# Physics 880.05: Problem Set #3

The problems are due on Tuesday, November 24. See the online lecture notes for details of any of the individual topics covered in this problem set. Check the 880.05 webpage for suggestions and hints. Please give feedback early and often (and stop by to ask about anything).

1. **Noninteracting partition function in discrete form.** (In the following, $\zeta = +1$ for bosons and $\zeta = -1$ for fermions and units such that $\hbar = 1$). This problem is based on the class notes pages 107–109. We derived (quickly!) the partition function as

$$
\begin{aligned}
Z &= \mathrm{Tr} e^{-\beta(\widehat{H}-\mu\widehat{N})} = \int \prod_\alpha d\phi_\alpha^* d\phi_\alpha \, e^{-\sum_\alpha \phi_\alpha^* \phi_\alpha} \langle \zeta\phi_\alpha | e^{-\beta(\widehat{H}-\mu\widehat{N})} | \phi_\alpha \rangle \\
&= \lim_{N_\tau\to\infty} \int \prod_{k=1}^{N_\tau} \prod_\alpha \frac{1}{C} d\phi_{\alpha,k}^* d\phi_{\alpha,k} \, e^{-S(\phi^*,\phi)} \ , \qquad (1)
\end{aligned}
$$

where

$$
\begin{aligned}
S(\phi^*,\phi) &= \epsilon \sum_{k=2}^{N_\tau} \Bigl[ \sum_\alpha \phi_{\alpha,k}^* \Bigl\{ \frac{(\phi_{\alpha,k} - \phi_{\alpha,k-1})}{\epsilon} - \mu\phi_{\alpha,k-1} \Bigr\} + H(\phi_{\alpha,k}^*, \phi_{\alpha,k-1}) \Bigr] \\
&\quad + \epsilon \Bigl[ \sum_\alpha \phi_{\alpha,1}^* \Bigl\{ \frac{(\phi_{\alpha,1} - \zeta\phi_{\alpha,N_\tau})}{\epsilon} - \mu\zeta\phi_{\alpha,N_\tau} \Bigr\} + H(\phi_{\alpha,1}^*, \zeta\phi_{\alpha,N_\tau}) \Bigr] \ . \qquad (2)
\end{aligned}
$$

and $\epsilon = \beta/N_\tau$. (See the Negele/Orland excerpt for comparison.)

   (a) Go through the derivation again and answer the following questions:

      i. What does $\alpha$ label?

      ii. Why is the factor $e^{-\sum_\alpha \phi_\alpha^* \phi_\alpha}$ in the first line of Eq. (1)?

      iii. Where do the $\phi_{\alpha,k}^* \phi_{\alpha,k-1}$ terms come from? (These are the ones where one $\phi$ has $k$ and the other has $k-1$.)

      iv. Where does the second line in Eq. (2) come from? Why is there a $\zeta$ factor?

   (b) Now we let $H \to H_0 = \sum_\alpha \epsilon_\alpha a_\alpha^\dagger a_\alpha$, which is for a system of non-interacting particles with single-particle energies $\epsilon_\alpha$. The discrete expression for $Z$ may be written

$$
Z_0 = \lim_{N_\tau\to\infty} \prod_\alpha \Bigl[ \prod_{k=1}^{N_\tau} \int \frac{1}{C} d\phi_{\alpha,k}^* d\phi_{\alpha,k} \, e^{-\sum_{i,j=1}^{N_\tau} \phi_i^* S_{ij}^{(\alpha)} \phi_j} \Bigr] = \lim_{N_\tau\to 0} \prod_\alpha [\det S^{(\alpha)}]^{-\zeta} \ . \qquad (3)
$$

   Construct the matrix $S^{(\alpha)}$ (the answer is given in the Negele/Orland excerpt but you need to explain where the entries come from).

(c) Evaluate the determinant of $S^{(\alpha)}$ (expanding by minors along some row is one way) to show

$$\lim_{N_\tau\to\infty} \det S^{(\alpha)} = \lim_{N_\tau\to\infty}\left[1-\zeta\left(1-\frac{\beta(\epsilon_\alpha-\mu)}{N_\tau}\right)^{N_\tau}\right] = 1-\zeta e^{-\beta(\epsilon_\alpha-\mu)} . \tag{4}$$

(d) Show that this determinant yields the familiar result

$$Z_0 = \prod_\alpha (1-\zeta e^{-\beta(\epsilon_\alpha-\mu)})^{-\zeta} . \tag{5}$$

(e) The matrix $S^{(\alpha)}$ is said to be "sparse". What does that mean? Show that its inverse $[S^{(\alpha)}]^{-1}$ is "dense" (either analytically or via MATLAB or Mathematica examples).

2. **Spin-dependent force.** Revisit the diagrammatic calculation of the energy density in three-dimensions through second order, but now with a spin-dependent potential:

$$V_s(\mathbf{x}_1,\mathbf{x}_2)_{\alpha\beta,\lambda\mu} = \lambda_s\,\vec{\sigma}(1)_{\alpha\beta}\cdot\vec{\sigma}(2)_{\lambda\mu}\,\delta^3(\mathbf{x}_1-\mathbf{x}_2) ,$$

where the spin indices have been given explicitly.

(a) What is the corresponding interaction term in the Lagrangian?

(b) What is the Feynman rule for this vertex?

(c) Calculate the first-order correction to the energy density for a dilute Fermi gas with this interaction (bow-tie diagram).

(d) Bonus: Calculate the second-order correction to the energy density for a dilute Fermi gas with this interaction.

3. **One-dimensional system.** Repeat the calculations done in class but for a Fermi "fluid" in *one* spatial dimension with two-body interaction $V(x-x') = \lambda\delta(x-x')$. [Note: For an attractive $\delta$ function ($\lambda < 0$), this system can be solved to all orders numerically in terms of some relatively simple coupled integral equations.]

(a) Find the ground-state energy through first-order in the interaction for general $\nu$, using appropriate modifications of the Feynman rules.

(b) Compare the one- and three-dimensional cases qualitatively for both attractive and repulsive interactions.

(c) Bonus: Find and evaluate (all the way!) the second-order energy density in one dimension. Compare to the first-order result and sketch as a function of density for attractive and repulsive potentials.

4. **The Hubbard-Stratonovich trick.** In class we studied the following integral

$$\mathcal{Z} = \int_{-\infty}^{\infty} dx \exp\left(-\frac{1}{2}x^2 + \frac{\lambda}{4}x^4\right) . \tag{6}$$

Now consider the following identity:

$$\int_{-\infty}^{\infty} d\sigma \exp\left[-\frac{1}{2}\left(\sigma - cx^2\right)^2\right] = \sqrt{2\pi} \tag{7}$$

where $c$ is a constant. Quick question: Is the identity valid for any value of $x$?

(a) Using this identity, with an appropriate value of $c$, re-write the first integral as a double integral of the form

$$\mathcal{Z} = \int_{-\infty}^{\infty} dx\ d\sigma\ e^{-S[\sigma,x]} \tag{8}$$

where $S$ is *at most quadratic* in $x$ and $\sigma$. What is $S$ (find $c$ explicitly)?

(b) Generalize this result to the multidimensional case:

$$\mathcal{Z} = \int \mathcal{D}x_i \exp\left(-\frac{1}{2}\sum_{i,j=1}^{N_\tau} x_i A_{ij} x_j + \lambda \sum_{i=1}^{N_\tau} x_i^4\right) . \tag{9}$$

(Hint: What role does the $A_{ij}$ term play?) If you did everything correctly, at this point you should have the anticipated quadratic result for $S$. You should now be able to do the integral over $\{x_i\}$, obtaining a final answer as an integral over $\sigma$.

(c) In this case it may seem pointless to exchange the original integral over $x$ for a potentially much more complicated integral over $\sigma$. However, consider now the following Grassmann integral for a spin 1/2 system:

$$\mathcal{Z} = \int \mathcal{D}\eta_1^*\mathcal{D}\eta_1\mathcal{D}\eta_2^*\mathcal{D}\eta_2 \exp\left(-\sum_{i,j=1}^{N_\tau}({\eta^*}_{1,i}M_{i,j}\eta_{1,j} + {\eta^*}_{2,i}M_{i,j}\eta_{2,j}) + g\eta_1^*\eta_1\eta_2^*\eta_2\right) \tag{10}$$

This looks really complicated, but it's worse than that: we cannot even put it on a computer! The only Grassmann integrals we know how to do are Gaussian:

$$\int \mathcal{D}\eta^*\mathcal{D}\eta \exp\left(-\sum_{i,j=1}^{N_\tau} {\eta^*}_i M_{i,j}\eta_j\right) = \mathrm{Det}[M] \tag{11}$$

which is only useful when $g = 0$. Using (11), what is $\mathcal{Z}$ when $g = 0$?

(d) Now consider the following identity, valid for any $i$, where $\sigma_i$ is a real variable.

$$\int_{-\infty}^{\infty} d\sigma_i \exp\left[-\frac{1}{2}\left(\sigma_i - c(\eta^*_{1,i}\eta_{1,i} + \eta^*_{2,i}\eta_{2,i})\right)^2\right] = \sqrt{2\pi} \tag{12}$$

where $c$ is a constant. Using this identity, with an appropriate value of $c$, re-write the integral in Eq.(10) as an integral of the form

$$\begin{aligned}\mathcal{Z} \; &= \int_{-\infty}^{\infty} \mathcal{D}\sigma \exp(-S[\sigma]) \\ &\times \int \mathcal{D}\eta^*_1 \mathcal{D}\eta_1 \mathcal{D}\eta^*_2 \mathcal{D}\eta_2 \exp\left(-\frac{1}{2}\sum_{i,j=1}^{N_\tau} ({\eta^*}_{1,i}\widetilde{M}_{i,j}[\sigma]\eta_{1,j} + {\eta^*}_{2,i}\widetilde{M}_{i,j}[\sigma]\eta_{2,j})\right)\end{aligned} \tag{13}$$

Find $S[G]$ and $\widetilde{M}[\sigma]$. Do the integral over the Grassmann variables, expressing the final result as an integral that, though complicated, depends solely on real variables. Now you are ready for computers!

(e) Bonus: These are the simplest and most common forms of Hubbard-Stratonovich transformations. In the 80's Hirsch came up with a discrete flavor:

$$\exp(g\eta^*_{1,i}\eta_{1,i}\eta^*_{2,i}\eta_{2,i}) = \frac{1}{2}\sum_{\sigma_i=\pm 1}(1 + A\sigma_i\eta^*_{1,i}\eta_{1,i})(1 + A\sigma_i\eta^*_{2,i}\eta_{2,i}) \tag{14}$$

Given $g$, find the value of $A$ that satisfies the above identity.

(f) Bonus: Very recently Dean Lee (young faculty member at NCSU) derived yet another form that is very useful in practical simulations:

$$\exp(g\eta^*_{1,i}\eta_{1,i}\eta^*_{2,i}\eta_{2,i}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\sigma_i(1 + A\sin(\sigma_i)\eta^*_{1,i}\eta_{1,i})(1 + A\sin(\sigma_i)\eta^*_{2,i}\eta_{2,i}) \tag{15}$$

Prove this identity by finding the value of $A$ that makes it true.