NOTE: Tuesday's colloquium (J. Thomas from Duke) is very relevant!

3/31/03

Second Quarter of 880.05

- For now, we'll simply continue the format used in the Winter quarter
  - but please send me feedback recommending changes
  - same web page — just add new notes, etc. on top of old — notes available before class
  - 3 or 4 problem sets — first one (PS#4) will cover material from last quarter as review/reinforcement
  - I (Furnstahl) will will three lectures because of conferences in Germany (May 12, 14) and at Duke (June 9), where I'll give talks on nuclear many-body physics(!) Achim will take over at those times.

- Some of the topics that are planned:
  - review/finish effective action with auxiliary fields applied to large N → Bose limit, using effective field theory (EFT)
  - introduction to pairing and using effective action to calculate
    - pairing in nuclear physics
  - finite systems
    - Skyrme-type energy functional for nuclei (warm-up in PS#4)
    - density functional theory (DFT) and EFT
  - renormalization group for nuclear physics
    - low-momentum nucleon-nucleon potential
    - application to many-body physics
      - superfluid gap in neutron stars

- In covering these topics, we will also review and extend material discussed last quarter. Examples:
  - coherent states and other basis states (besides plane waves)
  - spin dependence
  - functional determinants

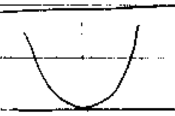
3/31/03

- Returning to effective actions and saddlepoint expansions...

- Before diving back in, let's use our model partition function to illustrate the power of saddlepoint expansions compared to ordinary perturbation theory. Then back to

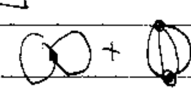
- Way back on 57 +, we considered

$$\boxed{Z(\lambda) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi/a}}\, e^{-\frac{ax^2}{2} - \frac{\lambda}{4}x^4}}$$

with $a>0, \lambda>0$  $\leftarrow$ -exponent : $\frac{ax^2}{2} + \frac{\lambda}{4}x^4$

- We calculated this in perturbation theory by expanding

$$\boxed{e^{-\frac{\lambda}{4}x^4} = 1 - \frac{\lambda}{4}x^4 + \frac{1}{2!}\left(\frac{\lambda}{4}\right)^2 x^8 + \ldots}$$

and derived a diagrammatic expansion:  + ...

- We evaluated the approximation numerically for various values of λ and found that it is an asymptotic series.
  - For $\lambda = 0.1$ (for example), the most accurate result was with only two terms ($O(\lambda^2)$) and could only achieve a relative error of about 25%.
  - Aside: Here we see the importance of combinatoric factors: Really we should use $\tilde{\lambda} = 6\lambda$ so that the exponent is $e^{-\frac{\tilde{\lambda}}{4!}x^4}$? Then $\tilde{\lambda} = .6$ doesn't seem very good anymore!
- How can we do a saddlepoint evaluation?
  - Recall the general idea for

$$\boxed{I(g) \equiv \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi g}}\, e^{-S(x)/g}}$$

where we'll expand with the idea that $g \to 0$ (so g is like $\hbar$ in the path integral or $1/\ell$ in our discussion from 222+).

3/31/03

- The dominant contribution comes from the minimum x_0 of $S(x)$ <u>if</u> the following conditions hold:

$$\boxed{\text{i) } \frac{S(x_0)}{g} \underset{g\to 0}{<} \infty} \quad \boxed{\text{ii) } \left.\frac{dS}{dx}\right|_{x=x_0} = 0} \quad \boxed{\text{iii) } \left.\frac{d^2S}{dx^2}\right|_{x=x_0} > 0}$$

- we didn't mention condition i) in our previous discussion, but it will be important below.

- Now expand $S(x)$:

$$\boxed{S(x) = S(x_0) + \frac{1}{2!} S^{(2)}(x_0)(x-x_0)^2 + \frac{1}{3!} S^{(3)}(x_0)(x-x_0)^3 + \ldots}$$

insert into $I(g)$ and change variables to $y = (x-x_0)/\sqrt{g}$ (so that the quadratic term has no g dependence):

$$I(g) \doteq e^{-S(x_0)/g} \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}}\, e^{-\frac{1}{2} S^{(2)}(x_0) y^2}\, e^{-\frac{\sqrt{g}}{3!} S^{(3)}(x_0) y^3 + O(g)}$$

$$\boxed{S^{(n)}(x_0) \equiv S_0^{(n)}} \quad \doteq e^{-S_0/g} \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}}\, e^{-S_0^{(2)} y^2/2} \left\{ 1 - \frac{g}{4!} S_0^{(4)} y^4 + \frac{g}{2}\left(\frac{S_0^{(3)}}{3!}\right)^2 y^6 + O(g^2) \right\}$$

where we've eliminated odd powers, which vanish when integrated.

These are all gaussian integrals we can do, yielding

$$\boxed{I(g) = e^{-S_0/g} \frac{1}{\sqrt{S_0^{(2)}}} \left(1 + O(g)\right)}$$

where

$$O(g) = \frac{\displaystyle\int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}} \left\{ -\frac{g}{4!} S_0^{(4)} y^4 + \frac{g}{2}\left(\frac{S_0^{(3)}}{3!}\right)^2 y^6 + O(g^2) \right\} e^{-S_0^{(2)} y^2/2}}{\displaystyle\int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}}\, e^{-S_0^{(2)} y^2/2}}$$

$$= -\frac{g}{8} \frac{S_0^{(4)}}{\left(S_0^{(2)}\right)^2} + \frac{5g}{24} \frac{\left(S_0^{(3)}\right)^2}{\left(S_0^{(2)}\right)^3} + O(g^2)$$

3/31/03

Note that the expansion in $O(g)$ is perturbative, but the overall expansion is not, as signalled by $e^{-S_0/g}$ ← (essential singularity in g)

- We can see that $Z(\lambda)$ has the appropriate form for $\lambda \to 0$ by switching to $y = \sqrt{\lambda}\, x$

$$\Rightarrow \boxed{Z(\lambda) = \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi\lambda/a}}\, e^{-\frac{1}{\lambda}\left(\frac{ay^2}{2} + \frac{1}{4}y^4\right)}}$$

- If we expand about the minimum for real x (or y), which is $y_0 = 0$, then we simply reproduce the perturbative expansion from before:

$$\boxed{S(y) = \frac{ay^2}{2} + \frac{1}{4}y^4 \underset{y_0=0}{\Rightarrow} S_0 = 0,\ S_0^{(1)} = S_0^{(3)} = 0,\ S_0^{(2)} = a,\ S_0^{(4)} = 6}$$

$$\Rightarrow \boxed{Z(\lambda) = e^{0} \frac{1}{\sqrt{a/a}}\left(1 - \frac{\lambda}{8}\frac{6}{a^2} + O(\lambda^2)\right) = 1 - \frac{3\lambda}{4a^2} + O(\lambda^2)}$$

cf. 62

- Can we expand around something else? Yes, and we'll do so by introducing an "auxiliary field" z.
  - The idea, in most cases, is to introduce an additional integral that has a term that cancels non-gaussian terms (like $-\frac{\lambda}{4}x^4$ in $Z(\lambda)$).
  - Usually do this by inserting unity. Here:

$$\boxed{1 = \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}}\, e^{-\left(z + i\sqrt{\frac{\lambda}{2}}x^2\right)^2/2} = \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}}\, e^{-\frac{z^2}{2} - i\sqrt{\frac{\lambda}{2}}zx^2}\, e^{+\frac{\lambda}{4}x^2}}$$

$$\Rightarrow \boxed{\begin{aligned} Z(\lambda) &= \int_{-\infty}^{\infty}\frac{dz}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi/a}}\, e^{-(a + i\sqrt{2\lambda}z)x^2}\, e^{-z^2/2} \\ &= \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} \frac{1}{\sqrt{1 + i\sqrt{2\lambda}/a}}\, e^{-z^2/2} = \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}}\, e^{-\frac{z^2}{2} - \frac{1}{2}\ln\left(1 + i\sqrt{2\lambda}/a\right)} \end{aligned}}$$

3/31/03

If we expand the log in powers of λ,

$$\boxed{e^{-\frac{1}{2}\ln\left(1+i\sqrt{\frac{2\lambda}{a^2}}z\right)} = 1 - \frac{1}{2}i\sqrt{\frac{2\lambda}{a}}z - \frac{3}{8}z^2\left(\frac{2\lambda}{a^2}\right) + i\frac{5}{16}z^3\left(\frac{2\lambda}{a^2}\right)^{3/2} + \frac{35}{128}z^4\left(\frac{2\lambda}{a^2}\right)^2 + O(\lambda^{5/2})}$$

and integrate term-by-term, the imaginary terms don't contribute (odd in z) and we get back the perturbative expansion for $Z(\lambda)$.

However, if we instead set

$$\boxed{S(z) \equiv \frac{z^2}{2} + \frac{1}{2}\ln\left(1 + i\sqrt{\frac{2\lambda}{a^2}}z\right)}$$

then $\boxed{S'(z_0) = z_0 + \frac{i\sqrt{2\lambda/a^2}}{2(1+i\sqrt{2\lambda/a^2}\,z_0)} = 0}$ to find zeros $z_0^{(\pm)}$

$$\Rightarrow \boxed{i\sqrt{\frac{2\lambda}{a^2}}\,z_0^{(\pm)} = \frac{-1 \pm \sqrt{1+4\lambda/a^2}}{2}}$$

which as $\lambda \to 0$ behave as

$$\boxed{i z_0^{(\pm)} \xrightarrow{\lambda\to 0} \begin{cases} \sqrt{\frac{\lambda}{2a^2}} \\ -a/\sqrt{2\lambda} \end{cases}}$$

We can only expand about the $z_0^{(+)}$ root, since $S(z_0^{(-)}) \xrightarrow{\lambda \to 0} -\infty$.

So with $S^{(n)}(z_0^{(+)}) \equiv S_0^{(n)}$, we get from the expansion

$$\boxed{S_0 = -\frac{a^2}{16\lambda}\left(\sqrt{1+4\lambda/a^2}-1\right)^2 + \frac{1}{2}\ln\left(\frac{1+\sqrt{1+4\lambda/a^2}}{2}\right)}$$

$$\boxed{S_0^{(2)} = \frac{2\sqrt{1+4\lambda/a^2}}{1+\sqrt{1+4\lambda/a^2}} > 0} \qquad \boxed{S_0^{(3)} = -\frac{8i\left(\sqrt{2\lambda}/a\right)^3}{1+\sqrt{1+4\lambda/a^2}}}$$

$$\boxed{S_0^{(4)} = \frac{-48\left(\sqrt{2\lambda}/a\right)^4}{\left(1+\sqrt{1+4\lambda/a^2}\right)^4}} \qquad \boxed{S_0^{(n)} = O(\lambda^{n/2})}$$

So for $n \geq 3$ we have a perturbative expansion (but higher powers of λ included).

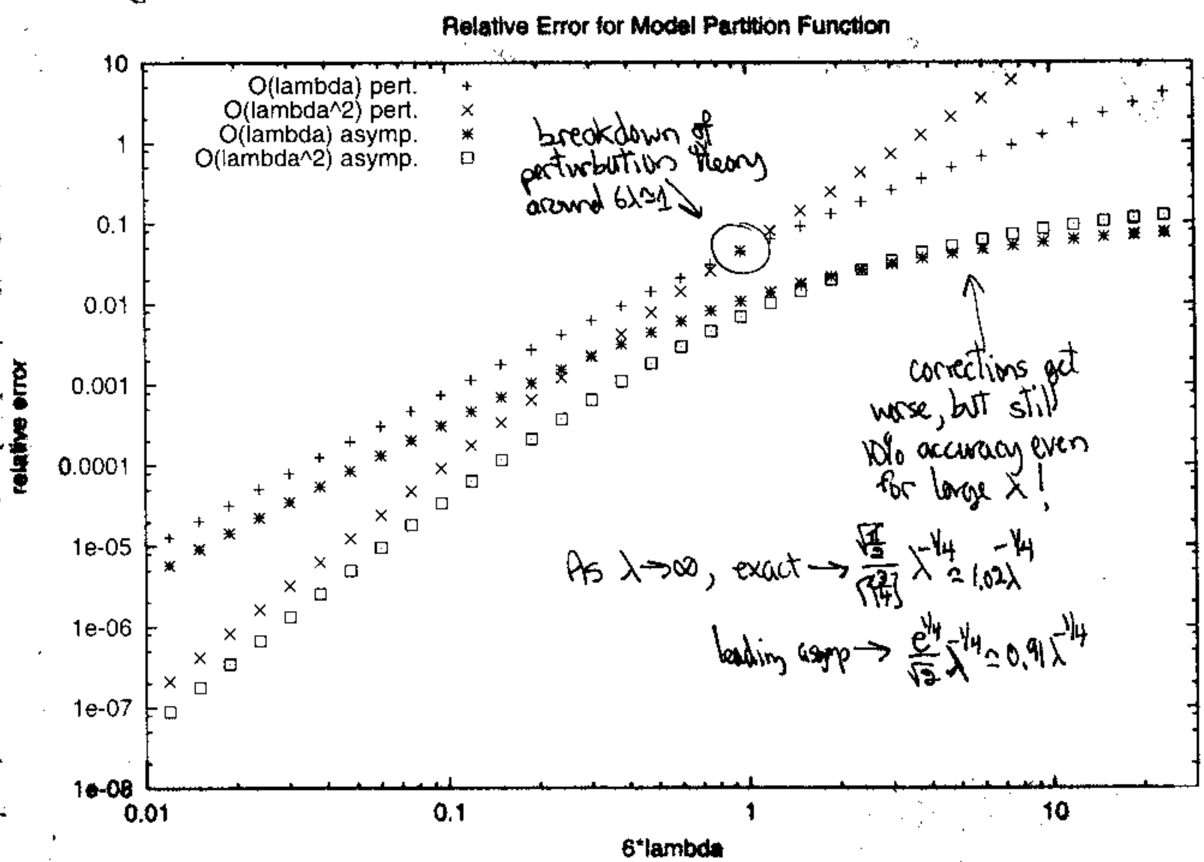
3/31/03

Plugging back into $Z(\lambda)$ and shifting variables (also take $a=1$ at this point for convenience),

$$\boxed{\begin{aligned} Z(\lambda) &= e^{\frac{1}{16\lambda}(\sqrt{1+4\lambda}-1)^2}\, e^{-\frac{1}{2}\ln\left(\frac{1+\sqrt{1+4\lambda}}{2}\right)} \int_{-\infty}^{\infty} \frac{dz'}{\sqrt{2\pi}}\, e^{-S_0^{(2)} z'^2/2} \left[1 + \frac{48(2\lambda)^4}{(1+\sqrt{1+4\lambda})^2}\frac{z'^4}{4!} + O(z'^6)\right] \\ &\doteq \frac{e^{(\sqrt{1+4\lambda}-1)^2/16\lambda}}{(1+4\lambda)^{1/4}}\left(1 + \frac{12\lambda^2}{2(1+4\lambda)(1+\sqrt{1+4\lambda})^2} + \ldots\right) \end{aligned}}$$

- The expansion has an essential singularity as $\lambda \to 0$, so not perturbation theory, but the truncation in the ( )'s is justified at small λ.

- If we compare the expansion numerically to perturbation theory, it's remarkably superior!



3/31/03

Recap of Bose Limit Discussion

If we take degeneracy $g \to \infty$ and Fermi momentum $k_F \to 0$ with the density constant: $\rho = g k_F^3 / 6\pi^2$, then we expect to recover the energy density of a dilute (spinless) Bose gas from the fermion energy density

$$\boxed{\mathcal{E} = \frac{2\pi a_s}{M}\rho^2 + \frac{2\pi a_s}{M}\rho^2 \frac{128}{15\sqrt{\pi}}\sqrt{\rho a_s^3} + \ldots}$$

$\Rightarrow$ look at large g expansion

- Plan: introduce an auxiliary field and do a saddlepoint evaluation
  - Start with

$$\boxed{\begin{aligned} Z &= \int \mathcal{D}(\psi^\dagger\psi)\, e^{i\int d^4x \left[\psi_\alpha^\dagger\left(i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m} + \mu\right)\psi_\alpha - \frac{1}{2}C_0(\psi_\alpha^\dagger\psi_\alpha)^2\right]} \\ &\to \int \mathcal{D}(\psi^\dagger\psi)\mathcal{D}\sigma\, e^{i\left[\int d^4x\, \psi_\alpha^\dagger \underbrace{\left(i\frac{\partial}{\partial t} - \frac{\nabla^2}{2m} + \mu - C_0\sigma(x)\right)}_{G^{-1}}\psi_\alpha(x) + \frac{1}{2}C_0(\sigma(x))^2\right]} \end{aligned}}$$

after inserting unity in the form:

$$\boxed{1 = \frac{\int \mathcal{D}\sigma\, e^{\frac{i}{2}C_0\int d^4x\,(\sigma(x) - \psi_\alpha^\dagger(x)\psi_\alpha(x))^2}}{\int \mathcal{D}\sigma\, e^{\frac{i}{2}C_0\int d^4x\,[\sigma(x)]^2}}}$$

- Adding a source term $J(x)\sigma(x)$ and doing the $\psi^\dagger\psi$ integrals:

$$\boxed{Z[J] = e^{iW[J]} = \int \mathcal{D}\sigma\, e^{g\,\mathrm{Tr}\ln G^{-1}(x,y)}\, e^{\frac{i}{2}C_0\int d^4x\,(\sigma(x))^2}\, e^{i\int d^4x\, J(x)\sigma(x)}}$$

  - This has a natural large g expansion (after scaling C_0 and σ).

- We want to Legendre transform to $\boxed{\Gamma[\sigma_c] \equiv W[J] - \int d^4x\, J(x)\sigma_c(x)}$ and find

$$\boxed{\frac{\delta\Gamma[\sigma_c]}{\delta\sigma_c} = 0} \quad \text{where} \quad \boxed{\sigma_c(x) = \langle\sigma(x)\rangle_J = \frac{\delta W[J]}{\delta J(x)}}$$

3/31/03

The plan is to do a stationary phase (saddlepoint for Minkowski space!) expansion of $Z[J]$ about $\sigma_c(x)$.

- This is somewhat tricky since σ_c is not the stationary point of the exponent in $Z[J]$
- However we can split the exponent into two pieces: a part which is stationary about σ_c (by construction) and the rest (which includes any counterterms), which we treat perturbatively.

- This is described in detail on (231). The end result is

$$\boxed{\Gamma[\sigma_c] = \frac{g}{i}\,\mathrm{Tr}\ln\left[G_H^{-1}(x,y)\right] + \frac{C_0}{2}\int d^4x\,(\sigma_c(x))^2 + \frac{i}{2}\mathrm{Tr}\ln\left[D_\sigma^{-1}(x,y)\right] + (\text{2PI diagrams in } D_\sigma)}$$

where

$$\boxed{G_H^{-1}(x,y) = \left[i\frac{\partial}{\partial t} + \frac{\nabla^2}{2m} + \mu - C_0\sigma_c(x)\right]\delta^4(x-y)}$$

$$\boxed{D_\sigma^{-1}(x,y) = -iC_0\delta^4(x-y) + g\,C_0^2\,G_H(y,x)\,G_H(x,y)}$$

The leading order (LO) term, which comes from the $\frac{C_0}{2}\sigma^2$ term and the $\mathrm{Tr}\ln[G_H^{-1}]$ are evaluated on (235)-(236), yielding

$$\boxed{\mathcal{E}_{LO} = \rho\left(\frac{3}{5}\frac{k_f^2}{2m} + \frac{C_0}{2}\rho\right) = \rho\frac{k_f^2}{2m}\left(\frac{3}{5} + g\frac{2}{3\pi}k_f a_s\right)}$$

which is the noninteracting piece plus the Hartree (not Fock) term.

- Evaluating $\mathcal{E}_{NLO}$ is much harder (the $\mathrm{Tr}\ln D_\sigma^{-1}$) but it simplifies dramatically in the Bose limit, generating exactly the diagrams we need, which is the sum of the ring diagrams.
  - These are evaluated on (237)-(239), yielding

$$\boxed{\mathcal{E}_1^{Bose} = \frac{2\pi a_s}{m}\rho^2\,\frac{128}{15\sqrt{\pi}}\sqrt{\rho a_s^3}}$$

3/31/03

- Next we return to the dilute Fermi gas and consider whether our perturbative calculation is really finding the correct ground state ⇒ leads us to pairing.

- Advertisement for Prof. John Thomas' colloquium Tuesday: "Universal Dynamics in a Strongly Interacting Fermi Gas"

Here's what is abstract says:

"Recent theory suggests that strongly interacting Fermi systems exhibit universal behavior. Hence, experiments which explore the dynamics of strongly interacting atomic Fermi gases provide measurements of parameters relevant to systems ranging from compact stellar objects to strongly correlated electrons. We use all-optical methods to produce a highly degenerate, two-component gas of fermionic ^6Li atoms in an applied magnetic field (910G) near a Feshbach resonance where strong interactions are observed. In this case, the S-wave scattering length is estimated to be -10^4 bohr, which is large compared to the interparticle spacing. This system provides an excellent starting point for studies of universal interactions and the onset of resonance superfluidity at very high transition temperatures. I will describe measurements of novel expansion dynamics which may be a sign of superfluidity and measurements of the interaction energy which are in reasonable agreement with predictions for nuclear matter."

PLEASE ATTEND IF AT ALL POSSIBLE!

- we'll discuss the colloquium talk on Wednesday.

3/31/03

- Let's recall what we've learned for the 1-D attractive delta-function potential $\boxed{V(x-x') = \lambda\delta(x-x')}$.
  - $\lambda < 0$ for attractive
  - g spin-flavor-isospin degeneracy

- Introduce a length scale a_λ to characterize the strength of the potential:

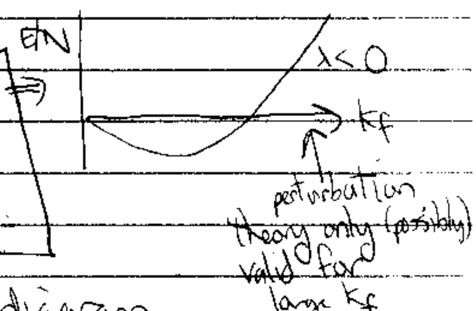
$$\boxed{\lambda \equiv \frac{\hbar^2}{m a_\lambda}}$$

- In 1-D, $\boxed{\rho = g k_f/\pi = 1/l_0}$ where l_0 is the interparticle spacing.

- So plotting against k_f is the same as against density, up to a constant factor.

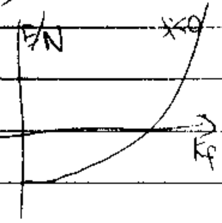
- From PS#1, kinetic energy plus , is a <u>variational</u> estimate of E/N ($\hbar = 1$)

$$\Rightarrow \boxed{\begin{aligned} E/N &= \frac{1}{3}\frac{k_f^2}{2m} + (g-1)\frac{k_f\lambda}{2\pi} + \dots \\ &= \frac{1}{2m}\left(\frac{1}{3}k_f^2 + (g-1)\frac{k_f}{\pi a_\lambda} + \dots\right) \end{aligned}} \Rightarrow$$



- From PS#3, added in beachball diagram
  $\Rightarrow$ 2nd order perturbation theory $\rightarrow$ lowers the energy

$$\boxed{\begin{aligned} E/N &= \frac{1}{3}\frac{k_f^2}{2m} + (g-1)\frac{k_f\lambda}{2\pi} - \frac{1}{24}\lambda^2 m(g-1) + \dots \\ &= \frac{1}{2m}\left(\frac{1}{3}k_f^2 + (g-1)\frac{k_f}{\pi a_\lambda} - (g-1)\frac{1}{12 a_\lambda^2} + \dots\right) \end{aligned}}$$



- Note that all terms have $1/m$ factor and one less power of k_f and one more inverse power of a_λ with each order in the perturbative expansion.

3/31/03

Let's verify this trend with a power counting argument as on page 166 for the 3-D case:

i) for every propagator we get a factor of m/k_F^2 (1/kinetic energy)

ii) for every loop integration a factor of $k_F \cdot \frac{k_F^2}{m} = k_F^3/m$ [cf. k_F^5/m]

iii) for every n-body vertex with 2i derivatives, a factor $k_F^{2i}/m\Lambda^{2i+n-3}$ $\rightarrow$ $\Lambda \sim 1/a_s$, 2-body δ function $\rightarrow \frac{4}{m} \sim \frac{1}{m a_s}$
(cf. $k_F^{2i}/m\Lambda^{2i+3n-5}$)

If an energy density diagram has L loops and I external lines and V_{2i}^n n-body vertices with 2i derivatives, it scales precisely as k_F^ν where we use the rules to count powers of k_F.

$$\boxed{\nu = 3 \times L - 2 \times I + \sum_{n=2}^{\infty}\sum_{i=0}^{\infty}(2i)V_{2i}^n}$$

But these are not independent:

$$\boxed{L = I - \sum_{n=2}^{\infty}\sum_{i=0}^{\infty} V_{2i}^n + 1} \quad \text{(topological identity)}$$

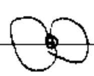
and

$$\boxed{I + \overset{0 \text{ for energy density}}{\cancel{\frac{E}{2}}} = \sum_{n=2}^{\infty}\sum_{i=0}^{\infty} n_i V_{2i}^n} \quad \text{(topological identity)}$$

Eliminating L and I, we find

$$\boxed{\nu = 3 - \sum_{n=2}^{\infty}\sum_{i=0}^{\infty}(2i+n-3)V_{2i}^n} \qquad \mathcal{E} \propto k_F^\nu$$

Check: all λ vertices ⇒ i=0, n=2

 ⇒ $V_0^2 = 1$ ⇒ $\nu = 3-1 = 2$ ✓ (and $E/N \sim k_F^{\nu-1}$)

 ⇒ $V_0^2 = 2$ ⇒ $\nu = 3-2 = 1$ ✓

If α vertices then $\nu = 3 - \alpha$ ⇒ E/N has a negative power of k_F with 3 or more vertices. (But if only $2i \geq 2$ vertices, then ok)

3/31/03

- So the perturbative solution becomes plausible at high density, but cannot be correct at very low density (recall that only the Hartree-Fock term is a variational estimate.

- One possible low-density ground state would be uniformly distributed, essentially zero energy isolated fermions.
  - The energy per particle is about zero.
  - But if there is a two-body bound state, it will have negative energy and the ground state will be a dilute gas of these bound states.

- To address this possibility, we'll first review bound states in a delta-function potential in free space
  - there's always exactly one in one dimension

- Then consider what happens in the medium
  ⇒ reformulate the bound state problem so that "Pauli blocking" can be taken into account

- Then use the effective action formalism to address the question of the true ground state and its energy.
  - We'll introduce an auxiliary field (or fields) again, but instead of being "coupled" to a particle-hole combination $\psi^+_\alpha \psi_\alpha$, we'll couple to the "pairing" combinations $\psi_\uparrow \psi_\downarrow$ and $\psi^+_\uparrow \psi^+_\downarrow$.
  - The "mean field" approximation to the effective action corresponds to the weak-coupling BCS ground state.