PHYSICS 880.05 PROBLEM SET #5 SOLUTIONS

1. "Effective action in one dimension".

a) Our first goal is to derive an expression for the leading order action based on the effective action.

If we start with the partition function back on 227 in the notes, the only difference now is the space-time integration is d^2x and not d^4x:

$$\boxed{Z = \int \mathcal{D}(\psi^\dagger,\psi)\, e^{i\int d^2x \left[\psi^\dagger_\alpha\left(i\frac{\partial}{\partial t}+\frac{1}{2m}\frac{d^2}{dx^2}+\mu\right)\psi_\alpha - \frac{1}{2}C_0\left(\psi^\dagger_\alpha\psi_\alpha\right)^2\right]}} \quad (1)$$

and the introduction of the σ field goes through precisely the same. We again introduce $J(x)\sigma(x)$ and do the $\psi,\psi^\dagger$ integration:

$$\boxed{Z[J] = e^{iW[J]} = \int \mathcal{D}\sigma\, e^{g\,\mathrm{Tr}\ln G^{-1}(x,y)}\, e^{\frac{i}{8}C_0\int d^2x(\sigma(x))^2}\, e^{i\int d^2x J(x)\sigma(x)}} \quad (2)$$

- The Tr over space-time is now over 1+1 dimensions.

- The classical expectation value $\sigma_c(x)$ and the Legendre transformation follow as before:

$$\boxed{\sigma_c(x) \equiv \langle\Psi_0|\sigma(x)|\Psi_0\rangle_J = \frac{\delta W[J]}{\delta J(x)}} \quad (3)$$

$$\boxed{\Gamma[\sigma_c] \equiv W[J] - \int d^2x\, J(x)\sigma_c(x)} \quad (4)$$

When minimized, we have the usual result with $V \to L$ in 1d.

$$\boxed{\frac{\delta\Gamma[\sigma_c^0]}{\delta\sigma_c(x)} = 0} \quad (5) \Rightarrow \boxed{\Gamma[\sigma_c^0]\Big|_{J=0} = -LT\mathcal{E}} \quad (6)$$

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- We expand $\sigma(x) = \sigma_c + \eta(x)$ as in the notes and arrive at (exact same steps as in the notes!)

$$\boxed{\Gamma[\sigma_c] = \frac{g}{i}\,\mathrm{Tr}\ln\left[G_H^{-1}(x,y)\right] + \frac{C_0}{2}\int d^2x\,[\sigma_c(x)]^2 + \frac{i}{2}\mathrm{Tr}\ln\left[D_\sigma^{-1}(x,y)\right]} \quad (7)$$

with

$$\boxed{G_H^{-1}(x,y) = \left[i\frac{\partial}{\partial t} + \frac{1}{2m}\frac{d^2}{dx^2} + \mu - C_0\sigma_c(x)\right]\delta^2(x-y)} \quad (8)$$

and

$$\boxed{D_\sigma^{-1}(x,y) = C_0\delta^2(x-y) + i g C_0^2 G_H(y,x) G_H(x,y)} \quad (9)$$

- We've got a uniform system, so we evaluate $\mathrm{Tr}\ln[G_H^{-1}]$ by diagonalizing it in momentum/frequency space

$$\Rightarrow \boxed{\mathrm{Tr}\ln G_H^{-1} = LT\int\frac{dp}{2\pi}\int\frac{dp_0}{2\pi}\ln\left[p_0 - e_p + i\eta\,\mathrm{sgn}(|p|-k_F)\right]} \quad (10)$$

instead of using μ

with

$$\boxed{e_p = \frac{p^2}{2m} + C_0\sigma_c} \quad (11)$$

- We'll follow the notes and use the fact that with $C_0 = 0$,

$$\boxed{\mathrm{Tr}\ln G_H^{-1}\Big|_{C_0=0} = -LT\,\mathcal{E}_{\text{noninteracting}} = -LT\,\frac{1}{3}\frac{k_F^2}{2m}\rho} \quad (12)$$

$$\Rightarrow \Delta g\,\mathrm{Tr}\ln G_H^{-1} = g\int_0^1 d\lambda\,\frac{d}{d\lambda}\,LT\int\frac{dp}{2\pi}\int\frac{dp_0}{2\pi}\ln\left(p_0 - \frac{p^2}{2m} - \lambda C_0\sigma_c + i\eta\,\mathrm{sgn}(|p|-k_F)\right)$$

$$= g\,LT\int\frac{dp}{2\pi}\int_0^1 d\lambda\int_{-\infty}^{\infty}\frac{dp_0}{2\pi}\left(\frac{\theta(|p|-k_F)}{p_0 - e_p + i\eta} + \frac{\theta(k_F-|p|)}{p_0 - e_p - i\eta}\right)e^{ip_0\eta}\left(-C_0\sigma_c\right) \quad (13)$$

$$= -LT C_0\sigma_c\,g\int\frac{dp}{2\pi}\,i\theta(k_F-|p|) = -iLT\,C_0\sigma_c\,\rho$$

So up to this point, essentially no difference and

$$\boxed{\Gamma_{LO}[\sigma_c] = LT\left(-\frac{1}{3}\frac{k_F^2}{2m}\rho - C_0\sigma_c\rho + \frac{1}{2}C_0\sigma_c^2\right)} \quad (14)$$

$$\Rightarrow \boxed{\left.\frac{\delta\Gamma_{LO}}{\delta\sigma_c}\right|_{\sigma_c=\sigma_c^0} = 0} \; (15) \Rightarrow \boxed{\sigma_c^0 = \rho} \quad (16)$$

$$\Rightarrow \boxed{\mathcal{E}_{LO} = -\frac{1}{LT}\Gamma_{LO}[\sigma_c^0] = \rho\left(\frac{1}{3}\frac{k_F^2}{2m} + \frac{C_0}{2}\rho\right)} \quad (17)$$

In perturbation theory, e.g. from PS#3, we had

$$\boxed{\mathcal{E}_1^{pert} = \rho\left(\frac{1}{3}\frac{k_F^2}{2m} + \left(1-\frac{1}{g}\right)\frac{C_0}{2}\rho\right)} \quad (18)$$

which included both Hartree and Fock, while (17) has only the Hartree piece.

b) Continuing with the class notes on pg. 237,

$$\boxed{G_H(x_1, x_2) = \int\frac{d^2p}{(2\pi)^2}G_H(p)\,e^{-ip\cdot(x_1-x_2)}} \quad (19)$$

with

$$\boxed{G_H(p) = \frac{\Theta(|p|-k_F)}{p_0 - e_p + i\epsilon} + \frac{\Theta(k_F-|p|)}{p_0 - e_p - i\epsilon}} \quad (20)$$

$$\Rightarrow \Gamma_{NLO} = \frac{i}{2}\mathrm{Tr}\ln[D_0^{-1}] = \frac{i}{2}\mathrm{Tr}_{x_1x_2}\ln\left[\int\frac{d^2q}{(2\pi)^2}e^{-iq\cdot(x_1-x_2)}\left[-iC_0 + gC_0^2\int\frac{d^2p}{(2\pi)^2}G_H(p+q)G_H(p)\right]\right]$$

$$= \frac{i}{2}LT\int\frac{d^2q}{(2\pi)^2}\ln\left(-iC_0 + gC_0^2\int\frac{d^2p}{(2\pi)^2}G_H(p+q)G_H(p)\right) \quad (21)$$

and define

$$\boxed{\Pi_0(q) \equiv -ig\int\frac{d^2p}{(2\pi)^2}G_0(p+q)G_0(p) = -ig\int\frac{d^2p}{(2\pi)^2}G_H(p+q)G_H(p)} \quad (22)$$

We can still apply

$$\boxed{\int \frac{d^D k}{(2\pi)^D} k^n = 0} \quad (23)$$

and

$$\boxed{\int \frac{d^D k}{(2\pi)^D} \frac{k^2}{k\sqrt{k^2+\Lambda^2}} = \frac{\Gamma(-D/2)\Gamma(1/2+D/2)}{4\pi^{D/2}\Gamma(D/2)\Gamma(1/2)}(\Lambda^2)^{D/2}} \quad (24)$$

We use the λ trick once again:

$$\Gamma_{NLO}[\sigma_c] = \frac{i}{2}LT\int\frac{d^2q}{(2\pi)^2}\int_0^1 d\lambda \frac{d}{d\lambda}\left(\ln\left(-i\lambda C_0 + i\lambda^2 C_0^2 \Pi_0(q)\right)\right)$$

$$= \frac{i}{2}LT\int_0^1 d\lambda\int\frac{d^2q}{(2\pi)^2}\frac{1}{-i\lambda C_0 + i\lambda^2 C_0^2\Pi_0(q)}\left(-iC_0 + 2i\lambda C_0^2\Pi_0\right)$$

$$= -\frac{i}{2}LT\int_0^1\frac{d\lambda}{\lambda}\int\frac{d^2q}{(2\pi)^2}\frac{\lambda C_0\Pi_0(q)}{1-\lambda C_0\Pi_0(q)} \quad (25)$$

dropping $\int \frac{d^2q}{(2\pi)^2} 1 \to 0$ in DR, by (23),

To find $\mathcal{E}_{NLO}$, we need to minimize $\Gamma[\sigma_c]$ with respect to $\sigma_c \Rightarrow \sigma_{c_0}$, but this is trivial since Γ_{NLO} is independent of σ_c for the uniform system! Then

$$\boxed{\mathcal{E}_{NLO} = -\frac{\Gamma_{NLO}[\sigma_{c_0}]}{LT}} \quad (26)$$

Consider

$$\Pi_0(q) = -ig\int\frac{d^2p}{(2\pi)^2}G_0(p+q)G_0(p)$$

$$= -ig\int\frac{dp}{2\pi}\int_{-\infty}^{\infty}\frac{dp_0}{2\pi}\left(\frac{\theta(|p+q|-k_f)}{p_0+q_0-\epsilon_{p+q}+i\epsilon} + \frac{\theta(k_f-|p+q|)}{p_0+q_0-\epsilon_{p+q}-i\epsilon}\right)\left(\frac{\theta(|p|-k_f)}{p_0-\epsilon_p+i\epsilon} + \frac{\theta(k_f-|p|)}{p_0-\epsilon_p-i\epsilon}\right)$$

only cross terms contribute

$$= -ig\int\frac{dp}{2\pi}\, i\left[\frac{\theta(k_f-|p|)\theta(|p+q|-k_f)}{q_0+\epsilon_p-\epsilon_{p+q}+i\epsilon} - \frac{\theta(|p|-k_f)\theta(k_f-|p+q|)}{q_0+\epsilon_p-\epsilon_{p+q}-i\epsilon}\right]$$

$p' = -p-q$ in second term, then $p' \to p$

$$= g\int_{-\infty}^{\infty}\frac{dp}{2\pi}\,\theta(k_f-|p|)\left[\frac{\theta(|p+q|-k_f)}{q_0+\epsilon_p-\epsilon_{p+q}+i\epsilon} - \frac{\theta(|p+q|-k_f)}{q_0+\epsilon_{p+q}-\epsilon_p-i\epsilon}\right] \quad (27)$$

Note that the Θ functions are the same in each term.

We can evaluate $\Pi_0(q_0, q)$ by considering the real and imaginary parts separately and using

$$\boxed{\frac{1}{x \pm i\epsilon} = \mathbb{P}\frac{1}{x} \mp i\pi\delta(x)} \quad (28)$$

and some geometry. This translates to 1-D the discussion in Fetter and Walecka, sect. 12.

Let $$\boxed{\epsilon_{pq} \equiv \epsilon_{p+q} - \epsilon_p = \frac{1}{2m}\left[(p+q)^2 - p^2\right] = \frac{1}{m}\left(pq + \tfrac{1}{2}q^2\right)} \quad (29)$$

Then

$\Theta(z) = 1 - \Theta(-z)$

$$\boxed{\begin{aligned} \mathrm{Re}\,\Pi_0(q_0, q) &= g\,\mathbb{P}\int_{-\infty}^{\infty} \frac{dp}{2\pi}\,\Theta(k_F - |p|)\left(1 - \Theta(k_F - |p+q|)\right) \frac{2\epsilon_{pq}}{q_0^2 - \epsilon_{pq}^2} \\ &= g\,\mathbb{P}\int_{-\infty}^{\infty} \frac{dp}{2\pi}\,\Theta(k_F - |p|)\,\frac{2\epsilon_{pq}}{q_0^2 - \epsilon_{pq}^2} \end{aligned}} \quad (30)$$

since ϵ_{qk} is odd under $p \leftrightarrow p+q$ while the product of Θ functions is even

$$\boxed{\begin{aligned} \Rightarrow \mathrm{Re}\,\Pi_0(q_0, q) &= g\,\mathbb{P}\int_{-k_F}^{k_F} \frac{dp}{2\pi}\,\frac{2\epsilon_{pq}}{q_0^2 - \epsilon_{pq}^2} = g\,\mathbb{P}\int_{-k_F}^{k_F} \frac{dp}{2\pi}\left(\frac{1}{q_0 - \epsilon_{pq}} - \frac{1}{q_0 + \epsilon_{pq}}\right) \\ \text{mathematica} \quad &= \frac{gm}{\pi q}\left(\mathrm{Tanh}^{-1}\left[\frac{mq_0}{qk_F} + \frac{1}{2}\frac{q}{k_F}\right] - \mathrm{Tanh}^{-1}\left[\frac{mq_0}{qk_F} - \frac{1}{2}\frac{q}{k_F}\right]\right) \\ &= \frac{gm}{2\pi q}\left[\ln\left|\frac{1 + \frac{mq_0}{qk_F} + \frac{1}{2}\frac{q}{k_F}}{1 - \frac{mq_0}{qk_F} - \frac{1}{2}\frac{q}{k_F}}\right| - \ln\left|\frac{1 + \frac{mq_0}{qk_F} - \frac{1}{2}\frac{q}{k_F}}{1 - \frac{mq_0}{qk_F} + \frac{1}{2}\frac{q}{k_F}}\right|\right] \end{aligned}} \quad (31)$$

For the imaginary part, we have

$$\boxed{\mathrm{Im}\,\Pi_0(q_0,q) = -g\int_{-\infty}^{\infty}\frac{dp}{2\pi}\,\theta(k_f-|p|)\,\theta(|p+q|-k_f)\left[\delta(q_0-\epsilon_{pq})+\delta(q_0+\epsilon_{pq})\right]} \quad (32)$$

This is proportional to the absorption probability for transferring (q_0, q) to a free Fermi gas.

- We need only consider $q_0>0$ (symmetric or even in q_0), which means only the first δ-function is relevant,

$$\boxed{\delta(q_0-\epsilon_{pq}) = \delta\left(\frac{q}{m}\left(p+\frac{1}{2}q-\frac{q_0 m}{q}\right)\right) = \frac{m}{q}\,\delta\left(p-\left(\frac{q_0 m}{q}-\frac{1}{2}q\right)\right)} \quad (33)$$

So we can just do the integral using the δ function.

$$\Rightarrow \boxed{\mathrm{Im}\,\Pi_0(q_0,q) = \begin{cases} -\dfrac{gm}{2\pi q} & \left|\dfrac{q_0 m}{q}-\dfrac{1}{2}q\right|\le k_f \text{ and } \left|\dfrac{q_0 m}{q}+\dfrac{1}{2}q\right|\ge k_f \\ 0 & \text{otherwise}\end{cases}} \quad (34)$$

c) In the Bose limit, $g\to\infty$, $k_f\to 0$, and $\rho=\frac{g k_f}{\pi}$ is fixed,

$$\Rightarrow \boxed{\mathcal{E}_{L0}\to \tfrac{1}{2}C_0\rho^2} \quad (35)$$

The kinetic energy vanishes, since in this limit we have a Bose condensate $\Rightarrow$ zero momentum fermions,

We can simplify Π_0 from (27)

$$\boxed{\Pi_0(q_0,q)\xrightarrow[k_f\to 0]{} g\int\frac{dp}{2\pi}\,\theta(k_f-p)\left[\frac{1}{q_0-\epsilon_q+i\varepsilon}-\frac{1}{q_0+\epsilon_q-i\varepsilon}\right] = \frac{2\epsilon_q}{(q_0-\epsilon_q+i\varepsilon)(q_0+\epsilon_q-i\varepsilon)}} \quad (36)$$

So

$$\boxed{\mathcal{E}_2^{Bose} = -\frac{\Gamma_{NLO}}{LT} = \frac{\rho}{2}\int_0^1 d\lambda\, C_0 \int \frac{dp}{2\pi}\frac{\epsilon_p}{\tilde{\epsilon}_p} = \frac{\rho}{2}C_0\int_0^1 d\lambda \int \frac{dp}{2\pi}\frac{p^2}{p\sqrt{p^2+4m\rho\lambda C_0}}} \quad (37)$$

with $\boxed{\tilde{\epsilon}_p \equiv \sqrt{\epsilon_p^2 + 2\epsilon_p \rho C_0 \lambda}}$ (38)

Applying the DR formula

$$\boxed{\begin{aligned}\int_{-\infty}^{\infty}\frac{dp}{2\pi}\frac{p^2}{p\sqrt{p^2+4m\rho C_0\lambda}} &= \frac{\Gamma(-1/2)\Gamma(1)}{\sqrt{4\pi}\,\Gamma(1/2)\Gamma(1/2)}\sqrt{4m\rho C_0\lambda} \\ &= -\sqrt{\frac{4m\rho C_0}{\pi^2}}\,\lambda^{1/2}\end{aligned}} \quad (39)$$

and $\boxed{\int_0^1 d\lambda\, \lambda^{1/2} = \frac{2}{3}}$ (40)

$$\Rightarrow \boxed{\mathcal{E}_2^{Bose} = -\frac{2}{3\pi}\left(m\rho^3 C_0^3\right)^{1/2}} \quad (41)$$

2. "Number fluctuations in the BCS ground state."

The number operator is

$$\boxed{\sum_{k\alpha} n_{k\alpha} = \sum_{k\alpha} a^\dagger_{k\alpha} a_{k\alpha} = \sum_k \left(a^\dagger_{k\uparrow} a_{k\uparrow} + a^\dagger_{k\downarrow} a_{k\downarrow}\right)} \quad (1)$$

We'll write these in terms of

$$\boxed{\alpha_k = u_k a_{k\uparrow} - v_k a^\dagger_{-k\downarrow}} \quad (2) \qquad \boxed{\beta_{-k} = u_k a_{-k\downarrow} + v_k a^\dagger_{k\uparrow}} \quad (3)$$

because $\boxed{\alpha_k|BCS\rangle = \beta_k|BCS\rangle = 0}$ (4)

Since

$$\boxed{a_{k\uparrow} = u_k \alpha_k + v_k \beta^\dagger_{-k}} \quad \text{and} \quad \boxed{a_{-k\downarrow} = u_k \beta_{-k} - v_k \alpha^\dagger_k}$$

$$\sum_{k\alpha} n_{k\alpha} = \sum_k \left[(u_k \alpha^\dagger_k + v_k \beta_{-k})(u_k \alpha_k + v_k \beta^\dagger_{-k}) + (u_k \beta^\dagger_{-k} - v_k \alpha_k)(u_k \beta_{-k} - v_k \alpha^\dagger_k)\right]$$

$$\Rightarrow \boxed{\begin{aligned} \langle \hat N\rangle &= \langle BCS|\hat N|BCS\rangle = \sum_k \langle BCS|\sum_\alpha n_{k\alpha}|BCS\rangle \\ &= \sum_k v_k^2 \langle BCS|\beta_{-k}\beta^\dagger_{-k} + \alpha_k \alpha^\dagger_k|BCS\rangle = \sum_k v_k^2 \end{aligned}}$$

(since $\langle \beta_{-k}\beta^\dagger_{-k}\rangle = \langle \alpha_k \alpha^\dagger_k\rangle = 1$)

Then $\langle \hat N^2\rangle = \sum_k \sum_{k'} \langle BCS|\sum_{\alpha\alpha'} n_{k\alpha} n_{k'\alpha'}|BCS\rangle$

$$= \sum_k \sum_{k'} \langle BCS|\left[v_k \beta_{-k}(u_k \alpha_k + v_k \beta^\dagger_{-k}) + v_k \alpha_k (u_k \beta_{-k} - v_k \alpha^\dagger_k)\right] \left[(u_{k'} \alpha^\dagger_{k'} + v_{k'} \beta_{-k'}) v_{k'} \beta^\dagger_{-k'} + (u_{k'} \beta^\dagger_{-k'} - v_{k'} \alpha_{k'}) v_{k'} \alpha^\dagger_{k'}\right]|BCS\rangle$$

$$= \sum_k \sum_{k'} \langle BCS|\left(2v_k^2 + u_k v_k [\beta_{-k}, \alpha_k]\right)\left(2v_{k'}^2 + u_{k'} v_{k'} [\alpha^\dagger_{k'}, \beta^\dagger_{-k'}]\right)|BCS\rangle$$

all cross terms vanish, and α's, β's anticommute

$$= \left(\sum_k 2v_k^2\right)\left(\sum_{k'} 2v_{k'}^2\right) + 4\sum_{kk'} u_k v_k u_{k'} v_{k'} \langle BCS|(\beta_{-k}\alpha_k)(\alpha^\dagger_{k'} \beta^\dagger_{-k'})|BCS\rangle$$

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The first term is just $(\langle\hat{N}\rangle)^2$ and the second requires $\delta_{kk'}$ to be nonzero and then yields a factor of 4 since α's and β's anticommute

$$\Rightarrow \langle\hat{N}^2\rangle = (\langle\hat{N}\rangle)^2 + 4\sum_k (U_k V_k)^2$$

$$\Rightarrow \boxed{\frac{\langle\hat{N}^2\rangle - \langle\hat{N}\rangle^2}{\langle\hat{N}\rangle^2} = \frac{\sum_k (U_k V_k)^2}{\left(\sum_k V_k^2\right)^2}}$$ as desired.

b) $|F\rangle$ is an eigenstate of $\hat{N}$, so $\hat{N}^2|F\rangle = N^2|F\rangle$
or $\langle\hat{N}^2\rangle - \langle\hat{N}\rangle^2 = N^2 - N^2 = 0$
$\Rightarrow$ no fluctuations!

c) If we calculate ^{202}Pb, we will mix in components of other nearby (even-even) nuclei such as ^{198}Pb, ^{200}Pb, ^{204}Pb and ^{206}Pb. If we want ^{202}Pb only, we would need to project this component.

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3. "Another BCS ground state problem".

a) To compare $a^\dagger_{k\uparrow}|BCS\rangle$ and $a_{-k\downarrow}|BCS\rangle$, it is again useful to write the a's in terms of α's and β's. Then

$$\boxed{a^\dagger_{k\uparrow}|BCS\rangle = (u_k\alpha_k^\dagger + v_k\beta_{-k})|BCS\rangle = u_k\alpha_k^\dagger|BCS\rangle}$$

while

$$\boxed{a_{-k\downarrow}|BCS\rangle = (u_k\beta_{-k} - v_k\alpha_k^\dagger)|BCS\rangle = -v_k\alpha_k^\dagger|BCS\rangle}$$

so both states are equal to $\alpha_k^\dagger|BCS\rangle$, up to normalization. Also $\langle BCS|\alpha_k^\dagger = 0$, so both are orthogonal to $|BCS\rangle$.

b) Since U is a c-number and $\hat{H}_1 = \sum_k E_k(\alpha_k^\dagger\alpha_k + \beta_k^\dagger\beta_k)$,

$\Delta\langle\hat{K}\rangle$ is simply $\Delta\langle\hat{H}_1\rangle = \langle\hat{H}_1\rangle$ since $\langle BCS|\hat{H}_1|BCS\rangle = 0$.

$$\Rightarrow \Delta\langle\hat{K}\rangle = \frac{\sum_{k'}\langle BCS|\alpha_k E_{k'}(\alpha_{k'}^\dagger\alpha_{k'} + \beta_{k'}^\dagger\beta_{k'})\alpha_k^\dagger|BCS\rangle}{\langle BCS|\alpha_k\alpha_k^\dagger|BCS\rangle}$$

$$= \sum_{k'} E_{k'}\delta_{kk'} = E_k \quad \text{as desired}$$

4. Feynman-Hellmann proof.

a) We have that

$$\left(-\frac{\nabla^2}{2m} + C_0 \sigma_c(\vec{x})\right) \varphi_\alpha(\vec{x}) = \epsilon_\alpha \varphi_\alpha(\vec{x})$$

and $\int |\varphi_\alpha(\vec{x})|^2 d^3x = 1$

Left multiply the first by $\varphi^\dagger_\alpha(\vec{x})$ and integral over $\vec{x}$:

$$\Rightarrow \epsilon_\alpha = \int d^3x' \, \varphi^\dagger_\alpha(\vec{x}') \left(-\frac{\nabla^2}{2m} + C_0 \sigma_c(\vec{x}')\right) \varphi_\alpha(\vec{x}')$$

$$\Rightarrow \frac{\delta \epsilon_\alpha}{\delta \sigma_c(\vec{x})} = \int d^3x' \, \frac{\delta \varphi^\dagger_\alpha(\vec{x}')}{\delta \sigma_c(\vec{x})} \underbrace{\left(-\frac{\nabla^2}{2m} + C_0 \sigma_c(\vec{x}')\right) \varphi_\alpha(\vec{x}')}_{\epsilon_\alpha \varphi_\alpha(\vec{x}')}$$

$$+ \; C_0 \varphi^*_\alpha(\vec{x}) \varphi_\alpha(\vec{x})$$

$$+ \int d^3x' \, \underbrace{\varphi^*_\alpha(\vec{x}') \left(-\frac{\nabla^2}{2m} + C_0 \sigma_c(\vec{x}')\right)}_{\epsilon_\alpha \varphi^\dagger_\alpha(\vec{x}')} \frac{\delta \varphi_\alpha(\vec{x}')}{\delta \sigma_c(\vec{x})}$$

$$= C_0 \varphi^*_\alpha(\vec{x}) \varphi_\alpha(\vec{x}) + \epsilon_\alpha \int d^3x' \left[\frac{\delta \varphi^\dagger_\alpha(\vec{x}')}{\delta \sigma_c(\vec{x})} \varphi_\alpha(\vec{x}') + \varphi^\dagger_\alpha(\vec{x}') \frac{\delta \varphi_\alpha(\vec{x}')}{\delta \sigma_c(\vec{x})} \right]$$

$$= C_0 \varphi^\dagger_\alpha(\vec{x}) \varphi_\alpha(\vec{x}) + \epsilon_\alpha \frac{\delta}{\delta \sigma_c(\vec{x})} \underbrace{\int d^3x' \, \varphi^\dagger_\alpha(\vec{x}') \varphi_\alpha(\vec{x}')}_{1}$$

$$= C_0 \varphi^\dagger_\alpha(\vec{x}) \varphi_\alpha(\vec{x}) \qquad \text{QED.}$$

b) The proof in a) shows that $\frac{\delta \phi_\alpha(\vec{x}')}{\delta \sigma_c(\vec{x})}$ does not need to be zero. The normalization implies that the variations of $\phi_\alpha(\vec{x}')$ and $\phi^\dagger_\alpha(\vec{x}')$ cancel.

- Changing $\sigma_c(\vec{x})$ certainly changes the eigenstate $\phi_\alpha(\vec{x})$, so the individual functional derivatives are not zero.