2/12/03

Let's step back a moment and summarize the formalism for the $T=0$ calculations.

- While we arrived at the Feynman rules for the energy density at $T=0$ by way of the path integral for the partition function in imaginary or Euclidean time $\tau$, we could also start formally in Minkowski space (e.g. real time) in the canonical ensemble (which means in practice that we don't use a chemical potential).
- We won't rederive the results here, but simply summarize the formulas that lead to the Feynman rules we've given.
- We will find it useful to switch between the formalisms, depending on the problem under consideration.

- Instead of the partition function with sources, we have the closely related generating functional,

$$\boxed{Z[\eta,\eta^\dagger] = \int \mathcal{D}(\psi^\dagger\psi)\, e^{i\int d^4x\,[\mathcal{L}(x) + \eta^\dagger(x)\psi(x) + \psi^\dagger(x)\eta(x)]}}$$

where the integration $\int d^4x = \int d^3x \int dt$ is over real time and the Lagrangian (density) $\mathcal{L}(x)$ is (for the delta-function potential)

$$\boxed{\mathcal{L} = \psi^\dagger_\alpha(x)\left(i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m}\right)\psi_\alpha(x) - \frac{1}{2}\lambda\,\psi^\dagger_\alpha(x)\psi^\dagger_\beta(x)\psi_\beta(x)\psi_\alpha(x)}$$

- So this is essentially a continuation from $\tau$ to $it$.

Check: $-\int d\tau \to -i\int dt$, $\frac{\partial}{\partial\tau} \to -i\frac{\partial}{\partial t}$ so we cancel the minus signs and have $+i\int d^4x\, \psi^\dagger(i\frac{\partial}{\partial t})\psi$.

Similarly $-\int d\tau\, \psi^\dagger\left(-\frac{\nabla^2}{2m}\right)\psi \to +i\int d\tau\, \psi^\dagger \frac{\nabla^2}{2m}\psi$

- If we want to include a chemical potential $\mu$, we take

$$\boxed{i\frac{\partial}{\partial t} \to i\frac{\partial}{\partial t} + \mu}$$

2/12/03

- The development of a perturbation expansion follows closely that of the partition function: remove the interaction term from the Lagrangian using

$$\boxed{\psi^\dagger \to i\frac{\delta}{\delta\eta}} \quad \left[\text{since } \boxed{i\frac{\delta}{\delta\eta(x)} e^{i\int d^4y\, \psi^\dagger(y)\eta(y)} = e^{i\int d^4y\, \psi^\dagger(y)\eta(y)}\psi^\dagger(x)}\right]$$

$$\boxed{\psi \to -i\frac{\delta}{\delta\eta^\dagger}} \quad \left[\text{since } \boxed{-i\frac{\delta}{\delta\eta^\dagger(x)} e^{i\int d^4y\, \eta^\dagger(y)\psi(y)} = e^{i\int d^4y\, \eta^\dagger(y)\psi(y)}\psi(x)}\right]$$

and complete the square in the path integral using

$$\boxed{\psi^\dagger G^{0\,-1}\psi + \psi^\dagger\eta + \eta^\dagger\psi = (\psi^\dagger + \eta^\dagger G^0)G^{0\,-1}(\psi + G^0\eta) - \eta^\dagger G^0\eta}$$

(where all of the integrations are suppressed for simplicity).

The result is

$$\boxed{Z[\eta,\eta^\dagger] = Z_0\, e^{-\frac{i\lambda}{2}\int d^4x\left(\frac{i\delta}{\delta\eta_\alpha(x)}\frac{i\delta}{\delta\eta_\beta(x)}\frac{-i\delta}{\delta\eta^\dagger_\beta(x)}\frac{-i\delta}{\delta\eta^\dagger_\alpha(x)}\right)} e^{-i\int d^4x_1\, d^4x_2\, \eta^\dagger_\gamma(x_1)G^0_{\gamma\delta}(x_1,x_2)\eta_\delta(x_2)}}$$

where $x \equiv (t,\vec{x})$ and

$$\boxed{\begin{aligned} iG^0_{\alpha\beta}(x_1,x_2) &= \delta_{\alpha\beta}\frac{1}{\Omega}\sum_k e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}e^{-i\omega_k(t_1-t_2)} \\ &\qquad \times\left[\Theta(t_1-t_2-\eta)\Theta(|\vec{k}|-k_F) - \Theta(t_2-t_1+\eta)\Theta(k_F-|\vec{k}|)\right] \\ &\to \delta_{\alpha\beta}\int\frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}e^{-i\omega_k(t_1-t_2)}\left[\Theta(t_1-t_2-\eta)\Theta(|\vec{k}|-k_F) - \Theta(t_2-t_1+\eta)\Theta(k_F-|\vec{k}|)\right]\end{aligned}}$$

- We can see where the Feynman rules originate:
  - $-i\lambda$ for the vertex (the $\frac{1}{2}$ gets cancelled) and $\int d^4x_i$ for each vertex from the term we pulled out.
  - same symmetry factor and spin sum
  - $\left(-i\frac{\delta}{\delta\eta^\dagger_\alpha(x_1)}\right)\left(\frac{i\delta}{\delta\eta_\beta(x_2)}\right)\left[-i\int d^4x_3\, d^4x_4\, \eta^\dagger_\gamma(x_3)G^0_{\gamma\delta}(x_3,x_4)\eta_\delta(x_4)\right] = iG^0_{\alpha\beta}(x_1,x_2)$ for each line.

2/12/03

There are two different direct derivations (as opposed to going through the finite temperature formalism) of perturbation theory at $T=0$ for fixed particle number.

- The first is detailed in Negele and Orland chap. 3 and the 2nd in Fetter and Walecka section 6.
- Here we'll simply motivate the Negele/Orland construction.

First introduce the (normalized) N-particle eigenfunctions of the one-body Hamiltonian $\hat{H}_0$ and the full Hamiltonian $\boxed{\hat{H} = \hat{H}_0 + \hat{V}}$:

$$\boxed{\hat{H}_0 |\Phi_n^N\rangle = E_0^{(n)} |\Phi_n^N\rangle} \quad \text{with} \quad \boxed{\langle \Phi_n^N | \Phi_n^N \rangle = 1}$$

and

$$\boxed{(\hat{H}_0 + \hat{V}) |\Psi_n^N\rangle = E^{(n)} |\Psi_n^N\rangle} \quad \text{with} \quad \boxed{\langle \Psi_n^N | \Psi_n^N \rangle = 1}$$

- Completeness of the full eigenstates says

$$\boxed{1 = \sum_{n,N} |\Psi_n^N\rangle\langle\Psi_n^N|}$$

(we have to sum over the excited states labeled by n for each particle number N).

The basic idea is that a ratio like

$$\boxed{\frac{\langle \Phi_0^N | e^{-i(\hat{H}_0+\hat{V})T_0} | \Phi_0^N \rangle}{\langle \Phi_0^N | e^{-i\hat{H}_0 T_0} | \Phi_0^N \rangle} \underset{\substack{\text{no sum} \\ \text{over } N \text{ if } \hat{H} \\ \text{commutes with } \hat{N}}}{=} \frac{\sum_n \langle \Phi_0^N | e^{-i\hat{H}T_0} | \Psi_n^N \rangle \langle \Psi_n^N | \Phi_0^N \rangle}{e^{-E_0^{(0)}T_0} \langle \Phi_0^N | \Phi_0^N \rangle} = \sum_n |\langle \Phi_0^N | \Psi_n^N \rangle|^2 \, e^{-i(E^{(n)} - E_0^{(0)})T_0}}$$

goes to $e^{2\ln\langle \Phi_0^N | \Psi_0^N \rangle - i(E^{(0)} - E_0^{(0)})T_0}$ if $\operatorname{Im} T_0 \to -\infty$.

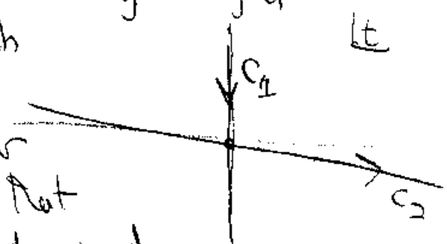
That is, if we take $\frac{i}{T_0} \ln(\text{ratio}) \to E - E_0$ (dropping the (0) superscript), which is what we want.

2/12/03

Note the close similarity to $\Omega-\Omega_0 = -\frac{1}{\beta}\ln\frac{Z}{Z_0}$.

- Important assumptions: $|\Phi_0^N\rangle$ is not orthogonal to $|\Psi_0\rangle$ and that $|\Psi_0\rangle$ is <u>non-degenerate</u>. If these are not fulfilled, we have to adjust our formalism.

- We can accomplish the condition $\mathrm{Im}\,T_0 \to -\infty$ by taking a <u>contour</u> in the complex time $t$ plane on which we evaluate the matrix element with a path integral. In imaginary time, we had a contour that ran vertically ($C_1$). Now we take one that runs for $-T_0/2$ to $+T_0/2$ with a slight downward slope: $t=(1-i\hat{\eta})\tilde{t}$ where $\tilde{t}$ is real and $\hat{\eta}$ is real and arbitrarily small.




  - We just need to remember that slope occasionally when it acts to make integrands converge that would otherwise simply oscillate.
  - We take $T_0 \to \infty$ after we cancel the overall $T_0$ factor.
  - We can break up the interval into little pieces of size $\epsilon$ and rework our path integral construction.
  - The basic substitutions are $\tau \to it$, $\int_0^\beta d\tau \to \int_{-T/2}^{T/2} dt$, $(-1)^n \to (-i)^n$, as implemented in the Feynman rules given earlier.

- Since the replica proof goes through as before, the log of the ratio is given by the sum of totally connected ("linked") diagrams:

$$\boxed{E-E_0 = \lim_{T_0\to\infty} \frac{i}{T_0}\sum(\text{all linked diagrams})}$$

- The 2nd direct derivation involves introducing an adiabatic (slow!) switching on of the potential: $\hat{H}_\epsilon(t) = \hat{H}_0 + e^{-\epsilon|t|}\hat{V}$ and applying the so-called Gell-Mann and Low theorem. See Fetter and Walecka for details.

2/12/03

· So by now we have diagrammatic methods to calculate the ground state energy or finite temperature thermodynamic functions in perturbation theory.

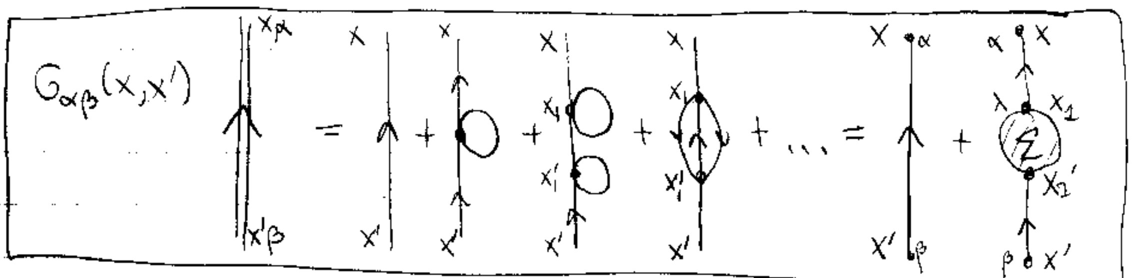
- What else can we learn about?
- How do we make non-perturbative approximations?

· To address both of these questions (in part), let's examine further the one-body Green's function

$$\boxed{G_{\alpha\beta}(x,x') = \frac{\int \mathcal{D}(\psi^\dagger\psi)\, \psi_\alpha(x)\psi^\dagger_\beta(x')\, e^{i\int d^4x\, \mathcal{L}}}{\int \mathcal{D}(\psi^\dagger\psi)\, e^{i\int d^4x\, \mathcal{L}}} = \langle \Psi_0^N | T[\psi_\alpha(\vec{x}t)\psi^\dagger_\beta(\vec{x}'t')] | \Psi_0^N \rangle}$$

for which we have a diagrammatic expansion.

· Every diagram in the expansion is either $G^0_{\alpha\beta}(x,x')$ or starts with one $G^0$ at $x'$ and ends with another at $x$. So




is the general structure, where the "self-energy"  stands for all of the possible (connected) diagram insertions.

· In equation form, this is an integral equation:

$$\boxed{G_{\alpha\beta}(x,x') = G^0_{\alpha\beta}(x,x') + \int d^4x_1 \int d^4x_1'\, G^0_{\alpha\lambda}(x,x_1)\,\Sigma(x_1,x_1')_{\lambda\mu}\, G^0_{\mu\beta}(x_1',x')}$$

which serves to define the self energy.

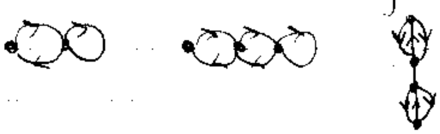
2/12/03

But we can go further. Compare two 2nd order contributions to $\Sigma$:

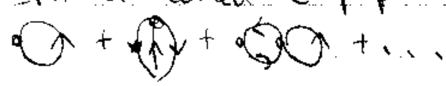
$\Rightarrow$ each part of the left diagram looks like a self-energy piece, because there is a single line joining them.

$\Rightarrow$ call it 1PR "one-particle reducible"

- The right diagram is 1PI "one-particle irreducible" because it does not fall into two pieces when a single line is cut (unlike the left diagram).
  - Check: are the following 1PR or 1PI?

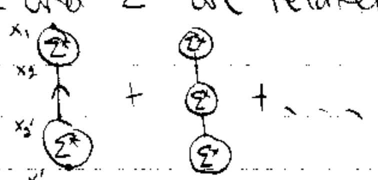


- The diagrams in $\Sigma$ that are 1PI are called the "proper" self-energy and designated $\Sigma^*$ =



- Diagrammatically, $\Sigma$ and $\Sigma^*$ are related by

$\Sigma$ = $\Sigma^*$ +

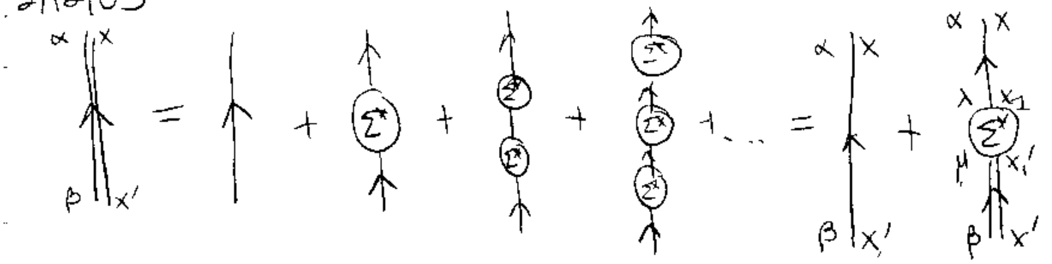


or, in equations (suppressing spin indices)

$$\boxed{\Sigma(x_1,x_1') = \Sigma^*(x_1,x_1') + \int d^4x_2\, d^4x_2'\, \Sigma^*(x_1,x_2)\, G^0(x_2,x_2')\, \Sigma^*(x_2',x_1') + \ldots}$$

- Now we can insert this equation back into our original equation for $G$ to derive another integral equation:

2/12/03



$$\Rightarrow \boxed{G_{\alpha\beta}(x,x') = G^0_{\alpha\beta}(x,x') + \int d^4x_1\, d^4x_1'\, G^0_{\alpha\lambda}(x,x_1)\, \Sigma^*(x_1,x_1')_{\lambda\mu}\, G_{\mu\beta}(x_1',x')}$$

· This is "Dyson's Equation" for the propagator ("2-point function").

· If the system of interest is translationally invariant, it almost always is best to introduce fourier transforms in three-momentum and frequency $\Rightarrow$ four-momentum

$$\boxed{\Sigma^*(x,x')_{\alpha\beta} = \Sigma^*(x-x')_{\alpha\beta} = \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot(x-x')}\, \Sigma^*(k)_{\alpha\beta}}$$

$$\boxed{G_{\alpha\beta}(x,x') = \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot(x-x')}\, G_{\alpha\beta}(k)}$$

$$\boxed{\begin{aligned} G^0_{\alpha\beta}(x,x') &= \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot(x-x')}\, G^0_{\alpha\beta}(k) \\ &= \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot(x-x')}\, \delta_{\alpha\beta}\left[\frac{\theta(|\vec{k}|-k_F)}{\omega-\omega_k+i\eta} + \frac{\theta(k_F-|\vec{k}|)}{\omega-\omega_k-i\eta}\right] \end{aligned}}$$

· Where $k\cdot x \equiv \vec{k}\cdot\vec{x} - \omega t$, $d^4k \equiv d^3k\, d\omega$

: We've substituted for $G^0_{\alpha\beta}(k)$ $\Rightarrow$ note the pole structure.

· To transform the equation for $G_{\alpha\beta}$, integrate over $(x-x')$:

$$\boxed{\int d^4(x-x')\, e^{-ik\cdot(x-x')}\left[G_{\alpha\beta} = G^0_{\alpha\beta} + G^0_{\alpha\lambda}\Sigma^*_{\lambda\mu}G_{\mu\beta}\right]}$$

$$\Rightarrow \boxed{\begin{aligned} G_{\alpha\beta}(k) = G^0_{\alpha\beta}(k) + \int d^4(x-y)\, d^4x_1\, d^4x_1' \int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4}\, e^{-ik\cdot(x-x')}\, e^{ik_1(x-x_1)}\, e^{ik_2(x_1-x_1')}\, e^{ik_3(x_1'-x')} \\ \times G^0_{\alpha\lambda}(k_1)\, \Sigma^*(k_2)_{\lambda\mu}\, G_{\mu\beta}(k_3) \end{aligned}}$$

2/12/03

Now carry out the integrals:

$x_1$ integral $\Rightarrow (2\pi)^4 \delta^4(k_1 - k_2)$

$k_1$ integral $\Rightarrow$ sets $k_1 = k_2$

$x_1'$ integral $\Rightarrow (2\pi)^4 \delta^4(k_2 - k_3)$

$k_2$ integral $\Rightarrow$ sets $k_2 = k_3$

$(x-x')$ integral $\Rightarrow \int d^4(x-x')\, e^{-ik\cdot(x-x')}\, e^{ik_3\cdot(x-x')} \to (2\pi)^4 \delta^4(k-k_3)$

$k_3$ integral $\Rightarrow$ sets $k_3 = k$

When the dust settles, we get

$$\boxed{G_{\alpha\beta}(k) = G^0_{\alpha\beta}(k) + G^0_{\alpha\lambda}(k)\Sigma^*(k)_{\lambda\mu} G_{\mu\beta}(k)}$$

$\Rightarrow$ an algebraic (matrix) equation

If, as in the case of the $\lambda\delta^3(\vec{x}-\vec{x}')$ interaction, $G, G^0$ and $\Sigma^*$ are diagonal in the spin indices, then

$$\Rightarrow G(k)\delta_{\alpha\beta} = G^0(k)\delta_{\alpha\beta} + G^0(k)\Sigma^*(k)G(k)\underbrace{\delta_{\alpha\lambda}\delta_{\lambda\mu}\delta_{\mu\beta}}_{\delta_{\alpha\beta}}$$

$$\Rightarrow \boxed{G(k) = G^0(k) + G^0(k)\Sigma^*(k)G(k).}$$

so we can just solve it!

$$\boxed{[1 - G^0(k)\Sigma^*(k)]\,G(k) = G^0(k)} \Rightarrow \boxed{G(k) = \frac{G^0(k)}{1 - G^0(k)\Sigma^*(k)} = \frac{1}{[G^0(k)]^{-1} - \Sigma^*(k)}}$$

Since $[G^0(k)]^{-1} \equiv [G^0(\vec{k},\omega)]^{-1} = \omega - \omega_{\vec{k}} = \omega - \frac{\epsilon_k^0}{\hbar}$ (now put $\hbar = 1$ again!)

$$\Rightarrow \boxed{G_{\alpha\beta}(k) \equiv G_{\alpha\beta}(\vec{k},\omega) = \frac{1}{\omega - \epsilon_k^0 - \Sigma^*(\vec{k},\omega)}\,\delta_{\alpha\beta}}$$

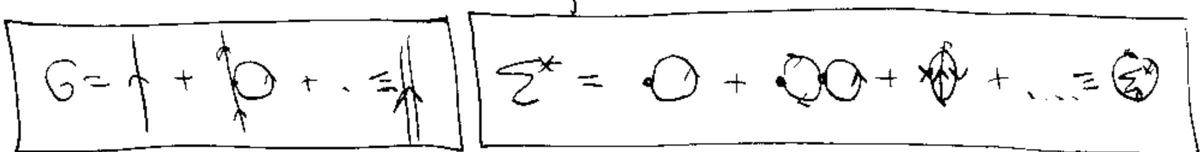
So instead of poles at $\omega = \epsilon_k^0$, they are at the solutions to $\omega_{pole} = \epsilon_k^0 + \Sigma^*(\vec{k}, \omega_{pole})$. What does this mean?

2/12/03

What good is Dyson's equation?

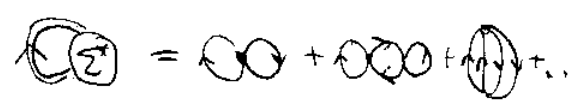
- If we approximate $\Sigma^*(\vec{k},\omega)$, e.g. to some order in the perturbation expansion in $\lambda$, then we get an <u>infinite</u> order approximation to $G$. (That is, <u>all</u> powers of $\lambda$ contribute.)
- For example, we could take $\Sigma^* \doteq \Sigma^*_1 = \left(1-\frac{1}{g}\right)\lambda\rho$ and the $G$ would have poles at $\epsilon_k^{(1)} = \epsilon_k^0 + \hbar\Sigma_1^*$.

- If we have an approximation to $\Sigma^*$ and $G$, how do we get a new approximation to the energy?
  - If we think of the diagrams



it looks like we can generate the diagrams in the $E-E_0$ expansion by using close the $G$ ends on the free ends of $\Sigma^*$

$\Rightarrow$



- This does, in fact, generate the relevant diagrams. But that's not <u>enough</u> $\Rightarrow$ we need to get the factors in front correct, and $G\Sigma^*$ doesn't do that.
- However, if we introduce a dimensionless parameter $\alpha$ that ranges from 0 to 1, and multiply $\hat{V}$ by $\alpha$:

$$\boxed{\hat{H}(\alpha) \equiv \hat{H}_0 + \alpha\hat{H}_1 = \hat{H}_0 + \alpha\hat{V}}$$

Then $\alpha$ interpolates from $\hat{H}_0$ to $\hat{H}$. Claim:

$$\boxed{\begin{aligned} E-E_0 &= -\frac{i}{2}\int_0^1 \frac{d\alpha}{\alpha}\int dx\,dx'\,dt'\,\Sigma^{*\alpha}(xt;x't')\,G^\alpha(x't';xt^+) \\ &= -\frac{i}{2}\Omega\int_0^1\frac{d\alpha}{\alpha}\int\frac{d^4k}{(2\pi)^4}\,e^{i\omega\eta}\,\Sigma^{*\alpha}_{\sigma\sigma'}(\vec{k},\omega)\,G^\alpha_{\sigma'\sigma}(\vec{k},\omega)\end{aligned}}$$

$x$ includes $\vec{x}$ and $\alpha$

2/12/03

- The notation $\Sigma^{*\alpha}$ and $G^\alpha$ simply means the proper self-energy and Green's function for a particular value of $\alpha$.
  - This is actually trivial in perturbation theory since we just have $\lambda \to \alpha\lambda$. So if we are working at $O(\lambda^3)$, all contributions have $\alpha^3$ [so $\Sigma^{*\alpha}$ could be $O(\alpha)$ while $G^\alpha$ is $O(\alpha^2)$ and all other combinations.)
  - In this case, the $\alpha$ integral is trivial.

- To show how the formula for $E-E_0$ comes about, we will go back to the field operators in the Heisenberg picture, which appear in the definition of $G$:

$$\boxed{iG_{\alpha\beta}(\vec{x}t, x't') = \langle \Psi_0^N | T[\hat{\psi}_{\alpha_H}(\vec{x}t)\hat{\psi}^\dagger_{\beta H}(\vec{x}'t')] | \Psi_0^N \rangle}$$

where the subscript $H$ is a reminder of the Heisenberg picture time dependence

$$\boxed{\hat{O}_H(t) = e^{i\hat{H}t}\hat{O}_S e^{-i\hat{H}t}} \Rightarrow \boxed{i\frac{\partial}{\partial t}\hat{O}_H(t) = [\hat{O}(t), \hat{H}]}$$

and $|\Psi_0^N\rangle$ is the (normalized) fully interacting ground state.

- Find the Heisenberg equation of motion for $\hat{\psi}_{\alpha H}(\vec{x}t)$

$$i\frac{\partial}{\partial t}\hat{\psi}_{\alpha H}(\vec{x}t) = [\hat{\psi}(\vec{x},t), \hat{H}]$$

$$= \Big[\hat{\psi}_{\alpha H}(\vec{x},t), \int d^3z\, \hat{\psi}^\dagger_{\alpha'}(\vec{z})\, T(\vec{z})_{\alpha'\beta'}\, \hat{\psi}_{\beta'}(\vec{z})$$

$$+ \frac{1}{2}\int d^3z\, d^3z'\, \hat{\psi}^\dagger_{\alpha'}(\vec{z})\hat{\psi}^\dagger_{\alpha''}(\vec{z}')\, V(\vec{z},\vec{z}')_{\alpha\beta',\alpha''\beta''}\, \hat{\psi}_{\beta''}(\vec{z}')\hat{\psi}_{\beta'}(\vec{z})$$

$V \to \lambda\delta^3(\vec{z}-\vec{z}')$

$$= \Big[\hat{\psi}_{\alpha H}(\vec{x},t), \int d^3z\, \hat{\psi}^\dagger_\alpha(\vec{z})\, T(\vec{z})\, \hat{\psi}_\alpha(\vec{z}) + \frac{\lambda}{2}\int d^3z\, \hat{\psi}^\dagger_\alpha(\vec{z})\hat{\psi}^\dagger_\beta(\vec{z})\hat{\psi}_\beta(\vec{z})\hat{\psi}_\alpha(\vec{z})\Big]$$

2/12/03

We can carry out the commutators using

$$\boxed{\begin{aligned} \{\hat{\psi}^{+}_{\alpha}(\vec{x}), \psi^{+}_{\beta}(\vec{x}')\} &= \{\hat{\psi}_{\alpha}(\vec{x}), \hat{\psi}_{\beta}(\vec{x}')\} = 0 \\ \{\hat{\psi}_{\alpha}(\vec{x}), \psi^{+}_{\beta}(\vec{x}')\} &= \delta_{\alpha\beta}\,\delta^3(\vec{x}-\vec{x}') \end{aligned}}$$

and

$$\boxed{[\hat{A}, \hat{B}\hat{C}] = \{A, B\}\hat{C} + B\{\hat{A}, \hat{C}\}}$$

When the dust settles,

$$\boxed{\begin{aligned} i\frac{\partial}{\partial t}\hat{\psi}_{\alpha_H}(\vec{x},t) &= T_x\,\hat{\psi}_{\alpha_H}(\vec{x}t) + \int d^3z'\,\hat{\psi}^{+}_{\alpha''_H}(\vec{z}'t)\,V(\vec{x},\vec{z}')_{\alpha\beta',\alpha''\beta''}\,\hat{\psi}_{H\beta''}(\vec{z}',t)\,\hat{\psi}_{H\beta'}(\vec{x},t) \\ &\overset{V\Rightarrow\lambda\delta^3(\,)}{=} T_x\,\hat{\psi}_{\alpha H}(\vec{x}t) + \lambda\,\hat{\psi}^{+}_{H\alpha'}(\vec{x}t)\,\hat{\psi}_{H\alpha'}(\vec{x}t)\,\psi_{H\alpha}(\vec{x}t) \end{aligned}}$$

So by taking a time derivative on the field, we generate the potential. $\Rightarrow$ exploits that to find $\langle\Psi_0|\hat{V}|\Psi_0\rangle$ and then $\langle\Psi_0|H|\Psi_0\rangle = E$ in terms of $G$.

- The idea is that we can evaluate <u>any</u> one-body observable if we know $G$. In second quantization,

$$\boxed{\hat{O} = \int d^3x\, d^3x'\, \hat{\psi}^{+}_{\alpha}(\vec{x}')\, O(\vec{x}',\vec{x})_{\alpha\alpha'}\, \hat{\psi}_{\alpha'}(\vec{x})}$$

$$\Rightarrow \boxed{\langle\Psi_0|\hat{O}|\Psi_0\rangle \equiv \langle\hat{O}\rangle = -i\int d^3x\, d^3x'\, O(\vec{x},\vec{x}')_{\beta\alpha}\, G_{\alpha\beta}(\vec{x}t;\vec{x}'t^{+})}$$

where the $t^{+}$ ensures the correct ordering of $\hat{\psi}^{+}$ and $\hat{\psi}$.
- Note that we take a <u>trace</u> in the spin indices.

- We can also accomodate time derivatives and local operators by taking a limiting procedure on the arguments of $G_{\alpha\beta}(\vec{x}t';\vec{x}'t')$. (see below!)

2/12/03

Examples:

- The kinetic energy

$$\boxed{\langle T \rangle = -i\int d^3x \lim_{\vec{x}'\to\vec{x}}\left[-\frac{\hbar^2\nabla^2}{2m}\,\mathrm{tr}\,G(\vec{x}t,\vec{x}'t^+)\right]}$$

where $\mathrm{tr}\,G \to G_{\alpha\alpha}$

- If $\boxed{J = \int d^3x\ \hat{J}(\vec{x})}$ with $\boxed{\hat{J}(\vec{x}) = \hat{\psi}^\dagger_\beta(\vec{x})\,J_{\beta\alpha}(\vec{x})\,\hat{\psi}_\alpha(\vec{x})}$

the

$$\boxed{\begin{aligned}\langle J(\vec{x})\rangle &= -i\lim_{t'\to t^+}\lim_{\vec{x}'\to\vec{x}} J_{\beta\alpha}(\vec{x})\,G_{\alpha\beta}(\vec{x}t,\vec{x}'t') \\ &= -i\lim_{\vec{x}'\to\vec{x}}\mathrm{tr}\,J(\vec{x})\,G_{\alpha\beta}(\vec{x}t,\vec{x}'t^+)\end{aligned}}$$

$\Rightarrow$ the density $\rho(\vec{x})$ [most interesting for a non-uniform system]

$$\Rightarrow \boxed{\hat{J}_{\beta\alpha}(\vec{x}) = \hat{\psi}^\dagger_\beta(\vec{x})\hat{\psi}_\beta(\vec{x})\,\delta_{\alpha\beta}} \Rightarrow \boxed{\rho(\vec{x}) = -i\,\mathrm{tr}\,G(\vec{x}t,\vec{x}t^+)}$$

- We can check this result using $G^0$ for a uniform system:

$$\boxed{\begin{aligned}\rho &\overset{?}{=} -\mathrm{tr}\,\delta_{\alpha\beta}\int\frac{d^3k}{(2\pi)^3}\,\cancelto{1}{e^{i\vec{k}\cdot(\vec{x}-\vec{x})}}\ \cancelto{1}{e^{-i\omega_k(t-t^+)}}\left[-\theta(k_F-|\vec{k}|)\right] \\ &= g\int\frac{d^3k}{(2\pi)^3}\,\theta(k_F-|\vec{k}|) = \frac{gk_F^3}{6\pi^2} \quad \checkmark\end{aligned}}$$

- Note that we can calculate any n-body operator expectation value using the n-body Green's function:

$$\boxed{\begin{aligned}&G^{(n)}(x_1t_1\cdots x_nt_n;\ x_1't_1\cdots x_n't_n') \\ &\equiv (-i)^n\langle\Psi_0|T[\hat{\psi}(x_1t_1)\cdots\hat{\psi}(x_nt_n)\hat{\psi}^\dagger(x_n't_n')\cdots\hat{\psi}^\dagger(x_1't_1')]|\Psi_0\rangle\end{aligned}}$$

2/12/03

So now we can use our result for $i\frac{\partial}{\partial t}\hat{\psi}_{\alpha H}(\vec{x}t)$ to write the expectation value of $\hat{V}$ and $\hat{H}$ in the ground state in terms of $\hat{G}$.

$$\Rightarrow \boxed{\begin{aligned}\langle \hat{V} \rangle &= \frac{1}{2}\int d^3x\, d^3x' \langle \Psi_0 | \hat{\psi}^\dagger_\alpha(\vec{x}) \hat{\psi}^\dagger_\beta(\vec{x}') V(\vec{x},\vec{x}')_{\alpha\alpha',\beta\beta'} \hat{\psi}_{\beta'}(\vec{x}') \hat{\psi}_{\alpha'}(\vec{x}) | \Psi_0 \rangle \\ &= -\frac{1}{2}i \int d^3x \lim_{t' \to t^+} \lim_{\vec{x}' \to \vec{x}} \left[ i\frac{\partial}{\partial t} - T(\vec{x}) \right] G_{\alpha\alpha}(\vec{x}t, \vec{x}'t')\end{aligned}}$$

and

$$\boxed{\begin{aligned}E = \langle \hat{H} \rangle &= \langle \hat{T} + \hat{V} \rangle \\ &= -\frac{1}{2}i \int d^3x \lim_{t' \to t^+} \lim_{\vec{x}' \to \vec{x}} \left[ i\frac{\partial}{\partial t} + T(\vec{x}) \right] G_{\alpha\alpha}(\vec{x}t, \vec{x}'t')\end{aligned}}$$

(recall that $T(\vec{x}) = -\frac{\nabla^2}{2m}$ in our examples so far)

- We relate these expressions to the self energy using Dyson's equation:

$$\boxed{G^{-1} = G_0^{-1} - \Sigma = i\frac{\partial}{\partial t} - T_x - \Sigma}$$

so we can write $\boxed{\left(i\frac{\partial}{\partial t} - T_x\right)G = (G^{-1} + \Sigma)G = 1 + \Sigma G}$.

- The final ingredient is

$$\boxed{\begin{aligned}\frac{d}{d\alpha}\langle \Psi_0^N(\alpha) | \hat{H}(\alpha) | \Psi_0^N(\alpha) \rangle &= E(\alpha) \\ &= \left(\frac{d}{d\alpha}\langle \Psi_0^N(\alpha)|\right) \hat{H}(\alpha) |\Psi_0^N(\alpha)\rangle + \langle \Psi_0^N(\alpha) | \hat{H}(\alpha) \left(\frac{d}{d\alpha}|\Psi_0^N(\alpha)\rangle\right) \\ &\quad + \langle \Psi_0^N(\alpha) | \frac{d\hat{H}(\alpha)}{d\alpha} | \Psi_0^N(\alpha) \rangle \\ &= E(\alpha) \underbrace{\cancel{\frac{d}{d\alpha}\langle \Psi_0^N(\alpha) | \Psi_0^N(\alpha) \rangle}}_{0} + \langle \Psi_0^N(\alpha) | \hat{V} | \Psi_0^N(\alpha) \rangle\end{aligned}}$$

2/12/03

So put it all together:

$$\boxed{\begin{aligned} E-E_0 &= \int_0^1 d\alpha \frac{d}{d\alpha}E(\alpha) = \int_0^1 \frac{d\alpha}{\alpha}\langle \Psi_0^N(\alpha)|\alpha\hat{V}|\Psi_0^N(\alpha)\rangle \\ &= -\frac{i}{2}\int_0^1 \frac{d\alpha}{\alpha}\int d^3x \left[\left(i\frac{\partial}{\partial t}-T(\vec{x})\right)G^{\alpha}_{\sigma\sigma}(\vec{x}t,\vec{x}'t^+)\right]_{\vec{x}=\vec{x}'} \\ &= -\frac{i}{2}\int_0^1 \frac{d\alpha}{\alpha}\int d^3x \left[\int d^3x''dt'\left([G^{\alpha}]^{-1}(\vec{x}t,\vec{x}''t') + \Sigma^{\alpha}(\vec{x}t,\vec{x}''t')\right)G^{\alpha}(\vec{x}''t';\vec{x}t^+)\right] \\ &= -\frac{i}{2}\int_0^1 \frac{d\alpha}{\alpha}\int d^3x\, d^3x'\, dt'\, \Sigma^{\alpha}_{\sigma\sigma}(\vec{x}t;\vec{x}'t')\, G^{\alpha}_{\sigma\sigma}(\vec{x}t,\vec{x}'t^+) \end{aligned}}$$

as desired.