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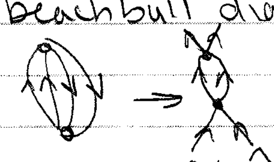
<u>Monday 880.05 Class</u>

- Questions on PS#3?

- Recap from last time:
  - beachball diagram $\propto \int d^3s\, d^3t\, d^3u \frac{1}{u^2-t^2}\,\theta(1-|\vec{s}\pm\vec{t}|)\,\theta(|\vec{s}\pm\vec{u}|-1)$ $\rightarrow \infty$!



divergence originates in free space $\Rightarrow$ consider scattering

$|\vec{k}| = |\vec{k}'|$

Scattering amplitude

convenient normalization

$$f(k,\theta) = \sum_{l=0}^{\infty} \frac{2l+1}{k\cot\delta_l - ik} P_l(\cos\theta) \equiv \frac{M}{4\pi} T(k,\cos\theta)$$

- effective range expansion $k^{2l+1}\cot\delta_l(k)$ has power series

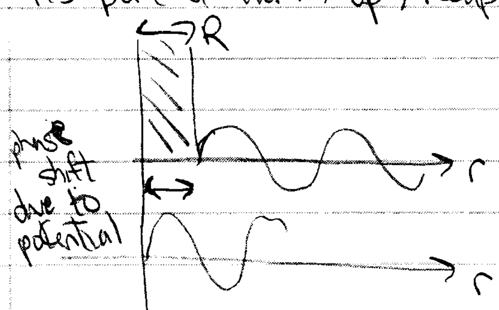
$$k\cot\delta_0(k) = -\frac{1}{a_s} + \frac{1}{2} r_s k^2 + \ldots \qquad k^3\cot\delta_1(k) = -\frac{3}{a_p^3} + \ldots$$

- Go over (164)-(169) Want to reproduce $k \ll \Lambda$:

$$T(k,\cos\theta) = -\frac{4\pi a_s}{M}\left[1 - i a_s k + (a_s r_s/2 - a_s^2)k^2\right] - \frac{4\pi}{M} a_p^3 k^2 \cos\theta + O(k^3/\Lambda^3)$$

- Then continue from (171), moving as quickly as possible without losing the class!

- As part of warm-up / recap, consider hard sphere scattering



equate

$$\sin(k(r-R)) = \sin(kr + \delta(k))$$

$$\Rightarrow \delta(k) = -kR$$

Tell mathematica to expand

$$k\cot(-kR) = -\frac{1}{R} + \frac{1}{3}Rk^2 + \frac{1}{45}R^3k^4 + O(k^6)$$

$$\Rightarrow a_s = R,\quad r_s = \frac{2}{3}R.$$

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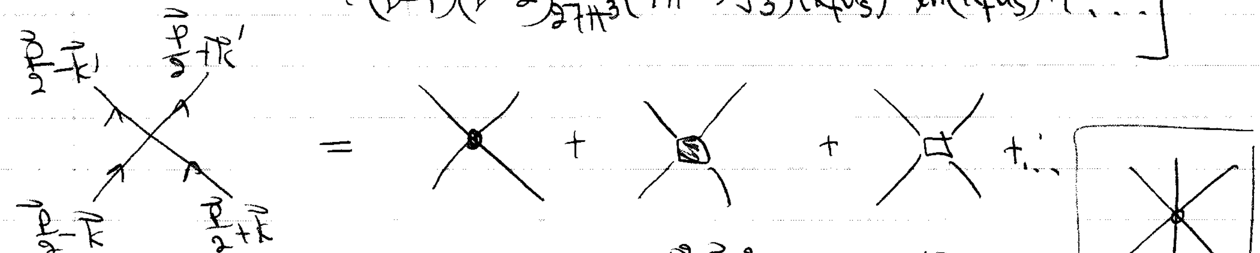
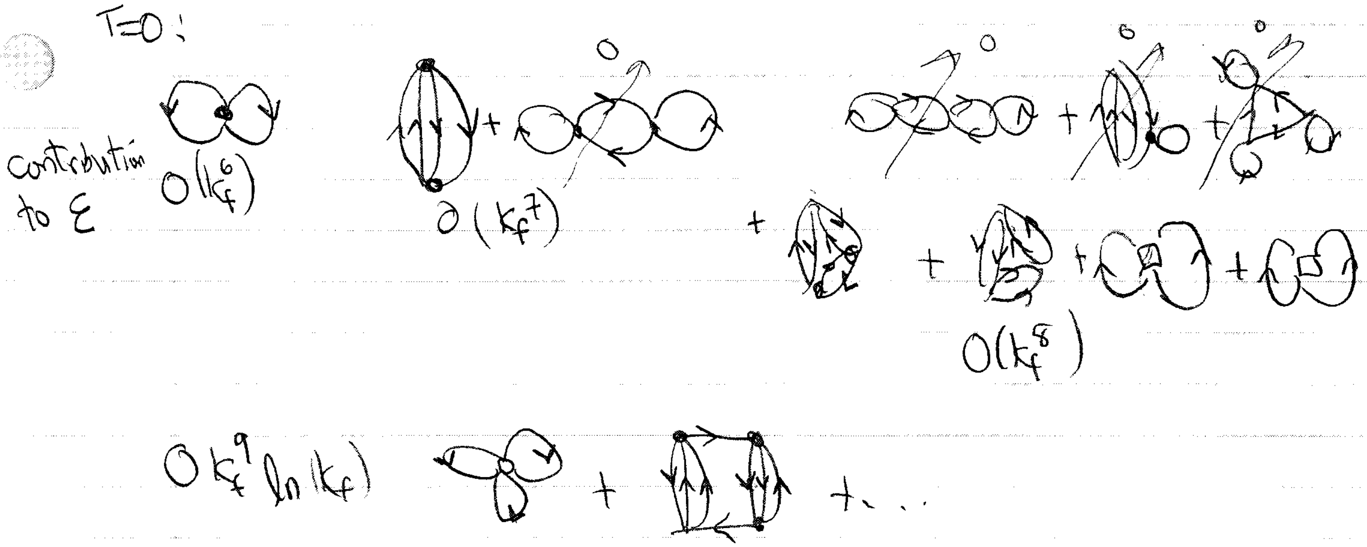
Result for left blackboard: (dilute Fermi system, natural $a_s > 0$)

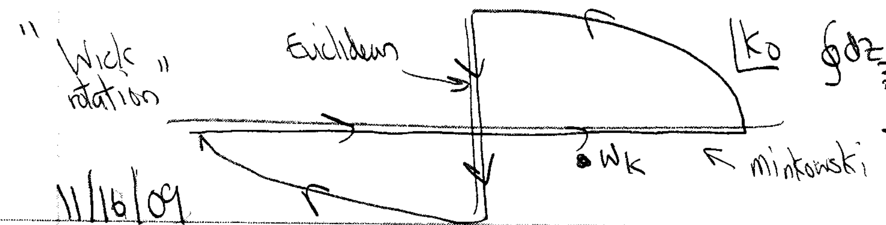
$$\frac{\mathcal{E}}{\rho} = \frac{E}{N} = \frac{k_f^2}{2m}\Big[\frac{3}{5} + (\nu-1)\Big\{\frac{2}{3\pi}(k_f a_s) + \frac{4}{35\pi^2}(11-2\ln 2)(k_f a_s)^2 + \frac{1}{10\pi}(k_f r_s)(k_f a_s)^2$$
$$+ (0.076 + 0.057(\nu-3))(k_f a_s)^3\Big\} + (\nu+1)\frac{1}{5\pi}(k_f a_p)^3$$
$$+ (\nu-1)(\nu-2)\frac{16}{27\pi^3}(4\pi - 3\sqrt{3})(k_f a_s)^4 \ln(k_f a_s) + \ldots\Big]$$



Minkowski: $-i\langle \vec{k}'|V_{EFT}|\vec{k}\rangle$ $\quad -iC_0 \quad -iC_2\frac{\vec{k}^2+\vec{k}'^2}{2} \quad -iC_2'\,\vec{k}\cdot\vec{k}' \quad -iD_0$

$T=0$:





$$\oint dz \frac{f(z)}{z-w_k} = 0 = \int_{-\infty}^{\infty} dk_0 \frac{f(k_0)}{k_0-w_k} - i\int_{-\infty}^{\infty} dk_0 \frac{f(-ik_0)}{ik_0-w_k} + \text{arcs} \quad (174)$$

$\uparrow$ switch between

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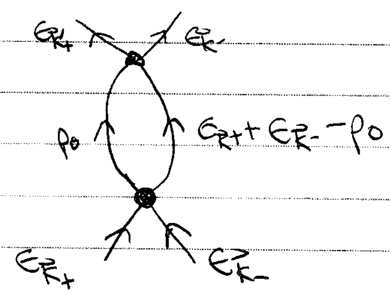
Now for (b) — we'll combine the Feynman rules:

- we have one important addition to our diagram: frequencies.
- The incoming frequencies correspond to "on-shell" free particles, e.g. $w_k = k^2/2m$. Conserving frequency at the vertices. $\leftarrow$ but Euclidean version would have i's in front



- Note that the sum of frequencies in to the lower vertex equals the sum out:

$$\epsilon_{\vec{k}_+} + \epsilon_{\vec{k}_-} = p_0 + (\epsilon_{\vec{k}_+} + \epsilon_{\vec{k}_-} - p_0)$$
$$= (\tfrac{\vec{P}}{2}+\vec{k})^2/2m + (\tfrac{\vec{P}}{2}-\vec{k})^2/2m$$
$$= P^2/4 + \vec{k}^2/m$$

- When we apply the Feynman rule for $iG^0_{\alpha\beta}(k_i)$, we are in free space $\Rightarrow$ take $k_F = 0 \Rightarrow$

$$\boxed{iG^0_{\alpha\beta}(k_i) = i\delta_{\alpha\beta}\frac{1}{k_{i_0} - \epsilon^0_{k_i} + i\varepsilon}}$$

cf. $\mathcal{G}^0_{\alpha\beta}(k) = \dfrac{-\delta_{\alpha\beta}}{ik_0 - (\epsilon^0_k - \cancel{\mu_0})}$ $\leftarrow$ 0 at zero density

and put i, in front of frequencies

- other factors of i are also present.

$$\Rightarrow \boxed{\frac{i}{2}(\delta_{\alpha_1\beta_3}\delta_{\alpha_2\beta_4} + \delta_{\alpha_1\beta_4}\delta_{\alpha_2\beta_3})(\delta_{\alpha_3\beta_1}\delta_{\alpha_4\beta_2} + \delta_{\alpha_3\beta_2}\delta_{\alpha_4\beta_1}) \times (-iC_0^{(0)})^2 \int\frac{d^3p}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{dp_0}{2\pi}\left(\frac{i\delta_{\alpha_3\beta_3}}{p_0 - \epsilon_{p_+} + i\varepsilon}\right)\left(\frac{i\delta_{\alpha_4\beta_4}}{\epsilon_{k+}+\epsilon_{k-} - p_0 - \epsilon_{p-} + i\varepsilon}\right)}$$

symmetry factor (1 pair equivalent lines)

- The spin sum just yields $2(\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2} + \delta_{\alpha_1\beta_2}\delta_{\alpha_2\beta_1})$. This 2 cancels the symmetry factor $\Rightarrow$ this will happen to all orders.

- We can close in the lower half plane $\Rightarrow -2\pi i \times$ residue at $p_0 = \epsilon_{p_+}$

$$\Rightarrow \boxed{= (C_0^{(0)})^2 \int\frac{d^3p}{(2\pi)^3}\frac{1}{\epsilon_{k+}+\epsilon_{k-} - (\epsilon_{p+}+\epsilon_{p-}) + i\varepsilon} = (C_0^{(0)})^2\int\frac{d^3p}{(2\pi)^3}\frac{M}{k^2 - p^2 + i\varepsilon}}$$

$\leftarrow$ looks infinite!

This is the "VG_0V" part alluded to earlier!

This is important since no restriction on p.

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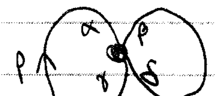
Check Minkowski vs. Euclidean versions.



⇐ contribution to energy density E/V at $T=0$

symmetry factor has i

same spin factors

Minkowski	Euclidean
$\frac{i}{2}(\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma})\delta_{\alpha\gamma}\delta_{\beta\delta}$	$\frac{1}{2}(\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma})\delta_{\alpha\gamma}\delta_{\beta\delta}$
$\times(-iC_0)$	$\times(C_0)$ (vertex factor of i or $-i$) e^{iS} vs. e^{-S_E}
$\times\int\frac{d^4p}{(2\pi)^4}\, i\left[\frac{\theta(\lvert\vec{p}\rvert-k_F)}{p_0-\epsilon_p^0+i\varepsilon}+\frac{\theta(k_F-\lvert\vec{p}\rvert)}{p_0-\epsilon_p^0-i\varepsilon}\right]e^{ip_0\eta}$	$\times\int\frac{d^4p}{(2\pi)^4}\left[\frac{-1}{ip_0-(\epsilon_p^0-\mu_0)}e^{ip_0\eta}\right]$
$\times\int\frac{d^4p'}{(2\pi)^4}\, i\left[\frac{\theta(\lvert\vec{p}'\rvert-k_F)}{p_0'-\epsilon_{p'}^0+i\varepsilon}+\frac{\theta(k_F-\lvert\vec{p}'\rvert)}{p_0'-\epsilon_{p'}^0-i\varepsilon}\right]e^{ip_0'\eta}$	$\times\int\frac{d^4p'}{(2\pi)^4}\left[\frac{-1}{ip_0'-(\epsilon_{p'}^0-\mu_0)}e^{ip_0'\eta}\right]$

$$= \frac{C_0}{2}\nu(\nu-1)\left[\int\frac{d^3p}{(2\pi)^3}\theta(k_F-\lvert\vec{p}\rvert)\right]\left[\int\frac{d^3p'}{(2\pi)^3}\theta(\mu_0-\epsilon_{p'}^0)\right]$$

These are the same; use either!

$$= \frac{C_0}{2}\left(1-\frac{1}{\nu}\right)\rho^2 \quad \text{where } \rho=\frac{\nu k_F^3}{6\pi^2},\ \mu_0=\frac{k_F^2}{2m}$$

- Note that we are just taking $p_0 \to ip_0$, as implied by the Wick rotation.

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Review: Contour Integrals

- We will very often find ourselves doing integrals over frequency (rather than time).
  - At zero temperature, these are integrals from $-\infty$ to $+\infty$, which are usually evaluated as contour integrals.
  - At finite temperature, the "time" (really τ) integration is over a finite interval (e.g. 0 to β), which leads to a frequency sum rather than integral. These require additional techniques, which we won't describe here.

- So let's review by example the sort of contour integral we need.

- The lowest order energy:  will involve this integral:

$$\boxed{\int_{-\infty}^{\infty} \frac{dk_0}{2\pi}\left[\underbrace{\frac{\Theta(|\vec{k}|-k_F)}{k_0-\omega_k+i\epsilon}}_{\text{Ⓐ}}+\underbrace{\frac{\Theta(k_F-|\vec{k}|)}{k_0-\omega_k-i\epsilon}}_{\text{Ⓑ}}\right]e^{ik_0\eta}=\frac{1}{2\pi}(2\pi i)\,\Theta(k_F-|\vec{k}|)}$$

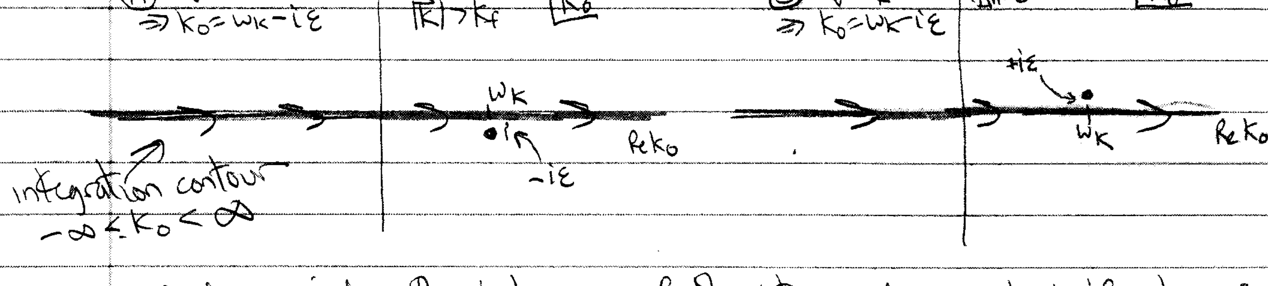
(Note $\omega_k \equiv \epsilon_k^0$ here)

let's "deconstruct" this result.

- First, identify where the poles are in the complex k_0 plane

Ⓐ $k_0-\omega_k+i\epsilon=0$ $\Rightarrow k_0=\omega_k-i\epsilon$

Ⓑ $k_0-\omega_k+i\epsilon=0$ $\Rightarrow k_0=\omega_k-i\epsilon$



- 2nd, consider the behavior of the integrand in each half plane:

upper half plane: $\boxed{\text{Im } k_0>0 \Rightarrow e^{ik_0\eta}=e^{i(\text{Re } k_0)\eta}\,e^{-(\text{Im } k_0)\eta}\xrightarrow[|k_0|\to\infty]{}0}$

lower half plane: $\boxed{\text{Im } k_0<0 \Rightarrow e^{ik_0\eta}=e^{i(\text{Re } k_0)\eta}\,e^{+|\text{Im } k_0|\eta}\xrightarrow[|k_0|\to\infty]{}\infty}$

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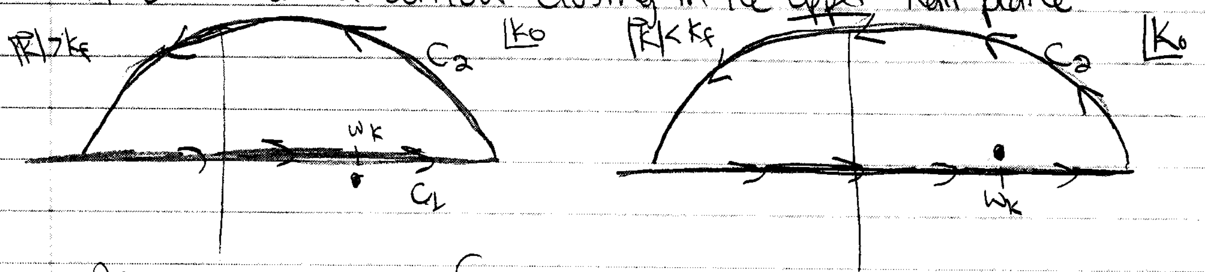
- 3rd, based on these results extends the integration to a <u>closed</u> <u>contour</u> (to which we can apply Cauchy's magic!)

Here: Draw a contour closing in the <u>upper</u> half plane



The contour integral $\oint_{C_1+C_2}$ is given by

$$\boxed{\oint_{C_1+C_2} \frac{dk_0}{2\pi}\left(\frac{\theta(|\vec{k}|-k_f)}{k_0-\omega_k+i\varepsilon}+\frac{\theta(k_f-|\vec{k}|)}{k_0-\omega_k-i\varepsilon}\right)e^{ik_0\eta} = 2\pi i \times \left(\begin{array}{c}\text{sum of residues}\\ \text{at } \underline{\text{enclosed}} \text{ poles}\end{array}\right)}$$

$+2\pi i$ because counterclockwise contour
($-2\pi i$ if clockwise contour)

But this integral is also the sum of the integral we want (contour C_1) and one we can evaluate separately (C_2).

- In most case we will consider, the integral on C_2 will <u>vanish</u>. Here it does because of the $e^{ik_0\eta}$ factor.

- 4th evaluate the residues.

$\theta(|\vec{k}|-k_f) \Rightarrow$ no poles $\Rightarrow \oint(\ \ ) = 0.$

$\theta(k_f-|\vec{k}|) \Rightarrow$ residue is $\frac{\theta(k_f-|\vec{k}|)}{2\pi}$ $\Rightarrow \boxed{\oint_{C_1+C_2}(\ \ ) = \frac{1}{2\pi}(2\pi i)\,\theta(k_f-|\vec{k}|)}$

so the final result is

$$\boxed{\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\Big[\qquad\qquad\Big]e^{ik_0\eta} = i\,\theta(k_f-|\vec{k}|)}$$ as advertised.

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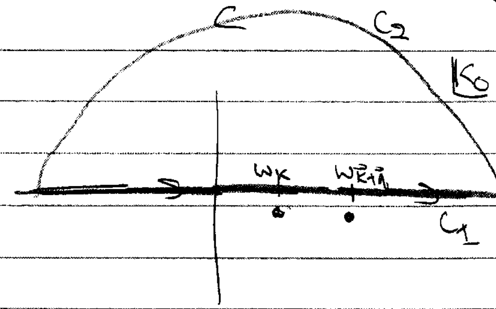
Now suppose we have

$$\boxed{\int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \frac{\theta(|\vec{k}|-k_F)}{k_0-\omega_k+i\varepsilon} \frac{\theta(|\vec{k}+\vec{q}|-k_F)}{k_0-\omega_{\vec{k}+\vec{q}}+i\varepsilon}}$$



① The poles are in the lower half plane, as shown.

② The integral converges in *either* half plane (by this we mean that the half-circle integral C_2 as $|k_0|\to\infty$ is finite $\Rightarrow$ vanishes here)

for $|k_0|$ large: $$\boxed{\oint_{C_2} \frac{dk_0}{2\pi}\frac{1}{k_0^2} \xrightarrow{k_0=r_0e^{i\theta}} \int_0^{\pi} \frac{1}{2\pi}\frac{r_0e^{i\theta}\,d\theta}{r_0^2e^{2i\theta}} = \frac{1}{2\pi r_0}\int_0^{\pi} d\theta\, e^{-i\theta} \propto \frac{(\text{constant})}{r_0} \xrightarrow[r_0\to\infty]{} 0}$$

③ Since we can close in the upper half plane, choose that one (as in the figure) because there are no poles enclosed.

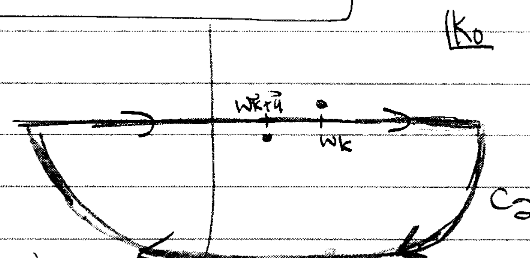
④ no residues $\Rightarrow$ $\boxed{\text{integral is zero}}$!

- If we wanted to work hard to get the same answer, we could close in the lower half plane

$$\Rightarrow \boxed{\frac{1}{2\pi}(-2\pi i)\left(\frac{1}{\omega_{|\vec{k}+\vec{q}|}-\omega_k} + \frac{1}{\omega_k-\omega_{\vec{k}+\vec{q}}}\right) = 0}$$

Finally, consider

$$\boxed{\int_{-\infty}^{\infty} \frac{dk_0}{2\pi} \frac{\theta(k_F-|\vec{k}|)}{k_0-\omega_k-i\varepsilon} \frac{\theta(|\vec{k}+\vec{q}|-k_F)}{k_0-\omega_{\vec{k}+\vec{q}}+i\varepsilon}}$$



① poles in *opposite* half planes, as shown

② converges in *either* half plane

③ choose to close in lower

④ $$\Rightarrow \boxed{\frac{1}{2\pi}\underset{\text{note!}}{(-2\pi i)} \frac{1}{\omega_{\vec{k}+\vec{q}}-\omega_k-i\varepsilon}\,\theta(k_F-|\vec{k}|)\,\theta(|\vec{k}+\vec{q}|-k_F)}$$ That's it!

- You should check that the same result is obtained when we choose to close in the upper half plane.

° note that we can drop the $i\varepsilon$ for $q>0$

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- Now we have to make explicit our regularization procedure, since otherwise this expression doesn't make sense (i.e. it is infinite)

i) Cutoff regulator. We can apply a momentum cutoff in various ways. For example,

- add it to $\langle \vec{k}'|V|\vec{k}\rangle \longrightarrow C_0 e^{-k^2/2\Lambda_c^2} e^{-k'^2/2\Lambda_c^2}$

  or $\longrightarrow C_0 e^{-(\vec{k}-\vec{k}')^2/2\Lambda_c^2}$

  (note that these do not contribute to <u>leading</u> order in the momentum expansion.)
- simply cut off the integral with a sharp cutoff
  $\Rightarrow$ we'll use this one for simplicity.

$$\Rightarrow \boxed{\begin{aligned}(C_0^{(0)})^2 \frac{M}{2\pi^2}\int_0^{\Lambda_c} dp\, \frac{p^2}{k^2-p^2+i\epsilon} &= -(C_0^{(0)})^2\frac{M}{2\pi^2}\int_0^{\Lambda_c} dp + (C_0^{(0)})^2 \frac{Mk^2}{2\pi^2}\int_0^{\Lambda_c}\frac{dp}{k^2-p^2+i\epsilon} \\ &= -(C_0^{(0)})^2 \frac{M}{2\pi^2}\Lambda_c + (C_0^{(0)})^2\frac{Mk^2}{2\pi^2}\, P\!\int_0^{\Lambda_c}\frac{dp}{k^2-p^2} - i\pi (C_0^{(0)})^2 \frac{Mk^2}{2\pi^2}\int_0^{\Lambda_c} dp\,\delta(k^2-p^2) \\ &= -(C_0^{(0)})^2 \frac{M\Lambda_c}{2\pi^2}\left(1+O\!\left(\frac{k^2}{\Lambda_c^2}\right)\right) - \frac{M}{4\pi}(C_0^{(0)})^2(ik)\end{aligned}}$$

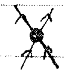
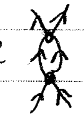
- we used $\boxed{\int_0^{\Lambda_c} dp\,\delta(k^2-p^2) = \frac{1}{2k}\int_0^{\Lambda_c} dp\,\delta(k-p) = \frac{1}{2k}}$ on the last integral.
  and $\boxed{\frac{1}{x\pm i\epsilon} = P\frac{1}{x} \mp i\pi\delta(x)}$
- it was not necessary to evaluate the middle integral in detail: we only needed to establish that the leading term was <u>independent</u> of k^2 and the rest could be expanded in powers of k^2/Λ_c^2. (At higher order we'd need those powers, however.)

$* \Rightarrow$ the form of this "extra" piece means that it <u>can</u> be absorbed into redefinitions of the couplings.

$\therefore$ we <u>can</u> renormalize! (meaning we can remove dependence on Λ_c.)

- In particular, if we take

$$\boxed{C_0 = C_0^{(0)} + C_0^{(1)} = \frac{4\pi a_s}{M} + (C_0^{(0)})^2\frac{M}{2\pi^2}\Lambda_c = \frac{4\pi a_s}{M}\left(1+\frac{2a_s\Lambda_c}{\pi}\right)}$$

Then the contribution of $C_0^{(1)}$ from  will cancel the leading Λ_c part of  !

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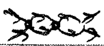
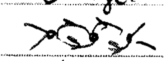
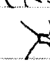
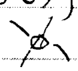
- We are left with the piece proportional to ik. To order k/Λ_c, ka_s:

$$\boxed{\begin{aligned} -T(k,\cos\theta) &= \frac{4\pi a_s}{M} - \frac{M}{4\pi}\left(C_0^{(0)}\right)^2 ik + O(k^2) \\ &= \frac{4\pi a_s}{M}\left[1 - \frac{\cancel{M}}{\cancel{4\pi}}\frac{\cancel{4\pi} a_s}{\cancel{M}} ik + O(k^2)\right] \\ &= \frac{4\pi a_s}{M}\left[1 - ia_s k + O(k^2)\right] \end{aligned}}$$

so we get the $O(k)$ term correct!

Comments:

- Note that we used the lowest order C_0 (that is, $C_0^{(0)}$) in evaluating the second-order diagram . If we go to the next order in the expansion, which means including , where we use $C_0^{(0)} + C_0^{(1)}$ in , and so on.
  $\Rightarrow$ it's clear that this is rather awkward in practice.
- To get the $O(k^2)$ terms in $T(k,\cos\theta)$, we would calculate  with C_0 vertices and  + , with C_2 and C_2' vertices. To remove all divergences in C_2', we'll have to add $C_0^{(2)}$ to C_0 and so on, at each order.
- $T(k,\cos\theta)$ is independent of Λ_c to the order we calculated. It must be, since it is a physical, measurable quantity (an "observable") and Λ_c is an arbitrary momentum. If we change Λ_c, then $C_0(\Lambda_c)$ must change to keep $T(k,\cos\theta)$ independent of Λ_c (and we have calculated the leading change.
  - We say that $C_0(\Lambda_c)$ "runs" with Λ_c.
  - The observable results are independent of Λ_c only to the accuracy of our truncation.
- We have assumed that $a_s\Lambda_c$ is small, so that we can neglect $(a_s\Lambda_c)^2$ at this order. If the scattering length is not small, we will have to perform a nonperturbative resummation (solve the S-equation!).

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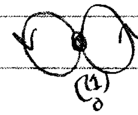
- We will return to consider the case of large a_s later.
- For now let's consider the divergence in the beach-ball diagram. By the way, since this divergence goes like a <u>linear</u> power of the cutoff Λ_c, we call this a "linear divergence". Such divergences that go like positive powers of Λ_c are called "power divergences".
  - "log divergences" go like $\ln \Lambda_c$. They do not appear in 2-to-2 scattering in odd spatial dimensions (e.g. 1 or 3), but do appear in even dimensions in 2-to-2 scattering or in 3-3 scattering.

- Our expression for $\mathcal{E}_2$ from the beachball diagram from 161, with $\lambda \to C_0$, is (note $-\frac{1}{u^2 - t^2} = \frac{1}{t^2 - u^2}$)

$$\boxed{\begin{aligned} \mathcal{E}_2 = +4Mg(g-1)\left(C_0^{(0)}\right)^2 k_c^7 \int \frac{d^3 s}{(2\pi)^3} \int \frac{d^3 t}{(2\pi)^3} \int \frac{d^3 u}{(2\pi)^3} \frac{1}{t^2 - u^2 + i\eta} \\ \times \left[\Theta(1 - |\vec{s}+\vec{t}|)\,\Theta(1 - |\vec{s}-\vec{t}|)\,\Theta(|\vec{s}+\vec{u}|-1)\,\Theta(|\vec{s}-\vec{u}|-1) \right] \end{aligned}}$$

which has the same integral (over u) that appears in the scattering diagram we just analyzed. It has a linear divergence.
  - But we also have a new contribution, from 

with $C_0^{(1)}$ at the vertex.

- The point of renormalization in the systematic fashion we have discussed, is that determining $C_0^{(1)}$ so that if we fix the high-momentum behavior in one place (e.g. 2-2 freespace scattering) fixes it <u>everywhere</u>.

- Let's check that it works...

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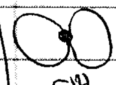
- From our earlier calculations,



$C_0^{(1)}$

$$\boxed{\equiv \Delta\mathcal{E}_1^{(1)} = \frac{1}{2}C_0^{(1)}\left(1-\frac{1}{g}\right)\rho^2 = \frac{1}{2}\left(\left(C_0^{(0)}\right)^2\frac{M}{2\pi^2}\Lambda_c\right)g(g-1)\left(\int\frac{d^3k}{(2\pi)^3}\theta(k_f-|\vec{k}|)\right)^2}$$

where we've substituted for $C_0^{(1)}$ and used $\boxed{\rho = g\int\frac{d^3k}{(2\pi)^3}\theta(k_f-|\vec{k}|)}$

- Applying the cutoff Λ_c/k_f to the u integral in $\mathcal{E}_2$, we can find the leading Λ_c dependence from the region of integration near Λ_c

$$\Rightarrow \boxed{\frac{1}{t^2-u^2+i\eta} \longrightarrow -\frac{1}{u^2}\left(1+\frac{t^2}{u^2}+\frac{t^4}{u^4}+\dots\right)}$$

and $\boxed{\theta(|\vec{s}+\vec{u}|-1)\rightarrow 1}$ $\boxed{\theta(|\vec{s}-\vec{u}|-1)\rightarrow 1}$

- The subleading terms from $\frac{1}{t^2-u^2}$ will be suppressed by $\frac{t^2}{\Lambda_c^2}\ll 1$.

$$\Rightarrow \boxed{\begin{aligned}\mathcal{E}_2 \longrightarrow\ & 4M g(g-1)\left(C_0^{(0)}\right)^2 k_f^7\left(\int\frac{d^3s}{(2\pi)^3}\int\frac{d^3t}{(2\pi)^3}\theta(1-|\vec{s}+\vec{t}|)\,\theta(1-|\vec{s}-\vec{t}|)\right)\\ & \times\frac{1}{(2\pi)^3}4\pi\int_0^{\Lambda_c/k_f}du\left(-\frac{u^2}{u^2}\right)\\ =\ & -\frac{2M}{\pi^2}\Lambda_c\left(C_0^{(0)}\right)^2 g(g-1)\,k_f^6\int\frac{d^3s}{(2\pi)^3}\int\frac{d^3t}{(2\pi)^3}\theta(1-|\vec{s}+\vec{t}|)\,\theta(1-|\vec{s}-\vec{t}|)\\ =\ & -\frac{M}{4\pi^2}\Lambda_c\left(C_0^{(0)}\right)^2 g(g-1)\int\frac{d^3k}{(2\pi)^3}\int\frac{d^3k'}{(2\pi)^3}\theta(k_f-|\vec{k}|)\,\theta(k_f-|\vec{k}'|)\end{aligned}}$$

Factor of $\frac{8}{k_f^6}$ from Jacobian

which is precisely cancelled by $\Delta\mathcal{E}_1^{(1)}$!

- Ok, so it works, but the cancellation between different diagrams and the fact that each diagram has to 'mix' with lower-order diagrams means that it is annoying at best to carry out the calculation to a specified order.

- Can we do better? Yes! Dimensional regularization!

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- We can motivate dimensional regularization based on our experience with delta functions in three and one spatial dimensions. We could think of the energy, or just the integral in the energy that diverges in 3-d as a function of the dimension.
  - The integral tells us how to define this function at isolated values of the dimension D (here I mean the spatial dimension — in the literature you will sometimes find that D refers to the space-time dimension — these differ by one).
  - If we can express the result of the integral in terms of functions defined for complex D that agree at the integral values, then this result is an analytic continuation that we can use to define (and thereby regulate) our integrals.
  - This will be clearer with an example...

- Return to the scattering graph  and our result from 157:

$$\boxed{(C_0^{(0)})^2 \int \frac{d^3p}{(2\pi)^3} \frac{M}{k^2-p^2+i\varepsilon}}$$

- Let's define the integral in D dimensions:

$$\boxed{L_0 \equiv \left(\frac{\mu}{2}\right)^{3-D} \int \frac{d^Dp}{(2\pi)^D} \frac{1}{k^2-p^2+i\varepsilon}}$$

- The parameter μ has dimensions of p $\Rightarrow$ it keeps the dimension of the integral unchanged as we vary from $D=3$ dimensions.
  - It is an auxiliary parameter, like a cutoff, that must not contribute in the end to physical quantities. It won't contribute in our initial discussion.

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- So how does the integral differ for $D=1,2,3,\ldots$?
  - The integrand depends only on p^2
    $\Rightarrow$ we can work in "spherical" coordinates generalized to D dimensions.
  - So there are $D-1$ angular integrals that we can do for free!

$$\boxed{\begin{aligned}\int \frac{d^Dp}{(2\pi)^D} f(p) &= \frac{1}{(2\pi)^D}\int_0^\infty p^{D-1}\,dp \int_0^{2\pi} d\theta_1 \int_0^\pi \sin\theta_2\, d\theta_2 \int_0^\pi \sin^2\theta_3\, d\theta_3 \cdots \times \int_0^\pi \sin^{D-2}\theta_{D-1}\, d\theta_{D-1}\, f(p) \\ &= \frac{2}{(4\pi)^{D/2}} \frac{1}{\Gamma(D/2)} \int_0^\infty p^{D-1}\, dp\, f(p)\end{aligned}}$$

- The formula $\boxed{\int_0^\pi \sin^m\theta\, d\theta = \frac{\sqrt{\pi}\,\Gamma((m+1)/2)}{\Gamma((m+2)/2)}}$

  helps to evaluate the "solid angle" integration.

- We still have the radial integral, which depends on D, but note how expressing the angular integration in terms of a gamma function lets us extend that part to any complex D.
  - The gamma function $\Gamma(z)$ is single-valued and analytic over the entire complex z plane except at $z=0,-1,-2,\ldots$ where it has simple poles with residue $(-1)^n/n!$ for $z=-n$. The reciprocal $1/\Gamma(z)$ is, in fact, an entire function with simple zeros at $z=-n$, $\{n=0,1,2,\ldots\}$.

- The plan is to do the rest of the integral the same way, expressing it in terms of gamma functions.

- The formula for the beta function

$$\boxed{B(x,y) = 2\int_0^\infty dt\, t^{2x-1}(1+t^2)^{-x-y} = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}}$$

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leads with a change of variables to

$$\boxed{\int_0^\infty \frac{\rho^\beta \, d\rho}{(c^2+\rho^2)^\alpha} = \frac{\Gamma\left((1+\beta)/2\right)\Gamma\left(\alpha-\frac{1}{2}(1+\beta)\right)}{2(c^2)^{\alpha-(1+\beta)/2}\,\Gamma(\alpha)}}$$

. This is almost the integrand we need, which is $\int_0^\infty \rho^{D-1} d\rho \, \frac{1}{k^2-\rho^2+i\varepsilon}$

$\Rightarrow$ replace $k^2+i\varepsilon$ by $-z$ and evaluate the integral for $c^2=z$, and then "continue" the result to $z=-k^2-i\varepsilon$.

$$\Rightarrow \boxed{-\int_0^\infty \frac{\rho^{D-1} d\rho}{(z+\rho^2)^1} \underset{\alpha=1}{\overset{\beta=D-1}{=}} -\frac{\Gamma(D/2)\,\Gamma\left(\frac{2-D}{2}\right)}{2(z)^{(2-D)/2}\,\cancel{\Gamma(1)}\, 1}}$$

Combining with the angular integral, which cancels the $\frac{\Gamma(D/2)}{2}$,

$$\boxed{L_0 = \left(\frac{\mu}{2}\right)^{3-D}\int \frac{d^D\rho}{(2\pi)^D}\frac{1}{k^2-\rho^2+i\varepsilon} = -\left(\frac{\mu}{2}\right)^{3-D}\frac{1}{(4\pi)^{D/2}}\Gamma\left(\frac{2-D}{2}\right)\left(-k^2-i\varepsilon\right)^{(D-2)/2}}$$

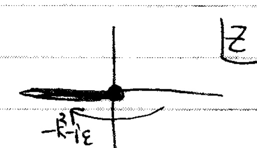
• This is defined for any D but has poles where the Γ function argument is zero or is a negative integer.

• For $D=3$, however, we can just take the limit, noting that $\boxed{\Gamma(-\tfrac{1}{2}) = -2\sqrt{\pi}}$

$$\boxed{L_0 = -\frac{1}{(4\pi)^{3/2}}\left(-2\sqrt{\pi}\right)\sqrt{-k^2-i\varepsilon} = -\frac{ik}{4\pi}}$$

• We evaluated $\sqrt{z} \xrightarrow[z\to -k^2-i\varepsilon]{} -ik$ by noting in the z-plane that we want to go to the under side of the square-root so with $z=re^{i\theta}$, we want $r=k^2$ and $\theta=-\pi$

$\Rightarrow \sqrt{z} = \sqrt{k^2}\, e^{-i\pi/2} = -ik$.



(we can see the need for a branch cut, by comparing $\sqrt{z}=\sqrt{r}\,e^{i\theta/2}$ for $\theta=\pi$ and $\theta=-\pi$ $\Rightarrow +i\sqrt{r}$ vs. $-i\sqrt{r}$ $\Rightarrow$ not single-valued if no cut.)

• We can easily generalize to $\boxed{L_n = \left(\frac{\mu}{2}\right)^{3-D}\int \frac{d^D\rho}{(2\pi)^D}\frac{\rho^{2n}}{k^2-\rho^2+i\varepsilon} \xrightarrow{D=3} -\frac{ik^{2n+1}}{4\pi}}$ which can be applied for C_2, C_4, etc. vertices.

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So now
$$\boxed{\begin{aligned} -T(k,\cos\theta) &= C_0 - \frac{M}{4\pi}(C_0)^2\, ik + O(k^2) \\ &= \frac{4\pi a_s}{M}\left[1 - i a_s k + O(k^2)\right] \end{aligned}}$$

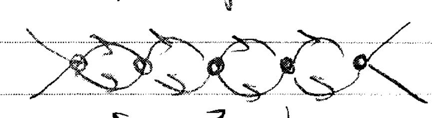
by choosing $C_0 = \frac{4\pi a_s}{M}$ as the "subtractions" that we did by hand to fix up the cutoff regulated result, the subtraction with dimensional regularization here is done automatically (invisibly!).

- More generally, there is an explicit renormalization prescription: the minimal subtraction prescription is to subtract poles in the Γ function at $D=3$ (that is, terms proportional to $1/D-3$).
- For 2-to-2 scattering there are no such poles, but for 3-to-3 scattering they appear, first at 4^{th} order (diagrams with 4 successive two-body scatterings, such as the fish diagram:)
- It is in these cases that the $(\mu/2)^{3-D}$ factor contributes. You might have thought it always goes to 1 as $D\to 3$, but not if it multiplies a pole in 3-D:

$$\boxed{(\mu/2)^{3-D} = e^{\ln(\mu/2)^{3-D}} = e^{(3-D)\ln(\mu/2)} \xrightarrow[D\to 3]{} (3-D)\ln(\mu/2)}$$

to multiply by $1/3-D$ leaves a finite piece proportional to $\ln \mu$.

- Because μ only appears in ln's if at all, the dimensions of any given diagram come only from a_s's (and other effective range parameters) in the vertices, balanced by k's coming from the loop integrals.
  - This is in stark contrast to the case of cutoff regularization, where factors of k/Λ_c to any power could appear.
  - The consequence is very clean "power counting," which means identifying for each diagram where it contributes in the expansion.
  - The loop integrals in a scattering diagram also decouple:



$\Rightarrow$ a product of $\left(-\frac{ik}{4\pi}\right)^4$ and the other factors.

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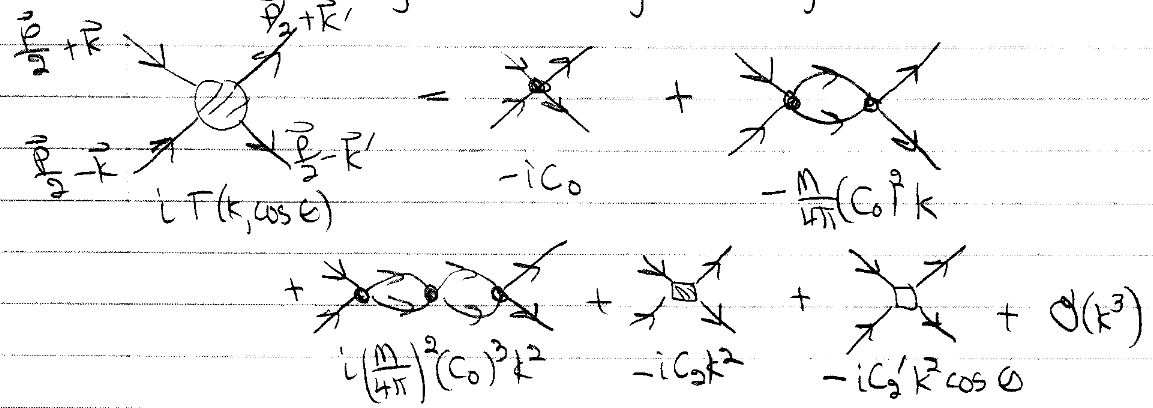
So the results for $T(k,\cos\theta)$ through $O(k^2)$ can be written down directly from the Feynman diagrams:



or

$$\boxed{\begin{aligned} T(k,\cos\theta) &= -C_0\left[1 - i\frac{m}{4\pi}C_0 k + \frac{C_2}{C_0}k^2 - \left(\frac{m}{4\pi}C_0\right)^2 k^2 + \frac{C_2'}{C_0}k^2\cos\theta\right] + O\left(\frac{k^3}{\Lambda^3}\right) \\ &= -\frac{4\pi a_s}{m}\left[1 - i a_s k + \left(a_s r_s/2 - a_s^2\right)k^2\right] - \frac{4\pi a_p^3}{m}k^2\cos\theta + O\left(\frac{k^3}{\Lambda^3}\right) \end{aligned}}$$

which identifies (by "matching") that

$$\boxed{C_0 = \frac{4\pi a_s}{m}} \quad \boxed{C_2 = C_0\frac{a_s r_s}{2}} \quad \boxed{C_2' = \frac{4\pi a_p^3}{m}}$$

with no renormalization! (All done automatically by dim. reg.).

- For any given scattering diagram, one can count:
  - i) for every propagator a factor of $\boxed{m/k^2}$ (1/kinetic energy)
  - ii) for every loop integration a factor of $\boxed{k^3\cdot\frac{k^2}{m} = k^5/m}$
  - iii) for every n-body vertex with $2i$ derivatives, a factor $\boxed{k^{2i}/m\Lambda^{2i+3n-5}}$

- If a diagram has L loops or E external lines and V_{2i}^n n-body vertices with $2i$ derivatives, it scales precisely as k^ν where

$$\boxed{\nu = 3L + 2 + \sum_{n=2}^{\infty}\sum_{i=0}^{\infty}(2i-2)V_{2i}^n = 5 - \frac{3}{2}E + \sum_{n=2}^{\infty}\sum_{i=0}^{\infty}(2i+3n-5)V_{2i}^n}$$

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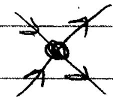
- Let's try a few to see how it works

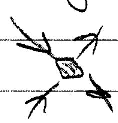


1 2-body vertex with no derivatives $\Rightarrow V_0^2 = 1$ $(n=2, i=0)$
no loops: $L=0$, 4 external lines: $E=4$

$$\Rightarrow \boxed{\begin{aligned} \nu &= 3\cdot\phi + 2 + (2\cdot 0 - 2)\cdot 1 = 0 \Rightarrow k^0 \\ \text{or} \quad &= 5 - \tfrac{3}{2}\cdot 4 + (2\cdot 0 + 3\cdot 2 - 5)\cdot 1 = 0 \Rightarrow k^0 \end{aligned}}$$

and, in fact, it contributes as $-iC_0$.

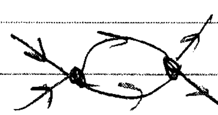
- Try a vertex with two derivatives:



1 2-body vertex with 2 derivatives $\Rightarrow V_2^2 = 1$ $(n=2, i=1)$
no loops: $L=0$, 4 external lines: $E=4$

$$\Rightarrow \boxed{\begin{aligned} \nu &= 3\cdot\phi + 2 + (2\cdot 1 - 2)\cdot 1 = 2 \Rightarrow k^2 \\ \text{or} \quad &= 5 - \tfrac{3}{2}\cdot 4 + (2\cdot 1 + 3\cdot 2 - 5)\cdot 1 = 0 \Rightarrow k^2 \end{aligned}} \Bigg\} \text{ as expected from (166)}$$

- Try a diagram with a loop!



2 2-body vertices with no derivatives $\Rightarrow V_0^2 = 2$ $(n=2, i=0)$
1 loop: $L=1$, 4 external lines

$$\Rightarrow \boxed{\begin{aligned} \nu &= 3\cdot 1 + 2 + (2\cdot 0 - 2)2 = 1 \Rightarrow k^1 \checkmark \\ &= 5 - \tfrac{3}{2}\cdot 4 + (2\cdot 0 + 3\cdot 2 - 5)2 = 1 \Rightarrow k^1 \checkmark \end{aligned}} \Bigg\} \text{ as expected from our calculation}$$

$\Rightarrow$ With dimensional regularization and minimal subtraction, each diagram contributes to precisely one power of k.

- Remember, we have assumed $\boxed{k \ll \frac{1}{a_s} \hat{=} \frac{1}{r_s} \hat{=} \frac{1}{R} \hat{=} \Lambda}$
- When this is not true, we have to work harder!
- Since the only scales are k and Λ, as we go to higher order, they must appear in the combination k/Λ to keep the units straight.

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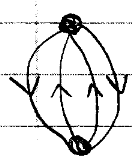
Ok, so once more return to the beachball diagram, now with dimensional regularization:



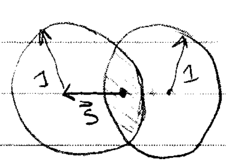
$$\Rightarrow \boxed{\begin{aligned}\mathcal{E}_g = -4\lambda^2 M g(g-1) k_F^7 \times \int \frac{d^3 s}{(2\pi)^3} \int \frac{d^3 t}{(2\pi)^3} \int \frac{d^D u}{(2\pi)^D} \frac{1}{u^2 - t^2 - i\eta} \\ \times\, \Theta(1-|\vec{s}+\vec{t}|)\,\Theta(1-|\vec{s}-\vec{t}|)\,\Theta(|\vec{s}+\vec{u}|-1)\,\Theta(|\vec{s}-\vec{u}|-1)\end{aligned}}$$

- Only the divergent integral (over u) has been extended to D dimensions; the $D \to 3$ limit doesn't change anything in the bounded s and t integrals.
  - we've suppressed the $(\mu/2)^{D-3}$ factor, since it doesn't contribute here.

- We can't apply our DR (dimensional regularization) integral formula immediately because of the $\Theta(|\vec{s}+\vec{u}|-1)$ and $\Theta(|\vec{s}-\vec{u}|-1)$ functions;
  - $\Rightarrow$ do a bit more work on the integral.

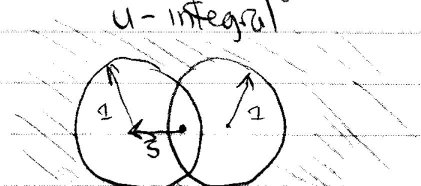
- Consider the regions of integration for the t and u integrals, each with $\vec{s}$ held fixed. They are volumes of overlapping spheres (remember way back at the beginning of the quarter?):

t-integral



$\Theta(1-|\vec{s}+\vec{t}|)\,\Theta(1-|\vec{s}-\vec{t}|)$

u-integral



$\Theta(|\vec{s}+\vec{u}|-1)\,\Theta(|\vec{s}-\vec{u}|-1)$

- The shaded regions are the ones defined by the Θ functions, where $\vec{t}$ and $\vec{u}$ start from the center (which is the origin of these systems)
  - each Θ function boundary is one of the spheres.

- We can use these pictures to define the limits of integration.

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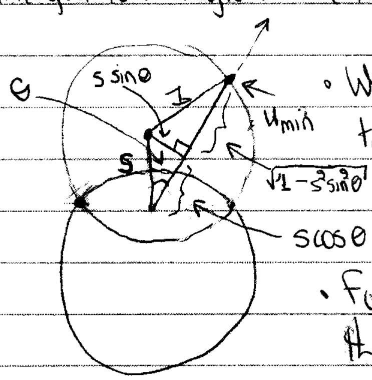
- First the $\vec{s}$ integration:
  - The magnitude of $\vec{s}$ is limited by the requirement that the spheres overlap. They just touch for $|\vec{s}|=1$ , so $0 \le s \le 1$.
  - The direction of $\vec{s}$ is unconstrained and $\frac{1}{u^2 t^2 - i\eta}$ doesn't depend on $\vec{s}$, so we can do the angular integral immediately $\Rightarrow 4\pi$ factor

$$\therefore \boxed{\int \frac{d^3 s}{(2\pi)^3} \longrightarrow \frac{4\pi}{(2\pi)^3} \int_0^1 s^2\, ds}$$

- When doing the t and u integrations, we'll choose the current direction of $\vec{s}$ as the $\hat{z}$-axis.
  - So the θ-functions and integrand are independent of the azimuthal angles φ and can do both of those integrals immediately $\Rightarrow (2\pi)\times(2\pi)$ factor.
  - Now we have to work a bit harder! Let's draw the u integration region with $\vec{s}$ (z-axis) pointing upward:



- We only need to integrate θ from 0 to $\pi/2$ and then multiply by 2 (using the symmetry of the geometry).
- For any angle θ, the u integral goes the edge of the top circle to infinity.

- If we define $y \equiv \hat{s}\cdot\hat{u} = \cos\theta_{su}$, then we see from the diagram that

$$\boxed{u_{min} = sx + \sqrt{1-s^2(1-x^2)} \equiv z_+(y)}$$

is the lower limit for the u integration.

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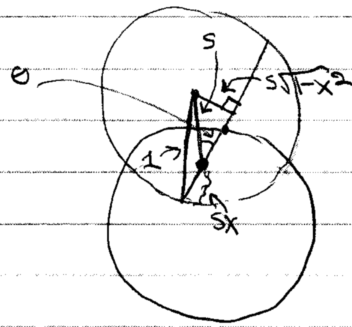
- Similarly, we can find the limits of the t-integration:

Let $x \equiv \cos\theta$



Then t goes from 0 to the edge of the lower circle

$$\Rightarrow \boxed{t_{max} = \sqrt{1-s^2(1-x^2)} - sx \equiv Z_-(x)}$$

- Putting this all together, we have

$$\boxed{\mathcal{E}_2 = -4\lambda^2 m\, g(g-1) k_F^7 \frac{8}{(2\pi)^6} \int_0^1 s^2 ds \int_0^1 dx \int_0^{Z_-(x)} t^2 dt \int_0^1 dy \int_{Z_+(y)}^{\infty} u^{D-1} du \frac{1}{u^2-t^2-i\eta}}$$

- We need an integral from 0 to ∞ to apply our DR formula, so simply add and subtract:

$$\boxed{\int_{Z_+(y)}^{\infty} du \frac{u^{D-1}}{u^2-t^2-i\eta} = \int_0^{\infty} du \frac{u^{D-1}}{u^2-t^2-i\eta} + \int_0^{Z_+(y)} du \frac{u^2}{u^2-t^2-i\eta}}$$

- The first integral on the right side is now do-able and yields a purely imaginary result
  - But the second integral also has a imaginary part (evaluated using $\frac{1}{u^2-t^2-i\eta} = P\frac{1}{u^2-t^2} + \pi i \delta(u^2-t^2)$), that exactly cancels it.
  - This is good, because it means the energy will be real!

$\Rightarrow$ we can simply drop the first integral and use the principal value of the second.

- Now $\mathcal{E}_2$ is finite, so it can be evaluated numerically or analytically (with a few judicial partial integrations).