PHYSICS 880.05 PROBLEM SET #1 SOLUTIONS

1. Perturbation Theory for one-dimensional delta-function system.

· We'll repeat the development in the class notes but with

$$\boxed{V(x-x') = \lambda\delta(x-x')} \quad (1)$$

in one spatial dimension, rather than in three dimensions. We still have:

i) spin independent interaction
ii) $\lambda > 0 \Rightarrow$ repulsive and $\lambda < 0$ attractive
iii) $g \equiv 2S+1$ spin states

· Note that the restriction to one dimension has nothing to do with the "spin" degeneracy, which could also be an internal degeneracy such as "color".

· Goal: Find the ground state energy/particle as a function of the "density". Compare to the 3-d results
Plan: Do perturbation theory about the noninteracting system.

· Put the system in a one-d box of length L, with N particles
· Take $L \to \infty$ at the end
· Translational-invariance plus periodic boundary conditions $\Rightarrow$ single-particle wave functions are plane waves:

$$\boxed{\psi_{k\alpha}(x) = \frac{1}{\sqrt{L}} e^{ikx}\eta_\alpha} \quad \boxed{\eta^\dagger_\alpha \eta_\beta = \delta_{\alpha\beta}} \quad (2)$$

and

$$\boxed{k_n = \frac{2\pi n}{L}, \quad n = 0, \pm1, \pm2, \ldots} \quad (3)$$

Note that $\boxed{\begin{aligned}\int \psi^{\dagger}_{k\alpha}(x)\psi_{k\alpha}(x)\,dx &= \frac{1}{L}\int e^{-i(k-k)x}dx\,\eta^{\dagger}_{\alpha}\eta_{\alpha}\\ &= \frac{1}{L}\int dx = 1 \Rightarrow \text{normalized}\end{aligned}}$ (4)

The Hamiltonian is $\boxed{H = \sum_{i=1}^{N}\frac{p_i^2}{2m} + \frac{1}{2}\lambda\sum_{i\neq j}^{N}\delta(x_i - x_j)}$ (5)

$\Rightarrow$ convert to 2nd quantization.

$$\begin{aligned}\langle k_1\alpha_1|\frac{p^2}{2m}|k_2\alpha_2\rangle &= \frac{1}{2mL}\int dx\; e^{-ik_1x}\eta^{\dagger}_{\alpha_1}\left(-\hbar^2\frac{d^2}{dx^2}\right)e^{ik_2x}\eta_{\alpha_2}\\ &= \frac{\hbar^2k_2^2}{2mL}\delta_{\alpha_1\alpha_2}\underbrace{\int dx\; e^{i(k_2-k_1)x}}_{L\delta_{k_1k_2}}\\ &= \frac{\hbar^2k_2^2}{2m}\delta_{\alpha_1\alpha_2}\delta_{k_1k_2}\end{aligned}$$

$$\Rightarrow \boxed{\hat{T} = \sum_{k\alpha}\frac{\hbar^2k^2}{2m}a^{\dagger}_{k\alpha}a_{k\alpha}} \quad (6)$$

$$\boxed{\begin{aligned}\langle k_1\alpha_1, k_2\alpha_2|V|k_3\alpha_3\,k_4\alpha_4\rangle = \frac{1}{L^2}\int dx_1\int dx_2\, &e^{-ik_1x_1}\eta^{\dagger}_{\alpha_1}(1)e^{-ik_2x_2}\eta^{\dagger}_{\alpha_2}(2)\\ &\times\lambda\delta(x_1-x_2)e^{ik_3x_1}\eta_{\alpha_3}(1)e^{ik_4x_2}\eta_{\alpha_4}(2)\end{aligned}} \quad (7)$$

$$\Rightarrow\boxed{\begin{aligned}\langle k_1\alpha_1\, k_2\alpha_2|V|k_3\alpha_3\,k_4\alpha_4\rangle &= \frac{\lambda}{L^2}\int dx_1\, e^{-i(k_1+k_2-k_3-k_4)x_1}\delta_{\alpha_1\alpha_3}\delta_{\alpha_2\alpha_4}\\ &= \frac{\lambda}{L}\delta_{\alpha_1\alpha_3}\delta_{\alpha_2\alpha_4}\delta_{k_1+k_2,\,k_3+k_4}\end{aligned}} \quad (8)$$

$$\Rightarrow\boxed{\hat{H} = \sum_{k\alpha}\frac{\hbar^2k^2}{2m}a^{\dagger}_{k\alpha}a_{k\alpha} + \frac{\lambda}{2L}\sum_{\substack{k_1k_2k_3k_4\\ \alpha_1\alpha_2\alpha_3\alpha_4}}a^{\dagger}_{k_1\alpha_1}a^{\dagger}_{k_2\alpha_2}a_{k_4\alpha_4}a_{k_3\alpha_3}\delta_{\alpha_1\alpha_3}\delta_{\alpha_2\alpha_4}\delta_{k_1+k_2,k_3+k_4}} \quad (9)$$

PS1-3

As in 3-d, we'll change variables to

$$\boxed{\begin{array}{ll} k_1 = k+q & k_3 = k \\ k_2 = p-q & k_4 = p \end{array}} \quad (10)$$

$$\Rightarrow \boxed{\hat{H} = \sum_{k\alpha} \frac{\hbar^2 k^2}{2m} a^\dagger_{k\alpha} a_{k\alpha} + \frac{\lambda}{2L} \sum_{kpq} \sum_{\alpha_1\alpha_2} a^\dagger_{k+q\,\alpha_1} a^\dagger_{p-q\,\alpha_2} a_{p\alpha_2} a_{k\alpha_1}} \quad (11)$$

(a)

- Now do dimensional analysis

$$\boxed{\frac{\text{volume}}{\text{particle}} = \frac{L}{N} \equiv l_0} \quad (12)$$

- Check the dimensions of λ: $\frac{[\lambda]}{[L]} = \frac{\hbar^2}{[m][L]^2}$ since $\delta(x_1-x_2) \sim \frac{1}{[L]}$

(we know $\int \delta(x_1-x_2)dx_1 = 1$)

$$\Rightarrow \boxed{\lambda \equiv \frac{\hbar^2}{m a_\lambda}} \quad (13) \quad \text{(we \underline{could} put } 4\pi \text{ or whatever in as well)}$$

$$\Rightarrow \boxed{\bar{L} = \frac{L}{l_0}} \quad \boxed{\bar{k} = l_0 k} \quad \boxed{\bar{p} = l_0 p} \quad \boxed{\bar{q} = l_0 q} \quad (14)$$

$$\Rightarrow \boxed{\hat{H} = \frac{\hbar^2}{m l_0^2}\left(\sum_{\bar{k}\alpha} \frac{1}{2} \bar{k}^2 a^\dagger_{\bar{k}\alpha} a_{\bar{k}\alpha} + \frac{l_0}{a_\lambda} \frac{1}{2\bar{L}} \sum_{\bar{k}\bar{p}\bar{q}} \sum_{\alpha_1\alpha_2} a^\dagger_{\bar{k}+\bar{q}\,\alpha_1} a^\dagger_{\bar{p}-\bar{q}\,\alpha_2} a_{\bar{p}\alpha_2} a_{\bar{k}\alpha_1} \right)} \quad (15)$$

- So at <u>low</u> density, $l_0/a_\lambda \to \infty$ and the interaction dominates.
- At high density, $l_0/a_\lambda \to 0$ and the interaction is small (kinetic energy dominates)

$\Rightarrow$ we expect perturbation theory to work at <u>high</u> density.

(b) Define $E = E^{(0)} + E^{(1)} + \ldots$ and we'll find $E^{(0)}$ and $E^{(1)}$

The $L \to \infty$ limit takes $\boxed{\sum_{k\alpha} f_\alpha(k) \xrightarrow[L\to\infty]{} L \sum_\alpha \int_{-\infty}^{\infty} \frac{dk}{2\pi} f_\alpha(k)} \quad (16)$

Call the ground state $|F\rangle$, fill up to k_F to get lowest energy.

$$N = \langle F|\hat{N}|F\rangle = \sum_{k\alpha} \langle F|\hat{n}_{k\alpha}|F\rangle = \sum_{k\alpha} \theta(k_F - k)$$

$$= \frac{L}{2\pi}\sum_{\alpha}\int_{-\infty}^{\infty} dk\, \theta(k_F - k) = \frac{L}{2\pi} g \cdot 2k_F$$

$$\Rightarrow \boxed{\rho \equiv \frac{N}{L} = \frac{gk_F}{\pi} = \frac{1}{l_0}} \quad (17) \Rightarrow L = N\frac{\pi}{gk_F}$$

$\Rightarrow$ expand in $\frac{1}{k_F a_\lambda}$!

Now for $E^{(0)}$:

$$E^{(0)} = \langle F|\hat{H}_0|F\rangle = \frac{\hbar^2}{2m}\sum_{k\alpha} k^2 \langle F|\hat{n}_{k\alpha}|F\rangle$$

$$= \frac{\hbar^2}{2m}\frac{L}{2\pi}\int_{-\infty}^{\infty} dk\, k^2\, \theta(k_F - |k|)$$

$$= \frac{\hbar^2}{2m}\frac{L}{2\pi}\int_{-k_F}^{k_F} k^2\, dk$$

$$= \frac{\hbar^2}{2m}\frac{L}{\pi}\frac{k_F^3}{3} = \frac{\hbar^2}{2m} N \frac{\pi}{gk_F}\frac{1}{\pi}\frac{k_F^{3\,2}}{3}$$

$$= \frac{1}{3}\frac{\hbar^2 k_F^2}{2m} N$$

$$\text{or} \quad \boxed{\frac{E^{(0)}}{N} = \frac{1}{3}\frac{\hbar^2 k_F^2}{2m}} \quad (18) \qquad \text{or} \quad \frac{E^{(0)}}{N} = \frac{1}{3}\epsilon_F$$

- 1^{st} order shift $E^{(1)}$:

$$E^{(1)} = \langle F|\hat{H}_1|F\rangle = \frac{\lambda}{2L}\sum_{kpq}\sum_{\alpha_1\alpha_2} \langle F|a^\dagger_{k+q\,\alpha_1} a^\dagger_{p-q\,\alpha_2} a_{p\alpha_2} a_{k\alpha_1}|F\rangle$$

$\Rightarrow \quad k+q, \alpha_1 = k\,\alpha_1$ and $p-q, \alpha_2 = p, \alpha_2$

or

$k+q\,\alpha_1 = p\,\alpha_2$ and $p-q, \alpha_2 = k\,\alpha_1$

Direct: $\delta_{q0} \langle F | a^\dagger_{k\alpha_1} a^\dagger_{p\alpha_2} a_{p\alpha_2} a_{k\alpha_1} | F \rangle$

$$= \delta_{q,0} \langle F | \hat{n}_{k\alpha_1} \hat{n}_{p\alpha_2} | F \rangle = \theta(k_f - |k|)\,\theta(k_f - |p|)\,\delta_{q,0}$$

$$\Rightarrow E^{(1)}_{direct} = \frac{\lambda}{2L} \sum_{\alpha_1\alpha_2} \sum_{kpq} \delta_{q0}\, \theta(k_f - |k|)\,\theta(k_f - |p|)$$

$$= \frac{\lambda}{2L} g^2 \left(\frac{L}{2\pi} \int_{-k_f}^{k_f} dk \right) \left(\frac{L}{2\pi} \int_{-k_f}^{k_f} dp \right)$$ ← note that the integrals are from $-k_f$ to $+k_f$.

$$= \frac{\lambda}{2L} N^2$$

energy density (E per length)

$$\Rightarrow \boxed{\frac{E^{(1)}_{direct}}{N} = \frac{\lambda}{2}\rho = \frac{g k_f \lambda}{2\pi}} \quad (19)$$

and

$$\boxed{\mathcal{E}^{(1)}_{direct} = \frac{\lambda}{2L^2} N^2 = \frac{\lambda}{2}\rho^2 = \frac{g^2 \lambda}{2\pi^2} k_f^2} \quad (20)$$

The exchange term is messier, as in 3-D

exchange

$$\boxed{\begin{aligned} \langle F | a^\dagger a^\dagger a a | F \rangle &\to \delta_{k+q,p}\, \delta_{\alpha_1\alpha_2} \langle F | a^\dagger_{k+q,\alpha_1} a^\dagger_{k\alpha_1} a_{k+q\alpha_1} a_{k\alpha_1} | F \rangle \\ &= -\delta_{k+q,p}\, \delta_{\alpha_1\alpha_2} \langle F | \hat{n}_{k+q,\alpha_1} \hat{n}_{k,\alpha_1} | F \rangle \\ &= -\delta_{k+q,p}\, \delta_{\alpha_1\alpha_2}\, \theta(k_f - |k+q|)\,\theta(k_f - k) \end{aligned}} \quad (21)$$

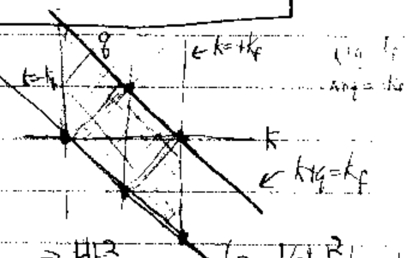
$$\Rightarrow E^{(1)}_{exchange} = \frac{\lambda}{2L} \sum_{\alpha_1\alpha_2} \sum_{kpq} (-\delta_{k+q,p})\, \delta_{\alpha_1\alpha_2}\, \theta(k_f - |k+q|)\,\theta(k_f - k)$$

$$= -\frac{\lambda}{2L} g \frac{L^2}{(2\pi)^2} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dq\; \theta(k_f - |k+q|)\,\theta(k_f - k)$$

$$= -\frac{\lambda}{2L} g \frac{L^2}{4\pi^2} 4k_f^2 = -\frac{\lambda}{2\pi^2} g k_f^2 L$$

$$= -\frac{\lambda}{2\pi^2} g k_f^2 N \frac{\pi}{g k_f}$$

$$\Rightarrow \boxed{\frac{E^{(1)}_{exchange}}{N} = -\frac{g k_f \lambda}{2\pi}} \quad (22)$$



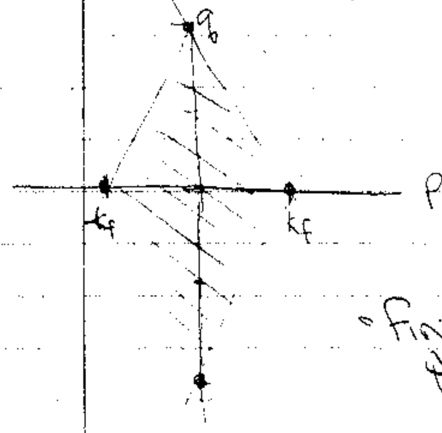
$\Rightarrow 4k_f^2$ is the area ($2 \times \frac{1}{2}(2k_f)^2$ triangles)

region in the k-q plane

Let's try the change of variables:

$$p = k + \tfrac{1}{2} q$$

$$\Rightarrow E^{(1)}_{\text{exchange}} = -\frac{g\lambda}{2}\frac{L^2}{4\pi^2}\int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dp \, \Theta\left(k_f - \left|p+\tfrac{1}{2}q\right|\right)\Theta\left(k_f - \left|p-\tfrac{1}{2}q\right|\right)$$



area is $4 \times \frac{1}{2}(k_f \cdot 2k_f) = 4k_f^2$ as before.

Here the symmetry means we only need to consider 1/4 of the area.

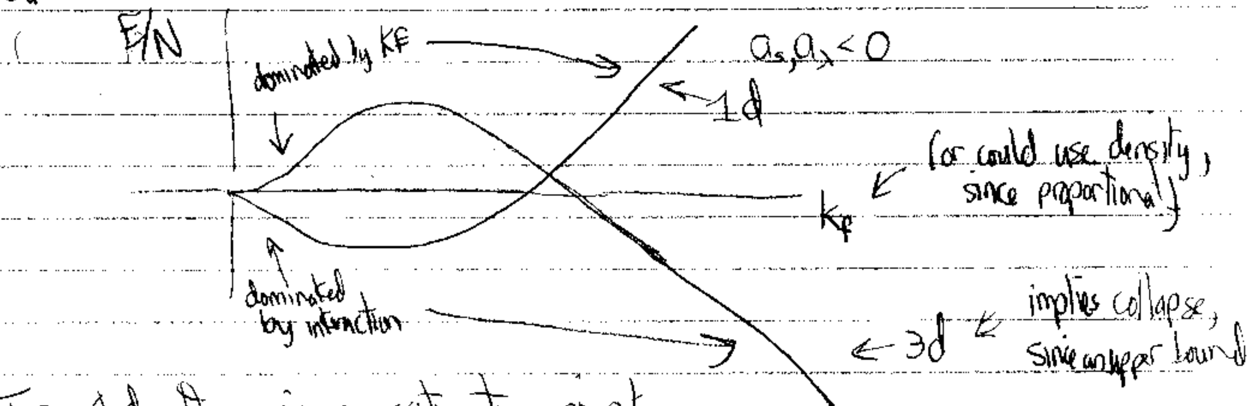
- Finally, we could also change to $q' = k+q$ and get the answer immediately!

Putting direct and exchange together: $\lambda = \frac{\hbar^2}{m a_1}$

$$\frac{E^{(1)}}{N} = \frac{\lambda}{2}\rho\left(1-\frac{1}{g}\right) = (g-1)\frac{k_F\lambda}{2\pi} = \frac{\hbar^2 k_F^2}{2m}(g-1)\frac{1}{\pi k_F a_1}$$

$$\Rightarrow \frac{E}{N} = \frac{E^{(0)}}{N} + \frac{E^{(1)}}{N} + \dots = \frac{\hbar^2 k_F^2}{2m}\left[\frac{1}{3} + (g-1)\frac{1}{\pi k_F a_1} + \dots\right] \quad \text{(1d)}$$

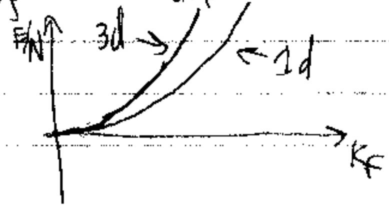
Compare to 3d
$$\frac{E}{N} = \frac{\hbar^2 k_F^2}{2m}\left[\frac{3}{5} + (g-1)\frac{2k_F a_s}{3\pi} + \dots\right] \quad \text{(3d)}$$



In 1d, there is a saturation point

$$\Rightarrow \frac{dE(\rho)}{d\rho} = 0 = \frac{dE(k_F)}{dk_F} = 0 \quad \text{(PS\#2)}.$$

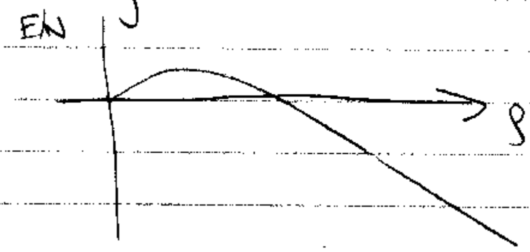
- For $\lambda > 0$



2.(a) A variational calculation at any given density will give an upper bound to the true energy. Our first-order perturbation theory calculation is a variational calculation because it is the expectation value of the full Hamiltonian in a trial state.

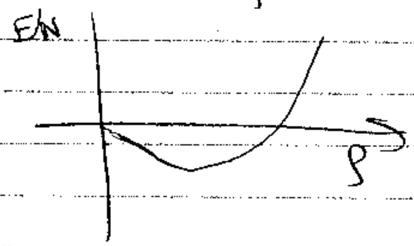
- note that even in regions of density where perturbation theory is a terrible approximation, the variational principle still holds.

- In the 3-d case, the energy per particle as a function of density looks like



For an attractive interaction. So, except at very low density, where the kinetic energy dominates and the pressure is positive, the pressure will be negative → higher density and collapse. Since the true energy/particle must lie below this curve, the exact case collapses.

b) In one-d, we have rather a different picture:



The upper bound here has an absolute minimum, so the exact energy could have one as well. However, we cannot exclude the collapse of the exact ground state.

3. 2^{nd} Order Perturbation Theory

- We've got $H = H_0 + H_1$ with $H_0|j\rangle = E_j|j\rangle$, so 2^{nd} order perturbation theory says

$$\boxed{E^{(2)} = \sum_{j\neq 0} \frac{|\langle 0|H_1|j\rangle|^2}{E_0 - E_j} = \langle 0|H_1 \frac{P}{E_0 - H_0} H_1|0\rangle} \quad (1)$$

- We just want to find the dependence on large energy states, so we really don't need a detailed evaluation of integrals.

- Now $$\boxed{H_1 = \sum_{\vec{k},\vec{p},\vec{q}} \sum_{\alpha_1\alpha_2} a^{\dagger}_{\vec{k}+\vec{q}\alpha_1} a^{\dagger}_{\vec{p}-\vec{q}\alpha_2} a_{\vec{p}\alpha_2} a_{\vec{k}\alpha_1}} \quad (2)$$

and we have $|0\rangle = |F\rangle$.

- That means that each $|j\rangle$ must differ ($j \neq 0$) from $|F\rangle$ by either one or two states. That is, one or two that are occupied in $|F\rangle$ must be unoccupied in $|j\rangle$ and vice versa.
- We can say that $\vec{p}$ and $\vec{k}$ take out those two states, while the two put back are $\vec{p}-\vec{q}$ and $\vec{k}+\vec{q}$ in $\langle j|H_1|F\rangle$.
  - if one is different and one is the same, then $\vec{p} = \vec{k}+\vec{q}$, but that makes both the same.

$\boxed{\text{• So } |j\rangle \text{ must be a "2 particle – 2 hole" state}}$

- So what is $E_0 - E_j$? $\Rightarrow$ $\boxed{\frac{\hbar^2 k^2}{2m} + \frac{\hbar^2 p^2}{2m} - \frac{\hbar^2(\vec{k}+\vec{q})^2}{2m} - \frac{\hbar^2(\vec{p}-\vec{q})^2}{2m}}$

$$\boxed{E_0 - E_j = \frac{-\hbar^2}{2m}\left(q^2 + 2k\cdot q + q^2 - 2p\cdot q\right) = -\frac{\hbar^2}{m}\left(q^2 + 2\,k\cdot q - p\cdot q\right) \xrightarrow{q\to\infty} -\frac{\hbar^2 q^2}{2m}}$$

- What do we sum over? $|\vec{p}|$ and $|\vec{k}|$ have to be in the Fermi sea, so they are bounded by k_F. $\Rightarrow$ finite integrals over $|\vec{p}|$ and $|\vec{k}|$
- But q can be anything, as long as $|\vec{p}-\vec{q}| > k_F$ and $|\vec{k}+\vec{q}| > k_F$

  Since $|j\rangle$ must be the same state on either side,

$$\langle 0|H_1|j\rangle\langle j|H_1|0\rangle \sim \theta(k_F - k)\,\theta(k_F - p)\,\theta(|\vec{p}-\vec{q}| - k_F)\,\theta(|\vec{k}+\vec{q}| - k_F)$$

and we have to pick out the same terms in each, so double sums reduce to single sums.

PS1-9

(a) $\Rightarrow \boxed{E^{(2)} \underset{q\to\infty}{\sim} -\int d^3q \int^{k_F} d^3k \int^{k_F} d^3p \, \frac{1}{q^2} \sim (\text{const}) \int_0^\infty dq}$ is the leading behavior.

$\Rightarrow \boxed{q\Big|_0^\infty \text{ diverges } \underline{\text{linearly}} \text{ as } q\to\infty}$

In one-d, the analysis is similar, but it is

(b) $\boxed{E^{(2)} \xrightarrow[q\to\infty]{} -\int_{-\infty}^{\infty} dq \, (\text{const}) \, \frac{1}{q^2}}$ which is finite in the ultraviolet ($q\to\infty$ region)

Note that when q is small, the denominator $\sim \frac{1}{q}$ which is no problem in 3-d but may be IR divergent in 1-d! However, in the case of small q, we need to put back the k and p dependence. $\Rightarrow$ finite.

(PS1-10)

1/12/03

4. [FtW 1.7] Polarized Fermi gas

- Now $g=2$, but we don't assume equal population of spin up and spin down.
  So we need to redo both $E^{(0)}$ and $E^{(1)}$.

- We write
$$\boxed{N = N_+ + N_-}$$
$$\boxed{\zeta = (N_+ - N_-)/N}$$
$$\Rightarrow \boxed{N_+ = N(1+\zeta)/2} \quad \boxed{N_- = N(1-\zeta)/2}$$

- We can use the existing results with $N \to N_+$ or N_- to get $E^{(0)}$. This will require us to define k_{F+} and k_{F-}
$$\frac{N_+}{V} = \frac{k_{F_+}^3}{6\pi^2} \Rightarrow \boxed{k_{F_+} = (6\pi^2 N_+/V)^{1/3}}$$
$$\frac{N_-}{V} = \frac{k_{F_-}^3}{6\pi^2} \Rightarrow \boxed{k_{F_-} = (6\pi^2 N_-/V)^{1/3}}$$
$$\boxed{\Rightarrow E^{(0)} = \frac{3}{5}\frac{\hbar^2 k_{F_+}^2}{2m}N_+ + \frac{3}{5}\frac{\hbar^2 k_{F_-}^2}{2m}N_- = \frac{3}{5}\frac{\hbar^2}{2m}\left(\frac{6\pi^2}{V}\right)^{2/3}\left[\left(\frac{N(1+\zeta)}{2}\right)^{5/3} + \left(\frac{N(1-\zeta)}{2}\right)^{5/3}\right] = \frac{3}{5}\frac{\hbar^2}{2m}\left(\frac{6\pi^2 N}{V}\right)^{2/3}\left[[(1+\zeta)/2]^{5/3} + [(1-\zeta)/2]^{5/3}\right]N}$$

- Now look at the $E^{(1)}$ matrix elements
  - first direct: $\delta_{\vec{q},0}\langle F|\hat{n}_{k\alpha_1}\hat{n}_{p\alpha_2}|F\rangle$
    so the spins are independent. There are four possible terms, corresponding to α_1, α_2 being $\uparrow\uparrow$ $\downarrow\downarrow$ $\uparrow\downarrow$ $\downarrow\uparrow$
$$\Rightarrow N_+N_+ + N_-N_- + 2N_+N_- = (N_+ + N_-)^2 \Rightarrow N^2 \text{ as before!}$$
$$\Rightarrow E^{(1)}_{\text{direct}} = \frac{\lambda}{2V}(N_+ + N_-)^2 = \frac{\lambda}{2V}N^2$$

  - Now exchange. The matrix element vanishes unless the α_1 and α_2 spins are the same. If it's spin up, then it's k_{F+}, spin down then k_{F-}.
    There's no cross term, unlike the direct case:
$$\boxed{E^{(1)}_{\text{exchange}} = -\frac{\lambda}{2}\left(\frac{N_+}{V}\right)^2 \cdot V - \frac{\lambda}{2}\left(\frac{N_-}{V}\right)^2 V = -\frac{\lambda}{2V}(N_+^2 + N_-^2) = \frac{\lambda}{2V}N^2\left(\frac{(1+\zeta)^2}{4} + \frac{(1-\zeta)^2}{4}\right)}$$

Combining,

$$\frac{E}{N} = \frac{3}{5}\frac{\hbar^2}{2m}\left(\frac{6\pi^2 N}{\Omega}\right)^{2/3}\left[\left[(1+\xi)/2\right]^{5/3} + \left[(1-\xi)/2\right]^{5/3}\right]$$

$$+ \frac{\lambda N}{2\Omega}\left(1 - \frac{(1+\xi)^2}{4} - \frac{(1-\xi)^2}{4}\right) \qquad 1 - \left(\frac{1}{4} + \frac{\xi}{2} + \frac{\xi^2}{4} + \frac{1}{4} - \frac{\xi}{2} + \frac{\xi^2}{4}\right)$$

$$= \mp 2^{2/3}\left(\left((1+\xi)/2\right)^{5/3} + \left((1-\xi)/2\right)^{5/3}\right) + \frac{\lambda N}{\Omega}\frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}\xi^2\right)$$

$$= \mp \frac{1}{2}\left[(1+\xi)^{5/3} + (1-\xi)^{5/3}\right] + \frac{\lambda N}{4\Omega}\left(1-\xi^2\right)$$

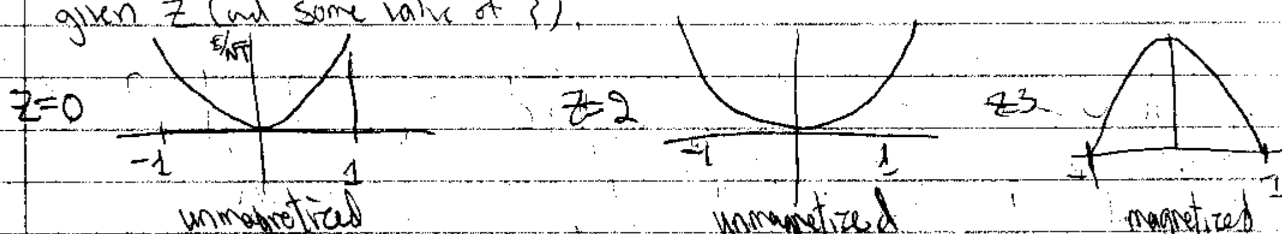
let $z \equiv \frac{\lambda N}{\Omega}\frac{1}{\mp}$

a) $\Rightarrow \boxed{\frac{E}{N}/\mp = \frac{1}{2}\left[(1+\xi)^{5/3} + (1-\xi)^{5/3}\right] + \frac{z}{4}(1-\xi^2)}$

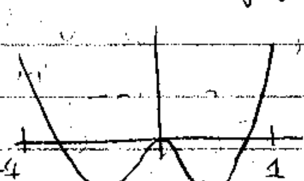
b) Plot the rhs for $-1<\xi<+1$ to see what is going on, for z from ϕ to 5 (or so)
You'll see a switch from $\xi=0$ being the minimum to $\xi \neq 0$ and then eventually $\xi=1$.

$$\boxed{\frac{\partial (E/N)/\mp}{\partial \xi} = \frac{1}{2}\left(-\frac{5}{3}(1-\xi)^{2/3} + \frac{5}{3}(1+\xi)^{2/3}\right) - \frac{\xi z}{2}}$$

Magnetized means $\xi \neq 0$ has lower energy than $\xi = 0$ for a given z (and some value of ξ).



So check in between $z=2$ and 3:



$\Rightarrow$ partially magnetized

Symmetric in ξ and $-\xi$ $\Rightarrow$ either sign gives equivalent physics.

b) The transition will be when $\frac{E}{N}/\mp$ has no curvature at $\xi = 0$

$\Rightarrow \left.\frac{\partial^2 (E/N)/\mp}{\partial \xi^2}\right|_{\xi=0} = 0 \Rightarrow \frac{10}{9} + \frac{10}{9} - z = 0$ or $\boxed{z = 20/9}$

Fully magnetized happens first when $\xi = 1$ is a minimum

$\Rightarrow$ find z so that $\left.\frac{\partial (E/N)/T}{\partial \xi}\right|_{\xi=1} = 0$

$\Rightarrow \left. -\frac{5}{3}(1-\xi)^{2/3} + 5/3(1+\xi)^{2/3} - \xi z \right|_{\xi=1} = 0$

$\Rightarrow \frac{5}{3} 2^{2/3} - z = 0$ or $\boxed{z = \frac{5}{3} 2^{2/3}}$

$\Rightarrow$ partially magnetized between these limits, or

$$\boxed{\frac{20}{9} < \frac{\lambda N}{\Omega T} < (5/3) 2^{2/3}}$$

c) The physics of the ground state is a competition between the kinetic energy and the repulsive (so positive) potential energy. The kinetic energy is minimized when the lowest k-states are filled $\Rightarrow$ equal numbers $N_+ = N_-$. But because of the Pauli principle, the potential energy is minimized (equal to zero!) if all the fermions are in a single spin state.

· The relative weight of KE and PE depends on the value of $\frac{\lambda N}{\Omega T}$, which is why there is a window of partial magnetization.
For $\frac{\lambda N}{\Omega T} < \frac{20}{9}$, kinetic energy wins $\Rightarrow \xi = 0$.
For $\frac{\lambda N}{\Omega T} > (5/3) 2^{2/3}$, PE wins absolutely $\Rightarrow \xi = 1$.!