Chapter 0

The Many-Body Problem for Everybody

0.0 What the many-body problem is about

The many-body problem has attracted attention ever since the philosophers of old speculated over the question of how many angels could dance on the head of a pin. In the angel problem, as in all many-body problems, there are two essential ingredients. First of all, there have to be many bodies present—many angels, many electrons, many atoms, many molecules, many people, etc. Secondly, for there to be a problem, these bodies have to interact with each other. To see why this is so, suppose the bodies did not interact. Then each body would act independently of all the others, so that we could simply investigate the behaviour of each body separately. In other words, without interaction, instead of having one many-body problem, we would have many one-body problems. Thus, interactions are essential, and in fact the many-body problem may be defined as *the study of the effects of interaction between bodies on the behaviour of a many-body system.*

(It might be noted here, for the benefit of those interested in exact solutions, that there is an alternative formulation of the many-body problem, i.e., how many bodies are required before we have a problem? G. E. Brown points out that this can be answered by a look at history. In eighteenth-century Newtonian mechanics, the three-body problem was insoluble. With the birth of general relativity around 1910 and quantum electrodynamics in 1930, the two- and one-body problems became insoluble. And within modern quantum field theory, the problem of zero bodies (vacuum) is insoluble. So, if we are out after exact solutions, no bodies at all is already too many!)

The importance of the many-body problem derives from the fact that almost any real physical system one can think of is composed of a set of interacting particles. For example, nucleons in a nucleus interact by nuclear forces, electrons in an atom or metal interact by Coulomb forces, etc. Some examples are shown schematically in Fig. 0.1. Furthermore, it turns out that in the calculation of physical properties of such systems—for example, the energy levels of the atom, or magnetic susceptibility of the metal—interactions between particles play a very important role.

It should be clear from the variety of systems in Fig. 0.1 that the many-body problem is *not* a branch of solid state, or nuclear, or atomic physics, etc. It deals rather with *general* methods applicable to *all* many-body systems.

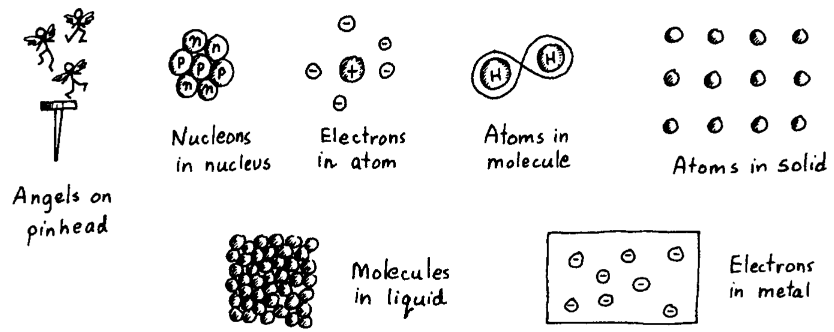

Fig. 0.1 *Some Many-body Systems*

The many-body problem is an extraordinarily difficult one because of the incredibly intricate motions of the particles in an interacting system. In Fig. 0.2 we contrast the simple behaviour of non-interacting particles with the complicated behaviour of interacting ones. Because of the complexity of the many-body problem, not much progress was made with it for a long time. In fact one of the preferred methods for solving the problem was simply to ignore it, i.e., pretend there were no interactions present. (Surprisingly enough, in some cases this 'method' produced good results anyway, and one of the great mysteries was how this could be possible!)

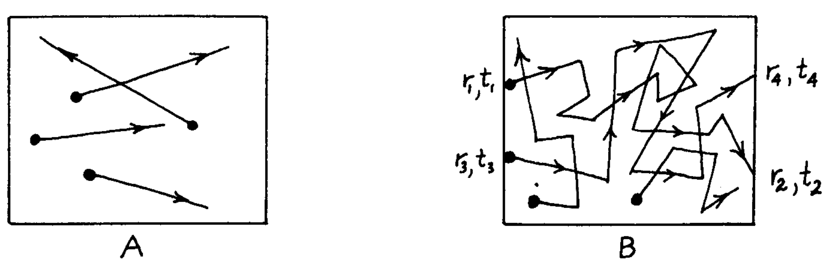

Fig. 0.2 A. *Non-interacting Particles*
B. *Interacting Particles*

Another of the early approaches to the problem, and one which is still used extensively today is the *canonical transformation* technique, described in appendix $\mathscr{A}$. This involves transforming the basic equations of the many-body system to a new set of coordinates in which the interaction term becomes small. Although considerable success has been achieved with this technique, it is not as systematic as one would like, and this sometimes makes it difficult to apply. It was this lack of a systematic method which kept the many-body field in its cradle well up into the 1950s.

The situation changed radically in 1956–7. In a series of pioneering papers, it was shown that the methods of *quantum field theory*, already famous for its success in elementary particle physics, provided a powerful, unified way of attacking the many-body problem. The new key opened many doors, and in rapid succession the idea was applied to nuclei, electrons in metals, ferromagnets, atoms, superconductors, plasmas, molecules—virtually everything in sight.

From that time on, much of the most exciting and fundamental research into the nature of matter has been based on the quantum field theory method. One of the things emerging from this research is a new simple picture of matter in which systems of interacting real particles are described in terms of approximately non-interacting fictitious bodies called 'quasi particles' and 'collective excitations'. Another thing is new results for calculated physical quantities which are in excellent agreement with experiment—for example, energy levels of light atoms, binding energy of nuclear matter, Fermi energy and effective electron mass in a variety of metals.

In this introductory chapter, we will give a physical picture of quasi particles and collective excitations. Then in the next chapter we show qualitatively how to describe quasi particles and calculate their properties by means of the quantum field theoretical technique known as the method of *Feynman diagrams*.

0.1. Simple example of non-interacting fictitious bodies

As mentioned at the beginning, one of nature's little surprises is that many-body systems often behave as if the bodies of which they are composed hardly interact at all! The reason for this is that the 'bodies' involved are not real but *fictitious*. That is, the system composed of *strongly* interacting *real* bodies acts *as if* it were composed of *weakly* interacting (or non-interacting) *fictitious* bodies. We consider now a very simple example of how this can occur.

Suppose we have two masses, m_1 and m_2 held together by a strong spring as shown in Fig. 0.3. That is, our system here consists of two strongly coupled real bodies. If this contraption is tossed up in a gravitational field, the motion of each body considered separately is very complicated because of the strong interaction (spring force) between the bodies.

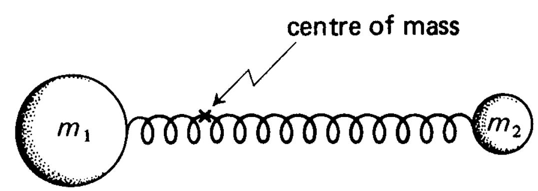

Fig. 0.3 *Two-body System*

However, we can break up the complicated motion into two independent simple motions: motion of the centre of mass and motion about the centre of mass. The centre of mass moves exactly as if it were an independent body of mass m_1+m_2, so it is one of the non-interacting fictitious bodies here. The other fictitious body is a body of mass $m_1 m_2/(m_1+m_2)$—the so-called 'reduced mass'—which moves independently relative to the centre of mass. Thus the system acts as if it were composed of two non-interacting fictitious bodies: the 'centre of mass body' and the 'reduced mass body'. (See appendix $\mathscr{A}$, eqs. ($\mathscr{A}$.11)–($\mathscr{A}$.14) for details.)

0.2 Quasi particles and quasi horses

The above two-body example is easy enough to understand, but finding the weakly interacting fictitious bodies in a set of *many* strongly interacting real bodies is a bit harder. We consider first the fictitious bodies called 'quasi particles'. These arise from the fact that when a real particle moves through the system, it pushes or pulls on its neighbours and thus becomes surrounded by a 'cloud' of agitated particles similar to the dust cloud kicked up by a galloping horse in a western. The real particle plus its cloud is the quasi particle (Fig. 0.4).

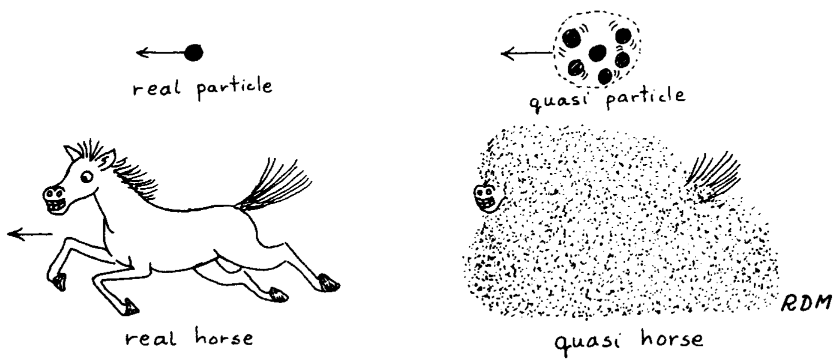

Fig. 0.4 *Quasi Particle Concept*

Just as the dust cloud hides the horse, the particle cloud 'shields' or 'screens' the real particles so that quasi particles interact only weakly with one another. The presence of the cloud also makes the properties of the quasi particle different from that of the real particle—it may have an '*effective mass*' different from the real mass, and a '*lifetime*'. These properties of quasi particles are directly observable experimentally.

It should be remarked that the quasi particle is in an excited energy level of the many-body system. Hence it is referred to as an '*elementary excitation*' of the system. (See appendix $\mathscr{A}$, §$\mathscr{A}$.2.) We now consider some examples of quasi particles.

1 *Quasi ion in a classical liquid*

Imagine that we have an electrolyte solution composed of an equal number of positive and negative ions moving about and colliding with each other as illustrated in Fig. 0.5. Let us focus our attention on a typical (+) ion in the

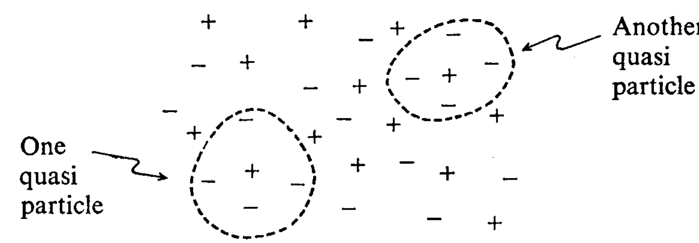

Fig. 0.5 *Quasi Particles in a Liquid of Positive and Negative Ions*

system. As this ion moves, on account of the strong Coulomb interaction, it will attract (−) ions to it. Some of these (−) ions will stick to the (+) for a while, then fall off due to collisions, then be replaced by other (−) ions, etc. Thus, on the average, because of the interaction, this typical (+) ion (and therefore every (+) ion) will be surrounded by a 'coat' or 'cloud' of (−) ions as shown in Fig. 0.5 inside the dotted lines. And of course each (−) ion will similarly have a coat of (+) ions. This coat of opposite charge will shield the ion's own charge so that its interaction with other similarly shielded ions will be much weaker than in the unshielded case. Thus the ions wearing their coats will act approximately independently of each other and constitute the quasi particles of this particular system. Many different types of systems of interacting particles may be described in this manner, and in general we have

$$\text{real particle} + \begin{matrix}\text{'coat' or 'cloud'}\\ \text{of other particles}\end{matrix} = \text{quasi particle.} \tag{0.1}$$

Sometimes this same equation is stated in a more powerful terminology coming from quantum field theory:

$$\text{'bare' particle} + \begin{matrix}\text{'clothing'}\\ \text{or 'cloud'}\end{matrix} = \begin{matrix}\text{'dressed' or 'clothed'}\\ \text{or 'physical' or}\\ \text{'renormalized' particle.}\end{matrix} \tag{0.2}$$

For example, in quantum electrodynamics a 'bare' electron interacting with a field of photons acquires a cloud of virtual photons around it, converting it into the 'dressed' electron. In a similar manner, the interaction between real particles is called the 'bare' interaction, while the weak interaction between quasi particles is referred to as the 'effective' or 'dressed' or 'renormalized' interaction.

It should be noted that each bare particle is simultaneously the 'core' of a quasi particle and a transient 'member' of the cloud of several other quasi particles. Therefore, if we try to visualize the whole system here as composed of quasi particles, we have to be careful, since each particle will have been counted more than once. For this reason, the quasi particle concept is valid only if one talks about a few quasi particles at a time, i.e., few in comparison with the total number of particles. In order to avoid this problem and concentrate attention on just a single quasi particle at a time, it is convenient to define quasi particles in terms of an experiment in which one adds an extra particle to the system, and observes the behaviour of this extra particle as it moves through the system. This is shown in Fig. 0.6 for a (+) ion.

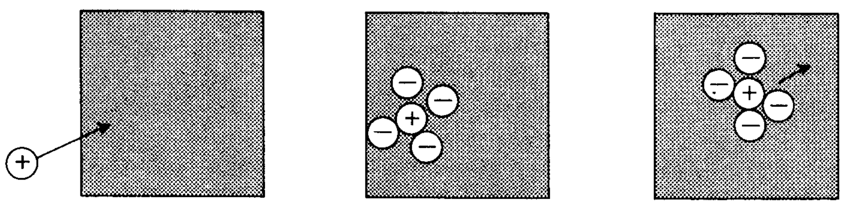

Fig. 0.6 *Moving Quasi Ion. (a) Extra (+) Ion Shot into Liquid. (b) (+) Ion Acquires Cloud of (−) Ions, Turning it into Quasi Ion. (c) Quasi Ion Moves Through System*

With this intuitive picture in mind, it is possible to guess at some of the properties of quasi particles. First, because there is in general still a small interaction left between quasi particles, a quasi particle of momentum **p** will only keep this momentum for an average time τ_p. This can be understood from Figs. 0.6 and 0.5. If the quasi ion in Fig. 0.6 (*b*) has momentum **p**, it will propagate undisturbed an average time τ_p before undergoing a collision with another quasi ion in the system (that is, a quasi ion which *belongs* to the system, like those shown in Fig. 0.5, *not* one which we shoot into the system) which scatters it out of momentum state **p**. Hence

$$\text{quasi particles have a lifetime, } \tau_p. \tag{0.3}$$

The lifetime must be reasonably long for us to say that the quasi particle approximation is a good one. It can also be seen that because of the average coat of particles on its back, the quasi particle may have an 'effective' or 'renormalized' mass which is different from that of the bare particle. (The effective mass concept is not always applicable however.) This implies that free quasi particles (i.e., not in an externally applied field) have a new energy law

$$\epsilon' = \frac{p^2}{2m^*} \quad \text{instead of} \quad \epsilon = \frac{p^2}{2m} \tag{0.4}$$

where m^* is the effective mass. The difference

$$\epsilon_{\substack{\text{quasi}\\\text{particle}}} - \epsilon_{\substack{\text{bare}\\\text{particle}}} = \epsilon_{\text{self}} \tag{0.5}$$

is called the '*self-energy*' of the quasi particle. This comes from the interpretation that the bare particle interacts with the many-body system, creating the cloud, and the cloud in turn reacts back on the particle, disturbing its motion. Thus the particle is, in a sense, interacting with itself via the many-body system, and changing its own energy.

2 *Quantum system: quasi electron in electron gas*

The 'electron gas' is a simple model often used to describe many-body effects in metals. It consists of a box containing a large number of electrons interacting by means of the Coulomb force. In addition, there is a uniform, fixed, positive charge 'background' put into the box in order to keep the whole system electrically neutral. In the ground state, the electrons are spread out uniformly in the box, as shown schematically in Fig. 0.7.

Fig. 0.7 *'Electron Gas': Interacting Electrons Spread Out Uniformly in Box, plus Uniform, Fixed, Positive Charge Background*

Suppose now that we have a single, well-localized electron which we shoot into the electron gas (Fig. 0.8). Because of the repulsive Coulomb interaction between electrons, this extra electron repels other electrons away from it, so

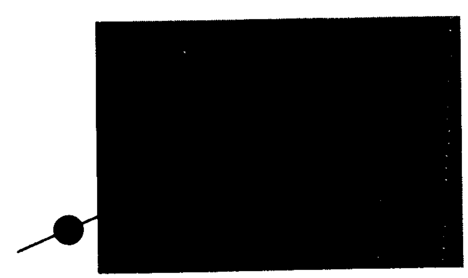

Fig. 0.8 *Extra Electron Shot into Electron Gas*

we get an 'empty space' near the extra electron, and repelled electrons further away (Fig. 0.9). The empty space has positive charge, since the positive charge background is exposed in this region. This empty region may be viewed in a more detailed or 'microscopic' way as composed of 'holes' in the electron gas. That is, the extra electron has 'lifted out' electrons from the uniform charge distribution in its vicinity, thus creating 'holes' in this charge distribution, and has 'put down' these lifted-out electrons further away. This is shown in Fig. 0.10. Because of the exposed positive background, these holes have positive charge.

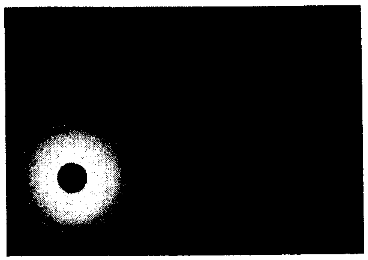

Fig. 0.9 *Extra Electron Pushes Other Electrons Away, Creating 'Empty' Region in its Immediate Vicinity*

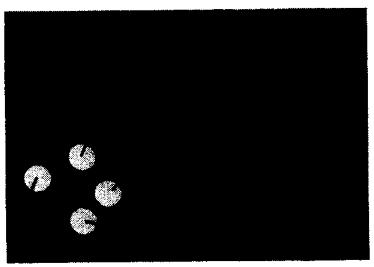

Fig. 0.10 *'Microscopic' View of Fig. 0.9 Showing Electrons Lifted out from Vicinity of the Extra Electron, thus Creating 'Holes'*

The above definition of hole in the sense of 'empty place' is the one commonly used in solid state physics. However, later on we shall re-define things so that the hole becomes an 'anti-particle' analogous to those of elementary particle physics (see §4.2).

The holes and lifted out electrons are constantly being destroyed by interaction with the extra electron and with the other electrons in the system, and new holes and lifted out electrons take their place. The sum of these microscopic processes, which go on all the time, is Fig. 0.9. Thus Fig. 0.9 may be visualized as an extra electron surrounded by a 'cloud' of constantly changing holes and lifted out electrons. This combination is called the *quasi electron.*

The quasi electron moves or 'propagates' through the system as shown in Fig. 0.11.

We now notice that the positive hole cloud immediately around the extra electron partially shields the electron's own negative charge. Hence, if we have two quasi electrons as shown in Fig. 0.12, and these are far enough

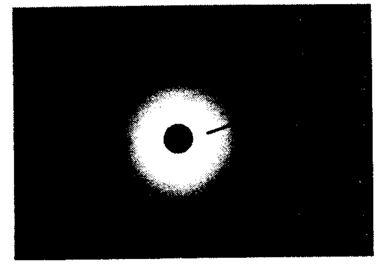

Fig. 0.11 *Quasi Electron Propagates Through System*

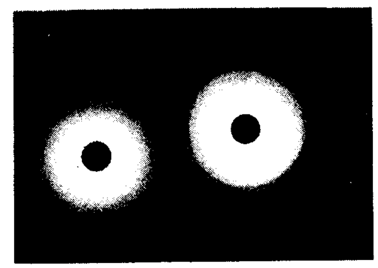

Fig. 0.12 *Two Quasi Electrons Interact only Weakly Because of Shielding*

apart so that their clouds do not overlap very much, then we see that because of the shielding the two quasi electrons will interact only weakly. That is, quasi electrons act nearly independently of one another. This is why metals generally behave as if their electrons were independent: it is not real electrons but rather quasi electrons we are looking at.

3 *Single electron in a metal*

Actually, the simplest quantum example of the quasi particle idea occurs not in a true many-body system, but rather in a system containing one particle moving in an external potential, i.e., a conduction electron in a metal. In a perfect metal the positive ions form a regular periodic lattice (we ignore lattice vibrations for the moment) so that the electron moves in a periodic force field due to the attractive Coulomb interaction between the ions and the electron (see Fig. 0.13a). In an imperfect metal, the periodicity is spoiled by the presence of a more or less random distribution of some impurity ions in the lattice, or the presence of some displaced ions (Fig. 0.13b).

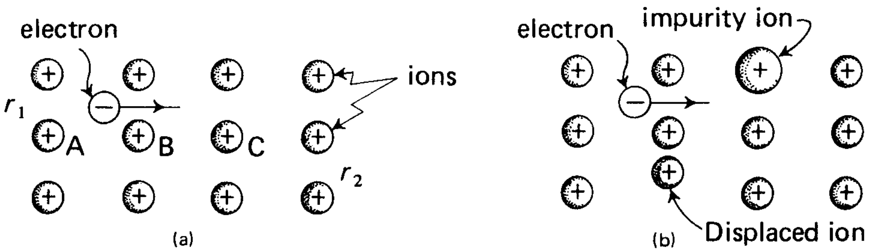

Fig. 0.13 *(a) Conduction Electron in Perfect Metal. (b) Imperfect Metal*

Since the lattice here is assumed fixed, there is no 'moving cloud' of lattice ions following the electron. Nevertheless, it turns out that even these stationary lattice ions are capable of 'clothing' the electron, and we find that for a perfect lattice, there is an effective mass, m^*, and an infinite lifetime. Addition of imperfections causes the lifetime to become finite.

4 *Quasi nucleon*

Despite powerful short-range forces between nucleons in a nucleus, they behave in many respects as if they were independent of each other, as is indicated by the success of the nuclear shell model. The nearly independent particles here are not the nucleons themselves, but the nucleons each surrounded by a cloud of other nucleons, i.e., the quasi nucleons.

5 *Bogoliubov quasi particles* ('*bogolons*')

These are the elementary excitations in a superconductor. We include them here since they are called quasi particles, but actually their structure is quite different from the 'particle plus cloud' picture described above. They consist of a linear combination of an electron in state $(+k, \uparrow)$ and a 'hole' in $(-k, \downarrow)$.

0.3 Collective excitations

As we have seen, the quasi particle consists of the original real, individual particle, plus a cloud of disturbed neighbours. It behaves very much like an individual particle, except that it has an effective mass and a lifetime. But there also exist other kinds of fictitious particles in many-body systems, i.e., 'collective excitations'. These do not centre around individual particles, but instead involve collective, wavelike motion of *all* the particles in the system simultaneously. Here are some examples:

1 *Plasmons*

If a thin metal foil is bombarded with high energy electrons, it is possible to set up sinusoidal oscillations in the density of the electron gas in the foil. This is known as a 'plasma wave', and it has a frequency ω_p and a wavelength λ_p (see Fig. 0.14a). The plasma wave may be visualized as built up of 'holes' in the low-density regions and extra electrons in the high-density regions as shown in Fig. 0.14(*b*). Just as light waves are quantized into units having energy $E=\hbar\omega$ called photons, plasma waves are quantized into units with energy $E_p=\hbar\omega_p$ called *plasmons*.

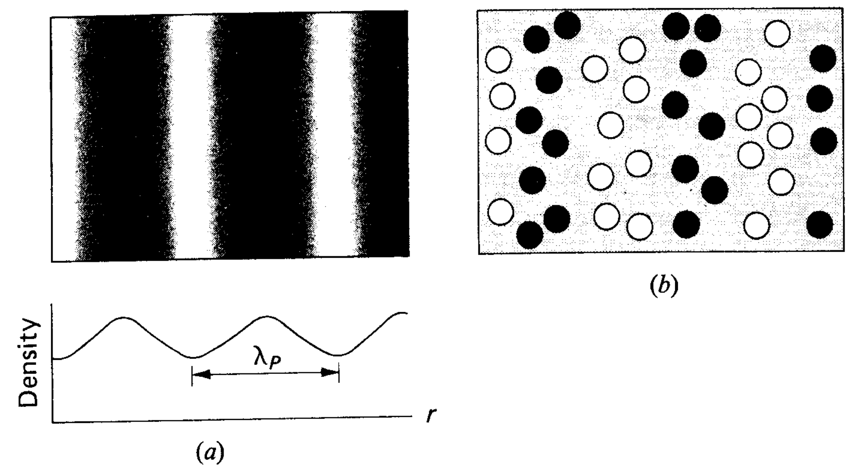

Fig. 0.14 *(a) Plasma Wave in Electron Gas. (b) Particle-hole Picture of Plasma Wave*

2 *Phonons*

Sound waves are sinusoidal oscillations in the crystal lattice of a solid. They are quantized into collective excitations called 'phonons'. (See appendix $\mathscr{A}$.)

3 *Magnons*

In ferromagnets there are regular fluctuations in the density of spin angular momentum known as 'spin waves'. The collective excitation here is the spin wave quantum known as the 'magnon'.

4 *Nuclear quanta*

In nuclei, one finds various vibrational and rotational motions; the associated quanta are the collective excitations in this case.

In the next chapter, we will describe in a very qualitative way how to find the properties of quasi particles and collective excitations by means of 'propagators' and 'Feynman diagrams'.

Further reading

Appendix $\mathscr{A}$
Patterson (1964).
Pines (1963), chap. 1.