PHYSICAL REVIEW C VOLUME 19, NUMBER 4 APRIL 1979

Problems in determining nuclear bound state wave functions

R. D. Amado
Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania 19104
(Received 20 December 1978)

The interior part of a bound state is difficult to determine empirically. We use momentum space singularity structure to show that final state interactions, rescattering, exchange currents, etc., always act so as to obscure the bound state interior. The asymptotic part of the bound state can be measured for sufficiently "light" components. We apply these results to explain why the deuteron D-state probability cannot be fixed and why it is difficult to give meaning to the idea of π-N resonances in the nucleus. We treat bound state breakup, elastic scattering, and form factors.

[NUCLEAR STRUCTURE Difficulties in empirical determination of bound state interiors. Application to deuteron D-state and N^* in nuclei.]

I. INTRODUCTION

Determining the nuclear bound state wave function is one of the goals of nuclear physics, but even for the deuteron, the simplest of nuclei, this goal is far from realized. The 50% uncertainty in P_D, the deuteron D-state probability, after 40 years of deuteron studies is a striking example of the problem.[1] Is this kind of difficulty an empirical problem, that can be solved with better data, or is it theoretical? We will show that the problem is basically theoretical, and that determining the structure of bound state wave functions cannot be separated from the problem of having a full hadron dynamics. We will show that the problem gets worse as one goes to shorter and shorter distances, but that at the other extreme there are asymptotic (long range) features of the bound state wave function that can be determined purely empirically. As examples we will show why P_D cannot be determined empirically, but the D-wave deuteron normalization can be, and why the idea of π-N resonances as bound state components is so difficult to make precise.[2]

We will present our arguments in terms of the k-space wave functions since statements about ranges, etc., can then be made quantitative. In particular, we will explout the k-space singularity structure of the wave function and of amplitudes used to "measure" the wave function. This permits direct contact with experiment, which is usually analyzed in terms of momentum variables. We will only concern ourselves with the position of singularities and with situations in which nearby singularities from complex processes obscure wave function information carried on more distant singularities. In general, we will not discuss the more difficult, model dependent, question of the strength of the singularities and whether the obscuring singularities are strong enough to be important. We will find that in all situations final state interactions and related dynamical processes have singularities directly on top of the singularities of the interior (inside the potential range) part of the bound state. Thus, only the asymptotic part of the wave function can be determined empirically without making a detailed dynamical analysis that is in some sense equivalent to assuming the interior bound state structure. If the bound state components being examined involve "heavy" particles (such as π-N resonances) even the asymptotic part is obscured by dynamical singularities. Hence, these components have no empirical meaning but can only be discussed in terms of nuclear dynamics that should, to be consistent, include the pion degrees of freedom. Unfortunately, no very good theory of this kind presently exists. Similar difficulties obscure the disentangling of exchange current effects from those of wave function interiors.

In Sec. II we review the analytic structure of a two-body Schrödinger wave function for a one- and a two-component system. This helps to establish our vocabulary. In Sec. III we discuss a break-up experiment as a way to "measure" the bound state wave function. We study breakup with a weak probe of known dynamics in order not to confuse the bound state structure with probe dynamics. We show that final state interactions obscure our ability to see "inside" the bound state in this reaction, but that the asymptotic or "outside" part of the bound state can be determined. Going to high energy so as to suppress final state interactions gives problems with exchange currents. We apply these ideas to the deuteron and show why P_D cannot be fixed empirically, but the asymptotic D-wave normalization can be. For the case of π-N resonance in the nucleus we argue that even the singularity associated with the asymptotic part of the state (spectator N^*) is obscured by resonances produced only by the final state interactions. However, such a discussion should be done relativistically and in Sec. IV we present such a treatment of bound state breakup. We apply relativistic kinematics to N^* exchange in nucleon-deuteron backward elastic scattering and show that rescattering obscures the exchange pole. We also give a brief discussion of form factors and exchange currents. Section V is a brief discussion of results and directions for further research. A technical discussion of the singularity analysis is given in the Appendix.

II. REVIEW OF BOUND STATE PROPERTIES

Before discussing multicomponent wave functions and how to measure them, let us review briefly some general features of one- and two-component nonrelativistic bound states. Our treatment will be primarily in momentum space and use analytic function methods since our results can be framed more precisely in that way, but it is clear that these results can all be translated, via Fourier transformation, to r space.

A. One channel bound state

Consider a two-body bound state $|s\rangle$ in a potential V with binding energy B $(B>0)$. The Schrödinger equation $(\hbar=1)$ is

$$\frac{k^2}{2m}\langle\vec{k}|s\rangle+\langle\vec{k}|V|s\rangle=-B\langle\vec{k}|s\rangle\,, \tag{1}$$

$$\langle\vec{k}|s\rangle=-\frac{\langle\vec{k}|V|s\rangle}{k^2/2m+B}, \tag{2}$$

where $\vec{k}$ is the appropriate relative momentum and m the reduced mass. The energy denominator of (2) has a zero at the on-shell point $k^2=-2mB \equiv-\alpha^2$, while the numerator, which we call the vertex function, depends on the potential. This form for the bound state of interaction dependent vertex divided by an energy denominator that vanishes at the on-shell point is very general.

The pole of (2) at $k^2=-\alpha^2$ is related to the large r $(r=|\vec{r}_1-\vec{r}_2|)$ behavior of the r-space wave function. For example, for an s-wave bound state we have

$$\langle\vec{r}|s\rangle=\frac{N}{(4\pi)^{1/2}}\frac{e^{-\alpha r}}{r}+C(r)\,, \tag{3}$$

where $C(r)$ is the short range or interior part of $\langle\vec{r}|s\rangle$ that for large r falls off more rapidly than $e^{-\alpha r}/r$. N is the asymptotic normalization of $\langle\vec{r}|s\rangle$ and is a measure of the strength of the asymptotic part of the properly normalized bound state. The asymptotic part of $\langle\vec{r}|s\rangle$ comes only from the pole of (2) so that N is related to the residue at that pole by

$$N=-\frac{2m}{(4\pi)^{1/2}}\langle\vec{k}|V|s\rangle\big|_{k^2=-\alpha^2}\,. \tag{4}$$

In terms of the momentum space wave function, the shorter range parts of $\langle\vec{r}|s\rangle$, $C(r)$, translate into singularities of $\langle\vec{k}|s\rangle$ that are further from the physical region $(k^2>0)$, than the pole. From (2) we see that these come from the vertex function and therefore must carry the potential range. If V is a superposition of Yukawas with maximum range μ_0, these vertex singularities will be branch cuts in the interval $-\infty<k^2<-(\mu_0+\alpha)^2$. For an s wave we can write, therefore,

$$\langle\vec{k}|s\rangle=\frac{(4\pi)^{1/2}N}{k^2+\alpha^2}+\frac{1}{\pi}\int_{(\alpha+\mu_0)^2}^{\infty}\frac{\rho(p^2)dp^2}{k^2+p^2}\,, \tag{5}$$

where ρ is the discontinuity across the vertex cuts and contains all information on the interior parts of the wave function.

Knowing the wave function completely is therefore equivalent to knowing N and $\rho(p^2)$. They are related dynamically, but the only kinematic constraint between them is the normalization condition. As we shall see, N is easy to determine empirically while information on the interior region, that is knowledge of $C(r)$ or equivalently $\rho(p^2)$ is far more difficult to get without knowing a good deal about the dynamics.

B. Two channels

Suppose the particles forming the bound state have two distinct states $|a\rangle$ and $|b\rangle$, separated by an energy ϵ $(\epsilon\geq 0)$. These states can be thought of as differing in internal symmetry with $\epsilon=0$ as in the S and D components of the deuteron, or as corresponding to internal excitation of one of the constituent particles by an energy ϵ as in the case of nuclear resonances in the bound state. Suppose also that the potential V is a matrix in this space. The bound state $|s\rangle$ with binding energy B $(B>0)$ can now be expanded as

$$|s\rangle=\int\frac{d^3k}{(2\pi)^3}(|\vec{k}a\rangle\langle\vec{k}a|s\rangle+|\vec{k}b\rangle\langle\vec{k}b|s\rangle)\,. \tag{6}$$

The Schrödinger equation for the two amplitudes is

$$\langle \vec{k}a \mid s\rangle = -\frac{\langle \vec{k}a \mid V \mid s\rangle}{k^2/2m + B}, \tag{7a}$$

$$\langle \vec{k}b \mid s\rangle = -\frac{\langle \vec{k}b \mid V \mid s\rangle}{k^2/2m + \epsilon + B}. \tag{7b}$$

These are again of the form of vertex over denominator, with the zero of the denominator coming at the appropriate on shell point for "dissociation" of the bound state. The corresponding r-space amplitudes can be written (for simplicity we again take s waves)

$$\langle \vec{r}a \mid s\rangle = \frac{N_a}{(4\pi)^{1/2}} \frac{e^{-\alpha r}}{r} + C_a(r), \quad \alpha^2 = 2mB \tag{8a}$$

$$\langle \vec{r}b \mid s\rangle = \frac{N_b}{(4\pi)^{1/2}} \frac{e^{-\beta r}}{r} + C_b(r), \quad \beta^2 = 2m(B+\epsilon) \tag{8b}$$

where the C's are the short range or interior parts. The asymptotic normalizations N_a and N_b and the residues of (7) are related by the appropriate generalizations of (4). The fact that the range β^{-1} is shorter than α^{-1} corresponds to the extra energy needed to excite b. For Yukawa potentials of maximum range μ_0, the branch cut of $\langle \vec{k}a \mid V \mid s\rangle$ begins at $k^2 = -(\alpha+\mu_0)^2$ corresponding to the longest range of C_α, just as we expect from the one channel case. However, the branch cut of $\langle \vec{k}b \mid V \mid s\rangle$ *does not* begin at $-(\beta+\mu_0)^2$ as one might expect. From off diagonal terms in the potential, there are contributions to $\langle \vec{k}b \mid V \mid s\rangle$ of the form

$$\int \langle \vec{k}b \mid V \mid \vec{k}'a\rangle \langle \vec{k}'a \mid s\rangle \frac{d^3k'}{(2\pi)^3}. \tag{9}$$

This gives a branch cut starting at $k^2 = -(\mu_0+\alpha)^2$ coming from the largest range (pole term) part of $\langle \vec{k}'a \mid s\rangle$. (We assume for simplicity that the off-diagonal potential also has maximum range μ_0.) If $\beta > \alpha + \mu_0$ we have the anomalous situation where even the asymptotic part of the b-component bound state falls inside the potential range. We will encounter just this problem for π-N resonances "in" nuclear bound states.

III. "MEASURING" THE WAVE FUNCTION BY BREAKUP

To specify the structure of a bound state, we must determine its binding energy (we will assume there is no difficulty with that), its constituents, and their wave function. We have seen from the discussion of Sec. II that, given the constituents, determining the wave function is equivalent to determining the residue at the bound state pole and the discontinuity across the vertex branch cut. In r space this corresponds to fixing the asymptotic normalization and determining the interior wave function.

The most direct way to determine the bound state structure is to break the bound state up and study the spectrum of emitted particles. However, if this is done with a hadronic probe, uncertainties associated with the probe dynamics will cloud the bound state information. Hence, we will concentrate on break-up by a weak probe, that is, a probe of fully understood dynamics and one that interacts only once. A simple example of this is a virtual photon from inelastic electron scattering. Let us begin by considering the breakup of our simple one-channel model by such a probe.

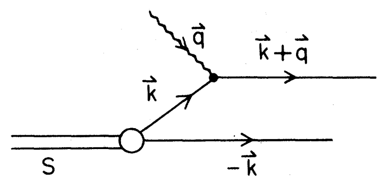

FIG. 1. Diagrammatic representation of lowest order breakup of the bound state $\mid s\rangle$ by a photon of momentum $\vec{q}$.

A. Breakup one-channel model

Consider breakup of the bound state $\mid s\rangle$ of Sec. II A by a virtual photon of momentum $\vec{q}$ as shown in Fig. 1. We take the "photon" to be scalar and to couple to only one of the bound particles with a coupling or charge e, but with no internal structure (i.e., no form factor). The breakup amplitude for producing final particles of momentum $\vec{k}+\vec{q}$ and $-\vec{k}$ corresponding to Fig. 1 is

$$\begin{aligned} \mathfrak{M}_0 &= e\langle \vec{k} \mid s\rangle \\ &= -e\frac{\langle \vec{k} \mid V \mid s\rangle}{k^2/2m + B}, \end{aligned} \tag{10}$$

using (2). $\mathfrak{M}_0$ carries just the information we seek. If $\mathfrak{M}_0$ were the full amplitude for the breakup, we could manipulate $\vec{q}$ and $\vec{k}+\vec{q}$ in an experiment so as to extract $\langle \vec{k} \mid s\rangle$ for all $\vec{k}$. However, $\mathfrak{M}_0$ is not the only contribution. While we can imagine arbitrarily weak probe coupling, we cannot switch off the potential that produces the bound state itself and that will lead to final state interactions (FSI) neglected in $\mathfrak{M}_0$. These FSI will act on the outgoing particle only if it is within range of the potential, hence FSI obscure our view of the "inside" part of the bound state. The long range part, the pole, is unobscured by FSI. To see this, it is enough to consider the simplest contribution to FSI—single scattering. The graph for this process is shown in Fig. 2 and the corresponding amplitude is

$$\mathfrak{M}_{(1)} = e\int \frac{d^3k'}{(2\pi)^3} \frac{\langle \vec{k}+\vec{q}, -\vec{k} \mid V \mid \vec{k}'+\vec{q}, -\vec{k}'\rangle \langle \vec{k}' \mid s\rangle}{E - (\vec{k}'+\vec{q})^2/2m_1 - k'^2/2m_2}, \tag{11}$$

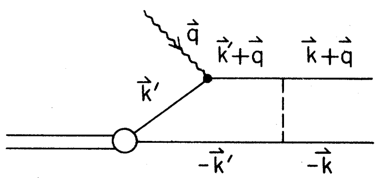

FIG. 2. Diagrammatic representation of first order rescattering contribution to bound state breakup, Eq. (11).

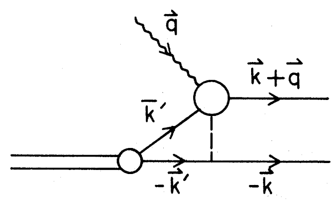

FIG. 3. Diagrammatic representation of exchange current contribution to breakup.

where m_1 is the mass of the charged constituent, m_2 of the other. E is the final energy, $E=(\vec{k}+\vec{q})^2/2m_1+k^2/2m_2$. In Appendix A we show explicitly, what is already clear from Fig. 2, that the leading k^2 singularity of $\mathfrak{M}_{(1)}$ is a branch cut beginning at $k^2=(\alpha+\mu_0)^2$. Hence, the leading final state interaction singularity lies directly on top of the leading vertex singularity in $\mathfrak{M}_0$. Therefore, vertex information cannot be extracted unless the FSI cut in $\mathfrak{M}_{(1)}$ is explicitly removed.

Higher order FSI graphs will have shorter ranges than $\mathfrak{M}_{(1)}$. Generalizing the analysis of $\mathfrak{M}_{(1)}$ we see that $\mathfrak{M}_{(n)}$ will have a branch cut beginning at $k^2=-(\alpha+n\mu_0)^2$. These cuts correspond to shorter and shorter range for increasing n. There are corresponding cuts in the vertex that come from iterating the Schrödinger equation (2). Hence, the deeper we try to penetrate into the vertex, the more complex becomes the FSI. This corresponds to the well-known fact that r-space features of the wave function are more and more difficult to extract as one goes to smaller and smaller r.

From this discussion and the result of Appendix A, we see that FSI do not obscure the pole at $k^2=-\alpha^2$ or its residue. Hence, that residue is a model independent observable that can be extracted in a weak breakup experiment, for example, by some extrapolation procedure, with not more dynamical input that the value of the binding energy and knowledge that the potential has finite range. To get at interior information requires disentangling the effect of FSI from bound state information. This is a dynamical problem that requires knowledge of the potential. The FSI terms interfere destructively with $\mathfrak{M}_0$ as can be seen by noting that as $\vec{q}\to 0$, the full breakup amplitude including FSI must vanish by the orthoganality of the scattering and bound states,[3] while $\mathfrak{M}_0$ clearly does not vanish as $\vec{q}\to 0$. This orthogonality can be exploited to develop general features of the breakup amplitude.[4]

Within the context of the nonrelativistic model being considered here, there is an alternative for disentangling the effects of FSI. They can be suppressed by going to high final state energy. Because of the energy denominator in (11), $\mathfrak{M}_{(1)}$ decreases compared with $\mathfrak{M}_0$, with increasing final state energy E. To keep k^2 fixed while increasing E requires that $\vec{q}$ increase. In our simple model with structureless photon vertex that is allowed, but in realistic situations, large q^2 introduces structure at the photon vertex (form factors) that suppresses $\mathfrak{M}_0$ and worse still exchange currents. A typical exchange current graph coupled to the potential of range μ_0 is shown in Fig. 3. It is easy to see by the arguments given in Appendix A (in fact, one needs also the discussion of relativistic dynamics given in Sec. IV) that this graph has a branch cut at $k^2\simeq-(\alpha+\mu_0)^2$, again obscuring our ability to see the short range behavior of the bound state without further dynamical input.

B. Breakup in two-channel model

Let us turn now to the weak probe breakup of the two-channel model discussed in Sec. II B. For the lower energy amplitude $\langle\vec{k}a\,|\,s\rangle$, the discussion will be identical to that given in III A above. However, for $\langle\vec{k}b\,|\,s\rangle$ there is a new feature. In lowest order, the amplitude for breakup by a weak probe leading to the state b corresponding to Fig. 4 is given by

$$\begin{aligned}\mathfrak{M}_{0b}&=e\langle\vec{k}b\,|\,s\rangle\\&=-e\frac{\langle\vec{k}b\,|\,V\,|\,s\rangle}{k^2/2m+\epsilon+B},\end{aligned}\tag{12}$$

which as discussed in II B has a pole at $k^2=-\beta^2=-2m(\epsilon+B)$, but a branch cut starting at $k^2=-(\alpha+\mu_0)^2$; $(\alpha^2=2mB)$. There are now two first

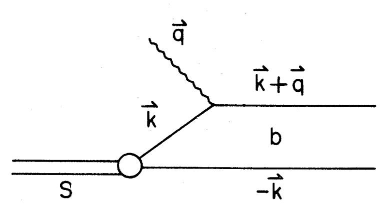

FIG. 4. Diagrammatic representation of lowest order breakup of the bound state $|\,s\rangle$ leading to the state b.

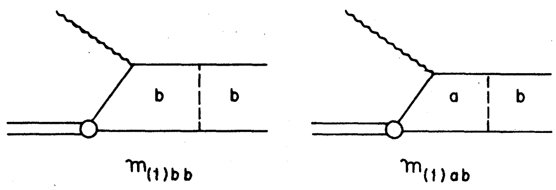

FIG. 5. Diagrammatic representations of the first order rescattering contribution to bound state breakup leading to the state b from a diagonal interaction $\mathfrak{M}_{(1)bb}$ from an off-diagonal interaction $\mathfrak{M}_{(1)ab}$.

order FSI graphs, as shown in Fig. 5, $\mathfrak{M}_{(1)bb}$ and $\mathfrak{M}_{(1)ab}$. One involves diagonal final state rescattering via V_{bb}, but the other involves off-diagonal rescattering via V_{ab}. This second term does not involve the $|kb\rangle$ content of the bound state at all, but rather b is "created" by the FSI. Because the $|\vec{k}a\rangle$ component has longer range than the $|\vec{k}b\rangle$ component, the $\mathfrak{M}_{(1)ab}$ term has the longer range. It is given by

$$\mathfrak{M}_{(1)ab}=e\int\frac{d^3k'}{(2\pi)^3}\frac{\langle\vec{k}b\,|\,V\,|\,\vec{k}'a\rangle\langle k'a\,|\,s\rangle}{E-\epsilon-(k'+q)^2/2m_1-k'^2/2m_2}\,. \tag{13}$$

By precisely the same arguments as given in Appendix A, we see that $\mathfrak{M}_{(1)ab}$ has a branch cut for $k^2<0$ beginning at $k^2=-(\alpha+\mu_0)^2$. Again this cut starts at the same place as the vertex cut because it has the same origin. Quantum mechanics requires them both. Not only does this cut obscure the interior region, but it confuses the issue of whether a b state coming from that region is part of the bound state or is formed in the rescattering.

If $\beta^2<(\alpha+\mu_0)^2$ we can at least fix the pole residue of the b component without interference from FSI, but if $\beta^2>(\alpha+\mu_0)^2$ the FSI cut will obscure even the pole. In that case one needs dynamics to decide how much b there is in the bound state even asymptotically, and to distinguish this from b components formed in the rescattering.

C. Two examples

We conclude this section by applying the arguments developed here to two simple examples, the deuteron D state and resonances in the nucleus.

In the case of the deuteron D state, $\epsilon=0$, and a and b are distinguished by their symmetry. Since $\epsilon=0$, N_a and N_b, the S- and D-wave asymptotic normalization constants, can be directly determined empirically. (In fact, since the D wave is small, this determination requires experiments that suppress the S wave and single out the D wave.[5]) However, determining the interior part of the S- or D-wave wave function requires a dynamical model. P_D, the percent D wave in the deuteron is the normalization integral of the entire D wave, and it receives considerable contribution from the interior of the wave function. Its extraction is therefore dependent on dynamical assumption and it should be no surprise that after 40 years, it still cannot be fixed to 40%. Given some particular dynamical framework, or potential, P_D has a well defined value, but different dynamics, adjusted to the same empirical input, give different values of P_D.[1] (In particular, the quadrupole moment, since it depends primarily on the exterior of the wave function, puts little constraint on P_D.[6]) As Friar has shown,[7] if one includes the meson degrees of freedom in the dynamics, the determination of P_D becomes even more model dependent. Thus short of our having a well defined and universally agreed to hadron dynamics, P_D should not be considered an empirical feature of the deuteron. On the other hand, the asymptotic normalization of the D state is a well defined empirical quantity and its nonzero value along with the quadrupole moment, assure that although P_D is not unique, it cannot be put equal to zero in any model.

Let us now turn to the example of nucleon resonance in the nucleus. We have shown previously that, in a nonrelativistic framework, the amplitude for finding a resonance in a bound state can be unambiguously defined,[8] as in (7b), with ϵ taken as the complex resonance energy. (In fact, for π-N resonances the problem should be discussed relativistically, and we do this in the next section.) These components are often studied in a breakup reaction with a "spectator" resonance.[2] This amounts to studying the pole terms or asymptotic normalization of the resonance in the breakup reaction. But as we emphasized above if $\beta^2>(\alpha+\mu_0)^2$ (suitably generalized in the case of complex ϵ), this pole lies behind the cut corresponding to the possibility of resonance excitation after breakup by FSI. From threshold considerations, it is easy to see (and we will show it explicitly in the next section) that this always happens in the π-N case. Hence, the extraction of these resonance components, even their asymptotic part, depends on some understanding of the long range part of π-N dynamics.

In summary, we have seen that only the asymptotic normalization of a bound state is "observable" in a weak probe breakup reaction, and in multichannel cases possibly not even that. Information on the interior part can be obtained only by adding dynamical input that is in some sense equivalent to the interior information itself. It is clear that this result is not a special feature of our simple model, and that the situation is at least as bad in other reactions that might be used to study the bound state. Hadronic probes only make matters more complicated, and we will examine elastic scattering and form factors explicitly in Sec. IV, since they are more conveniently dealt with relativistically. It is possible, of course, through many complementary experiments to develop the dynamics needed to understand both the FSI and the bound state. This can either lead to a partial picture dealing only with the longest range part of the interaction, or it can be a full dynamical scheme as in quantum electrodynamics. But in these cases information about the bound state interior is only as good as the dynamics used to extract it.

IV. RELATIVISTIC KINEMATICS AND OTHER REACTIONS

In this section we will report briefly on weak probe breakup by using relativistic kinematics. This will make our treatment appropriate for π-N resonances and for other aspects of π degrees of freedom. We will then turn to a short discussion of other reactions; elastic scattering, form factors, exchange currents, etc. that also involve the π degrees of freedom.

A. Breakup

Consider the breakup of a stable object of mass M and four-momentum Q by a virtual weak probe of four-momentum q leading to a particle of momentum $p+q$ and mass m and a particle of momentum Q-p and mass m' as shown in Fig. 6. Since we are interested only in the singularity structure of the amplitude, and since that comes entirely from denominators, it is enough to take all particles as scalar. Furthermore, our treatment makes no special distinction between "bound states" and "elementary particles." The stable particle of mass M can be either. Nor do resonances have to be specially treated, by the relativistic analog of the work of Ref. 8, it is only necessary to make m' appropriately complex. The mass shell conditions are

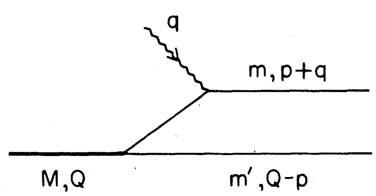

FIG. 6. Diagrammatic representation of the lowest order breakup process using relativistic kinematics. The four momenta and masses of lines are shown.

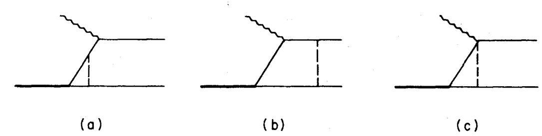

FIG. 7. Diagrammatic representation of the first order internal structure contribution (a) and the first order rescattering contribution (b) to breakup. Both graphs have the same reduced graph (c).

$$Q^2=M^2, \quad (p+q)^2=m^2, \quad (Q-p)^2=m'^2. \tag{14}$$

The diagram of Fig. 6 is proportional to the single propagator $(p^2-m^2)^{-1}$. The relativistic analog, k^2, of the relative momentum $|k|^2$ of Sec. II is the invariant relative momentum of the particle of moment p and of Q-p defined as

$$k^2=[\tfrac{1}{2}(Q-2p)]^2. \tag{15}$$

Using the mass shell conditions (14), the propagator denominator can be written

$$p^2-m^2=\tfrac{1}{2}k^2+\tfrac{1}{2}M^2-m'^2-m^2, \tag{16}$$

hence, the propagator pole $(p^2-m^2=0)$ comes at

$$k^2=\tfrac{1}{2}(m'^2+m^2-\tfrac{1}{2}M^2), \tag{17}$$

as one might expect. The stability of the object of mass M puts this condition at $k^2>0$, while the physical region corresponds to $k^2<0$ (spacelike). For example, if $m'=m$ and $M=2m-b$ $(b\ll m)$ corresponding to a deuteronlike state, we find the propagator pole at $k^2=mb\equiv\alpha^2$, as in the nonrelativistic result of Sec. II (with appropriate metric change), while the physical region begins at $k^2=0$.

Contributions to the internal part of the "wave function" or vertex singularities come from "internal" meson exchange, as shown in leading order in Fig. 7(a), while the leading order final state interactions come from "external" exchange as in Fig. 7(b). The left-hand cut singularities in k^2 for both graphs come from the same reduced graph, Fig. 7(c).

The graph of Fig. 7(c) is a triangle graph and, for the cases we are interested in, it has an anomalous threshold. The general analysis of the singularity structure of such graphs is technical and complicated. Since it is discussed at great length in the literature,[9] we will not repeat the discussion here, but rather will simply record the results. The graph has a branch cut in k^2 beginning at

$$k_{\mathrm{cut}}{}^2=m'^2-\frac{M^2m'^2}{4m^2}+\frac{M^2\mu^2}{4m^2} +M\mu\left\{\left(1-\frac{M^2}{4m^2}\right)\left[1-\left(\frac{m'^2-m^2-\mu^2}{2m\mu}\right)^2\right]\right\}^{1/2}. \tag{18}$$

If $m' = m$ and $M = 2m - b$ $(b \ll m)$ and if we neglect terms of order μ/m, this reduces to $k_{\text{cut}}{}^2 = (\mu + \alpha)^2$ just as we expect from the nonrelativistic example of Sec. II.

For the case of a π-N resonance in the deuteron we should again take $M = 2m - b$ $(b \ll m)$ and now take $m' = m + \mu + \Delta$, where Δ has a positive real part and is the resonance position above the $\pi - N$ threshold. This gives

$$k_{\text{cut}}{}^2 = \frac{b}{m}(m + \mu + \Delta)^2 + \mu^2 + 2m\mu\left\{\frac{b}{m}\left[1 - \left(\frac{(m + \mu + \Delta)^2 - m^2 - \mu^2}{2m\mu}\right)^2\right]\right\}^{1/2}, \tag{19}$$

while for the pole position for finding the asymptotic part of the resonance in the deuteron, we have [Eq. (17)]

$$k_{\text{pole}}{}^2 = \tfrac{1}{2}[(m + \mu + \Delta)^2 - m^2]. \tag{20}$$

It is easy to see that $k_{\text{cut}}{}^2 < k_{\text{pole}}{}^2$ for all interesting Δ. For example, at threshold $(\Delta = 0)$ $k_{\text{cut}}{}^2 = 0.022$ GeV^2, while $k_{\text{pole}}{}^2 = 0.97$ GeV^2. Thus, the cut is always much closer to the physical region $(k^2 < 0)$ than even the pole as we discussed in Sec. III C. Considerable control over the dynamics is required to be able to see past the resonance production by final state interactions and the interior wave function cut and see even the asymptotic component for the resonance "in" the deuteron. In general, we are very far from having the understanding required to do that.

B. Elastic scattering

It has been hoped that a N^* (π-N resonance) component in the deuteron could be observed through its effect on backward angle elastic nucleon-deuteron scattering.[10] In our language that means seeing the u-channel N^* exchange pole, where u is the cross momentum transfer. Consider the single particle exchange graph for elastic nucleon-deuteron scattering as shown in Fig. 8(a). With the variable assignment of this figure, u is defined as $u = (P - p - p')^2$ and the mass shell conditions are $(P - p)^2 = (P - p')^2 = M^2$, $p^2 = p'^2 = m^2$. The relative momentum variable k^2 is $k^2 = [\frac{1}{2}(P - p') - p]^2 = [\frac{1}{2}(P - p) - p']^2$ which can be expressed in terms of u and the mass shell conditions by

$$k^2 = \tfrac{1}{2}(m^2 + u - \tfrac{1}{2}M^2). \tag{21}$$

If the particle exchange in Fig. 8(a) is a nucleon, the u pole comes at $u = m^2$ which, for $M = 2m - b$, gives $k^2 = mb = \alpha^2$, just as we expect. If the exchanged particle is a resonance of mass m', the pole condition for k^2 is exactly the same as (20). (Recall that again the physical region is $k^2 \leqslant 0$.) It

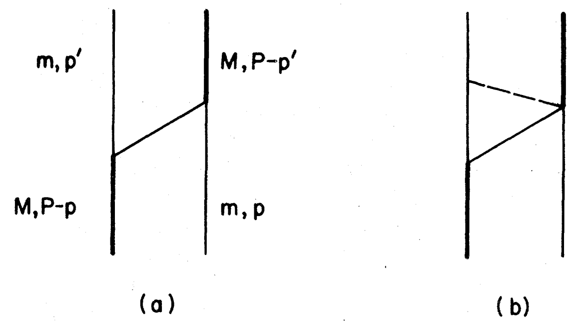

FIG. 8. The single particle exchange graph for nucleon deuteron scattering, (a), and the first order rescattering contribution (b).

is clear that the leading u-channel branch cut will be associated with nucleon and pion exchange, as shown in Fig. 8(b). This graph is kinematically equivalent to the breakup graph of Fig. 7(c), and its k^2 cut is again given by (8). Recall that now $m' = m$, and hence we find $k_{\text{cut}}{}^2 = (\mu + \alpha)^2$. Even if we put the N^* at π-N threshold $(m' + m + \mu)$ we find $k_{\text{pole}}{}^2$ for N^* exchange at $k^2 = 0.14$ GeV^2, while the cut begins at $k_{\text{cut}}{}^2 = 0.035$ GeV^2. The single particle exchange pole is at $k^2 = 0.0021$ GeV^2. Since the cut associated with Fig. 8(b) is fixed, while putting in a true N^* mass for m' only moves the N^* exchange pole farther away; using realistic masses only makes matters worse.

Thus, to see the effect of the N^* exchange pole in elastic nuelceon deuteron scattering requires, first, that the nuclear exchange pole be removed completely. This can be done, but must be done carefully since that pole is both very close and very strong. Secondly, it requires that one control the π-N continuum so well that one can extrapolate quite far past the cut from that continuum. This is usually done by asserting that the continuum is nothing but the resonance pole. From a phenomenological point of view this assumes the answer. It is clear that better analysis would require a far more sophisticated dynamical framework.[11]

C. Form factors and exchange currents

Another way to probe a bound state structure is via its form factor, which is the Fourier transform of the spatial density. Again we will concentrate on the deuteron for simplicity and specificity. The leading order graph for the form factor is represented in Fig. 9(a), while a graph that corresponds to structure in the wave function vertex is shown in Fig. 9(b) and an exchange current graph (corresponding to pion exchange) is shown in Fig. 9(c). Both Figs. 9(b) and 9(c) yield the

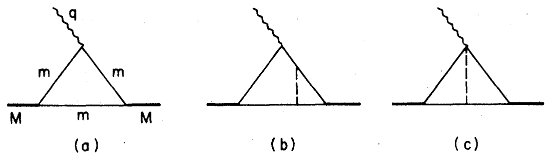

FIG. 9. The lowest order form factor graph (a), the leading internal structure contribution (b) and the longest range exchange current (c).

same reduced graph. Figures 9(a), 9(b), and 9(c) can be expressed together in Fig. 10(b), where for Fig. 9(a) we use $m' = m$ and for the threshold of 9(b) or 9(c) we use $m' = m + \mu$. Triangle graph analysis of these graphs gives for the beginning of the branch cut in q^2

$$q_{\text{cut}}{}^2 = \frac{3}{2}M^2 - \frac{M^4}{2m^2} + \frac{M^2 m'^2}{2m^2} + 2Mm'\left\{\left(1 - \frac{M^2}{4m^2}\right)\left[1 - \left(\frac{M^2 - m^2 - m'^2}{2mm'}\right)^2\right]\right\}^{1/2}. \tag{22}$$

For $m' = m$, and $M = 2m - b$, this reduced to $q^2 = 16\,mb = 16\alpha^2 = 0.034\ \text{GeV}^2$ as we expect. The continuum cut $(m' = m + \mu)$ begins at $q^2 = 0.719\ \text{GeV}^2$ and the contribution from a resonance at $\Delta = 0.6$ GeV has a cut starting at $q^2 = 4.20\ \text{GeV}^2$. What is significant here is not so much the numbers as the fact that cuts coming from vertex structure or the interior of the wave function, Fig. 9(b), and the cut from exchange currents, Fig. 9(c), begin at the same place, while the contribution from resonance structures, lie much further in. In few-body systems there has been much effort to calculate the wave function and the exchange current contribution. It may well be that these are well enough understood that their contributions to the form factor can be disentangled,[12] at least for the longest range parts, but clearly this is a problem that must be approached with care and skepticism. Furthermore, it is not at all clear that a reduction of the relativistic formalism can be made in which wave function and exchange current contributions can be unambiguously distinguished.[7]

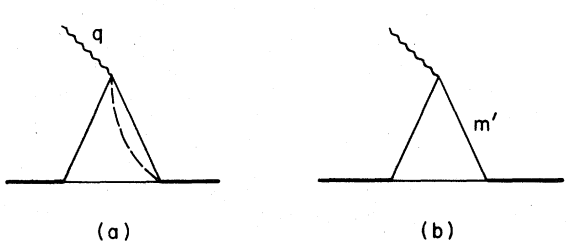

FIG. 10. (a) The reduced graph corresponding to Figs. 9(b) and 9(c). (b) A further reduction.

Exchange current contributions to other processes will suffer similar problems to those of the form factor. The longest range exchange current contributions will in general have the same range as the vertex part, and hence disentangling them will depend on dynamical understanding. For many systems we may have some understanding of the longest range part of the dynamics, but beyond that the issue is much confused and may in fact have a large degree of arbitrariness.

V. DISCUSSION

We have shown that the "interior" of a bound state wave function cannot be determined empirically without understanding of continuum processes, but that for sufficiently "light" components the asymptotic part of the bound state can be "measured." This helps to explain why the deuteron D-state probability cannot be measured and why a precise, model independent meaning of the strength of π-N resonance components in bound states is so difficult to give. In a sense all this is not surprising. We know that only the asymptotic part of the scattering state (the phase shift or on-shell t-matrix) can be measured. The interior of the scattering wave function can only be determined if we have a theory. We have given a discussion of the corresponding property for bound states. In fact, the on-shell t matrix carries bound state information only in the pole position and residue and these correspond to the binding energy and asymptotic normalization respectively.

The fact that final state interactions and similar dynamical features cloud our ability to see into bound states is known in a number of particular situations,[13] and the fact that it leads to destructive interference essentially due to orthogonality has been pointed out recently.[3,4] We believe, however, that the k-space singularity analysis provides a very general method for investigating this question in all cases.

Since our analysis here is given in terms of k-space singularities, and since we do have good theoretical understanding of the long range part of the force (nearby singularities), we can hope to extend our understanding of the bound state into this region. However, analysis of the truly short range part of the bound state would appear to be very difficult. Unfortunately, even the medium range contributions require some understanding of pion degrees of freedom and therefore a relativistic dynamics. It would be useful to formulate such a dynamics so that the medium range singularity structure was model independent, like the asymptotic part now is. We could then use such a dynamical framework to determine the intermediate part of the bound states.

Note added in proof. The surface contributions resulting from the kinetic energy operator not being self-adjoint has been discussed previously by E. Gerjuoy in Phys. Rev. 109, 1806 (1958). His analysis is based on the Lippmann-Schwinger integral equations. His conclusion that the surface contributions vanish if the energy is given a small positive imaginary part is very similar to our use of an average over a small energy interval to eliminate the surface contributions in the context of the R-matrix formalism.

ACKNOWLEDGMENTS

I am grateful to the Theory Group at the Swiss Institute of Nuclear Research for conversations that helped me focus the issues in the problem, to J. Friar for an interesting conversation and a preprint, and to D. Sparrow for an illuminating reference. This work was supported in part by a grant from the National Science Foundation.

APPENDIX A

To find the leading singularity of $\mathfrak{M}_{(1)}$, Eq. (11), as a function of k^2, we insert the longest range part of $\langle \vec{k}' | s \rangle$, the pole, and the longest range part of the potential. Removing factors that do not affect the position of the singularity, we are left with the integral

$$\mathfrak{M}_{(1)} \propto \int \frac{d^3k'}{[(\vec{k}-\vec{k}')^2+\mu_0{}^2]\{(\vec{k}-\vec{k}')\cdot[(\vec{k}+\vec{k}')/2m^2+\vec{q}/m_1]\}(k'^2+\alpha^2)} . \tag{A1}$$

This is of the usual form of an integral over denominator factors, and its singularity structure can be determined by application of the Landau rules.[14] The middle denominator in (A1) is an energy denominator and has a normal threshold branch cut for $k^2 > 0$ beginning at $k^2 = 0$. We are not concerned with this cut, but rather with singularities introduced by FSI that come for $k^2 < 0$ and interfere with the wave function information. These come from the reduced graph obtained from (A1) by dropping the middle propagator. This is equivalent to putting the Feynman parameter associated with this factor to zero. Calling the remaining Feynman parameters X_1 and X_2, we obtain for the Landau equations for the singularities of (A1) corresponding to Eq. 2.2.6 of Ref. 9

$$k'^2+\alpha^2=0\,, \tag{A2}$$

$$(\vec{k}-\vec{k}')^2+\mu_0{}^2=0\,, \tag{A3}$$

$$\vec{k}'X_1+(\vec{k}'-\vec{k})X_2=0, \quad X_1, X_2 \neq 0\,. \tag{A4}$$

These are most easily solved by dotting (A4) successively with $\vec{k}$ and $\vec{k}'$ and using (A2) and (A3) to express k'^2, and $\vec{k}\cdot\vec{k}'$ in terms of k^2, α^2, and $\mu_0{}^2$. The requirement that X_1 and X_2 be nonzero is then that the determinant of the coefficients X_1, X_2 vanish. This leads to $k^2=-(\alpha+\mu_0)^2$ or $k^2=-(\alpha-\mu_0)^2$. The second root is easily seen to be on the unphysical sheet (it corresponds to $X_2<0$), and hence the singularity (a logarithmic branch point) comes at $k^2=-(\alpha+\mu_0)^2$.

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[2] For a review of π-N resonances in bound states, see A. M. Green, Rep. Prog. Phys. 39, 1109 (1976).

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[5] Cf. R. D. Amado, M. P. Locher, and M. Simonius, Phys. Rev. C 17, 403 (1978); and R. D. Amado, M. P. Locher, Martorell, Konig, White, Schmelzbach, Gruebler, Burg, and Jenny, Phys. Lett. 79B, 368 (1978).

[6] Cf. G. E. Grown and A. D. Jackson, *The N-N Interaction* (North-Holland, Amsterdam, 1976), p. 74.

[7] J. Friar (unpublished).

[8] R. D. Amado, Phys. Rev. C (to be published).

[9] Cf. Eden, Landshoff, Olive, and Polkinghorne, *The Analytic S-Matrix* (Cambridge Univ. Press, Cambridge, 1966), pp. 57–73.

[10] A. Kerman and L. Kisslinger, Phys. Rev. 180, 1483 (1969).

[11] The subject of backward nucleon deuteron scattering and the need for N^* components has been investigated in far greater detail than we give here. (For a review and references see Ref. 2, p. 1138.) Our treatment is only meant to illustrate the singularity analysis method.

[12] Cf. I. Sick, in *Lecture Notes in Physics*, *87*, edited by Zingl, Haftel, and Zankel (Springer, Berlin, 1978), p. 236.

[13] Cf. D. J. S. Findlay and R. O. Owens, Nucl. Phys. A279, 385 (1977), see the footnote on p. 392.

[14] Cf. Ref. 9, pp. 50–57.