Symmetry constraints and microscopic origin of the nucleon-nucleon interaction I

For the two-body system, the Hamiltonian is a sum of center-of-mass and relative motion

$$H_{2\text{-body}} = \frac{\vec{P}_{cm}^{\,2}}{4m_N} + \underbrace{\frac{\vec{p}_{12}^{\,2}}{m_N} + V_{12}\,(\text{nuclear part}) + V_{12}\,(\text{Coulomb part})}_{H_{12}}$$

Therefore, the wave function factorizes

$$|\Psi\rangle = |\vec{K}\rangle \otimes |\text{relative motion and internal quantum numbers}\rangle$$

with $|\vec{k}\rangle$ a plane wave $\langle \vec{R}_{cm} | \vec{k}\rangle = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\vec{R}}$

$$\left(\vec{R}_{cm} = \frac{\vec{r}_1 + \vec{r}_2}{2} \right)$$

and a basis of angular momentum eigenstates and spin/isospin for the relative part of the wave function

$$|\Psi\rangle = |\vec{k}\rangle \otimes |n\,\underbrace{\ell\rangle\,|S}_{J,\,M_J}\rangle \;\; |T, M_T\rangle$$

relative space   2-body spin   ~~2-body isospin~~

coupled to total angular mom. eigenstates $\vec{J} = \vec{\ell} + \vec{S}$

Matrix elements of H_{12} "charge independence"

$$\langle nJM\ell S; TM_T | H_{12} | n'J'M'\ell' S', T'M_T'\rangle = \delta_{JJ'}\,\delta_{MM'}\,\delta_{SS'}\,\overbrace{\delta_{TT'}\,\delta_{M_T M_T'}}^{\text{"charge independence"}} \times$$

$$\langle \underline{n}\,\underline{J}\,\underline{\ell}\,S; TM_T | H_{12} | \underline{n'}\,J\,\underline{\ell'}\,S; TM_T\rangle$$

General form of the nucleon-nucleon interaction

$$H_{NN} = T_{cm} + T_{rel} + \underset{\substack{\uparrow \\ \text{nuclear force}}}{V_{NN}} + V_{Coulomb}$$

$$V_{Coulomb} = \frac{e^2}{|\vec{r}_1 - \vec{r}_2|} \underbrace{\frac{1}{2}\left(1 + \underset{\substack{\| \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}}{\tau_1^z}\right)\frac{1}{2}\left(1 + \tau_2^z\right)}_{\text{projects on protons in isospin space}}$$

(can be treated as a perturbation!)

Since we are not able to derive V_{NN} from the underlying theory of the strong interactions, Quantum Chromodynamics (QCD), we first constrain the nucleon-nucleon interaction by its symmetries and then introduce a model for it, based on heavy-boson exchange forces.

Symmetry constraints on V_{NN}

1.) hermiticity: $V_{NN} = V_{NN}^{+}$

*) 2.) translational invariance in time: V_{NN} must be time independent

3.) symmetric under exchange of particle labels: $V_{NN}(1,2) = V_{NN}(2,1)$

*) 4.) translational invariance: $V_{NN}(\vec{r}_1, \vec{r}_2, \ldots) = V_{NN}(\vec{r}_1 - \vec{r}_2, \ldots)$

*) 5.) rotational invariance: V_{NN} is a scalar under rotations in space and spin. $\vec{r}_{12}$, $\vec{p}_{12}$, $\vec{\sigma}_1$, $\vec{\sigma}_2$ have to come in scalar products

6.) Galilean invariance: $V_{NN}(\vec{r}_{12}, \vec{p}_1, \vec{p}_2, \ldots) = V_{NN}(\vec{r}_{12}, \frac{\vec{p}_1 - \vec{p}_2}{2}, \ldots)$

7.) time reversal invariance: $\vec{r}_{12} \to \vec{r}_{12}$, $\vec{p}_{12} \to -\vec{p}_{12}$, $\vec{\sigma}_i \to -\vec{\sigma}_i$
(reverse motion) $\vec{\tau}_i \to -\vec{\tau}_i$

<u>for a Hermitian operator:</u> $V_{NN}(\vec{r}_{12}, \vec{p}_{12}, \vec{\sigma}_1, \vec{\sigma}_2, \ldots) = V_{NN}(\vec{r}_{12}, -\vec{p}_{12}, -\vec{\sigma}_1, -\vec{\sigma}_2, \ldots)$

1)–7) also hold for the Coulomb interaction $V_{Coulomb}$. In addition, we consider the strong interaction part of V_{NN} and ignore weak interactions. Thus, we also have

8) parity invariance: $V_{NN}(\vec{r}_{12}, \vec{p}_{12}, \vec{\sigma}_1, \vec{\sigma}_2, \ldots) = V_{NN}(-\vec{r}_{12}, -\vec{p}_{12}, \vec{\sigma}_1, \vec{\sigma}_2, \ldots)$

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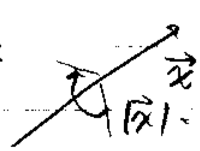
*) Recall: an operator O is invariant under a transformation U if

$$O = U^{-1} O U$$

explicit representations for the transformation are

(i) spatial translation: $U(\vec{a}) = e^{-\frac{i}{\hbar}\vec{a}\cdot\vec{p}_{12}}$ — $\vec{p}_{12}$: momentum operator; $\vec{a}$: spatial translation vector

(ii) temporal translation: $U(t) = e^{-\frac{i}{\hbar} t H_{12}}$

(iii) rotation: 

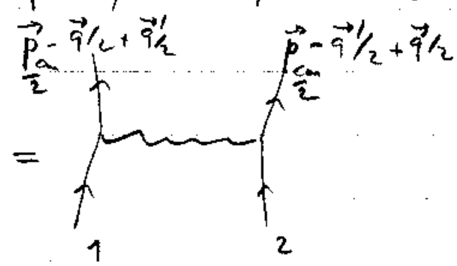
$U(\vec{x}) = e^{-\frac{i}{\hbar}\vec{x}\,\vec{J}}$ — $\vec{J}$: angular momentum operator; $\vec{x}$: angle of rotation

Furthermore, to a good approximation, the NN interaction may be taken to be charge independent: $V_{NN} = V_{NN}(\ldots, \vec{\tau}_1 \cdot \vec{\tau}_2, \ldots)$

(Scalar in isospin-space $[V_{NN}, T = \frac{\vec{\tau}_1 + \vec{\tau}_2}{2}]$, but $[V_{Coulomb}, T] \neq 0$)

In momentum space, the symmetries are realized (with $\vec{P}_{cm} = \vec{p}_1 + \vec{p}_2$ (center of mass))

$$\langle \vec{p}_3\, \vec{p}_4 | V_{NN} | \vec{p}_1\, \vec{p}_2 \rangle =$$



$$\vec{p}_1 = \frac{\vec{P}_{cm}}{2} + \vec{q}'/2 + \vec{q}/2 \qquad \frac{\vec{P}_{cm}}{2} - \vec{q}'/2 - \vec{q}/2 = \vec{p}_2$$

Galilean invariance: $V_{NN}(\vec{P}_{cm}, \vec{q}, \vec{q}', \vec{\sigma}_1, \vec{\sigma}_2, \ldots) = V_{NN}(\vec{q}, \vec{q}', \vec{\sigma}_1, \vec{\sigma}_2, \ldots)$

*) 1) hermiticity: $\vec{q} \to -\vec{q}$, complex conjugation of matrix element "$i \to -i$"

7) time reversal inv.: provided V_{NN} is hermitian

$$\vec{q}, \vec{q}' \to -\vec{q}, -\vec{q}', \quad \vec{\sigma}_1, \vec{\sigma}_2 \to -\vec{\sigma}_1, -\vec{\sigma}_2, \quad i \to -i$$

8) parity inv.: $\vec{q}, \vec{q}' \to -\vec{q}, -\vec{q}'$

3.) sym. under exchange of particle labels: $\vec{\sigma}_1 \leftrightarrow \vec{\sigma}_2$, $\vec{q}, \vec{q}' \to -\vec{q}, -\vec{q}'$

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*) hermiticity $\langle \vec{p}_3, \vec{p}_4 | V_{NN} | \vec{p}_1\, \vec{p}_2 \rangle = \langle \vec{p}_1\, \vec{p}_2 | V_{NN} | \vec{p}_3\, \vec{p}_4 \rangle^*$

$\vec{q} \to -\vec{q}$ "$i \to -i$"

Therefore the <u>most general form</u> of V_{NN} reads:

<u>Coordinate space</u>	<u>Momentum space</u>
scalar interaction $\mathcal{O}_1 = \mathbb{1}$	$\mathcal{O}_1 = \mathbb{1}$
spin-spin interaction $\mathcal{O}_2 = \vec{\sigma}_1 \cdot \vec{\sigma}_2$	$\mathcal{O}_2 = \vec{\sigma}_1 \cdot \vec{\sigma}_2$
tensor interaction $\mathcal{O}_3 = \vec{\sigma}_1 \cdot \hat{r}_{12}\, \vec{\sigma}_2 \cdot \hat{r}_{12} - \frac{1}{3} \vec{\sigma}_1 \cdot \vec{\sigma}_2$	$\mathcal{O}_3 = \vec{\sigma}_1 \cdot \hat{q}\, \vec{\sigma}_2 \cdot \hat{q} - \frac{1}{3} \vec{\sigma}_1 \cdot \vec{\sigma}_2$
spin-orbit interaction $\mathcal{O}_4 = \vec{L} \cdot \vec{S}$	$\mathcal{O}_4 = i(\vec{\sigma}_1 + \vec{\sigma}_2) \cdot \vec{q} \times \vec{q}\,'$
quadratic spin-orbit $\mathcal{O}_5 = \frac{1}{2}(\vec{\sigma}_1 \cdot \vec{L}\, \vec{\sigma}_2 \cdot \vec{L} + \vec{\sigma}_2 \cdot \vec{L}\, \vec{\sigma}_1 \cdot \vec{L})$	$\mathcal{O}_5 = \vec{\sigma}_1 \cdot (\vec{q} \times \vec{q}\,')\, \vec{\sigma}_2 \cdot (\vec{q} \times \vec{q}\,')$

With these set of operators

$$V_{NN} = \sum_{i=1}^{5} V_0^{(i)}(\vec{r}_{12}^{\,2}, \vec{p}_{12}^{\,2}, \vec{L}^2)\, \mathcal{O}_i$$

$$+ \sum_{i=1}^{5} V_\tau^{(i)}(\vec{r}_{12}^{\,2}, \vec{p}_{12}^{\,2}, \vec{L}^2)\, \mathcal{O}_i\, \vec{\tau}_1 \cdot \vec{\tau}_2$$

In momentum space, the $V^{(i)}$ depend on $V(\vec{q}^{\,2}, \vec{q}\,'^2, (\vec{q}-\vec{q}\,')^2)$.

<u>Notice</u>: a) V_{NN} doesn't connect different (iso-)spins $\langle k' J \ell S T | V_{NN} | k J \ell' S' T' \rangle = 0$ for $S \neq S'$, $T \neq T'$

b) spin-orbit and tensor forces vanish in $S=0$.

We have already discussed the spin-orbit interaction in the context of the nuclear shell model, but the tensor force is new to us. It operates like an interaction between between dipole-magnets. We will discuss the one-pion exchange potential in the next section. This interaction has a tensor force, which is of the form

$$V_{\pi}^{tensor} = \frac{g_\pi^2}{4m_N^2}\,\frac{1}{4\pi}\left(\frac{m_\pi^2}{3}+\frac{m_\pi}{r}+\frac{1}{r^2}\right)\frac{e^{-m_\pi r}}{r}\,\underbrace{\left(3\,\vec{\sigma}_1\cdot\hat{r}\,\vec{\sigma}_2\cdot\hat{r}-\vec{\sigma}_1\cdot\vec{\sigma}_2\right)}\vec{\tau}_1\vec{\tau}_2$$

position

tensor operator is attractive or repulsive depending on the spin direction relative to $\vec{r}_{12}$.

$\vec{\tau}_1\vec{\tau}_2$ attraction or repulsion depending on isospin $\langle pn|\vec{\tau}_1\cdot\vec{\tau}_2|pn\rangle = -1$

Therefore, the tensor force between a proton and neutron operates as follows