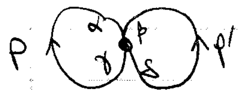
11/9/09 (160)

Monday 880.05 Class

Today: · Problem Set #3 → The idea is to reinforce (and in one case preview) what we have done in class with path integrals.
· Brief discussion of Euclidean ⟷ Minkowski space
· Momentum feynman rules for ∞ at $T=0$
· Finish up beachball → motivate start of effective field theory

· Bowtie diagram in momentum space for energy density



· Same rules for spin and symmetry factor parts and vertex $\Rightarrow$ yield $\frac{1}{2}(\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma})\delta_{\alpha\gamma}\delta_{\beta\delta}\cdot\lambda$

· The lines become momentum space Green's functions: ($T=0$)

$$\boxed{\mathcal{G}^0_{\alpha\beta}(p) = \frac{-\delta_{\alpha\beta}}{ip_0-(\epsilon^0_p-\mu_0)}} \quad \text{with} \quad \boxed{\epsilon^0_p = \frac{\hbar^2 p^2}{2m}}$$

(but not relativistic!)

instead of $\mathcal{G}^0_{\alpha\beta}(x,x')$ from before. Each line gets a four-momentum, which is conserved (no constraint here because p and p' both go in and out of the vertex).

· Integrate $\int\frac{d^4p}{(2\pi)^4}e^{ip_0\eta}$ over every four-momentum. $e^{ip_0\eta}$ is a convergence factor that tells us how to close contours, $\eta\to 0$ at end.

· ~~At $T\neq 0$, $p_0 \to \omega_n$ a discrete frequency and we have "Matsubara sums".~~

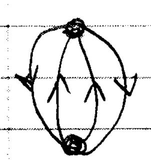
$$\boxed{\begin{aligned}\Rightarrow \infty &= \tfrac{1}{2}(\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma})\delta_{\alpha\gamma}\delta_{\beta\delta}\,\lambda\left[\int\frac{d^4p}{(2\pi)^4}\frac{-e^{ip_0\eta}}{ip_0-(\epsilon^0_p-\mu_0)}\right]\left[\int\frac{d^4p'}{(2\pi)^4}\frac{-e^{ip_0'\eta}}{ip_0'-(\epsilon^0_{p'}-\mu_0)}\right]\\ &= \frac{\lambda}{2}\nu(\nu-1)\left[-\int\frac{d^3p}{(2\pi)^3}\Theta(\mu_0-\epsilon^0_p)\right]\left[-\int\frac{d^3p'}{(2\pi)^3}\Theta(\mu_0-\epsilon^0_{p'})\right]\\ &= \frac{\lambda}{2}\left(1-\frac{1}{\nu}\right)\rho^2 \quad \text{which is what we got before for the energy density}\end{aligned}}$$

· note that we close in the upper half plane so $e^{i(\mathrm{Re}\,p_0+i\,\mathrm{Im}\,p_0)\eta} \propto e^{-(\mathrm{Im}\,p_0)\eta}$ makes the integral converge. We pick up a pole if $\mu_0 > \epsilon^0_p$, otherwise 0.

Euclidean ⟷ Minkowski
$t \leftrightarrow -i\tau$ so $\mathcal{L}_E(x,\tau) = -\mathcal{L}(x,-i\tau)$ $Z=\int e^{iS}$
or $\tau \leftrightarrow it$ or $\mathcal{L}(x,t) = -\mathcal{L}_E(x,it)$ $Z=\int e^{-S_E}$

11/9/09

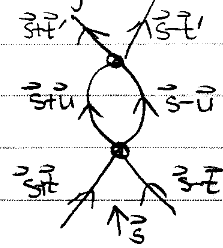
Further consideration of the beachball diagram:



$$\Rightarrow \boxed{\begin{aligned} \mathcal{E}_2 = -4\lambda^2 M\nu(\nu-1)\,k_F^7 \times \int\frac{d^3s}{(2\pi)^3}\int\frac{d^3t}{(2\pi)^3}\int\frac{d^3u}{(2\pi)^3}\,\frac{1}{u^2-t^2-i\eta} \\ \times\,\theta(1-|\vec{s}+\vec{t}|)\,\theta(1-|\vec{s}-\vec{t}|)\,\theta(|\vec{s}+\vec{u}|-1)\,\theta(|\vec{s}-\vec{u}|-1) \end{aligned}}$$

- Because the θ functions restrict the sum and difference of $\vec{s}$ and $\vec{t}$ to be less than 1, both of these integrals are bounded.
  - However, the $\vec{u}$ integration runs to $|\vec{u}|\to\infty$. For large u, $\theta(|\vec{s}+\vec{u}|-1) = \theta(|\vec{s}-\vec{u}|-1) = 1$ and $\frac{1}{u^2-t^2} \to \frac{1}{u^2}$, so the integral goes like $\int^\infty \frac{u^2du}{u^2} \sim \int^\infty du \to \infty$.

- To analyze the divergence and see how to renormalize it, we will "unfold" the diagram into the closely related scattering diagram:



  - We've labeled the final legs with t', but to generate the energy, we set $\vec{t}=\vec{t}'$ or $\vec{t}=-\vec{t}'$ and close the legs.
  - The variables in the $\mathcal{E}_2$ expression (s,t,u) are dimensionless, but we can always put back the powers of k_F and make them momenta again.

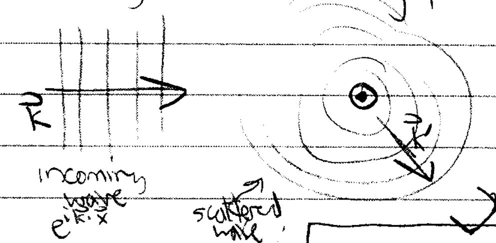
- In the scattering case, the intermediate state, specified by momenta $\vec{s}+\vec{u}$ and $\vec{s}-\vec{u}$ is unrestricted. In the contribution to $\mathcal{E}_2$, these momenta must be greater than k_F. But the divergence is for large $\vec{u}$, where there is no restriction
  - $\therefore$ If we fix the divergence for the scattering case, we'll fix it for finite density!

11/9/09

- The last conclusion is a general result about ultraviolet (short wavelength $\Rightarrow$ large momentum) divergences. The Pauli blocking effects at finite density are low momentum (e.g. compared to k_F) effects and so the same UV divergences in free space will appear in the same form at finite density.
  $\Rightarrow$ if we renormalize in free space, we should automatically renormalize in the many-body system.
  (This better work, because we have no choice!)

- So considering the scattering problem . . . first a one-body picture:

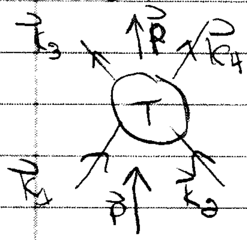
lab frame:



incoming $\vec{k}$ scatters to $\vec{k}'$
elastic: $|\vec{k}| = |\vec{k}'|$

Scattering wave function $\boxed{\psi^{(+)}_{\vec{k}}(\vec{x}) \sim e^{i\vec{k}\cdot\vec{x}} + f(\vec{k}',\vec{k})\dfrac{e^{ikx}}{x}}$ as $x \to \infty$

- We consider the scattering of two particles with potential V, and use k_F to scale the momenta (so we make a connection to the finite density problem — k_F doesn't have any particular meaning!).



- The total momentum is $\boxed{\vec{P} = \vec{k}_1 + \vec{k}_2 = 2k_F\vec{s}}$

- The relative momenta are

$$\boxed{\vec{k} = \tfrac{1}{2}|\vec{k}_1 - \vec{k}_2|} \quad \boxed{\vec{k}' = \tfrac{1}{2}|\vec{k}_3 - \vec{k}_4|} \Rightarrow \begin{array}{l} \vec{k}_1,\vec{k}_2 = \vec{P} \pm \vec{k} \\ \vec{k}_3,\vec{k}_4 = \vec{P} \pm \vec{k}' \end{array}$$

which correspond to the variables in the equivalent one-body picture above.

- Galilean invariance requires the interaction between the particles to be independent of their center-of-mass momentum $\vec{P}$. That is, you cannot tell what reference frame you are in.
- In contrast, in a finite density system, the other particles define a preferred reference frame,

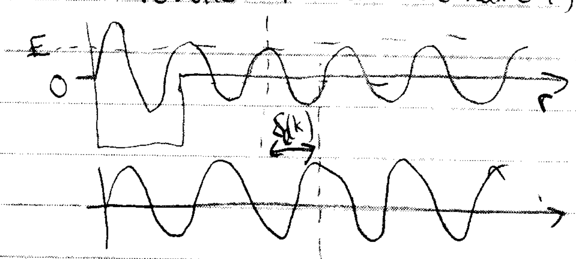
11/9/09

- The scattering amplitude f for elastic scattering depends only on the magnitude $k \equiv |\vec{k}| = |\vec{k}'|$ and the angle between $\vec{k}$ and $\vec{k}'$ (the scattering angle θ). The differential cross section $d\sigma/d\Omega$ is basically $|f|^2$.
  - f can be expanded in Legendre polynomials in $\cos\theta$:

$$\boxed{\begin{aligned} f(k,\theta) &= \sum_{l=0}^{\infty} \frac{2l+1}{k} e^{i\delta_l(k)} \sin\delta_l(k) P_l(\cos\theta) \\ &= \sum_{l=0}^{\infty} \frac{2l+1}{k\cot\delta_l - ik} P_l(\cos\theta) \end{aligned}}$$

where $\delta_l(k)$ is the "phase shift" for the l^{th} partial wave.

- Recall that one can determine $\delta_l(k)$ by comparing the phase of the radial wavefunction outside the range of the potential to the phase of a free wavefunction (hence the name!)



- We'll use a slightly different normalization, indicated by T:

$$\boxed{T(k,\cos\theta) = \frac{4\pi}{M} \sum_{l=0}^{\infty} \frac{2l+1}{k\cot\delta_l(k) - ik} P_l(\cos\theta)}$$

- For short-range interaction at low momentum, $k^{2l+1}\cot\delta_l$ has a power series expansion, called the "effective range expansion":

$$\boxed{k\cot\delta_0(k) = -\frac{1}{a_s} + \frac{1}{2} r_s k^2 + \ldots} \qquad \boxed{k^3\cot\delta_1(k) = -\frac{3}{a_p^3} + \ldots}$$

which defines the s-wave ($l=0$) scattering length a_s and effective range r_s, and the p-wave ($l=1$) scattering length a_p.

11/9/09

- Unless the system has a bound-state (for given l) near zero energy (either just bound, or just missing being bound), the expected size of a_s, r_s, a_p is about R, the range of the potential.
  - For example, for "hard spheres" of radius R (ie, the potential is zero for $r > R$ but is infinitely repulsive at $r = R$), $a_s = a_p = R$ and $r_s = 2R/3$.
  - We define Λ to be the momentum scale $\Lambda = 1/R$.

- If we consider momenta $k \ll \Lambda$, then we can expand $T(k, \cos\theta)$ in a perturbation series using the effective range expansion for $k \cot \delta_l$:

$$\boxed{T(k,\cos\theta) = -\frac{4\pi a_s}{m}\left[1 - i a_s k + (a_s r_s/2 - a_s^2)k^2\right] - \frac{4\pi a_p^3}{m} k^2 \cos\theta + \mathcal{O}(k^3/\Lambda^3)}$$

- So at very low momentum, the scattering is described to good accuracy by specifying just the scattering length a_s.
  - The next correction requires r_s and a_p, and so on.

** - There are an *infinite* number of potentials V (which means $\langle \vec{k}'|V|\vec{k}\rangle$) that have the same $a_s, r_s, \ldots$ (effective range parameters). (We mean, a *finite number* of the same parameters.)
  $\Rightarrow$ we can reproduce this momentum expansion *systematically* (order-by-order in k^2/Λ^2)

- In an *effective field theory* (EFT), we carry out this expansion using a local Lagrangian density to define the EFT.
  - For low momentum $k \ll \Lambda = \frac{1}{R}$, all interactions in the EFT are short-ranged and we have only contact interactions (eg. delta functions) between the fermions.

11/9/09

- The possible terms in the Lagrangian are constrained by some symmetries: Galilean, parity, and time-reversal invariances.
  - That means that we only include terms that don't change (up to total derivatives) under these transformations.
  - The Lagrangian is organized by the number of $\vec{\nabla}$'s that appear in an interaction term. Each $\vec{\nabla}$ acting on a field gives a momentum, so more $\vec{\nabla}$'s means higher order in the momentum expansion.

- Here is a Lagrangian we can use: (this is Euclidean)

$$\boxed{\begin{aligned}\mathcal{L}_E^{EFT} = \psi^\dagger\left[\frac{\partial}{\partial\tau} - \frac{\vec{\nabla}^2}{2m}\right]\psi + \frac{C_0}{2}(\psi^\dagger\psi)^2 - \frac{C_2}{16}\left[(\psi\psi)^\dagger(\psi\overleftrightarrow{\nabla}^2\psi) + \text{h.c.}\right] \\ - \frac{C_2'}{8}(\psi\overleftrightarrow{\nabla}\psi)^\dagger\cdot(\psi\overleftrightarrow{\nabla}\psi) + \frac{D_0}{6}(\psi^\dagger\psi)^3 + \ldots\end{aligned}}$$

where "h.c." mean Hermitian conjugate (so that the combined sum in [ ]'s is Hermitian) and $\overleftrightarrow{\nabla}$ is the Galilean invariant derivative:

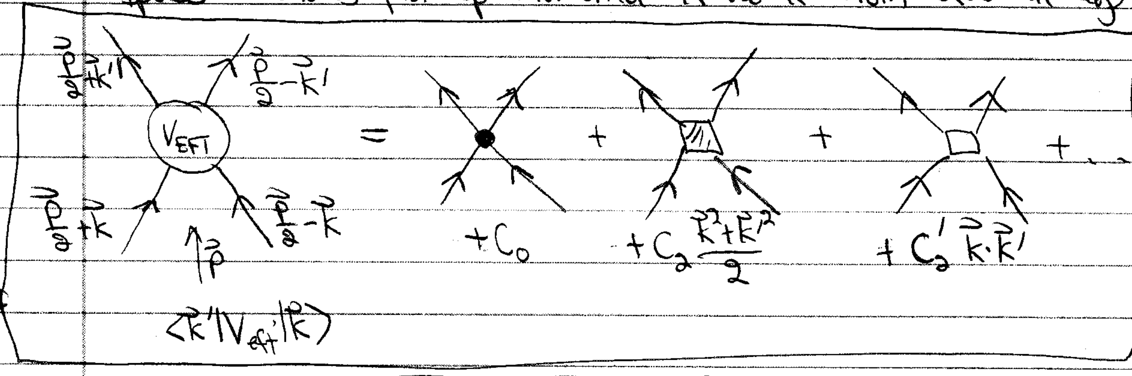
$$\boxed{\overleftrightarrow{\nabla} = \overleftarrow{\nabla} - \overrightarrow{\nabla}}$$

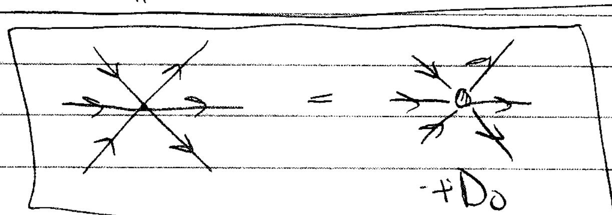
Using this derivative ensures that the Lagrangian is unchanged if all of the particle momenta are "boosted" by $\vec{v}$: $\vec{p} \to \vec{p} + m\vec{v}$.

- Our favorite δ-function Lagrangian corresponds to $C_0 = \lambda$, $C_2 = C_2' = D_0 = \ldots = 0$.
- This is a general, but not unique, form of the Lagrangian for short-range, spin-independent, interactions. One can perform "field redefinitions" which are basically changes of variable for ψ, which lead to different, but physically equivalent, forms (eg. more time derivatives.)
- We've including the leading ($\Rightarrow$ no derivatives) <u>three-body</u> interaction as well.

11/9/09

- The Feynman rules for the vertices beyond C_0 can be derived from the path integral expression. The momentum space versions pick up momenta $\vec{k}$ and $\vec{k}'$ from external legs





- Note that $\vec{p}$ drops out $\Rightarrow$ this is Galilean invariance at work!
- The terms involving only four fermion fields are simple polynomial vertices, equivalent to a momentum expansion of a potential V_{EFT} for particle-particle scattering:

$$\boxed{\langle \vec{k}' | V_{EFT} | \vec{k} \rangle = C_0 + C_2(\vec{k}'^2+\vec{k}^2)/2 + C_2'(\vec{k}\cdot\vec{k}') + \ldots}$$

- For example, if the "true" underlying physics were due to a Yukawa potential (the tree-level exchange of a meson ϕ with mass Λ and coupling g, for example):

$$\boxed{V(\vec{x}) = -\frac{g^2}{4\pi}\frac{e^{-\Lambda x}}{x}}$$

with $\boxed{x \equiv |\vec{x}|}$, then

$$\boxed{\begin{aligned}\langle \vec{k}' | V | \vec{k} \rangle &= \int d^3x \, \langle \vec{k}' | \vec{x} \rangle V(\vec{x}) \langle \vec{x} | \vec{k} \rangle = \frac{-g^2}{4\pi}\frac{1}{(2\pi)^3}\int d^3x \, e^{-i(\vec{k}'-\vec{k})\cdot\vec{x}} \frac{e^{-\Lambda x}}{x} \\ &= -\frac{g^2}{4\pi}\frac{1}{(2\pi)^2}\int_0^\infty x^2 dx \frac{e^{-\Lambda x}}{x}\int_{-1}^{1} d\chi \, e^{-iqx\chi} = \frac{-g^2}{4\pi}\frac{2}{(2\pi)^2}\int_0^\infty x^2 dx \frac{\sin qx}{qx}\frac{e^{-\Lambda x}}{x}\end{aligned}}$$

11/9/09

or, finally,

$$\boxed{\langle \vec{k}'|V|\vec{k}\rangle = -\left(\frac{g^2}{4\pi}\right)\frac{2}{(2\pi)^2}\frac{1}{\mu^2+q^2} = -\left(\frac{g^2}{4\pi}\right)\frac{2}{(2\pi)^2}\frac{1}{\mu^2+(\vec{k}'-\vec{k})^2}}$$

where $\vec{q}$ is the momentum transfer.

- From the calculation of $\langle \vec{k}'|V|\vec{k}\rangle$ in this case, we can generalize that any local potential (ie., V only depends on $\vec{x}$) will have $\langle \vec{k}'|V|\vec{k}\rangle$ be a function of $\vec{q}=\vec{k}'-\vec{k}$ alone.
  - More generally, $\boxed{\langle \vec{x}'|V|\vec{x}\rangle \neq V(\vec{x})\,\delta^3(\vec{x}-\vec{x}')}$ and therefore depends separately on $\vec{k}$ and $\vec{k}'$.
  - The Schrodinger equation:

$$\boxed{(\hat{H}_0+\hat{V})|\psi\rangle = E|\psi\rangle}$$

in coordinate space is

$$\int d^3x'\left(\underbrace{\langle\vec{x}|\hat{H}_0|\vec{x}'\rangle}_{=\,\frac{-\nabla^2}{2m}\delta^3(\vec{x}-\vec{x}')}\langle \vec{x}'|\psi\rangle + \langle\vec{x}|V|\vec{x}'\rangle\langle\vec{x}'|\psi\rangle\right) = E\langle\vec{x}|\psi\rangle$$

or

$$\boxed{-\frac{\nabla^2}{2m}\psi(\vec{x}') + \int d^3x'\,\langle\vec{x}|\hat{V}|\vec{x}'\rangle\psi(\vec{x}') = E\psi(\vec{x})}$$

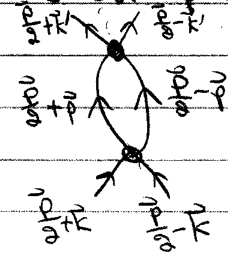
- The Yukawa potential, for $|\vec{k}|, |\vec{k}'| \ll \Lambda$, has a simple power series expansion.
  - One might expect that we determine $C_0, C_2, C_2', \ldots$ by simply matching this expansion to the expression for $\langle \vec{k}'|V_{EFT}|\vec{k}\rangle$.
  - BUT THIS WOULD BE WRONG!
  - The complication is that the scattering amplitude is given by more than just the "tree level" diagrams (ie., more than just first-order perturbation theory).

$\Rightarrow$ include diagrams with "loops"

$\Rightarrow$ summation over intermediate states with high momentum.

11/9/09

- So our problem is when we calculate diagrams such as



where we sum over intermediate states, with $|\vec{p}|$ getting arbitrarily large.
The vertices are $\langle\vec{p}|V_{eff}|\vec{k}\rangle$ and $\langle\vec{k}'|V_{eff}|\vec{p}\rangle$, which are simply incorrect for large $\vec{p}$.
- For example, note that the Yukawa potential dies off at large $|\vec{p}-\vec{k}|$ or $|\vec{k}'-\vec{p}|$, but the EFT vertices grow without bound!

- We can fix the problem, however, by noting that these vertices are correct for low momentum, and for high momentum the intermediate state is at high energy
  $\Rightarrow$ it is highly virtual
  - The uncertainty principle $\Delta E\,\Delta t \gtrsim \hbar$ implies that these high energy intermediate states (which need large ΔE) can only propagate for short times $\lesssim \Delta t$.
  - So the two vertices cannot be very far apart (or else the contribution is very small).

  $\Rightarrow$ the high momentum part of the diagram behaves like a local vertex  so we can "fix" the incorrect part by just adjusting the value of the constants $C_0, C_2, \ldots$ order-by-order in the momentum expansion.
    - This is called "renormalization".

- To carry out the renormalization program, we need to first make the divergent integrals finite
  - This is called "regularization"
  - There are many possible ways to do this — if our analysis is correct, it shouldn't matter in the end
    $\Rightarrow$ observables should be independent of the regularization scheme
  - We'll consider a momentum cutoff and dimensional regularization

11/9/09

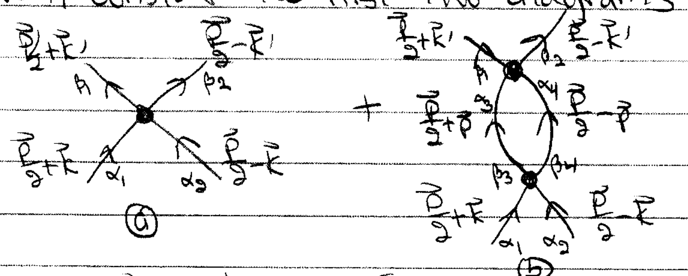
- We can apply our momentum-space Feynman rules
  $\Rightarrow$ This will generate the "T-matrix" Born series:

$$\boxed{T = V + VG_0V + VG_0VG_0V + \ldots = V + VG_0T}$$

in operator form, with $G_0 = \frac{1}{E - H_0}$. For scattering, we want matrix elements $\langle \vec{k}' | T(E) | \vec{k} \rangle$ with $E = k^2/m = k'^2/m$.

- This matrix element of T is equal to $(-1) * T(k, \cos\theta)$ from (150) and (151) (yes, I know: rotten notation!).

- We'll consider the first two diagrams:



Follow the rules on (127), except only apply iG_0 to internal lines
- Use the abbreviations $\vec{k}_+ \equiv \frac{\vec{P}}{2} + \vec{k}$, $\vec{k}_- \equiv \frac{\vec{P}}{2} - \vec{k}$, and so on.

(a) a), b), c): $\boxed{(\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2} + \delta_{\alpha_1\beta_2}\delta_{\alpha_2\beta_1})}$ This will be an overall factor, that multiplies $-T(k, \cos\theta)$.

vertex: $-iC_0^{(0)} \leftarrow$ to be explained later!

d) no integrations to do

e) $\times i$

$$\Rightarrow \boxed{-T^{(0)}(k, \cos\theta) = C_0^{(0)} = \frac{4\pi a_s}{m}} \quad \text{from (151)}$$

- So we get the leading-order value for C_0, which corresponds to the value we used for λ previously.
- In general, this value will be <u>modified</u> at the next order (and beyond) by renormalization.