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Path Integrals (cont.)

Recap from last time...

· The partition function in the coordinate basis

$$\boxed{Z = \mathrm{Tr}\, e^{-\beta\hat{H}} = \int dx \langle x|e^{-\beta\hat{H}}|x\rangle}$$

can be expressed using a path integral reprentation of the evolution operator in Euclidean time U_E:

$$U_E(x_f\tau_f, x_i\tau_i) = \langle x_f|e^{-(\tau_f-\tau_i)\hat{H}/\hbar}|x_i\rangle = \int_{(x_i,\tau_i)}^{(x_f,\tau_f)} \mathcal{D}[x(\tau)]\, e^{-\frac{1}{\hbar}S_E[x(\tau)]}$$

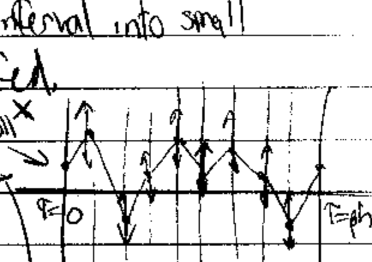
erratum: had $\int_0^{\beta\hbar} S_E[x(\tau)]d\tau$ in notes from last time. The $\int d\tau$ is included in S_E.

where $\boxed{S_E[x(\tau)] \equiv \int_0^{\beta\hbar} d\tau\, \frac{m}{2}\left(\frac{dx(\tau)}{d\tau}\right)^2 + V(x(\tau))}$

is the Euclidean action [this case applies when $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$].

· The path integral follows by dividing the $[\tau_i, \tau_f]$ interval into small subintervals in which $e^{-i\delta\tau\hat{H}/\hbar}$ can be approximated.

· The final expression for Z is



$$\boxed{Z = \int_{x(\beta\hbar)=x(0)} \mathcal{D}[x(\tau)]\, e^{-\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau \left(\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x(\tau))\right)}}$$

where the path integral is over periodic trajectories (those for which $x(\beta\hbar)=x(0)$).

· We could extend this expression to a many-particle system by replacing $|x\rangle$ by a symmetrized (bosons) or antisymmetrized set of N-particle states (see Negele and Orland eqs. (2.55)-(2.57)). Instead we'll use the 2nd quantized representation.

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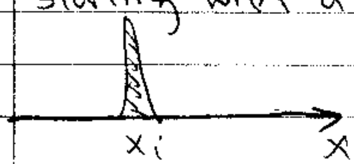
Time out to recall the Heisenberg picture ("H picture")...

If we return to the transition amplitude

$$\boxed{U(x_f t_f, x_i t_i) = \langle x_f | e^{-i(t_f - t_i)\hat{H}/\hbar} | x_i \rangle}$$

- We can interpret this in the Schrödinger picture as starting with a position eigenstate at $t = t_i$



and letting it evolve according to the S-equation until time $t = t_f$:

$$\boxed{i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = \hat{H} |\Psi(t)\rangle}$$

$$\Rightarrow \boxed{|\Psi_{x_i}(t)\rangle = e^{-i/\hbar \hat{H}(t - t_i)} |x_i\rangle}$$

with initial condition $\boxed{|\Psi_{x_i}(t_i)\rangle = |x_i\rangle}$

So the transition amplitude is simply the overlap of $\langle x_f|$ with $|\Psi_{x_i}(t_f)\rangle \Rightarrow$ probability amplitude to find the particle at position x_f at time t_f, if it started at x_i at time t_i.

$$\Rightarrow \boxed{\langle x_f | \Psi_{x_i}(t_f)\rangle = \langle x_f | e^{-\frac{i}{\hbar}\hat{H}(t_f - t_i)} | x_i \rangle}$$

- We interpret this in the Heisenberg picture by introducing the time-independent state

$$\boxed{|x_i(t)\rangle \equiv e^{\frac{i}{\hbar}\hat{H}t} |\Psi_{x_i}(t)\rangle}$$

Note that the t on the left is deceptive, since $|x_i(t)\rangle$ is time independent!

[check $i\hbar \frac{\partial}{\partial t} |x_i(t)\rangle = \left(i\hbar \frac{\partial}{\partial t} e^{\frac{i}{\hbar}\hat{H}t}\right) |\Psi_{x_i}(t)\rangle + e^{i/\hbar \hat{H}t}\, i\hbar \frac{\partial}{\partial t} |\Psi_{x_i}(t)\rangle =$

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So we'll label the state $|x_i t_i\rangle$ instead, since

$$\boxed{|x_i t_i\rangle = e^{\frac{i}{\hbar}\hat{H}t}|\psi_{x_i}(t)\rangle = e^{\frac{i}{\hbar}\hat{H}t} e^{-\frac{i}{\hbar}\hat{H}(t-t_i)}|x_i\rangle = e^{\frac{i}{\hbar}\hat{H}t_i}|x_i\rangle}$$

This is a <u>complicated</u> state, which is an eigenstate of the H-picture ("Heisenberg picture") operator $\hat{x}(t_i)$ with eigenvalue x_i:

$$\boxed{\begin{aligned}\hat{x}(t_i)|x_i t_i\rangle &= \left(e^{i/\hbar \hat{H}t_i}\,\hat{x}\,e^{-\frac{i}{\hbar}\hat{H}t_i}\right)e^{\frac{i}{\hbar}\hat{H}t_i}|x_i\rangle \\ &= e^{i/\hbar \hat{H}t_i}\hat{x}|x_i\rangle = x_i e^{i/\hbar \hat{H}t_i}|x_i\rangle = x_i|x_i t_i\rangle\end{aligned}}$$

and these states are complete and orthonormal:

$$\boxed{\begin{aligned}&\int dx_1\, |x_1 t_1\rangle\langle x_1 t_1| = 1 \\ &\langle x_2 t_1|x_1 t_1\rangle = \langle x_2|x_1\rangle = \delta(x_2 - x_1)\end{aligned}}$$

So, finally, we see that the transition amplitude can also be written as the overlap of H-picture states:

$$\boxed{U(x_f t_f, x_i t_i) = \left(\langle x_f|e^{-\frac{i}{\hbar}\hat{H}t_f}\right)\left(e^{\frac{i}{\hbar}\hat{H}t_i}|x_i\rangle\right) = \langle x_f t_f|x_i t_i\rangle}$$

so our path integral expression is for the overlap of Heisenberg states.

Furthermore, we can reconstruct the path integral as

$$\langle x_f t_f|x_i t_i\rangle = \int\left(\prod_{k=1}^{m-1} dx_k\right)\underbrace{\langle x_m t_m|x_{m-1} t_{m-1}\rangle}_{\langle x_m|e^{-i\epsilon/\hbar \hat{H}}|x_{m-1}\rangle}\langle x_{m-1} t_{m-1}|x_{m-2} t_{m-2}\rangle \cdots \langle x_1 t_1|x_0 t_0\rangle$$

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What is we put Heisenberg operators in the overlap matrix element?

Claim:

$$\boxed{\langle x_f t_f | T[\hat{x}(t_1) \cdots \hat{x}(t_n)] | x_i t_i \rangle = \int_{(x_i t_i)}^{(x_f t_f)} \mathcal{D}[x(t)]\, x(t_1) \cdots x(t_n) e^{\frac{i}{\hbar} S[x(t)]}}$$

- We can see this by splitting up $[t_i, t_f]$ again into small intervals and inserting $\int dx_k |x_k\rangle\langle x_k|$ again. The time ordering operator T ensures that the $\hat{x}(t_j)$ operators appear in just the right order that they can be evaluated at intermediate times (which we'll choose equal to the t_i for simplicity):

value of x at time slice $t_2 \Rightarrow x_2$

$$\langle x_1 t_1 | \hat{x}(t_2) | x_2 t_2 \rangle = x(t_2) \langle x_1 t_1 | x_2 t_2 \rangle$$

$\ast$ - So the natural objects calculated as path integrals are time-ordered matrix elements of Heisenberg operators.

- When we generalize to field theory, these will be the n-point Green's functions.

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- As noted before our Heisenberg aside, we could generalize our path integral expression for Z almost immediately

$$\boxed{Z = \frac{1}{N!}\int \prod_{i=1}^{N} dx_i \,\{x_1 \cdots x_n| e^{-\beta H} | x_1 \cdots x_n\}}$$

(these are symmetrized or antisymmetrized states $|x_1 \cdots x_n\}$)
and then breaking the β interval into small time steps and inserting

$$\boxed{\frac{1}{N!}\sum_{\alpha_1 \cdots \alpha_N} |\alpha_1 \cdots \alpha_N\}\{\alpha_1 \cdots \alpha_N| = 1}$$

at each time step. [See Negele and Orland (2.55)-(2.57) and Chap. 8].
This will have a similar appearence for a Hamiltonian

$$\boxed{H = \sum_{i=1}^{N} \frac{\hat{p}_i^2}{2m} + \frac{1}{2}\sum_{i\neq j} v(\hat{x}_i - \hat{x}_j)}$$

$$\Rightarrow \boxed{Z = \frac{1}{N!}\sum_{P} \zeta^{P} \int_{\substack{x_1(\beta)=x_{P_1}(0)\\ x_N(\beta)=x_{P_N}(0)}} \mathcal{D}[x_1(\tau)]\cdots\mathcal{D}[x_N(\tau)]\, e^{-\int_0^\beta d\tau \left[\sum_{i=1}^{N} \frac{m}{2}\left(\frac{dx_i(\tau)}{d\tau}\right)^2 + \frac{1}{2}\sum_{i\neq j} v(x_i(\tau)-x_j(\tau))\right]}}$$

[Note: I've skipped some details to write this down, such as ζ^P and the permutations P. We won't need them.]

- This form of the many-body path integral may be particularly useful for numerical evaluation (N+O chap. 8).

- We will work primarily in a representation in terms of fields.

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[This page repeats from page 29, which we didn't cover.]

- Jump back to second quantization and switch to the "x" basis. We can do this by forming the "field operators" $\hat{\psi}(x), \hat{\psi}^\dagger(x)$:

$$\boxed{\hat{\psi}(\vec{x}) \equiv \sum_k \psi_k(\vec{x}) c_k} \quad \boxed{\hat{\psi}^\dagger(\vec{x}) = \sum_k \psi_k^*(\vec{x}) c_k^\dagger} \qquad c_k = \underset{\text{bosons}}{b_k} \text{ or } \underset{\text{fermions}}{a_k}$$

where $k = \{\vec{k}, s_z\}$ or $\{E, L, J, m\}$ or ... for spin-1/2 fermions and so on.

- For spin-1/2 fermions, $\boxed{\psi_k(\vec{x}) = \begin{bmatrix} \psi_k(\vec{x})_1 \\ \psi_k(\vec{x})_2 \end{bmatrix} \equiv \psi_k(\vec{x})_\alpha}$ and $\hat{\psi}_\alpha(x)$ is defined analogously

- Using the $c_k, c_k^\dagger$ commutation or anticommutation relations,

$$\boxed{\begin{aligned} \left[\hat{\psi}_\alpha(\vec{x}), \hat{\psi}_\beta^\dagger(\vec{x}\,')\right]_\pm &= \sum_k \psi_k(\vec{x})_\alpha \psi_k(\vec{x}\,')^*_\beta = \delta_{\alpha\beta}\, \delta(\vec{x} - \vec{x}\,') \\ \left[\hat{\psi}_\alpha(\vec{x}), \hat{\psi}_\beta(\vec{x}\,')\right]_\pm &= \left[\hat{\psi}_\alpha^\dagger(\vec{x}), \hat{\psi}_\beta^\dagger(\vec{x}\,')\right]_\pm = 0 \end{aligned}} \quad \text{(completeness)}$$

- Operators become integrals over field operators:

$$\boxed{\hat{H} = \int d^3x\, \hat{\psi}^\dagger(\vec{x}) T(\vec{x}) \hat{\psi}(\vec{x}) + \frac{1}{2} \iint d^3x\, d^3x'\, \hat{\psi}^\dagger(\vec{x}) \hat{\psi}^\dagger(\vec{x}\,') V(\vec{x}, \vec{x}\,') \hat{\psi}(\vec{x}\,') \hat{\psi}(\vec{x})}$$

$\Rightarrow$ "second quantization": looks like expectation values wrt operators.
- The integrations over $\vec{x}, \vec{x}\,'$ generate matrix elements $\Rightarrow$ recover previous results

- If $J = \sum_{i=1}^N J(\vec{x}_i)$, then
$$\boxed{\begin{aligned} \hat{J} &= \sum_{rs} \langle r | J | s \rangle c_r^\dagger c_s \quad \text{from before} \\ &= \int d^3x \sum_{rs} \psi_r(\vec{x})^\dagger J(\vec{x}) \psi_s(\vec{x})\, c_r^\dagger c_s \\ &= \int d^3x\, \hat{\psi}^\dagger(\vec{x}) J(\vec{x}) \hat{\psi}(\vec{x}) \end{aligned}}$$

- number density $\rho(\vec{x}) = \sum_{i=1}^N \delta(\vec{x} - \vec{x}_i) \Rightarrow \boxed{\hat{n}(\vec{x}) = \sum_{rs} \psi_r(\vec{x})^\dagger \psi_s(\vec{x})\, c_r^\dagger c_s = \hat{\psi}^\dagger(\vec{x}) \hat{\psi}(\vec{x})}$

and total number $\boxed{\hat{N} = \int d^3x\, \hat{n}(\vec{x}) = \sum_r c_r^\dagger c_r = \sum_r \hat{n}_r = \int d^3x\, \hat{\psi}^\dagger(\vec{x}) \hat{\psi}(\vec{x})}$

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- What does the "field" version of the many-body path integral for the partition look like?

- For now we'll just jump to the answer and then gradually backtrack to understand where it came from and how to approximate it.

- There are <u>many</u> different notations used in the texts and in the literature $\Rightarrow$ no choice but to get used to them!

- Negele and Orland write (for either bosons or fermions)

$$\boxed{\begin{aligned} Z_G = \int_{\phi(\vec{x},\beta) = \pm\phi(\vec{x},0)} \mathcal{D}[\phi^*(\vec{x},\tau)\phi(\vec{x},\tau)]\, & e^{-\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau \int d^3x\, \phi^*(\vec{x},\tau)\left(\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu\right)\phi(\vec{x},\tau)} \\ & \times e^{-\frac{1}{2\hbar}\int_0^{\beta\hbar} d\tau \int d^3x\, d^3x'\, \phi^*(\vec{x},\tau)\phi^*(\vec{x}',\tau) V(\vec{x}-\vec{x}')\phi(\vec{x}',\tau)\phi(\vec{x},\tau)} \end{aligned}}$$

- The Euclidean action appears in the exponent (along with $-\mu\phi^*\phi$, as expected from $\mathrm{Tr}\, e^{-\beta(\hat{H}-\mu\hat{N})}$).
- This comes from breaking up the β "time" interval as before, only inserting this time "coherent" states (later!).

- For the special case of $\boxed{V(\vec{x},\vec{x}') = \lambda\,\delta^3(\vec{x}-\vec{x}')}$

$$\Rightarrow \boxed{\hat{H} = \int d^3x\, \hat{\psi}^\dagger_\alpha(\vec{x})\left(-\frac{\hbar^2\nabla^2}{2m}\right)\hat{\psi}_\alpha(\vec{x}) + \frac{\lambda}{2}\int d^3x\, \hat{\psi}^\dagger_\beta(\vec{x})\hat{\psi}^\dagger_\alpha(\vec{x})\hat{\psi}_\alpha(\vec{x})\hat{\psi}_\beta(\vec{x})}$$

the path integral for the fermionic partition function is often written (by me, for example!) [$\hbar = 1$ here]

$$\boxed{\begin{aligned} Z_G &= \mathrm{Tr}\left(e^{-\beta(\hat{H}-\mu\hat{N})}\right) \\ &= \int \mathcal{D}(\psi,\psi^\dagger)\, e^{-\int_0^\beta d\tau \int d^3x\, \psi^\dagger_\alpha\left(\frac{\partial}{\partial\tau} - \frac{1}{2m}\nabla^2 - \mu\right)\psi_\alpha - \frac{\lambda}{2}\psi^\dagger_\alpha\psi^\dagger_\beta\psi_\beta\psi_\alpha} \end{aligned}}$$

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- These are Grassman numbers $\psi^\dagger_\alpha \psi_\beta = -\psi_\beta \psi^\dagger_\alpha$ (not operators) and there is an implied $\vec{x}, \tau \Rightarrow \psi_\alpha \to \psi_\alpha(\vec{x},\tau)$.
- The path integral is over trajectories with anti-periodic boundary conditions
- $\mathcal{D}(\psi,\psi^\dagger)$ once again hides various measure and normalization factors.
- The entire expression is just a short-hand for the same type of discretization applied to the one-particle quantum mechanics case.

- Most of these details will not be relevant in applying the path integral formulation, except when we're trying something new.
  - Instead, if we learn some basic rules (mostly about Gaussian integration), we'll be able to proceed.

- We will be considering the $\beta \to \infty$ (zero temperature) limit. In this limit, the ground state $|\Psi_0\rangle$ dominates the trace and we will typically go back to real-time path integrals and focus on Green's functions: (this is FtW notation now!)

$$\boxed{\begin{aligned} iG_{\alpha\beta}(\vec{x}t, \vec{x}'t') &= \frac{\langle \Psi_0 | T[\psi_{H\alpha}(\vec{x}t)\psi^\dagger_{H\beta}(\vec{x}'t')] | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle} \\ &= \frac{\int \mathcal{D}\psi^\dagger \mathcal{D}\psi \; \psi_\alpha(\vec{x},t)\psi^\dagger_\beta(\vec{x}'t') \, e^{\frac{i}{\hbar}\int dt \int d^3x \left[\psi^\dagger_\alpha\left(i\frac{\partial}{\partial t} + \frac{\hbar^2 \nabla^2}{2m} + \mu\right)\psi_\alpha - \frac{\lambda}{2}\psi^\dagger_\alpha \psi^\dagger_\beta \psi_\beta \psi_\alpha\right]}}{\int \mathcal{D}\psi^\dagger \mathcal{D}\psi \; e^{i/\hbar \int dt \int d^3x \, \mathcal{L}(\vec{x},t)}} \end{aligned}}$$

"H" means Heisenberg operators

α, β are spin indices (e.g. run over 1,2 for spin 1/2)

- Whew! That's a lot of obscure notation. What does it mean and how do we do anything with it?
  - We answer these questions first by analogy...

1/22/03 0-d Model Partition Function

Before proceeding, we'll introduce some important issues and techniques by considering a simple integral that is analogous in a classical partition function for a particle in a potential:

$$\boxed{Z = \int dx\, e^{-V(x)}}$$ (see Negele + Orland)

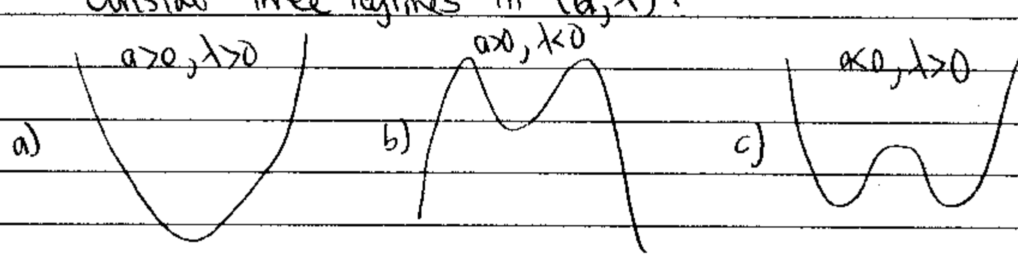
In particular, we consider the analog of a path integral of a particle in a quadratic plus quartic potential.

$$\boxed{Z(\lambda) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi/a}}\, e^{-\frac{ax^2}{2} - \frac{\lambda}{4}x^4}}$$

$\left[\begin{array}{l} \text{quadratic} \to \psi^\dagger[\ \ ]\psi \\ \text{quartic} \to \frac{\lambda}{2}\psi^\dagger\psi^\dagger\psi\psi \end{array}\right]$

- we could rescale x to get rid of a, but we'll keep it since it serves as the analog of an inverse propagator (see below!) and lets us consider $a<0$.
- Approximation strategies
  - So far we've considered perturbation theory for simple many-body systems (up to 1st order).
  - The underlying idea is that the behavior of a system can be varied continuously between a solvable problem ($\hat{H}_0$) and the system of interest ($\hat{H}_0 + \hat{H}_1$) in terms of a small parameter in $\hat{H}_1$.
  - In general, perturbation theory does not yield a <u>convergent</u> series, but an <u>asymptotic</u> series.

- Can we use λ as an expansion parameter in $Z(\lambda)$?
  - Consider three regimes in (a, λ):



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From case a) to case b), we change the sign of λ. We know that both the classical and quantum behavior completely changes — eg. localized vs. unconfined

- if the expansion in λ converged, then there is a finite radius of convergence and the behavior would not abruptly change from $+|\lambda|$ to $-|\lambda|$ (for small enough $|\lambda|$.

$\Rightarrow$ zero radius of convergence $\Rightarrow$ expect nonanalyticity at $\lambda=0$.

- By inspection, the integral diverges for $\lambda<0$

  $\Rightarrow$ nonanalytic at $\lambda=0$.

- This argument is applicable in quantum field theory. Dyson used it in the 50's to show that QED perturbation theory is asymptotic (changing the sign of e^2 leads to vacuum unstable to decay into e^+e^- pairs, since like particles attract and unlike repel).

$\Rightarrow$ given the incredible successes of QED perturbation theory, we should not be too discouraged!

- Let's analyze $Z(g)$ explicitly.
  - The perturbation series is generated explicitly by expanding the exponent:

$$\boxed{e^{-\frac{\lambda}{4}x^4} = 1 - \frac{\lambda}{4}x^4 + \frac{1}{2!}\left(\frac{\lambda}{4}\right)^2 x^8 + \ldots}$$

and doing the integrals term-by-term:

$$\boxed{Z(g) = \sum_{n=0}^{\infty} Z_n \lambda^n}$$

where

$$\lambda^n Z_n = \frac{(-\lambda)^n}{n!}\frac{1}{4^n}\int_{-\infty}^{\infty}\frac{dx}{\sqrt{2\pi}}\, e^{-x^2/2}\, x^{4n}$$

$$= \frac{(-\lambda)^n (4n-1)!!}{n!\,4^n}$$

$$= \frac{(-\lambda)^n (4n)!}{n!\,16^n (2n)!} \sim \frac{1}{\sqrt{n\pi}}\left(\frac{4\lambda n}{e}\right)^n$$

$$\boxed{\int_{-\infty}^{\infty} x^{4n} e^{-x^2} dx = \frac{1}{2}\int_0^{\infty} x^{4n} e^{-x^2} dx \overset{x=t^{1/2}}{=} \int_0^{\infty} t^{(2n+1/2)-1} e^{-t} dt = \Gamma(2n+1/2)}$$

from $\boxed{n! \doteq \sqrt{2\pi}\, n^{n+1/2} e^{-n}}$

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So Z_n grows like $n!$ $\Rightarrow$ series diverges for large n
$\Rightarrow$ zero radius of convergence.

- We can define $Z(\lambda)$ by analytic continuation to $\lambda<0$, but there is a branch cut on the negative real axis and a branch point at the origin.

- How well does a finite number of terms work? Define (take $a=1$ for simplicity)

$$\boxed{R_n \equiv \left|Z(\lambda) - \sum_{m=0}^{n} \lambda^m Z_m\right| \leq \int \frac{dx}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{(n+1)!}\left(\frac{\lambda x^4}{4}\right)^{n+1} = \lambda^{n+1}|Z_{n+1}|}$$

so as long as successive terms decrease, the approximation gets better.

- The minimum occurs for $n \sim 1/4\lambda$ so the minimum error goes like $\boxed{(\lambda^n Z_z)_{min} \sim \sqrt{\frac{4\lambda}{\pi}}\, e^{-1/4\lambda}}$ ← error in truncation is the order of the first term omitted. $\Rightarrow$ could be small

- In QED, with $\alpha = \frac{e^2}{4\pi} \sim \frac{1}{137}$, this works great.

- Use Mathematica to see how well it works here for different λ (with $a=1$). $\Rightarrow$ see Figure of $R_n(\lambda)/R_0$ vs. n (# of terms kept)
  - we see that for very small $\lambda \sim .01$, we can get very accurate results from perturbation theory
  - but for $\lambda = 0.1$, which might seem quite small, we have to stop at the third term and the accuracy is only 25%

- How to proceed if perturbation theory is inadequate
  - even if asymptotic, it organizes how we attack the problem and identifies or isolates different parts of the physics, such as short-range or long-range correlations
    $\Rightarrow$ suggests partial resummations of the series $\Rightarrow$ nonperturbative
  - other approaches suggested by integral: [later!]
    - strong coupling (perturb about quartic potential) or for case c), stationary phase

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- In problems of interest, we can't generate the perturbation series in closed form $\Rightarrow$ evaluate $Z(\lambda)$ by other means.

- Our alternative calculation of $Z(\lambda)$ is based on the idea that we know how to calculate Gaussian integrals

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} dx\, e^{-\frac{ax^2}{2}} = \sqrt{\frac{2\pi}{a}}}$$

- we'll generalize this to multiple x's $\Rightarrow ax^2 \to x_i A_{ij} x_j$ and use these Gaussian integrals (and their complex and Grassman extensions) as the basis for our path integral expansions.

- A fruitful technique to evaluate

$$\boxed{Z_\lambda \equiv N \int_{-\infty}^{\infty} dx\, e^{-\frac{ax^2}{2} - \frac{\lambda x^4}{4}}} \quad (N \Rightarrow \text{normalization, measure, etc})$$

is to add a source term jx in the exponent:

$$\Rightarrow \boxed{Z_\lambda[j] = N \int_{-\infty}^{\infty} dx\, e^{-\frac{ax^2}{2} - \frac{\lambda x^4}{4} + jx}}$$

so that

$$\boxed{Z_\lambda = Z_\lambda[j]\big|_{j=0}}$$

and we recover Z_λ after any "j" manipulations by setting $j=0$ at the end.

- In particular, since $\frac{\partial}{\partial j} e^{jx} = x e^{jx}$, we can replace functions of x by functions of j (and take them out of the integral:

$$\boxed{f(x) e^{jx} = f\left(\frac{\partial}{\partial j}\right) e^{jx}}$$

where $f\left(\frac{\partial}{\partial j}\right)$ means to replace x by $\frac{\partial}{\partial j}$ in the (assumed) Taylor series of $f(x)$.

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$$\Rightarrow \boxed{Z_\lambda[j] = N e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4} \int_{-\infty}^{\infty} dx\, e^{-\frac{a}{2}x^2 + jx}}$$

· Now complete the square in the exponent:

$$Z_\lambda[j] = N e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4} \int_{-\infty}^{\infty} dx\, e^{-\frac{a}{2}\left(x^2 + \frac{2j}{a}x\right)}$$

$$= N e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4} \int_{-\infty}^{\infty} dx\, e^{-\frac{a}{2}\underbrace{\left(x^2 + \frac{2j}{a}x + \frac{j^2}{a^2}\right)}_{(x+j/a)^2}}\, e^{+\frac{a}{2}\frac{j^2}{a^2}}$$

switch to $\begin{cases} x' = x + j/a \\ dx' = dx \end{cases}$

$$= N e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4} e^{\frac{1}{2} j a^{-1} j} \underbrace{\int_{-\infty}^{\infty} dx'\, e^{-\frac{a}{2}x'^2}}_{Z_0/N}$$

$$\Rightarrow \boxed{Z_\lambda[j] = Z_0\, e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4} e^{\frac{1}{2} j a^{-1} j}}$$

and $$\boxed{\frac{Z_\lambda}{Z_0} = \left[e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4} e^{1/2\, j a^{-1} j} \right]\Big|_{j=0}}$$

· Including the known result Z_0 divides out all of the nasty normalization and measure factors.

- Z_0 is the noninteracting partition function, which we can calculate by other means
- When we take the logarithm to find Ω, we'll have an expression for $\Omega - \Omega_0$.

· What is we want an expectation value, such as $\langle x^2 \rangle$?

$$\boxed{\begin{aligned} \langle x^2 \rangle &= \frac{\cancel{N}\int_{-\infty}^{\infty} dx\, x^2 e^{-\frac{a}{2}x^2 - \frac{\lambda}{4}x^4}}{\cancel{N}\int_{-\infty}^{\infty} dx\, e^{-\frac{a}{2}x^2 - \frac{\lambda}{4}x^4}} = \frac{\frac{\partial}{\partial j}\frac{\partial}{\partial j} Z_\lambda[j]\big|_{j=0}}{Z_\lambda[j]\big|_{j=0}} \\ &= \frac{\left(\frac{\partial}{\partial j}\frac{\partial}{\partial j}\right) e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4} e^{\frac{1}{2} j a^{-1} j}\Big|_{j=0}}{e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4} e^{\frac{1}{2} j a^{-1} j}\Big|_{j=0}} \end{aligned}}$$

· $\langle x^2 \rangle$ is analogous to a Green's function. Notice how factors again cancel out.

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- To generate a perturbative expansion for Z_λ or $\langle x^2 \rangle$ in powers of λ, expand $e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4}$ to the relevant power of λ and expand $e^{\frac{1}{2} j a^{-1} j}$ so there are just enough j's for the $\left(\frac{\partial}{\partial j}\right)$'s to kill. (Since we set $j=0$ at the end, any terms with leftover j's will vanish.)

$$\Rightarrow \frac{Z_\lambda}{Z_0} = \left[1 - \frac{1}{4}\lambda\left(\frac{\partial}{\partial j}\right)^4 + \frac{1}{2!}\left(\frac{\lambda}{4}\right)^2\left(\frac{\partial}{\partial j}\right)^4\left(\frac{\partial}{\partial j}\right)^4 + \frac{1}{3!}\left(\frac{\lambda}{4}\right)^3\left(\frac{\partial}{\partial j}\right)^4\left(\frac{\partial}{\partial j}\right)^4\left(\frac{\partial}{\partial j}\right)^4 + \dots\right]$$

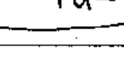
$$\times \left[1 + \frac{1}{2}(j a^{-1} j) + \frac{1}{2!}\left(\frac{1}{2} j a^{-1} j\right)\left(\frac{1}{2} j a^{-1} j\right) + \dots\right]$$

- Let's do λ^1 and λ^2. From earlier, $\boxed{\frac{Z_\lambda}{Z_0} = \sum_{n=0}^{\infty} \lambda^n Z_n}$ with $\boxed{Z_n = \frac{(-1)^n}{n!}\frac{(4n-1)!!}{4^n}\frac{1}{a^{2n}}}$ $\Rightarrow \boxed{\lambda Z_1 = -\frac{3}{4}\frac{\lambda}{a^2}}$ and $\boxed{\lambda^2 Z_2 = \frac{105}{32}\frac{\lambda^2}{a^4}}$

λ^1: $-\frac{1}{4}\lambda\left(\frac{\partial}{\partial j}\right)^4 \frac{1}{2!}\left(\frac{1}{2} j a^{-1} j\right)\left(\frac{1}{2} j a^{-1} j\right)$ is the only term that survives $j=0$

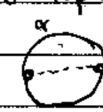
$$= -\frac{\lambda}{a^2}\frac{1}{32}\left(\frac{\partial}{\partial j}\right)^4 (jjjj) = -\frac{\lambda}{a^2}\frac{1}{32}4! = -\frac{3\lambda}{4a^2} \checkmark$$

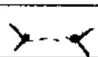
- Note the $4!$, which comes from all the ways for $\left(\frac{\partial}{\partial j}\frac{\partial}{\partial j}\frac{\partial}{\partial j}\frac{\partial}{\partial j}\right)$ to anhilate $j \times j \times j \times j$.
- The basic calculation is $\frac{\partial}{\partial j}(j) = 1$. In the path integral case it will almost be as simple!

- If we associate the a^{-1} in $\frac{1}{2} j a^{-1} j$ with a line and the "interaction" $\frac{\lambda}{4}$ with a $\bullet$, we can represent our result with a diagram: 

- Suppose j came in two flavors ("spins") j_1 and $j_2 \Rightarrow j_\alpha$ and we were really calculating (with sums over repeated indices)

$$\left(\frac{\partial}{\partial j_\alpha}\frac{\partial}{\partial j_\alpha}\right)\left(\frac{\partial}{\partial j_\beta}\frac{\partial}{\partial j_\beta}\right)\left(\frac{1}{2} j_\delta a^{-1}_{\delta\gamma} j_\gamma\right)\left(\frac{1}{2} j_\lambda a^{-1}_{\lambda\rho} j_\rho\right) \Rightarrow$$

 α β $\propto g^2$ +  α α $\propto g$

with $\bullet$ becoming  α β

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Now continue with the next order:

$$\lambda^2 : \frac{1}{2}\left(\frac{\lambda}{4}\right)^2 \left(\frac{\partial}{\partial j}\right)^4 \left(\frac{\partial}{\partial j}\right)^4 \frac{1}{4!}\left(\frac{1}{2}\right)^4 \frac{1}{a^4}(jj)(jj)(jj)(jj) = \frac{1}{2}\frac{1}{16}\frac{1}{16}\frac{1}{4!}8!\frac{\lambda^2}{a^4} = \frac{105\lambda^2}{32a^4} \checkmark$$

- Lots of combinatoric factors but we can organize it according to how the $\left(\frac{\partial}{\partial j}\right)^4$ attack the $\frac{1}{2}ja^{-1}j$ terms, as represented by diagrams:

each $\left(\frac{\partial}{\partial j}\right)^4$ hits two separate $\frac{1}{2}ja^{-1}j$ terms

each $\frac{\partial}{\partial j}$ in $\left(\frac{\partial}{\partial j}\right)^4$ picks up 1 j from each $\frac{1}{2}ja^{-1}j$ term

in between: in each $\left(\frac{\partial}{\partial j}\right)^4$, two hit one $ja^{-1}j$ term and the other two hit different $ja^{-1}j$'s

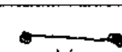
- Let's see how it works for $\langle x^2\rangle$

$$\langle x^2\rangle = \frac{\frac{\partial}{\partial j}\frac{\partial}{\partial j}Z_\lambda[j]\Big|_{j=0}}{Z_\lambda[j]_{j=0}} = \frac{\frac{\partial}{\partial j}\frac{\partial}{\partial j}e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4}e^{\frac{1}{2}ja^{-1}j}\Big|_{j=0}}{e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4}e^{\frac{1}{2}ja^{-1}j}\Big|_{j=0}}$$

$$\Rightarrow \langle x^2\rangle = \frac{\frac{\partial}{\partial j}\frac{\partial}{\partial j}\left[1-\frac{1}{4}\lambda\left(\frac{\partial}{\partial j}\right)^4+\ldots\right]\left[1+\frac{1}{2}(ja^{-1}j)+\ldots\right]\Big|_{j=0}}{\left[1-\frac{1}{4}\lambda\left(\frac{\partial}{\partial j}\right)^4+\ldots\right]\left[1+\frac{1}{2}(ja^{-1}j)+\ldots\right]\Big|_{j=0}}$$

- Now the expansion has a numerator and denominator, with two extra $\frac{\partial}{\partial j}$'s in the numerator.

$$\lambda^0 : \frac{\frac{\partial}{\partial j}\frac{\partial}{\partial j}[1]\left[\frac{1}{2}ja^{-1}j\right]}{[1][1]} = \frac{1}{a}$$



- This is the "noninteracting propagator". Our "Feynman rule" is to let the $\frac{\partial}{\partial j}$'s represent endpoints on a line.

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Next order:

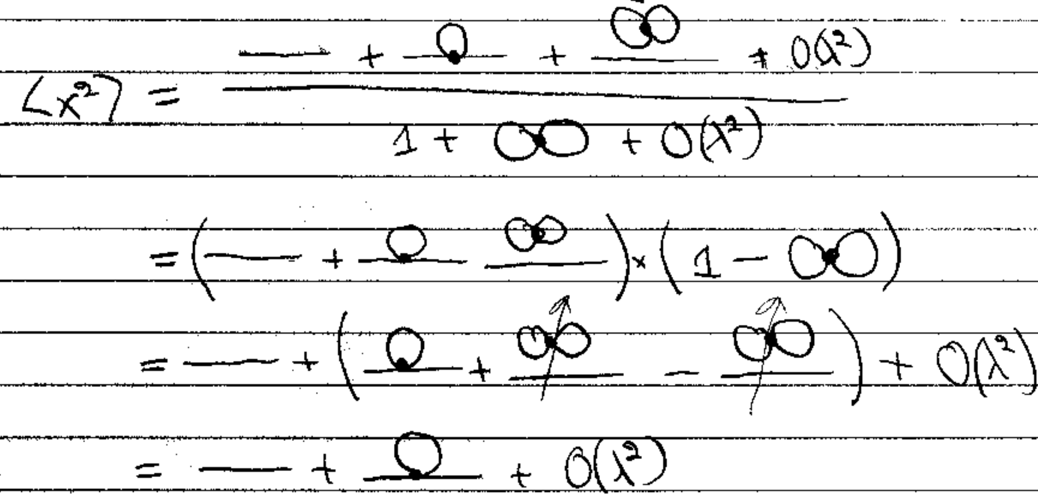
$$\lambda^1: \frac{\frac{1}{a} + \frac{\partial}{\partial j}\frac{\partial}{\partial j}\left(-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4\right)\frac{1}{3!}\left(\frac{1}{2}ja^{-1}j\right)\left(\frac{1}{2}ja^{-1}j\right)\left(\frac{1}{2}ja^{-1}j\right)}{1 - \frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4\frac{1}{2!}\left(\frac{1}{2}ja^{-1}j\right)\left(\frac{1}{2}ja^{-1}j\right)}$$

$$= \left(\frac{1}{a} - \frac{\lambda}{4}\frac{1}{3!}\left(\frac{1}{2}\right)^3\frac{1}{a^3}6!\right)\left(1 + \frac{\lambda}{4}\frac{1}{2!}\left(\frac{1}{2}\right)^2\frac{1}{a^2}4! + O(\lambda^2)\right)$$

$$= \frac{1}{a} - \frac{\lambda}{a^3}\left[\frac{6\cdot5\cdot4}{4\cdot8} - \frac{4\cdot3}{4\cdot4}\right] + O(\lambda^2) = \frac{1}{a} - \frac{3\lambda}{a^3} + O(\lambda^2)$$

• What does this look like in diagrams?

$\boxed{\text{——} \Rightarrow \frac{1}{a}}$ $\boxed{\times \Rightarrow -\frac{\lambda}{4}4!}$ (since we get a 4! from $\left(\frac{\partial}{\partial j}\right)^4 j^4$)



• The "disconnected" parts cancel. This turns out to be a general result (later!)

• Make sure you understand what each diagram corresponds to. Try the λ^2 term.

• The Feynman rules applied to give us the correct answer except for an extra 2 in the second term. This means we need another rule: the "symmetry factor." Coming up next!

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ASIDE: Gaussian integrals...

Start with $\boxed{\int_{-\infty}^{\infty} dx\, e^{-\frac{ax^2}{2}} = \sqrt{\frac{2\pi}{a}}}$

[e.g. switch $\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, e^{-\frac{ax^2}{2}} e^{-\frac{ay^2}{2}}$ to polar coordinates
$= 2\pi \int_0^\infty r\,dr\, e^{-\frac{a}{2}r^2} = \frac{2\pi}{a}\int_0^\infty dz\, e^{-z} = \frac{2\pi}{a} \Rightarrow \sqrt{\frac{2\pi}{a}}$]

and then generalize to n variables $x_1, \ldots, x_n$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} dx_1 \cdots dx_n\, e^{-\frac{a_1 x_1^2}{2}} e^{-\frac{a_2 x_2^2}{2}} \cdots e^{-\frac{a_n x_n^2}{2}} = \int_{-\infty}^{\infty} dx_1 \cdots dx_n\, e^{-\sum_{m=1}^{n} \frac{1}{2} a_m x_m^2} = \prod_{m=1}^{n} \sqrt{\frac{2\pi}{a_m}}}$$

but now make it more complicated: allow $x_1 x_2$, $x_1 x_5$, and so on

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} dx_1 \cdots dx_n\, e^{-\frac{1}{2} x_i A_{ij} x_j}}$$

with i, j implicitly summed from 1 to n. A_{ij} is an $n \times n$ matrix.

If we require $\underline{A}$ to be a real, symmetric, positive definite matrix, then there is an orthogonal transformation $\mathcal{O}$ that diagonalizes $\underline{A}$

$$\Rightarrow \boxed{\underline{\underline{\mathcal{O}}}\,\underline{\underline{A}}\,\underline{\underline{\mathcal{O}}}^T = \underline{\underline{A}}_D = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix}} \quad \text{where } \boxed{\mathcal{O}\mathcal{O}^T = \mathbb{1}},\ \boxed{\det \mathcal{O} = 1}$$

$$\Rightarrow \boxed{x_i A_{ij} x_j = \underline{x}^T \underline{\underline{A}}\, \underline{x} = x^T \mathcal{O}^T \mathcal{O} A \mathcal{O}^T \mathcal{O} x = (\mathcal{O}x)^T A_D (\mathcal{O}x)}$$

So let $\boxed{y_j = \mathcal{O}_{ji} x_i}$ $\Rightarrow$ $\boxed{\underline{x}^T \underline{\underline{A}}\, \underline{x} = \underline{y}^T \underline{\underline{A}}^D \underline{y} = y_j A^D_{ji} y_i = \sum_{i=1}^{n} a_i y_i^2}$

Note that $dx_1 \cdots dx_n \to dy_1 \cdots dy_n$ (i.e. Jacobian is 1 for orthogonal trans.)

Now the problem is reduced to one already solved:

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} dx_1 \cdots dx_n\, e^{-\frac{1}{2} x_i A_{ij} x_j} = \prod_{m=1}^{n} \sqrt{\frac{2\pi}{a_m}} = \frac{(2\pi)^{n/2}}{(\det A)^{1/2}}} \quad \text{since } \boxed{\det \underline{A}^D = \det \underline{A} = \prod_{i=1}^{n} a_i}$$

Finally, by completing the square (left as an exercise for the reader!):

$$\boxed{\int_{-\infty}^{\infty} dx_1 \cdots dx_n\, e^{-\frac{1}{2} x_i A_{ij} x_j + x_i J_i} = (2\pi)^{n/2} [\det A]^{-1/2}\, e^{\frac{1}{2} J_i A^{-1}_{ik} J_k}}$$

and $\boxed{\det A = e^{\operatorname{tr} \ln A}}$ can be used.