AJT notes (9/16/21)

- Assume scalar external probe as in Dickhoff (7.6):

$$\rho(\vec{q}) = \sum_{\vec{p}} a^\dagger_{\vec{p}} a_{\vec{p}-\vec{q}} \qquad (\hbar = 1) \tag{1}$$

Assume $|\Psi_i\rangle = |\Psi_0^A\rangle$ (ground state of A-body nucleus) and $|\Psi_f\rangle = a^\dagger_{\vec{p}} |\Psi_\alpha^{A-1}\rangle$ (excited state of A-1 nucleus and free proton / No FSI's). Transition matrix element for a particular α is

$$\langle \Psi_f | \hat{\rho}(\vec{q}) | \Psi_i \rangle = \sum_{\vec{p}'} \langle \Psi_\alpha^{A-1} | a_{\vec{p}} a^\dagger_{\vec{p}'} a_{\vec{p}'-\vec{q}} | \Psi_0^A \rangle$$

$$= \sum_{\vec{p}'} \langle \Psi_\alpha^{A-1} | \delta_{\vec{p},\vec{p}'} a_{\vec{p}'-\vec{q}} - a^\dagger_{\vec{p}'} \underbrace{a_{\vec{p}} a_{\vec{p}'-\vec{q}}}_{= -a_{\vec{p}'-\vec{q}} a_{\vec{p}}} | \Psi_0^A \rangle$$

$$= \langle \Psi_\alpha^{A-1} | a_{\vec{p}-\vec{q}} | \Psi_0^A \rangle + \sum_{\vec{p}'} \langle \Psi_\alpha^{A-1} | a^\dagger_{\vec{p}'} a_{\vec{p}'-\vec{q}} a_{\vec{p}} | \Psi_0^A \rangle \tag{2}$$

- Use $\langle \Psi_\alpha^{A-1} | = \langle \Psi_0^A | a^\dagger_\alpha$ (3)

$$\Rightarrow \; = \langle \Psi_0^A | a^\dagger_\alpha a_{\vec{p}-\vec{q}} | \Psi_0^A \rangle + \sum_{\vec{p}'} \langle \Psi_0^A | a^\dagger_\alpha a^\dagger_{\vec{p}'} a_{\vec{p}'-\vec{q}} a_{\vec{p}} | \Psi_0^A \rangle \tag{4}$$

Then use $a(\vec{q}') = \sum_\alpha \phi_\alpha(\vec{q}') a_\alpha \Rightarrow a_\alpha = \int d^3q' \, \phi^*_\alpha(\vec{q}') \, a(\vec{q}')$

For discretized version, $a_\alpha = \sum_{\vec{q}'} \Phi^*_\alpha(\vec{q}')\, a_{\vec{q}'}$ (5)

Then Eq. (4) becomes

$$= \sum_{\vec{q}'} \Phi_\alpha(\vec{q}') \langle \Psi_0^A | a^\dagger_{\vec{q}'} a_{\vec{p}-\vec{q}} | \Psi_0^A \rangle + \sum_{\vec{q}',\vec{p}'} \Phi_\alpha(\vec{q}') \langle \Psi_0^A | a^\dagger_{\vec{q}'} a^\dagger_{\vec{p}'} a_{\vec{p}'-\vec{q}}\, a_{\vec{p}} | \Psi_0^A \rangle \qquad (6)$$

Now we insert $U_\lambda^\dagger U_\lambda$ in each matrix element twice.

Let $U_\lambda |\Psi_0^A\rangle \equiv |\Phi\rangle$.

Evolved

$$\Rightarrow \sum_{\vec{q}'} \Phi_\alpha(\vec{q}') \langle \Phi | U_\lambda\, a^\dagger_{\vec{q}'} a_{\vec{p}-\vec{q}}\, U_\lambda^\dagger | \Phi \rangle$$

$$+ \sum_{\vec{q}',\vec{p}'} \Phi_\alpha(\vec{q}') \langle \Phi | U_\lambda\, a^\dagger_{\vec{q}'} a^\dagger_{\vec{p}'} a_{\vec{p}'-\vec{q}}\, a_{\vec{p}}\, U_\lambda^\dagger | \Phi \rangle \qquad (7)$$

Expand $$U_\lambda = I + \frac{1}{4} \sum_{\vec{k}\vec{k}'\vec{K}} \delta\tilde{U}(\vec{k},\vec{k}')\, a^\dagger_{\frac{\vec{K}}{2}+\vec{k}}\, a^\dagger_{\frac{\vec{K}}{2}-\vec{k}}\, a_{\frac{\vec{K}}{2}-\vec{k}'}\, a_{\frac{\vec{K}}{2}+\vec{k}'} + \cdots \qquad (8)$$

and $$U_\lambda^\dagger = I + \frac{1}{4} \sum_{\vec{k}''\vec{k}'''\vec{K}'} \delta\tilde{U}^\dagger(\vec{k}'',\vec{k}''')\, a^\dagger_{\frac{\vec{K}'}{2}+\vec{k}''}\, a^\dagger_{\frac{\vec{K}'}{2}-\vec{k}''}\, a_{\frac{\vec{K}'}{2}-\vec{k}'''}\, a_{\frac{\vec{K}'}{2}+\vec{k}'''} + \cdots \qquad (9)$$

where $\delta\tilde{U}$ is antisymmetrized.

Substitute (8) and (9) into (7) and split into eight terms of the following form:

① $I \, (a^\dagger_{\vec{q}}, a_{\vec{p}-\vec{q}}) \, I$

② $\delta\tilde{U} (a^\dagger_{\vec{q}}, a_{\vec{p}-\vec{q}}) \, I$

③ $I \, (a^\dagger_{\vec{q}}, a_{\vec{p}-\vec{q}}) \, \delta\tilde{U}^\dagger$

④ $\delta\tilde{U} (a^\dagger_{\vec{q}}, a_{\vec{p}-\vec{q}}) \, \delta\tilde{U}^\dagger$

⑤ $I \, (a^\dagger_{\vec{q}}, a^\dagger_{\vec{p}}, a_{\vec{p}+\vec{q}} \, a_{\vec{p}}) \, I$

⑥ $\delta\tilde{U} (a^\dagger_{\vec{q}}, a^\dagger_{\vec{p}}, a_{\vec{p}+\vec{q}} \, a_{\vec{p}}) \, I$

⑦ $I \, (a^\dagger_{\vec{q}}, a^\dagger_{\vec{p}}, a_{\vec{p}+\vec{q}} \, a_{\vec{p}}) \, \delta\tilde{U}^\dagger$

⑧ $\delta\tilde{U} (a^\dagger_{\vec{q}}, a^\dagger_{\vec{p}}, a_{\vec{p}+\vec{q}} \, a_{\vec{p}}) \, \delta\tilde{U}^\dagger$

Strategy: Evaluate operator at 2-body level by applying Wick's theorem with respect to true vacuum $|0\rangle$. This gives only 2-body terms $a^\dagger a^\dagger a a$.

Then evaluate $\sum \langle \Phi | a^\dagger_\rho a^\dagger_\sigma a_\nu a_\mu | \Phi \rangle$ as before

$\Rightarrow \theta(F-\rho)\,\theta(F-\sigma)$

1 $$\sum_{\vec{q}'} \Phi_\alpha(\vec{q}') \langle \Phi | a^\dagger_{\vec{q}'} a_{\vec{p}-\vec{q}} | \Phi \rangle$$

$$= \sum_{\vec{q}'} \Phi_\alpha(\vec{q}') \delta_{\vec{q}', \vec{p}-\vec{q}} \langle \Phi | a^\dagger_{\vec{p}-\vec{q}} a_{\vec{p}-\vec{q}} | \Phi \rangle$$

$$\boxed{= \Phi_\alpha(\vec{p}-\vec{q}) \Theta(k_F - |\vec{p}-\vec{q}|)} \tag{10}$$

2 $$\frac{1}{4} \sum_{\vec{k}\vec{k}'\vec{K}\vec{q}'} \Phi_\alpha(\vec{q}') \delta\tilde{U}(\vec{k},\vec{k}') \langle \Phi | a^\dagger_{\frac{\vec{K}}{2}+\vec{k}} a^\dagger_{\frac{\vec{K}}{2}-\vec{k}} a_{\frac{\vec{K}}{2}-\vec{k}'} a_{\frac{\vec{K}}{2}+\vec{k}'} a^\dagger_{\vec{q}'} a_{\vec{p}-\vec{q}} | \Phi \rangle \tag{11}$$

Two possible contractions w.r.t. $|0\rangle$. But gives identical contribution in the end since $\delta\tilde{U}$ is antisymmetrized. Take first contraction $(a^\dagger a^\dagger a a a^\dagger a)$ and multiply by 2.

(skipping details which are currently in handwritten notes)

$$\boxed{= 8 \sum_{\vec{k}} \Phi_\alpha(\vec{p}-\vec{q}) \delta\tilde{U}(\vec{k},\vec{k}) \Theta(k_F - |\vec{p}-\vec{q}|) \Theta(k_F - |\vec{p}-\vec{q}-2\vec{k}|)} \tag{12}$$

3 $$\frac{1}{4} \sum_{\vec{q}'\vec{k}''\vec{k}'''\vec{K}'} \Phi_\alpha(\vec{q}') \delta\tilde{U}^\dagger(\vec{k}'',\vec{k}''') \langle \Phi | a^\dagger_{\vec{q}'} a_{\vec{p}-\vec{q}} a^\dagger_{\frac{\vec{K}'}{2}+\vec{k}'''} a^\dagger_{\frac{\vec{K}'}{2}-\vec{k}'''} a_{\frac{\vec{K}'}{2}-\vec{k}''} a_{\frac{\vec{K}'}{2}+\vec{k}''} | \Phi \rangle \tag{13}$$

$$\downarrow$$

$$\boxed{= 8 \sum_{\vec{k}'''} \Phi_\alpha(\vec{p}-\vec{q}) \delta\tilde{U}^\dagger(\vec{k}''',\vec{k}''') \Theta(k_F - |\vec{p}-\vec{q}|) \Theta(k_F - |\vec{p}-\vec{q}-2\vec{k}'''|)} \tag{14}$$

④ $\frac{1}{16}\sum_{\vec{q}'\vec{k}\vec{k}'\vec{k}''\vec{k}'''\vec{K}\vec{K}'} \Phi_\alpha(\vec{q}')\,\delta\tilde{U}(\vec{k},\vec{k}')\,\delta\tilde{U}^+(\vec{k}'',\vec{k}''')$

$$\times \langle \Phi | a^+_{\frac{\vec{K}}{2}+\vec{k}} a^+_{\frac{\vec{K}}{2}-\vec{k}} a_{\frac{\vec{K}}{2}-\vec{k}'} a_{\frac{\vec{K}}{2}+\vec{k}'} a^+_{\vec{q}'} a_{\vec{p}-\vec{q}}\, a^+_{\frac{\vec{K}'}{2}+\vec{k}''} a^+_{\frac{\vec{K}'}{2}-\vec{k}''} a_{\frac{\vec{K}'}{2}-\vec{k}'''} a_{\frac{\vec{K}'}{2}+\vec{k}'''} |\Phi\rangle \quad (15)$$

$$\downarrow$$

$$\boxed{= \frac{1}{2}\sum_{\vec{k}\vec{K}} \Phi_\alpha(\vec{p}-\vec{q})\,\delta\tilde{U}(\vec{k},\, \vec{p}-\vec{q}-\tfrac{\vec{K}}{2})\,\delta\tilde{U}^+(\vec{p}-\vec{q}-\tfrac{\vec{K}}{2},\, \vec{k}) \times \Theta(k_F - |\tfrac{\vec{K}}{2}+\vec{k}|)\,\Theta(k_F - |\tfrac{\vec{K}}{2}-\vec{k}|)} \quad (16)$$

⑤ $$\sum_{\vec{q}',\vec{p}'} \Phi_\alpha(\vec{q}')\langle \Phi | a^+_{\vec{q}'} a^+_{\vec{p}'} a_{\vec{p}'-\vec{q}} a_{\vec{p}} |\Phi\rangle \quad (17)$$

$$= \sum_{\vec{q}'\vec{p}'} \Phi_\alpha(\vec{q}')\langle \Phi | \delta_{\vec{q}',\vec{p}}\,\delta_{\vec{p}',\vec{p}'-\vec{q}}\, a^+_{\vec{q}'} a_{\vec{p}}\, a^+_{\vec{p}'} a_{\vec{p}'-\vec{q}}$$

$$- \delta_{\vec{q}',\vec{p}'-\vec{q}}\,\delta_{\vec{p}',\vec{p}}\, a^+_{\vec{q}'} a_{\vec{p}'-\vec{q}}\, a^+_{\vec{p}'} a_{\vec{p}} |\Phi\rangle$$

$$= \sum_{\vec{p}'} \Phi_\alpha(\vec{p})\,\delta_{\vec{q},0}\langle \Phi | \hat{n}_{\vec{p}}\,\hat{n}_{\vec{p}'} |\Phi\rangle$$

$$- \Phi_\alpha(\vec{p}-\vec{q})\langle \Phi | \hat{n}_{\vec{p}-\vec{q}}\,\hat{n}_{\vec{p}} |\Phi\rangle$$

$$\boxed{= \delta_{\vec{q},0}\sum_{\vec{p}'} \Phi_\alpha(\vec{p})\,\Theta(k_F - p)\,\Theta(k_F - p') - \Phi_\alpha(\vec{p}-\vec{q})\,\Theta(k_F - p)\,\Theta(k_F - |\vec{p}-\vec{q}|)} \quad (18)$$

6 $$\frac{1}{4}\sum_{\vec{k}\vec{k}'\vec{K}\vec{q}'\vec{p}'} \Phi_\alpha(\vec{q}')\,\delta U(\vec{k},\vec{k}')$$

$$\times \langle \Phi | a^\dagger_{\frac{\vec{K}}{2}+\vec{k}} a^\dagger_{\frac{\vec{K}}{2}-\vec{k}} a_{\frac{\vec{K}}{2}-\vec{k}'} a_{\frac{\vec{K}}{2}+\vec{k}'} a^\dagger_{\vec{q}'} a_{\vec{p}}\, a^\dagger_{\vec{p}'} a_{\vec{p}'-\vec{q}} | \Phi \rangle \tag{19}$$

$$\downarrow$$

$$\boxed{= 8 \sum_{\vec{k}} \Phi_\alpha(\vec{p}-\vec{q})\,\delta U(\vec{k},\vec{k}-\vec{q})\,\theta(k_F - p)\,\theta(k_F - |\vec{p}-2\vec{k}|)} \tag{20}$$

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7 $$\frac{1}{4}\sum_{\vec{k}''\vec{k}'''\vec{K}'\vec{q}'\vec{p}'} \Phi_\alpha(\vec{q}')\,\delta U^\dagger(\vec{k}'',\vec{k}''')$$

$$\times \langle \Phi | a^\dagger_{\vec{q}'} a_{\vec{p}}\, a^\dagger_{\vec{p}'} a_{\vec{p}'-\vec{q}}\, a^\dagger_{\frac{\vec{K}'}{2}+\vec{k}''} a^\dagger_{\frac{\vec{K}'}{2}-\vec{k}''} a_{\frac{\vec{K}'}{2}-\vec{k}'''} a_{\frac{\vec{K}'}{2}+\vec{k}'''} | \Phi \rangle \tag{21}$$

$$\downarrow$$

$$\boxed{= 8 \sum_{\vec{k}''} \Phi_\alpha(\vec{p}-\vec{q})\,\delta U^\dagger(\vec{k}'''+\vec{q},\vec{k}''')\,\theta(k_F - |\vec{p}-\vec{q}|)\,\theta(k_F - |\vec{p}-\vec{q}-2\vec{k}'''|)} \tag{22}$$

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8 $$\frac{1}{16}\sum_{\vec{k}\vec{k}'\vec{k}''\vec{k}'''\vec{K}\vec{K}'\vec{q}'\vec{p}'} \Phi_\alpha(\vec{q}')\,\delta U(\vec{k},\vec{k}')\,\delta U^\dagger(\vec{k}'',\vec{k}''')$$

$$\times \langle \Phi | a^\dagger_{\frac{\vec{K}}{2}+\vec{k}} a^\dagger_{\frac{\vec{K}}{2}-\vec{k}} a_{\frac{\vec{K}}{2}-\vec{k}'} a_{\frac{\vec{K}}{2}+\vec{k}'} a^\dagger_{\vec{q}'} a^\dagger_{\vec{p}'} a_{\vec{p}'-\vec{q}}\, a_{\vec{p}}$$

$$\times\, a^\dagger_{\frac{\vec{K}'}{2}+\vec{k}''} a^\dagger_{\frac{\vec{K}'}{2}-\vec{k}''} a_{\frac{\vec{K}'}{2}-\vec{k}'''} a_{\frac{\vec{K}'}{2}+\vec{k}'''} | \Phi \rangle \tag{23}$$

$$\boxed{\begin{aligned} &= \sum_{\vec{k}\vec{K}} \phi_\alpha(\vec{p}-\vec{q})\, \delta\tilde{U}\left(\vec{k},\, \vec{p}-\vec{q}-\frac{\vec{K}}{2}\right) \delta\tilde{U}^{+}\left(\vec{p}-\frac{\vec{K}}{2},\, \vec{k}\right) \\ &\times \Theta\left(k_F - \left|\frac{\vec{K}}{2}+\vec{k}\right|\right) \Theta\left(k_F - \left|\frac{\vec{K}}{2}-\vec{k}\right|\right) \end{aligned}} \qquad (24)$$