* Here we derive the SRG-evolved single-nucleon momentum distribution $n_\lambda^\tau(q)$.

* Start by defining the operators in second quantization.

$$n^\tau(\vec{q}) = \sum_\sigma a^\dagger_{\vec{q}\sigma\tau} a_{\vec{q}\sigma\tau} \qquad (1)$$

$\sigma$ spin projection
$\tau$ isospin projection
$\uparrow$ NOT relative momentum

$$U_\lambda = \mathbb{1} + \frac{1}{4} \sum_{\sigma_1\sigma_2\sigma_3\sigma_4} \sum_{\tau_1\tau_2\tau_3\tau_4} \sum_{\vec{K}\vec{k}\vec{k}'} \langle \vec{k}\sigma_1\tau_1\sigma_2\tau_2 | \delta\tilde{U} | \vec{k}'\sigma_3\tau_3\sigma_4\tau_4 \rangle \times$$
$$a^\dagger_{\frac{\vec{K}}{2}+\vec{k},\sigma_1\tau_1} a^\dagger_{\frac{\vec{K}}{2}-\vec{k},\sigma_2\tau_2} a_{\frac{\vec{K}}{2}-\vec{k}',\sigma_4\tau_4} a_{\frac{\vec{K}}{2}+\vec{k}',\sigma_4\tau_4} + \text{3-body} \cdots \qquad (2)$$

where $\delta\tilde{U}$ is antisymmetrized ($\delta\tilde{U}_{1234} = -\delta\tilde{U}_{1243}$).

Our task is to evaluate $U_\lambda n^\tau(\vec{q}) U_\lambda^\dagger$ truncating up through 2-body. We will use simplified notation for momentum, spin, isospin basis.

$\sum_\alpha \equiv \sum_\sigma$ NOT $\sum_{\vec{q}\sigma\tau}$

$(\vec{q},\sigma,\tau) \to \alpha$, $(\frac{\vec{K}}{2}+\vec{k},\sigma_1,\tau_1) \to 1$, ...

$(\frac{\vec{K}}{2}-\vec{k}',\sigma_4,\tau_4) \to 4$, $(\frac{\vec{K}'}{2}+\vec{k}'',\sigma_5,\tau_5) \to 5$, ...

$(\frac{\vec{K}'}{2}-\vec{k}''',\sigma_8,\tau_8) \to 8$

So we have

$$\left(\mathbb{1} + \frac{1}{4}\sum_{1234} \langle 12|\delta\tilde{U}|34\rangle a_1^\dagger a_2^\dagger a_4 a_3\right)\left(\sum_\alpha a_\alpha^\dagger a_\alpha\right) \times$$

$$\left(\mathbb{1} + \frac{1}{4}\sum_{5678} \langle 56|\delta\tilde{U}|78\rangle a_5^\dagger a_6^\dagger a_8 a_7\right)^\dagger$$

$$= \sum_\alpha a_\alpha^\dagger a_\alpha \qquad \text{1 term}$$

$$\left.\begin{aligned} &+ \frac{1}{4}\sum_\alpha \sum_{1234} \langle 12|\delta\tilde{U}|34\rangle a_1^\dagger a_2^\dagger a_4 a_3 a_\alpha^\dagger a_\alpha \\ &+ \frac{1}{4}\sum_\alpha \sum_{5678} \langle 78|\delta\tilde{U}^\dagger|56\rangle a_\alpha^\dagger a_\alpha a_7^\dagger a_8^\dagger a_6 a_5 \end{aligned}\right] \delta U \text{ term}$$

$$\underbrace{+ \frac{1}{16}\sum_\alpha \sum_{1234}\sum_{5678} \langle 12|\delta\tilde{U}|34\rangle\langle 78|\delta\tilde{U}^\dagger|56\rangle a_1^\dagger a_2^\dagger a_4 a_3 a_\alpha^\dagger a_\alpha a_7^\dagger a_8^\dagger a_6 a_5}_{\delta U \delta U^\dagger \text{ term}} \qquad (3)$$

---

- Evaluate contractions with respect to $|0\rangle$ first

1 term: No possible contractions

---

$\delta U$ term: $\quad a_1^\dagger a_2^\dagger a_4 a_3 a_\alpha^\dagger a_\alpha$

$$\rightarrow a_1^\dagger a_2^\dagger a_4 a_3 a_\alpha^\dagger a_\alpha + a_1^\dagger a_2^\dagger a_4 a_3 a_\alpha^\dagger a_\alpha$$

Antisymmetrized so evaluate first one and multiply by 2

$$\frac{1}{4}\sum_{\alpha}\sum_{1234}\langle 12|\delta\tilde{U}|34\rangle\, 2\,\delta_{3\alpha}\, a_1^{\dagger}a_2^{\dagger}a_4 a_\alpha \qquad \frac{\vec{K}}{2}+\vec{k}' = \vec{q}$$

$$= \frac{1}{2}\sum_{\alpha}\sum_{124}\langle 12|\delta\tilde{U}|\alpha 4\rangle\, a_1^{\dagger}a_2^{\dagger}a_4 a_\alpha \qquad \vec{k}' \to \vec{q}-\frac{\vec{K}}{2}$$

$\delta U^{\dagger}$ term gives $\frac{1}{2}\sum_{\alpha}\sum_{568}\langle \alpha 8|\delta\tilde{U}^{\dagger}|56\rangle\, a_\alpha^{\dagger}a_8^{\dagger}a_6 a_5$ (4)

$\frac{\vec{K}'}{2}+\vec{k}''' = \vec{q} \qquad \vec{k}''' = \vec{q}-\frac{\vec{K}'}{2}$

$\delta U \delta U^{\dagger}$ term: $a_1^{\dagger}a_2^{\dagger}a_4 a_3\, a_\alpha^{\dagger}a_\alpha\, a_7^{\dagger}a_8^{\dagger}a_6 a_5$

$$\to a_1^{\dagger}a_2^{\dagger}a_4 a_3 a_\alpha^{\dagger}a_\alpha a_7^{\dagger}a_8^{\dagger}a_6 a_5$$
$$+\ a_1^{\dagger}a_2^{\dagger}a_4 a_3 a_\alpha^{\dagger}a_\alpha a_7^{\dagger}a_8^{\dagger}a_6 a_5$$
$$+\ a_1^{\dagger}a_2^{\dagger}a_4 a_3 a_\alpha^{\dagger}a_\alpha a_7^{\dagger}a_8^{\dagger}a_6 a_5$$
$$+\ a_1^{\dagger}a_2^{\dagger}a_4 a_3 a_\alpha^{\dagger}a_\alpha a_7^{\dagger}a_8^{\dagger}a_6 a_5$$

$\frac{\vec{K}}{2}+\vec{k}' = \vec{q} \to \frac{\vec{K}}{2} = \vec{q}-\vec{k}'$

$\frac{\vec{K}'}{2}+\vec{k}''' = \vec{q} \to \frac{\vec{K}'}{2} = \vec{q}-\vec{k}'''$

$\frac{\vec{K}}{2}-\vec{k}' = \frac{\vec{K}'}{2}-\vec{k}'''$

$\vec{q}-2\vec{k}' = \vec{q}-2\vec{k}''' \to \vec{k}' = \vec{k}'''$

$\Rightarrow \vec{K}' = \vec{K} \qquad \vec{k}' = \vec{q}-\frac{\vec{K}}{2}$

Antisymmetrized $\to$ Factor of 4 (do last one only)

$$= \frac{1}{16}\sum_{\alpha}\sum_{1234}\sum_{5678}\langle 12|\delta\tilde{U}|34\rangle\langle 78|\delta\tilde{U}^{\dagger}|56\rangle\, 4\,\delta_{3\alpha}\,\delta_{\alpha 7}\,\delta_{48}\; a_1^{\dagger}a_2^{\dagger}a_6 a_5$$

$$= \frac{1}{4}\sum_{\alpha}\sum_{12456}\langle 12|\delta\tilde{U}|\alpha 4\rangle\langle \alpha 4|\delta\tilde{U}^{\dagger}|56\rangle\, a_1^{\dagger}a_2^{\dagger}a_6 a_5 \qquad (5)$$

So in total, we have

$$n_\lambda^{\tau}(\vec{q}) = \sum_{\alpha} a_\alpha^{\dagger}a_\alpha + \frac{1}{2}\sum_{\alpha}\sum_{124}\langle 12|\delta\tilde{U}|\alpha 4\rangle a_1^{\dagger}a_2^{\dagger}a_4 a_\alpha +$$

$$\frac{1}{2}\sum_{\alpha}\sum_{568}\langle \alpha 8|\delta\tilde{U}^{\dagger}|56\rangle a_{\alpha}^{\dagger}a_{8}^{\dagger}a_{6}a_{5} \quad +$$

$$\frac{1}{4}\sum_{\alpha}\sum_{12456}\langle 12|\delta\tilde{U}|\alpha 4\rangle\langle \alpha 4|\delta\tilde{U}^{\dagger}|56\rangle a_{1}^{\dagger}a_{2}^{\dagger}a_{6}a_{5} \qquad (6)$$

---

Rewrite then take continuum limit

$$= \sum_{\sigma} a_{\vec{q}\sigma\tau}^{\dagger}a_{\vec{q}\sigma\tau} + \frac{1}{2}\sum_{\sigma}\sum_{\sigma_1\sigma_2\sigma_4}\sum_{\tau_1\tau_2\tau_4}\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3K}{(2\pi)^3} \times$$

$$\langle \vec{k}\,\sigma_1\tau_1\sigma_2\tau_2|\delta\tilde{U}|\vec{q}-\tfrac{\vec{K}}{2}\,\sigma\tau\sigma_4\tau_4\rangle\, a^{\dagger}_{\frac{\vec{K}}{2}+\vec{k},\sigma_1\tau_1}a^{\dagger}_{\frac{\vec{K}}{2}-\vec{k},\sigma_2\tau_2}a_{\vec{K}-\vec{q},\sigma_4\tau_4}a_{\vec{q}\sigma\tau}$$

$$+\frac{1}{2}\sum_{\sigma}\sum_{\sigma_5\sigma_6\sigma_8}\sum_{\tau_5\tau_6\tau_8}\int\frac{d^3k'}{(2\pi)^3}\int\frac{d^3K'}{(2\pi)^3}\langle \vec{q}-\tfrac{\vec{K}'}{2}\,\sigma\tau\sigma_8\tau_8|\delta\tilde{U}^{\dagger}|\vec{k}'\sigma_1\tau_1\sigma_2\tau_2\rangle \times$$

$$a^{\dagger}_{\vec{K}'-\vec{q}\,\sigma_8\tau_8}a^{\dagger}_{\vec{q}\sigma\tau}a_{\frac{\vec{K}'}{2}-\vec{k}'\sigma_6\tau_6}a_{\frac{\vec{K}'}{2}+\vec{k}'\sigma_5\tau_5} \quad + \frac{1}{4}\sum_{\sigma}\sum_{\sigma_1\sigma_2\sigma_4\sigma_5\sigma_6} \times$$

$$\sum_{\tau_1\tau_2\tau_4\tau_5\tau_6}\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3k'}{(2\pi)^3}\int\frac{d^3K}{(2\pi)^3}\langle \vec{k}\,\sigma_1\tau_1\sigma_2\tau_2|\delta\tilde{U}|\vec{q}-\tfrac{\vec{K}}{2}\sigma\tau\sigma_4\tau_4\rangle \times$$

$$\langle \vec{q}-\tfrac{\vec{K}}{2}\sigma\tau\sigma_4\tau_4|\delta\tilde{U}^{\dagger}|\vec{k}'\sigma_5\tau_5\sigma_6\tau_6\rangle\, a^{\dagger}_{\frac{\vec{K}}{2}+\vec{k},\sigma_1\tau_1}a^{\dagger}_{\frac{\vec{K}}{2}-\vec{k},\sigma_2\tau_2} \times$$

$$a_{\frac{\vec{K}}{2}-\vec{k}',\sigma_6\tau_6}a_{\frac{\vec{K}}{2}+\vec{k}',\sigma_5\tau_5} \qquad (7)$$

---

Now do contractions with respect to a filled Fermi sea $|F\rangle$ where $\langle F|a_{\beta}^{\dagger}a_{\beta}|F\rangle = \theta(k_F^{\beta}-k_{\beta})$.

1 term : $\sum_\sigma \langle F| a^\dagger_{\vec{q}\sigma\tau} a_{\vec{q}\sigma\tau} |F\rangle \quad (2\pi)^3 \delta^3(\vec{q}-\vec{q})$

$\downarrow$

One contraction : $\overbrace{a^\dagger_{\vec{q}\sigma\tau} a_{\vec{q}\sigma\tau}} = V\,\theta(k_F^\tau - q)$

$\times \sum_\sigma \to 2$

$$\boxed{= V\, 2\, \theta(k_F^\tau - q)} \qquad (8)$$

$\delta U$ term : $\frac{1}{2}\sum_\sigma \sum_{\sigma_1\sigma_2\sigma_4} \sum_{\tau_1\tau_2\tau_4} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3K}{(2\pi)^3} \langle \vec{k}\,\sigma_1\tau_1\sigma_2\tau_2 | \delta\tilde{U} | \vec{q}-\frac{\vec{K}}{2}\,\sigma\tau\sigma_4\tau_4 \rangle \times$

$a^\dagger_{\frac{\vec{K}}{2}+\vec{k},\sigma_1\tau_1}\; a^\dagger_{\frac{\vec{K}}{2}-\vec{k},\sigma_2\tau_2}\; a_{\vec{K}-\vec{q},\sigma_4\tau_4}\; a_{\vec{q}\sigma\tau}$

Two contractions : $a^\dagger a^\dagger a a$ and $a^\dagger a^\dagger a a$

Do the first and multiply by 2

$$= \sum_\sigma \sum_{\sigma_1\sigma_2\sigma_4} \sum_{\tau_1\tau_2\tau_4} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3K}{(2\pi)^3} \langle \vec{k}\,\sigma_1\tau_1\sigma_2\tau_2 | \delta\tilde{U} | \vec{q}-\frac{\vec{K}}{2}\,\sigma\tau\sigma_4\tau_4 \rangle \times$$

$$\delta_{\sigma_2,\sigma_4}\,\delta_{\tau_2,\tau_4}\,(2\pi)^3\delta^3\!\left(\frac{\vec{K}}{2}-\vec{k}-\vec{K}+\vec{q}\right)\delta_{\sigma_1\sigma}\,\delta_{\tau_1\tau}\;\delta^3\!\left(\frac{\vec{K}}{2}+\vec{k}-\vec{q}\right)(2\pi)^3$$

$$\langle F|\, a^\dagger_{\frac{\vec{K}}{2}+\vec{k},\sigma_1\tau_1}\, a_{\vec{q}\sigma\tau}\, a^\dagger_{\frac{\vec{K}}{2}-\vec{k}\,\sigma_2\tau_2}\, a_{\vec{K}-\vec{q}\,\sigma_4\tau_4} |F\rangle$$

Do $\int \frac{d^3K}{(2\pi)^3}(2\pi)^3\delta^3\!\left(\frac{\vec{K}}{2}+\vec{k}-\vec{q}\right)[\ldots]$

$\to \frac{\vec{K}}{2}+\vec{k}-\vec{q} = 0 \implies \frac{\vec{K}}{2} = \vec{q}-\vec{k}$

$$= \sum_{\sigma\sigma_2\sigma_4} \sum_{\tau_2\tau_4} \int \frac{d^3k}{(2\pi)^3} \langle \vec{k}\,\sigma\tau\sigma_2\tau_2 | \delta\tilde{U} | \vec{k}\,\sigma\tau\sigma_4\tau_4 \rangle \times$$

$$\delta_{\sigma_2,\sigma_4}\,\delta_{\tau_2,\tau_4}\,\underbrace{(2\pi)^3\,\delta^3(\vec{q}-\vec{k}-\vec{q}+\vec{k})}_{\substack{=(2\pi)^3\delta^3(0)\\ \equiv V}} \times$$

$$\langle F |\, \hat{n}_{\vec{q}\sigma\tau}\, a^\dagger_{\vec{q}-2\vec{k}\,\sigma_2\tau_4}\, a_{\vec{q}-2\vec{k}\,\sigma_4\tau_4} | F \rangle$$

Relabel $\sigma_2, \tau_2 \to \sigma'\tau'$ and evaluate

$$\langle F | \hat{n}_{\vec{q}\sigma\tau}\, \hat{n}_{\vec{q}-2\vec{k}\,\sigma'\tau'} | F \rangle = \Theta(k_F^\tau - q)\,\Theta(k_F^{\tau'} - |\vec{q}-2\vec{k}|)$$

$$= V \sum_{\sigma\sigma'} \sum_{\tau'} \int \frac{d^3k}{(2\pi)^3} \langle \vec{k}\,\sigma\tau\sigma'\tau' | \delta\tilde{U} | \vec{k}\,\sigma\tau\sigma'\tau' \rangle \times$$

$$\Theta(k_F^\tau - q)\,\Theta(k_F^{\tau'} - |\vec{q}-2\vec{k}|)$$

$\delta U^\dagger$ term gives identical contribution since
$\langle \vec{k} | \delta\tilde{U} | \vec{k} \rangle = \langle \vec{k} | \delta U^\dagger | \vec{k} \rangle$. Factor of 2

$$\boxed{\begin{aligned} &= V\, 2 \sum_{\sigma\sigma'} \sum_{\tau'} \int \frac{d^3k}{(2\pi)^3} \langle \vec{k}\,\sigma\tau\sigma'\tau' | \delta\tilde{U} | \vec{k}\,\sigma\tau\sigma'\tau' \rangle \times \\ &\Theta(k_F^\tau - q)\,\Theta(k_F^{\tau'} - |\vec{q}-2\vec{k}|) \qquad (9) \end{aligned}}$$

$\delta U \delta U^\dagger$ term: $\frac{1}{4} \sum_{\sigma\sigma_1\sigma_2\sigma_4\sigma_5\sigma_6} \sum_{\tau_1\tau_2\tau_4\tau_5\tau_6} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \int \frac{d^3K}{(2\pi)^3} \times$

$\langle \vec{k}\, \sigma_1\tau_1\, \sigma_2\tau_2 | \delta\tilde{U} | \vec{q} - \frac{\vec{K}}{2}\, \sigma\tau\, \sigma_4\tau_4 \rangle \langle \vec{q} - \frac{\vec{K}}{2}\, \sigma\tau\, \sigma_4\tau_4 | \delta\tilde{U}^\dagger | \vec{k}'\, \sigma_5\tau_5\, \sigma_6\tau_6 \rangle \times$

$a^\dagger_{\frac{\vec{K}}{2}+\vec{k}\,\sigma_1\tau_1}\, a^\dagger_{\frac{\vec{K}}{2}-\vec{k}\,\sigma_2\tau_2}\, a_{\frac{\vec{K}}{2}-\vec{k}'\,\sigma_6\tau_6}\, a_{\frac{\vec{K}}{2}+\vec{k}'\,\sigma_5\tau_5}$

Two contractions: $a^\dagger a^\dagger a a$ and $a^\dagger a^\dagger a a$

Do first one and multiply by 2.

$$= \frac{1}{2} \sum_{\sigma\sigma_1\sigma_2\sigma_4\sigma_5\sigma_6} \sum_{\tau_1\tau_2\tau_4\tau_5\tau_6} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} \int \frac{d^3K}{(2\pi)^3} \langle \vec{k}\, \sigma_1\tau_1\, \sigma_2\tau_2 | \delta\tilde{U} | \vec{q} - \frac{\vec{K}}{2}\, \sigma\tau\, \sigma_4\tau_4 \rangle \times$$

$$\langle \vec{q} - \frac{\vec{K}}{2}\, \sigma\tau\, \sigma_4\tau_4 | \delta\tilde{U}^\dagger | \vec{k}'\, \sigma_5\tau_5\, \sigma_6\tau_6 \rangle \delta_{\sigma_1\sigma_5} \delta_{\tau_1\tau_5} (2\pi)^3 \delta^3\left(\frac{\vec{K}}{2} + \vec{k} - \frac{\vec{K}}{2} - \vec{k}'\right)$$

$$\times\, \delta_{\sigma_2\sigma_6} \delta_{\tau_2\tau_6} (2\pi)^3 \delta^3\left(\frac{\vec{K}}{2} - \vec{k} - \frac{\vec{K}}{2} + \vec{k}'\right) \langle F | a^\dagger a\, a^\dagger a | F \rangle$$

(Similar to evaluation in $\delta U$ term)

$$= V \frac{1}{2} \sum_{\sigma\sigma'\sigma''\sigma'''} \sum_{\tau'\tau''\tau'''} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3K}{(2\pi)^3} \langle \vec{k}\, \sigma'\tau'\, \sigma''\tau'' | \delta\tilde{U} | \vec{q} - \frac{\vec{K}}{2}\, \sigma\tau\, \sigma'''\tau''' \rangle$$

$$\times \langle \vec{q} - \frac{\vec{K}}{2}\, \sigma\tau\, \sigma'''\tau''' | \delta\tilde{U}^\dagger | \vec{k}\, \sigma'\tau'\, \sigma''\tau'' \rangle\, \theta\left(k_F^{\tau'} - \left|\frac{\vec{K}}{2} + \vec{k}\right|\right) \times$$

$$\theta\left(k_F^{\tau''} - \left|\frac{\vec{K}}{2} - \vec{k}\right|\right) \tag{10}$$

Go to partial waves for each term separately

$2V\Theta(k_F^\tau - q)$ does not change.

$\delta U$ term (includes $\delta U + \delta U^\dagger$):

$$= V\,2\sum_{\sigma\sigma'}\sum_{\tau'}\int\frac{d^3k}{(2\pi)^3}\langle \vec{k}\,\sigma\tau\sigma'\tau'|\,\delta\tilde{U}\,|\vec{k}\,\sigma\tau\sigma'\tau'\rangle\,\Theta(k_F^\tau - q)\times$$

$$\Theta(k_F^{\tau'} - |\vec{q} - 2\vec{k}|)$$

Insert complete set of states using

$$\sum_{S=0}^{1}\sum_{M_S=-S}^{S}|S M_S\rangle\langle S M_S| = 1 \qquad (11)$$

$$\sum_{T=0}^{1}\sum_{M_T=-T}^{T}|T M_T\rangle\langle T M_T| = 1 \qquad (12)$$

$$\sqrt{\frac{2}{\pi}}\sum_{L M_L}|k L M_L\rangle\langle k L M_L| = 1 \qquad (13)$$

$$\sum_{J=|L-S|}^{L+S}\sum_{M_J=-J}^{J}|J M_J L S\rangle\langle J M_J L S| = 1 \qquad (14)$$

$$= V\,2\frac{1}{(2\pi)^3}\frac{2}{\pi}\sum_{\sigma\sigma'}\sum_{\tau'}\sum_{SM_S}\sum_{S'M_S'}\sum_{TM_T}\sum_{T'M_T'}\sum_{LM_L}\sum_{L'M_L'}\sum_{JM_J}\sum_{J'M_J'}\int_0^\infty dk\,k^2\;\times$$

$$\int d\Omega_{\vec{k}}\,\langle\sigma\sigma'|S M_S\rangle\langle\tau\tau'|T M_T\rangle\langle\hat{k}|k L M_L\rangle\langle S M_S L M_L|J M_J L S\rangle\times$$

$$\langle kJLST|\delta\tilde{U}|kJ'L'S'T'\rangle\langle kL'M_L'|\hat{k}\rangle\langle T'M_T'|\tau\tau'\rangle\langle S'M_S'|\sigma\sigma'\rangle\,\Theta's$$

Apply $\sum_{\sigma\sigma'}\langle\sigma\sigma'|SM_S\rangle\langle S'M_S'|\sigma\sigma'\rangle = \delta_{SS'}\delta_{M_SM_S'}$

$$\int d\Omega_{\hat{k}}\,\langle\hat{k}|kLM_L\rangle\langle kL'M_L'|\hat{k}\rangle = \delta_{LL'}\delta_{M_LM_L'}$$

$$\sum_{M_LM_S}\langle SM_SLM_L|JM_J\rangle\langle J'M_J'|SM_SLM_L\rangle = \delta_{JJ'}\delta_{M_JM_J'}$$

$$\sum_{M_J}[\dots] = (2J+1)$$

$$\delta\tilde{U} \propto \delta_{TT'}\delta_{M_TM_T'}$$

$$= V\;2\frac{1}{(2\pi)^3}\frac{2}{\pi}\sum_{\tau\tau'}\sum_{TM_T}\sum_S\sum_L\sum_J\int_0 dk\,k^2\langle\tau\tau'|TM_T\rangle\times$$

$$\langle kJLST|\delta\tilde{U}|kJLST\rangle\langle TM_T|\tau\tau'\rangle\,\Theta(k_F^\tau-q)\,\Theta(k_F^{\tau'}-|\vec{q}-2\vec{k}|) \qquad (15)$$

But we still have dependence on $|\vec{q}-2\vec{k}| = \sqrt{q^2+4k^2-4qkx}$ where $x=\hat{q}\cdot\hat{k}$. Average by evaluating

$$\int_{-1}^{1}\frac{dx}{2}[\dots]$$

$$= V\;2\frac{1}{(2\pi)^3}\frac{2}{\pi}\sum_{\tau\tau'}\sum_{TM_T}\sum_S\sum_L\sum_J (2J+1)\int_0^\infty dk\,k^2\int_{-1}^{1}\frac{dx}{2}\langle\tau\tau'|TM_T\rangle\times$$

$$\langle kJLST|\delta\tilde{U}|kJLST\rangle\langle TM_T|\tau\tau'\rangle\,\Theta(k_F^\tau-q)\,\Theta(k_F^{\tau'}-|\vec{q}-2\vec{k}|) \qquad (16)$$

$\delta U \delta U^{\dagger}$ term

$$= V \frac{1}{2} \sum_{\sigma\sigma'\sigma''\sigma'''} \sum_{\tau'\tau''\tau'''} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3K}{(2\pi)^3} \langle \vec{k}\, \sigma'\tau'\sigma''\tau'' | \delta\tilde{U} | \vec{q} - \tfrac{\vec{K}}{2}\, \sigma\tau\sigma'''\tau''' \rangle$$

$$\times \langle \vec{q} - \tfrac{\vec{K}}{2}\, \sigma\tau\sigma'''\tau''' | \delta\tilde{U}^{\dagger} | \vec{k}\, \sigma'\tau'\sigma''\tau'' \rangle\, \theta\left(k_F^{\tau'} - \left|\tfrac{\vec{K}}{2} + \vec{k}\right|\right) \times$$

$$\theta\left(k_F^{\tau''} - \left|\tfrac{\vec{K}}{2} - \vec{k}\right|\right)$$

Insert complete set of states for $S M_S T M_T$ etc.

$$= \frac{V}{2} \left(\frac{2}{\pi}\right)^2 \frac{1}{(2\pi)^6} \sum_{\sigma\sigma'\sigma''\sigma'''} \sum_{\tau'\tau''\tau'''} \sum_{T M_T T' M_T' T'' M_T'' T''' M_T'''} \sum_{S M_S S' M_S' S'' M_S'' S''' M_S'''} \sum_{L M_L L' M_L' L'' M_L'' L''' M_L'''}$$

$$\sum_{J M_J J' M_J' J'' M_J'' J''' M_J'''} \int_0^{\infty} dk\, k^2 \int_0^{\infty} dK\, K^2 \int d\Omega_k\, d\Omega_K\, \langle \sigma'\sigma'' | S M_S \rangle \langle \tau'\tau'' | T M_T \rangle \times$$

$$\langle \vec{k} | k L M_L \rangle \langle S M_S L M_L | J M_J L S \rangle \langle k J L S T | \delta\tilde{U} | |\vec{q} - \tfrac{\vec{K}}{2}|\, J' L' S' T' \rangle \times$$

$$\langle J' M_J' L' S' | S' M_S' L' M_L' \rangle \langle |\vec{q} - \tfrac{\vec{K}}{2}|\, L' M_L' | \vec{q} - \tfrac{\vec{K}}{2} \rangle \langle T' M_T' | \tau\tau''' \rangle \times$$

$$\langle S' M_S' | \sigma\sigma''' \rangle \langle \sigma\sigma''' | S'' M_S'' \rangle \langle \tau\tau''' | T'' M_T'' \rangle \langle \vec{q} - \tfrac{\vec{K}}{2} | |\vec{q} - \tfrac{\vec{K}}{2}|\, L'' M_L'' \rangle \times$$

$$\langle S'' M_S'' L'' M_L'' | J'' M_J'' L'' S'' \rangle \langle |\vec{q} - \tfrac{\vec{K}}{2}|\, J'' L'' S'' T'' | \delta\tilde{U}^{\dagger} | k\, J''' L''' S''' T''' \rangle \times$$

$$\langle J''' M_J''' L''' S''' | S''' M_S''' L''' M_L''' \rangle \langle k L''' M_L''' | \vec{k} \rangle \langle T''' M_T''' | \tau'\tau'' \rangle \langle S''' M_S''' | \sigma\sigma'' \rangle$$

$\times\ \theta$'s

Apply $\sum_{\sigma'\sigma''} \langle \sigma'\sigma'' | S M_S \rangle \langle S''' M_S''' | \sigma\sigma'' \rangle = \delta_{SS'''} \delta_{M_S M_S'''}$ (11)

$$\sum_{\sigma\sigma'''} \langle S' M_S' | \sigma\sigma''' \rangle \langle \sigma\sigma''' | S'' M_S'' \rangle = \delta_{S'S''} \delta_{M_S' M_S''}$$

$$\int d\Omega_{\vec{k}} \langle \vec{k} | k L M_L \rangle \langle k L''' M_L''' | \vec{k} \rangle = \delta_{LL'''} \delta_{M_L M_L'''}$$

(avg.) $$\int \frac{d\Omega_{\hat{q}\cdot\hat{K}}}{4\pi} \langle |\vec{q} - \tfrac{1}{2}\vec{K}| L' M_L' | \vec{q} - \tfrac{1}{2}\vec{K} \rangle \langle \vec{q} - \tfrac{1}{2}\vec{K} | |\vec{q} - \tfrac{1}{2}\vec{K}| L'' M_L'' \rangle$$
$$= \frac{1}{4\pi} \delta_{L'L''} \delta_{M_L' M_L''}$$

$$\sum_{M_L M_S} \langle S M_S L M_L | J M_J L S \rangle \langle J''' M_J''' L S | S M_S L M_L \rangle = \delta_{JJ'''} \delta_{M_J M_J'''}$$

$$\sum_{M_L' M_S'} \langle J' M_J' L' S' | S' M_S' L' M_L' \rangle \langle S' M_S' L' M_L' | J'' M_J'' L' S' \rangle = \delta_{J'J''} \delta_{M_J' M_J''}$$

$$\int d\Omega_{\vec{K}} = 4\pi$$

$\delta U, \delta U^\dagger$ - diagonal in $S, M_S, T, M_T, J, M_J$

$$\sum_{M_J} [\ldots] = (2J+1)[\ldots]$$

Do $\int_{-1}^{1} \frac{dy}{2} [\ldots]$ where $|\tfrac{1}{2}\vec{K} \pm \vec{k}| = \sqrt{\frac{K^2}{4} + k^2 \pm K k y}$

$$= V \frac{1}{2} \left(\frac{2}{\pi}\right)^2 \frac{1}{(2\pi)^6} \sum_{\tau\tau'\tau''\tau'''} \sum_{T M_T T' M_T'} \sum_{S} \sum_{LL'} \sum_{J} \underset{(2J+1)}{\wedge} \int_0^\infty dk\, k^2 \int_0^\infty dK\, K^2 \times$$
$$\int_{-1}^{1} \frac{dy}{2} \langle \tau'\tau'' | T M_T \rangle \langle k J L S T | \delta U | |\vec{q} - \tfrac{1}{2}\vec{K}| J L' S T \rangle \langle T M_T | \tau\tau''' \rangle \times$$

$$\langle \tau\tau''' | T'M_T' \rangle \langle |\vec{q}-\tfrac{1}{2}\vec{K}| JL'ST' | \delta\tilde{U}^\dagger | kJLST' \rangle \langle T'M_T' | \tau'\tau'' \rangle \times$$

$$\Theta\left(k_F^{\tau'} - |\tfrac{1}{2}\vec{K}+\vec{k}|\right) \Theta\left(k_F^{\tau''} - |\tfrac{1}{2}\vec{K}-\vec{k}|\right) \quad (17)$$

Overall we have :

1 term

$\delta U$ term

$$\Lambda_\lambda^\tau(q) = V\Big[\, 2\,\Theta(k_F^\tau - q)$$

$$+ 2\frac{1}{(2\pi)^3}\frac{2}{\pi}\sum_{\tau'}\sum_{TM_T}\sum_S\sum_L\sum_J \int_0^\infty dk\, k^2 \int_{-1}^{1}\frac{dx}{2} \overset{(2J+1)}{\langle \tau\tau' | TM_T \rangle} \times$$

$$\langle kJLST | \delta\tilde{U} | kJLST \rangle \langle TM_T | \tau\tau' \rangle\, \Theta(k_F^\tau - q)\, \Theta\left(k_F^{\tau'} - |\vec{q}-2\vec{k}|\right)$$

$$+ \frac{1}{2}\frac{1}{(2\pi)^6}\left(\frac{2}{\pi}\right)^2 \sum_{\tau'\tau''\tau'''}\sum_{TM_T}\sum_{T'M_T'}\sum_S\sum_{LL'}\sum_J \int_0^\infty dk\, k^2 \int_0^\infty dK\, K^2 \times$$

$$\int_{-1}^{1}\frac{dx}{2} \underset{(2J+1)}{\langle \tau''\tau''' | TM_T \rangle} \langle kJLST | \delta\tilde{U} | |\vec{q}-\tfrac{1}{2}\vec{K}| JL'ST \rangle \langle TM_T | \tau\tau' \rangle \times$$

$$\langle \tau\tau' | T'M_T' \rangle \langle |\vec{q}-\tfrac{1}{2}\vec{K}| JL'ST' | \delta\tilde{U}^\dagger | kJLST' \rangle \langle T'M_T' | \tau''\tau''' \rangle \times$$

$$\Theta\left(k_F^{\tau''} - |\tfrac{1}{2}\vec{K}+\vec{k}|\right) \Theta\left(k_F^{\tau'''} - |\tfrac{1}{2}\vec{K}-\vec{k}|\right) \Big] \quad (18)$$

$\delta U \delta U^\dagger$

where $|\vec{q}-2\vec{k}| = \sqrt{q^2+4k-4qkx}$ ,

$$\left|\vec{q} - \tfrac{1}{2}\vec{K}\right| \approx q^2 + \frac{K^2}{4}, \quad \text{and}$$

$$\left|\tfrac{1}{2}\vec{K} \pm \vec{k}\right| = \sqrt{\frac{K^2}{4} + k^2 \pm Kky}$$

---

Let's evaluate this up through $L=2$.
Strategy: start with $\tau = +\frac{1}{2}$ then consider cases for $\tau'$.

$\delta V$ term:

(A) $\tau' = +\frac{1}{2}$ $\quad M_T = 1, \; T = 1: \quad CG's = 1$

A1) $S = 0, \; J = L$ $\begin{cases} L \text{ even} & {}^1S_0, \; {}^1D_2, \ldots \\ L \text{ odd} & \text{not allowed} \end{cases}$

A2) $S = 1, \; J = |L-S|, \ldots, L+S$ $\begin{cases} L \text{ even} & \text{not allowed} \\ L \text{ odd} & {}^3P_0, {}^3P_1, {}^3P_2, \ldots \end{cases}$

(B) $\tau' = -\frac{1}{2}$ $\quad M_T = 0, \; T = 0, 1: \quad CG's = \frac{1}{\sqrt{2}}$

A1) $T = 1$

i) $S = 0, \; J = L$ $\begin{cases} L \text{ even} & {}^1S_0, \; {}^1D_2, \ldots \\ L \text{ odd} & \text{not allowed} \end{cases}$

ii) $S=1$, $J=|L-S|,\dots,L+S$ $\begin{cases} L \text{ even not allowed} \\ L \text{ odd } {}^3P_0, {}^3P_1, {}^3P_2-{}^3F_2 \end{cases}$

A2) $T=0$

i) $S=0$, $J=L$ $\begin{cases} L \text{ even not allowed} \\ L \text{ odd } {}^1P_1, \dots \end{cases}$

ii) $S=1$, $J=|L-S|\dots L+S$ $\begin{cases} L \text{ even } {}^3S_1, {}^3D_1, {}^3D_2, {}^3D_3 \\ L \text{ odd not allowed}\dots \end{cases}$

$$\Big[\Big(\delta U_{^1S_0} + \delta U_{^3P_0} + 3\,\delta U_{^3P_1} + 5\,\delta U_{^3P_2-^3F_2} + 5\,\delta U_{^1D_2}\Big)$$
$$\times\,\Theta\big(k_F^p - |\vec{q}-2\vec{k}|\big) + \Big(\frac{1}{2}\delta U_{^1S_0} + \frac{1}{2}\delta U_{^3P_0} + \frac{3}{2}\delta U_{^1P_1}$$
$$\frac{3}{2}\delta U_{^3P_1} + \frac{5}{2}\delta U_{^3P_2-^3F_2} + \frac{5}{2}\delta U_{^1D_2} + \frac{3}{2}\delta V_{^3S_1-^3S_1} + \frac{3}{2}\delta V_{^3D_1-^3D_1} +$$
$$\frac{5}{2}\delta U_{^3D_2} + \frac{7}{2}\delta U_{^3D_3-^3D_3}\Big)\,\Theta\big(k_F^n - |\vec{q}-2\vec{k}|\big)\Big] \times$$
$$\Theta\big(k_F^p - q\big) \tag{19}$$

pp underlined in green

pn underlined in blue

$SUSU^{+}$ term:

A) $\tau' = +\frac{1}{2}$ $\quad M_T = 1,\ T = 1,\ M_T' = 1,\ T' = 1 \quad CG's = 1$

$\tau'' = \tau''' = +\frac{1}{2}$

A1) $S = 0,\ J = L,\ L' = L$

- $L$ even $\ {}^1S_0,\ {}^1D_2, \ldots$
- $L$ odd not allowed

A2) $S = 1,\ J = |L-S|, \ldots, L+S$

- $L$ even not possible
- $L$ odd $\ {}^3P_0,\ {}^3P_1,\ {}^3P_2 - {}^3P_2,\ {}^3P_2 - {}^3F_2$

B) $\tau' = -\frac{1}{2}$ $\quad M_T = 0$ $\quad \tau'' = \pm\frac{1}{2} \quad \tau''' = \mp\frac{1}{2}$

$M_T' = 0$ $\quad \Rightarrow (G_p G_n + G_n G_p)$

A1) $T = 1,\ T' = 1 \quad CG's = \frac{1}{\sqrt{2}}$

i) $S = 0,\ J = L$
- $L$ even $\ {}^1S_0,\ {}^1D_2$
- not possible

ii) $S = 1,\ J = |L-S|, \ldots, L+S$
- $L$ even not possible
- $L$ odd $\ {}^3P_0,\ {}^3P_1,\ {}^3P_2 - {}^3P_2,\ {}^3P_2 - {}^3F_2$

A2) $T=0,\ T'=0 \quad CGs = \frac{1}{\sqrt{2}}$

i) $S=0,\ J=L$ $\begin{cases} L \text{ even not pos.} \\ L \text{ odd } {}^1P_1 \end{cases}$

ii) $S=1,\ J=|L-S|\dots L+S$ $\qquad {}^3D_1 - {}^3D_1$

$\begin{cases} L \text{ even} & {}^3S_1 - {}^3S_1,\ {}^3S_1 - {}^3D_1,\ {}^3D_2,\ {}^3D_3 - {}^3D_3,\ {}^3D_3 - {}^3G_3 \\ L \text{ odd} & \text{not possible} \end{cases}$

$$\Big[ \Big( \delta U_{^1S_0} \delta U^{\dagger}_{^1S_0} + \delta U_{^3P_0} \delta U^{\dagger}_{^3P_0} + 3\, \delta U_{^3P_1} \delta U^{\dagger}_{^3P_1} + 5\, \delta U_{^3P_2 - ^3P_2} \times \delta U^{\dagger}_{^3P_2 - ^3P_2} + 5\, \delta U_{^3P_2 - ^3F_2} \delta U_{^3F_2 - ^3P_2} + 5\, \delta U_{^1D_2} \delta U^{\dagger}_{^1D_2} \Big) \times$$

$$\Theta^{+}_{p} \Theta^{-}_{p} + \frac{1}{4} \Big( \delta U_{^1S_0} \delta U^{\dagger}_{^1S_0} + 5\, \delta U_{^1D_2} \delta U^{\dagger}_{^1D_2} + \delta U_{^3P_0} \delta U^{\dagger}_{^3P_0} + 3\, \delta U_{^3P_1} \delta U^{\dagger}_{^3P_1} + 5\, \delta U_{^3P_2 - ^3P_2} \delta U^{\dagger}_{^3P_2 - ^3P_2} +$$

$$5\, \delta U_{^3P_2 - ^3F_2} \delta U^{\dagger}_{^3F_2 - ^3P_2} + 3\, \delta U_{^1P_1} \delta U^{\dagger}_{^1P_1} + 3\, \delta U_{^3S_1 - ^3S_1} \delta U^{\dagger}_{^3S_1 - ^3S_1}$$

$$+ 3\, \delta U_{^3S_1 - ^3D_1} \delta U^{\dagger}_{^3D_1 - ^3S_1} + 3\, \delta U_{^3D_1 - ^3D_1} \delta U^{\dagger}_{^3D_1 - ^3D_1} + 5\, \delta U_{^3D_2} \delta U^{\dagger}_{^3D_2}$$

$$+ 7\, \delta U_{^3D_3 - ^3D_3} \delta U^{\dagger}_{^3D_3 - ^3D_3} + 7\, \delta U_{^3D_3 - ^3G_3} \delta U^{\dagger}_{^3G_3 - ^3D_3} \Big) \times \left( \Theta^{+}_{p} \Theta^{-}_{n} + \Theta^{+}_{n} \Theta^{-}_{p} \right) \Big] \quad (20)$$

where $G_x^{\pm} \equiv G(k_F^x - |\tfrac{1}{2}\vec{K} \pm \vec{k}|)$

This gives pp, pn, and np contributions to $n_\lambda^p(q)$ from $SUSU^\dagger$ term.

---

In LDA, we simply average over $k_F^p(r)$ and $k_F^n(r)$ by evaluating

$$\langle n_\lambda^p(q) \rangle_A = 4\pi \int_0^\infty dr\, r^2\, n_\lambda^p\big(q; k_F^p(r), k_F^n(r)\big)$$

where $k_F^N(r) = \left[3\pi^2 \rho_A^N(r)\right]^{1/3}$

(swap $p \to n$, $n \to p$ for neutron distribution)