q_i\rangle\langle q_i|T(E)|k\rangle + \frac{2}{\pi}k^2\langle q_j|V|k\rangle\langle k|T(E)|k\rangle\sum_{i=1}^{N}\frac{w_i}{q_i^2 - E}, \qquad \text{V(85.1)}$$

which may be solved, as before, for the unknowns $\langle q_j|T(E)|k\rangle$. Such equations are objectionable in the sense that, as N increases, this matrix equation develops two singularities which cancel. Clearly, in the limit of large N round-off errors will make eqs. (85.1) useless for numerical calculations. In practice, however, these objections do not destroy the usefulness of eqs. (85.1) for $N \leqslant 100$ and for potentials and values of E and k of interest in nuclear physics.

It is possible to rewrite the LS equation in a form which is manifestly non-singular for positive energies*. Let us again consider the case $\langle p|T(k^2)|k\rangle$. We first write the LS equation (80) for $\langle p|T(k^2)|k\rangle$ and $\langle k|T(k^2)|k\rangle$, multiply the latter by $\langle p|V|k\rangle/\langle k|V|k\rangle$ and subtract to yield

$$\langle p|T(k^2)|k\rangle = \frac{\langle p|V|k\rangle}{\langle k|V|k\rangle}\langle k|T(k^2)|k\rangle - \frac{2}{\pi}\mathrm{P}\int_0^\infty \frac{q^2\,dq}{q^2 - k^2}\left[\langle p|V|q\rangle - \frac{\langle p|V|k\rangle\langle k|V|q\rangle}{\langle k|V|k\rangle}\right]\langle q|T(k^2)|k\rangle. \qquad \text{V(86)}$$

The integrand is perfectly regular as q approaches k and the principal value prescription may thus be dropped. Making the replacement of eq. (81) yields linear equations which, unlike eqs. (85.1), are completely free of singularities. Such equations are not immediately useful since the resulting N equations involve the $(N+1)$ unknowns $\langle q_i|T(k^2)|k\rangle$ and $\langle k|T(k^2)|k\rangle$ and eq. (86) for $p = k$ is trivial.

Thus we define the quantity $f(p,k)$ through the relation

$$\langle p|T(k^2)|k\rangle = f(p,k)\,\langle k|T(k^2)|k\rangle \qquad \text{V(87)}$$

with the obvious condition $f(k,k) = 1$. Substitution of eq. (87) in eq. (86) yields the non-singular equation

$$f(p,k) = \frac{\langle p|V|k\rangle}{\langle k|V|k\rangle} - \frac{2}{\pi}\int_0^\infty \frac{q^2\,dq}{q^2 - k^2}\left[\langle p|V|q\rangle - \frac{\langle p|V|k\rangle\langle k|V|q\rangle}{\langle k|V|k\rangle}\right]f(q,k). \qquad \text{V(88)}$$

The replacement eq. (81) may now be made and the equations solved for $f(q_i,k)$. In

* K.L. Kowalski, Phys. Rev. Lett. 15 (1965) 798 (erratum 15 (1965) 908).

order to determine $\langle k|T(k^2)|k\rangle$ we use eqs. (87) and (80) to yield

$$\langle k|T(k^2)|k\rangle = \frac{\langle k|V|k\rangle}{1 + \frac{2}{\pi} \mathrm{P} \int_0^\infty \frac{q^2\,dq}{q^2 - k^2} \langle k|V|q\rangle f(q,k)} \,. \qquad \text{V(89)}$$

Eq. (89) involves a principal value integral which may be performed numerically with the use of the fact that $f(k,k) = 1$.

Eqs. (87) - (89) represent a completely non-singular and numerically practical approach to solving the LS equation for $E = k^2$ and may be generalized easily to the case of arbitrary positive E.

CHAPTER VI

RELATIVISTIC FORMALISM, NUCLEON FORM FACTORS

VI.A. Relativistic scattering equations

In this section we shall consider modifications of the Lippmann-Schwinger equation necessitated by the kinematic requirements of relativity. At first glance such modifications might seem to be of little relevance to the problem of nucleon-nucleon scattering below the first inelastic threshold since the energy in the c.m. system is always a modest fraction of the nucleon rest mass. This is not the case, however, due to the ubiquitous short-range repulsion familiar from phenomenological descriptions of the nucleon-nucleon interaction using local potentials. In field theoretic discussions of the NN interaction (presented in subsequent chapters) this repulsion is associated with the exchange of ρ, ω and ϕ vector mesons. (Observed VNN coupling constants suggest that the isoscalar ω-meson provides the most significant vector meson exchange contribution to the force at short distances. That this exchange yields repulsion is reasonable by analogy with the exchange of a more familiar neutral vector particle, the photon, between particles of like charge. In the case of ω-meson exchange the baryon number assumes the role of the "strong charge".) This strong repulsion forces the two nucleon wave function in S-states to decrease rapidly at distances less then ~0.5 fm and thus builds high momentum components into the wave function at all scattering energies. There is no reason to believe that these high momentum components can be described adequately by the non-relativistic scattering equations considered in the preceding chapter. Within the context of phenomenological descriptions of the NN interaction, the inadequacies of the non-relativistic approach are of little importance since the parameterizations of the interaction currently employed do have sufficient flexibility to provide a quantitative fit to experimental data. To the extent that our goal is to provide a quantitative description of NN scattering in terms of the exchange of bosons with coupling constants and masses determined from other experiments, we are justified in using non-relativistic scattering equations only so long as they represent a numerically reliable alternative to a fully relativistic description of the scattering process.

In the absence of a complete theory of strong interactions, the investigation of suitable approximate relativistic equations is to some extent speculative. We shall thus begin by modifying the Lippmann-Schwinger equation slightly in order to satisfy one unambiguous requirement of any relativistic theory; that the scattering amplitude should incorporate relativistic unitarity along the elastic cut. While the