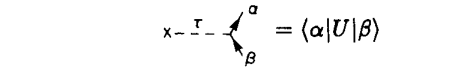
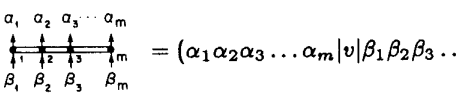
treat a residual Hamiltonian $V = H - H_0$ containing arbitrary many-body operators. In addition to two-body vertices, a labeled diagram may have one-body vertices represented by



$$= \langle \alpha | U | \beta \rangle \tag{2.117a}$$

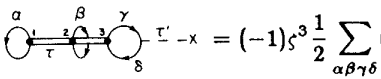
and m-body vertices represented by



$$= (\alpha_1 \alpha_2 \alpha_3 \ldots \alpha_m | v | \beta_1 \beta_2 \beta_3 \ldots \beta_m) \ . \tag{2.117b}$$

The numbered points at which incoming lines meet outgoing lines will be denoted as joints. The n^{th} order contribution to $\frac{Z}{Z_0}$ is given by all distinct unlabeled diagrams composed of n vertices, where now these n vertices involve all possible combinations of the one and many-body vertices included in the theory. The symmetry factor S is now equal to the number of transformations composed of time permutations among identical vertices and permutations of the numbered joints of each many-body vertex which yield a deformation.

As an example, for a residual Hamiltonian, V, containing a one-body potential U and a three-body potential W, one possible graph is



$$= (-1)\zeta^3 \frac{1}{2} \sum_{\alpha\beta\gamma\delta} (\alpha\beta\gamma | W | \alpha\beta\delta) \langle \delta | U | \gamma \rangle g_\alpha(0) g_\beta(0) \times \int_0^\beta d\tau d\tau' g_\gamma(\tau' - \tau) g_\delta(\tau - \tau') \ . \tag{2.118}$$

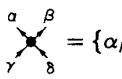
The symmetry factor is 2 since the only transformation which produces a deformation of the original diagram is the exchange of joints 1 and 3.

HUGENHOLTZ DIAGRAMS

The unlabeled Feynman diagrams derived above treat direct and exchange matrix elements separately. For many purposes it is simpler and more convenient to combine them in a single symmetrized or antisymmetrized matrix element. The resulting diagrams, called Hugenholtz diagrams, are readily derived starting from the symmetrized or antisymmetrized version of the residual Hamiltonian, V. For simplicity, we again consider the case of a two-body potential.

$$\begin{aligned} V &= \frac{1}{4} \sum_{\alpha\beta\gamma\delta} \{\alpha\beta | v | \gamma\delta\} a_\alpha^\dagger a_\beta^\dagger a_\delta a_\gamma \\ &= \frac{1}{4} \sum_{\alpha\beta\gamma\delta} [(\alpha\beta | v | \gamma\delta) + \zeta(\alpha\beta | v | \delta\gamma)] a_\alpha^\dagger a_\beta^\dagger a_\delta a_\gamma \ . \end{aligned} \tag{2.119}$$

Since we no longer wish to distinguish direct and exchange diagrams, this vertex will now be represented graphically by a dot with two incoming and two outgoing lines



$$= \{\alpha\beta | v | \gamma\delta\} \ . \tag{2.120}$$

According to Wick's theorem, we again construct diagrams by drawing n vertices and connecting all incoming lines with outgoing lines. Note, however, that by the symmetry or antisymmetry of the vertex, two contractions corresponding to the exchange of the incoming or outgoing lines associated with a given vertex are equal:

$$\sum_{\gamma\delta} \{\alpha\beta | v | \gamma\delta\} a_\alpha^\dagger a_\beta^\dagger \overbrace{a_\delta a_\gamma \ldots a_{\alpha'}^\dagger} a_{\beta'}^\dagger = \sum_{\gamma\delta} \{\alpha\beta | v | \gamma\delta\} a_\alpha^\dagger a_\beta^\dagger \overbrace{a_\delta a_\gamma \ldots a_{\alpha'}^\dagger} a_{\beta'}^\dagger \ . \tag{2.121}$$

Thus, in contrast to the previous Feynman diagrams, we will no longer need to distinguish the two incoming or outgoing lines of a vertex.

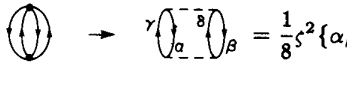
A diagram which does not distinguish the two incoming lines or two outgoing lines at each vertex represents many sets of contractions. To count the number of contractions associated with a given Hugenholtz diagram it is useful to define an equivalent pair of lines as two directed propagators which begin at the same vertex, end at the same vertex (possibly the original vertex), and point in the same direction.

First, consider the two outgoing propagators from a given vertex v_i, and assume that they are not an equivalent pair. Then they terminate at different vertices, say v_j and v_k, and in addition other propagators terminating at v_j and v_k cannot belong to equivalent pairs. Now associate the factor $\sqrt{2}$ with each end of every line which does not belong to an equivalent pair. Then the product of the $\sqrt{2}$'s accounts for the two choices for contracting non-equivalent outgoing lines at any vertex, such as v_i and the two choices for contracting non-equivalent incoming lines at any vertex, such as v_j and v_k. It is convenient to use the four factors of $\sqrt{2}$ associated with the non-equivalent pair of lines originating from a vertex to cancel the factor $\frac{1}{4}$ associated with that vertex from Eq. (2.119).

Secondly, consider the case in which the two outgoing propagators from a given vertex v_i are an equivalent pair. This pair of lines must terminate at the same vertex and therefore has only 2 equal contractions. When combined with the factor $\frac{1}{4}$ associated with the vertex at which the pair originated, we see that each equivalent pair produces an overall factor $\frac{1}{2}$. Hence all the contractions arising from Wick's theorem are correctly counted if for each Hugenholtz diagram one eliminates the factor $\frac{1}{4^n}$ arising from the n interactions and includes a factor of $\frac{1}{2}$ for each equivalent pair of propagators.

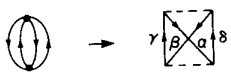
Since we do not distinguish pairs of lines entering or leaving vertices, labeled Hugenholtz diagrams only possess time labels and arrows denoting the direction of propagators. The symmetry factor S is then the number of permutations of the time labels which yield a deformation of the original labeled Hugenholtz diagram, and the number of distinct labeled Hugenholtz diagrams corresponding to a given unlabeled diagram is $\frac{n!}{S}$.

Finally, the sign of a Hugenholtz diagram is most conveniently determined by selecting any order of assigning the propagators to the left and right sides of a conventional vertex in a corresponding Feynman diagram, and applying the previous sign rule $(-1)^n \zeta^{n_L}$ to the resulting diagram. By the equivalence with respect to interchange at a vertex, although different assignments may produce different Feynman diagrams, they must yield the same contribution. For example



$$= \frac{1}{8} \zeta^2 \{\alpha\beta | v | \gamma\delta\} \{\gamma\delta | v | \alpha\beta\} \tag{2.122a}$$

or



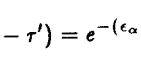
$$= \frac{1}{8}\varsigma\{\alpha\beta|v|\gamma\delta\}\{\gamma\delta|v|\beta\alpha\} \qquad (2.122b)$$
$$= \frac{1}{8}\varsigma^2\{\alpha\beta|v|\gamma\delta\}\{\gamma\delta|v|\alpha\beta\} \ .$$

These arguments give rise to the following rules, designated by the subscript H, for calculating the n^{th} order contribution to the perturbation expansion of $\frac{Z}{Z_o}$ using Hugenholtz diagrams:

1_H Draw all distinct unlabeled diagrams composed of n vertices connected by directed lines. Two diagrams are distinct if they cannot be deformed so as to coincide completely, including the direction of arrows on propagators. For each distinct unlabeled Hugenholtz diagram, evaluate the contribution as follows.

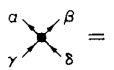
2_H Calculate the symmetry factor S for the diagram. This may be done by adding time labels to each vertex in the diagram. Then S is equal to the number of time permutations which transform the labeled diagram into a deformation of itself.

3_H Assign a time label τ_i to each vertex and a single-particle index to each directed line. For each directed line include the factor



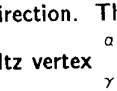
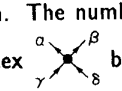
$$= g_\alpha(\tau - \tau') = e^{-(\epsilon_\alpha - \mu)(\tau - \tau')}[(1 + \varsigma n_a\theta(\tau - \tau' - \eta) + \varsigma n_a\theta(\tau' - \tau + \eta)] \ .$$

4_H For each vertex, include the factor



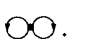
$$= \{\alpha\beta|v|\gamma\delta\} = (\alpha\beta|v|\gamma\delta) + \varsigma(\alpha\beta|v|\delta\gamma) \ .$$

5_H Sum over all single-particle indices and integrate all times over the interval $[0, \beta]$.

6_H Multiply the result by the factor $\frac{(-1)^n\varsigma^{n_L}}{2^{n_e}S}$ The symmetry factor S is defined above in (2_H). The number of equivalent pairs of lines, n_e, is the number of pairs of lines beginning at the same vertex, terminating at the same vertex, and oriented in the same direction. The number of closed loops n_L is obtained by replacing each Hugenholtz vertex  by a conventional vertex  and counting the number of closed loops as for a Feynman diagram. Note that the order of the labels on the conventional vertex must agree with the matrix elements in (4_H).

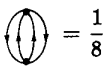
It is an instructive exercise to generalize these Hugenholtz diagram rules to the case of n-body interactions (see Problem 2.9).

At this point, it is useful to recalculate in terms of Hugenholtz diagrams some of the contributions we previously evaluated using Feynman diagrams. The only diagram for $n = 1$ is

 (2.123)

There is one equivalent pair of lines and because no time permutations are possible, $S = 1$. Thus the contribution is equal to that in Eq. (2.99).

In second order, whereas there are 24 labeled diagrams and eight unlabeled Feynman diagrams, there are three Hugenholtz diagrams. The diagram



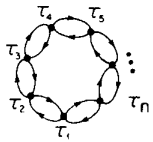
$$= \frac{1}{8}\int_0^\beta d\tau_1 \int_0^\beta d\tau_2 g_\alpha(\tau_1 - \tau_2)g_\beta(\tau_1 - \tau_2)g_\gamma(\tau_2 - \tau_1)g_\delta(\tau_2 - \tau_1) \qquad (2.124a)$$
$$\times [(\alpha\beta|v|\gamma\delta) + \varsigma(\alpha\beta|v|\delta\gamma)][(\gamma\delta|v|\alpha\beta) + \varsigma(\gamma\delta|v|\beta\alpha)]$$

corresponds to the two unlabeled Feynman diagrams G and H of Fig. (2.4). It has two equivalent pairs and the symmetry factor is 2 since interchange of the upper and lower interactions is a deformation of the original diagram. Note that the two contributions from $(\alpha\beta|v|\gamma\delta)(\gamma\delta|v|\alpha\beta)$ and $(\alpha\beta|v|\delta\gamma)(\gamma\delta|v|\beta\alpha)$ are equal, reproducing the direct graph (H) of Fig. (2.4) with the correct weight $\frac{1}{4}$ and that the contributions from $(\alpha\beta|v|\gamma\delta)(\gamma\delta|v|\beta\alpha)$ and $(\alpha\beta|v|\delta\gamma)(\gamma\delta|v|\alpha\beta)$ are equal, reproducing the exchange graph (G) of Fig. (2.4) with weight $\frac{1}{4}$. Similarly, the graph

 (2.124b)

has no equivalent pairs and symmetry factor 2 so that when all the combinations of direct and exchange matrix elements are enumerated, the three Feynman graphs (D), (E), and (F) of Fig. (2.4) are reproduced with the proper weight.

The counting for higher-order ring diagrams proceeds as for the case of Feynman diagrams. The generic n^{th} order ring diagram

 (2.125)

has no equivalent pairs and a symmetry factor $S = 2n$ to account for the cyclic permutations of $(\tau_1\tau_2\ldots\tau_n)$ and $(\tau_n\ldots\tau_2\tau_1)$ which yield deformations of the original diagram. Thus, for $n = 1$ and $n > 2$, the factors $\frac{1}{2n}$ for Hugenholtz ring diagrams are identical to those we derived before for Feynman direct ring diagrams. However, for $n = 2$, because of the presence of two pairs of equivalent lines, the correct factor of $\frac{1}{8}$ is one half of the general ring factor $\frac{1}{2n}$. Hence, with antisymmetrized matrix elements, the formula $-\frac{1}{2}\mathrm{Tr}[\ln(1 - \varsigma vgg)]$ overcounts the second-order diagram, and a correction term must therefore be subtracted explicitly.

As in the case of unlabeled Feynman diagrams, it is valuable to derive a sum rule which provides a consistency check on the enumeration of Hugenholtz diagrams and the calculation of symmetry factors. We have already seen that the number of distinct labeled diagrams corresponding to a given unlabeled diagram is $\frac{n!}{S}$. For each distinct labeled diagram, there correspond $\frac{4^n}{2^{n_e}}$ contractions, where a factor of 4 arises for each vertex from which a non-equivalent pair of lines originates and a factor of 2 occurs for each vertex from which an equivalent pair originates. Therefore, the total number of

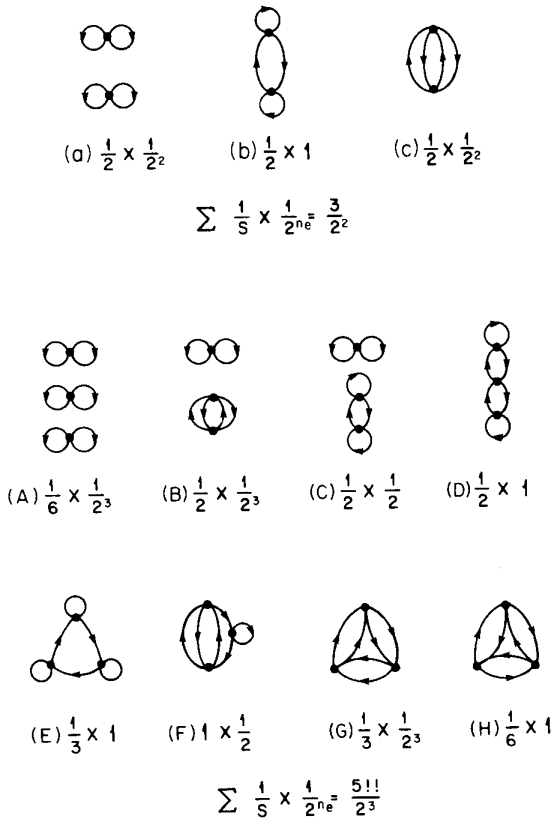


Fig. 2.5 Second and third order Hugenholtz diagrams with factors $\frac{1}{S}\times\frac{1}{2^{n_e}}$.

contractions, which is $(2n)!$, must be obtained by summing $\frac{4^n}{2^{n_e}}\times\frac{n!}{S}$ over all distinct unlabeled diagrams. Thus, we obtain

$$\sum \frac{1}{S2^{n_e}}=\frac{(2n)!}{4^n n!}=\frac{(2n-1)!!}{2^n} \tag{2.126}$$

where the sum is over all distinct unlabeled Hugenholtz diagrams.

In Fig. 2.5 all the second-order and third-order Hugenholtz diagrams are shown, together with the factor $\frac{1}{S}\times\frac{1}{2^{n_e}}$. Again, it is an instructive exercise to verify the diagrams, symmetry factors and sum rule. In comparison with the 720 original contractions, the 8 third-order Hugenholtz diagrams represent a significant simplification.

FREQUENCY AND MOMENTUM REPRESENTATION

For systems which are homogeneous in space such as a liquid or gas, it is frequently advantageous to perform the perturbation expansion in momentum representation. Similarly, for problems which are homogeneous in time, it is often useful to Fourier transform from time to frequency.

The following conventions will be used in defining Fourier series and Fourier transforms. In one dimension for a periodic function $F(x)$ in a box of length L, we write the Fourier series as

$$\tilde{f}(k_n)=\int_0^L dx e^{-ik_n x}f(x) \tag{2.127a}$$

$$f(x)=\frac{1}{L}\sum_{k_n} e^{ik_n x}\tilde{f}(k_n) \tag{2.127b}$$

$$k_n=\frac{2\pi n}{L} \tag{2.127c}$$

and the corresponding Fourier transform for an infinite system as

$$\tilde{f}(k)=\int_{-\infty}^{\infty} dx e^{-ikx}f(x) \tag{2.128a}$$

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ikx}\tilde{f}(k)\ . \tag{2.128b}$$

For a function of time $g(\tau)$ which is periodic or antiperiodic in the interval $(0,\beta)$, we write the Fourier series as

$$\tilde{g}(\omega_n)=\int_0^\beta d\tau e^{i\omega_n\tau}g(\tau) \tag{2.129a}$$

$$g(\tau)=\frac{1}{\beta}\sum_{\omega_n} e^{-i\omega_n\tau}\tilde{g}(\omega_n) \tag{2.129b}$$

where for periodic functions

$$\omega_n=\frac{2\pi n}{\beta} \tag{2.129c}$$

and for antiperiodic functions

$$\omega_n=\frac{(2n+1)\pi}{\beta}\ . \tag{2.129d}$$

Consider the evaluation of a diagram for a spatially homogeneous system. In this case H_0 will be translationally invariant so that the eigenstates will be plane waves, and it is most convenient to formulate the problem in a finite box with periodic boundary conditions. Denoting the discrete momentum $\vec{k}_n\equiv(k^x_{n_x},k^y_{n_y},k^z_{n_z})$ and volume $\mathcal{V}=L_xL_yL_z$, the matrix element of the potential evaluated with normalized eigenstates is

$$\begin{aligned}(\vec{k}_{n_1}\vec{k}_{n_2}|v|\vec{k}_{n_3}\vec{k}_{n_4}) &= \int \frac{d^3x_1 d^3x_2}{\mathcal{V}^2} v(\vec{x}_1-\vec{x}_2)e^{i[(\vec{k}_{n_1}-\vec{k}_{n_3})\cdot\vec{x}_1+(\vec{k}_{n_2}-\vec{k}_{n_4})\cdot\vec{x}_2]}\\ &= \int \frac{d^3r d^3R}{\mathcal{V}^2} v(\vec{r})e^{i\vec{R}(\vec{k}_{n_1}+\vec{k}_{n_2}-\vec{k}_{n_3}-\vec{k}_{n_4})+\frac{i\vec{r}}{2}(\vec{k}_{n_1}-\vec{k}_{n_3}+\vec{k}_{n_4}-\vec{k}_{n_2})}\\ &= \frac{1}{\mathcal{V}}\delta^{(3)}_{\vec{k}_{n_1}+\vec{k}_{n_2},\vec{k}_{n_3}+\vec{k}_{n_4}}\tilde{v}(\vec{k}_{n_1}-\vec{k}_{n_3})\ .\end{aligned} \tag{2.130}$$

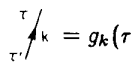
Thus, each interaction conserves momentum, with the Kronecker δ requiring that the sum of the momenta of propagators entering the vertex equal the sum of the momenta leaving the vertex, and carries a factor of $\frac{1}{\mathcal{V}}$.

Every diagram can be decomposed into connected parts, that is parts in which an uninterrupted chain of propagators connects any vertex to any other vertex. For example in Fig. 2.5, diagram A has three connected parts, diagrams B and C have two connected parts, and diagrams D through H have one connected part.

The diagram rules for a connected diagram with m interactions give rise to the following contributions. There will be $2m$ propagators $g_{k_n}(\tau_i - \tau_j)$, each summed independently over a complete set of momentum states k_n. The m interactions will give rise to the product of m momemtum-conserving Kronecker δ's and a factor of $\mathcal{V}^{-m}$. Since the diagram is linked and momentum is conserved at each vertex, one of the m Kronecker δ's is redundant. Thus, there are $(m-1)$ constraints on the $2m$ momenta for the propagator, leading to a sum over $(m+1)$ independent momenta. In the continuum limit, each sum over an independent mometum is replaced by an integral $\sum_{k_n} F(k_n) \to \frac{\mathcal{V}}{(2\pi)^3}\int d^3k F(k)$. Combining the factor $\mathcal{V}^{m+1}$ from these independent momentum integrals with the factor $\mathcal{V}^{-m}$ from the m matrix elements thus yields an overall factor of the volume $\mathcal{V}$, for each connected diagram. A diagram having n_c connected parts yields a contribution proportional to $\mathcal{V}^{n_c}$ and it is evident that only completely connected diagrams, having $n_c = 1$, produce extensive contributions.

The diagram rules presented previously for Feynman and Hugenholtz diagrams are thus modified as follows in mometum representation:

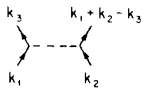
$3_{F,H}$ Assign momentum labels to each directed line as follows. Within each connected part of the diagram containing m interactions, select $m+1$ propagators to label independently and use conservation of momentum at each vertex to label the remaining propagators in terms of combinations of the independent momenta. Assign time labels as before and for each directed line include the factor



$$= g_k(\tau - \tau') = e^{-(\epsilon_k - \mu)(\tau - \tau')}[(1+\varsigma n_k)\theta(\tau - \tau' - \eta) + \varsigma n_k \theta(\tau' - \tau + \eta)] \ .$$

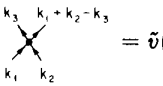
The single-particle energy ϵ_k may include the contribution of a general non-local one-body potential in addition to the kinetic energy $\frac{k^2}{2m}$.

4_F For each vertex include the factor



$$= \tilde{v}(\vec{k}_1 - \vec{k}_3) \ .$$

4_H For each vertex include the factor



$$= \tilde{v}(\vec{k}_1 - \vec{k}_3) + \varsigma\tilde{v}(\vec{k}_2 - \vec{k}_3) \ .$$

$5_{F,H}$ For each independent momentum, perform the integral $\int \frac{d^3k}{(2\pi)^3}$ and integrate all times over the interval $[0, \beta]$.

$6_{F,H}$ Multiply the previously defined factor by $\mathcal{V}^{n_c}$ where n_c is the number of connected parts in the diagram.

Fourier transformation from τ to ω for a problem which is homogeneous in time proceeds analogously, except that periodic or antiperiodic functions of τ are integrated over the finite interval $[0, \beta]$. The propagator $g(\tau_1, \tau_2)$ defined in Eq. (2.98) is periodic or antiperiodic with period β in each of the times and therefore also in the relative time. Although the relative time ranges from $-\beta$ to β, it is sufficient to calculate the Fourier coefficients by integrating over the range $[0, \beta]$. Using Eq. (2.74b) for n_α, the definitions Eqs. (2.129) with $\omega_n = \frac{2\pi n}{\beta}$ for Bosons and $\omega_n = \frac{(2n+1)\pi}{\beta}$ for Fermions, and noting that $e^{i\beta\omega_n} = \varsigma$, we obtain

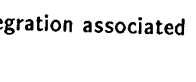
$$\begin{aligned}
\tilde{g}_\alpha &= \int_0^\beta d\tau e^{(i\omega_n - (\epsilon_\alpha - \mu))\tau}[\theta(\tau)(1+\varsigma n_\alpha) + \varsigma\theta(-\tau)n_\alpha] \\
&= \frac{e^{\beta(i\omega_n - (\epsilon_\alpha - \mu))} - 1}{i\omega_n - (\epsilon_\alpha - \mu)} \frac{e^{\beta(\epsilon_\alpha - \mu)}}{e^{\beta(\epsilon_\alpha - \mu)} - \varsigma} \\
&= \frac{\varsigma - e^{\beta(\epsilon_\alpha - \mu)}}{[(i\omega_n - \epsilon_\alpha - \mu)](e^{\beta(\epsilon_\alpha - \mu)} - \varsigma)} \\
&= \frac{-1}{i\omega_n - (\epsilon_\alpha - \mu)} \qquad (2.131a)
\end{aligned}$$

and hence

$$g_\alpha(\tau) = \sum_{\omega_n} \frac{-1}{\beta} e^{-i\omega_n\tau} \frac{1}{i\omega_n - (\epsilon_\alpha - \mu)} \ . \qquad (2.131b)$$

One additional consideration is required to treat the special case of a propagator which begins and ends at the same vertex. In order to invoke the proper θ-function in $g_\alpha(\tau)$ corresponding to the occupation probability n_k, the argument is always shifted by an infinitesimal η to $g_\alpha(\tau - \eta)$. (See,for example, Eq.(2.78)). This same result may be obtained by multiplying $\tilde{g}_\alpha(\omega_n)$ by the factor $e^{i\omega_n\eta}$, in which case

$$\begin{aligned}
\frac{1}{\beta}\sum_{\omega_n} e^{-i\omega_n(\tau-\eta)}\tilde{g}_\alpha(\omega_n) &= \frac{-1}{\beta}\sum_{\omega_n} e^{-i\omega_n(\tau-\eta)} \frac{1}{i\omega_n - (\epsilon_\alpha - \mu)} \\
&= g_\alpha(\tau - \eta) \ . \qquad (2.131c)
\end{aligned}$$

The time integration associated with the vertex 

yields the integral

$$\begin{aligned}
&\int d\tau g_{\alpha_1}(\tau - \tau_1) g_{\alpha_2}(\tau - \tau_2) g_{\alpha_3}(\tau_3 - \tau) g_{\alpha_4}(\tau_4 - \tau) \\
&= \int_0^\beta d\tau e^{-i(\omega_{n_1} + \omega_{n_2} - \omega_{n_3} - \omega_{n_4})\tau} \prod_{i=1}^4 \frac{\tilde{g}(\omega_{n_i})}{\beta} e^{i(\omega_{n_1}\tau_1 + \omega_{n_2}\tau_2 - \omega_{n_3}\tau_3 - \omega_{n_4}\tau_4)} \\
&= \beta\delta_{\omega_{n_1} + \omega_{n_2}, \omega_{n_3} + \omega_{n_4}} \prod_{i=1}^4 \frac{\tilde{g}(\omega_{n_i})}{\beta} e^{i(\omega_{n_1}\tau_1 + \omega_{n_2}\tau_2 - \omega_{n_3}\tau_3 - \omega_{n_4}\tau_4)} \ . \qquad (2.132)
\end{aligned}$$

Thus, at every vertex, we obtain a factor β and a Kronecker δ requiring that the sum of the frequencies associated with the propagators entering the vertex equal the sum of the frequencies of propagators leaving the vertex. As was the case with momentum conservation, a connected diagram with m interactions will have $m+1$ independent frequencies. Since in a general n^{th} order diagram each of the $2n$ propagators has factor of $\frac{1}{\beta}$ and each of the n time integrals yields a factor β, the overall contribution is proportional to β^{-n}. The diagram rules presented previously for Feynman and Hugenholtz diagrams are then modified as follows in the frequency representation.

$3_{F,H}$ Assign frequency labels to each directed line as follows. Within each connected part of a diagram containing m interactions, select $m+1$ propagators to label independently and use frequency conservation at each vertex to label the remaining propagators in terms of combinations of independent momenta. Assign single-particle indices as before. For each directed line include the factor

$$\alpha,\omega_n \quad = \tilde{g}_\alpha(\omega_n) = \frac{-1}{i\omega_n - (\epsilon_\alpha - \mu)} \ .$$

For propagators beginning and ending at the same vertex, include an additional factor $e^{i\omega_n \eta}$.

$5_{F,H}$ Sum over all single-particle labels and sum over all independent frequencies ω_n.

$6_{F,H}$ Multiply the previously defined factor by $\frac{1}{\beta^m}$, where m is the number of interactions.

The generalization of these rules using momentum and frequency representation to n-body interactions is straightforward. It is often convenient to use momentum and frequency representation simultaneously in which case one associates with each propagator a four-momentum (ω_n, k) which is conserved at each vertex. In the zero temperature limit which will be discussed in detail in Chapter 3, $\beta \to \infty$, the Fourier series in ω_n becomes a Fourier integral, and the diagram rules can be expressed in the covariant form familiar in relativistic field theory.

THE LINKED CLUSTER THEOREM

We have seen that the expansion for Z contains all powers of the volume $\mathcal{V}$, with an individual diagram with n_c connected parts being proportional to $\mathcal{V}^{n_c}$. In contrast, the ground potential $\Omega = -\frac{1}{\beta}\ln Z = -P\mathcal{V}$ is an extensive quantity so it must be possible to regroup the expansion to obtain an extensive expansion for Ω. Whereas it is plausible that Ω may be expressed in terms of connected diagrams, the combinatorial factors are not *a priori* obvious. The linked cluster theorem states that $\ln Z$ is in fact given by the sum of all connected diagrams.

We will derive this theorem using the replica technique, both for the sake of brevity and to introduce this useful method. A standard derivation is given in Problem 2.10. The basic idea of the replica method is to evaluate Z^n for integer n by replicating the system n times and expand the result as follows.

$$Z^n = e^{n\ln Z} = 1 + n\ln Z + \sum_{m=2}^{\infty} \frac{(n\ln Z)^m}{m!} \ . \tag{2.133}$$

Thus, if we evaluate Z^n for integer n in perturbation theory, $\ln Z$ will be given by the coefficients of the terms proportional to n. A more general statement of the method is to calculate Z^n for integer n, continue the function to $n=0$ (which is valid by Carlson's theorem) and evaluate an appropriate expression involving the continued function to calculate the observable of interest. In the present case, we calculate

$$\begin{aligned} \lim_{n\to 0} \frac{d}{dn} Z^n &= \lim_{n\to 0} \frac{d}{dn}(e^{n\ln Z}) \\ &= \ln Z \ . \end{aligned} \tag{2.134}$$

We will first evaluate Z^n for integer n by perturbation theory, and according to equation (2.133), Log Z will be given by the coefficient of the graphs proportional to n. Since

$$\frac{Z}{Z_0} = \frac{1}{Z_0} \int_{\substack{\psi_\alpha(\beta) = \\ \varsigma\psi_\alpha(0)}} \mathcal{D}(\psi^*_\alpha(\tau), \psi_\alpha(\tau)) e^{-\int_0^\beta dt \left(\sum_\alpha \psi_\alpha(\tau)^*(\partial_t + \epsilon_\alpha - \mu)\psi_\alpha(\tau) - V(\psi^*_\alpha(\tau), \psi_\alpha(\tau))\right)} \ . \tag{2.135}$$

We may write Z^n as a functional integral over n sets of fields $\{\psi^{*\sigma}_\alpha(\tau), \psi^\sigma_\alpha(\tau)\}$ where the index σ runs from 1 to n.

$$\left(\frac{Z}{Z_0}\right)^n = \frac{1}{Z_0^n} \int_{\psi^\sigma_\alpha(\beta) = \varsigma\psi^\sigma_\alpha(0)} \prod_{\sigma=1}^{n} \mathcal{D}(\psi^{*\sigma}_\alpha(\tau), \psi^\sigma_\alpha(\tau)) \times e^{-\int_0^\beta dt \sum_{\sigma=1}^n \left(\sum_\alpha \psi^{*\sigma}_\alpha(\partial_t + \epsilon_\alpha - \mu)\psi^\sigma_\alpha - V(\psi^{*\sigma}_\alpha, \psi^\sigma_\alpha)\right)} \ . \tag{2.136}$$

The Feynman rules for $(\frac{Z}{Z_0})^n$ are the same as those for $(\frac{Z}{Z_0})$, except that each propagator now carries an index σ, all propagators entering or leaving a given vertex have the same index σ, and all σ's are summed from 1 to n. It is evident that each connected part of a diagram must carry a single index σ, which when summed from 1 to n, yields a factor n. Thus, a graph with n_c connected parts is proportional to n^{n_c} and the graphs proportional to n are those with only one connected part, that is the connected graphs. As a consequence, we obtain the linked cluster theorem:

$$\Omega - \Omega_0 = -\frac{1}{\beta} \sum (\text{all connected graphs}) \tag{2.137a}$$

where Ω_0 is the grand potential of the unperturbed system

$$\Omega_0 = \frac{\varsigma}{\beta} \sum_\alpha \ln(1 - \varsigma e^{-\beta(\epsilon_\alpha - \mu)}) \ . \tag{2.137b}$$

CALCULATION OF OBSERVABLES AND GREENS FUNCTIONS

The linked expansion for the expectation value of any n-body operator R is analogous to that for Ω. One way to proceed, as outlined in Problem 2.11, is to calculate $\Omega(\lambda)$ for the Hamiltonian $\hat{H}_\lambda = \hat{H} + \lambda\hat{R}$ and evaluate $\langle R\rangle = \frac{d}{d\lambda}\Omega(\lambda)$. An alternative

method, which has the advantage of showing how symmetry factors are simplified, is to use the replica technique again. Using the definition for the unperturbed thermal average, Eq. (2.81b), the expectation value of R may be written

$$\langle R\rangle = \frac{\int \mathcal{D}(\psi_\alpha^*,\psi_\alpha) e^{-\int dt \sum_\alpha \psi_\alpha^*(\tau)(\partial_\tau+\epsilon_\alpha-\mu)\psi_\alpha(\tau)+V(\psi_\alpha^*(\tau),\psi_\alpha(\tau))} R(\psi_\alpha^*(0),\psi_\alpha(0))}{\int \mathcal{D}(\psi_\alpha^*,\psi_\alpha) e^{-\int dt \sum_\alpha \psi_\alpha^*(\tau)(\partial_\tau+\epsilon_\alpha-\mu)\psi_\alpha(\tau)+V(\psi_\alpha^*(\tau),\psi_\alpha(\tau))}}$$
$$= \frac{\langle e^{-\int dt V(\psi_\alpha^*(\tau),\psi_\alpha(\tau))} R(\psi_\alpha^*(0)\psi_\alpha(0))\rangle_0}{\langle e^{-\int dt V(\psi_\alpha^*(\tau),\psi_\alpha(\tau))}\rangle_0} \quad . \tag{2.138}$$

Let us again introduce n fields $\{\psi_\alpha^{*\sigma}(\tau),\psi_\alpha^\sigma(\tau)\}$ where the index σ runs from 1 to n and define

$$R_n = Z_0^{-n}\int \prod_{\sigma=1}^n \mathcal{D}(\psi_\alpha^{\sigma*}(\tau),\psi_\alpha^\sigma(\tau)) R(\psi_\alpha^{1*}(0)),\psi_\alpha^1(0))$$
$$\times e^{-\sum_{\sigma=1}^n \int_0^\beta dt\left(\sum_\alpha \psi_\alpha^{\sigma*}(\tau)(\partial_\tau+\epsilon_\alpha-\mu)\psi_\alpha^\sigma(\tau)+V(\psi_\alpha^{*\sigma}(\tau),\psi_\alpha^\sigma(\tau)\right)} \quad . \tag{2.139a}$$

Note that the operator R is calculated with the field $\psi_\alpha^{1*},\psi_\alpha^1$ associated with $\sigma=1$ and is evaluated at $\tau=0$. By separating the $\sigma=1$ component from the $(n-1)$ other components we observe that

$$R_n = \left\langle e^{-\int_0^\beta dt V(\psi_\alpha^*(\tau),\psi_\alpha(\tau))} R(\psi_\alpha^*(0),\psi_\alpha(0))\right\rangle_0 \left\langle e^{-\int_0^\beta dt V(\psi_\alpha^*(\tau)\psi_\alpha(\tau))}\right\rangle_0^{n-1} \tag{2.139b}$$

and thus the desired expectation value is obtained for $n=0$:

$$R_0 = \langle R\rangle \quad . \tag{2.140}$$

The perturbation expansion for R_n is obtained by expanding Eq. (2.139a) in powers of V

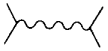
$$R_n = \sum_{p=0}^\infty \frac{(-1)^p}{p!}\sum_{\sigma_1=1}^n \cdots \sum_{\sigma_p=1}^n \int_0^\beta d\tau_1 \ldots d\tau_p \tag{2.141}$$
$$\times \langle V(\psi_\alpha^{*\sigma_1}(\tau_1),\psi_\alpha^{*\sigma_1}(\tau_1)) \ldots V(\psi_\alpha^{*\sigma_p}(\tau_p),\psi_\alpha^{\sigma_p}(\tau_p)) R(\psi_\alpha^{*1}(0)\psi_\alpha^1(0))\rangle_0$$

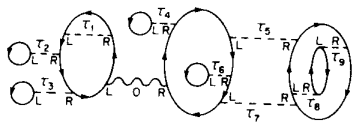
and diagram rules for unlabeled diagrams, labeled Feynman diagrams, Hugenholtz diagrams, or the momentum frequency transform of any of these diagrams are developed as a straightforward generalization of the rules in the preceding sections. We will describe here the case of a two-body operator R and unlabeled Feynman diagrams.

The expansion of Eq. (2.141) consists of all distinct diagrams containing one vertex $(\alpha\beta|R|\gamma\delta)$ at time $\tau=0$ and any number of vertices $(\alpha\beta|v|\gamma\delta)$ at other times which are integrated from 0 to β. As in the case of the linked cluster proof, all propagators now carry an index σ, all propagators entering or leaving an interaction

vertex have the same index, and all indices are summed from 1 to n. In addition, the index associated with R is one so that all the propagators entering and leaving R are restricted to have $\sigma=1$. Now, consider a diagram composed of a set of interaction vertices v connected to R and one or more additional unconnected parts. Since R is constrained to carry $\sigma=1$ and all propagators and v vertices conserve σ, σ must be 1 everywhere in the portion of the diagram linked to R. In the additional unconnected parts, however, there is at least one free summation over σ leading to an overall factor of at least one power of n. Hence all diagrams with disconnected parts vanish when n is set equal to zero and $\langle R\rangle$ is given by the sum of all connected diagrams linked to R. Having played their role in the proof that only connected diagrams contribute to $\langle R\rangle$, the σ indices which are constrained to be 1 in all components linked to the vertex R are superfluous and may be omitted from the final diagram rules.

The symmetry factors for diagrams contributing to $\langle R\rangle$ are much simpler than for corresponding contributions to the grand potential. Since the symmetry of a diagram is reduced by singling out a particular vertex to be labeled R, it is clear that S, which specifies the number of combinations of time perturbations and vertex extremity exchanges which transform a labeled diagram into a deformation of itself, will be reduced. In fact, as we shall now demonstrate, the symmetry is reduced so much that the only possible values of S are 1 or 2.

To see how the symmetry factors are constrained, consider the typical labeled diagram in which nine v-vertices at times $\tau_1 \ldots \tau_9$ are denoted by and the R-vertex at time 0 is denoted by 



. (2.142a)

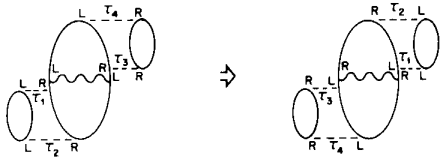
One may see that there are no permutations of time labels which produce a deformation of this diagram by the following argument. Each diagram may be considered as a series of closed Fermion loops connected by one or more vertices, and in this example there are eight such loops. Consider any loop which has one time fixed. Since the propagators comprising the loop are directed, there are no permutations of the vertex time labels within the loop which yield a deformation. Therefore, all the time labels in the loop are fixed. If another loop is connected to the first loop by at least one interaction, then at least one of its time labels is fixed and there is no freedom to permute time labels within that loop. By induction, one observes that no permutations of time labels within closed loops are possible. In this example, the time 0 fixes the labels $\tau_4,\tau_5,\tau_6,\tau_7$ which in turn fix the labels τ_8 and τ_9 in the next loop. The only remaining possibility, then, is to permute times between loops. However, since in the case of the diagram (2.142a) the sub-diagrams connected to the left and right sides of the R-vertex are topologically inequivalent, none of these permutations produce a deformation.

The following diagram is an example in which exchanging times between different

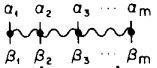
loops will produce a deformation.

(2.142b)

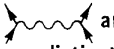
Note that the subdiagram connected to the left side of the R-vertex is topologically equivalent to that connected to the right side. Interchanging τ_1 with τ_3 and τ_2 with τ_4 and simultaneously interchanging the extremities of all the vertices thus yields a deformation of the original diagrams as shown at the right. This simultaneous extremity exchange and exchange of all the times in the topologically equivalent subdiagrams connected to the R-vertex is in fact the only operation which can produce a deformation of the diagram. We have seen that no other operations which change time labels produce deformations, and by analyzing the L-R labels in each closed propagator loop of a general diagram, one can also see that pure extremity exchange without time permuations can never produce a deformation. Thus, the symmetry factor for diagram (2.142b) is $S = 2$. Another example is the exchange diagram shown in below which has symmetry factor $S = 2$ because of the time and extremity permutations shown at the right



(2.142c)

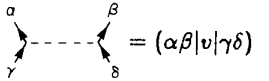
The general rule for the symmetry factor resulting from this analysis is simple. For a two-body operator R, $S = 2$ if the exchange of the extremities of R combined with some time permutation and exchange of interaction extremities yields a deformation; otherwise $S = 1$. For an m-body operator R,  the argument is easily generalized. The symmetry factor S is equal to the number of permutations of the m joints of R, which, when combined with time and interaction extremity exchanges, yields a deformation.

The rules for calculating the p^{th} order contribution to the perturbation expansion of the expectation value of a two-body operator $\langle R \rangle$ using unlabeled Feynman diagrams may be summarized as follows:

1. Draw all distinct unlabeled connected diagrams composed of one R-vertex  and p v-vertices connected by directed lines . Two diagrams are distinct if they cannot be deformed so as to coincide completely including the direction of arrows on propagators. For each distinct unlabeled diagram, evaluate the contribution as follows.
2. Calculate the symmetry factor S for the diagram. If the exchange of the extremities of R combined with some time permutation and exchange of interaction extremities yields a deformation of the original diagram, $S = 2$. Otherwise, $S = 1$.
3. Assign a time label τ_i to each of the p v-vertices, associate the time $\tau = 0$ with the R-vertex, and assign a single-particle index to each directed line. For each directed line include the factor

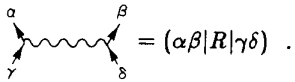
$$\overset{\tau}{\underset{\tau'}{\nearrow}}\!\!\alpha = g_\alpha(\tau - \tau') = e^{-(\epsilon_\alpha - \mu)(\tau - \tau')}[(1 + \zeta n_\alpha)\theta(\tau - \tau' - \eta) + \zeta n_\alpha \theta(\tau' - \tau - \eta)] \ .$$

4. For each v-vertex include the factor



$$= (\alpha\beta|v|\gamma\delta)$$

and for the R-vertex include the factor



$$= (\alpha\beta|R|\gamma\delta) \ .$$

5. Sum over all single particle indices and integrate the p times over the interval $[0, \beta]$.
6. Multiply the result by the factor $\frac{(-1)^F}{S}\zeta^{n_L}$ where n_L is the number of closed loops and S is the symmetry factor.

The eight unlabeled diagrams contributing to $\langle R \rangle$ in orders $p = 0$ and 1 are shown in Fig. 2.6, together with the factors $\frac{1}{S}$.

The contribution at order 0 is given by diagrams A and B

$$\langle R \rangle^{(0)} = \frac{1}{2}\sum_{\alpha,\beta}[(\alpha\beta|R|\alpha\beta) + \zeta(\alpha\beta|R|\beta\alpha)]n_\alpha n_\beta \qquad (2.143)$$

and a typical contribution at order 1 is that of diagram E

$$\langle R \rangle^{(E)} = -\frac{1}{2}\sum_{\alpha\beta\gamma\delta}\int_0^\beta d\tau(\alpha\beta|R|\gamma\delta)(\gamma\delta|v|\alpha\beta)g_\alpha(-\tau)g_\beta(-\tau)g_\gamma(\tau)g_\delta(\tau) \ . \qquad (2.144)$$

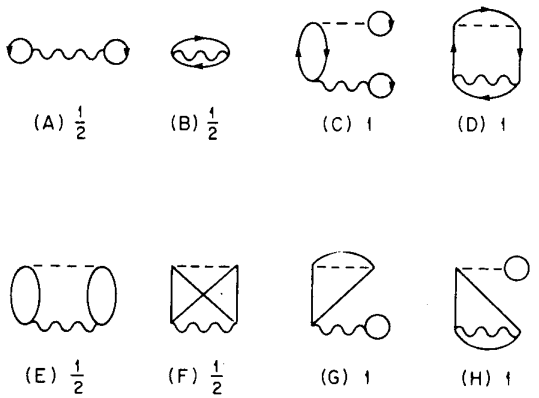


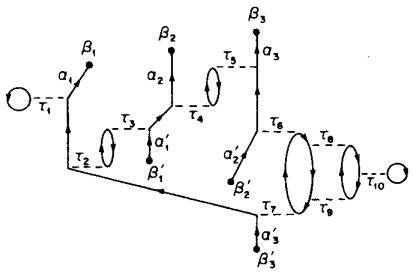
Fig. 2.6 Unlabeled Feynman diagrams for $\langle R \rangle$ with factor $\frac{1}{S}$

The derivation of the diagrammatic expansion for the imaginary-time Green's function

$$\mathcal{G}^{(n)}(\alpha_1\beta_1,\ldots\alpha_n\beta_n|\alpha'_1\beta'_1,\ldots\alpha'_n\beta'_n) \tag{2.145}$$
$$= \frac{\left\langle e^{-\int d\tau V(\psi^*_\alpha(\tau),\psi_\alpha(\tau))}\psi_{\alpha_1}(\beta_1)\ldots\psi_{\alpha_n}(\beta_n)\psi^*_{\alpha'_n}(\beta'_n)\ldots\psi^*_{\alpha'_1}(\beta'_1)\right\rangle_0}{\left\langle e^{-\int d\tau V(\psi^*_\alpha(\tau),\psi_\alpha(\tau))}\right\rangle_0}$$

is completely analogous to that for the expectation value $\langle R\rangle$ with $R(\psi^*_\alpha(0)\psi_\alpha(0))$ replaced by $\psi_{\alpha_1}(\beta_1)\ldots\psi_{\alpha_n}(\beta_n)\psi^*_{\alpha'_n}(\beta'_n)\ldots\psi^*_{\alpha'_1}(\beta'_1)$. Let us denote each of the n external points $\psi_{\alpha_i}(\beta_i)$ by a solid dot at time (β_i) with a propagator for state α_i entering it, $\overset{\beta_i}{\swarrow}_{\alpha_i}$, and each of the n external points $\psi^*_{\alpha'_i}(\beta'_i)$ as a solid dot at time β'_i with a propagator for state α'_i leaving it, $\overset{\alpha'_i}{\searrow}_{\beta'_i}$. These external points are to be treated analogous to the R-vertex in the previous derivation. Introduction of m replicas labeled by indices σ_i with the constraint that $\psi_{\alpha_i}(\beta_i)$ and $\psi^*_{\alpha'_i}(\beta'_i)$ be associated with σ_1, yields, in the limit $m\to 0$, the sum of all linked diagrams in which the interaction vertices, v, are linked to the $2n$ external points.

An example of a diagram contributing to the three-particle Green's function $G_3^{(I)}$ is the following.

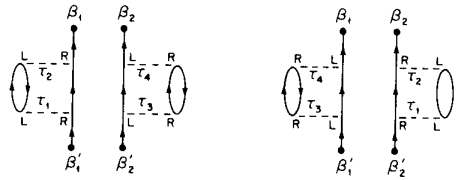


(2.146)

It consists of three directed lines beginning at the external points β'_1,β'_2 and β'_3 and terminating at the external points β_1,β_2 and β_3 as well as an arbitrary number, in this case six, of closed loops.

In contrast to a diagram for a three-body operator, where the propagators entering and leaving each of the three joints may be exchanged, each of the external points in (2.146) is fixed. Hence, its symmetry factor is even lower than that of the corresponding contribution to a three-body operator, and as we shall now demonstrate, $S=1$. The analysis is similar to that for the expectation value $\langle R\rangle$. In the present case, because there is no freedom to exchange propagators at the external points, there are no permutations involving the times along the three propagators starting and terminating at these external points which can produce a deformation. Since these times are now uniquely fixed, propagator loops connected to these lines by interaction vertices in turn have all their times fixed and by induction all times are fixed. For example, in the diagram (2.146), propagators beginning at β'_3 and β'_2 fix times τ_6 and τ_7 which in turn fix τ_8 and τ_9 which then fix τ_{10}. Similarly, the fact that propagators cannot be exchanged

at external points prevents any exchange of vertex extremities from producing a deformation. Since no time permutations, extremity exchanges, or combinations thereof can yield a deformation of the original diagram, $S=1$. The difference between the case of the expectation value $\langle R\rangle$ and a Green's function is illustrated by the diagram below which shows the contribution to $G_2^{(I)}$ analogous to diagram (2.142b).



(2.147)

The simultaneous interchange of τ_1 with τ_3 and τ_2 with τ_4 and the exchange of all interaction extremities which produced a deformation in diagram (2.142b) yields the distinct labeled diagram in the right of (2.147).

Whereas the symmetry factor is simpler for Green's functions than for any other quantity we have evaluated, the sign of the contractions contributing to a diagram is slightly more complicated. First, consider the sign associated with the contractions of $\langle\psi_{\alpha_1}(\beta_1)\ldots\psi_{\alpha_n}(\beta_n)\psi^*_{\alpha'_n}(\beta'_n)\cdots\psi^*_{\alpha'_1}(\beta'_1)\rangle_0$ without any interaction vertices. The contraction $\langle\psi_{\alpha_1}(\beta_1)\psi_{\alpha_2}(\beta_2)\ldots\psi_{\alpha_n}(\beta_n)\psi^*_{\alpha_{n'}}(\beta'_n)\ldots\psi^*_{\alpha_{2'}}(\beta'_2)\psi_{\alpha'_1}(\beta'_1)\rangle$ in which $\psi^*_{\alpha'_m}(\beta'_m)$ is contracted with $\psi_{\alpha_m}(\beta_m)$ for all m has sign $+1$, and for any permutation P such that $\psi^*_{\alpha'_m}(\beta'_m)$ is contracted with $\psi_{\alpha_{Pm}}(\beta_{Pm})$ the overall sign is $(\varsigma)^P$. Adding any number of interaction vertices to produce a diagram with n_L closed loops modifies the overall sign by the factor $(\varsigma)^{n_L}$ by the same argument as presented previously for diagrams for the grand potential.

An equivalent sign rule is to close up a Green's functions diagram by deforming it so that each external point β'_m coincides with β_m. The diagram then looks like the expectation value of $\varsigma^n\langle\psi^*_{\alpha'_n}\cdots\psi^*_{\alpha'_1}\psi_{\alpha_1}\cdots\psi_{\alpha_n}\rangle$ where the factor ς^n arises from placing the $\psi^*\psi$ in normal order. The sign is then $\varsigma^{n+\tilde{n}_L}$ where $\tilde{n}_L$ is the number of closed loops in the deformed diagram.

Application of these rules to the diagram (2.146), yields $+1$. Using the first method, we note that (123) is an even permutation of (312) and the six closed loops yield an additional factor $\varsigma^6=1$. According to the second rule, joining the external points β'_m with β_m produces a seventh closed loop, so that $\varsigma^{n+\tilde{n}_L}=\varsigma^{3+7}=1$

In summary, the rules for calculating the r^{th} order contribution to the perturbation expansion of $\mathcal{G}^{(n)}(\alpha_1\beta_1,\ldots\alpha_n,\beta_n|\alpha'_1\beta'_1,\ldots\alpha'_n\beta'_n)$ using unlabeled Feynman diagrams are as follows:

1. Draw all distinct unlabeled connected diagrams composed of n external points $\psi_{\alpha_i}(\beta_i)\swarrow$, n external points $\psi^*_{\alpha'_i}(\beta'_i)\searrow$, and r interaction vertices $\rangle\text{----}\langle$ connected by directed lines $\dashv$. Two diagrams are distinct if, holding the external points fixed, the vertices and internal propagators cannot be deformed so as to coincide completely including the direction of arrows on propagators. The contribution for each distinct unlabeled diagram is evaluated as follows:

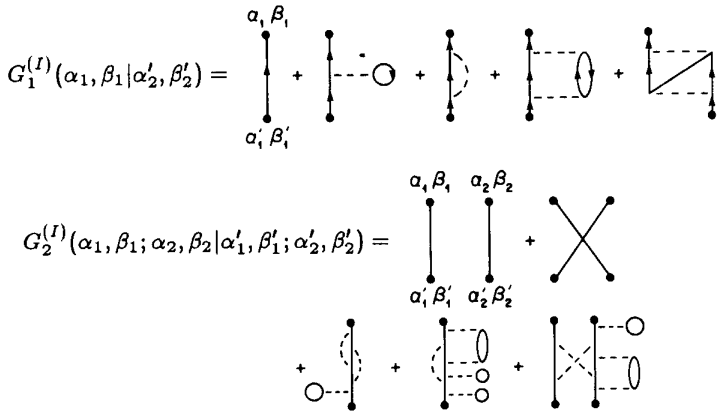


Fig. 2.7 Diagrams contributing to one- and two-particle Green's functions with overall sign factors.

2. Each external point $\overset{\beta_i}{\underset{\alpha_i}{\nearrow}}$ corresponds to a specified state α_i and time β_i. Assign an internal time label τ_i to each of the r interaction vertices and for any propagator which is not connected to an external point assign an internal single-particle index. For each directed line include the factor

$$\overset{\tau,\alpha}{\underset{\tau',\alpha'}{\nearrow}} = \delta_{\alpha'\alpha} g_\alpha(\tau - \tau')$$
$$= \delta_{\alpha'\alpha} e^{-(\epsilon_\alpha - \mu)(\tau - \tau')}[(1 + \zeta n_\alpha)\theta(\tau - \tau' - \eta) + \zeta n_\alpha \theta(\tau' - \tau + \eta)]$$

where τ and τ' denote either internal times τ_i or external times β_i. Propagators connected to an interaction vertex will only have one single-particle index α, in which case the factor $\delta_{\alpha'\alpha}$ is superfluous.

3. For each interaction vertex include the factor

$$\overset{\alpha}{\underset{\gamma}{\rangle}}\!\!- - -\!\!\overset{\beta}{\underset{\delta}{\langle}} = (\alpha\beta|v|\gamma\delta) \quad .$$

4. Sum over all internal single-particle indices and integrate the r internal times τ_i over the interval $[0, \beta]$.
5. Multiply the result by the factor $(-1)^r \zeta^P \zeta^{n_L}$ where n_L is the number of closed propagator loops and ζ^P is the sign of the permutation P such that each propagator line originating at the external point $\psi^*_{\alpha'_m}(\beta'_m)$ terminates at the external point $\psi_{\alpha_{Pm}}(\beta_{Pm})$.

Examples of graphs contributing to the one- and two-particle Green's functions, together with the overall sign factor are given in Fig. (2.7).

2.4 IRREDUCIBLE DIAGRAMS AND INTEGRAL EQUATIONS

With the foundations of diagrammatic perturbation theory presented in the preceding section, it is now possible to derive exact integral equations relating connected Green's functions and irreducible vertex functions. Since these equations include contributions of all orders of perturbation theory, they are useful in defining consistent approximations involving infinite resummations of diagrams and in treating the renormalization of divergent field theories. They also lead naturally to the effective potential, Γ, which will be useful in understanding spontaneous symmetry breaking in Chapter 3, and to the self energy, Σ, which governs the propagation of a single particle in a many-body medium.

GENERATING FUNCTION FOR CONNECTED GREEN'S FUNCTIONS

As seen in the preceding sections, it is often useful to define a generating function by adding to the physical Hamiltonian under consideration additional terms in which field operators are coupled to external sources. The simplest possibility is to couple the field operators to an external source J by adding the term

$$S = \sum_\alpha \int d\tau [(J^*_\alpha(\tau) a_\alpha(\tau) + a^\dagger_\alpha(\tau) J_\alpha(\tau))] \tag{2.148}$$

where the sources $J_\alpha(\tau)$ are complex or Grassman variables for Bosons and Fermions respectively. Other more general sources, such as the bilinear form

$$S = \sum_{\alpha\beta} \int d\tau d\tau' [a^\dagger_\alpha(\tau_1) a^\dagger_\beta(\tau_2) \eta_{\alpha\beta}(\tau_1, \tau_2) + a^\dagger_\alpha(\tau_1) a_\beta(\tau_2) \bar{\eta}_{\alpha\beta}(\tau_1, \tau_2) + a_\alpha(\tau_1) a_\beta(\tau_2) \eta^*_{\alpha\beta}(\tau_1, \tau_2)] \tag{2.149}$$

are useful for specific applications.

Similar to the generating function, Eq. (2.85) used to derive Wick's theorem, the generating function for imaginary-time Green's Function is defined as the partition function for the full Hamiltonian plus the source term Eq. (2.148). Using the functional integral representation for the partition function, Eq. (2.64), the generating function may be written

$$\mathcal{G}(J^*_\alpha(\tau), J_\alpha(\tau)) \equiv \frac{1}{Z} \int \mathcal{D}[\psi^*_\alpha(\tau)\psi_\alpha(\tau)] e^{-\int_0^\beta d\tau [\sum_\alpha \psi_\alpha(\tau)(\partial_\tau - \mu)\psi_\alpha(\tau) + H[\psi^*_\alpha(\tau), \psi_\alpha(\tau)]]}$$
$$\times e^{-\int_0^\beta d\tau \sum_\alpha [J^*_\alpha(\tau)\psi_\alpha(\tau) + \psi^*_\alpha(\tau) J_\alpha(\tau)]}$$
$$= \langle e^{-\int_0^\beta d\tau \sum_\alpha [J^*_\alpha(\tau)\psi_\alpha(\tau) + \psi^*_\alpha(\tau) J_\alpha(\tau)]} \rangle \tag{2.150a}$$