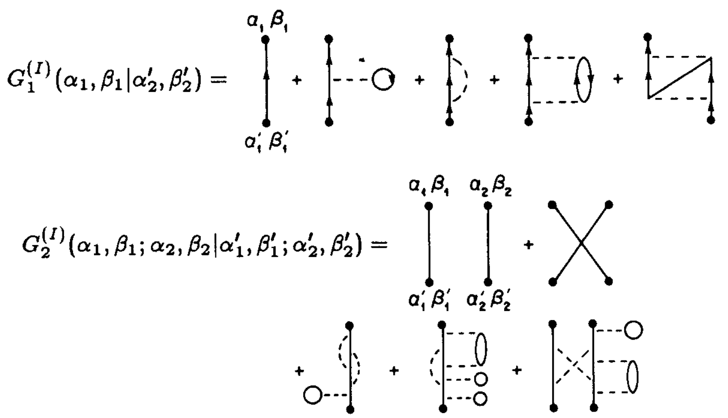
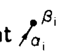
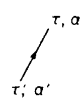



Fig. 2.7 Diagrams contributing to one- and two-particle Green's functions with overall sign factors.

2. Each external point  corresponds to a specified state $\alpha_i$ and time $\beta_i$. Assign an internal time label $\tau_i$ to each of the $r$ interaction vertices and for any propagator which is not connected to an external point assign an internal single-particle index. For each directed line include the factor

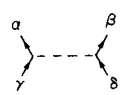



$$= \delta_{\alpha'\alpha} g_\alpha(\tau - \tau')$$

$$= \delta_{\alpha'\alpha} e^{-(\epsilon_\alpha - \mu)(\tau - \tau')}[(1 + \varsigma n_\alpha)\theta(\tau - \tau' - \eta) + \varsigma n_\alpha \theta(\tau' - \tau + \eta)]$$

where $\tau$ and $\tau'$ denote either internal times $\tau_i$ or external times $\beta_i$. Propagators connected to an interaction vertex will only have one single-particle index $\alpha$, in which case the factor $\delta_{\alpha'\alpha}$ is superfluous.

3. For each interaction vertex include the factor




$$= (\alpha\beta|v|\gamma\delta) \quad .$$

4. Sum over all internal single-particle indices and integrate the $r$ internal times $\tau_i$ over the interval $[0, \beta]$.
5. Multiply the result by the factor $(-1)^r \varsigma^P \varsigma^{n_L}$ where $n_L$ is the number of closed propagator loops and $\varsigma^P$ is the sign of the permutation $P$ such that each propagator line originating at the external point $\psi^*_{\alpha'_m}(\beta'_m)$ terminates at the external point $\psi_{\alpha_{P_m}}(\beta_{P_m})$.

Examples of graphs contributing to the one- and two-particle Green's functions, together with the overall sign factor are given in Fig. (2.7).



## 2.4 IRREDUCIBLE DIAGRAMS AND INTEGRAL EQUATIONS

With the foundations of diagrammatic perturbation theory presented in the preceding section, it is now possible to derive exact integral equations relating connected Green's functions and irreducible vertex functions. Since these equations include contributions of all orders of perturbation theory, they are useful in defining consistent approximations involving infinite resummations of diagrams and in treating the renormalization of divergent field theories. They also lead naturally to the effective potential, $\Gamma$, which will be useful in understanding spontaneous symmetry breaking in Chapter 3, and to the self energy, $\Sigma$, which governs the propagation of a single particle in a many-body medium.

### GENERATING FUNCTION FOR CONNECTED GREEN'S FUNCTIONS

As seen in the preceding sections, it is often useful to define a generating function by adding to the physical Hamiltonian under consideration additional terms in which field operators are coupled to external sources. The simplest possibility is to couple the field operators to an external source $J$ by adding the term

$$S = \sum_\alpha \int d\tau [(J^*_\alpha(\tau) a_\alpha(\tau) + a^\dagger_\alpha(\tau) J_\alpha(\tau))] \tag{2.148}$$

where the sources $J_\alpha(\tau)$ are complex or Grassman variables for Bosons and Fermions respectively. Other more general sources, such as the bilinear form

$$\begin{aligned} S = \sum_{\alpha\beta} \int d\tau d\tau' [& a^\dagger_\alpha(\tau_1) a^\dagger_\beta(\tau_2) \eta_{\alpha\beta}(\tau_1, \tau_2) \\ & + a^\dagger_\alpha(\tau_1) a_\beta(\tau_2) \bar{\eta}_{\alpha\beta}(\tau_1, \tau_2) + a_\alpha(\tau_1) a_\beta(\tau_2) \eta^*_{\alpha\beta}(\tau_1, \tau_2)] \end{aligned} \tag{2.149}$$

are useful for specific applications.

Similar to the generating function, Eq. (2.85) used to derive Wick's theorem, the generating function for imaginary-time Green's Function is defined as the partition function for the full Hamiltonian plus the source term Eq. (2.148). Using the functional integral representation for the partition function, Eq. (2.64), the generating function may be written

$$\begin{aligned} \mathcal{G}(J^*_\alpha(\tau), J_\alpha(\tau)) &\equiv \frac{1}{Z} \int \mathcal{D}[\psi^*_\alpha(\tau) \psi_\alpha(\tau)] e^{-\int_0^\beta d\tau [\sum_\alpha \psi_\alpha(\tau)(\partial_\tau - \mu)\psi_\alpha(\tau) + H[\psi^*_\alpha(\tau), \psi_\alpha(\tau)]]} \\ &\quad \times e^{-\int_0^\beta d\tau \sum_\alpha [J^*_\alpha(\tau)\psi_\alpha(\tau) + \psi^*_\alpha(\tau) J_\alpha(\tau)]} \\ &= \langle e^{-\int_0^\beta d\tau \sum_\alpha [J^*_\alpha(\tau)\psi_\alpha(\tau) + \psi^*_\alpha(\tau) J_\alpha(\tau)]} \rangle \end{aligned} \tag{2.150a}$$

where, analagous to the thermal average with respect to $H_0$ in Eq. (2.81), we define the thermal average with respect to the full Hamitonian $H$ as

$$\begin{aligned}\langle F(\psi^*,\psi)\rangle \\ = \frac{1}{Z}\int \mathcal{D}[\psi^*_\alpha(\tau)\psi_\alpha(\tau)]F(\psi^*_\alpha,\psi_\alpha)e^{-\int_0^\beta d\tau[\sum_\alpha \psi_\alpha(\tau)(\partial_\tau-\mu)\psi_\alpha(\tau)+H[\psi^*_\alpha(\tau),\psi_\alpha(\tau)]]} .\end{aligned} \tag{2.150b}$$

Differentiation with respect to the sources $J^*_\alpha(\tau)$ and $J_\alpha(\tau)$ yields

$$\frac{\delta \mathcal{G}(J^*_\alpha(\tau),J_\alpha(\tau))}{\delta J^*_{\alpha_1}(\tau_1)} = -\langle\psi_{\alpha_1}(\tau_1)e^{-\int_0^\beta d\tau\sum_\alpha[J^*_\alpha(\tau)\psi_\alpha(\tau)+\psi^*_\alpha(\tau)J_\alpha(\tau)]}\rangle \tag{2.151a}$$

and

$$\frac{\delta \mathcal{G}(J^*_\alpha(\tau),J_\alpha(\tau))}{\delta J_{\alpha_1}(\tau_1)} = -\varsigma\langle\psi^*_{\alpha_1}(\tau_1)e^{-\int_0^\beta d\tau\sum_\alpha[J^*_\alpha(\tau)\psi_\alpha(\tau)+\psi^*_\alpha(\tau)J_\alpha(\tau)]}\rangle \tag{2.151b}$$

so that the $n$-particle imaginary-time Green's function may be written

$$\begin{aligned}\mathcal{G}^{(n)}(\alpha_1,\tau_1;\ldots\alpha_n,\tau_n|\alpha'_1,\tau'_1;\ldots;\alpha'_n,\tau'_n) \\ = \frac{1}{\varsigma^n}\left.\frac{\delta^{2n}\mathcal{G}(J^*_\alpha(\tau),J_\alpha(\tau))}{\delta J^*_{\alpha_1}(\tau_1)\ldots\delta J^*_{\alpha_n}(\tau_n)\delta J_{\alpha'_n}(\tau'_n)\ldots\delta J_{\alpha'_1}(\tau'_1)}\right|_{J=J^*=0} .\end{aligned} \tag{2.152}$$

Although the diagrams for Green's functions derived in Section 2.3 have all the interaction vertices linked to the external legs, the diagrams are not all connected. For example, the first four diagrams in Fig. (2.7) for $\mathcal{G}^{(2)}$ are disconnected; that is, they may be separated into distinct subdiagrams in which the external points and vertices of one subdiagram are not connected by any interactions or propagators to any other subdiagrams. Since the sum of all disconnected diagrams simply corresponds to combinations of products of fewer-particle Green's functions, it is useful to deal with connected Green's functions, where $\mathcal{G}_c^{(n)}(\alpha_1,\tau_1;\ldots\alpha_n,\tau_n|\alpha'_1,\tau'_1;\ldots\alpha'_n,\tau'_n)$ is defined as the sum of all connected diagrams linked to the external points $(\alpha_1,\tau_1;\ldots\alpha_n,\tau_n)$ and $(\alpha'_1,\tau'_1;\ldots\alpha'_n,\tau'_n)$. The last diagram in Fig. (2.7) is an example of a contribution to $\mathcal{G}_c^{(2)}$.

The generating function for connected Green's Functions, which we shall denote $W(J^*_\alpha(\tau),J_\alpha(\tau))$, may be obtained from the generating function $\mathcal{G}(J^*_\alpha(\tau),J_\alpha(\tau))$ by using the replica technique once again. The functional $[\mathcal{G}(J^*_\alpha(\tau),J_\alpha(\tau))]^p$ may be written as a functional integral over $p$ distinct fields $\{\psi^{*\sigma}_\alpha,\psi^\sigma_\alpha\}$ and the resulting Green's function diagrams will have the property that all connected diagrams will be proportional to $p$ and all disconnected diagrams will contain at least two factors of $p$. The terms proportional to $p$ are singled out by

$$W(J^*_\alpha(\tau),J_\alpha(\tau)) = \lim_{p\to 0}\frac{\partial}{\partial p}(\mathcal{G}(J^*_\alpha(\tau),J_\alpha(\tau))^p$$



so that

$$W(J^*_\alpha(\tau),J_\alpha(\tau)) = \ln\ \mathcal{G}(J^*_\alpha(\tau),J_\alpha(\tau)) \tag{2.153}$$

and

$$\begin{aligned}\mathcal{G}_c^{(n)}(\alpha_1,\tau_1;\ldots\alpha_n,\tau_n|\alpha'_1,\tau'_1;\ldots\alpha'_n,\tau'_n) \\ = \varsigma^n\left.\frac{\delta^{2n}W(J^*_\alpha(\tau),J_\alpha(\tau))}{\delta_\alpha J^*_{\alpha_1}(\tau_1)\ldots\delta J^*_{\alpha_n}(\tau_n)\delta J_{\alpha'_n}(\tau'_n)\ldots\delta J_{\alpha'_1}(\tau'_1)}\right|_{J^*=J=0} .\end{aligned} \tag{2.154}$$

Note that by Eqs. (2.150) and (2.153), $W(J^*,J) = -\beta(\Omega(J^*,J)-\Omega(0,0))$ so that physically, $W$ represents the difference between the grand canonical potential in the presence and absence of sources.

The structure of the terms produced by this generating function is illustrated by the examples of one- and two-particle connected Green's functions. Abbreviating $\sum_\alpha\int d\tau[J^*_\alpha(\tau)\psi_\alpha(\tau)+\psi^*_\alpha(\tau)J_\alpha(\tau)]$ by $J^*\psi+\psi^*J$ and $\{J^*_{\alpha_1}(\tau_1),\psi^*_{\alpha_1}(\tau_1),J_{\alpha'_1}(\tau'_1),\psi_{\alpha'_1}(\tau'_1)\}$ by $\{J^*_1,\psi^*_1,J_{1'},\psi_{1'}\}$ in obvious notation, we obtain:

$$\begin{aligned}\mathcal{G}_c^{(n)}(1|1') &= \varsigma\frac{\delta^2}{\delta J^*_1\delta J_{1'}}\left[\ln\langle e^{-(J^*\psi+\psi^*J)}\rangle\right]_{J=J^*=0} \\ &= -\frac{\delta}{\delta J^*_1}\left[\langle e^{-(J^*\psi+\psi^*J)}\rangle^{-1}\langle\psi^*_{1'}e^{-(J^*\psi+\psi^*J)}\rangle\right]_{J=J^*=0} \\ &= \Big[\langle e^{-(J^*\psi+\psi^*J)}\rangle^{-1}\langle\psi_1\psi^*_{1'}e^{-(J^*\psi+\psi^*J)}\rangle \\ &\quad -\langle e^{-(J^*\psi+\psi^*J)}\rangle^{-2}\langle\psi_1 e^{-(J^*\psi+\psi^*J)}\rangle\langle\psi^*_{1'}e^{-(J^*\psi+\psi^*J)}\rangle\Big]_{J=J^*=0} .\end{aligned} \tag{2.155}$$

Sucessive derivatives act on both the numerator and denominator of each term, yielding the familiar structure of a cumulant expansion in which all possible linked and unlinked combinations of $n\psi$'s and $n\psi^*$'s are generated. Note that because of the presence of Grassman variables, the order of the factors is important in calculating derivatives. Except in the presence of spontaneous symmetry breaking, as arises for example in the case of Bose condensation treated in Section 2.5, all expectation values involving unequal numbers of creation and annihilation operators vanish, greatly simplifying the general result. For the one-body connected Green's Function in the absence of symmetry breaking we thus obtain:

$$\mathcal{G}_c^{(1)}(1|1') = \langle\psi_1\psi^*_{1'}\rangle = \mathcal{G}^{(1)}(1,1') . \tag{2.156}$$

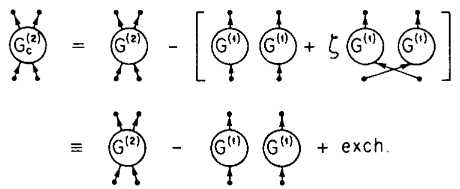
The two-body connected Green's Function is evaluated similarly:

$$\begin{aligned}\mathcal{G}_c^{(2)}(1,2|1',2') &= \frac{\delta^4}{\delta J^*_1\delta J^*_2\delta J_{2'}\delta J_{1'}}\left[\ln\langle e^{-(J^*\psi+\psi^*J)}\rangle\right]_{J=J^*=0} \\ &= \frac{\delta^2}{\delta J^*_1\delta J^*_2}\Big[\langle e^{-(J^*\psi+\psi^*J)}\rangle^{-1}\langle\psi^*_{2'}\psi^*_{1'}e^{-(J^*\psi+\psi^*J)}\rangle \\ &\quad -\langle e^{-(J^*\psi+\psi^*J)}\rangle^{-2}\langle\psi^*_{2'}e^{-(J^*\psi+\psi^*J)}\rangle\langle\psi^*_{1'}e^{-(J^*\psi+\psi^*J)}\rangle\Big]_{J=J^*=0} .\end{aligned} \tag{2.157}$$

The only non-vanishing terms generated by the two remaining derivatives, assuming no symmetry breaking, are those in which derivatives acting on numerator terms add two $\psi$'s to $\langle\psi^*\psi^*\rangle$ and one $\psi$ to each of the factors $\langle\psi^*\rangle$, yielding

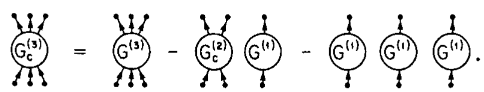
$$\begin{aligned}\mathcal{G}_c^{(2)}(1,2|1',2') &= \langle\psi_1\psi_2\psi^*_{2'}\psi^*_{1'}\rangle - \langle\psi_2\psi^*_{2'}\rangle\langle\psi_1\psi^*_{1'}\rangle - \varsigma\langle\psi_1\psi^*_{2'}\rangle\langle\psi_2\psi^*_{1'}\rangle \\ &= \mathcal{G}^{(2)}(1,2|1'2') - \left[\mathcal{G}^{(1)}(1|1')\mathcal{G}^{(1)}(2|2') + \varsigma\mathcal{G}^{(1)}(1|2')\mathcal{G}^{(1)}(2|1')\right] \quad .\end{aligned} \tag{2.158a}$$

Note that the factor $\varsigma$ in the last term arises from the fact that $\frac{\delta}{\delta J_2^*}$ had to be commutted through an odd number of Grassman variables to act on $\langle\psi_1^* e^{-(J^*\psi+\psi^*J)}\rangle$. A compact graphical representation of Eq. (2.158a) is



(2.158b)

where ⁄ denotes an internal propagator connected to an external point $\psi_\alpha(\tau)$ and the abbreviation *exch.* for exchange indicates the sum of all possible permutations, $P$, of the external points, in this case 2, with associated factor $\varsigma^P$. Higher order connected Green's functions are evaluated similarly, and it is straightforward to verify that the three-body connected Green's function satisfies the following equation, which is also evident from the diagram rules for Green's functions;



(2.159)

Since the Green's functions only involve equal numbers of $\psi$ and $\psi^*$, a generating function with a bilinear source, Eq. (2.149), may also be used, and it is an instructive exercise to reproduce the preceding results with a bilinear source (see Problem 2.12).

### THE EFFECTIVE POTENTIAL

In the presence of sources, the operators $\{a^\dagger_\alpha(\tau), a_\alpha(\tau)\}$ acquire non-zero expectation values. Let us define the average field

$$\begin{aligned}\phi_\alpha &= \langle a_\alpha\rangle_{J^*,J} \\ &= \langle\psi_\alpha\rangle_{J^*,J} \\ &= \frac{\int \mathcal{D}[\psi^*_\alpha\psi_\alpha]\psi_\alpha e^{-\int_0^\beta d\tau[\sum_\alpha\psi_\alpha(\partial_\tau-\mu)\psi_\alpha + H[\psi^*_\alpha\psi_\alpha] + \sum_\alpha[J^*_\alpha\psi_\alpha+\psi^*_\alpha J_\alpha]]}}{\int \mathcal{D}[\psi^*_\alpha\psi_\alpha] e^{-\int_0^\beta d\tau[\sum_\alpha\psi_\alpha(\partial_\mu-\mu)\psi_\alpha + H[\psi^*_\alpha\psi_\alpha] + \sum_\alpha[J^*_\alpha\psi_\alpha+\psi^*_\alpha J_\alpha]]}} \\ &= -\frac{\delta}{\delta J^*_\alpha(\tau)} W[J^*_\alpha(\tau), J_\alpha(\tau)]\end{aligned} \tag{2.160a}$$



and its complex conjugate field

$$\begin{aligned}\phi^*_\alpha(\tau) &= \langle a^\dagger_\alpha(\tau)\rangle_{J^*,J} \\ &= -\varsigma\frac{\delta}{\delta J_\alpha(\tau)} W[J^*_\alpha(\tau), J_\alpha(\tau)] \quad .\end{aligned} \tag{2.160b}$$

Here, in an obvious notation $\langle\ \ \rangle_{J^*,J}$ denotes a thermal average with respect to $H$ plus the source term of the form introduced in Eqs. (2.80) - (2.82) for $H_0$ and in Eq. (2.150b) for $H$ and the $\tau$ arguments of $\psi_\alpha$ and $J_\alpha$ have been suppressed for compactness.

Instead of dealing with the generating function $W$ as a function of the sources $J^*, J$, it is useful to perform a Legendre transformation to obtain a function of the fields $\phi^*, \phi$. The motivation for performing this transformation is evident from considering the familiar example of a spin system in a magnetic field, which will be treated in detail in Chapter 4. Denoting the Hamiltonian of the spin variable as $\mathcal{H}(s)$, the free energy as a function of an external magnetic field $\vec{H}$ is given by

$$\mathrm{Tr}\, e^{-\beta(\mathcal{H}(s)-\vec{H}\cdot\sum_i\vec{s}_i)} = e^{-\beta F(H)} \tag{2.161a}$$

from which it follows that the magnetization is given by

$$M = -\frac{\partial F(H)}{\partial H} \quad . \tag{2.161b}$$

A state function which depends upon the magnetization instead of the external magnetic field is obtained by inverting the relation (2.161b) to obtain $H(M)$ and defining the Legendre transform

$$G(M) = F(H(M)) + MH(M) \quad . \tag{2.162}$$

It follows that $G$ satisfies the reciprocity relation

$$\frac{\partial G}{\partial M} = \frac{\partial F}{\partial H}\frac{\partial H}{\partial M} + H + M\frac{\partial H}{\partial M} = H \quad . \tag{2.163}$$

Whereas both $F(H)$ and $G(M)$ contain the same physical information, in the case of the first-order ferromagnetic phase transition, $G(M)$ has better analytic properties and will thus be preferable to approximate. Below the critical temperature, since $M$ is non-vanishing and has the same sign as $H$, equation (2.161b) implies that $F(H)$ has non-zero positive slope for positive $H$ and a non-zero negative slope for negative $H$ so that it has a cusp at the origin. In contrast, $G(M)$ is a smooth double well, and the discontinuity associated with the first-order phase transition arises from moving from the solution to Eq. (2.163) in one well to the solution in the second well.

In the case of the generating function $W[J^*_\alpha(\tau), J_\alpha(\tau)]$, the equations (2.160) for $\phi_\alpha(J^*_\alpha, J_\alpha)$ and $\phi^*_\alpha(J^*_\alpha, J_\alpha)$ are inverted to obtain the sources as functions of the fields $J^*_\alpha(\phi^*_\alpha, \phi_\alpha)$ and $J_\alpha(\phi^*_\alpha, \phi_\alpha)$ and the effective potential (or effective action) is defined as the Legendre transform

$$\Gamma[\phi^*_\alpha(\tau), \phi_\alpha(\tau)] = -W[J^*_\alpha(\tau), J_\alpha(\tau)] - \sum_\gamma\int_0^\beta d\tau'[\phi^*_\gamma(\tau')J_\gamma(\tau') + J^*_\gamma(\tau')\phi_\gamma(\tau')] \quad . \tag{2.164}$$

As in the example $G(M)$, the effective potential satisfies the reciprocity relation

$$\frac{\partial}{\partial\phi^*_\alpha(\tau)}\Gamma[\phi^*_\alpha(\tau),\phi_\alpha(\tau)] = \sum_\gamma\int_0^\beta d\tau'\left[-\frac{\partial W}{\partial J^*_\gamma(\tau')}\frac{\partial J^*_\gamma(\tau')}{\partial\phi^*_\alpha(\tau)} - \frac{\partial W}{\partial J_\gamma(\tau')}\frac{\partial J_\gamma(\tau')}{\partial\phi^*_\alpha(\tau)}\right.$$
$$\left. -\delta_{\alpha\gamma}\delta(\tau-\tau')J_\gamma(\tau') - \varsigma\phi^*_\gamma(\tau')\frac{\partial J_\gamma(\tau')}{\partial\phi^*_\alpha(\tau)} - \frac{\partial J^*_\gamma(\tau')}{\partial\phi^*_\alpha(\tau)}\phi_\gamma(\tau')\right]$$
$$= -J_\alpha(\tau) \tag{2.165a}$$

and the companion equation

$$\frac{\partial}{\partial\phi_\alpha(\tau)}\Gamma[\phi^*_\alpha(\tau),\phi_\alpha(\tau)] = -\varsigma J^*_\alpha(\tau)\quad . \tag{2.165b}$$

When the sources are set equal to zero, Eqs. (2.165) show that the effective potential is stationary. That is, if we denote the fields in the absence of sources by $\tilde\phi^*_\alpha(\tau)$ and $\tilde\phi_\alpha(\tau)$, then

$$\frac{\delta\Gamma(\tilde\phi^*_\alpha(\tau),\tilde\phi_\alpha(\tau))}{\delta\tilde\phi^*_\alpha(\tau)} = \frac{\delta\Gamma(\tilde\phi^*_\alpha(\tau),\tilde\phi_\alpha(\tau))}{\delta\tilde\phi_\alpha(\tau)} = 0\quad . \tag{2.166}$$

As we have already mentioned, in the case of Bose condensation Eqs. (2.166) have non-zero solutions $\{\tilde\phi^*_\alpha(\tau),\tilde\phi_\alpha(\tau)\}$ and these solutions will be studied in Section 2.5. In the absence of symmetry breaking, the fields $\{\tilde\phi^*_\alpha(\tau),\tilde\phi_\alpha(\tau)\}$ are zero and all Green's functions which do not have equal numbers of creation and annihilation operators vanish.

The effective potential is a generating function for vertex functions. These vertex functions are generated by differentiating the effective potential $\Gamma[\phi^*_\alpha(\tau)\phi_\alpha(\tau)]$ in the same way as connected Green's functions are generated from $W[J^*_\alpha(\tau)J_\alpha(\tau)]$:

$$\Gamma_{m\phi^*,n\phi}(\alpha_1\tau_1,\ldots\alpha_m\tau_m|\alpha'_1\tau'_1,\ldots\alpha'_n\tau'_n) \tag{2.167}$$
$$= \left.\frac{\delta^{m+n}}{\delta\phi^*_{\alpha_1}(\tau_1)\ldots\delta\phi^*_{\alpha_m}(\tau_m)\delta\phi_{\alpha'_n}(\tau'_n)\ldots\delta\phi_{\alpha'_1}(\tau'_1)}\Gamma[\phi^*_\alpha(\tau),\phi_\alpha(\tau)]\right|_{J^*_\alpha=J_\alpha=0}\quad .$$

Note that evaluation at $J^*_\alpha = J_\alpha = 0$ is equivalent to evaluation at the stationary solutions $\{\tilde\phi^*_\alpha,\tilde\phi_\alpha\}$ of Eq. (2.166).

The vertex functions $\Gamma_{m\phi^*,n\phi}$ defined in this way have several important properties. One feature is that they are one-particle irreducible and thus cannot be disconnected by removing a single internal propagator. Another significant property is the fact that connected Green's functions may be constructed from vertex functions using only tree diagrams, that is, diagrams containing no closed propagator loops. This property is extremely useful in renormalization of field theories, since all the divergences arise from loop integrals which are isolated in the vertex functions $\Gamma$, and in the definition of consistent truncated expansions. These properties are most easily seen by deriving a hierarchy of integral equations satisfied by the vertex functions and Green's functions.



### THE SELF-ENERGY AND DYSON'S EQUATION

Before proceeding to the general case, it is useful to study the vertex function $\Gamma_{\phi^*\phi}$. An economical method to derive the desired integral equations for vertex functions is to take successive derivatives of the generating function $W[J^*,J]$ with respect to $\phi_{\alpha'}(\tau')$ and $\phi^*_{\alpha'}(\tau')$ using the chain rule and Eqs. (2.165) to write functional derivatives of a functional $F$

$$\frac{\delta F(J^*,J)}{\delta\phi_{\alpha_1}(\tau_1)} = \sum_{\alpha_2}\int_0^\beta d\tau_2\left[\frac{\delta F}{\delta J^*_{\alpha_2}(\tau_2)}\frac{\delta J^*_{\alpha_2}(\tau_2)}{\delta\phi_{\alpha_1}(\tau_1)} + \frac{\delta F}{\delta J_{\alpha_2}(\tau_2)}\frac{\delta J_{\alpha_2}(\tau_2)}{\delta\phi_{\alpha_1}(\tau_1)}\right]$$
$$= \sum_{\alpha_2}\int_0^\beta d\tau_2\left[-\varsigma\frac{\delta F}{\delta J^*_{\alpha_2}(\tau_2)}\frac{\delta^2\Gamma}{\delta\phi_{\alpha_1}(\tau_1)\delta\phi_{\alpha_2}(\tau_2)}\right.$$
$$\left. - \frac{\delta F}{\delta J_{\alpha_2}(\tau_2)}\frac{\delta^2\Gamma}{\delta\phi_{\alpha_1}(\tau_1)\delta\phi^*_{\alpha_2}(\tau_2)}\right] \tag{2.168a}$$

and similarly

$$\frac{\delta F(J^*,J)}{\delta\phi^*_{\alpha_1}(\tau'_1)} = \sum_{\alpha_2}\int_0^\beta d\tau_2\left[-\varsigma\frac{\delta F}{\delta J^*_{\alpha_2}(\tau_2)}\frac{\delta^2\Gamma}{\delta\phi^*_{\alpha_1}(\tau_1)\delta\phi_{\alpha_2}(\tau_2)}\right.$$
$$\left. - \frac{\delta F}{\delta J_{\alpha_2}(\tau_2)}\frac{\delta^2\Gamma}{\delta\phi^*_{\alpha_1}(\tau_1)\delta\phi^*_{\alpha_2}(\tau_2)}\right]\quad . \tag{2.168b}$$

The lowest order equation, a general matrix form of Dyson's equation, is obtained by differentiating each of the quantities $\phi_{\alpha_3}(\tau_3) = -\frac{\delta W}{\delta J^*_{\alpha_3}(\tau_3)}$ and $\phi^*_{\alpha_3}(\tau_3) = -\varsigma\frac{\delta W}{\delta J_{\alpha_3}(\tau_3)}$ with respect to $\phi_{\alpha_1}(\tau_1)$ and $\phi^*_{\alpha_1}(\tau_1)$. Calculating $\frac{\delta\phi_{\alpha_3}(\tau_3)}{\delta\phi_{\alpha_1}(\tau_1)}$ in detail, we obtain

$$\delta_{\alpha_3\alpha_1}\delta(\tau_3,\tau_1) = \frac{\delta\phi_{\alpha_3}(\tau_3)}{\delta\phi_{\alpha_1}(\tau_1)} = \frac{\delta}{\delta\phi_{\alpha_1}(\tau_1)}\left[-\frac{\delta W}{\delta J^*_{\alpha_3}(\tau_3)}\right]$$
$$= \sum_{\alpha_2}\int_0^\beta d\tau_2\left[\varsigma\frac{\delta^2 W}{\delta J^*_{\alpha_2}(\tau_2)\delta J^*_{\alpha_3}(\tau_3)}\frac{\delta^2\Gamma}{\delta\phi_{\alpha_1}(\tau_1)\delta\phi_{\alpha_2}(\tau_2)}\right.$$
$$\left. + \frac{\delta^2 W}{\delta J_{\alpha_2}(\tau_2)\delta J^*_{\alpha_3}(\tau_3)}\frac{\delta^2\Gamma}{\delta\phi_{\alpha_1}(\tau_1)\delta\phi^*_{\alpha_2}(\tau_2)}\right] \tag{2.169}$$

which may be rewritten in the more compact notation

$$\delta(31) = \int d2\left[\varsigma\frac{\delta^2 W}{\delta J^*(2)\delta J^*(3)}\frac{\delta^2\Gamma}{\delta\phi^*(1)\delta\phi(2)} + \frac{\delta^2 W}{\delta J(2)\delta J^*(3)}\frac{\delta^2\Gamma}{\delta\phi(1)\delta\phi^*(2)}\right] \tag{2.170a}$$

where 1 denotes the variables $\{\alpha_1,\tau_1\}$ and $\int d2$ implies a sum over $\alpha_2$ and an integral over $\tau_2$. The remaining three derivatives yield the equations

$$\delta(31) = \frac{\delta}{\delta\phi^*(1)}\phi^*(3) = \frac{\delta}{\delta\phi^*(1)}\left[-\varsigma\frac{\delta W}{\delta J(3)}\right] \tag{2.170b}$$
$$= \int d2\left[\frac{\delta^2 W}{\delta J^*(2)\delta J(3)}\frac{\delta^2\Gamma}{\delta\phi^*(1)\delta\phi(2)} + \varsigma\frac{\delta^2 W}{\delta J(2)\delta J(3)}\frac{\delta^2\Gamma}{\delta\phi^*(1)\delta\phi^*(2)}\right]$$

$$\begin{aligned}0 &= \frac{\delta}{\delta\phi^*(1)}\phi(3) = \frac{\delta}{\delta\phi^*(1)}[-\frac{\delta W}{\delta J^*(3)}] \\ &= \int d2\left[\varsigma\frac{\delta^2 W}{\delta J^*(2)\delta J^*(3)}\frac{\delta^2\Gamma}{\delta\phi^*(1)\delta\phi(2)} + \frac{\delta^2 W}{\delta J(2)\delta J^*(3)}\frac{\delta^2\Gamma}{\delta\phi^*(1)\delta\phi^*(2)}\right]\end{aligned} \tag{2.170c}$$

and

$$\begin{aligned}0 &= \frac{\delta}{\delta\phi(1)}\phi^*(3) = \frac{\delta}{\delta\phi(1)}\left[-\varsigma\frac{\delta W}{\delta J(3)}\right] \\ &= \int d2\left[\frac{\delta^2 W}{\delta J^*(2)\delta J(3)}\frac{\delta^2\Gamma}{\delta\phi(1)\delta\phi(2)} + \varsigma\frac{\delta^2 W}{\delta J(2)\delta J(3)}\frac{\delta^2\Gamma}{\delta\phi(1)\delta\phi^*(2)}\right] .\end{aligned} \tag{2.170d}$$

The equations (2.170) may be expressed in the matrix form

$$\int d2\begin{pmatrix}\frac{\delta^2 W}{\delta J^*(3)\delta J(2)} & \varsigma\frac{\delta^2 W}{\delta J^*(3)\delta J^*(2)} \\ \varsigma\frac{\delta^2 W}{\delta J(3)\delta J(2)} & \frac{\delta^2 W}{\delta J(3)\delta J^*(2)}\end{pmatrix}\begin{pmatrix}\frac{\delta^2\Gamma}{\delta\phi^*(2)\delta\phi(1)} & \frac{\delta^2\Gamma}{\delta\phi^*(2)\delta\phi^*(1)} \\ \frac{\delta^2\Gamma}{\delta\phi(2)\delta\phi(1)} & \frac{\delta^2\Gamma}{\delta\phi(2)\delta\phi^*(1)}\end{pmatrix} = \delta(31)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{2.171}$$

which shows that the matrix of second derivatives of $\Gamma$ is the inverse of the matrix of second derivatives of $W$. Thus, the matrix composed of $\Gamma_{\phi^*\phi}, \Gamma_{\phi^*\phi^*}, \Gamma_{\phi\phi}$ and $\Gamma_{\phi\phi}$ is the inverse of the matrix of connected Green's functions

$$\begin{pmatrix}\Gamma_{\phi^*\phi} & \Gamma_{\phi^*\phi^*}\\ \Gamma_{\phi\phi} & \Gamma_{\phi\phi^*}\end{pmatrix} = \varsigma\begin{pmatrix}\langle\psi\psi^*\rangle & \langle\psi\psi\rangle\\ \langle\psi^*\psi^*\rangle & \langle\psi^*\psi\rangle\end{pmatrix}^{-1} . \tag{2.172}$$

In order to understand the properties and physical significance of $\Gamma_{\phi^*\phi}$, we now consider a system with no symmetry breaking, in which case Green's functions with unequal numbers of $\psi^*$'s and $\psi$'s vanish and Eqs. (2.170– 2.172) simplify to

$$\int d2\,\mathcal{G}_c^{(1)}(1,2)\Gamma_{\phi^*\phi}(2,3) = \int d2\,\Gamma_{\phi^*\phi}(1,2)\mathcal{G}_c^{(1)}(2,3) = \delta(1,3) \tag{2.173}$$

and

$$\Gamma_{\phi^*\phi}(1,2) = [\mathcal{G}_c^{(1)}]^{-1}(1,2) . \tag{2.174}$$

For a non-interacting system, rewriting Eq. (2.78) for the single-particle Green's functions with explicit $\{\alpha,\tau\}$ dependence yields

$$\sum_{\alpha_2}\left(\delta_{\alpha_1\alpha_2}\left(\frac{\partial}{\partial\tau_1}-\mu\right) + \langle\alpha_1|H_0|\alpha_2\rangle\right)\mathcal{G}_{0,c}^{(1)}(\alpha_2,\tau_1|\alpha_3,\tau_3) = \delta_{\alpha_1\alpha_3}\delta(\tau_1-\tau_3) \tag{2.175}$$

where $H_0$ is the single-particle Hamiltonian. Thus, for a non-interacting system

$$\begin{aligned}\Gamma_{\phi^*\phi}^{(0)}(\alpha_1,\tau_1|\alpha_2,\tau_2) &= [\mathcal{G}_{0,c}^{(1)}]^{-1}(\alpha_1\tau_1|\alpha_2\tau_2) \\ &= \left(\delta_{\alpha_1\alpha_2}\left(\frac{\partial}{\partial\tau_1}-\mu\right) + \langle\alpha_1|H_0|\alpha_2\rangle\right)\delta(\tau_1-\tau_2) .\end{aligned} \tag{2.176}$$



The one-particle vertex function is a special case, and rather than deal with it directly, it is conventional to define the self energy $\Sigma$ as the difference between the vertex function for the interacting and non-interacting systems

$$\Gamma_{\phi^*\phi}(1,2) \equiv \Gamma^{(0)}_{\phi^*\phi}(1,2) + \Sigma(1,2) . \tag{2.177}$$

Simplifying the notation by writing $\mathcal{G}_c^{(1)}(1,2)$ as the matrix $\mathcal{G}$, with the arguments and labels suppressed, equation (2.177) may be expressed

$$\mathcal{G}^{-1} = [\mathcal{G}_0]^{-1} + \Sigma \tag{2.178}$$

which when multiplied on the left by $\mathcal{G}_0$ and on the right by $\mathcal{G}$ yields the Dyson equation

$$\begin{aligned}\mathcal{G} &= \mathcal{G}_0 - \mathcal{G}_0\Sigma\mathcal{G} \\ &= \mathcal{G}_0 - \mathcal{G}_0\Sigma\mathcal{G}_0 + \mathcal{G}_0\Sigma\mathcal{G}_0\Sigma\mathcal{G}_0\ldots\end{aligned} \tag{2.179a}$$

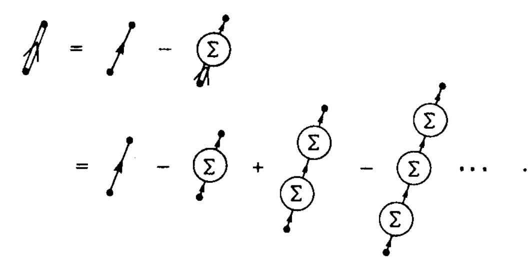
or, exhibiting the explicit $\{\alpha,\tau\}$ dependence,

$$\begin{aligned}\mathcal{G}_c^{(1)}(\alpha_1\tau_1|\alpha_4\tau_4) &= \mathcal{G}_{0,c}^{(1)}(\alpha_1,\tau_1|\alpha_4,\tau_4) \\ &\quad - \sum_{\alpha_2\alpha_3}\int_0^\beta d\tau_2\,d\tau_3\,\mathcal{G}_{0,c}^{(1)}(\alpha_1,\tau_1|\alpha_2\tau_2)\Sigma(\alpha_2\tau_2,\alpha_3\tau_3)\mathcal{G}_c^{(1)}(\alpha_3,\tau_3|\alpha_4,\tau_4) .\end{aligned} \tag{2.179b}$$

The graphical expansion of the self-energy $\Sigma$ is evident from expressing the Dyson equation and its series expansion in diagrams. With the following graphical notation for the one-particle Green's functions non-interacting Green's functions, and self energy,

 $= \mathcal{G}_c^{(1)}(2,1)$  $= \mathcal{G}_{0,c}^{(1)}(2,1)$  $= \Sigma(2,1)$

equation (2.179a) yields

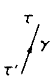



(2.179c)

The following two definitions are required to specify the diagrams contributing to the self energy $\Sigma$. A diagram is $n$-particle irreducible if it cannot be separated into two or more disconnected diagrams by cutting $n$ internal propagators. An amputated diagram attached to the points $(\alpha_1\tau_1), (\alpha_2\tau_2), \ldots (\alpha_n\tau_n)$ has no free propagator $G^{(0)}$ attached to these points; hence each point must connect directly to an interaction vertex. With these definitions, it is evident from (2.179c) that $-\Sigma(\alpha_2\tau_2, \alpha_1\tau_1)$ is the sum of all one-particle irreducible amputated diagrams connecting the points $(\alpha_1, \tau_1)$ and $(\alpha_2, \tau_2)$. To see this, imagine enumerating all the diagrams for the one-particle Green's function according to the rules given in Section 2.3. Amputate the external propagators, and let the set of one-particle irreducible diagrams define $-\Sigma$. Finally, consider the set of graphs generated by substituting this set of diagrams into the last line of Eq. (2.179c). By construction, each original Green's function diagram which could have been disconnected by cutting $n$ different propagators will be generated once and only once by the $n$th term in this expansion (2.179c)

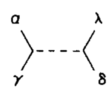
The rules for calculating the self energy $\Sigma(\alpha\beta, \alpha'\beta')$ thus follow directly from thoses enumerated in Section 2.4 for the one-particle Green's functions. The rules for the $n$th order contribution using unlabeled Feynman diagrams are summarized as follows:

1. Draw all distinct, unlabeled, one-particle irreducible, amputated diagrams composed of $n$ interaction vertices )----( with the label $\{\alpha, \beta\}$ assigned to one outgoing arrow of an interaction vertex, the label $\{\alpha', \beta'\}$ assigned to one ingoing arrow of an interaction vertex, and all other arrows of the interaction vertices connected by directed lines. Two diagrams are distinct if, holding the points $\{\alpha, \beta\}$ and $\{\alpha', \beta'\}$ fixed, the lines and propagators cannot be deformed to coincide completely including the direction of arrows on propagators. The contribution for each distinct unlabeled diagram is evaluated as follows:
2. Assign an internal time label $\tau_i$ to each interaction vertex which is not assigned to one of the external time values $\beta$ or $\beta'$. For every directed line, assign a single-particle index $\gamma$ and include the factor



$$= g_\gamma(\tau - \tau') = e^{-(\epsilon_\gamma - \mu)(\tau - \tau')}[(1 + \varsigma n_\gamma)\theta(\tau - \tau' - \eta) + \varsigma n_\gamma \theta(\tau' - \tau + \eta)]$$

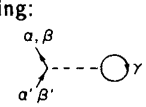
   where $\tau, \tau'$ denote either internal or external times.
3. For each interaction vertex include the factor



$$= (\alpha\lambda|v|\gamma\delta) \ .$$

   Note that if the external points $\{\alpha, \beta\}$ and $\{\alpha', \beta'\}$) are associated with the same interaction vertex, since the interaction is instantaneous the factor $\delta(\beta - \beta')$ must also be included.
4. Sum over all internal single-particle indices and integrate all internal times $\tau_i$ over the interval $[0, \beta]$.
5. Multiply the result by the factor $(-1)^{n-1}\varsigma^{n_L}$ where $n_L$ is the number of closed propagator loops and the extra minus sign accounts for the fact that (2.179c) specifies $-\Sigma$.

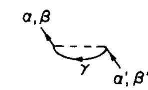
To illustrate these rules, the first and second order contributions to the self-energy are the following:



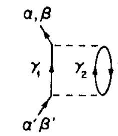
$$= \varsigma\delta(\beta - \beta')\sum_\alpha (\alpha\gamma|v|\alpha'\gamma)g_\gamma(0)$$



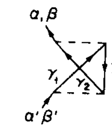
$$= \delta(\beta - \beta')\sum_\alpha (\alpha\gamma|v|\alpha'\gamma)n_\gamma \qquad (2.180a)$$



$$= \delta(\beta - \beta')\varsigma\sum_\alpha (\alpha\gamma|v|\gamma\alpha')n_\gamma \qquad (2.180b)$$



$$= -\varsigma \sum_{\gamma_1\gamma_2\gamma_3} (\alpha_1\gamma_3|v|\gamma_1\gamma_2)(\gamma_1\gamma_2|v|\alpha'\gamma_3)$$
$$\times g_{\gamma_1}(\beta - \beta')g_{\gamma_2}(\beta - \beta')g_{\gamma_3}(\beta' - \beta) \qquad (2.180c)$$



$$= -\sum_{\gamma_1\gamma_2\gamma_3} (\alpha_1\gamma_3|v|\gamma_2\gamma_1)(\gamma_1\gamma_2|v|\alpha'\gamma_3)$$
$$\times g_{\gamma_1}(\beta - \beta')g_{\gamma_2}(\beta - \beta')g_{\gamma_3}(\beta' - \beta) \ . \qquad (2.180d)$$

Note that amputated diagrams, such as (2.180) are drawn with incoming and outgoing lines which do not begin or end with solid dots, indicating that the external labels are assigned to the vertex and no propagator is present. In contrast, diagrams having propagators in the external legs such as the Green's function diagrams in Fig. 2.7 are drawn with solid dots for the external points and it is understood that a propagator is to be included between each external point and the interaction to which it is connected.

The physical interpretation of $\Sigma$ as a self energy or effective one-body potential is evident from using Eq. (2.171) to rewrite (2.178) in the form

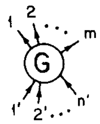
$$[\mathcal{G}_c^{(1)}]^{-1}(\alpha_1\tau_1|\alpha_2\tau_2) = \left(\delta_{\alpha_1\alpha_2}\left(\frac{\partial}{\partial\tau_1} - \mu\right) + \langle\alpha_1|H_0|\alpha_2\rangle\right)\delta(\tau_1 - \tau_2) + \Sigma(\alpha_1\tau_1|\alpha_2\tau_2) \ . \qquad (2.181)$$

The full Green's functions specifying the propagation of a particle in the many-body medium is obtained from the propagator in a non-interacting system by adding to $\langle\alpha_1|H_0|\alpha_2\rangle$ all the irreducible graphs describing the particle's interaction with the rest of the system. The contributions (2.180a and b) are instantaneous like $H_0$ and specify the shift in energy due to the Hartree Fock mean field. These and the higher order contributions will be discussed in detail in Chapter 5.

## HIGHER-ORDER VERTEX FUNCTIONS

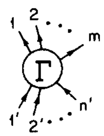
The essential features of $\Sigma$, which differs from the one-particle vertex function $\Gamma_{\phi^*\phi}$ only by the trivial term $\Gamma^0_{\phi^*\phi}$, are that it is one-particle irreducible and that the full one-particle Green's function is obtained from it by Eq. (2.179) which involves no loop integrals. We will now demonstate that these two features generalize to $n$-particle vertex functions.

We begin by introducing an economical graphical representation for differentiations of the form which led to (2.170). Let



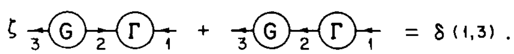
$$\equiv \zeta^n \frac{\delta^{m+n} W(J^*, J)}{\delta J^*(1)\delta J^*(2)\ldots\delta J^*(m)\delta J(n')\ldots\delta J(2')\delta J(1')} \qquad (2.182a)$$

and



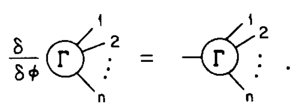
$$\equiv \frac{\delta^{m+n}\Gamma(\phi^*, \phi)}{\delta\phi^*(1)\delta\phi(2)\ldots\delta\phi^*(m)\delta\phi(n')\ldots\delta\phi(2')\delta\phi(1')} \;. \qquad (2.182b)$$

With this notation, Eq. (2.170a) may be written



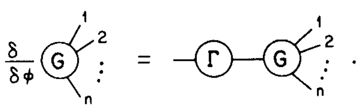
(2.183)

To emphasize the essential structure, we shall condense the notation still further by omitting signs, disregarding the direction of the arrows, letting $\frac{\delta}{\delta\phi}$ represent either $\frac{\delta}{\delta\phi(i)}$ or $\frac{\delta}{\delta\phi^*(i)}$ and letting $\frac{\delta}{\delta J}$ represent either $\frac{\delta}{\delta J(i)}$ or $\frac{\delta}{\delta J^*(i)}$. Then a functional derivative $\frac{\delta}{\delta\phi}$ applied to $\frac{\delta^n}{\delta\phi^n}\Gamma$ increases the number of legs by one:



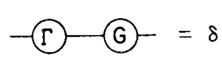
(2.184)

Using the chain rule $\frac{\delta}{\delta\phi} = \frac{\delta J}{\delta\phi}\frac{\delta}{\delta J}$, Eq. (2.168), the functional derivative $\frac{\delta}{\delta\phi}$ applied to $\frac{\delta^n}{\delta J^n}W$ adds a leg containing $\frac{\delta J}{\delta\phi} = \frac{\delta^2\Gamma}{\delta\phi^2}$:



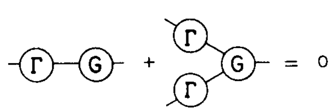
(2.185)

With this compact notation, evaluation of $\frac{\delta^n}{\delta\phi^n}[\phi] = \frac{\delta^n}{\delta\phi^n}[\frac{\partial W}{\partial J}]$ for successive values of $n$ yields the desired hierarchy of equations. For $n = 1$, we recover the abbreviated form of (2.183)



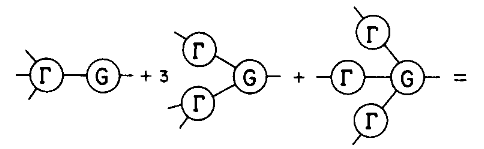
(2.186a)

Evaluating successive derivatives by letting $\frac{\delta}{\partial\phi}$ act on $\Gamma$'s using (2.184) or on $G$'s using (2.185) yields

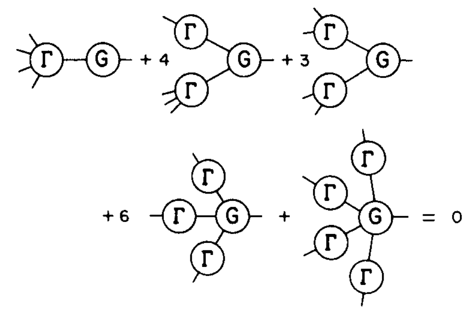


(2.186b)

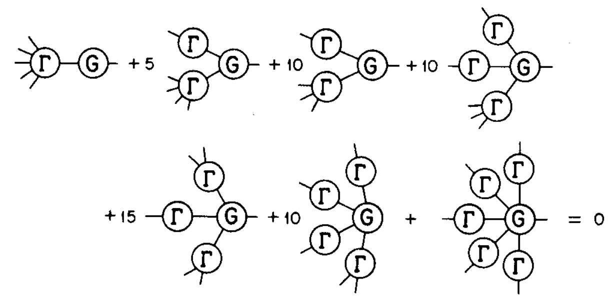




(2.186c)

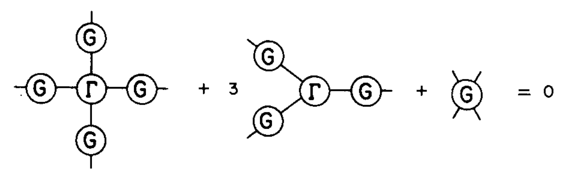


(2.186d)



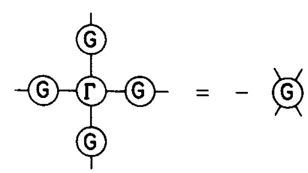
(2.186e)

where the integers denote the number of distinct ways of arranging the external labels. Multiplying each of the external legs associated with $\frac{\delta}{\delta\phi}$ in Eq. (2.186c) by $G_1 = (\Gamma_1)^{-1}$, we obtain



(2.187)

In the case of no symmetry breaking, this simplifies to



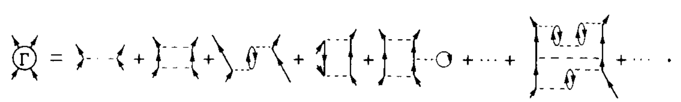
(2.188a)

or explicitly

$$\mathcal{G}_c^{(2)}(\alpha_1\beta_1,\alpha_2\beta_2|\alpha_1'\beta_1',\alpha_2'\beta_2') \tag{2.188b}$$
$$= -\sum_{\substack{\alpha_3\alpha_4\\ \alpha_3'\alpha_4'}}\int_0^\beta d\tau_3 d\tau_4 d\tau_3' d\tau_4' \mathcal{G}_c^{(1)}(\alpha_1\beta_1|\alpha_3\tau_3)\mathcal{G}_c^{(1)}(\alpha_2\beta_2|\alpha_4\tau_4)$$
$$\times \Gamma_{2\phi^*,2\phi}(\alpha_3\tau_3,\alpha_4\tau_4|\alpha_3'\tau_3',\alpha_4'\tau_4')\mathcal{G}_c^{(1)}(\alpha_3'\tau_3'|\alpha_1'\beta_1')\mathcal{G}_c^{(1)}(\alpha_4'\tau_4'|\alpha_2'\beta_2') \; .$$

Note that in spite of our condensed notation, the signs and factors of $\varsigma$ are obvious: In the absence of symmetry breaking, only terms with equal numbers of derivatives with respect to $\phi^*, \phi$ or $J^*, J$ are non vanishing by Eq. (2.168), so that ingoing or outgoing legs of the form $\rightarrow\!\!(\Gamma)\!-$ and $-\!(\Gamma)\!\rightarrow$ attached to $\mathcal{G}_c^{(2)}$ in Eq. (2.186d) have factors $(-1)$ and $(-\varsigma)$ respectively. Since $\mathcal{G}_c^{(2)}$ has two incoming lines and two outgoing lines and, by Eq. (2.154), 2 factors of $\varsigma$, the terms containing $\mathcal{G}_c^{(2)}$ and $\Gamma_{2\phi^*2\phi}$ in Eqs. (2.186d) and (2.187) have coefficient $+1$ leading to the overall minus sign in (2.188a).

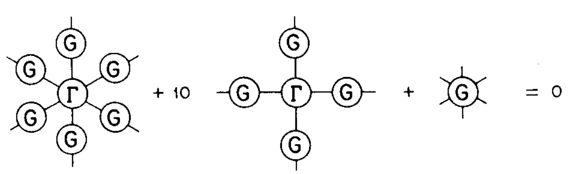
By the same arguments as applied to Eq. (2.179c) for the self energy, equation (2.188) shows that $\Gamma_{2\phi^*,2\phi}$ is the sum of all one-particle irreducible amputated connected diagrams with four external legs. The diagram rules for $\Gamma_{2\phi^*2\phi}$ are identical to those given for the self-energy, except for the obvious modifications that there are two external labels $\{\alpha_1,\beta_1\},\{\alpha_2,\beta_2\}$ assigned to outgoing arrows of interaction vertices, two external labels $\{\alpha_1',\beta_1'\},\{\alpha_2',\beta_2'\}$ assigned to incoming arrows of interaction vertices, and the factor for a diagram with $n$ interactions and $n_L$ closed propagator loops is $(-1)^{n-1}\varsigma^{n_L}\varsigma^P$ where $\varsigma^P$ is the sign of the permutation such that each propagator originating at the vertex with external label $\{\alpha_m',\beta_m'\}$ terminates at the vertex with external label $\{\alpha_{Pm},\beta_{Pm}\}$. Examples of contributions to the two-particle vertex function $\Gamma_{2\phi^*2\phi}$ are the following



(2.189)

Just as $\Sigma$ specifies the self energy or effective one-body potential for a particle propagating in a many-particle system, $\Gamma_{2\phi^*2\phi}$ corresponds to the effective two-body interaction between two particles propagating in a many-particle medium.

In addition to showing that $\Gamma_{2\phi^*2\phi}$ is composed of one-particle irreducible amputated connected diagrams, Eq. (2.188a) also demonstrates that the two-particle Green's function is obtained from a tree diagram composed of Green's functions and vertex functions of the same and lower order. These two properties are completely general, as is evident from the structure of the hierarchy Eq. (2.186). As a final example, the corresponding result for the three-particle Green's function and vertex function from Eq. (2.184e) in the absence of symmetry breaking is the following:



(2.190)



again demonstrating the one-particle irreducibility and tree-diagram structure.

Analogous equations in which vertex functions are $n$-particle irreducible amputated connected diagrams may be derived straightforwardly by the same approach we have used starting with sources coupled to all combinations of $n$ creation and annihiltion operators. For example using the bilinear source, (2.149), the principal steps in deriving the two-particle irreducible theory are the following. The generating function for connected Green's functions is

$$W(\eta,\bar\eta,\eta^*) = ln\langle e^{-\int_0^\beta d\tau \sum_{\alpha,\beta}\eta_{\alpha\beta}\psi_\alpha^*\psi_\beta^* + \bar\eta_{\alpha\beta}\psi_\alpha^*\psi_\beta + \eta^*_{\alpha\beta}\psi_\alpha\psi_\beta}\rangle \tag{2.191}$$

from which we define the expectation values

$$\begin{aligned}\xi_{\alpha\beta} &= -\frac{\delta W}{\delta\eta^*_{\alpha\beta}} = \langle\psi_\alpha\psi_\beta\rangle\\ \xi^*_{\alpha\beta} &= -\frac{\delta W}{\delta\eta_{\alpha\beta}} = \langle\psi^*_\alpha\psi^*_\beta\rangle\\ \bar\xi_{\alpha\beta} &= -\frac{\delta W}{\delta\bar\eta_{\alpha\beta}} = \langle\psi^*_\alpha\psi_\beta\rangle\end{aligned} \tag{2.192}$$

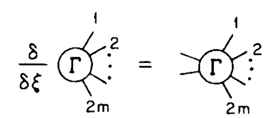
and Legendre transform

$$\Gamma(\xi,\bar\xi,\xi^*) = -W(\eta,\bar\eta,\eta^*) - \sum_{\alpha\beta}\int_0^\beta d\tau[\xi^*_{\alpha\beta}\eta_{\alpha\beta} + \bar\xi_{\alpha\beta}\bar\eta_{\alpha\beta} + \xi_{\alpha\beta}\eta^*_{\alpha\beta}] \tag{2.193}$$

with the reciprocity relations

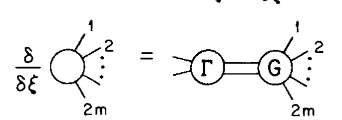
$$\frac{\partial\Gamma}{\delta\xi^*_{\alpha\beta}} = -\eta_{\alpha\beta} \qquad \frac{\delta\Gamma}{\delta\bar\xi_{\alpha\beta}} = -\bar\eta_{\alpha\beta} \qquad \frac{\delta\Gamma}{\delta\xi_{\alpha\beta}} = -\eta^*_{\alpha\beta} \; . \tag{2.194}$$

Using the schematic notation of Eq. (2.184 − 2.188) in which $\frac{\delta}{\delta\xi}$ represents $\frac{\delta}{\delta\xi_{\alpha\beta}}$, $\frac{\delta}{\delta\xi^*_{\alpha\beta}}$, or $\frac{\delta}{\delta\bar\xi_{\alpha\beta}}$, $\frac{\delta}{\delta\eta}$ represents $\frac{\delta}{\delta\eta_{\alpha\beta}}$, $\frac{\delta}{\delta\eta^*_{\alpha\beta}}$, or $\frac{\delta}{\delta\bar\eta_{\alpha\beta}}$ and indices and directions of propagators are suppressed, a functional derivative $\frac{\delta}{\delta\xi}$ applied to $\frac{\delta^m}{\delta\xi^m}\Gamma$ increases the number of legs by two



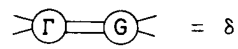
(2.195a)

and applied to $\frac{\delta^m}{\delta\eta^m}W$ adds a leg containing $\frac{\delta\eta}{\delta\xi} = \frac{\delta^2\Gamma}{\delta\xi^2}$:

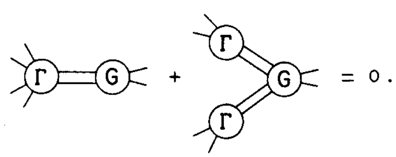


(2.195b)

Evaluation of successive deriatives $\frac{\delta^n}{\delta\xi}[\xi] = \frac{\delta^n}{\delta\xi}[\frac{\partial W}{\partial\eta}]$ yields a hierarchy of the form of Eq. (2.186)



(2.196a)



(2.196b)

The first equation shows that the matrix of four-point vertex functions is the inverse of the matrix of two-particle Green's functions. In the absence of symmetry-breaking, all $\mathcal{G}$'s and $\Gamma$'s must have equal numbers of incoming and outgoing lines and the pairs of lines in these equations represent all combinations of aligned or anti-aligned propagators consistent with this restriction. By the same arguments used for Eqs. (2.186), Eqs. (2.196) show that the $\Gamma$'s are composed of two-particle irreducible amputated connected diagrams and that the $n$-particle Green's functions is obtained from tree graphs involving fewer particle Green's functions and $2n$-point and fewer point vertex functions. It is an instructive exercise to derive these equations in detail (see Problem 2.12).

## 2.5 STATIONARY-PHASE APPROXIMATION AND LOOP EXPANSION

Whereas perturbation theory is valuable for the formal developments of Section 2.3 and is directly applicable to a limited class of physical problems with weak interactions characterized by a small expansion parameter, many problems of physical interest involve strong many-body interactions for which a perturbation expansion in the interaction strength is inappropriate. A natural approach for such problems is to reorganize the perturbation expansion into a series in powers of a new small parameter. We have already seen examples of such infinite resummations of perturbation theory in Section 2.4, where solution of Dyson's equations with some finite set of self energy diagrams or, in general, calculation of Green's functions with any finite set of diagrams for $n$-particle irreducible vertex functions $\Gamma$ resums an infinite number of terms of the original perturbation series. Other physically motivated resummations will be presented subsequently for specific systems. In view of the general lack of mathematical control on the convergence properties of the original series and the obvious ambiguity associated with regrouping divergent series, any such resummations must be understood ultimately on physical grounds.

In this section, we consider a specific systematic regrouping of terms obtained by applying the stationary-phase approximation to the functional integral for the partition function. This regrouping will be seen to be ordered in the number of loops occuring in the Feynman diagrams. For certain systems, this approximation generates an asymptotic expansion in a small parameter, such as $\hbar$ or the number of degrees of freedom associated with an internal symmetry.

We first review the stationary-phase approximation in the case of a one-dimensional integral, and then generalize to the cases of the Feynman path integral and the partition function for many-particle systems. As usual, the primary emphasis will be on the essence of the method and its physical interpretation, rather than mathematical rigor.



### ONE-DIMENSIONAL INTEGRAL

The stationary-phase approximation, also referred to as the saddle point approximation or method of steepest descent, is a method for developing an asymptotic expansion in powers of $\frac{1}{\ell}$ for an integral of the form

$$I(\ell) = \int_{-\infty}^{\infty} dt e^{-\ell f(t)} \tag{2.197}$$

where $\ell$ is a real parameter and in general $f(t)$ is an analytic function in the complex $t$-plane.

For simplicity, we will first consider the special case of a real function $f(t)$ with an absolute minimum at $t = t_0$. As $\ell$ increases, the integral becomes sharply peaked around the point $t_0$, and the dominant contribution to the integral arises from the vicinity of $t_0$. Expanding $f(t)$ around $t_0$, recognizing the fact that $f'(t_0) = 0$ and $f''(t_0) > 0$ since $t_0$ is the minimum, the integral may be rewritten:

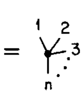
$$\begin{aligned} I(\ell) &= e^{-\ell f_0} \int_{-\infty}^{\infty} dt\, e^{-\frac{\ell}{2} f_0''(t-t_0)^2 - \ell \sum_{n=3}^{\infty} \frac{(t-t_0)^n}{n!} f_0^{(n)}} \\ &= \sqrt{\frac{2\pi}{\ell f_0''}}\, e^{-\ell f_0} \int_{-\infty}^{\infty} \frac{d\tau}{\sqrt{2\pi}} e^{-\frac{\tau^2}{2} - \sum_{n=3}^{\infty} \frac{\tau^n}{n!} \frac{f_0^{(n)}}{\ell^{(\frac{n}{2}-1)} (f_0'')^{\frac{n}{2}}}} \end{aligned} \tag{2.198}$$

where derivatives evaluated at $t_0$ are denoted $f_0''$ and $f_0^{(n)}$ and the change of variables $\tau = (t - t_0)\sqrt{\ell f_0''}$ has been introduced to rescale the Gaussian to unit width. As $\ell \to \infty$, the terms with $n \geq 3$ go to zero and we may expand $I(\ell)$ in powers of $\frac{1}{\ell}$. For the problems of physical interest, we will usually be interested in expanding the logarithm of $I(\ell)$, and in the present example, it is straightforward to expand the exponential in Eq. (2.149), perform the Gaussian integrals, and exponentiate the result to the desired power of $\frac{1}{\ell}$ (see Problem 2.13).

A more economical derivation, however, may be obtained by utilizing our knowledge of Wick's theorem and the linked cluster expansion. Just as we defined contractions in connection with Eq. (2.84), we may define the contraction of $\tau$ as

$$\overbrace{\tau \cdot \tau} = \int_{-\infty}^{\infty} \frac{d\tau}{\sqrt{2\pi}} \tau \cdot \tau e^{-\frac{\tau^2}{2}} = 1 \ . \tag{2.199}$$

The coefficients of $\tau^n$ for $n \geq 3$ in Eq. (2.198) are regarded as vertices with $n$ lines

$$V_n = \frac{1}{n!} \frac{f_0^{(n)}}{\ell^{(\frac{n}{2}-1)} (f_0'')^{\frac{n}{2}}} =$$  . (2.200)

Diagrams representing all possible contractions contributing to Eq. (2.198) are obtained by drawing any number of vertices $V_{n_1} V_{n_2} \dots V_{n_N}$ and connecting them with propagators equal to 1. In addition to the vertices, $V_n$, a diagram of order $N$ has the overall factor $\frac{(-1)^N}{N!}$. By the linked cluster theorem,

$$I(\ell) = e^{-\ell f_0} \sqrt{\tfrac{2\pi}{\ell f_0''}}\, e^{(\text{sum of all linked diagrams})} \ . \tag{2.201}$$