PHYSICS 880.05: Problem Set #2 Solutions

1. Stability and equilibrium conditions for uniform matter.
We have N particles in volume $\Omega \Rightarrow$ density $\rho = N/\Omega$.
The energy per particle is denoted $\epsilon(\rho) = E/N$.

a) From thermodynamics, $\boxed{P = -\left(\frac{\partial E}{\partial V}\right)_N = -\left(\frac{\partial (E/N)}{\partial (V/N)}\right)_N}$

which follows since N is held constant.

$$\Rightarrow \boxed{P = -\left(\frac{\partial \epsilon(\rho)}{\partial 1/\rho}\right)_N = +\rho^2 \frac{\partial \epsilon(\rho)}{\partial \rho}} \quad \text{since} \quad \boxed{\frac{\partial}{\partial(1/\rho)} = \frac{\partial}{\partial \rho}\left(\frac{\partial \rho}{\partial 1/\rho}\right) = \rho^2 \frac{\partial}{\partial \rho}}$$

At equilibrium density, the pressure is zero (no net force!).

$\Rightarrow \frac{\partial \epsilon(\rho)}{\partial \rho} = 0$ from part a) $\Rightarrow \epsilon(\rho)$ is an extremum.
If $\epsilon(\rho)$ is minimized, it must be a minimum!

b) The chemical potential at T=0, S=0 is found from

$$\boxed{\begin{aligned}\mu = \left(\frac{\partial E}{\partial N}\right)_V &= \left(\frac{\partial}{\partial N}(N\epsilon(\rho))\right)_V = \left(\epsilon(\rho) + N\frac{\partial \epsilon}{\partial N}\right)_V \\ &= \epsilon(\rho) + \left(\frac{N}{V}\frac{\partial \epsilon}{\partial N/V}\right)_V = \epsilon + \rho\frac{\partial \epsilon}{\partial \rho} \\ &= \epsilon + \frac{1}{\rho}P \quad \text{from a).}\end{aligned}}$$

The 2nd line followed since V is held constant.
Note that $\boxed{\mu = E(N+1) - E(N) \xrightarrow[N\to\infty]{} \left(\frac{\partial E}{\partial N}\right)_V (\Delta N = 1)}$

c) From problem set #1, in one-d

$$\epsilon(\rho) = \frac{E}{N} = \frac{1}{3}\frac{\hbar^2 k_F^2}{2m} + \frac{\lambda}{2}\left(1-\frac{1}{g}\right)\rho \qquad \text{with } \rho = g k_F/\pi \Rightarrow k_F = \pi\rho/g$$

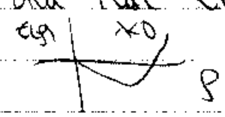
$$\Rightarrow \boxed{\epsilon(\rho) = \frac{1}{3}\frac{\hbar^2\pi^2}{2m}\frac{1}{g^2}\rho^2 + \frac{\lambda}{2}\left(1-\frac{1}{g}\right)\rho}$$

So $\frac{\partial\epsilon}{\partial\rho} = 0 \Rightarrow \frac{1}{3}\frac{\hbar^2\pi^2}{m}\frac{1}{g^2}\rho = -\frac{\lambda}{2}\left(1-\frac{1}{g}\right)$

or $\boxed{\rho_{sat} = -\frac{3\lambda}{2}\frac{m}{\hbar^2\pi^2}(g-1)g}$ is the saturation density

and $\boxed{\epsilon(\rho_{sat}) = -\frac{3(g-1)^2\lambda^2 m}{8\hbar^2\pi^2}}$ (I used Mathematica to do the algebra!)

Note that $\rho_{sat} > 0$ requires $\lambda < 0 \Rightarrow$ attractive potential (how else could it saturate!) and that $\epsilon(\rho_{sat}) < 0$ (that latter is clear from sketching $\epsilon(\rho)$):



d) It's easiest to work in terms of the energy density, $E/\Omega = (E/N)(N/\Omega) = \rho\,\epsilon(\rho)$, since that is constant in each half volume.

· Then the condition for stability is a comparison of energies:

$$\underbrace{(\rho+\delta)\epsilon(\rho+\delta)\Omega/2 + (\rho-\delta)\epsilon(\rho-\delta)\Omega/2}_{\text{total energy of nonuniform state}} > \underbrace{\rho\,\epsilon(\rho)\,\Omega}_{\text{total energy of uniform state}}$$

· Expand to 2nd order:

$$\boxed{\frac{1}{2}(\rho+\delta)\left(\epsilon(\rho)+\delta\frac{d\epsilon}{d\rho}+\frac{1}{2}\delta^2\frac{d^2\epsilon}{d\rho^2}\right) + \frac{1}{2}(\rho-\delta)\left(\epsilon(\rho)-\delta\frac{d\epsilon}{d\rho}+\frac{1}{2}\delta^2\frac{d^2\epsilon}{d\rho^2}\right) > \rho\,\epsilon(\rho)}$$

or δ^0: $\frac{1}{2}\rho\epsilon + \frac{1}{2}\rho\epsilon = \rho\epsilon$ ✓

δ^1: $\frac{1}{2}\rho\epsilon + \frac{1}{2}\rho\frac{d\epsilon}{d\rho} - \frac{1}{2}\rho\epsilon - \frac{1}{2}\rho\frac{d\epsilon}{d\rho} = 0$ ✓

δ^2: $\frac{1}{2}\frac{d\epsilon}{d\rho} + \frac{1}{4}\rho\frac{d^2\epsilon}{d\rho^2} + \frac{1}{2}\frac{d\epsilon}{d\rho} + \frac{1}{4}\rho\frac{d^2\epsilon}{d\rho^2} > 0$

$$\Rightarrow \rho\frac{\partial^2\epsilon}{\partial\rho^2} + 2\frac{\partial\epsilon}{\partial\rho} > 0 \Rightarrow \boxed{\frac{\partial^2}{\partial\rho^2}(\rho\epsilon(\rho)) > 0}$$

[Check $\frac{\partial^2}{\partial\rho^2}(\rho\epsilon) = \frac{\partial}{\partial\rho}\left(\epsilon + \rho\frac{\partial\epsilon}{\partial\rho}\right) = \frac{\partial\epsilon}{\partial\rho} + \frac{\partial\epsilon}{\partial\rho} + \rho\frac{\partial^2\epsilon}{\partial\rho^2} = \rho\frac{\partial^2\epsilon}{\partial\rho^2} + 2\frac{\partial\epsilon}{\partial\rho}$ ✓]

Since $\frac{\partial P}{\partial\rho} = \frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial\epsilon}{\partial\rho}\right) = 2\rho\frac{\partial\epsilon}{\partial\rho} + \rho^2\frac{\partial^2\epsilon}{\partial\rho^2} = \rho\left(\rho\frac{\partial^2\epsilon}{\partial\rho^2} + 2\frac{\partial\epsilon}{\partial\rho}\right)$,
the equivalent condition is

$$\boxed{\frac{\partial P}{\partial\rho} > 0} \quad \text{for stability.}$$

The one-d result from above is $\epsilon(\rho) = \frac{1}{3}\frac{\hbar^2\pi^2}{2m}\frac{1}{g^2}\rho^2 + \frac{\lambda}{2}\left(1-\frac{1}{g}\right)\rho$

$$\Rightarrow \frac{\partial^2}{\partial\rho^2}\rho\epsilon(\rho) = \frac{\hbar^2\pi^2}{mg^2}\rho + \lambda\left(1-\frac{1}{g}\right) > 0$$

$$\Rightarrow \boxed{\rho > -\frac{\lambda g(g-1)m}{\hbar^2\pi^2} \quad \text{if } \lambda < 0}$$

(if $\lambda > 0$, $\frac{\partial^2}{\partial\rho^2}\rho\epsilon(\rho) > 0$ always).

Since ρ_{sat} is greater than this bound, it would appear (to this order in perturbation theory) that the one-d saturation point is stable to very long wavelength fluctuations.
(Later we'll consider shorter wavelengths!)

2. a) degeneracy factor g = (spin up or spin down) × (proton or neutron) $= 4$

b) $R = r_0 A^{1/3}$ with $r_0 \doteq 1.2 \times 10^{-13}$ cm $= 1.2$ fm.

The volume is $\frac{4}{3}\pi R^3 = \frac{4}{3}\pi r_0^3 A$ so the density is $\boxed{\rho = \frac{A}{\frac{4}{3}\pi R^3} = \frac{3}{4\pi}\frac{1}{r_0^3}}$ or

$\boxed{\rho = 0.14 \text{ fm}^{-3}}$ <u>independent</u> of A.

Now $\rho = \frac{4 k_f^3}{6\pi^2}$ or $\boxed{k_f = \left(\frac{3\pi^2 \rho}{2}\right)^{1/3} = \left(\frac{9\pi}{8}\right)^{1/3}\frac{1}{r_0} = 1.27 \text{ fm}^{-1}}$

and $\boxed{\epsilon_F = \frac{\hbar^2 k_F^2}{2m} = \frac{(\hbar c)^2 k_F^2}{2mc^2} = 33.4 \text{ MeV}}$ (using $\hbar c \doteq 197.33$ MeV-fm and $mc^2 \doteq 939$ MeV)

As noted, both k_F and ϵ_F are independent of A.

c) At this point, we have only the kinetic energy. Since these are fermions, even at $T=0$ they must have non zero momentum because of the Pauli exclusion principle. The pressure is from momentum transfer to the walls of the container. (If there is no container, the gas expands!)

The magnitude of the pressure can be found using $P = \rho^2 \frac{\partial \epsilon}{\partial \rho}$ with $\epsilon = \frac{3}{5} k_f^2/2m$. $\boxed{P = \rho\left(\rho \frac{\partial}{\partial \rho}\epsilon\right) = \rho\left(\frac{1}{3}k_F \frac{\partial \epsilon}{\partial k_F}\right) = \rho \frac{1}{3}\frac{2 \cdot 3 k_F^2}{5 \cdot 2m} = \frac{2}{5}\rho\epsilon_F = 1.85 \text{ MeV/fm}^3}$

d) The binding energy per nucleon is about -16 MeV, and this must be the kinetic energy per nucleon $= \frac{3}{5}\epsilon_F \doteq 20$ MeV plus the potential energy per nucleon. Thus $\boxed{|V_0| = 36 \text{ MeV}}$ (and the potential energy is negative.

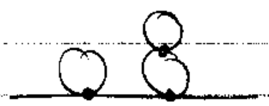
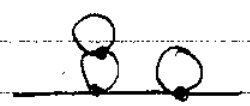
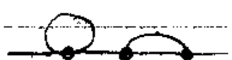
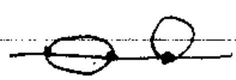
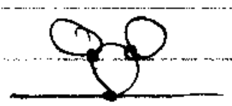
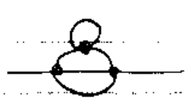
e) Roughly speaking, the quantum statistics don't matter when the occupation number is much less than unity. Or $e^{\beta(\epsilon_i - \mu)} \gg 1$. Let's arbitrarily pick $-\beta\mu = 3$. If we're in the classical limit, then $\beta\mu = \ln\left[\frac{1}{4}\rho\left(\frac{2\pi}{mT}\right)^{3/2}\right] \Rightarrow \boxed{T \approx 200 \text{ MeV}}$. This is roughly the deconfinement temperature!

3 The Feynman rules for $\langle x^2 \rangle$ where

$$\boxed{\langle x^2 \rangle = \frac{\int dx\, x^2\, e^{-ax^2/2 - \lambda x^4/4}}{\int dx\, e^{-ax^2/2 - \lambda x^4/4}}}$$

tells us that we find the λ^3 contribution by summing the contributions from all connected diagrams with two external lines and three vertices (one for each λ). The disconnected diagrams from the numerator cancel with those from the denominator.

We'll draw the diagram, symmetry factor, and contribution in three colums:

a)	b)	c)
	$\frac{1}{2^3} \times 1 \times 1$	$\frac{1}{8}(-6\lambda)^3 \frac{1}{a^7}$
	$\frac{1}{2^2} \times \frac{1}{2} \times 1$	$\frac{1}{8}(-6\lambda)^3 \frac{1}{a^7}$
	$\frac{1}{2^2} \times \frac{1}{2} \times 1$	$\frac{1}{8}(-6\lambda)^3 \frac{1}{a^7}$
	$\frac{1}{2} \times \frac{1}{3!} \times 1$	$\frac{1}{12}(-6\lambda)^3 \frac{1}{a^7}$
	$\frac{1}{2} \times \frac{1}{3!} \times 1$	$\frac{1}{12}(-6\lambda)^3 \frac{1}{a^7}$
	$\frac{1}{2^2} \times 1 \times \frac{1}{2}$	$\frac{1}{8}(-6\lambda)^3 \frac{1}{a^7}$
	$\frac{1}{2} \times \left(\frac{1}{2}\right)^2 \times 1$	$\frac{1}{8}(-6\lambda)^3 \frac{1}{a^7}$
	$1 \times \left(\frac{1}{2}\right)^2 \times 1$	$\frac{1}{4}(-6\lambda)^3 \frac{1}{a^7}$
	$\frac{1}{2} \times \frac{1}{2} \times 1$	$\frac{1}{4}(-6\lambda)^3 \frac{1}{a^7}$
	$1 \times \frac{1}{3!} \times \frac{1}{2}$	$\frac{1}{12}(-6\lambda)^3 \frac{1}{a^7}$

total $\boxed{-297\lambda^3/a^7}$

4 a) We have $\boxed{\hat{N} = \int \hat{\psi}^\dagger(\vec{x}_1)\hat{\psi}(\vec{x}_1)\, d^3x_1 \equiv \int \psi_1^\dagger \psi_1\, dx_1}$ for short

and $$\boxed{\hat{H} = \int dx_2\, \psi_2^\dagger T(x_2)\psi_2 + \int dx_2\, dx_3\, \psi_2^\dagger \psi_3^\dagger V(x_2,x_3)\psi_3\psi_2}$$

where we've dropped the hats and simplified the arguments. We'll also let the subscripts stand for spin.

We need to apply $\boxed{[\psi_1, \psi_2]_F = [\psi_1^\dagger, \psi_2^\dagger]_F = 0}$

and $\boxed{[\psi_1, \psi_2^\dagger]_F = \delta_{12}}$

where $\boxed{\delta_{12} \equiv \delta^3(\vec{x}_1 - \vec{x}_2)\delta_{\alpha_1\alpha_2}}$

Our goal is to show that $[\hat{N}, \hat{H}]_- = 0$. The plan is to simply move $\psi_1^\dagger\psi_1$ in the first term through the other field operators until we can cancel it with the 2nd term. Then we need to show that any terms generated on the way vanish or cancel.

We can simply our task by first calculating

$$\boxed{\psi_1^\dagger\psi_1\psi_2 = \pm \psi_1^\dagger\psi_2\psi_1 = \psi_2\psi_1^\dagger\psi_1 - \psi_1\delta_{12}}$$

and $$\boxed{\psi_1^\dagger\psi_1\psi_2^\dagger = \pm \psi_1^\dagger\psi_2^\dagger\psi_1 + \psi_1^\dagger\delta_{12} = \psi_2^\dagger\psi_1^\dagger\psi_1 + \psi_1^\dagger\delta_{12}}$$

So start with

$$\hat{N}\hat{H}_{a} = \int dx_1 dx_2\, \psi_1^\dagger\psi_1\psi_2^\dagger\psi_2\, T(x_2) = \int dx_1 dx_2\, T(x_2)\left(\psi_2^\dagger\psi_1^\dagger\psi_1 + \psi_1^\dagger\delta_{12}\right)\psi_2$$

$$= \int dx_1 dx_2\, T(x_2)\left(\psi_2^\dagger\left(\psi_2\psi_1^\dagger\psi_1 - \psi_1\delta_{12}\right) + \psi_1^\dagger\psi_2\delta_{12}\right)$$

$$= \int dx_2\, \psi_2^\dagger T(x_2)\psi_2 \int dx_1\, \psi_1^\dagger\psi_1 + \int dx_1\, T(x_1)\left(\cancel{\psi_1^\dagger\psi_1} + \psi_1^\dagger\psi_1\right)$$

$$\Rightarrow \hat{N}\hat{H}_a = \int dx_2\, \psi_2^+ T(x_2)\psi_2 \int dx_1\, \psi_1^+\psi_1 = \hat{H}_a\hat{N} \quad \text{or} \quad [\hat{N},\hat{H}_a]=0.$$

Next,

$$\hat{N}\hat{H}_b = \int \underbrace{dx_1 dx_2 dx_3\, T(x_1)V(x_2,x_3)}\ \psi_1^+\psi_1\,\psi_2^+\psi_3^+\,\psi_3\psi_2$$

$$= \sim (\psi_2^+\psi_1^+\psi_1 + \psi_1^+\delta_{12})\,\psi_3^+\psi_3\psi_2$$

$$= \sim (\psi_2^+(\psi_3^+\psi_1^+\psi_1 + \psi_1^+\delta_{13}) - \psi_3^+\psi_1^+\delta_{12})\psi_3\psi_2$$

$$= \sim (\psi_2^+\psi_3^+(\psi_3\psi_1^+\psi_1 - \psi_1\delta_{13})\psi_2 + \psi_2^+\psi_1^+\psi_3\psi_2\delta_{13} - \psi_3^+\psi_1^+\psi_3\psi_2\delta_{12}$$

$$= \sim (\psi_2^+\psi_3^+\psi_3(\psi_2\psi_1^+\psi_1 - \psi_1\delta_{12}) - \psi_2^+\psi_3^+\psi_1\psi_2\delta_{13} + \psi_2^+\psi_1^+\psi_3\psi_2\delta_{13}$$
$$+ \psi_3^+\psi_1^+\psi_2\psi_3\delta_{12}$$

$$= \int dx_2 dx_3\, \psi_2^+\psi_3^+ V(x_2,x_3)\psi_3\psi_2 \int dx_1\, \psi_1^+\psi_1$$
$$+ \int dx_2 dx_3\, T(x_2)V(x_2,x_3)\big(\overset{0}{\cancel{-\psi_2^+\psi_3^+\psi_3\psi_2 - \psi_2^+\psi_3^+\psi_3\psi_2 + \psi_2^+\psi_3^+\psi_3\psi_2 + \psi_3^+\psi_2^+\psi_2\psi_3}}\big) \quad \text{since } V(x_2,x_3)=V(x_3,x_2)$$

$$= \hat{H}_b\hat{N}$$

$$\Rightarrow \boxed{[\hat{N},\hat{H}]=0}$$

b) Start with

$$\hat{H} = \int d^3x\, \hat{\psi}^+(\vec{x})T(\vec{x})\hat{\psi}(\vec{x}) + \frac{1}{2}\int d^3x\, d^3x'\, \hat{\psi}^+(\vec{x})\hat{\psi}^+(\vec{x}')V(\vec{x},\vec{x}')\hat{\psi}(\vec{x}')\hat{\psi}(\vec{x})$$
$$+ \frac{1}{3!}\int d^3x\, d^3x'\, d^3x''\, \hat{\psi}^+(\vec{x})\hat{\psi}^+(\vec{x}')\hat{\psi}^+(\vec{x}'')V_3(\vec{x},\vec{x}',\vec{x}'')\hat{\psi}(\vec{x}'')\hat{\psi}(\vec{x}')\hat{\psi}(\vec{x})$$

substitute ~~and do~~ the x' and x'' integrals

$$= \int d^3x\, \hat{\psi}^+(\vec{x})T(\vec{x})\hat{\psi}(\vec{x}) - \frac{\Delta}{2}\int d^3x\, \hat{\psi}^+(\vec{x})\hat{\psi}^+(\vec{x})\hat{\psi}(\vec{x})\hat{\psi}(\vec{x})$$
$$+ \frac{\beta}{3!}\int d^3x\, \hat{\psi}^+(\vec{x})\hat{\psi}^+(\vec{x})\hat{\psi}^+(\vec{x})\hat{\psi}(\vec{x})\hat{\psi}(\vec{x})\hat{\psi}(\vec{x})$$

where the spin indices are implicit.

5. a) The transition amplitude U is the matrix element of the evolution operator $e^{-i\hat{H}t}$.

In particular,

$$\boxed{U(x_f t_f; x_i t_i) = \langle x_f | e^{-i\hat{H}(t_f - t_i)} | x_i \rangle}$$

but we can always write $e^{-i\hat{H}(t_f-t_i)} = e^{-i\hat{H}(t_f-t_m)} e^{-i\hat{H}(t_m-t_i)}$ and then insert a complete set of position states $\int dy\, |y\rangle\langle y|$ between the two operators. This yields:

$$\boxed{\begin{aligned} U(x_f t_f; x_i t_i) &= \int dy\, \langle x_f | e^{-i\hat{H}(t_f-t_m)} | y\rangle\langle y | e^{-i\hat{H}(t_m-t_i)} | x_i \rangle \\ &= \int dy\, U(x_f t_f; y t_m)\, U(y t_m; x_i t_i) \end{aligned}}$$

which is the composition rule.

b) If we now fix x_i and t_i and identify

$$\boxed{\psi(x,t) \equiv U(xt; x_i t_i)}$$

then the composition rule applied with $t_m = t$ and $t_f = t+\varepsilon$ is

$$\boxed{\psi(x, t+\varepsilon) = \int dy\, U(x\, t+\varepsilon; y t)\, \psi(y,t)}$$

in which we can insert Feynman's path integral representation for U:

$$\boxed{U(x\, t+\varepsilon; y t) = \frac{1}{A} e^{\frac{i}{\hbar}\left(\frac{m}{2}\frac{(x-y)^2}{\varepsilon} - \varepsilon V\left(\frac{x+y}{2}\right)\right)}}$$

We'll take $m=1$ and $\hbar = 1$ from here on (this is actually implied by the wording of the problem).

The phase $e^{\frac{i(x-y)^2}{2\varepsilon}}$ oscillates rapidly with y if $(x-y)^2 \gg \varepsilon$.

This oscillation means that the integral will average to zero (since the remaining integrand varies smoothly) unless the exponent is order unity or smaller

$$\Rightarrow \boxed{(x-y)^2 \lesssim \varepsilon} \text{ or } \boxed{|x-y| \lesssim \sqrt{\varepsilon}}$$

So we can write $\boxed{y = x+\eta}$ with $\eta \sim \mathcal{O}(\sqrt{\varepsilon})$ and expand everything except the phase.

$$\boxed{\begin{aligned}\psi(x,t+\varepsilon) &= \int dy \, \frac{1}{A} e^{\frac{i}{2}\frac{(x-y)^2}{\varepsilon}} e^{-i\varepsilon V\left(\frac{x+y}{2}\right)} \psi(y,t) \\ &= \frac{1}{A}\int d\eta \, e^{\frac{i}{2}\frac{\eta^2}{\varepsilon}} e^{-i\varepsilon V(x+\frac{\eta}{2})} \psi(x+\eta,t)\end{aligned}}$$

We get the lowest order term by setting $\varepsilon = \eta = 0$ everywhere except $e^{\frac{i}{2}\eta^2/\varepsilon}$

$$\Rightarrow \psi(x,t) = \frac{1}{A}\int d\eta \, e^{\frac{i}{2}\frac{\eta^2}{\varepsilon}} \psi(x,t)$$

$$\text{or } \boxed{A = \int_{-\infty}^{\infty} d\eta \, e^{\frac{i\eta^2}{2\varepsilon}}}$$

b) Odd terms in η will be integrated with the even term $e^{i\eta^2/\varepsilon}$, and so average to zero.

d) Now expand to $\mathcal{O}(\varepsilon) = \mathcal{O}(\eta^2)$:

$$\boxed{\psi(x,t) + \varepsilon\frac{\partial\psi(x,t)}{\partial t} = \frac{1}{A}\int d\eta \, e^{\frac{i\eta^2}{2\varepsilon}} \left(1 - i\varepsilon V(x)\right)\left(\psi(x,t) + \frac{1}{2}\eta^2 \frac{\partial^2\psi(x,t)}{\partial x^2}\right)}$$

Cancelling the zeroth order terms:

$$\boxed{\begin{aligned}\varepsilon\frac{\partial\psi(x,t)}{\partial t} &= -i\varepsilon V(x)\psi(x,t) + \frac{1}{A}\left(\int_{-\infty}^{\infty} d\eta \, e^{\frac{i\eta^2}{2\varepsilon}}\eta^2\right)\frac{1}{2}\frac{\partial^2\psi(x,t)}{\partial x^2} \\ &= \varepsilon\left(-iV(x)\psi(x,t) + i\frac{1}{2}\frac{\partial^2\psi(x,t)}{\partial x^2}\right)\end{aligned}}$$

take $\varepsilon' = i\varepsilon$ to evaluate the integrals $\Rightarrow$ result is $\varepsilon' \to i\varepsilon$.

putting back $\hbar$ and m $\Rightarrow$ $\boxed{i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{1}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t)}$ $\Rightarrow$ The Schrödinger equation!