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Since at finite temperature, we find the energy E from

$$\boxed{E = \Omega + \mu N + TS}$$

where μ is adjusted so that $N = -\frac{\partial \Omega}{\partial \mu}$ is the desired number of particles at the given temperature T, we find the zero temperature difference $(E - E_0)_{T=0}$ from

$$\boxed{(E-E_0)_{T=0} = \lim_{\substack{T\to 0 \\ (\text{or } \beta\to\infty)}} \left(\Omega - \Omega_0 + (\mu-\mu_0)N\right)}$$

- E_0 is the ground state energy of the "noninteracting" system (e.g. free Fermi gas or noninteracting fermions in a harmonic trap).
- $\Omega - \Omega_0$ is the thing we have a diagrammatic expansion of.

- At $T=0$, we'd like to work with fixed particle number, which entails using μ_0 in the non-interacting Green's functions. There are two differences with the $T \neq 0$ diagrammatic expansion:

1) $\boxed{\frac{\partial \Omega_0}{\partial \mu}\Big|_{\mu_0} = N}$ and $\boxed{\frac{\partial \Omega}{\partial \mu}\Big|_{\mu} = N}$ so in general $\mu \neq \mu_0$.

2) Graphs in which the same state appears like in  (called "anomalous" graphs) are strictly zero at $T=0$ but have a $n_k^0(1-n_k^0)$ factor that does not vanish at $T>0$ at the Fermi surface.

But, it turns out that for uniform systems (exception below!), these two differences cancel and we get the correct answer for the $T=0$ energy by:

i) using μ_0 in the propagators in the same diagrams as at $T>0$ except

ii) ignore the anomalous diagrams.

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· The key point about the μ dependence is that μ enters $\Omega-\Omega_0$ <u>only</u> through the occupation number

$$\boxed{n_k^0 = \frac{1}{e^{\beta(\epsilon_k^0-\mu)}+1}}$$

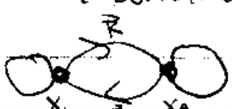
which goes to $\Theta(\mu-\epsilon_k^0)$ as $\beta\to\infty$ $(T\to 0)$.

· Therefore, we get corrections to $\frac{\partial\Omega_0}{\partial\mu}$ from $\Omega-\Omega_0$ (that is, from $\frac{\partial}{\partial\mu}(\Omega-\Omega_0)$) that contain μ only through

$$\boxed{\frac{\partial n_k^0}{\partial\mu} \xrightarrow[\beta\to\infty]{} \delta(\mu-\epsilon_k^0)}$$

· If we have a <u>finite</u>, periodic box, then this is never satisfied because the energies are discrete, so $\frac{\partial\Omega}{\partial\mu}=\frac{\partial\Omega_0}{\partial\mu}$ and we find $\mu=\mu_0$ just from the non-interacting Ω_0.

· If we consider the lowest order anomalous diagram:



we'll end up with a contribution like

$$\boxed{\begin{aligned}\Omega_A^{(2)} &= \sum_k \frac{1}{\beta}\int_0^\beta\!\!\int_0^\beta d\tau_2\, d\tau_2\,(1-n_k^0)\,n_k^0 \times (\text{constants from } \bigcirc, \text{etc.})\\ &\propto \sum_k \beta(1-n_k^0)\,n_k^0\end{aligned}}$$

We always get a factor like this from anomalous diagrams.

· In the zero temperature limit,

$$\boxed{\begin{aligned}\beta\, n_k^0(1-n_k^0) &= \beta\frac{e^{\beta(\epsilon_k^0-\mu)}}{\left(1+e^{\beta(\epsilon_k^0-\mu)}\right)^2} = -\frac{\partial}{\partial\epsilon_k^0}n_k^0\\ &\xrightarrow[\beta\to\infty]{} \delta(\epsilon_k^0-\mu)\end{aligned}}$$

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- So we get the same factor in these diagrams. If we consider the double limit of zero temperature and infinite volume:

$$\lim_{T\to 0}\{\lim_{\Omega\to\infty}\}$$

then both the anomalous diagrams and $N(\mu-\mu_0)$ are non-vanishing.

- In fact, they cancel unless the state that evolves from the noninteracting state at zero temperature evolves to the wrong state when the interaction is turned on.
    - A simple example of this is a system of spin-1/2 fermions in an external magnetic field.
    - If H_0 is just the kinetic energy, then the Fermi surface of the noninteracting system is just two equally filled "Fermi spheres", one for spin-up and one for spin down.
    - If we now have a one-body perturbation $-\mu B\sigma_z$ where B is the magnetic field, then the true interacting ground state will be magnetized. But we'll get zero magnetization in the zero temperature formulation, because the perturbation can't flip spins, so we can't get from the equal Fermi sphere non-interacting state to the true interacting state with unequal Fermi spheres!

- We'll prove the cancellation elsewhere. For now we note that we can carry over our rules to $T=0$, using μ_0 instead of μ (which means filling up to k_F and $\rho = N/\Omega = g k_F^3/6\pi^2$) and neglecting anomalous diagrams.
    - To avoid the problem above, we can always be smart and pick H_0 so that the noninteracting ground states has the correct symmetries and corresponds to the correct phase.

- Now we'll look at the coordinate and momentum space $T=0$ rules.

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Feynman Rules for the n^{th} Order of $\mathcal{E}-\mathcal{E}_0$ $(T=0)$

* for a single potential $V(\vec{x}-\vec{x}') = \lambda\delta^3(\vec{x}-\vec{x}')$ in a uniform system.

Coordinate space:

a. Draw all distinct, fully connected diagrams with n vertices. Distinct diagrams are those that cannot be deformed to coincide with each other, including arrows.

b. Assign a space-time point $x_i = (\vec{x}_i, t_i)$ to each vertex and a factor $-i\lambda$. Each internal line gets a factor of $iG^0_{\alpha\beta}(x_1,x_2)$ running from x_2 to x_1, where

$$\boxed{\begin{aligned} iG^0_{\alpha\beta}(x_1,x_2) &= \delta_{\alpha\beta}\frac{1}{\Omega}\sum_{\vec{k}} e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}\, e^{-i\omega_k(t_1-t_2)} \\ &\quad \times\left[\theta(t_1-t_2-\eta)\theta(|\vec{k}|-k_F) - \theta(t_2-t_1+\eta)\theta(k_F-|\vec{k}|)\right] \\ &\xrightarrow{\Omega\to\infty} \delta_{\alpha\beta}\int\frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}\, e^{-i\omega_k(t_1-t_2)}\left[\theta(t_1-t_2-\eta)\theta(|\vec{k}|-k_F) - \theta(t_2-t_1+\eta)\theta(k_F-|\vec{k}|)\right] \end{aligned}}$$

The vertex lines each have a spin index. For spin-independent interactions (like $\delta^3(\vec{x}-\vec{x}')$), the two-body vertices have the structure $(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})$ where α, β are incoming spins and γ, δ are outgoing spins. $\boxed{\omega_k \equiv k^2/2m}$

c. Do the spin summations and substitute $-g$ for each $\delta_{\alpha\alpha}$ (in each closed fermion loop).

d. Integrate $\int d^3\vec{x}_i \int dt_i$ over all x_i. Divide by one space-time volume ΩT.

e. Multiply by a symmetry factor appropriate for diagrams with arrows and an extra i: $i/(S\prod_{l=2}^{l_{max}}(l!)^m)$ where S is the number of vertex permutations that transform the diagram into itself, and m is the number of equivalent l-tuples of lines. Equivalent lines begin and end at the same vertices with the same direction of arrows.

° Anomalous diagrams, which have $\theta(|\vec{k}|-k_F)\times\theta(k_F-|\vec{k}|)$ with the same $|\vec{k}|$ are zero.

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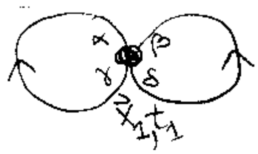
Check some low-order diagrams to see if the factors work out:

a) 

b) $iG^0_{\alpha\gamma}(x_1,x_1)\, iG^0_{\beta\delta}(x_1,x_1)$
$\times\ (\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma})$
$\times\ -i\lambda$

c) From the G^0's, we get $\delta_{\alpha\gamma}\delta_{\beta\delta}$

$$\Rightarrow \delta_{\alpha\gamma}\delta_{\beta\delta}(\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma}) = \delta_{\alpha\alpha}\delta_{\beta\beta}+\delta_{\gamma\delta}\delta_{\gamma\delta} = \delta_{\alpha\alpha}\delta_{\beta\beta}+\delta_{\gamma\gamma}$$

$\Rightarrow (g^2-g)$ is the spin factor.

d) $\int d^3\vec{x}_1 \int_{-\infty}^{\infty} dt_1 \to \Omega T$ since no x_1 dependence in the end. Divide this out to get energy density

e) $S=1$, $m=1$ 2-tuple $\Rightarrow \frac{i}{2}$

$\boxed{iG^0(x_1,x_1) = -\int\frac{d^3k}{(2\pi)^3}\,\theta(k_f-|\vec{k}|)}$, so putting it all together

$$\Rightarrow \boxed{\begin{aligned}\mathcal{E}_1 &= (i)^2(-1)^2\frac{-\lambda}{2}(g^2-g)\int\frac{d^3k_1}{(2\pi)^3}\theta(k_f-|\vec{k}_1|)\int\frac{d^3k_2}{(2\pi)^3}\theta(k_f-|\vec{k}_2|)\\ &= +\frac{\lambda}{2}\left(1-\frac{1}{g}\right)\left[g\int\frac{d^3k_1}{(2\pi)^3}\theta(k_f-|\vec{k}_1|)\right]^2 = \frac{\lambda}{2}\left(1-\frac{1}{g}\right)\rho^2\end{aligned}}$$

✓

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a) 

b) $iG^0_{\alpha_1\beta_1}(x_1,x_1)\, iG^0_{\alpha_2\beta_3}(x_1,x_2)\, iG^0_{\alpha_3\beta_2}(x_2,x_1)\, iG^0_{\alpha_4\beta_4}(x_2,x_2)$

$\times(\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2}+\delta_{\alpha_1\beta_2}\delta_{\alpha_2\beta_1})(\delta_{\alpha_3\beta_3}\delta_{\alpha_4\beta_4}+\delta_{\alpha_3\beta_3}\delta_{\alpha_4\beta_4})$

$\times(-i\lambda)^2$

c) $\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_3}\delta_{\alpha_3\beta_2}\delta_{\alpha_4\beta_4}(\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2}+\delta_{\alpha_1\beta_2}\delta_{\alpha_2\beta_1})(\delta_{\alpha_3\beta_3}\delta_{\alpha_4\beta_4}+\delta_{\alpha_3\beta_4}\delta_{\alpha_4\beta_3})$

$= -g(g-1)^2$ (using deltasimplify.m Mathematica package)

d) $\int d^4x_1 \int d^4x_2 \Rightarrow$ since 2 G's have the same argument and the other two depend only on $x_1-x_2 \Rightarrow$ switch one integral to $y \equiv x_1-x_2$ and the other gives $\Omega \cdot T \Rightarrow$ divide this out.

e) $S=2$, no l-tuples $\Rightarrow \frac{i}{2}$

If we plug in $iG^0(x_1,x_2)\, iG^0(x_2,x_1)$ with $\sum_{\vec{k}_1}$ and $\sum_{\vec{k}_2}$, we'll have the integral over $\vec{x}_1-\vec{x}_2 \equiv \vec{y}$

$$\int d^3y\; e^{i\vec{k}_1\cdot\vec{y}}\, e^{-i\vec{k}_2\cdot\vec{y}} = (2\pi)^3\,\delta^3(\vec{k}_1-\vec{k}_2)$$

which sets those momenta equal. The $\theta(t_1-t_2)$ and $\theta(t_2-t_1)$ factors leave us with $\theta(k_1-k_F)\,\theta(k_F-k_1)$ after $\vec{k}_1=\vec{k}_2$, which is zero (anomalous diagram).

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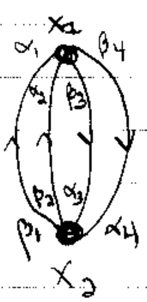
a) 

b) $iG^0_{\alpha_1\beta_1}(x_1,x_2)\, iG^0_{\alpha_2\beta_2}(x_1,x_2)\, iG^0_{\alpha_3\beta_3}(x_2,x_1)\, iG^0_{\alpha_4\beta_4}(x_2,x_1)$

$\times(\delta_{\alpha_1\beta_3}\delta_{\alpha_2\beta_4}+\delta_{\alpha_1\beta_4}\delta_{\alpha_2\beta_3})(\delta_{\alpha_3\beta_1}\delta_{\alpha_4\beta_2}+\delta_{\alpha_3\beta_2}\delta_{\alpha_4\beta_1})$

$\times(-i\lambda)^2$

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c) $\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2}\delta_{\alpha_3\beta_3}\delta_{\alpha_4\beta_4}\left(\delta_{\alpha_1\beta_3}\delta_{\alpha_2\beta_4}+\delta_{\alpha_1\beta_4}\delta_{\alpha_2\beta_3}\right)\left(\delta_{\alpha_3\beta_1}\delta_{\alpha_4\beta_2}+\delta_{\alpha_3\beta_2}\delta_{\alpha_4\beta_1}\right)$

$= 2g(g-1)$ (using deltasimplify in Mathematica package)

d) $\int d^4x_1 \int d^4x_2$; as above, the G^0's depend only on x_1-x_2
$\Rightarrow$ switch one integral to $y = x_1 - x_2$ and the other gives $\Omega \cdot T \Rightarrow$ divide this out.
The $\vec{y}$ integral will enforce momentum conservation, e.g. $\vec{k}_1+\vec{k}_2=\vec{k}_3+\vec{k}_4$. $\Rightarrow$ put in $\Omega\delta_{\vec{k}_1+\vec{k}_2,\vec{k}_3+\vec{k}_4} \to (2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2-\vec{k}_3-\vec{k}_4)$

e) $S=2$, 2 2-tuples $\Rightarrow (2!)^2 \Rightarrow \frac{i}{8}$

All of the time θ-functions are either $\theta(t_1-t_2)$ or $\theta(t_2-t_1)$ [we can let $\eta\to 0$]. So we only have two terms overall, with two "particles" ($k>k_f$) and two "holes" ($k<k_f$) in each. If we just exchange the variable labels, one term transforms into the other $\Rightarrow$ keep one with factor of 2

$$\Rightarrow \mathcal{E}_2 = 2\cdot\frac{i}{8}(-i\lambda)^2\, 2g(g-1)\int\frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3}\frac{d^3k_3}{(2\pi)^3}\frac{d^3k_4}{(2\pi)^3}(2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2-\vec{k}_3-\vec{k}_4)$$
$$\times\,\theta(k_1-k_f)\,\theta(k_2-k_f)\,\theta(k_f-k_3)\,\theta(k_f-k_4)$$
$$\times\underbrace{\int_{-\infty}^{\infty}dy\,\theta(y)\,e^{-i(\omega_{k_1}+\omega_{k_2}-\omega_{k_3}-\omega_{k_4}-i\eta)y}}_{\frac{1}{i}\frac{1}{\omega_{k_1}+\omega_{k_2}-\omega_{k_3}-\omega_{k_4}-i\eta}}$$

convergence factor

Let $k_f\vec{s}=\frac{1}{2}(\vec{k}_1+\vec{k}_2)=(\vec{k}_3+\vec{k}_4)/2$ be the total momentum $\Rightarrow$ enforced by delta function.
then let $k_f\vec{u}=\frac{1}{2}(\vec{k}_1-\vec{k}_2)$, $k_f\vec{t}=\frac{1}{2}(\vec{k}_3-\vec{k}_4)$ $\Rightarrow \omega_{k_1}+\omega_{k_2}=\frac{k_f^2}{2m}\left((s+u)^2+(s-u)^2\right)=\frac{k_f^2}{m}(s^2+u^2)$
and $\omega_{k_3}+\omega_{k_4}=\frac{k_f^2}{2m}\left((s+t)^2+(s-t)^2\right)=\frac{k_f^2}{m}(s^2+t^2)$ and factor of 8 from switching variables.

$$\boxed{\Rightarrow \mathcal{E}_2 = -4\lambda^2 m\, g(g-1)k_f^7\int\frac{d^3s}{(2\pi)^3}\int\frac{d^3t}{(2\pi)^3}\int\frac{d^3u}{(2\pi)^3}\,\theta(1-|\vec{s}+\vec{t}|)\,\theta(1-|\vec{s}-\vec{t}|)\,\theta(|\vec{s}+\vec{u}|-1)\,\theta(|\vec{s}-\vec{u}|-1)\,\frac{1}{u^2-t^2-i\eta}}$$

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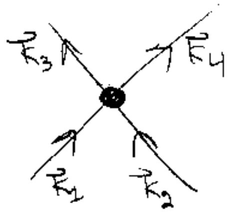
Aside: Variable changes and Jacobians...

- When we have the interaction:

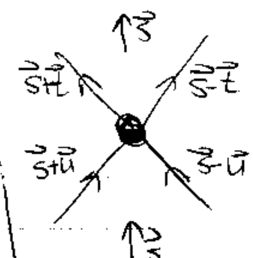


with total momentum conserved:
$$\vec{k}_1 + \vec{k}_2 = \vec{k}_3 + \vec{k}_4$$

then it is frequently advantageous to switch to total and relative momenta.
- These are often defined with varied factors of $\tfrac{1}{2}$: $\vec{k}_1 \pm \vec{k}_2$ vs. $\tfrac{1}{2}(\vec{k}_1 \pm \vec{k}_2)$
- The only difference is in the Jacobian.

- For example, we scale by k_f and define

$$\boxed{\begin{aligned} k_f \vec{s} &\equiv \tfrac{1}{2}(\vec{k}_1 + \vec{k}_2) = \tfrac{1}{2}(\vec{k}_3 + \vec{k}_4) \\ k_f \vec{u} &\equiv \tfrac{1}{2}(\vec{k}_1 - \vec{k}_2) \qquad k_f \vec{t} \equiv \tfrac{1}{2}(\vec{k}_3 - \vec{k}_4) \end{aligned}}$$



- How is the integral $\boxed{\int d^3k_1 \cdots d^3k_4 \, \delta^3(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4)} \boxed{= 8 k_f^9 \int d^3s \, d^3u \, d^3t}$ expressed in terms of $\vec{s}$, $\vec{t}$, and $\vec{u}$? Introduce $\vec{s}' \equiv \vec{k}_3 + \vec{k}_4$ so we can have an integral over d^3s' or $\delta^3(\vec{s} - \vec{s}')$ for balance.

- We can split it up into Cartesian components $\Rightarrow$ raise to 3rd power any Jacobian we get.

$$\delta(2k_f(s_x - s'_x)) = \frac{1}{2k_f}\delta(s_x - s'_x)$$

$$\int dk_{1x}\, dk_{2x}\, dk_{3x}\, dk_{4x}\, \delta(k_{1x} + k_{2x} - k_{3x} - k_{4x})$$

$$= k_f^4 \int ds_x\, ds'_x\, du_x\, dt_x \frac{1}{2k_f}\delta(s - s') \begin{vmatrix} \frac{\partial k_{1x}}{\partial s_x} & \frac{\partial k_{2x}}{\partial s_x} \\ \frac{\partial k_{1x}}{\partial u_x} & \frac{\partial k_{2x}}{\partial u_x} \end{vmatrix} \begin{vmatrix} \frac{\partial k_{3x}}{\partial s'_x} & \frac{\partial k_{4x}}{\partial s'_x} \\ \frac{\partial k_{3x}}{\partial t_x} & \frac{\partial k_{4x}}{\partial t_x} \end{vmatrix}$$

$$= 2k_f^3 \int ds_x\, ds'_x\, du_x\, dt_x\, \delta(s - s') \Rightarrow \boxed{2^3 = 8 \text{ factor overall}}$$

$$\begin{aligned} k_{1x} &= k_f(s_x + u_x) \\ k_{2x} &= k_f(s_x - u_x) \\ k_{3x} &= k_f(s'_x + t_x) \\ k_{4x} &= k_f(s'_x - t_x) \end{aligned}$$

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Feynman Rules for the n^{th} Order of $\mathcal{E}-\mathcal{E}_0$ $(T=0)$

* For a single potential $V(\vec{x}-\vec{x}') = \lambda\delta^3(\vec{x}-\vec{x}')$ in a uniform system.

Momentum space:

a. Draw all distinct, fully connected diagrams with n vertices.

b. Assign nonrelativistic four-momenta $\tilde{k}_i \equiv (k_0, \vec{k})$ to all lines and enforce four-momentum conservation at each vertex. Each internal line gets a factor of $iG^0_{\alpha\beta}(\tilde{k}_i)$, where

$$\boxed{iG^0_{\alpha\beta}(\tilde{k}_i) = i\delta_{\alpha\beta}\left(\frac{\theta(|\vec{k}_i|-k_f)}{k_{i_0}-\omega_{k_i}+i\varepsilon} + \frac{\theta(k_f-|\vec{k}_i|)}{k_{i_0}-\omega_{k_i}-i\varepsilon}\right)}$$

Each vertex gets a factor $-i\lambda$. (note the minus sign)
The vertex lines each have a spin index. For spin-independent interactions (like $\delta^3(\vec{x}-\vec{x}')$), the two-body vertices have the structure $(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})$ where α,β are incoming spins and γ,δ are outgoing spins. $\boxed{\omega_{k_i} \equiv \hbar^2 k_i^2/2m}$

c. Do the spin summations and substitute $-g$ for each $\delta_{\alpha\alpha}$ (in each closed fermion loop).

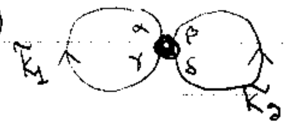
d. Integrate over all independent momenta (after momentum conservation applied) with $\int d^4\tilde{k}_i/(2\pi)^4$ where $d^4\tilde{k}_i \equiv dk_{i_0}\, d^3\vec{k}_i$. Divergent integrals will be discussed elsewhere.
For lines ending and originating at the same vertex, multiply by $e^{ik_{i_0}\eta}$ and take $\eta \to 0^+$ after the k_{i_0} integrals.

e. Multiply by a symmetry factor and i: $i/(S\prod_{l=2}^{l_{max}}(l!)^m)$ where S is the number of vertex permutations and m is the number of equivalent l-tuples of lines.

∘ Anomalous diagrams, with $\theta(|\vec{k}|-k_f)\theta(k_f-|\vec{k}|)$ are zero.

2/10/03

Once again, check some low-order diagrams to see that the factors work out:

a), b) 

$(\vec{k}_1+\vec{k}_2)_{in} = (\vec{k}_1+\vec{k}_2)_{out}$ ✓

$$iG^0_{\alpha\gamma}(\vec{k}_1)\, iG^0_{\beta\delta}(\vec{k}_2) \times (\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) \times (-i\lambda)$$

c). $\delta_{\alpha\gamma}\delta_{\beta\delta}(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}) = \delta_{\alpha\alpha}\delta_{\beta\beta} + \delta_{\gamma\gamma} \Rightarrow (g^2-g)$

d) $$\int\frac{d^3k_1}{(2\pi)^3}\int\frac{d^3k_2}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{dk_{10}}{2\pi}\int_{-\infty}^{\infty}\frac{dk_{20}}{2\pi}\, e^{ik_{10}\eta_1} e^{ik_{20}\eta_2}$$

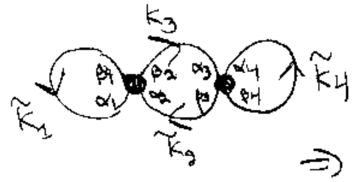
The factors $e^{ik_0\eta}$ tell us to close the k_{10} and k_{20} integrals in the upper-half plane $\Rightarrow$ pick up only the pole from the 2nd term:

$$\boxed{\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\left(\frac{\theta(|\vec{k}|-k_F)}{k_0-\omega_k+i\varepsilon} + \frac{\theta(k_F-|\vec{k}|)}{k_0-\omega_k-i\varepsilon}\right)e^{ik_0\eta} = \frac{1}{2\pi}(2\pi i)\,\theta(k_F-|\vec{k}|) = i\,\theta(k_F-|\vec{k}|)}$$

e) $S=1$, $m=1$ 2-tuple $\Rightarrow \frac{1}{2}$

$$\Rightarrow \boxed{\begin{aligned} \mathcal{E}_1 &= (i)^4\frac{-\lambda}{2}(g^2-g)\int\frac{d^3k_1}{(2\pi)^3}\, i\theta(k_F-|\vec{k}_1|)\int\frac{d^3k_2}{(2\pi)^3}\, i\theta(k_F-|\vec{k}_2|) \\ &= +\frac{\lambda}{2}\left(1-\frac{1}{g}\right)\left[g\int\frac{d^3k_1}{(2\pi)^3}\theta(k_F-|\vec{k}_1|)\right]^2 = \frac{\lambda}{2}\left(1-\frac{1}{g}\right)\rho^2 \end{aligned}}$$

2/10/03

a, b) 

$(\tilde{k}_1+\tilde{k}_2)_{in} = (\tilde{k}_1+\tilde{k}_3)_{out} \Rightarrow \tilde{k}_2 = \tilde{k}_3$

$(\tilde{k}_3+\tilde{k}_4)_{in} = (\tilde{k}_4+\tilde{k}_2)_{out} \Rightarrow \tilde{k}_2 = \tilde{k}_3$

$\Rightarrow \tilde{k}_1, \tilde{k}_2, \tilde{k}_4$ are independent momenta and $\tilde{k}_3 = \tilde{k}_2$

$$iG^0_{\alpha_1\beta_1}(\tilde{k}_1)\, iG^0_{\alpha_2\beta_3}(\tilde{k}_2)\, iG^0_{\alpha_3\beta_2}(\tilde{k}_2)\, iG^0_{\alpha_4\beta_4}(\tilde{k}_4)$$

$$\times\, (\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2} + \delta_{\alpha_1\beta_2}\delta_{\alpha_2\beta_1})(\delta_{\alpha_3\beta_3}\delta_{\alpha_4\beta_4} + \delta_{\alpha_3\beta_4}\delta_{\alpha_4\beta_3})$$

$$\times\, (-i\lambda)^2$$

c) $\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_3}\delta_{\alpha_3\beta_2}\delta_{\alpha_4\beta_4}(\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2} + \delta_{\alpha_1\beta_2}\delta_{\alpha_2\beta_1})(\delta_{\alpha_3\beta_3}\delta_{\alpha_4\beta_4} + \delta_{\alpha_3\beta_4}\delta_{\alpha_4\beta_3})$

$= -g(g-1)^2$ (using deltasimplify Mathematica package!)

d) $\int \frac{d^4k_1}{(2\pi)^4} \int \frac{d^4k_2}{(2\pi)^4} \int \frac{d^4k_4}{(2\pi)^4}\, e^{ik_{1_0}\eta_1}\, e^{ik_{4_0}\eta_4}$

e) $S=2$, no l-tuples $\Rightarrow \frac{1}{2}$

but $\int \frac{d^4k_2}{(2\pi)^4}\, iG^0(\tilde{k}_2)\, iG^0(\tilde{k}_2) \propto \int \frac{d^3k_2}{(2\pi)^3}\, \Theta(|\vec{k}_2| - k_F)\Theta(k_F - |\vec{k}_2|) = 0$

so this is one of the famous "anomalous" diagrams.

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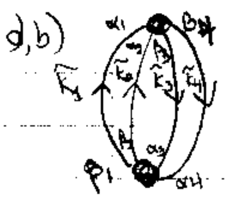
a, b) 

$(\tilde{k}_1+\tilde{k}_2)_{in} = (\tilde{k}_3+\tilde{k}_4)_{out} \Rightarrow \tilde{k}_1 \to \tilde{p},\ \tilde{k}_2 \to \tilde{k};\ \tilde{k}_3 \to \tilde{k}+\tilde{q},\ \tilde{k}_4 \to \tilde{p}-\tilde{q}$

$(\tilde{k}_1+\tilde{k}_2)_{out} = (\tilde{k}_3+\tilde{k}_4)_{in}$ ✓

$\Rightarrow \tilde{p}, \tilde{k}, \tilde{q}$ are the independent momenta

$$iG^0_{\alpha_1\beta_1}(\tilde{p})\, iG^0_{\alpha_2\beta_2}(\tilde{k})\, iG^0_{\alpha_3\beta_3}(\tilde{k}+\tilde{q})\, iG^0_{\alpha_4\beta_4}(\tilde{p}-\tilde{q})$$

$$\times\, (\delta_{\alpha_1\beta_3}\delta_{\alpha_2\beta_4} + \delta_{\alpha_1\beta_4}\delta_{\alpha_2\beta_3})(\delta_{\alpha_3\beta_1}\delta_{\alpha_4\beta_2} + \delta_{\alpha_3\beta_2}\delta_{\alpha_4\beta_1})$$

$$\times\, (-i\lambda)^2$$

c) $\delta_{\alpha_1\beta_1}\delta_{\alpha_2\beta_2}\delta_{\alpha_3\beta_3}\delta_{\alpha_4\beta_4}\left(\delta_{\alpha_1\beta_3}\delta_{\alpha_2\beta_4}+\delta_{\alpha_1\beta_4}\delta_{\alpha_2\beta_3}\right)\left(\delta_{\alpha_3\beta_1}\delta_{\alpha_4\beta_2}+\delta_{\alpha_3\beta_2}\delta_{\alpha_4\beta_1}\right)$

$= 2g(g-1)$ (using deltasimplify Mathematica package)

d) $\displaystyle\int\frac{d^4\tilde{k}}{(2\pi)^4}\int\frac{d^4\tilde{p}}{(2\pi)^4}\int\frac{d^4\tilde{q}}{(2\pi)^4}$

e) $S=2$, 2 2-tuples $\Rightarrow (2!)^2 \Rightarrow i/8$

$$\Rightarrow \boxed{\Sigma_2 = -\frac{i\lambda^2}{4}\, g(g-1)\int\frac{d^4\tilde{k}}{(2\pi)^4}\int\frac{d^4\tilde{p}}{(2\pi)^4}\int\frac{d^4\tilde{q}}{(2\pi)^4}\, G^0(\tilde{p})\,G^0(\tilde{k})\,G^0(\tilde{k}+\tilde{q})\,G^0(\tilde{p}-\tilde{q})}$$

Exercise for the reader: Work out the details!