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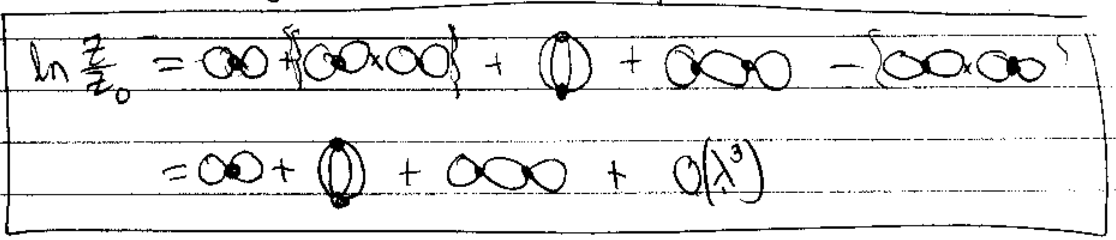
Ok, let's continue with our study of the expansions of Z and $\langle x^2 \rangle$.

- Actually, we want the expansion of $\ln Z$ and not Z, since it is $\ln Z$ that is proportional to the thermodynamic functions of interest.
- To find $\ln Z$ or $\ln Z/Z_0$ in perturbation theory, we just do the Taylor series expansion:

$$\boxed{\begin{aligned} \ln \frac{Z}{Z_0} &= \ln\left[1 - \frac{3\lambda}{4a^2} + \frac{105}{32}\frac{\lambda^2}{a^4} + O(\lambda^3)\right] \\ &= -\frac{3\lambda}{4a^2} + \frac{3\lambda^2}{a^4} + \ldots \end{aligned}}$$

- There are two parts here at $O(\lambda^2)$: the $\frac{105}{32}\frac{\lambda^2}{a^4}$ appearing once and $-\frac{3\lambda}{4a^2}$ appearing twice.
- If we go back to our diagrams, we find that



and the "disconnected" parts cancels out when we take the logarithm!

- This is not really convincing yet, because we don't know that the factors multiplying the cancelling terms are really the same.
- But, we can prove the result in general (which is called the "linked cluster theorem") using a very elegant technique called the "replica method".

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Replica method: Consider n copies of the integral and the product $\left(\frac{Z}{Z_0}\right)^n \equiv \tilde{Z}^n$. We rewrite $\tilde{Z}^n$ in a form that we can use in perturbation theory in n:

$$\boxed{\tilde{Z}^n = e^{\ln \tilde{Z}^n} = e^{n \ln \tilde{Z}} = 1 + n(\ln \tilde{Z}) + \tfrac{1}{2}n^2(\ln \tilde{Z})^2 + \dots}$$

So we can find $\ln \tilde{Z}$ if we can calculate $\tilde{Z}^n$ and pick off the linear term in n!

But what is $\tilde{Z}^n$? Just n copies of $\tilde{Z}$. We'll add the sources as well:

$$\boxed{\begin{aligned}\tilde{Z}^n &= \left(\int dx_1\, e^{-\frac{1}{2}a_1 x_1^2 - \frac{\lambda_1}{4}x_1^4 + j_1 x_1}\right)\left(\int dx_2\, e^{-\frac{1}{2}a_2 x_2^2 - \frac{\lambda_2}{4}x_2^4 + j_2 x_2}\right)\cdots\left(\int dx_n \cdots\right)\\ &= \int dx_1 \cdots dx_n\, e^{-\frac{1}{2}\sum_i a_i x_i^2 - \frac{1}{4}\sum_i \lambda_i x_i^4 + \sum_i j_i x_i}\end{aligned}}$$

- We've added indices to a and λ, even though all of the a_i are the same and all of the λ_i are the same.

- Perturbation theory is derived as before: use $\frac{\delta}{\delta j_i}$ to remove the interaction terms from all of the integrals.
  - each vertex has the same index i for all lines coming out of it, so the a_i's have the same i vertex at each end.
  - We sum over i from 1 to n.

$\Rightarrow$ any connected diagram as all of it's lines and vertices only one i at a time $\Rightarrow$ factor of n from the sum.
- So disconnected diagrams get n from each piece.

$\Rightarrow$ the terms linear in n, which is $\ln \tilde{Z}$, are precisely those that are connected. $\Rightarrow$ $\ln Z - \ln Z_0$ is the sum of connected diagrams

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Now for the $\langle x^2 \rangle$ evaluation. We'll use the replica method again.

- Note: We can also carry out these proofs directly, but they are not nearly as elegant!
  [See the references for alternative proofs.]

- We'll make n copies as before but define R_n to have only x_1^2 in the integral besides the $e^{-a_i x_i^2/2 - \lambda_i/4 x_i^4 + J_i x_i}$ factors.

$$\Rightarrow \boxed{\begin{aligned} R_n &= \frac{1}{Z_0^n}\int dx_1 \cdots dx_n \, x_1^2 \, e^{-\frac{1}{2}\sum_i a_i x_i^2 - \frac{1}{4}\sum_i \lambda_i x_i^4} \\ &= \left(\frac{1}{Z_0}\int dx_1 \, x_1^2 \, e^{-\frac{1}{2}a_1 x_1^2 - \frac{1}{4}\lambda_1 x_1^4}\right)\underbrace{\left(\frac{1}{Z_0}\int dx \, e^{(\quad)}\right)^{n-1}}_{n-1 \text{ copies}} \end{aligned}}$$

so R_0 is what we want with the 2nd part giving the denominator!
$\Rightarrow$ calculate R_n and continue to $n=0$.

- If we now expand R_n perturbatively (i.e. introducing J_i and $\frac{\delta}{\delta J_i}$ and so on), then all indices on the part of any diagram that is connected to the x_1^2 must also be 1's.
  $\Rightarrow$ no summation on the x_1^2 part and summations on all disconnected parts.
  $\therefore$ The $n=0$ part, which is $\langle x^2 \rangle$ is simply the sum of all connected diagrams.
  - Obviously at that point we can drop the 1. QED.

- Note that nothing in the argument depended on the operator averaged being x^2 $\Rightarrow$ $\langle \hat{O} \rangle$ for any operator $\hat{O}$ is given by the sum of connected diagrams.

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Time to talk about symmetry factors ...

- The $\frac{1}{n!}$ from the Taylor series exponsion of the exponentials almost always cancels the $n!$ ways of interchanging the vertices. The factor of $4!$ from $\left(\frac{\delta}{\delta j}\right)^4 j^4$ is taken into account in the Feynman rule $\bullet \; -\frac{\lambda}{4}\cdot 4!$

  $\Rightarrow$ the symmetry factor is the left-over correction to these cancellations.

- There are three types of factors for our diagrams:

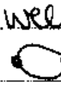
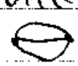
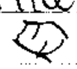
  ① Factor of $\frac{1}{2}$ from each line that starts and ends on the same vertex.
  - This goes away for lines with an arrow.
  - We'll write these factors as fractions, but you'll ~~of~~ see them collected first as a factor S and then $1/S$ applied to the diagram.
  - You can derive this factor by comparing  to 

    $$\left(\frac{\delta}{\delta j}\right)^4\left(-\frac{\lambda}{4}\right)\left(\frac{\delta}{\delta j}\right)^4\frac{1}{4!}\left(\frac{1}{2}\right)^4\left(j a^{-1} j\right)^4 = -\frac{6\lambda}{a^4}$$

    vs.

    $$\left(\frac{\delta}{\delta j}\right)^2\left(-\frac{\lambda}{4}\right)\left(\frac{\delta}{\delta j}\right)^4\frac{1}{3!}\left(\frac{1}{2}\right)^3\left(j a^{-1} j\right)^3 = -\frac{3\lambda}{a^4} \quad \Rightarrow \frac{1}{2} \text{ factor}$$

  ② Factor of $\frac{1}{n!}$ for each set of n "equivalent lines", which are lines that run (in the same direction, if arrows) between two different vertices.

   $\frac{1}{2!}$,  $\frac{1}{3!}$  1  $\frac{1}{2!}$

  ③ Factor of $1/P$ for permutations P of the vertices that leave the diagram unchanged (including any arrows).
  - external points stay fixed when considering permutations.

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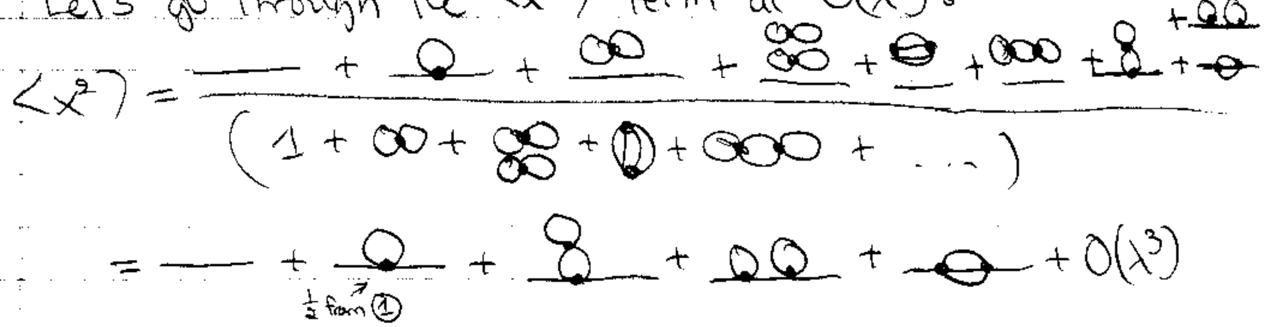
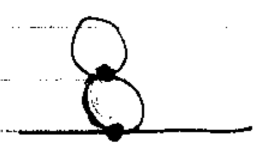
Let's go through the $\langle x^2 \rangle$ term at $O(\lambda^2)$:



so once again the disconnected pieces cancel when we account for the denominator.

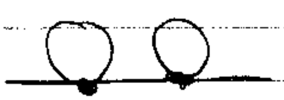
- Now let's go ahead and try the symmetry factors:



① lines ~~to~~ to same vertex:

② equivalent lines:

③ vertex permutations:



① lines to same vertex:

② equivalent lines:

③ vertex permutations:

① lines to same vertex:

② equivalent lines:

③ vertex permutations:

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- Calculation of $\ln \frac{Z}{Z_0}$ at $O(\lambda^3)$

$$\boxed{\frac{Z}{Z_0} = \sum_{n=0}^{\infty} \lambda^n Z_n} \quad \text{with} \quad \boxed{Z_n = \frac{(-1)^n}{n!}\frac{(4n-1)!!}{4^n}\cdot\frac{1}{a^{2n}}}$$

$$\boxed{\lambda Z_1 = -\frac{3}{4}\frac{\lambda}{a^2}} \qquad \boxed{\lambda^2 Z_2 = \frac{105}{32}\frac{\lambda^2}{a^4}}$$

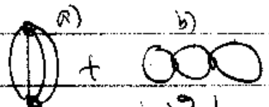
$\Rightarrow \boxed{\lambda^3 Z_3 = -\frac{3465}{128 a^6}\lambda^3}$ is what we expect.

$\Rightarrow \boxed{\ln \frac{Z}{Z_0} = -\frac{3\lambda}{4a^2} + \frac{3\lambda^2}{a^4} - \frac{99\lambda^3}{4a^6} + O(\lambda^4)}$ (from Mathematica)

λ^1:  $\left(-\frac{\lambda}{4}4!\right)\frac{1}{a^2}$

① $\frac{1}{2}\cdot\frac{1}{2}$
② $1/2$
③ 1

$\Rightarrow -\frac{\lambda}{a^2}\,6\cdot\frac{1}{8} = -\frac{3\lambda}{4a^2}$ ✓

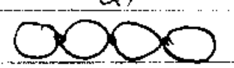
λ^2: a) b) 

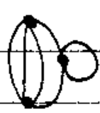
$(-3!\lambda)^2\frac{1}{a^4}$

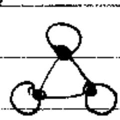
	a)	b)
①	1	$\frac{1}{2}\cdot\frac{1}{2}$
②	$\frac{1}{4!}$	$\frac{1}{2}$
③	$\frac{1}{2}$	$\frac{1}{2}$

$\frac{1}{12}$

$\Rightarrow \frac{36\lambda}{a^4}\left(\frac{1}{48}+\frac{1}{16}\right) = +\frac{3\lambda}{a^4}$ ✓

λ^3: a) 

b) 

c) 

d) 

$(-3!\lambda)^3\frac{1}{a^6}$

	a)	b)	c)	d)
①	$\frac{1}{2}\cdot\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$	1
②	$\frac{1}{2}\cdot\frac{1}{2}$	$\frac{1}{3!}$	1	$\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$
③	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{3!}$	$\frac{1}{3!}$

$\frac{1}{32} + \frac{1}{24} + \frac{1}{48} + \frac{1}{48}$

$\Rightarrow -\frac{\lambda^3}{a^6}\,6^3\left(\frac{11}{96}\right) = -\frac{99\lambda^3}{4a^6}$ ✓

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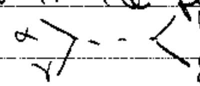
If the lines have <u>arrows</u> on them, then the symmetry factors are:

1. this is always 1
2. only equivalent lines in the <u>same</u> direction
3. the permutations can't mess up the direction of the lines  $\neq$ 

But we count  as $\frac{1}{2}$ under ② still.

- Now if the potential is not contracted to a point:
   for example, keeping track of spin, then we lose ② entirely $\Rightarrow$ only ③.
- In this case, we wouldn't pick up the $4!$ factor for the vertex
- For n point functions, the remaining symmetry factor contribution is at most $1/2$ and for many case there <u>are</u> no symmetry factors.

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Ok, let's continue to explore the "Feynman rules" for our simple model a bit...

- The Feynman rule for the "quadratic part" of the action, which is the part with a^{-1}, came from considering the "two-point" function $\langle x \cdot x \rangle$. [This is just $\langle x^2 \rangle$ here, but we write it this way it anticipation of having $\langle T[x(t_1)\, x(t_2)] \rangle$ or $\langle T[\psi(x_1,t_1)\psi^{\dagger}(x_2,t_2)] \rangle$ with operators at different times or different places and times.]
  - The leading contribution to $\langle x^2 \rangle$, of order λ^0, was $\frac{1}{a}$ when we include a $\frac{1}{2}$ in the original "action": $e^{-\frac{1}{2}ax^2}$.
  - If we had written e^{-ax^2} instead, we obviously would get $\frac{2}{a}$, so it's convenient (although never necessary) to could the constants in the action in such a way as to eliminate inconvenient numerical factors.

- We can do an analogous analysis for any "vertex", such as arises for $e^{-\frac{\lambda}{4!}x^4}$ or $e^{-\frac{\delta}{6!}x^6}$ or whatever.
  - The vertices are different from the propagator (the quadratic part), since the propagator is found by actually solving the quadratic part (or, rather, by completing the square), while the vertices come about from pulling the exponent outside the integral, replacing x by $\frac{\partial}{\partial J}$.
  - Consider the leading n-point vertex by looking at the leading (in λ, δ) contributions to the n-point function
    $\Rightarrow \langle x^4 \rangle$ for $e^{-\frac{\lambda}{4!}x^4}$
    $\Rightarrow \langle x^6 \rangle$ for $e^{-\delta x^6/6!}$
    and so on.
  - (note: there will be λ^1 and δ^0 contributions!)

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Generating our previous result for $\langle x^2\rangle$:

$$\boxed{\langle x^4\rangle = \frac{N\int_{-\infty}^{\infty}dx\, x^4\, e^{-\frac{a}{2}x^2-\frac{\lambda}{4}x^4}}{N\int_{-\infty}^{\infty}dx\, e^{-a/2\, x^2-\frac{\lambda}{4}x^4}} = \frac{\left(\frac{\partial}{\partial j}\frac{\partial}{\partial j}\frac{\partial}{\partial j}\frac{\partial}{\partial j}\right)e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4}e^{\frac{1}{2}ja^{-1}j}\Big|_{j=0}}{e^{-\frac{\lambda}{4}\left(\frac{\partial}{\partial j}\right)^4}e^{1/2\, ja^{-1}j}\Big|_{j=0}}}$$

• To do $\langle x^6\rangle$ in the theory with $e^{-\delta x^6/6!}$, just multiply $e^{-\delta x^6/6!}$ with the $e^{-a/2\,x^2-\frac{\lambda}{4}x^4}$ part (so the "action" is now $\frac{a}{2}x^2+\frac{\lambda}{4}x^4+\frac{\delta x^6}{6!}$) and have $\left(\frac{\partial}{\partial j}\right)^6$ in the numerator.

• The leading contributions to $\langle x^4\rangle$ are:

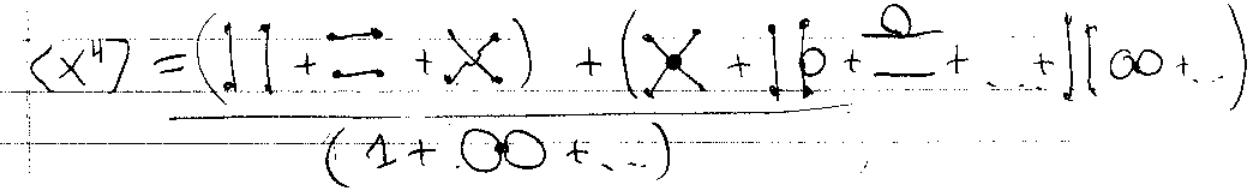
$$\Rightarrow \langle x^4\rangle = \frac{\left(\frac{\partial}{\partial j}\frac{\partial}{\partial j}\frac{\partial}{\partial j}\frac{\partial}{\partial j}\right)\left(1-\frac{1}{4}\lambda\left(\frac{\partial}{\partial j}\right)^4+\dots\right)\left[1+\frac{1}{1!}\left(\frac{ja^{-1}j}{2}\right)+\frac{1}{2!}\left(\frac{ja^{-1}j}{2}\right)^2+\frac{1}{3!}\left(\frac{ja^{-1}j}{2}\right)^3+\frac{1}{4!}\left(\frac{ja^{-1}j}{2}\right)^4+\dots\right]\Big|_{j=0}}{\left[1-\frac{1}{4}\lambda\left(\frac{\partial}{\partial j}\right)^4+\dots\right]\left[1+\frac{1}{1!}\left(\frac{ja^{-1}j}{2}\right)+\frac{1}{2!}\left(\frac{ja^{-1}j}{2}\right)^2+\dots\right]\Big|_{j=0}}$$

$$= \frac{\left(\frac{1}{2!}\frac{1}{4}\cdot 4!\,\frac{1}{a^2} - \frac{1}{4}\lambda\frac{1}{a^4}\frac{1}{4!}\frac{1}{2^4}8!\right)}{1-\frac{\lambda}{4}\frac{1}{2!}\frac{1}{4}4!\,\frac{1}{a^2}} \qquad \frac{8\cdot7\cdot6\cdot5}{4\cdot3\cdot2}=\frac{105}{4}$$

$$= \left(\frac{3}{a^2}-\frac{105\lambda}{4a^4}\right)\left(1+\frac{3}{4}\frac{\lambda}{a^2}\right) \doteq \frac{3}{a^2}-\frac{24\lambda}{a^4}$$

What is this in our diagrammatic expansion?

You should imagine the x^4 is represented by 4 fixed points : : to which we can join propagators $\text{—} \Rightarrow \frac{1}{a}$ and vertices $\bullet \Rightarrow \frac{\lambda}{4}\cdot 4!$



So look at $\langle x^4\rangle - \langle x^2\rangle^2$ to subtract off the type pieces.

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When the dust settles, $\langle x^4\rangle - \langle x^2\rangle\langle x^2\rangle$ is given by the set of <u>totally</u> connected diagrams.

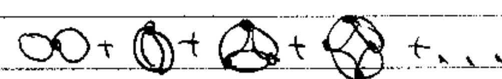
What next?

- When we go to path integral representations of boson and fermion partition functions, everything goes through analogously.
  - We'll have formulas for Gaussian integrals that are more complicated, but most of the rest goes through the same way.
  - Lines will generally have arrows $\Rightarrow$ slightly modified symmetry factors.

- Now we can think of partial summations.

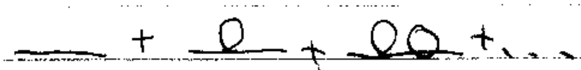
  Examples:

  ① For $\ln Z/Z_0$, sum

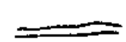
  

  which picks out one term to each order.

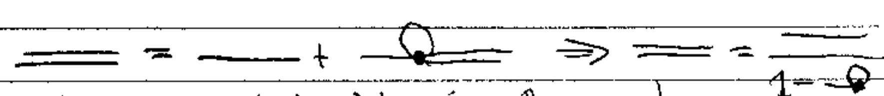
  ② For $\langle x^2\rangle$, consider

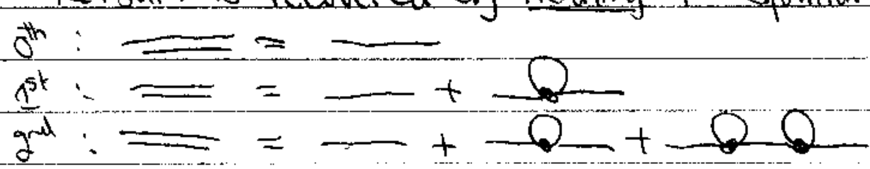
  If we designate the sum with a double line:

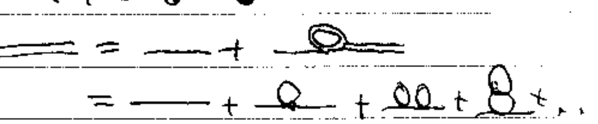
  then

  - The sum is recovered by iterating the equation:

  and so on. More general:

  ③ More generally, pick out the "one particle irreducible" (1PI) pieces: does the diagram fall apart when you cut a line?
  - Later: Sum these with Dyson's equation.