AJT notes (3/24/21) ①

Updated (5/13/21)

* Here we derive the SRG-evolved single-nucleon momentum distribution $n_\lambda^\tau(q)$.

* Start by defining the operators in second quantization.

$$n^\tau(\vec{q}) = \sum_\sigma a^\dagger_{\vec{q}\sigma\tau} a_{\vec{q}\sigma\tau} \qquad (1)$$

σ spin projection

τ isospin projection

$\uparrow$ NOT relative momentum

$$U_\lambda = \mathbb{1} + \frac{1}{4} \sum_{\sigma_1\sigma_2\sigma_3\sigma_4} \sum_{\tau_1\tau_2\tau_3\tau_4} \sum_{\vec{K}\vec{k}\vec{k}'} \langle \vec{k}\,\sigma_1\tau_1\sigma_2\tau_2 | \delta\tilde{U} | \vec{k}'\sigma_3\tau_3\sigma_4\tau_4 \rangle \times$$

$$a^\dagger_{\frac{\vec{K}}{2}+\vec{k},\sigma_1\tau_1} a^\dagger_{\frac{\vec{K}}{2}-\vec{k},\sigma_2\tau_2} a_{\frac{\vec{K}}{2}-\vec{k}',\sigma_4\tau_4} a_{\frac{\vec{K}}{2}+\vec{k}',\sigma_4\tau_4} + 3\text{-body} \cdots \qquad (2)$$

where $\delta\tilde{U}$ is antisymmetrized ($\delta\tilde{U}_{1234} = -\delta\tilde{U}_{1243}$).

Our task is to evaluate $U_\lambda n^\tau(\vec{q}) U_\lambda^\dagger$ truncating up through 2-body. We will use simplified notation for momentum, spin, isospin basis.

$\sum_\alpha \equiv \sum_\sigma$ NOT $\sum_{\vec{q}\sigma\tau}$

$(\vec{q},\sigma,\tau) \to \alpha$, $(\frac{\vec{K}}{2}+\vec{k},\sigma_1,\tau_1) \to 1$, ...

$(\frac{\vec{K}}{2}-\vec{k}',\sigma_4,\tau_4) \to 4$, $(\frac{\vec{K}'}{2}+\vec{k}'',\sigma_5,\tau_5) \to 5$, ...

$(\frac{\vec{K}'}{2}-\vec{k}''',\sigma_8,\tau_8) \to 8$

So we have

$$\left(\mathbb{1} + \frac{1}{4}\sum_{1234} \langle 12|\delta\tilde{U}|34\rangle a_1^\dagger a_2^\dagger a_4 a_3\right)\left(\sum_\alpha a_\alpha^\dagger a_\alpha\right) \times$$

$$\left(\mathbb{1} + \frac{1}{4}\sum_{5678} \langle 56|\delta\tilde{U}|78\rangle a_5^\dagger a_6^\dagger a_8 a_7\right)^\dagger$$

$$= \sum_\alpha a_\alpha^\dagger a_\alpha \qquad \text{1 term}$$

$$\left.\begin{aligned} &+ \frac{1}{4}\sum_\alpha \sum_{1234} \langle 12|\delta\tilde{U}|34\rangle a_1^\dagger a_2^\dagger a_4 a_3\, a_\alpha^\dagger a_\alpha \\ &+ \frac{1}{4}\sum_\alpha \sum_{5678} \langle 78|\delta\tilde{U}^\dagger|56\rangle a_\alpha^\dagger a_\alpha\, a_7^\dagger a_8^\dagger a_6 a_5 \end{aligned}\right\} \delta U \text{ term}$$

$$+ \underbrace{\frac{1}{16}\sum_\alpha \sum_{1234}\sum_{5678} \langle 12|\delta\tilde{U}|34\rangle\langle 78|\delta\tilde{U}^\dagger|56\rangle a_1^\dagger a_2^\dagger a_4 a_3\, a_\alpha^\dagger a_\alpha\, a_7^\dagger a_8^\dagger a_6 a_5}_{\delta U \delta U^\dagger \text{ term}} \qquad (3)$$

- Evaluate contractions with respect to $|0\rangle$ first

1 term: No possible contractions

δU term: $\quad a_1^\dagger a_2^\dagger a_4 a_3 a_\alpha^\dagger a_\alpha$

$$\rightarrow a_1^\dagger a_2^\dagger a_4 a_3 a_\alpha^\dagger a_\alpha + a_1^\dagger a_2^\dagger a_4 a_3 a_\alpha^\dagger a_\alpha$$

Antisymmetrized so evaluate first one and multiply by 2

$$\frac{1}{4}\sum_\alpha\sum_{1234}\langle 12|\delta\tilde U|34\rangle\, 2\,\delta_{3\alpha}\, a_1^\dagger a_2^\dagger a_4 a_\alpha \qquad \frac{\vec K}{2}+\vec k' = \vec q$$

$$= \frac{1}{2}\sum_\alpha\sum_{124}\langle 12|\delta\tilde U|\alpha 4\rangle\, a_1^\dagger a_2^\dagger a_4 a_\alpha \qquad \vec k' \to \vec q - \frac{\vec K}{2}$$

$\delta U^\dagger$ term gives $\frac{1}{2}\sum_\alpha\sum_{568}\langle \alpha 8|\delta\tilde U^\dagger|56\rangle\, a_\alpha^\dagger a_8^\dagger a_6 a_5$ (4)

$\frac{\vec K'}{2}+\vec k''' = \vec q \qquad \vec k''' = \vec q - \frac{\vec K'}{2}$

$\delta U \delta U^\dagger$ term: $\quad a_1^\dagger a_2^\dagger a_4 a_3\, a_\alpha^\dagger a_\alpha\, a_7^\dagger a_8^\dagger a_6 a_5$

$$\to \quad a_1^\dagger a_2^\dagger a_4 a_3 a_\alpha^\dagger a_\alpha a_7^\dagger a_8^\dagger a_6 a_5$$
$$+ \quad a_1^\dagger a_2^\dagger a_4 a_3 a_\alpha^\dagger a_\alpha a_7^\dagger a_8^\dagger a_6 a_5$$
$$+ \quad a_1^\dagger a_2^\dagger a_4 a_3 a_\alpha^\dagger a_\alpha a_7^\dagger a_8^\dagger a_6 a_5$$
$$+ \quad a_1^\dagger a_2^\dagger a_4 a_3 a_\alpha^\dagger a_\alpha a_7^\dagger a_8^\dagger a_6 a_5$$

$\frac{\vec K}{2}+\vec k' = \vec q \to \frac{\vec K}{2} = \vec q - \vec k'$

$\frac{\vec K'}{2}+\vec k''' = \vec q \to \frac{\vec K'}{2} = \vec q - \vec k'''$

$\frac{\vec K}{2} - \vec k' = \frac{\vec K'}{2} - \vec k'''$

$\vec q - 2\vec k' = \vec q - 2\vec k''' \to \vec k' = \vec k'''$

$\Rightarrow \vec K' = \vec K \qquad \vec k' = \vec q - \frac{\vec K}{2}$

Antisymmetrized $\to$ Factor of 4 (do last one only)

$$= \frac{1}{16}\sum_\alpha\sum_{1234}\sum_{5678}\langle 12|\delta\tilde U|34\rangle\langle 78|\delta\tilde U^\dagger|56\rangle\, 4\,\delta_{3\alpha}\,\delta_{\alpha 7}\,\delta_{48}\quad a_1^\dagger a_2^\dagger a_6 a_5$$

$$= \frac{1}{4}\sum_\alpha\sum_{12456}\langle 12|\delta\tilde U|\alpha 4\rangle\langle \alpha 4|\delta\tilde U^\dagger|56\rangle\, a_1^\dagger a_2^\dagger a_6 a_5 \qquad (5)$$

So in total, we have

$$n_\alpha^\tau(\vec q) = \sum_\alpha a_\alpha^\dagger a_\alpha + \frac{1}{2}\sum_\alpha\sum_{124}\langle 12|\delta\tilde U|\alpha 4\rangle a_1^\dagger a_2^\dagger a_4 a_\alpha +$$

$$\frac{1}{2}\sum_{\alpha}\sum_{568}\langle \alpha 8|\delta\tilde{U}^{\dagger}|56\rangle a_{\alpha}^{\dagger}a_{8}^{\dagger}a_{6}a_{5} \quad +$$

$$\frac{1}{4}\sum_{\alpha}\sum_{12456}\langle 12|\delta\tilde{U}|\alpha 4\rangle\langle \alpha 4|\delta\tilde{U}^{\dagger}|56\rangle a_{1}^{\dagger}a_{2}^{\dagger}a_{6}a_{5} \qquad (6)$$

Rewrite then take continuum limit

Removing $(2\pi)^3$ factors (5/13/21)

$$= \sum_{\sigma} a^{\dagger}_{\vec{q}\sigma\tau}a_{\vec{q}\sigma\tau} + \frac{1}{2}\sum_{\sigma}\sum_{\sigma_1\sigma_2\sigma_4}\sum_{\tau_1\tau_2\tau_4}\int d^3k\int d^3K \times$$

$$\langle \vec{k}\,\sigma_1\tau_1\,\sigma_2\tau_2|\delta\tilde{U}|\vec{q}-\tfrac{\vec{K}}{2}\,\sigma\tau\,\sigma_4\tau_4\rangle\, a^{\dagger}_{\frac{\vec{K}}{2}+\vec{k},\sigma_1\tau_1}a^{\dagger}_{\frac{\vec{K}}{2}-\vec{k},\sigma_2\tau_2}a_{\vec{K}-\vec{q},\sigma_4\tau_4}a_{\vec{q}\sigma\tau}$$

$$+\frac{1}{2}\sum_{\sigma}\sum_{\sigma_5\sigma_6\sigma_8}\sum_{\tau_5\tau_6\tau_8}\int d^3k'\int d^3K'\langle \vec{q}-\tfrac{\vec{K}'}{2}\,\sigma\tau\,\sigma_8\tau_8|\delta\tilde{U}^{\dagger}|\vec{k}'\sigma_1\tau_1\sigma_2\tau_2\rangle\times$$

$$a^{\dagger}_{\vec{K}'-\vec{q}\,\sigma_8\tau_8}a^{\dagger}_{\vec{q}\sigma\tau}a_{\frac{\vec{K}'}{2}-\vec{k}'\sigma_6\tau_6}a_{\frac{\vec{K}'}{2}+\vec{k}'\sigma_5\tau_5} \quad + \frac{1}{4}\sum_{\sigma}\sum_{\sigma_1\sigma_2\sigma_4\sigma_5\sigma_6}\times$$

$$\sum_{\tau_1\tau_2\tau_4\tau_5\tau_6}\int d^3k\int d^3k'\int d^3K\,\langle \vec{k}\,\sigma_1\tau_1\,\sigma_2\tau_2|\delta\tilde{U}|\vec{q}-\tfrac{\vec{K}}{2}\sigma\tau\,\sigma_4\tau_4\rangle\times$$

$$\langle \vec{q}-\tfrac{\vec{K}}{2}\sigma\tau\,\sigma_4\tau_4|\delta\tilde{U}^{\dagger}|\vec{k}'\,\sigma_5\tau_5\,\sigma_6\tau_6\rangle\, a^{\dagger}_{\frac{\vec{K}}{2}+\vec{k},\sigma_1\tau_1}a^{\dagger}_{\frac{\vec{K}}{2}-\vec{k},\sigma_2\tau_2}\times$$

$$a_{\frac{\vec{K}}{2}-\vec{k}',\sigma_6\tau_6}a_{\frac{\vec{K}}{2}+\vec{k}',\sigma_5\tau_5} \qquad (7)$$

Now do contractions with respect to a filled Fermi sea $|F\rangle$ where $\langle F|a^{\dagger}_{\beta}a_{\beta}|F\rangle = \theta(k_F^{\beta}-k_{\beta})$.

1 term: $\sum_\sigma \langle F | a^\dagger_{\vec q \sigma\tau} a_{\vec q \sigma\tau} | F \rangle \quad (2\pi)^3 \delta^3(\vec q - \vec q)$

$\downarrow$

One contraction: $\overbrace{a^\dagger_{\vec q \sigma\tau} \, a_{\vec q \sigma\tau}} = V\, \theta(k_F^\tau - q)$

$\times \sum_\sigma \to 2$

$$\boxed{= V\, 2\, \theta(k_F^\tau - q)} \qquad (8)$$

δU term: $\frac{1}{2} \sum_\sigma \sum_{\sigma_1 \sigma_2 \sigma_4} \sum_{\tau_1 \tau_2 \tau_4} \int d^3k \int d^3K \, \langle \vec k\, \sigma_1 \tau_1 \sigma_2 \tau_2 | \delta \tilde U | \vec q - \tfrac{\vec K}{2}\, \sigma \tau \sigma_4 \tau_4 \rangle \times$

$$a^\dagger_{\frac{\vec K}{2} + \vec k, \sigma_1 \tau_1} \, a^\dagger_{\frac{\vec K}{2} - \vec k, \sigma_2 \tau_2} \, a_{\vec K - \vec q, \sigma_4 \tau_4} \, a_{\vec q \sigma \tau}$$

Two contractions: $a^\dagger a^\dagger a a$ and $a^\dagger a^\dagger a a$

Do the first and multiply by 2

$$= \sum_\sigma \sum_{\sigma_1 \sigma_2 \sigma_4} \sum_{\tau_1 \tau_2 \tau_4} \int d^3k \int d^3K \, \langle \vec k\, \sigma_1 \tau_1 \sigma_2 \tau_2 | \delta \tilde U | \vec q - \tfrac{\vec K}{2}\, \sigma \tau \sigma_4 \tau_4 \rangle \times$$

$$\delta_{\sigma_2, \sigma_4} \delta_{\tau_2, \tau_4} \delta^3\!\left(\tfrac{\vec K}{2} - \vec k - \vec K + \vec q\right) \delta_{\sigma_1 \sigma} \delta_{\tau_1 \tau} \, \delta^3\!\left(\tfrac{\vec K}{2} + \vec k - \vec q\right)$$

$$\langle F | a^\dagger_{\frac{\vec K}{2} + \vec k, \sigma_1 \tau_1} \, a_{\vec q \sigma \tau} \, a^\dagger_{\frac{\vec K}{2} - \vec k\, \sigma_2 \tau_2} \, a_{\vec K - \vec q\, \sigma_4 \tau_4} | F \rangle$$

Do $\int d^3K \, \delta^3\!\left(\tfrac{\vec K}{2} + \vec k - \vec q\right) [\ldots]$

$$\to \tfrac{\vec K}{2} + \vec k - \vec q = 0 \Rightarrow \tfrac{\vec K}{2} = \vec q - \vec k$$

$$= \sum_{\sigma\sigma_2\sigma_4} \sum_{\tau_2\tau_4} \int d^3k \, \langle \vec{k}\,\sigma\tau\,\sigma_2\tau_2 | \delta\tilde{U} | \vec{k}\,\sigma\tau\,\sigma_4\tau_4 \rangle \times$$

$$\delta_{\sigma_2,\sigma_4}\,\delta_{\tau_2,\tau_4}\,\underbrace{\delta^3(\vec{q}-\vec{k}-\vec{q}+\vec{k})}_{=\,\delta^3(0)\,\equiv V} \times$$

$$\langle F | \hat{n}_{\vec{q}\sigma\tau}\, a^\dagger_{\vec{q}-2\vec{k}\,\sigma_2\tau_4}\, a_{\vec{q}-2\vec{k}\,\sigma_4\tau_4} | F \rangle$$

Relabel $\sigma_2, \tau_2 \to \sigma'\tau'$ and evaluate

$$\langle F | \hat{n}_{\vec{q}\sigma\tau}\, \hat{n}_{\vec{q}-2\vec{k}\,\sigma'\tau'} | F \rangle = \Theta(k_F^\tau - q)\,\Theta(k_F^{\tau'} - |\vec{q}-2\vec{k}|)$$

$$= V \sum_{\sigma\sigma'} \sum_{\tau'} \int d^3k \, \langle \vec{k}\,\sigma\tau\sigma'\tau' | \delta\tilde{U} | \vec{k}\,\sigma\tau\sigma'\tau' \rangle \times$$

$$\Theta(k_F^\tau - q)\,\Theta(k_F^{\tau'} - |\vec{q}-2\vec{k}|)$$

$\delta U^\dagger$ term gives identical contribution since $\langle \vec{k} | \delta\tilde{U} | \vec{k} \rangle = \langle \vec{k} | \delta U^\dagger | \vec{k} \rangle$. Factor of 2

$$\boxed{= V\, 2 \sum_{\sigma\sigma'} \sum_{\tau'} \int d^3k \, \langle \vec{k}\,\sigma\tau\sigma'\tau' | \delta\tilde{U} | \vec{k}\,\sigma\tau\sigma'\tau' \rangle \times \Theta(k_F^\tau - q)\,\Theta(k_F^{\tau'} - |\vec{q}-2\vec{k}|)} \quad (9)$$

$\delta U \delta U^\dagger$ term: $\frac{1}{4}\sum_{\sigma\sigma_1\sigma_2\sigma_4\sigma_5\sigma_6}\sum_{\tau_1\tau_2\tau_4\tau_5\tau_6}\int d^3k\int d^3k'\int d^3K \times$

$\langle \vec{k}\,\sigma_1\tau_1\,\sigma_2\tau_2|\delta\tilde{U}|\vec{q}-\frac{\vec{K}}{2}\,\sigma\tau\,\sigma_4\tau_4\rangle\langle\vec{q}-\frac{\vec{K}}{2}\,\sigma\tau\,\sigma_4\tau_4|\delta\tilde{U}^\dagger|\vec{k}'\,\sigma_5\tau_5\,\sigma_6\tau_6\rangle \times$

$a^\dagger_{\frac{\vec{K}}{2}+\vec{k}\,\sigma_1\tau_1}\,a^\dagger_{\frac{\vec{K}}{2}-\vec{k}\,\sigma_2\tau_2}\,a_{\frac{\vec{K}}{2}-\vec{k}'\,\sigma_6\tau_6}\,a_{\frac{\vec{K}}{2}+\vec{k}'\,\sigma_5\tau_5}$

Two contractions: $a^\dagger a^\dagger a a$ and $a^\dagger a^\dagger a a$

Do first one and multiply by 2.

$$= \frac{1}{2}\sum_{\sigma\sigma_1\sigma_2\sigma_4\sigma_5\sigma_6}\sum_{\tau_1\tau_2\tau_4\tau_5\tau_6}\int d^3k\int d^3k'\int d^3K\,\langle\vec{k}\,\sigma_1\tau_1\,\sigma_2\tau_2|\delta\tilde{U}|\vec{q}-\frac{\vec{K}}{2}\,\sigma\tau\,\sigma_4\tau_4\rangle \times$$

$$\langle\vec{q}-\frac{\vec{K}}{2}\,\sigma\tau\,\sigma_4\tau_4|\delta\tilde{U}^\dagger|\vec{k}'\,\sigma_5\tau_5\,\sigma_6\tau_6\rangle\,\delta_{\sigma_1\sigma_5}\delta_{\tau_1\tau_5}\,\delta^3\!\left(\frac{\vec{K}}{2}+\vec{k}-\frac{\vec{K}}{2}-\vec{k}'\right)$$

$$\times\,\delta_{\sigma_2\sigma_6}\delta_{\tau_2\tau_6}\,\delta^3\!\left(\frac{\vec{K}}{2}-\vec{k}-\frac{\vec{K}}{2}+\vec{k}'\right)\langle F|a^\dagger a\,a^\dagger a|F\rangle$$

(Similar to evaluation in δU term)

$$\boxed{\begin{aligned}&= V\,\frac{1}{2}\sum_{\sigma\sigma'\sigma''\sigma'''}\sum_{\tau'\tau''\tau'''}\int d^3k\int d^3K\,\langle\vec{k}\,\sigma'\tau'\sigma''\tau''|\delta\tilde{U}|\vec{q}-\frac{\vec{K}}{2}\,\sigma\tau\sigma'''\tau'''\rangle\\ &\times\langle\vec{q}-\frac{\vec{K}}{2}\,\sigma\tau\sigma'''\tau'''|\delta\tilde{U}^\dagger|\vec{k}\,\sigma'\tau'\sigma''\tau''\rangle\,\Theta\!\left(k_F^{\tau'}-\left|\frac{\vec{K}}{2}+\vec{k}\right|\right)\times\\ &\Theta\!\left(k_F^{\tau''}-\left|\frac{\vec{K}}{2}-\vec{k}\right|\right)\qquad (10)\end{aligned}}$$

Go to partial waves for each term separately

$2V\,\Theta(k_F^\tau - q)$ does not change.

δU term (includes $\delta U + \delta U^\dagger$):

$$= V\,2 \sum_{\sigma\sigma'} \sum_{\tau'} \int d^3k \, \langle \vec{k}\,\sigma\tau\sigma'\tau' | \delta\tilde{U} | \vec{k}\,\sigma\tau\sigma'\tau' \rangle \, \Theta(k_F^\tau - q) \times$$

$$\Theta(k_F^{\tau'} - |\vec{q} - 2\vec{k}|)$$

Insert complete set of states using

$$\sum_{S=0}^{1} \sum_{M_S=-S}^{S} |S M_S\rangle\langle S M_S| = 1 \qquad (11)$$

$$\sum_{T=0}^{1} \sum_{M_T=-T}^{T} |T M_T\rangle\langle T M_T| = 1 \qquad (12)$$

$$\sqrt{\frac{2}{\pi}} \sum_{L M_L} |k L M_L\rangle\langle k L M_L| = 1 \qquad (13)$$

$$\sum_{J=|L-S|}^{L+S} \sum_{M_J=-J}^{J} |J M_J L S\rangle\langle J M_J L S| = 1 \qquad (14)$$

$$= V\,2\,\frac{2}{\pi} \sum_{\sigma\sigma'} \sum_{\tau'} \sum_{S M_S} \sum_{S' M_S'} \sum_{T M_T} \sum_{T' M_T'} \sum_{L M_L} \sum_{L' M_L'} \sum_{J M_J} \sum_{J' M_J'} \int_0^\infty dk\,k^2 \times$$

$$\int d\Omega_{\hat{k}} \, \langle \sigma\sigma' | S M_S \rangle \langle \tau\tau' | T M_T \rangle \langle \hat{k} | k L M_L \rangle \langle S M_S L M_L | J M_J L S \rangle \times$$

$[1-(-1)^{L+S+T}]$

$\langle J'M_J'|L'S'|S'M_S'L'M_L'\rangle\,[1-(-1)^{L+S+T}]$ (5/13/21)

$$\langle kJLST|\delta\tilde{U}|kJ'L'S'T'\rangle\langle kL'M_L'|\vec{k}\rangle\langle T'M_T'|\tau\tau'\rangle\langle S'M_S'|\sigma\sigma'\rangle\,\Theta's$$

Apply $\sum_{\sigma\sigma'}\langle\sigma\sigma'|SM_S\rangle\langle S'M_S'|\sigma\sigma'\rangle = \delta_{SS'}\delta_{M_SM_S'}$

$$\int d\Omega_{\hat{k}}\,\langle\vec{k}|kLM_L\rangle\langle kL'M_L'|\vec{k}\rangle = \delta_{LL'}\delta_{M_LM_L'}$$

$$\sum_{M_LM_S}\langle SM_SLM_L|JM_J\rangle\langle J'M_J'|SM_SLM_L\rangle = \delta_{JJ'}\delta_{M_JM_J'}$$

$$\sum_{M_J}[\ldots] = (2J+1)$$

$$\delta\tilde{U} \propto \delta_{TT'}\delta_{M_TM_T'}$$

$$= V\;2\;\frac{2}{\pi}\sum_{\tau\tau'}\sum_{TM_T}\sum_S\sum_L\sum_J\int_0 dk\,k^2\langle\tau\tau'|TM_T\rangle\times$$

$$\langle kJLST|\delta\tilde{U}|kJLST\rangle\langle TM_T|\tau\tau'\rangle\,\Theta(k_F^{\tau}-q)\,\Theta(k_F^{\tau'}-|\vec{q}-2\vec{k}|) \qquad (15)$$

But we still have dependence on $|\vec{q}-2\vec{k}| = \sqrt{q^2+4k^2-4qkx}$ where $x = \hat{q}\cdot\hat{k}$. Average by evaluating

$$\int_{-1}^{1}\frac{dx}{2}[\ldots]$$

$[1-(-1)^{L+S+T}]^2$

$$\boxed{\begin{aligned}&= V\,2\,\frac{2}{\pi}\sum_{\tau\tau'}\sum_{TM_T}\sum_S\sum_L\sum_J 2^2(2J+1)\int_0^{\infty}dk\,k^2\int_{-1}^{1}\frac{dx}{2}\,\langle\tau\tau'|TM_T\rangle\times\\&\langle kJLST|\delta\tilde{U}|kJLST\rangle\langle TM_T|\tau\tau'\rangle\,\Theta(k_F^{\tau}-q)\,\Theta(k_F^{\tau'}-|\vec{q}-2\vec{k}|)\end{aligned}} \qquad (16)$$

$\delta U \delta U^\dagger$ term

$$= V\,\frac{1}{2}\sum_{\sigma\sigma'\sigma''\sigma'''}\sum_{\tau'\tau''\tau'''}\int d^3k \int d^3K \,\langle \vec{k}\,\sigma'\tau'\sigma''\tau''|\,\delta\tilde{U}\,|\vec{q}-\tfrac{\vec{K}}{2}\,\sigma\tau\sigma'''\tau'''\rangle$$

$$\times \langle \vec{q}-\tfrac{\vec{K}}{2}\,\sigma\tau\sigma'''\tau'''|\,\delta\tilde{U}^\dagger\,|\vec{k}\,\sigma'\tau'\sigma''\tau''\rangle\, \theta\!\left(k_F^{\tau'}-\left|\tfrac{\vec{K}}{2}+\vec{k}\right|\right) \times$$

$$\theta\!\left(k_F^{\tau''}-\left|\tfrac{\vec{K}}{2}-\vec{k}\right|\right)$$

Insert complete set of states for $S M_S T M_T$ etc.

$$= \frac{V}{2}\left(\frac{2}{\pi}\right)^2 \sum_{\sigma\sigma'\sigma''\sigma'''}\sum_{\tau'\tau''\tau'''}\sum_{TM_T T'M_T' T''M_T'' T'''M_T'''}\sum_{SM_S S'M_S' S''M_S'' S'''M_S'''}\sum_{LM_L L'M_L' L''M_L'' L'''M_L'''}$$

$$\sum_{JM_J J'M_J' J''M_J'' J'''M_J'''}\int_0^\infty dk\,k^2\int_0^\infty dK\,K^2\int d\Omega_k\, d\Omega_K\;\langle\sigma'\sigma''|SM_S\rangle\langle\tau'\tau''|TM_T\rangle \times$$

$$\langle\vec{k}|kLM_L\rangle\underbrace{\langle SM_S LM_L|JM_J LS\rangle}_{[1-(-1)^{L+S+T}]}\langle kJLST|\,\delta\tilde{U}\,|\,|\vec{q}-\tfrac{\vec{K}}{2}|\,J'L'S'T'\rangle \times$$

$$\langle J'M_J'L'S'|S'M_S'L'M_L'\rangle\underbrace{\langle|\vec{q}-\tfrac{\vec{K}}{2}|\,L'M_L'\,|\,\vec{q}-\tfrac{\vec{K}}{2}\rangle}_{[1-(-1)^{L'+S'+T'}]}\langle T'M_T'|\tau\tau'''\rangle \times$$

$$\langle S'M_S'|\sigma\sigma'''\rangle\langle\sigma\sigma'''|S''M_S''\rangle\langle\tau\tau'''|T''M_T''\rangle\langle\vec{q}-\tfrac{\vec{K}}{2}\,|\,|\vec{q}-\tfrac{\vec{K}}{2}|\,L''M_L''\rangle \times$$

$$[1-(-1)^{L''+S''+T''}]$$

$$\langle S''M_S''L''M_L''|J''M_J''L''S''\rangle\langle|\vec{q}-\tfrac{\vec{K}}{2}|\,J''L''S''T''|\,\delta\tilde{U}^\dagger\,|kJ'''L'''S'''T'''\rangle \times$$

$$[1-(-1)^{L'''+S'''+T'''}]$$

$$\langle J'''M_J'''L'''S'''|S'''M_S'''L'''M_L'''\rangle\langle kL'''M_L'''|\vec{k}\rangle\langle T'''M_T'''|\tau'\tau''\rangle\langle S'''M_S'''|\sigma\sigma''\rangle$$

$\times\ \theta$'s

Apply $\sum_{\sigma'\sigma''} \langle \sigma'\sigma'' | S M_S \rangle \langle S''' M_S''' | \sigma\sigma'' \rangle = \delta_{SS'''} \delta_{M_S M_S'''}$ (11)

$$\sum_{\sigma\sigma'''} \langle S' M_S' | \sigma\sigma''' \rangle \langle \sigma\sigma''' | S'' M_S'' \rangle = \delta_{S'S''} \delta_{M_S M_S''}$$

$$\int d\Omega_{\vec{k}} \, \langle \vec{k} | k L M_L \rangle \langle k L''' M_L''' | \vec{k} \rangle = \delta_{LL'''} \delta_{M_L M_L'''}$$

(avg.) $$\int \frac{d\Omega_{\hat{q}\cdot\hat{k}}}{4\pi} \langle |\vec{q}-\tfrac{1}{2}\vec{K}| L' M_L' | \vec{q}-\tfrac{1}{2}\vec{K} \rangle \langle \vec{q}-\tfrac{1}{2}\vec{k} \, | \, |\vec{q}-\tfrac{1}{2}\vec{K}| L'' M_L'' \rangle = \frac{1}{4\pi} \delta_{L'L''} \delta_{M_L' M_L''}$$

$$\sum_{M_L M_S} \langle S M_S L M_L | J M_J L S \rangle \langle J''' M_J''' L S | S M_S L M_L \rangle = \delta_{JJ'''} \delta_{M_J M_J'''}$$

$$\sum_{M_L' M_S'} \langle J' M_J' L' S' | S' M_S' L' M_L' \rangle \langle S' M_S' L' M_L' | J'' M_J'' L' S' \rangle = \delta_{J'J''} \delta_{M_J' M_J''}$$

$$\int d\Omega_{\vec{K}} = 4\pi$$

$\delta U, \delta U^{+}$ – diagonal in S, M_S, T, M_T, J, M_J

$$\sum_{M_J} [\ldots] = (2J+1)[\ldots]$$

Do $\int_{-1}^{1} \frac{dy}{2} [\ldots]$ where $|\tfrac{1}{2}\vec{K} \pm \vec{k}| = \sqrt{\frac{K^2}{4} + k^2 \pm K k y}$

$$= V \, \frac{1}{2} \left(\frac{2}{\pi}\right)^2 \sum_{\tau\tau''\tau'''} \sum_{T M_T T' M_T'} \sum_{S} \sum_{LL'} \sum_{J} \underset{2^4(2J+1)}{\wedge} \int_0^\infty dk\, k^2 \int_0^\infty dK\, K^2 \times$$

$$\int_{-1}^{1} \frac{dy}{2} \langle \tau'\tau'' | T M_T \rangle \langle k J L S T | \delta U | \, |\vec{q}-\tfrac{1}{2}\vec{K}| J L' S T \rangle \langle T M_T | \tau\tau''' \rangle \times$$

$$\langle \tau\tau''' | T'M_T' \rangle \langle |\vec{q}-\tfrac{1}{2}\vec{K}| JL'ST' | \delta\tilde{U}^\dagger | kJLST' \rangle \langle T'M_T' | \tau'\tau'' \rangle \times$$

$$\Theta(k_F^{\tau'} - |\tfrac{1}{2}\vec{K}+\vec{k}|)\, \Theta(k_F^{\tau''} - |\tfrac{1}{2}\vec{K}-\vec{k}|) \qquad (17)$$

Overall we have :

1 term

δU term

$$n_\lambda^\tau(q) = V\Big[\, 2\,\Theta(k_F^\tau - q)$$

$$+2\, 2^2 \frac{2}{\pi} \sum_{\tau'} \sum_{TM_T} \sum_S \sum_L \sum_J (2J+1) \int_0^\infty dk\, k^2 \int_{-1}^{1} \frac{dx}{2} \langle \tau\tau' | TM_T \rangle \times$$

$$\langle kJLST | \delta\tilde{U} | kJLST \rangle \langle TM_T | \tau\tau' \rangle\, \Theta(k_F^\tau - q)\, \Theta(k_F^{\tau'} - |\vec{q}-2\vec{k}|)$$

$$+ \frac{1}{2} 2^4 \left(\frac{2}{\pi}\right)^2 \sum_{\tau'\tau''\tau'''} \sum_{TM_T} \sum_{T'M_T'} \sum_S \sum_{LL'} \sum_J (2J+1) \int_0^\infty dk\, k^2 \int_0^\infty dK\, K^2 \times$$

$$\int_{-1}^{1} \frac{dx}{2} \langle \tau''\tau''' | TM_T \rangle \langle kJLST | \delta\tilde{U} | |\vec{q}-\tfrac{1}{2}\vec{K}|\, JL'ST \rangle \langle TM_T | \tau\tau' \rangle \times$$

$$\langle \tau\tau' | T'M_T' \rangle \langle |\vec{q}-\tfrac{1}{2}\vec{K}| JL'ST' | \delta\tilde{U}^\dagger | kJLST' \rangle \langle T'M_T' | \tau''\tau''' \rangle \times$$

$$\Theta(k_F^{\tau''} - |\tfrac{1}{2}\vec{K}+\vec{k}|)\, \Theta(k_F^{\tau'''} - |\tfrac{1}{2}\vec{K}-\vec{k}|) \Big] \qquad (18)$$

where $|\vec{q}-2\vec{k}| = \sqrt{q^2+4k-4qkx}$,

$\delta U \delta U^\dagger$

$$\left|\vec{q}-\tfrac{1}{2}\vec{K}\right| \approx \sqrt{q^2+\frac{K^2}{4}}, \quad \text{and}$$

$$\left|\tfrac{1}{2}\vec{K}\pm\vec{k}\right| = \sqrt{\frac{K^2}{4}+k^2\pm Kky}$$

Let's evaluate this up through $L=2$.
Strategy: start with $\tau=+\frac{1}{2}$ then consider cases for τ'.

δV term:

(A) $\tau'=+\frac{1}{2}$ $\quad M_T=1,\ T=1: \quad CG's=1$

A1) $S=0,\ J=L$ $\begin{cases} L \text{ even} & {}^1S_0,\ {}^1D_2,\ \ldots \\ L \text{ odd} & \text{not allowed} \end{cases}$

A2) $S=1,\ J=|L-S|,\ldots,L+S$ $\begin{cases} L \text{ even} & \text{not allowed} \\ L \text{ odd} & {}^3P_0,\ {}^3P_1,\ {}^3P_2,\ \ldots \end{cases}$

(B) $\tau'=-\frac{1}{2}$ $\quad M_T=0,\ T=0,1: \quad CG's=\frac{1}{\sqrt{2}}$

A1) $T=1$

i) $S=0,\ J=L$ $\begin{cases} L \text{ even} & {}^1S_0,\ {}^1D_2,\ \ldots \\ L \text{ odd} & \text{not allowed} \end{cases}$

ii) $S=1$, $J=|L-S|,\dots,L+S$ $\begin{cases} L \text{ even not allowed} \\ L \text{ odd } {}^3P_0, {}^3P_1, {}^3P_2-{}^3F_2 \end{cases}$

A2) $T=0$

i) $S=0$, $J=L$ $\begin{cases} L \text{ even not allowed} \\ L \text{ odd } {}^1P_1, \dots \end{cases}$

ii) $S=1$, $J=|L-S|\dots L+S$ $\begin{cases} L \text{ even } {}^3S_1, {}^3D_1, {}^3D_2, {}^3D_3 \\ L \text{ odd not allowed}\dots \end{cases}$

$$\Bigg[\Big(\delta U_{^1S_0} + \delta U_{^3P_0} + 3\,\delta U_{^3P_1} + 5\,\delta U_{^3P_2-{}^3P_2} + 5\,\delta U_{^1D_2} \Big) \times \Theta\big(k_F^p - |\vec{q}-2\vec{k}|\big) + \Big(\tfrac{1}{2}\delta U_{^1S_0} + \tfrac{1}{2}\delta U_{^3P_0} + \tfrac{3}{2}\delta U_{^1P_1} + \tfrac{3}{2}\delta U_{^3P_1} + \tfrac{5}{2}\delta U_{^3P_2-{}^3P_2} + \tfrac{5}{2}\delta U_{^1D_2} + \tfrac{3}{2}\delta V_{^3S_1-{}^3S_1} + \tfrac{3}{2}\delta V_{^3D_1-{}^3D_1} + \tfrac{5}{2}\delta U_{^3D_2} + \tfrac{7}{2}\delta U_{^3D_3-{}^3D_3} \Big)\,\Theta\big(k_F^n - |\vec{q}-2\vec{k}|\big) \Bigg] \times \Theta\big(k_F^p - q\big) \qquad (19)$$

pp underlined in green

pn underlined in blue

$SU\delta U^{+}$ term:

A) $\tau' = +\frac{1}{2}$ $M_T = 1, T = 1, M_T' = 1, T' = 1$ $CG's = 1$

$\tau'' = \tau''' = +\frac{1}{2}$

A1) $S = 0, J = L, L' = L$

$\begin{cases} L \text{ even} & {}^1S_0, {}^1D_2, \ldots \\ L \text{ odd} & \text{not allowed} \end{cases}$

A2) $S = 1, J = |L-S|, \ldots, L+S$

$\begin{cases} L \text{ even} & \text{not possible} \\ L \text{ odd} & {}^3P_0, {}^3P_1, {}^3P_2 - {}^3P_2, {}^3P_2 - {}^3F_2 \end{cases}$

B) $\tau' = -\frac{1}{2}$ $M_T = 0$, $M_T' = 0$ $\tau'' = \pm\frac{1}{2}$ $\tau''' = \mp\frac{1}{2}$

$\Rightarrow (\theta_p \theta_n + \theta_n \theta_p)$

A1) $T = 1, T' = 1$ $CG's = \frac{1}{\sqrt{2}}$

i) $S = 0, J = L$ $\begin{cases} L \text{ even} & {}^1S_0, {}^1D_2 \\ \text{not possible} \end{cases}$

ii) $S = 1, J = |L-S|, \ldots, L+S$

$\begin{cases} L \text{ even not possible} \\ L \text{ odd } {}^3P_0, {}^3P_1, {}^3P_2 - {}^3P_2, {}^3P_2 - {}^3F_2 \end{cases}$

A2) $T=0,\ T'=0$ $\quad CG's = \frac{1}{\sqrt{2}}$

i) $S=0,\ J=L$ $\begin{cases} L \text{ even not pos.} \\ L \text{ odd } {}^1P_1 \end{cases}$

ii) $S=1,\ J=|L-S|\ldots L+S$ $\quad {}^3D_1-{}^3D_1$

$\begin{cases} L \text{ even } \quad {}^3S_1-{}^3S_1,\ {}^3S_1-{}^3D_1,\ {}^3D_2,\ {}^3D_3-{}^3D_3,\ {}^3D_3-{}^3G_3 \\ L \text{ odd not possible} \end{cases}$

$$\Big[\big(\delta U_{^1S_0}\,\delta U^{\dagger}_{^1S_0} + \delta U_{^3P_0}\,\delta U^{\dagger}_{^3P_0} + 3\,\delta U_{^3P_1}\,\delta U^{\dagger}_{^3P_1} + 5\,\delta U_{^3P_2-^3P_2} \times \delta U^{\dagger}_{^3P_2-^3P_2} + 5\,\delta U_{^3P_2-^3F_2}\,\delta U_{^3F_2-^3P_2} + 5\,\delta U_{^1D_2}\,\delta U^{\dagger}_{^1D_2} \big) \times \Theta^{+}_{p}\,\Theta^{-}_{p} + \frac{1}{4}\big(\delta U_{^1S_0}\,\delta U^{\dagger}_{^1S_0} + 5\,\delta U_{^1D_2}\,\delta U^{\dagger}_{^1D_2} + \delta U_{^3P_0}\,\delta U^{\dagger}_{^3P_0} + 3\,\delta U_{^3P_1}\,\delta U^{\dagger}_{^3P_1} + 5\,\delta U_{^3P_2-^3P_2}\,\delta U^{\dagger}_{^3P_2-^3P_2} + 5\,\delta U_{^3P_2-^3F_2}\,\delta U^{\dagger}_{^3F_2-^3P_2} + 3\,\delta U_{^1P_1}\,\delta U^{\dagger}_{^1P_1} + 3\,\delta U_{^3S_1-^3S_1}\,\delta U^{\dagger}_{^3S_1-^3S_1} + 3\,\delta U_{^3S_1-^3D_1}\,\delta U^{\dagger}_{^3D_1-^3S_1} + 3\,\delta U_{^3D_1-^3D_1}\,\delta U^{\dagger}_{^3D_1-^3D_1} + 5\,\delta U_{^3D_2}\,\delta U^{\dagger}_{^3D_2} + 7\,\delta U_{^3D_3-^3D_3}\,\delta U^{\dagger}_{^3D_3-^3D_3} + 7\,\delta U_{^3D_3-^3G_3}\,\delta U^{\dagger}_{^3G_3-^3D_3} \big) \times \big(\Theta^{+}_{p}\Theta^{-}_{n} + \Theta^{+}_{n}\Theta^{-}_{p} \big) \Big] \tag{20}$$

where $G_x^{\pm} \equiv G(k_F^x - |\tfrac{1}{2}\vec{K} \pm \vec{k}|)$

This gives pp, pn, and np contributions to $n_\lambda^p(q)$ from $SUSU^+$ term.

In LDA, we simply average over $k_F^p(r)$ and $k_F^n(r)$ by evaluating

$$\langle n_\lambda^p(q)\rangle_A = 4\pi \int_0^\infty dr\, r^2\, n_\lambda^p(q; k_F^p(r), k_F^n(r))$$

where $k_F^N(r) = \left[3\pi^2 \rho_A^N(r)\right]^{1/3}$

(swap $p \to n$, $n \to p$ for neutron distribution)