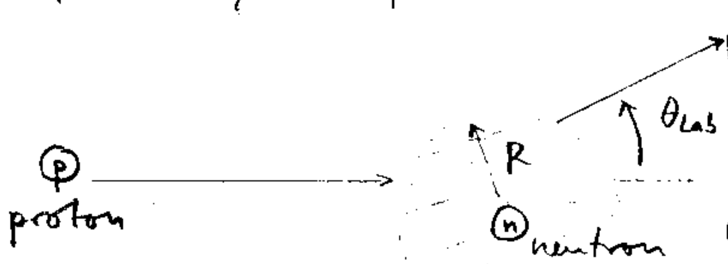
The nucleon-nucleon interaction

2 basic references: Brown + Jackson, The Nucleon-Nucleon Interaction, North Holland Amsterdam, 1976.
Machleidt, Adv. Nucl Phys. 19 (1989) 189.

We are now at a point, where we have several calculational tools at hand, e.g., perturbation theory, self-consistent Hartree-Fock, Density Functional Theory, etc., but we have so far used simple spin-independent contact interactions (delta functions) since them are model-independent and play a special role in Effective Field Theories

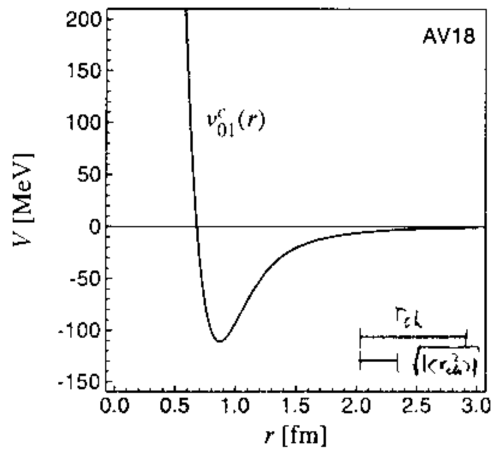
Therefore, we now take a step back and discuss, what we know about the nature of the nucleon-nucleon interaction. Since an interaction is not observable, we can only infer its properties from the fingerprints the interaction leaves in experiments, to be precise in elastic nucleon-nucleon scattering



R range of the interaction
θ_{Lab} scattering angle

Before we consider the general propeties of a two-body interaction, its symmetries and convenient classification, we discuss the basic phenomenology of the nuclear force.

At the beginning of these lectures, we have argued that due to the composite nature of the nucleon ($|n\rangle = |udd\rangle$, $|p\rangle = |uud\rangle$), the nucleon-nucleon interaction is very much like the interatomic potential, such as the Lennard-Jones potential between e.g. He atoms. In fact a realistic NN potential is shown below (finite range forces in contrast to the Coulomb interaction)



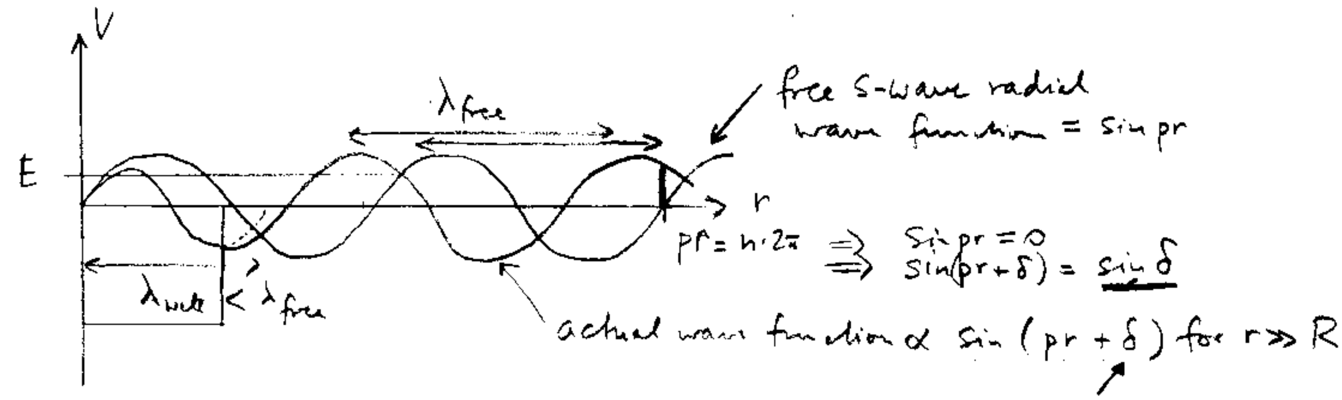
On the same scale, we have drawn the

proton charge radius $r_{ch} = 0{,}87$ fm and the
neutron mean-square charge radius $\sqrt{|\langle r_{ch}^2\rangle|} = 0{,}34$ fm

to set the scale of the "size" of the nucleon. One has to be somewhat careful, since there are "electromagnetic sizes" but the point we wish to make is that it is indeed very plausible that the NN interaction may be regarded as due to the polarization of one nucleon by the other one. The difference to electronic systems, like atoms, is that the polarization is on the quark level. Since quarks confine, it is of course more complicated and you might think of meson clouds.
We will discuss such microscopic details in the next lecture.

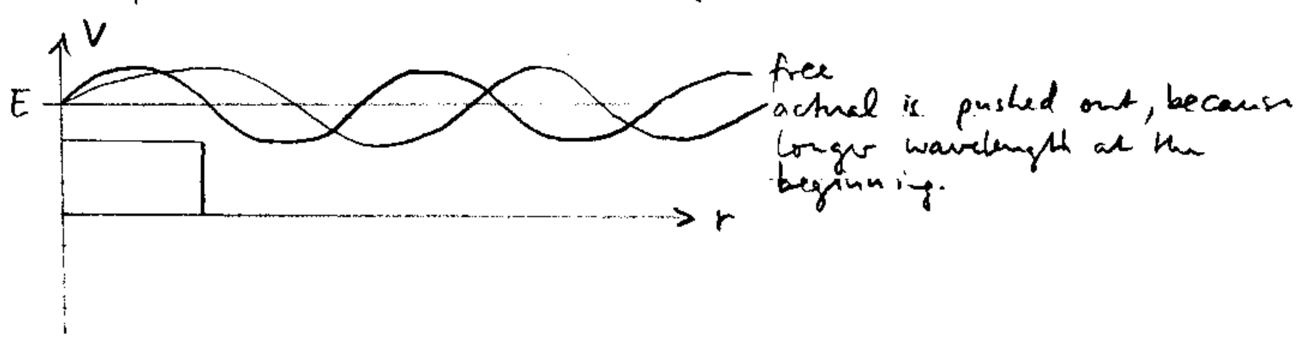
Let us now try to find to find the fingerprint of a Lennard-Jones potential in scattering experiments.

Recall the phase shift δ. For an attractive square well potential the scattering wave function is at large distance $r \gg R$ (R the range (here size) of the square well) shifted as compared to the free wave function:

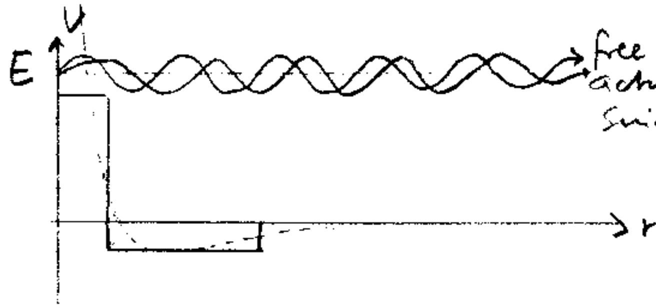


The phase shift is given by the phase difference of the free and actual wave at large distances. For attractive potentials the wave is pulled in and thus δ is positive.

For repulsive interactions, δ is negative.



And at higher energies for a mixture of both



actual is still pushed out since the attraction is weak in this case

In the figure below, we give the observed S and D-wave scattering phase shifts as function of the energy (in the lab frame)

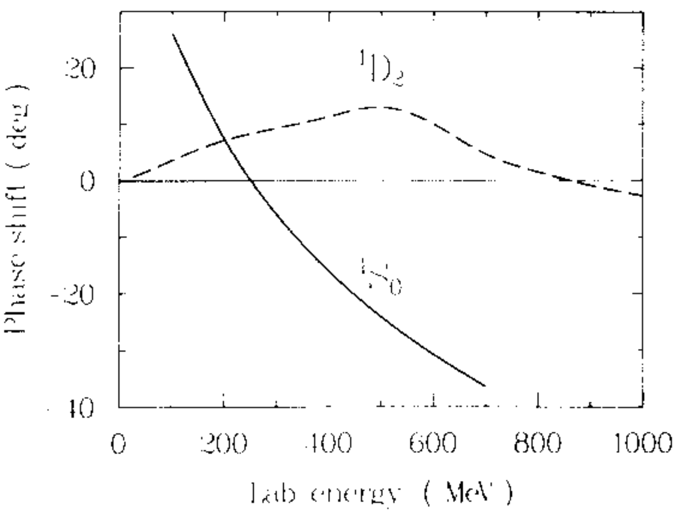


Fig. 3.1. ***NN* phase shifts for the $^1S_0$ and $^1D_2$ state.** Shown is the energy-dependent analysis by Arndt (Arn 87).

We find that the S-wave phase shift goes from attractive at low energies to repulsive at energies $E_{lab} > 250$ MeV, whereas the D-wave phase shift does not change sign.
(D-wave sees a centrifugal barrier and therefore probes outer region)
The maximum classical orbital angular momentum L_{max} for a range R_{core} is

$$L_{max} \approx R_{core}\, p \,,$$

where p = momentum in the cms frame $= \sqrt{\frac{m_n E_{lab}}{2}}$.

For $E_{lab} = 250$ MeV, we have $p = 1.7\ \mathrm{fm}^{-1}$ and with

$L_{max} \lesssim 1$ (no repulsion in D-wave) we obtain the size of the repulsive core $\underline{R_{core} \leq 0.6 \text{ fm}}$

We can moreover estimate the range of the intermediate attraction to be on the order of the internucleon spacing in large nuclei. The central density of heavy nuclei is about $0.17 \text{ fm}^{-3} \simeq R_{int}^{-3} \Rightarrow \underline{R_{int} \approx 1.8 \text{ fm}}$.

A microscopic picture of the nuclear force will be discussed in the next lecture.

We have already discussed in our brief introduction to the shell model, that the single particle basis states consist of

$$|\Psi\rangle = |\psi_{space}\rangle \otimes |\chi_{spin}\rangle \otimes |\chi_{isospin}\rangle$$

plane wave states

i.e., $$\left\{\begin{matrix} |\vec{k}\rangle \\ |k\, \ell m\rangle \\ |n\, \ell m\rangle \end{matrix}\right\} \otimes |s\, m_s\rangle \otimes |t\, m_t\rangle$$

angular momentum

radial quantum number

For spin ½ particles: $|\chi_{spin}\rangle = |\tfrac{1}{2}\; m_s = \tfrac{1}{2}\rangle = |\uparrow\rangle$

or $|\tfrac{1}{2}\; m_s = -\tfrac{1}{2}\rangle = |\downarrow\rangle$

and isospin $|n\rangle = |\tfrac{1}{2}\; m_t = -\tfrac{1}{2}\rangle$, $|p\rangle = |\tfrac{1}{2}\; m_t = \tfrac{1}{2}\rangle$.

We found that there is a spin-orbit force in nuclei and thus it is convenient to work in a basis of eigenfunctions of the total angular momentum $\vec{J} = \vec{L} + \vec{S}$. These are starting from $|\underset{\text{or } k}{n}\, l m\rangle\, |s\, m_s\rangle\, |t\, m_t\rangle$

$$|\underset{\text{or } k}{n}\; l s\, j\, m_j\, m_t\rangle = \sum_{m\, m_s} |\underset{\text{or } k}{n}\; l m\rangle\, |s\, m_s\rangle \underbrace{\langle l\, m\, s\, m_s \,|\, j\, m_j\rangle}_{\text{Clebsch-Gordan coefficients}} \otimes |t\, m_t\rangle$$

We say l and s ($=\frac{1}{2}$ for nucleons) are coupled to good j

Properties as for orbital angular momentum

$$\vec{J}^2 |l s\, j m_j\rangle = j(j+1)\, |l s\, j m_j\rangle \qquad (\hbar = 1)$$
$$J_z\, |l s\, j m_j\rangle = m_j\, |l s\, j m_j\rangle$$
$$\vec{L}^2 |l s\, j m_j\rangle = l(l+1)\, |l s\, j m_j\rangle$$
$$\vec{S}^2 |l s\, j m_j\rangle = s(s+1)\, |l s\, j m_j\rangle$$

Finally, we are at the stage for relating the nucleon-nucleon interaction to the observable scattering phase shifts. Since the two-body interaction is Galilean and translationally invariant, the centre-of-mass and the relative motion factorizes, i.e.,

$$H_{\substack{2\text{-body}\\ \text{system}}} = T_1 + T_2 + V_{12}$$

$$= -\frac{\hbar^2}{2m_N}\left(\nabla_1^2 + \nabla_2^2\right) + V\left(\vec{r}_1 - \vec{r}_2,\ \underbrace{\vec{\nabla}_1 - \vec{\nabla}_2}_{\parallel}\right)$$

$$= \underbrace{-\frac{\hbar^2}{4m_N}\nabla_{cm}^2}_{\substack{\nabla_{cm}\,=\,\nabla_1+\nabla_2\\ \text{plane wave in centre-of-mass}\\ \text{motion}}} - \frac{\hbar^2}{m_N}\nabla_{12}^2 + V\left(\vec{r}_{12}, \vec{\nabla}_{12}\right)$$

In the centre-of-mass system, $\vec{P}_{cm} = \vec{p}_1 + \vec{p}_2 = 0$, and we therefore have to solve an effectively one-body Schrödinger equation

$$H_{\vec{P}_{cm}=0} = -\frac{\hbar^2}{m_N}\nabla_{12}^2 + V\left(\vec{r}_{12}, \vec{\nabla}_{12}\right)$$

Equivalently, one can solve the Lippmann-Schwinger equation in momentum space for the scattering amplitude T

$$\langle \vec{k}'|T(E)|\vec{k}\rangle = \langle \vec{k}'|V|\vec{k}\rangle + \underset{\substack{\text{principal}\\ \text{value integral}}}{P}\int_{-\infty}^{\infty}\frac{d^3\vec{q}}{(2\pi)^3}\,\frac{\langle \vec{k}'|V|\vec{q}\rangle\langle\vec{q}|T(E)|\vec{k}\rangle}{E - \vec{q}^{\,2}},$$

where we have set $\frac{\hbar^2}{m_N} = 41.47\ \text{MeV fm}^2 = 1$.

Since the symmetries of the interaction V also hold for the scattering amplitude (as for any two-body operator), we will use the symmetries and expand V and T into so-called partial waves. This reduces the 3dim. Lippmann-Schwinger equation to many one-dim. scattering equations. We will also build in spin, but consider neutron-proton scattering, so that we have picked a particular isospin wave function 1)

$$\langle \vec{k}' S m_s | T(E) | \vec{k} S m_s \rangle$$

$$= \sum_{m,\ell} \langle \vec{k}' S m_s | T(E) | \ell m S m_s \rangle \underbrace{\langle \ell m | \vec{k} \rangle}_{|k\rangle\, Y^*_{\ell m}(\hat{k})\, 4\pi\, i^\ell}$$

$$= \sum_{\substack{m,\ell \\ M,J}} \langle \vec{k}' S m_s | T(E) | k J M \ell S \rangle \langle J M | \ell m S m_s \rangle Y^*_{\ell m}(\hat{k})\, 4\pi\, i^\ell$$

Clebsch-Gordan Coefficients

$$= \sum_{\substack{\ell,\ell' \\ m,m' \\ J,J' \\ M,M'}} \langle k' J' M' \ell' S | T(E) | k J M \ell S \rangle \langle J M | \ell m S m_s \rangle \langle \ell' m' S m_s | J' M' \rangle \, Y^*_{\ell m}(\hat{k})\, Y_{\ell' m'}(\hat{k}')\, (4\pi)^2\, i^{\ell-\ell'}$$

and the same for V with $\langle k' J' M' \ell' S | V | k J M \ell S \rangle$.

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1) The Pauli principle then constrains the state to be the $|T=1, m_T=0\rangle$ part of $|np\rangle$ isospin wave function if $|\Psi_{space}\rangle \otimes |\chi_{spin}\rangle$ is symmetric and the $|T=0, m_T=0\rangle$ part otherwise.

Since $T(E)$ and V are diagonal in J and independent of M, it reduces to

$$\langle \vec{k}' S m_s | T(E) | \vec{k} S m_s \rangle$$

$$= \sum_{\substack{l,l',\\ m,m'\\ J,M}} \langle k' J l' S | T(E) | k J l S \rangle \underbrace{\langle J M | l m S m_s \rangle \langle l' m' S m_s | J M \rangle} \, Y^*_{lm}(\hat{k}) \, Y_{l'm'}(\hat{k}') \, (4\pi)^2 \, i^{l-l'}$$

$$\sum_{M, m_s} \cdots = \frac{2J+1}{2l+1} \delta_{ll'} \delta_{mm'}$$

For spin-averaged processes, in particular

$$\frac{1}{2S+1} \sum_{m_s} \langle \vec{k}' S m_s | T(E) | \vec{k} S m_s \rangle$$

$$= \sum_{\substack{l,m\\ J}} \langle k' J l S | T(E) | k J l S \rangle \frac{2J+1}{2l+1} Y^*_{lm}(\hat{k}) \, Y_{lm} \, (4\pi)^2 \frac{1}{2S+1}$$

$$\sum_m \cdots = \frac{2l+1}{4\pi} P_l(\hat{k} \cdot \hat{k}')$$

$$= \sum_{l J} \frac{2J+1}{2S+1} \langle k' J l S | T(E) | k J l S \rangle \, 4\pi \, P_l(\hat{k} \cdot \hat{k}')$$

Partial wave expansion and we define $\langle k J l S | T(k^2) | k J l S \rangle = -\frac{1}{k} \tan \delta_{l,J,S}$

phase shifts in l, J, S partial wave,

e.g., ${}^1S_0$, in general ${}^{2S+1}l_J$

$S=0$ $\quad l=0$ $\quad J=0$

We can similarly expand the integral term

$$P\int_{-\infty}^{\infty} \frac{d^3\vec{q}}{(2\pi)^3} \frac{\langle \vec{k}' | V | \vec{q} \rangle \langle \vec{q} | T(E) | \vec{k} \rangle}{E - \vec{q}^{\,2}}$$

$$= P\int_{-\infty}^{\infty} \frac{d^3\vec{q}}{(2\pi)^3} \sum_{S' m_s'} \frac{\langle \vec{k}' S m_s | V | \vec{q}\, S' m_s' \rangle \langle \vec{q}\, S' m_s' | T(E) | \vec{k}\, S m_s \rangle}{E - \vec{q}^{\,2}}$$

V and T also diagonal in Spin S

$$= \frac{2}{\pi} P\int_0^{\infty} q^2 dq \sum_{\substack{\ell, \ell' \\ m, m' \\ J, M \\ \ell''}} \langle k' J \ell' S | V | q J \ell'' S \rangle \frac{1}{E - q^2} \langle q J \ell'' S | T(E) | k J \ell S \rangle$$
$$\langle J M | \ell m S m_s \rangle \langle \ell' m' S m_s | J M \rangle (4\pi)^2 \, i^{\ell - \ell'}$$
$$Y^*_{\ell m}(\hat{k}) Y_{\ell' m'}(\hat{k}')$$

(check this at home yourself)

Therefore, for given J, $\ell, \ell', \ell'' = J \pm 1$, if $S = 1$ only, and we have a matrix coupled-channel equation

$$T(k', k; E)^{JS}_{\ell' \ell} = V(k', k)^{JS}_{\ell' \ell} + \sum_{\ell''} \frac{2}{\pi} P\int_0^{\infty} q^2 dq \; V(k', q)^{JS}_{\ell' \ell''} \, T(q, k; E)^{JS}_{\ell'' \ell}$$

$$\begin{array}{c|cc} & \ell = J-1 & J+1 \\ \hline \ell' = J-1 & \cdot & \cdot \\ = J+1 & \cdot & \cdot \end{array} = \left(\begin{array}{c|c} \cdot & \cdot \\ \end{array}\right) + \left(\begin{array}{c|c} \cdot & \cdot \\ \hline \cdot & \cdot \end{array}\right) \cdot \left(\begin{array}{c|c} \cdot & \cdot \\ \hline \cdot & \cdot \end{array}\right)$$

The way one proceeds now in nuclear physics is to parametrize the potential (details in the next lecture), commonly as one-boson exchange interactions, and then to fit the parameters of such a model to the elastic nucleon-nucleon scattering phase shifts. In Fig 1 of the handout, we show the momentum-space matrix elements of different high precision potential models in the two S-waves. We observe that they are very different, although they are fitted to the same set of data, see e.g., Fig. 3 or below.

Next week, we will discuss an approach which removes these differences and involves the <u>renormalization group</u>.