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Monday 880.05 Class

- **Handouts**: None.
- **Problem Set**: Any questions?
- **Plan for This week**:

back to 107 - 110, 117 on

1. Rapidly get to the coherent state path integral formula for bosons
2. Repeat the process for fermions
   - in both cases put off all but the necessary details until later (or the notes or problem sets)
3. Consider the case of a dilute Fermi gas with weak interactions as a prototype example.
4. Apply the Feynman rules and evaluate in the $T=0$ limit (this is $\beta\to\infty$).

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we'll use $\sum_\alpha$ to mean $\sum_{\alpha_i}$ often

Summary of coherent state results we'll need: $\{|\alpha_i\rangle\}$ is orthonormal basis

Boson → complex numbers

$$|\phi\rangle = e^{\sum_\alpha \phi_\alpha a^\dagger_\alpha}|0\rangle,\quad \langle\phi| = \langle 0|e^{\sum_\alpha \phi^*_\alpha a_\alpha}$$

- linear combo of $|n_{\alpha_1} n_{\alpha_2}\cdots\rangle$ states.

$$a_\alpha|\phi\rangle = \phi_\alpha|\phi\rangle,\quad \langle\phi|a^\dagger_\alpha = \langle\phi|\phi^*_\alpha,\quad \langle\phi|\phi'\rangle = e^{\sum_\alpha \phi^*_\alpha\phi'_\alpha}$$

check: $\langle\phi|\phi'\rangle = \langle 0|0\rangle + \langle 0|\sum_\alpha \phi^*_\alpha a_\alpha \sum_\beta \phi'_\beta a^\dagger_\beta|0\rangle \to \sum_\alpha \phi^*_\alpha\phi'_\alpha$

$\leftarrow \langle 0|a_\alpha a^\dagger_\beta|0\rangle = \delta_{\alpha\beta}$

$$1 = \int \prod_\alpha \frac{d\phi^*_\alpha\, d\phi_\alpha}{2\pi i}\, e^{-\sum_\alpha \phi^*_\alpha\phi_\alpha}|\phi\rangle\langle\phi|$$

(proof on 106 → sub definitions)

$$\langle\phi|A(a^\dagger_\alpha, a_\alpha)|\phi'\rangle = A(\phi^*_\alpha, \phi'_\alpha)\,e^{\sum_\alpha \phi^*_\alpha\phi'_\alpha}$$

$$\langle\phi|V^{(2)}|\phi'\rangle = \frac{1}{2}\sum_{\lambda\mu\nu\rho}\langle\lambda\mu|V^{(2)}|\nu\rho\rangle\langle\phi|a^\dagger_\lambda a^\dagger_\mu a_\rho a_\nu|\phi'\rangle$$
$$= \frac{1}{2}\sum_{\lambda\mu\nu\rho}\langle\lambda\mu|V^{(2)}|\nu\rho\rangle\,\phi^*_\lambda\phi^*_\mu\phi'_\rho\phi'_\nu\, e^{\sum_\alpha \phi^*_\alpha\phi'_\alpha}$$

$$\int \prod_{i=1}^n \frac{dz^*_i\, dz_i}{2\pi i}\, e^{-z^*_i H_{ij} z_j + J^*_i z_i + z^*_i J_i} = [\det H]^{-1} e^{J^*_i H^{-1}_{ij} J_j}$$

Fermion ⇒ Grassmann numbers

$$|\xi\rangle = e^{-\sum_\alpha \xi_\alpha a^\dagger_\alpha}|0\rangle = \prod_\alpha(1-\xi_\alpha a^\dagger_\alpha)|0\rangle,\quad a_\alpha|\xi\rangle = \xi_\alpha|\xi\rangle$$

$$\langle\xi| = \langle 0|e^{-\sum_\alpha a_\alpha \xi^*_\alpha} = \langle 0|e^{\sum_\alpha \xi^*_\alpha a_\alpha},\quad \langle\xi|a^\dagger_\alpha = \langle\xi|\xi^*_\alpha$$

$$\langle\xi|\xi'\rangle = e^{\sum_\alpha \xi^*_\alpha\xi'_\alpha}$$

$$1 = \int \prod_\alpha d\xi^*_\alpha\, d\xi_\alpha\, e^{-\sum_\alpha \xi^*_\alpha\xi_\alpha}|\xi\rangle\langle\xi|$$

$$\mathrm{Tr}\,A = \int \prod_\alpha d\xi^*_\alpha\, d\xi_\alpha\, e^{-\sum_\alpha \xi^*_\alpha\xi_\alpha}\langle -\xi|A|\xi\rangle$$

$\leftarrow$ **

$$\langle\xi|A(a^\dagger_\alpha, a_\alpha)|\xi'\rangle = e^{\sum_\alpha \xi^*_\alpha\xi'_\alpha}A(\xi^*_\alpha, \xi'_\alpha)$$

$$\int \prod_{i=1}^n d\eta^*_i\, d\eta_i\, e^{-\eta^*_i H_{ij}\eta_j + \xi^*_i\eta_i + \xi_i\eta^*_i} = [\det H]\, e^{\xi^*_i H^{-1}_{ij}\xi_j}$$

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At this point it would be opportune to present the analogous formula for Grassmann (anti-commuting) variables. [The following is taken largely from Negele and Orland, sect 1.5]

- So we'll start with a quick introduction to the essentials of Grassmann numbers, integrals, and all that.
  - We are really dealing with algebras of anticommuting numbers, but for our purposes (for now at least) we can simply regard the following definitions and manipulations as a convenient and efficient mathematical construction that builds in all of the minus signs associated with antisymmetry.
    $\Rightarrow$ There is no need to intrepret them physically.
  - Standard reference: Brézin, LeGuillou, Zinn-Justin, Phys. Rev. D15 (1977) 1544; 1588.

- An n-dimensional Grassmann algebra is defined by a set of generators $\{\xi_\alpha\}$, $\alpha = 1, \dots, n$ that satisfy anti-commutation relations:

$$\boxed{\{\xi_\alpha, \xi_\beta\} \equiv \xi_\alpha \xi_\beta + \xi_\beta \xi_\alpha = 0} \quad \alpha, \beta = 1, 2, \dots, n$$

  - Note that these are not like field operator anti-commutation relation where one could get a non-zero result — these always precisely anti commute.

- An immediate consequence is that

$$\boxed{\xi_\alpha^2 = 0 \quad \text{for } \forall \alpha}$$

so these are somewhat unusual objects!

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- We form a basis of the algebra by considering all distinct products of the generators.

$\Rightarrow$ a number in the algebra is a linear combination of

$$\boxed{\{1, \xi_{\alpha_1}, \xi_{\alpha_1}\xi_{\alpha_2}, \ldots, \xi_{\alpha_1}\xi_{\alpha_2}\cdots\xi_{\alpha_n}\}}$$

with complex coefficients with a conventional ordering of the coefficients $\alpha_1 < \alpha_2 < \ldots < \alpha_n$.

- The dimension with n generators of the algebra is 2^n (exercise for the reader to verify!)

- We'll need to define complex conjugates, which we do when n is even by just assigning half of the generators ξ_α to have corresponding ξ^*_α among the other half.
  - The properties of complex conjugation are predictable:

$$\boxed{(\xi_\alpha)^* = \xi^*_\alpha \quad (\xi^*_\alpha)^* = \xi_\alpha}$$

If $\lambda \in \mathbb{C}$ (a complex number), then $\boxed{(\lambda\xi_\alpha)^* = \lambda^*\xi^*_\alpha}$

and $\boxed{(\xi_{\alpha_1}\cdots\xi_{\alpha_n})^* = \xi^*_{\alpha_n}\cdots\xi^*_{\alpha_1}}$

- Let's see what happens with only two generators, ξ and ξ^*, so the basis is $\boxed{\{1, \xi, \xi^*, \xi^*\xi\}}$.
  - A function of ξ must be linear:

$$\boxed{f(\xi) = f_0 + f_1\xi}$$

because $\xi^2 = 1$. So Taylor series always truncate!

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We define derivatives by

$$\boxed{\frac{\partial}{\partial \xi}\xi = 1}$$

but we have to anticommute the ξ until it is to the right of $\frac{\partial}{\partial \xi}$:

$$\boxed{\frac{\partial}{\partial \xi}(\xi^* \xi) = \frac{\partial}{\partial \xi}(-\xi \xi^*) = -\xi^*}$$

If $\boxed{A(\xi^*, \xi) = a_0 + a_1 \xi + \bar{a}_1 \xi^* + a_{12} \xi^* \xi}$, then

$$\boxed{\begin{aligned} \frac{\partial}{\partial \xi} A(\xi^*, \xi) &= a_1 - a_{12}\xi^* \\ \frac{\partial}{\partial \xi^*} A(\xi^*, \xi) &= \bar{a}_1 + a_{12}\xi \\ \frac{\partial}{\partial \xi^*}\frac{\partial}{\partial \xi} A(\xi^* \xi) &= -a_{12} = -\frac{\partial}{\partial \xi}\frac{\partial}{\partial \xi^*} A(\xi^*, \xi) \end{aligned}}$$

which illustrates that $\frac{\partial}{\partial \xi^*}$ and $\frac{\partial}{\partial \xi}$ (as well as their generalizations to n-dimensional algebra) anti-commute.

Ok, so what about integrals?

- We can't do anything like the Riemannian sum as for ordinary variables.
- So we define integrals as anti-derivatives, which has the effect of making integrals the same as derivatives!
  - i.e. since $\frac{\partial}{\partial \xi}\xi = 1$, the definite integral of 1 is zero:

$$\Rightarrow \boxed{\begin{aligned} \int d\xi\, 1 &= 0 \\ \int d\xi\, \xi &= 1 \end{aligned}}$$

(Don't worry if this seems weird; you just need to know the rules. But don't treat "$d\xi$" as an infinitesimal. It's not!)

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As with derivatives, you need to anti-commute $d\xi_\alpha$ and ξ_α so they are adjacent.

- Integration for complex conjugate variables is what you would imagine:

$$\boxed{\int d\xi^* \, 1 = 0 \quad \text{and} \quad \int d\xi^* \, \xi^* = 1}$$

- So the integral exactly removes the corresponding variable (0 if not available)

OK, we can apply these rules to $f(\xi)$ and $A(\xi^*,\xi)$ as defined earlier:

$$\boxed{\int d\xi \, f(\xi) = \int d\xi \, (f_0 + f_1 \xi) = f_0 \cdot 0 + f_1 \cdot 1 = f_1}$$

$$\boxed{\begin{aligned} &\int d\xi \, A(\xi^*,\xi) = \int d\xi \, (a_0 + a_1\xi + \bar{a}_1\xi^* + a_{12}\xi^*\xi) = a_1 - a_{12}\xi^* \\ &\int d\xi^* \, A(\xi^*,\xi) = \bar{a}_1 + a_{12}\xi \\ &\int d\xi^* \, d\xi \, A(\xi^*,\xi) = -a_{12} = -\int d\xi \, d\xi^* \, A(\xi^*,\xi) \end{aligned}}$$

- We can define a δ-function as (with η another Grassmann variable)

$$\boxed{\delta(\xi,\xi') \equiv \int d\eta \, e^{-\eta(\xi-\xi')} = \int d\eta \, (1 - \underbrace{\eta(\xi-\xi')}_{\text{stops since } \eta^2=0})) = -(\xi-\xi')}$$

Check: $\int d\xi' \, \delta(\xi,\xi') f(\xi') = -\int d\xi' (\xi-\xi')(f_0 + f_1\xi') = f_0 + f_1\xi = f(\xi)$ ✓

- Finally, we define the scalar product of Grassmann functions $\langle f|g\rangle$ by

$$\boxed{\begin{aligned} \langle f|g\rangle &\equiv \int d\xi^* \, d\xi \, e^{-\xi^*\xi} f^*(\xi) g(\xi^*) \\ &= \int d\xi^* d\xi \, (1 - \xi^*\xi)(f_0^* + f_1^* \underbrace{\xi}_{\text{note: not } \xi^*})(g_0 + g_1\xi^*) \\ &= -\int d\xi^* \, d\xi \, \xi^*\xi \, f_0^* g_0 + \int d\xi^* d\xi \, f_1^* g_1 \xi\xi^* = f_0^* g_0 + f_1^* g_1 \end{aligned}}$$

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Ok, now we can generalize to $2n$ generators and think about Gaussian integrals.

· Suppose we have just ξ and ξ^*, then (with "a" a number)

$$\boxed{\int d\xi^* d\xi\, e^{-\xi^* a \xi} = \int d\xi^* d\xi\, (1 - \xi^* a \xi) = a}$$

$\Rightarrow$ we get "a" instead of "1/a" as in the ordinary Gaussian integral.

· Now try $\xi_1, \xi_2, \xi_1^*, \xi_2^*$: [class try this!]

$$\int d\xi_1^* d\xi_1\, d\xi_2^* d\xi_2\, e^{-\xi_i^* H_{ij} \xi_j}$$

$$= \int d\xi_1^* d\xi_1\, d\xi_2^* d\xi_2\, e^{-[\xi_1^* H_{11} \xi_1 + \xi_2^* H_{21} \xi_1 + \xi_1^* H_{12} \xi_2 + \xi_2^* H_{22} \xi_2]}$$

expand to 2nd order: $$= \int d\xi_1^* d\xi_1\, d\xi_2^* d\xi_2\, \frac{(-1)^2}{2!} (\xi_1^* H_{11} \xi_1 + \xi_2^* H_{21} \xi_1 + \xi_1^* H_{12} \xi_2 + \xi_2^* H_{22} \xi_2)$$
$$\times (\xi_1^* H_{11} \xi_1 + \xi_2^* H_{21} \xi_1 + \xi_1^* H_{12} \xi_2 + \xi_2^* H_{22} \xi_2)$$

$$= \frac{1}{2!} (H_{11} H_{22} - H_{21} H_{12} - H_{12} H_{21} + H_{22} H_{11})$$

$$= H_{11} H_{22} - H_{12} H_{21} = \det \underline{\underline{H}}$$

· So, again, the determinant is in the numerator rather than the denominator.

· The generalization, including Grassmann "sources" η_i and η_i^*, is

$$\boxed{\int \prod_{i=1}^{n} d\xi_i^* d\xi_i\, e^{-\xi_i^* H_{ij} \xi_j + \eta_i^* \xi_i + \xi_i \eta_i^*} = [\det H]\, e^{\eta_i^* H_{ij}^{-1} \eta_j}}$$

This works for H Hermitian.

· The proof (which requires showing how to change Grassmann variables), is given in Negele + Orland section 1.5.

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- So we see that the boson and fermion Gaussian integral formulas are quite similar, with the main distinction being $[\det H]^{-1}$ in the former and $[\det H]$ in the latter.
  - We have restrictions on H for bosons (positive definite) to ensure that the integral converges
  - No restrictions on H for fermions, since we can expand the exponential $e^{a\xi^*\xi}$ and it terminates at first order, so we get a finite integral no matter what "a" is.

- This distinction embodies the difference that the Pauli principle makes — 0, or 1 occupation numbers for fermions versus 0, 1, 2, ... for bosons.

- This shows up in the noninteracting Grand Canonical partition function evaluated in the occupation number basis:

$$Z_G^0 = \operatorname{Tr} e^{-\beta(\hat{H}_0 - \mu\hat{N})} = \prod_\alpha \sum_{n_\alpha} e^{-\beta(\epsilon_\alpha - \mu)n_\alpha}$$

where $n_\alpha = 0, 1$ for fermions, and $n_\alpha = 0, 1, 2, \ldots$ for bosons.

$\Rightarrow$ fermions: $Z_F^0 = \prod_\alpha \left(1 + e^{-\beta(\epsilon_\alpha - \mu)}\right)$ ∴ just two terms

bosons: $Z_B^0 = \prod_\alpha \left(1 + \sum_{n=1}^{\infty} \left(e^{-\beta(\epsilon_\alpha - \mu)}\right)^n\right)$

- For fermions, $\epsilon_\alpha - \mu$ can be anything, but for bosons, $\epsilon_\alpha - \mu > 0$ for all α or the partition function is not finite. $\Rightarrow$ $\hat{H} - \mu\hat{N}$ must be positive definite for bosons but not fermions.

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Example: Dilute Fermi Gas with Short-Range Interaction

- This is a good example to use for practicing perturbation theory expansions and infinite summations. It ties in nicely with effective field theory and lets us explore pairing. When we consider large scattering length, we are forced to do stochastic evaluations. We can also do it in 1D.

- The physical system could be a collection of fermionic atoms confined magnetically or optically (it makes a difference, see below).
  - we consider the trap to be constant (therefore a constant energy contribution) and very big, so effectively infinite.

- We take the two-body interaction to be (in three dimensions)

$$\boxed{V(\vec{x}-\vec{x}') = \lambda \delta^3(\vec{x}-\vec{x}')}$$

- spin-independent for now (same matrix element between any combination of spin-up and spin down)
- if $\lambda > 0 \Rightarrow$ repulsive, $\lambda < 0 \Rightarrow$ attractive
- $\nu \equiv 2S+1$ "spin" states in general ($S > \frac{1}{2}$ is allowed — or spin and isospin)
- We'll find soon that there are problem with a delta function!

- Goal: Find the ground-state energy per particle as a function of density and/or the free-energy at finite temperature. Also find the pressure and consider stability.

Plan: First pass will be perturbation theory about the noninteracting system.
- No guarantee that this is a useful thing to do!

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- To treat a bulk, uniform medium, put the system in a large box of side L (could be 1D or 3D) and take $L \to \infty$ at the end.
  - Uniform at infinite $\Rightarrow$ physical properties are translationally invariant
  - apply periodic boundary conditions on the single-particle wave functions:

$$\boxed{\psi_{\vec{k}\alpha}(\vec{x}) = \frac{1}{\sqrt{V}} e^{i\vec{k}\cdot\vec{x}} \eta_\alpha}$$ (plane waves: momentum eigenstates)

with $V \equiv L^3$ the volume of the box and η_α is the spinor.

- For spin-1/2 quantized along a given z axis:

$$\boxed{\eta_\uparrow = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \quad \boxed{\eta_\downarrow = \begin{bmatrix} 0 \\ 1 \end{bmatrix}} \quad \text{and} \quad \eta^\dagger_\alpha \eta_\beta = \delta_{\alpha\beta} \text{ with } (\alpha,\beta) = \{0,1\}$$

- Periodic boundary conditions imply discrete momenta

$$\boxed{k_i = \frac{2\pi n_i}{L}, \quad i = x,y,z, \quad n_i = 0, \pm 1, \pm 2, \ldots}$$

- The states are normalized:

$$\boxed{\begin{aligned} \int \psi^\dagger_{\vec{k}\alpha}(\vec{x}) \psi_{\vec{k}',\alpha'} d^3x &= \frac{1}{V} \int e^{-i(\vec{k}-\vec{k}')\cdot\vec{x}} \eta^\dagger_\alpha \eta_{\alpha'} d^3x \\ &= \frac{1}{V} \delta_{\vec{k}\vec{k}'} \int d^3x \, \delta_{\alpha\alpha'} = \delta_{\vec{k}\vec{k}'} \delta_{\alpha\alpha'} \end{aligned}}$$

- The usual "first-quantized" Hamiltonian for N particles is

$$\boxed{\hat{H} = \sum_{i=1}^{N} \frac{\hat{p}_i^2}{2m} + \frac{1}{2}\lambda \sum_{i\neq j}^{N} \delta^3(\vec{x}_i - \vec{x}_j) = \sum_{i=1}^{N} -\frac{\hbar^2 \nabla_i^2}{2m} + \lambda \sum_{i<j=1}^{N} \delta^3(\vec{x}_i - \vec{x}_j)}$$

with the warning that this $\hat{H}$ is actually ill-defined without a regularization/renormalization scheme.

- In second quantization, this takes the form

$$\boxed{\hat{H} = \hat{T} + \hat{V} = \sum_{\vec{k}\alpha} \frac{\hbar^2 k^2}{2m} a^\dagger_{\vec{k}\alpha} a_{\vec{k}\alpha} + \frac{\lambda}{2V} \sum_{\substack{\vec{k}_1\alpha_1 \\ \vec{k}_2\alpha_2}} \sum_{\substack{\vec{k}_3\alpha_3 \\ \vec{k}_4\alpha_4}} a^\dagger_{\vec{k}_1\alpha_1} a^\dagger_{\vec{k}_2\alpha_2} a_{\vec{k}_4\alpha_4} a_{\vec{k}_3\alpha_3} \delta_{\alpha_1\alpha_3} \delta_{\alpha_2\alpha_4} \delta_{\vec{k}_1+\vec{k}_2,\, \vec{k}_3+\vec{k}_4}}$$

10/26/09 OK, Let's apply the path integral formalism to this problem...
Start with

$$\boxed{Z = \int \mathcal{D}[\psi^\dagger(x)\psi(x)]\, e^{-\int_0^\beta d\tau \int d^3x\, \psi^\dagger_\alpha(x)\left(\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu\right)\psi_\alpha(x)} \times e^{-\frac{\lambda}{2}\int_0^\beta d\tau \int d^3x\, \psi^\dagger_\alpha(x)\psi^\dagger_\beta(x)\psi_\beta(x)\psi_\alpha(x)}}$$

where $x \equiv \{\vec{x}, \tau\}$ and $\psi^\dagger_\alpha\psi_\alpha \equiv \sum_{\alpha=1}^{g} \psi^\dagger_\alpha\psi_\alpha$ (implied summation)
and we're working with $\hbar = 1$

- We'll deal with Fermions only here, so the ψ's and $\psi^\dagger$'s are Grassmann functions (think of them as Grassmann variables ψ_{ijkl}, $\psi^\dagger_{ijkl}$ on a spacetime lattice with i,j,k,l corresponding to discrete values of $\tau, x, y,$ and z).

- Do our "usual" procedure:

① generalize Z to include "sources" $\eta(x)$ and $\eta^\dagger(x)$ coupled to $\psi^\dagger$ and ψ, respectively.
- These are Grassmann sources, with spin indices

$$\boxed{Z[\eta_\alpha, \eta^\dagger_\beta] = \int \mathcal{D}[\psi^\dagger\psi]\, e^{-\left(S_E + \int_0^\beta d\tau \int d^3x\, \eta^\dagger_\alpha(x)\psi_\alpha(x) + \psi^\dagger_\alpha(x)\eta_\alpha(x)\right)}}$$

where S_E is the Euclidean action (including the chemical potential)

$$\boxed{S_E = \int_0^\beta d\tau \int d^3x \left\{\psi^\dagger_\alpha(x)\underbrace{\left(\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu\right)}_{\equiv \mathcal{G}^{0\,-1}}\psi_\alpha(x) + \frac{\lambda}{2}\psi^\dagger_\alpha(x)\psi^\dagger_\beta(x)\psi_\beta(x)\psi_\alpha(x)\right\}}$$

② Remove the interaction term in favor of functional derivatives with respect to the Grassmann sources: $\psi \to \frac{\delta}{\delta\eta^\dagger}$, $\psi^\dagger \to \frac{\delta}{\delta\eta}$

$$\boxed{Z[\eta_\alpha, \eta^\dagger_\alpha] = e^{-\int_0^\beta d\tau \int d^3x\, \frac{\lambda}{2}\left(\frac{\delta}{\delta\eta_\alpha(x)}\right)\left(\frac{\delta}{\delta\eta_\beta(x)}\right)\left(\frac{-\delta}{\delta\eta^\dagger_\beta(x)}\right)\left(\frac{-\delta}{\delta\eta^\dagger_\alpha(x)}\right)} \int \mathcal{D}[\psi^\dagger\psi]\, e^{-\int_0^\beta d\tau \int d^3x \left[\psi^\dagger_\alpha(x)\left(\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu\right)\psi_\alpha(x) + \eta^\dagger_\alpha(x)\psi_\alpha(x) + \psi^\dagger_\alpha(x)\eta_\alpha(x)\right]}}$$

- we're using $\boxed{\frac{\delta}{\delta\eta_\alpha(x)} e^{-\int \psi^\dagger_\beta\eta_\beta} = e^{-\int \psi^\dagger_\beta\eta_\beta} \, {+\psi^\dagger_\alpha(x)}}$ and $\boxed{\frac{-\delta}{\delta\eta^\dagger_\alpha(x)} e^{-\int \eta^\dagger_\beta\psi_\beta} = e^{-\int \eta^\dagger_\beta\psi_\beta}\,\psi_\alpha(x)}$

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- We make our lives easier here by moving the Grassmann fields brought down by the derivatives all the way to the right. We can do this since all of the Grassmann variables in the exponents appear in pairs.

(3) Complete the square in the remaining path integral

- We'll carry this out with "schematic" notation, which means we'll drop the explicit $x = (\vec{x}, \tau)$ indices. Since we will keep spin indices, one could also imagine those to stand for discrete time and space indices as well.

$$\boxed{\begin{aligned}
\Rightarrow &\int \mathcal{D}(\psi^\dagger\psi)\, e^{-\int \psi^\dagger_\alpha \mathcal{G}^{0\,-1}_{\alpha\beta}\psi_\beta + \eta^\dagger_\alpha\psi_\alpha + \psi^\dagger_\alpha\eta_\alpha} \\
&= \int \mathcal{D}(\psi^\dagger\psi)\, e^{-\int (\psi^\dagger_\alpha + \eta^\dagger_\gamma \mathcal{G}^0_{\gamma\alpha})\mathcal{G}^{0\,-1}_{\alpha\beta}(\psi_\beta + \mathcal{G}^0_{\beta\delta}\eta_\delta)}\, e^{\int \eta^\dagger_\gamma \mathcal{G}^0_{\gamma\delta}\eta_\delta} \\
&= e^{\int \eta^\dagger_\gamma \mathcal{G}^0_{\gamma\delta}\eta_\delta} \int \mathcal{D}(\psi^{\dagger\prime}\psi')\, e^{-\int \psi'^\dagger_\alpha \mathcal{G}^{0\,-1}_{\alpha\beta}\psi'_\beta} \\
&= e^{\int \eta^\dagger_\gamma \mathcal{G}^0_{\gamma\delta}\eta_\delta}\, Z_0
\end{aligned}}$$

- We get the third line by shifting variables — the "Jacobian" is 1 — to $\psi'_\alpha = \psi_\alpha + \mathcal{G}^0_{\alpha\beta}\eta_\beta$.
  A proof that we can change Grassmann variables like this is given in Negele and Orland, but it should be plausible from simple examples like: $\int d\psi^\dagger d\psi\, (\psi^\dagger + \eta^\dagger)(\psi + \eta) = \int d\psi^\dagger d\psi\, \psi^\dagger\psi$.

- We've introduced $\mathcal{G}^0_{\alpha\beta}$ as the inverse of

$$\boxed{\mathcal{G}^{0\,-1}_{\alpha\beta} \to \delta_{\alpha\beta}\left(\partial/\partial\tau - \frac{\nabla^2}{2m} - \mu\right)}$$

with anti-periodic boundary conditions.

(4) Now we can do perturbation theory in powers of λ as usual.

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- In order to do perturbation theory, we need the non-interacting single-particle Green's function $\mathcal{G}^0_{\alpha\beta}$.

- We will follow the sign convention in Negele and Orland (which is opposite to that in Fetter and Walecka).

- Then $\mathcal{G}^0_{\alpha\beta}$ is the solution to $\mathcal{G}^{0-1}\mathcal{G}^0 = 1$, which becomes

$$\boxed{\delta_{\alpha\beta}\left(\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu\right)\mathcal{G}^0_{\beta\gamma}(\vec{x}\tau;\vec{x}'\tau') = \delta_{\alpha\gamma}\,\delta^3(\vec{x}-\vec{x}')\,\delta(\tau-\tau')}$$

with the boundary condition $\boxed{\mathcal{G}^0_{\alpha\beta}(\vec{x}0;\vec{x}'\tau') = -\mathcal{G}_{\alpha\beta}(\vec{x}\beta;\vec{x}'\tau')}$
(note: β sum is implied).

- There are several different ways we can derive $\mathcal{G}^0$. One of the simplest is to "guess" the answer and verify that it works. We guess

$$\boxed{\begin{aligned}\mathcal{G}^0_{\beta\gamma}(\vec{x}\tau;\vec{x}'\tau') &= \delta_{\beta\gamma}\frac{1}{V}\sum_{\vec{k}} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\, e^{-(\epsilon^0_k-\mu)(\tau-\tau')} \\ &\quad\times\left[\theta(\tau-\tau'-\eta)(1-n^0_k) - \theta(\tau'-\tau+\eta)\,n^0_k\right] \\ &\rightarrow \delta_{\beta\gamma}\int\frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\, e^{-(\epsilon^0_k-\mu)(\tau-\tau')}\left[\theta(\tau-\tau'-\eta)(1-n^0_k) - \theta(\tau'-\tau+\eta)\,n^0_k\right]\end{aligned}}$$

- The corresponding function for bosons has $1-n^0_k \to 1+n^0_k$ and a plus sign between the two θ-functions.

* - The η is an infinitesimal that indicates that if $\tau=\tau'$, we should keep the second term.
  - This prescription follows from a careful evaluation of $\mathcal{G}^0$ as the inverse of the discrete version of $\left(\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu\right)$. See Negele and Orland section 2.2 for details.
  - We take $\eta \to 0$ as soon as we've used it to pick out which θ function to take at $\tau=\tau'$.

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Let's check that it works:

i) boundary condition

$$\boxed{\mathcal{G}^0_{\rho\delta}(\vec{x}\beta;\vec{x}'\tau') = \delta_{\rho\delta}\frac{1}{V}\sum_k e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\, e^{-(\epsilon^0_k-\mu)\beta}\, e^{(\epsilon^0_k-\mu)\tau'} \times (1-n^0_k)}$$

since the first Θ function will be satisfied for $0 \le \tau' < \beta$.

$$\boxed{\mathcal{G}^0_{\rho\delta}(\vec{x}\,0;\vec{x}'\tau') = \delta_{\rho\delta}\frac{1}{V}\sum_k e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\, e^{(\epsilon^0_k-\mu)\tau'}(-n^0_k)}$$

But
$$e^{-\beta(\epsilon^0_k-\mu)}(1-n^0_k) = e^{-\beta(\epsilon^0_k-\mu)}\left(1-\frac{1}{e^{\beta(\epsilon^0_k-\mu)}+1}\right)$$
$$= \cancel{e^{-\beta(\epsilon^0_k-\mu)}}\left(\frac{\cancel{e^{\beta(\epsilon^0_k-\mu)}}}{e^{\beta(\epsilon^0_k-\mu)}+1}\right) = n^0_k$$

$$\Rightarrow \boxed{\mathcal{G}^0_{\rho\delta}(\vec{x}\beta;\vec{x}'\tau') = -\mathcal{G}^0_{\rho\delta}(\vec{x}\,0;\vec{x}'\tau')} \quad \text{as expected.}$$

ii) satisfies the differential equation:

$$\boxed{\begin{aligned}
&\delta_{\alpha\beta}\left(\frac{\partial}{\partial\tau}-\frac{\nabla^2}{2m}-\mu\right)\mathcal{G}^0_{\rho\delta}(\vec{x}\tau;\vec{x}'\tau') \\
&= (\delta_{\alpha\beta}\delta_{\rho\delta})\frac{1}{V}\sum_k \underbrace{\left(-(\epsilon^0_k-\mu)+\frac{k^2}{2m}-\mu\right)}_{0} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\, e^{-(\epsilon^0_k-\mu)(\tau-\tau')} \\
&\quad + \delta_{\alpha\beta}\delta_{\rho\delta}\frac{1}{V}\sum_k e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\, e^{-(\epsilon^0_k-\mu)(\tau-\tau')}\Big[\underbrace{\delta(\tau-\tau')(1-n^0_k)+\delta(\tau-\tau')n^0_k}_{\delta(\tau-\tau')}\Big] \\
&= \delta_{\alpha\delta}\,\delta(\tau-\tau')\frac{1}{V}\sum_k e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \\
&= \delta_{\alpha\delta}\,\delta^3(\vec{x}-\vec{x}')\,\delta(\tau-\tau') \quad \checkmark
\end{aligned}}$$

where we've used $\boxed{\frac{\partial}{\partial\tau}\Theta(\tau-\tau') = \delta(\tau-\tau')}$ and $\boxed{\frac{\partial}{\partial\tau}\Theta(\tau'-\tau) = -\delta(\tau-\tau')}$

- In PS#2 you'll <u>derive</u> the result for $\mathcal{G}^0$ by solving the differential equation explicitly.

- For completeness, we include yet another derivation using <u>field operators</u>

- We define the single-particle Green's function as

$$\boxed{\mathcal{G}_{\alpha\beta}(\vec{x}\tau;\vec{x}'\tau') = \frac{\int \mathcal{D}(\psi^+\psi)\, \psi_\alpha(\vec{x}\tau)\psi^+_\beta(\vec{x}'\tau')\, e^{-S_E}}{\int \mathcal{D}(\psi^+\psi)\, e^{-S_E}} = \mathrm{Tr}\left[e^{-\beta(\hat{H}-\mu\hat{N})}\, \underbrace{T}_{\text{time-ordering operator}}\left[\hat{\psi}_\alpha(\vec{x}\tau)\hat{\psi}^+_\beta(\vec{x}'\tau')\right]\right] \Big/ \mathrm{Tr}\left[e^{-\beta(\hat{H}-\mu\hat{N})}\right]}$$

where the imaginary time dependence of the field operators is given by

$$\boxed{\begin{aligned} \hat{\psi}_\alpha(\vec{x}\tau) &= e^{(\hat{H}-\mu\hat{N})\tau}\, \hat{\psi}_\alpha(\vec{x})\, e^{-(\hat{H}-\mu\hat{N})\tau} \\ \hat{\psi}^+_\beta(\vec{x}\tau) &= e^{(\hat{H}-\mu\hat{N})\tau}\, \hat{\psi}^+_\beta(\vec{x})\, e^{-(\hat{H}-\mu\hat{N})\tau} \end{aligned}}$$

- Fetter and Walecka define $\boxed{\hat{K} \equiv \hat{H} - \mu\hat{N}}$ which is like a grand canonical Hamiltonian, so the field operators are in a modified Heisenberg picture (F+W call them "Keisenberg" operators).

- For τ real, $\hat{\psi}(\vec{x}\tau)$ and $\hat{\psi}^+(\vec{x},\tau)$ are not Hermitian adjoints, but they are if we continue to $\tau = it$.

- These definitions are useful because the grand canonical weighting operator $e^{-\beta(\hat{H}-\mu\hat{N})}$ is of the same form.

- The noninteracting version $\mathcal{G}^0_{\alpha\beta}(\vec{x}\tau;\vec{x}'\tau')$ is found by replacing $\hat{H}$ by $\hat{H}_0$:

$$\boxed{\mathcal{G}^0_{\alpha\beta}(\vec{x}\tau;\vec{x}'\tau') = \frac{\mathrm{Tr}\; e^{-\beta(\hat{H}_0-\mu\hat{N})}\, T\left[\hat{\psi}_\alpha(\vec{x}\tau)\hat{\psi}^+_\beta(\vec{x}'\tau')\right]}{\mathrm{Tr}\; e^{-\beta(\hat{H}_0-\mu\hat{N})}}}$$

(Recall $H_0 = \sum_{\vec{k}\lambda} \epsilon^0_k\, a^+_{\vec{k}\lambda} a_{\vec{k}\lambda}$.)

- The noninteracting field operators are

$$\boxed{\hat\psi(\vec{x}) = \frac{1}{\sqrt{V}}\sum_{\vec{k}\lambda} e^{i\vec{k}\cdot\vec{x}}\,\eta_\lambda a_{\vec{k}\lambda}}$$

and
$$\boxed{\hat\psi^\dagger(\vec{x}) = \frac{1}{\sqrt{V}}\sum_{\vec{k}\lambda} e^{-i\vec{k}\cdot\vec{x}}\,\eta_\lambda^\dagger a_{\vec{k}\lambda}^\dagger}$$

and it is the $a_{\vec{k}\lambda}$ and $a^\dagger_{\vec{k}\lambda}$ operators that pick up the time dependence.

- It is straightforward to show that ($\hbar = 1$ everywhere)

$$\boxed{a_{\vec{k}\lambda}(\tau) = a_{\vec{k}\lambda}e^{-(\epsilon_k^0-\mu)\tau} \quad \text{and} \quad a^\dagger_{\vec{k}\lambda}(\tau) = a^\dagger_{\vec{k}\lambda}e^{(\epsilon_k^0-\mu)\tau}}$$

by evaluating commutators (exercise for the reader...)

- Let
$$\boxed{\langle a^\dagger_{\vec{k}\lambda}a_{\vec{k}'\lambda'}\rangle_0 \equiv \frac{\mathrm{Tr}\left[e^{-\beta(\hat{H}_0-\mu\hat{N})}a^\dagger_{\vec{k}\lambda}a_{\vec{k}'\lambda'}\right]}{\mathrm{Tr}\left[e^{-\beta(\hat{H}_0-\mu\hat{N})}\right]} = \delta_{\lambda\lambda'}\delta_{\vec{k}\vec{k}'}n_k^0}$$

Why is it diagonal? (This is an assumption that might be invalid for some systems)

with $\boxed{n_k^0 = \frac{1}{e^{\beta(\epsilon_k^0-\mu)}+1}}$ our good old fermion occupation number.

- It follows that $\boxed{\langle a_{\vec{k}'\lambda'}a^\dagger_{\vec{k}\lambda}\rangle_0 = \delta_{\lambda\lambda'}\delta_{\vec{k}\vec{k}'}(1-n_k^0)}$ [anticommutation relations]

- Now just plug and chug into $\mathcal{G}^0_{\alpha\beta}(\vec{x}\tau;\vec{x}'\tau')$

$$\boxed{\begin{aligned}\tau>\tau':\ &\frac{1}{V}\sum_{\vec{k}\vec{k}'}\sum_{\lambda\lambda'} e^{i(\vec{k}\cdot\vec{x}-\vec{k}'\cdot\vec{x}')}(\eta_\lambda)_\alpha(\eta^\dagger_{\lambda'})_\beta\, e^{(-(\epsilon_k^0-\mu)\tau+(\epsilon_{k'}^0-\mu)\tau')}\langle a_{\vec{k}\lambda}a^\dagger_{\vec{k}'\lambda'}\rangle_0\\ &= \frac{1}{V}\delta_{\alpha\beta}\sum_k e^{ik(\vec{x}-\vec{x}')}e^{-(\epsilon_k^0-\mu)(\tau-\tau')}(1-n_{k_0})\end{aligned}}$$

$$\boxed{\tau<\tau':\ \frac{1}{V}\delta_{\alpha\beta}\sum_k e^{ik(\vec{x}-\vec{x}')}e^{-(\epsilon_k^0-\mu)(\tau-\tau')}(-n_k^0)}$$ ("−" from time ordering)

$\Rightarrow \mathcal{G}^0_{\alpha\beta}(\vec{x}\tau;\vec{x}'\tau')$ is the same as before!

10/26/09

- Let's try the perturbative expansion of $\ln Z/Z_0$, which we relate to our thermodynamic observables by

$$\boxed{\Omega(V,T,\mu) - \Omega_0(V,T,\mu) = -\frac{1}{\beta}(\ln Z - \ln Z_0) = -1/\beta \ln Z/Z_0}$$

- The replica method proof that $\ln Z/Z_0$ follows from keeping only the connected diagrams goes through just as in the model partition function case by considering $(Z/Z_0)^n$, which we construct simply by introducing duplicate Grassmann path integrals over ψ_i and $\psi_i^\dagger$ with $i=1,\dots,n$.

- To find out the precise Feynman rules, however, we'll need to carry out a couple orders of the expansion explicitly.

$$\Rightarrow \boxed{\frac{Z}{Z_0} = \frac{Z[\eta_\alpha, \eta_\alpha^\dagger]}{Z_0}\Big|_{\eta=\eta^\dagger=0} = e^{-\int_0^\beta d\tau \int d^3x \frac{\lambda}{2}\left(\frac{\delta}{\delta\eta_\alpha(x)}\frac{\delta}{\delta\eta_\beta(x)}\frac{-\delta}{\delta\eta_\beta^\dagger(x)}\frac{-\delta}{\delta\eta_\alpha^\dagger(x)}\right)} e^{\int \eta_\gamma^\dagger \mathcal{G}^0_{\gamma\delta}\eta_\delta}\Big|_{\eta=\eta^\dagger=0}}$$

where the integral in the second exponential is a shorthand for

$$\boxed{\int \eta_\gamma^\dagger \mathcal{G}^0_{\gamma\delta}\eta_\delta \rightarrow \int_0^\beta d\tau_1 \int d^3x_1 \int_0^\beta d\tau_2 \int d^3x_2\, \eta_\gamma^\dagger(x_1)\, \mathcal{G}^0_{\gamma\delta}(\vec{x}_1\tau, \vec{x}_2\tau)\eta_\delta(x_2)}$$

- The $\vec{x}$ and τ dependence is actually rather easy to follow, so we will use the schematic form and just trace the spin indices.

- The other expansion we'll want to do is the Green's function $(\mathcal{G}^{(1)})$, which we can write as

$$\boxed{\mathcal{G}_{\alpha\beta}(\vec{x}\tau;\vec{x}'\tau') = \frac{\frac{-\delta}{\delta\eta_\alpha^\dagger(\vec{x}\tau)}\frac{\delta}{\delta\eta_\beta(\vec{x}'\tau')}\, e^{-\frac{\lambda}{2}\int\left(\frac{\delta}{\delta\eta}\frac{\delta}{\delta\eta^\dagger}\right)^2} e^{\int \eta^\dagger \mathcal{G}^0 \eta}\Big|_{\eta=\eta^\dagger=0}}{e^{-\frac{\lambda}{2}\int\left(\frac{\delta}{\delta\eta}\frac{\delta}{\delta\eta^\dagger}\right)^2} e^{\int \eta^\dagger \mathcal{G}^0 \eta}\Big|_{\eta=\eta^\dagger=0}}}$$

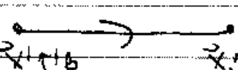
and only connected diagrams survive.

10/26/09

Let's do the leading order of $\mathcal{G}_{\alpha\beta}$ first:

$$\boxed{\begin{aligned}\mathcal{G}_{\alpha\beta}(\vec{x}\tau;\vec{x}'\tau') &= -\frac{\delta}{\delta\eta^\dagger_\alpha(\vec{x},\tau)}\frac{\delta}{\delta\eta_\beta(\vec{x}'\tau')}\left(1+\int\eta^\dagger_\gamma\mathcal{G}^0_{\gamma\delta}\eta_\delta+\dots\right)+O(\lambda)\\ &= +\mathcal{G}^0_{\alpha\beta}(\vec{x}\tau;\vec{x}'\tau')+O(\lambda)\end{aligned}}$$

- notice how the minus sign was eliminated after anticommuting $\frac{\delta}{\delta\eta}$ through $\eta^\dagger$.
- We didn't bother putting in the space and time coordinates in the expansion, since they just get set equal to those in the functional derivatives.
- Note that there is no $1/2$ in $\int\eta^\dagger\mathcal{G}^0\eta$, in contrast to the QM case where we had $\int jA^{-1}j$ with the same j's.
- Since we can tell the $\vec{x}$ end from the $\vec{x}'$ end, the Feynman rule will be to assign $\mathcal{G}^0_{\alpha\beta}$ to a line with an arrow:

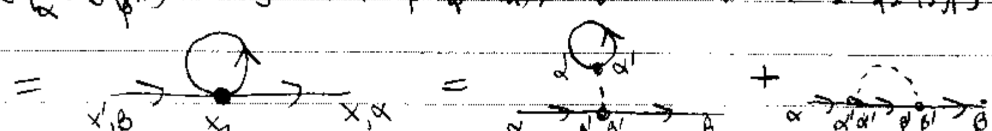


$$\Rightarrow \mathcal{G}^0_{\alpha\beta}(\vec{x}\tau;\vec{x}'\tau')\equiv\delta_{\alpha\beta}\mathcal{G}^0(\vec{x}\tau;\vec{x}'\tau')$$

* • [NOTE: Different conventions are used by different authors, which lead to minus signs (or worse) differing in the rules. The final answers for observables, of course, should be the same.]

- When we go to the next order, λ^1, there is only one connected diagram: (keeping only the terms surviving $\eta,\eta^\dagger\to0$ at the end)

$$-\frac{\delta}{\delta\eta^\dagger_\alpha(x)}\frac{\delta}{\delta\eta_\beta(x')}\left(-\frac{\lambda}{2}\int d^4x_1\left(\frac{\delta}{\delta\eta_{\alpha'}}\frac{\delta}{\delta\eta_{\beta'}}\frac{\delta}{\delta\eta^\dagger_{\beta'}}\frac{\delta}{\delta\eta^\dagger_{\alpha'}}\right)\right)\frac{1}{3!}\left(\int\eta^\dagger_{\alpha_1}\mathcal{G}^0_{\alpha_1\beta_1}\eta_{\beta_1}\right)\left(\int\eta^\dagger_{\alpha_2}\mathcal{G}^0_{\alpha_2\beta_2}\eta_{\beta_2}\right)\left(\int\eta^\dagger_{\alpha_3}\mathcal{G}^0_{\alpha_3\beta_3}\eta_{\beta_3}\right)$$



- The second set of diagrams is just a visual aid to help with the spin algebra. $\Rightarrow$ the two ends of $\cdot\cdots\cdot$ are the same space-time point x_1.
- Evaluate it after we establish Feynman rules...

10/26/09

Now do the $\mathcal{O}(\lambda)$ part of Z/Z_0 (or $\ln Z/Z_0$; it is the same to that order sin $\ln(1+\epsilon) = \epsilon$ for ϵ small).

$$\ln\frac{Z}{Z_0} = \left(-\frac{\lambda}{2}\int d^4x \frac{\delta}{\delta\eta_\alpha}\frac{\delta}{\delta\eta_\beta}\frac{\delta}{\delta\eta^\dagger_\beta}\frac{\delta}{\delta\eta^\dagger_\alpha}\right)\left(\frac{1}{2!}\int dx_1\, \eta^\dagger_{\gamma_1} \mathcal{G}^0_{\gamma_1\delta_1}\eta_{\delta_1}\int dx_2\, \eta^\dagger_{\gamma_2}\mathcal{G}^0_{\gamma_2\delta_2}\eta_{\delta_2}\right)$$

①

or ②

are the two distinct ways to match up the functional derivatives and the corresponding Grassmann variables.

- The net sign change from anticommuting is given by $(-1)^{\#\text{ of line crossings}}$ $\Rightarrow$ + for the first and − for the second.
- The first set is the "direct" term while the second set is the "exchange" term.
- There are two ways of doing each type of term $\Rightarrow$ kills one $1/2$.

$$\Rightarrow \boxed{① = 2\times\left(-\frac{\lambda}{2}\right)\frac{1}{2!}\,\delta_{\alpha\gamma_1}\delta_{\alpha\delta_1}\delta_{\beta\gamma_2}\delta_{\beta\delta_2}\int d^4x\, \mathcal{G}^0_{\gamma_1\delta_1}(x,x^+)\,\mathcal{G}^0_{\gamma_2\delta_2}(x,x^+) = -\frac{\lambda}{2}\underbrace{\left(\delta_{\alpha\gamma_1}\delta_{\gamma_1\delta_1}\delta_{\delta_1\alpha}\right)}_{\delta_{\alpha\alpha}=\nu}\underbrace{\left(\delta_{\beta\gamma_2}\delta_{\gamma_2\delta_2}\delta_{\delta_2\beta}\right)}_{\delta_{\beta\beta}=\nu}\left(\int_0^\beta d\tau\int d^3x\right)\mathcal{G}^0(0,0^+)\,\mathcal{G}^0(0,0^+)}$$

$$\boxed{② = -2\times\left(-\frac{\lambda}{2}\right)\frac{1}{2!}\,\delta_{\alpha\delta_1}\delta_{\alpha\gamma_2}\delta_{\beta\delta_2}\delta_{\beta\gamma_1}\int d^4x\, \mathcal{G}^0_{\gamma_1\delta_1}(x,x^+)\,\mathcal{G}^0_{\gamma_2\delta_2}(x,x^+) = +\frac{\lambda}{2}\underbrace{\left(\delta_{\alpha\delta_1}\delta_{\delta_1\gamma_1}\delta_{\gamma_1\beta}\delta_{\beta\delta_2}\delta_{\delta_2\gamma_2}\delta_{\gamma_2\alpha}\right)}_{\delta_{\alpha\alpha}=\nu}\left(\int_0^\beta d\tau\int d^3x\right)\mathcal{G}^0(0,0^+)\,\mathcal{G}^0(0,0^+)}$$

- So one spin sums gives ν^2 (direct) and the other gives ν (exchange) and there is a minus sign between them. (We've used $\mathcal{G}^0_{\alpha\beta} \propto \delta_{\alpha\beta}$.)
- The vertex gives $-\lambda$ and the two is cancelled.
- We've written $\mathcal{G}^0(0,0^+)$ as an alternative to using the η infinitesimal in $\mathcal{G}^0$ to decide what to do at equal time.

10/26/09

- From page ( ), $\mathcal{G}^0(0;0^+) = \int \frac{d^3k}{(2\pi)^3}(-n_k^0)$
- Since there are no remaining dependencies of x or τ, $\int_0^\beta d\tau \int d^3x = \beta \cdot V$

$$\Rightarrow \boxed{\ln\frac{Z}{Z_0} = (\beta V)\left(+\frac{\lambda}{2}\right)(\nu^2-\nu)\left(\int\frac{d^3k}{(2\pi)^3}n_k^0\right)^2 + O(\lambda^2) = +\beta(\Omega-\Omega_0)}$$

or

$$\boxed{\Omega = \Omega_0 + V\frac{\lambda}{2}\left(1-\frac{1}{\nu}\right)\left(\nu\int\frac{d^3k}{(2\pi)^3}n_k^0\right)^2 + O(\lambda^2)}$$

- but $\nu\int\frac{d^3k}{(2\pi)^3}n_k^0$ (or $\frac{1}{V}\sum_{k\alpha} n_k^0$) is just the density (the thermal averaged density).
- There is no μ dependence in the $O(\lambda)$ term, so in the $T=0$ limit,

$$\boxed{N = \frac{\partial\Omega}{\partial\mu} = \frac{\partial\Omega_0}{\partial\mu}} \quad \text{and} \quad \boxed{\Omega_0 = E_0 + \mu N}$$

$$\overset{T=0}{\Longrightarrow} E = E_0 + E_1 = \frac{3}{5}\frac{\hbar^2k_F^2}{2m}\cdot N + V\frac{\lambda}{2}\left(1-\frac{1}{\nu}\right)\rho^2$$

or

$$\boxed{\frac{E^{(0)}}{N} + \frac{E^{(1)}}{N} = \frac{3}{5}\frac{\hbar^2k_F^2}{2m} + \frac{\lambda}{2}\left(1-\frac{1}{\nu}\right)\rho}$$

- This result agrees with a simple perturbative calculation using 2nd quantization.

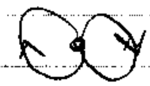
- Note that we got the correct answer immediately for the exchange part, without any change of variables or tricky integration regions.
  - The calculation would be considerably trickier if we didn't have a delta function interaction (i.e., if it had finite or infinite range) but the generalization of our procedure is relatively easy.

- Note also that we rather trivially ended up in momentum space because of the $\mathcal{G}^0(0,0^+)$'s.

- Generalizing from $\frac{-\nabla^2}{2m}$ to $\frac{-\nabla^2}{2m} + U(\vec{x})$, with a background field $U(\vec{x})$ is simple (see later).

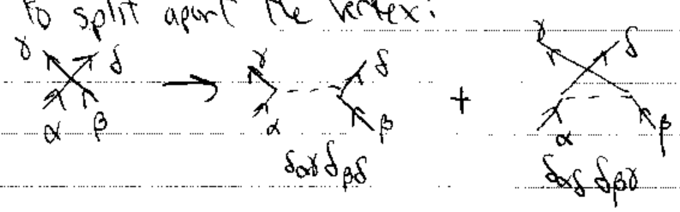
10/26/09

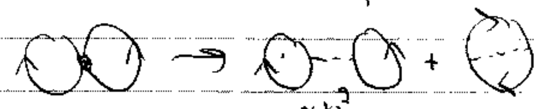
Ok, so what about the Feynman diagram and rules?

One vertex, two lines 

$\Rightarrow$ vertex $\bullet$ $-\lambda$

- The "$-$" comes from the minus sign in front of the action in the exponent of the path integral.
- The cancellation of the $\frac{1}{2}$ in front of the λ is like the 4! factor in our model. We get 2 instead of 4! because the two ends of each line (noninteracting Green's function) are different and we keep direct and exchange separate.
- Each line gets $\mathcal{G}^0_{\alpha\beta}(x,x')$ with x and x' determined by the vertices (or external points) it connects.
- Each vertex gets a space-time point x_i and we integrate $\int d^4x_i$ at the end.
- The vertices have two incoming lines and two outgoing lines.
- The spin sums follow the fermion lines around, until they close on themselves, yielding a net factor of ν. At each vertex there are two choices of directions to go. One way to follow the spin is to split apart the vertex:



so the diagrams are 

There is a relative sign, which we can account for in different ways. One convenient way is to just do the spin sums and substitute $-\nu$ for every $\delta_{\alpha\alpha}$ factor.

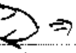
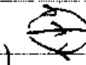
Note: If there are spin dependent interactions, one simply inserts the appropriate spin-matrices at the vertices.

10/26/09

Finally, to get the overall factor correct, we need a symmetry factor. These diagrams have lines with arrows so the rules are modified to:

① no factor anymore $\Rightarrow$ 1 always

② equivalent lines must have arrows in the same direction:

 $\Rightarrow 1$,  $\frac{1}{2!}$,  $\frac{1}{3!}$,  $\frac{1}{2!}$ and so on

③ permutations must transform the diagram into itself, including the arrows.

To summarize, the rules for the n^{th} order contribution to $\ln(Z/Z_0)$ at temperature $T = 1/\beta$:

a. Draw all distinct, full connected diagrams with n vertices. Distinct diagrams are those that cannot be deformed to coincide with each other, including the directions of arrows.

b. Assign a space time point $x_i = (\vec{x}_i, \tau_i)$ to each vertex and a factor $(-\lambda)$. Each internal line gets a factor $\mathcal{G}^0_{\alpha\beta}(x_1, x_2)$ running from x_2 to x_1. The vertex lines each have a spin index. For spin-independent interactions the two-body vertices have the structure $(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})$ where α, β are incoming spins and γ, δ are outgoing spins.

c. Do the spin summations and substitute $-\nu$ for each $\delta_{\alpha\alpha}$ in a closed fermion loop.

d. Integrate $\int dx_i$ over all x_i. (We'll discuss how to deal with divergences later.)

e. Multiply by a symmetry factor as indicated above.

Pretty easy, huh?

10/26/09

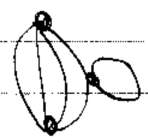
Diagram topology and symmetry factor practice:

λ^1 :  ② $1/2$ ③ $1 \Rightarrow \frac{1}{2}$

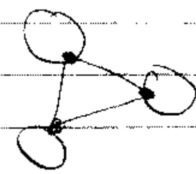
λ^2 :  ② $\frac{1}{2} \cdot \frac{1}{2}$ ③ $\frac{1}{2} \Rightarrow \frac{1}{8}$  ② 1 ③ $\frac{1}{2} \Rightarrow \frac{1}{2}$

② ③

λ^3 : 

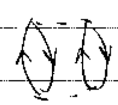
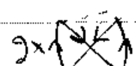


add arrows and factors



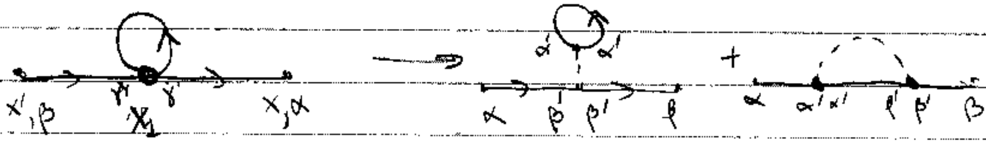
Try some spin sums. What are the factors with the λ^2 diagrams?

Use Mathematica notebook.

 $\Rightarrow 2 \times$  $2 \times$ 

10/26/09

- If we now return to calculating the single-particle Green's function (from page (110)):



$\Rightarrow(-\nu)$ $\qquad\qquad$ $\Rightarrow(+1)$

- The spin sums are indicated by the exploded diagrams.
  - In both cases we end up the $\delta_{\alpha\beta}$ connecting the outside lines, the outside lines but the direct term has an additional spin sum in the tadpole $\Rightarrow$ factor of $(-\nu)$ by our Feynman rules.

- The rest of the diagram is evaluated trivially:

$$\boxed{G^{(2)}_{\alpha\beta}(\vec{x}\tau, \vec{x}'\tau') = -\int d^4x_1\, G^0_{\alpha\gamma}(x,x_1)\left[\delta_{\gamma\delta}(-\lambda)(-\nu+1)\,G^0(x_1,x_1^+)\right]G^0_{\delta\beta}(x_1,x')}$$

- As before, $\boxed{G^0(x_1,x_1^+) = G^0(0,0^+) = \int \frac{d^3k}{(2\pi)^3}(-n_k^0) = -\frac{\rho}{\nu}}$ for a uniform system

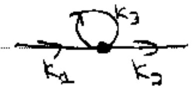
- The structure here is of the form $G^0(-\Sigma)G^0$, where we suppress the integrations and spin indices. Any contribution to G at any order will always have G^0's bracketing a piece in the middle, which we call the "self-energy". (It is conventionally defined with a minus sign at $T\neq 0$).
  - In the present case Σ is a constant and diagonal in spin. More generally it is a function of two space-time points and is spin dependent.
- We'll talk much more about the self-energy and the "proper" self energy below.

10/26/09

If we substitute

$$\boxed{\mathcal{G}^0_{\alpha\beta}(\vec{x}\tau;\vec{x}'\tau') = \delta_{\alpha\beta}\frac{1}{V}\sum_{\vec{k}} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\, e^{-(\epsilon_k-\mu)(\tau-\tau')} \times\left[\theta(\tau-\tau')(1-n_k^0) - \theta(\tau'-\tau)n_k^0\right]}$$

into the equation for $\mathcal{G}^{(1)}_{\alpha\beta}$, we initially have to sum over a different variable $\vec{k}_1, \vec{k}_2, \ldots$ for each propagator line.



- But all of the $\vec{x}_1$ dependence will then be explicit:

$$\boxed{\int d^3x_1\, e^{-i\vec{k}_2\cdot\vec{x}_1}\, e^{i\vec{k}_1\cdot\vec{x}_1} = \int d^3x_1\, e^{i(\vec{k}_1-\vec{k}_2)\cdot\vec{x}_1} = (2\pi)^3\delta^3(\vec{k}_1-\vec{k}_2).}$$

$\Rightarrow$ momentum conservation.

- we also have a $e^{i\vec{k}_3\cdot(\vec{x}_1-\vec{x}_1)} = 1$ factor, which just says that $\vec{k}_3$ is both entering and exiting the vertex, so it doesn't add any new constraint.

- This pattern will repeat with any diagram, which means we can eliminate all $\vec{x}$ integrations from the start and replace them directly with momentum sums (which become integrals as $V\to\infty$) in the propagators. The Feynman rules will include the prescription to conserve momentum at the vertices.

- To deal with the time dependence we will end up doing something similar and switch to frequency space.
  - We have to be a bit more careful, since the time interval is from 0 to β and we have periodic (or antiperiodic in this case) boundary conditions.
  - We'll come back to this.