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880.05 lecture 5

Handouts:

- 2 pages from a paper on EFT for dilute Fermi gas including pairing — perturbative. Note the diagrams and the form of the path integral.
  - Our next task is to understand where this comes from.
  - Note the comments on connected diagrams!

3.10 disconnected → connected

Announcements: • hints online. ϵ^2 an error?
- office hours friday afternoon 3pm+

Recap from last time:

- expectation values like $\langle \xi^2 \rangle$ or free energies / thermodynamic potentials like $\ln Z/Z_0$ are found in perturbation theory by calculating connected diagrams.
- We motivated diagrams by carrying out an expansion
- Replica method used to prove $\ln Z/Z_0$ is only from connected diagrams — "linked cluster expansion"

× Questions?

Today:

- go through symmetry factors
- Find Legendre transformation for model Z
- briefly consider alternative expansions
- start path integrals

$$\langle \xi^2 \rangle = \frac{\int_{-\infty}^{\infty} d\xi\, \xi\, e^{-(\ )}}{\int_{-\infty}^{\infty} d\xi\, e^{-(\ )}} = \frac{\frac{\partial}{\partial j}\frac{\partial}{\partial j}\, e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4} e^{\frac{1}{2} j a^{-1} j}\Big|_{j=0}}{e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4} e^{\frac{1}{2} j a^{-1} j}\Big|_{j=0}}$$

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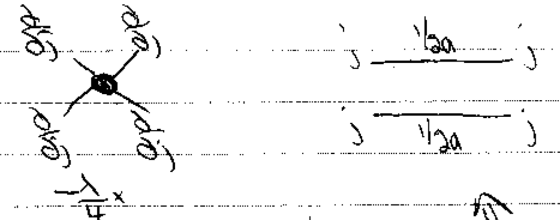
Model Partition Function: Recap and moving on

$$Z_\lambda[j] = \underbrace{\frac{1}{\sqrt{2\pi/a}}}_{N} \int_{-\infty}^{\infty} d\xi\, e^{-\frac{a}{2}\xi^2 - \frac{\lambda}{4}\xi^4 + j\xi} = \frac{1}{\sqrt{2\pi/a}}\, e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4} \int_{-\infty}^{\infty} d\xi\, e^{-\frac{a}{2}\xi^2 + j\xi}$$

$$= Z_0\, e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4} e^{\frac{1}{2} j a^{-1} j}$$

- set $j=0$ at end to get Z_λ / Z_0.

$\left(\frac{\partial}{\partial j}\right)^n$ ← in $\langle \xi^n \rangle$



from expanding $e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4}$ from expanding $e^{\frac{1}{2} j a^{-1} j}$

$\Rightarrow$ combine where any $\frac{\partial}{\partial j}$ and act on any j with $\frac{\partial}{\partial j} j = 1$

$\Rightarrow$ connected and disconnected diagrams

- $\ln Z_\lambda / Z_0$ and $\langle \xi^2 \rangle$ are given by the sum of connected diagrams (shown by replica methods).

- We need to keep track of the combinatoric factors.
  - $\Rightarrow$ Feynman rules.
  - We choose $\frac{1}{a}$ for ●—● and $-\frac{\lambda}{4} 4!$ for ✕

- Since $\langle \xi^2 \rangle =$ ●—● $+ \ldots = \frac{\partial}{\partial j}\frac{\partial}{\partial j} \frac{1}{2} j a^{-1} j = \frac{1}{a}$, this makes sense.

- For ✕, we include the $4!$ factor from $\left(\frac{\partial}{\partial j}\right)^4 jjjj = 4!$

- But then we calculate $\rightarrow -\frac{\lambda}{4} \cdot 4! \left(\frac{1}{a}\right)^3 = -\frac{6\lambda}{a^3}$ but should be $-\frac{3\lambda}{a^3}$

  $\Rightarrow$ need $1/2$

or $\rightarrow -\frac{\lambda}{4} \cdot 4! \frac{1}{a^2} = -\frac{6\lambda}{a}$ but should be $-\frac{3\lambda}{4a^2}$

$\Rightarrow$ need $\frac{1}{8}$ factor. $\Rightarrow$ symmetry factors.

$\Rightarrow$ back to (60)

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The precise symmetry factors will depend on the theory.

- Note that we've treated each end of the •—• 's equivalently, because the j's in $\frac{1}{2} j \bar{a}^{-1} j$ are the same.
- But this is not the case in most of our examples to follow ⇒ fermions, for example.

- Suppose our model partition function is generalized so that ξ is complex. Designate ξ^{+} the complex conjugate (transpose even though only one variable is still ok and generalizes!).
  - We think of the integration over real and imaginary parts as being $\int d\xi^{+} d\xi$

$$\Rightarrow Z_{\lambda} = \int d\xi^{+} d\xi \; e^{-\xi^{+} a \xi - \frac{\lambda}{4}(\xi^{+}\xi)^{2}}$$

- How would we do the perturbative expansion now?
  - Introduce j and j^{+}!

$$\Rightarrow Z_{\lambda}[j, j^{+}] = \int d\xi^{+} d\xi \; e^{-\xi^{+} a \xi - \frac{\lambda}{4}(\xi^{+}\xi)^{2} + j^{+}\xi + \xi^{+} j}$$

So $\frac{\delta}{\delta j^{+}} \rightarrow \xi$ and $\frac{\delta}{\delta j} \rightarrow \xi^{+}$

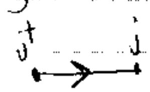
$$\Rightarrow Z_{\lambda}[j, j^{+}] = \underbrace{e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\frac{\partial}{\partial j^{+}}\right)^{2}}} \int d\xi^{+} d\xi \; e^{-\xi^{+} a \xi + j^{+}\xi + \xi^{+} j}$$

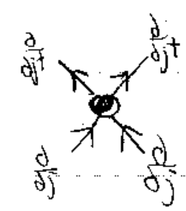
$$= \sim \int d\xi^{+} d\xi \; e^{-(\xi^{+} + j^{+}\bar{a}^{-1}) a (\xi + \bar{a}^{-1} j)} e^{+j^{+}\bar{a}^{-1} j}$$

$$= \left[e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\frac{\partial}{\partial j^{+}}\right)^{2}} e^{j^{+}\bar{a}^{-1} j}\right] \int d\xi'^{+} d\xi' \; e^{-\xi'^{+} a \xi'} \quad \leftarrow \text{shift variables}$$

$$= Z_{0}\left[e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\frac{\partial}{\partial j^{+}}\right)^{2}} e^{j^{+}\bar{a}^{-1} j}\right]$$

Now the two ends of $\bar{a}^{-1}$ are different ⇒ put arrows on the lines.





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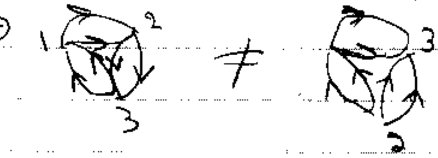
Now the symmetry factor rules are:

(1) this is always 1

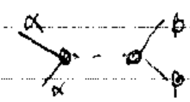
(2) only equivalent lines in the same direction, so  has no equivalent lines anymore.

·  still counts as 1/2 under this rule.

(3) the permutations of vertices are not allowed to violate two in - two out: 

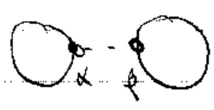
⇒  since now all out of 3 and into 2!

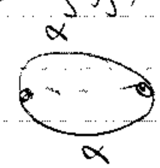
• If j come in ν "flavors" (or "spins"), then j_α $\alpha=1,\dots,\nu$ would keep track of them.

∘ Then we can write the interaction as  if it doesn't change the flavor.

· Then we'd have

$$\left(\frac{\partial}{\partial j_\alpha}\frac{\partial}{\partial j_\alpha}\right)\left(\frac{\partial}{\partial j_\beta}\frac{\partial}{\partial j_\beta}\right)\left(\frac{1}{2} j_\delta a^{-1}_{\delta\gamma} j_\gamma\right)\left(\frac{1}{2} j_\lambda a^{-1}_{\lambda\rho} j_\rho\right)$$

⇒  $\propto \nu^2$ "Hartree"

 $\propto \nu$ "Fock"

[more later!]

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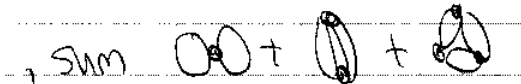
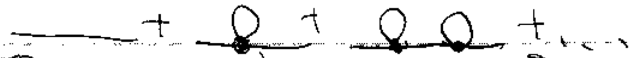
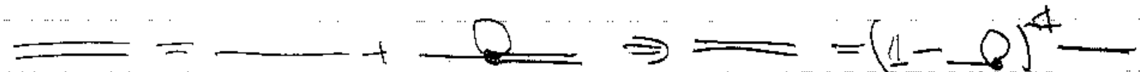
What kind of "partial summations" can we think of?

Examples:

① For $\ln Z/Z_0$, sum  + ...
which picks out one term to each order.

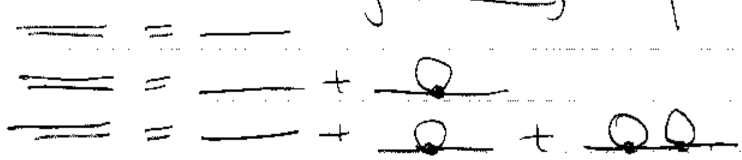
② For $\langle \xi^2 \rangle$, consider  + ...
How can we sum these? If we designate the sum with a double line: ═══, then



[later: $G = G^0 + G^0 \Sigma G$]

The sum is recovered by iterating the equation:

0th:
1st:
2nd:

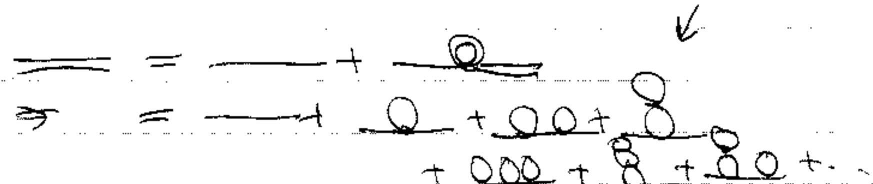


and so on.

sum of all "tadpoles"

More general:

You try: What do get ⇒



What do you miss?  for example

③ More generally, pick out the "one particle irreducible" (1PI) pieces: does the diagram fall apart when you cut a line? Or 2PI: does the diagram fall apart when you cut two lines?
- Later we'll see schemes to sum these.

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- Before moving on to quantum mechanics, let's put a couple of other things on the table for future reference. First: saddlepoint expansions (also known as "steepest descent" or "stationary phase" when considered generally — see any advanced mathematical physics book).

- Consider a somewhat generic integral with parameter g:

$$\boxed{I(g) \equiv \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi g}}\, e^{-S(x)/g}}$$

and imagine expanding for $g \to 0$. What do we expect?

- If $S(x)$ has a minimum at x_0 and if the following conditions hold:

$\boxed{\text{i) [additional condition to be discussed later!]}}$ $\boxed{\text{(ii) } \left.\frac{dS}{dx}\right|_{x=x_0} = 0}$ $\boxed{\left.\frac{d^2S}{dx^2}\right|_{x=x_0} > 0}$

then the dominant contribution comes from the region near x_0 as $g \to 0$.
$\Rightarrow$ expand $S(x)$ about x_0 (with $S^{(1)}(x_0) = 0$ from ii)):

(If more than one minimum: sum the contributions. They cannot be too close)

$$\boxed{S(x) = S(x_0) + \frac{1}{2!}S^{(2)}(x_0)(x-x_0)^2 + \frac{1}{3!}S^{(3)}(x_0)(x-x_0)^3 + \ldots}$$

and insert into $I(g)$. We switch variables to $\boxed{y = (x-x_0)/\sqrt{g}}$ to remove the g dependence of the quadratic $(x-x_0)^2$ term.
[let $S^{(n)}(x_0) \equiv S_0^{(n)}$]:

$$\boxed{\begin{aligned} I(g) &\doteq e^{-S_0/g}\int_{-\infty}^{\infty}\frac{dy}{\sqrt{2\pi}}\, e^{-S_0^{(2)}y^2/2}\, e^{-\frac{\sqrt{g}}{3!}S_0^{(3)}y^3 + \frac{g}{4!}S_0^{(4)}y^4 + \ldots} \\ &\doteq e^{-S_0/g}\int_{-\infty}^{\infty}\frac{dy}{\sqrt{2\pi}}\, e^{-S_0^{(2)}y^2/2}\left\{1 - \frac{g}{4!}S_0^{(4)}y^4 + \frac{g}{2}\left(\frac{S_0^{(3)}}{3!}\right)^2 y^6 + O(g^2)\right\}\end{aligned}}$$

where the odd powers of y vanish when integrated.

- So we have a term out front that has an essential singularity in g and then a perturbative expansion that we can integrate term by term (just gaussians!).

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The result is:

$$\boxed{I(g) \doteq e^{-S_0/g}\frac{1}{\sqrt{S_0^{(2)}}}\left[1-\frac{g}{8}\frac{S_0^{(4)}}{(S_0^{(2)})^2}+\frac{5g}{24}\frac{(S_0^{(3)})^2}{(S_0^{(2)})^3}+O(g^2)\right]}$$

Can we use this expansion?

· Return to Z_λ and consider $\lambda \to 0$ for the case where $a>0, \lambda>0$.

$$Z_\lambda = \int_{-\infty}^{\infty}\frac{d\xi}{\sqrt{2\pi}\sqrt{a}}\, e^{-\frac{a\xi^2}{2}-\frac{1}{4}\lambda\xi^4} \underset{x=\sqrt{\lambda}\,\xi}{=} \int_{-\infty}^{\infty}\frac{dx}{\sqrt{2\pi}\sqrt{a}}\, e^{-\frac{1}{\lambda}\left(\frac{ax^2}{2}+\frac{1}{4}x^4\right)}$$

so with this scaling of $\xi \to \frac{x}{\sqrt{\lambda}}$, we get an overall $1/\lambda$ out front.

· With $a>0, \lambda>0$,  $x_0=0$ and $S(x)=\frac{ax^2}{2}+\frac{1}{4}x^4$ so we have

$$S_0=0,\quad S_0^{(1)}=S_0^{(3)}=0,\quad S_0^{(2)}=a,\quad S_0^{(4)}=6$$

which yields $Z_\lambda = e^{0}\frac{1}{\sqrt{a/a}}\left(1-\frac{\lambda}{8}\frac{6}{a^2}+O(\lambda^2)\right) = 1-\frac{3\lambda}{4a^2}+O(\lambda^2)$

So we just get back the perturbative expansion we had before.

· (Can we apply this if $a<0, \lambda>0$, expanding around the two minima?)

· That wasn't very illuminating so we're going to change the integral in Z_λ by introducing an auxiliary integration variable z to eliminate the ξ^4 term.

· The basic idea, used frequently, is to insert "1" in a useful form.

· Consider the normalized Gaussian integral

$$1 = \int_{-\infty}^{\infty}\frac{dz}{\sqrt{2\pi}}\, e^{-(z+i\sqrt{\frac{\lambda}{2}}\xi^2)^2/2} = \int_{-\infty}^{\infty}\frac{dz}{\sqrt{2\pi}}\, e^{-\frac{z^2}{2}-i\sqrt{\frac{\lambda}{2}}z\xi^2}\, e^{+\frac{\lambda}{4}\xi^2}$$

· We need the i so that we get $e^{+\lambda/4\xi^2}$ (instead of $e^{-\frac{\lambda}{4}\xi^2}$) but the integral is well defined (just shift to $z'=z+i\sqrt{\frac{\lambda}{2}}\xi^2 \to$ there are no singularities in the complex z' plane so the integral can be moved to $\int_{-\infty}^{\infty}\frac{dz'}{\sqrt{2\pi}}=1$).

c

[Warning: Some of the operations on the next two pages are not carried out with mathematical rigor!]

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- Now we stick this "1" inside the Z_λ integral:

$$Z_\lambda = \int_{-\infty}^{\infty} \frac{d\xi}{\sqrt{2\pi/a}}\, e^{-\frac{a\xi^2}{2} - \cancel{\frac{\lambda}{4}\xi^4}} \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}}\, e^{-\frac{z^2}{2} - i\sqrt{\frac{\lambda}{2}}\, z\xi^2}\, \cancel{e^{+\frac{\lambda}{4}\xi^2}}$$

switch ξ and z integrals

$$= \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}}\, e^{-\frac{z^2}{2}} \int_{-\infty}^{\infty} \frac{d\xi}{\sqrt{2\pi/a}}\, e^{-(a+i\sqrt{2\lambda}\, z)\xi^2/2}$$ ← just a gaussian in ξ now!

$$= \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}}\, e^{-\frac{z^2}{2}}\, \frac{1}{\sqrt{1+i\sqrt{2\lambda}\, z/a}} = \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}}\, e^{-\frac{z^2}{2} - \frac{1}{2}\ln\left(1+i\frac{\sqrt{2\lambda}}{a}z\right)}$$

- We will use analogous versions of this "trick" to replace the "interaction" (meaning non-quadratic) terms in our integrals.
- We introduced the ln in "exponentiating" the result of the Gaussian integral over ξ. More generally when this is the Gaussian integral of a matrix we will get a determinant, which we exponentiate using the identity $\det A = e^{\mathrm{Tr} \ln A}$. [Note: No $1/\lambda$ out front now.]

- So now we've got a new integral to try expanding.
  - We get back the same old perturbative expansion if we just expand the ln in powers of λ:

$$e^{-\frac{1}{2}\ln\left(1+\sqrt{\frac{2\lambda}{a}}z\right)} = 1 - \frac{1}{2} i \sqrt{\frac{2\lambda}{a}}\, z - \frac{3}{8} z^2 \left(\frac{2\lambda}{a^2}\right) + i\frac{5}{16} z^3 \left(\frac{2\lambda}{a^2}\right)^{3/2} + \frac{35}{128} z^4 \left(\frac{2\lambda}{a^2}\right)^2 + O(\lambda^{5/2})$$

  and integrate term by term. The imaginary terms are odd in z and don't contribute.

- Instead, let's try the saddlepoint expansion.
  - We will not identify an overall factor analogous to $1/g$ in the exponent, but just apply the saddle point to

$$\boxed{S(z) = \frac{z^2}{2} + \frac{1}{2}\ln\left(1 + i\frac{\sqrt{2\lambda}}{a}z\right)}$$ with $g = 1$

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Calculate $S'(z_0)=0$ to find the zeros (which are potential minima)

$$\Rightarrow S'(z_0) = z_0 + \frac{i\sqrt{2\lambda/a^2}}{2(1+i\sqrt{2\lambda/a^2}\, z_0)} = 0 \quad \Rightarrow \text{zeros: } z_0^{(\pm)}$$

$$\Rightarrow \quad z_0^{(\pm)} = \left(\frac{-1\pm\sqrt{1+4\lambda/a^2}}{2i}\right)\frac{a}{\sqrt{2\lambda}}$$

as $\lambda \to 0$, these behave like

$$\boxed{i z_0^{(\pm)} \xrightarrow{\lambda\to 0} \begin{cases} \sqrt{\frac{\lambda}{2a^2}} \\ -a/\sqrt{2\lambda} \end{cases}}$$

· We'll expand about the $z_0^{(+)}$ root since the $z_0^{(-)}$ root makes $S_\lambda(z_0^{(-)}) \xrightarrow{\lambda\to 0} \infty$ [Is this correct to do?]

· So with $S^{(n)}(z_0^{(+)}) \equiv S_0^{(n)}$, we get the results:

$$\boxed{S_0 = -\frac{a^2}{16\lambda}\left(\sqrt{1+4\lambda/a^2}-1\right)^2 + \frac{1}{2}\ln\left(\frac{1+\sqrt{1+4\lambda/a^2}}{2}\right)} \checkmark$$

$$\boxed{S_0^{(2)} = 1 + \frac{4\lambda/a^2}{(1+\sqrt{1+4\lambda/a^2})^2} \geq 0} \; ? \qquad \boxed{S_0^{(3)} = \frac{-8i\left(\sqrt{2\lambda}/a\right)^3}{\left(1+\sqrt{1+4\lambda/a^2}\right)^3}} \checkmark$$

$$\boxed{S_0^{(4)} = \frac{-48\left(\sqrt{2\lambda}/a\right)^4}{\left(1+\sqrt{1+4\lambda/a^2}\right)^4}} \checkmark \qquad \boxed{S_0^{(n)} = O(\lambda^{n/2})} \qquad e^{-S_0}\frac{1}{\sqrt{S_0''}} \text{ together give } \frac{1}{(1+4\lambda)^{1/4}}$$

Plugging back into Z_λ, taking $a=1$ (for convenience), when the dust settles...

$$\boxed{Z_\lambda \doteq e^{\frac{1}{16\lambda}\left(\sqrt{1+4\lambda}-1\right)^2}\,\frac{1}{(1+4\lambda)^{1/4}}\left(1+\frac{6\lambda^2}{\left(1+4\lambda+\sqrt{1+4\lambda}\right)^2}+O(\lambda^3)\right)}$$

· This works better than perturbation theory and over a very large range!
· Even for large λ, where we would not expect great results, the error is only 10% asymptotically. (Could this be chance?)