VARIOUS DERIVATIONS
(MAGNETISM AND MICROMAGNETICS)

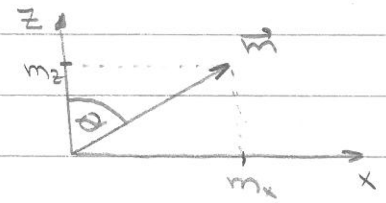
MARIJAN BEG

CONTENTS

1. CONFINED, INTERFACIAL DMI [Rohart and Thiaville, PRB 88, 184422 (2013).]

1.1 ONE-DIMENSIONAL CASE (domain wall and cycloid)

Interfacial DMI ⇒ Néel-type $\vec{m} \in (\hat{x}, \hat{z})$ plane



$$\left.\begin{aligned} m_x &= \sin\theta \\ m_y &= 0 \\ m_z &= \cos\theta \end{aligned}\right\} \quad \begin{aligned} &\vec{m} = \sin\theta\,\hat{x} + \cos\theta\,\hat{z} \\ &\theta = \theta(x) \\ &|\vec{m}| = 1 \end{aligned}$$

- EXCHANGE

$$w_{ex} = A\left[(\nabla m_x)^2 + (\nabla m_y)^2 + (\nabla m_z)^2\right]$$

$$\nabla m_x = \frac{\partial m_x}{\partial x}\hat{x} + \frac{\partial m_x}{\partial y}\hat{y} + \frac{\partial m_x}{\partial z}\hat{z} = \cos\theta \frac{d\theta}{dx}\hat{x}$$

$$\nabla m_y = 0$$

$$\nabla m_z = -\sin\theta \frac{d\theta}{dx}\hat{x}$$

$$w_{ex} = A\left[\cos^2\theta\left(\frac{d\theta}{dx}\right)^2 + \sin^2\theta\left(\frac{d\theta}{dx}\right)^2\right] = A\left(\frac{d\theta}{dx}\right)^2$$

- DMI

$$w_{DMI} = D\left[m_x\frac{\partial m_z}{\partial x} - m_z\frac{\partial m_x}{\partial x} + \cancel{m_y\frac{\partial m_z}{\partial y}}^{\to 0} - \cancel{m_z\frac{\partial m_y}{\partial y}}^{\to 0}\right]$$

$$w_{DMI} = D\left[\sin\theta\cdot(-\sin\theta)\frac{d\theta}{dx} - \cos\theta\cdot\cos\theta\frac{d\theta}{dx}\right]$$

$$= -D\frac{d\theta}{dx}$$

- ANISOTROPY

$$w_a = -K(\vec{m}\cdot\hat{z})^2 = -K\cos^2\theta$$

- TOTAL ENERGY DENSITY

$$w = A\left(\frac{d\theta}{dx}\right)^2 - D\frac{d\theta}{dx} - K\cos^2\theta$$

$$W = \int_{x_A}^{x_B} \underbrace{\left[A\left(\frac{d\theta}{dx}\right)^2 - D\frac{d\theta}{dx} - K\cos^2\theta\right]}_{w} dx$$

$$\frac{\delta w}{\delta\theta(x')} = 2A\frac{d\theta}{dx}\frac{\delta}{\delta\theta(x')}\frac{d\theta}{dx} - D\frac{\delta}{\delta\theta(x')}\frac{d\theta}{dx} + 2K\cos\theta\sin\theta\frac{\delta\theta}{\delta\theta(x')}$$

$$\frac{\delta w}{\delta\theta(x')} = 2A\frac{d\theta}{dx}\frac{d}{dx}\frac{\delta\theta}{\delta\theta(x')} - D\frac{d}{dx}\frac{\delta\theta}{\delta\theta(x')} + 2K\cos\theta\sin\theta\frac{\delta\theta}{\delta\theta(x')}$$

$$\frac{\delta w}{\delta\theta(x')} = 2A\frac{d\theta}{dx}\frac{d}{dx}\delta(x-x') - D\frac{d}{dx}\delta(x-x') + 2K\cos\theta\sin\theta\,\delta(x-x')$$

$$\frac{\delta W}{\delta\theta(x')} = \underbrace{\int_{x_A}^{x_B} 2A\frac{d\theta}{dx}\frac{d}{dx}\delta(x-x')\,dx}_{I_1} - \underbrace{\int_{x_A}^{x_B} D\frac{d}{dx}\delta(x-x')\,dx}_{I_2} + \underbrace{\int_{x_A}^{x_B} 2K\cos\theta\sin\theta\,\delta(x-x')\,dx}_{I_3}$$

$$I_1 = \int_{x_A}^{x_B} 2A\frac{d\theta}{dx}\frac{d}{dx}\delta(x-x')\,dx = \left| \begin{array}{ll} u = \frac{d\theta}{dx} & dv = \frac{d}{dx}\delta(x-x')\,dx \\ du = \frac{d^2\theta}{dx^2}dx & v = \delta(x-x') \end{array} \right|$$

$$= 2A\frac{d\theta}{dx}\delta(x-x')\Big|_{x_A}^{x_B} - 2A\int_{x_A}^{x_B}\frac{d^2\theta}{dx^2}\delta(x-x')\,dx$$

$$= 2A\frac{d\theta}{dx}\delta(x-x')\Big|_{x_A}^{x_B} - 2A\frac{d^2\theta(x')}{dx'^2}$$

$$I_2 = -\int_{x_A}^{x_B} D\frac{d}{dx}\delta(x-x')\,dx = -D\delta(x-x')\Big|_{x_A}^{x_B}$$

$$I_3 = \int_{x_A}^{x_B} 2K\cos\theta\sin\theta\,\delta(x-x')\,dx = 2K\cos\theta(x')\sin\theta(x')$$

$$\frac{\delta W}{\delta\theta(x')} = I_1 + I_2 + I_3 = 2A\frac{d\theta}{dx}\delta(x-x')\Big|_{x_A}^{x_B} - 2A\frac{d^2\theta(x')}{dx'^2} - D\delta(x-x')\Big|_{x_A}^{x_B}$$

$$+ 2K\cos\theta(x')\sin\theta(x') = 0$$

$$\cancel{2}A\frac{d^2\theta}{dx^2} = \cancel{2}K\cos\theta\sin\theta \qquad 2A\frac{d\theta}{dx}\cancel{\delta(x-x')} = D\cancel{\delta(x-x')}$$

ODE: $\dfrac{d^2\theta}{dx^2} = \dfrac{\cos\theta\sin\theta}{\Delta^2}$ $\quad x_A < x < x_B \quad$ BC: $\dfrac{d\theta}{dx} = \dfrac{1}{\xi}$, $x = x_A \wedge x = x_B$

$\Delta = \sqrt{\frac{A}{K}}$ Bloch wall width parameter $[\Delta] = m$

$\xi = \frac{2A}{D}$ parameter specific to DMI $[\xi] = m$

$$\frac{d^2\theta}{dx^2} = \frac{\cos\theta\sin\theta}{\Delta^2} = \frac{\sin 2\theta}{2\Delta^2}$$

$$z = \frac{d\theta}{dx} \Rightarrow \frac{d^2\theta}{dx^2} = \frac{dz}{dx} = \frac{dz}{d\theta}\overbrace{\frac{d\theta}{dx}}^{z} = z\frac{dz}{d\theta}$$

$$z\frac{dz}{d\theta} = \frac{1}{2\Delta^2}\sin 2\theta$$

$$\int z\,dz = \frac{1}{2\Delta^2}\int \sin 2\theta\,d\theta$$

$$\frac{z^2}{\cancel{2}} = \frac{1}{\cancel{2}\Delta^2}\cdot\left(-\frac{1}{2}\cos 2\theta\right) + C$$

$$z^2 = \frac{-\cos 2\theta + \overbrace{C\cdot 2\Delta^2}^{C}}{2\Delta^2} = \frac{-(1-2\sin^2\theta) + C}{2\Delta^2} = \frac{\cancel{2}\sin^2\theta\ \overbrace{-1+C}^{C}}{\cancel{2}\Delta^2} = \frac{C+\sin^2\theta}{\Delta^2}$$

$$\left(\frac{d\theta}{dx}\right)^2 = \frac{C+\sin^2\theta}{\Delta^2} \qquad (1)$$

The second order ODE can be written as a system of two first order ODEs.

$$\frac{d\theta}{dx} = \theta' \qquad \Rightarrow \frac{d\theta[0]}{dx} = \theta[1]$$

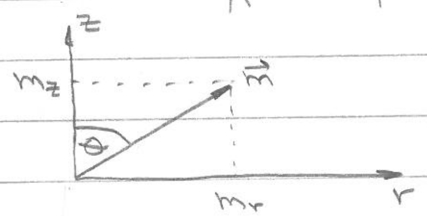
$$\frac{d\theta'}{dx} = \frac{\cos\theta\sin\theta}{\Delta^2} \qquad \frac{d\theta[1]}{dx} = \frac{\cos\theta[0]\sin\theta[0]}{\Delta^2}$$

From equation (1), the boundary conditions can be obtained for $C=0$.

$$\left(\frac{1}{\xi}\right)^2 = \frac{\sin^2\theta}{\Delta^2} \Rightarrow \sin\theta = \frac{\Delta}{\xi} \Rightarrow \theta = \pm\sin^{-1}\left(\frac{\Delta}{\xi}\right) \text{ at } x = x_A, x = x_B$$

1.2 TWO-DIMENSIONAL CASE (skyrmion)

Néel type skyrmion $\vec{m} \in (\hat{r}, \hat{z})$



$$\left.\begin{aligned} m_r &= \sin\theta \\ m_\varphi &= 0 \\ m_z &= \cos\theta \end{aligned}\right\} \quad \begin{aligned} &|\vec{m}| = 1 \\ &\vec{m} = \sin\theta\,\hat{r} + \cos\theta\,\hat{z} \\ &\theta = \theta(r) \end{aligned}$$

Conversion from cylindrical to Cartesian coordinate system:

$$\begin{bmatrix} \hat{r} \\ \hat{\varphi} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \qquad \hat{r} = \cos\varphi\,\hat{x} + \sin\varphi\,\hat{y}$$

$$\vec{m} = \sin\theta(\cos\varphi\,\hat{x} + \sin\varphi\,\hat{y}) + \cos\theta\,\hat{z}$$

$$\vec{m} = \underbrace{\sin\theta\cos\varphi}_{m_x}\,\hat{x} + \underbrace{\sin\theta\sin\varphi}_{m_y}\,\hat{y} + \underbrace{\cos\theta}_{m_z}\,\hat{z}$$

$$m_x = \sin\theta\cos\varphi$$
$$m_y = \sin\theta\sin\varphi$$
$$m_z = \cos\theta$$

$$r = \sqrt{x^2+y^2}$$

$$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x = \frac{x}{r} = \cos\varphi$$

$$\frac{\partial r}{\partial y} = \frac{1}{\cancel{2}\sqrt{x^2+y^2}} \cdot \cancel{2}y = \frac{y}{r} = \sin\varphi$$

$$\varphi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial \varphi}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right) = \frac{1}{\frac{x^2+y^2}{\cancel{x^2}}} \cdot \left(-\frac{y}{\cancel{x^2}}\right) = -\frac{y}{r^2} = -\frac{1}{r}\sin\varphi$$

$$\frac{\partial \varphi}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{1}{\frac{x^2+y^2}{x^{\cancel{2}}}} \cdot \frac{1}{\cancel{x}} = \frac{x}{r^2} = \frac{1}{r}\cos\varphi$$

$$\frac{\partial m_x}{\partial x} = \cos\theta \frac{d\theta}{dr}\cos^2\varphi + \sin\theta(-\sin\varphi)\left(-\frac{1}{r}\sin\varphi\right) = \cos\theta\cos^2\varphi\frac{d\theta}{dr} + \frac{1}{r}\sin\theta\sin^2\varphi$$

$$\frac{\partial m_x}{\partial y} = \cos\theta\frac{d\theta}{dr}\sin\varphi\cos\varphi - \frac{1}{r}\sin\theta\sin\varphi\cos\varphi$$

$$\frac{\partial m_x}{\partial z} = 0$$

$$\frac{\partial m_y}{\partial x} = \cos\theta\frac{d\theta}{dr}\cos\varphi\sin\varphi - \frac{1}{r}\sin\theta\cos\varphi\sin\varphi$$

$$\frac{\partial m_y}{\partial y} = \cos\theta\frac{d\theta}{dr}\sin^2\varphi + \frac{1}{r}\sin\theta\cos^2\varphi$$

$$\frac{\partial m_y}{\partial z} = 0$$

$$\frac{\partial m_z}{\partial x} = -\sin\theta\frac{d\theta}{dr}\cos\varphi$$

$$\frac{\partial m_z}{\partial y} = -\sin\theta\frac{d\theta}{dr}\sin\varphi$$

$$\frac{\partial m_z}{\partial z} = 0$$

EXCHANGE ENERGY

$$W_{ex} = A\left[(\nabla m_x)^2 + (\nabla m_y)^2 + (\nabla m_z)^2\right]$$

$$\left(\frac{\partial m_x}{\partial x}\right)^2 = \cos^2\theta\cos^4\varphi\left(\frac{d\theta}{dr}\right)^2 + \frac{1}{r^2}\sin^2\theta\sin^4\varphi + \frac{2}{r}\cos\theta\sin\theta\cos^2\varphi\sin^2\varphi\left(\frac{d\theta}{dr}\right)$$

$$\left(\frac{\partial m_x}{\partial y}\right)^2 = \cos^2\theta\sin^2\varphi\cos^2\varphi\left(\frac{d\theta}{dr}\right)^2 + \frac{1}{r^2}\sin^2\theta\sin^2\varphi\cos^2\varphi - \frac{2}{r}\sin\theta\cos\theta\sin^2\varphi\cos^2\varphi\left(\frac{d\theta}{dr}\right)$$

$$\left(\frac{\partial m_x}{\partial z}\right)^2 = 0$$

$$\left(\frac{\partial m_y}{\partial x}\right)^2 = \cos^2\theta\cos^2\varphi\sin^2\varphi\left(\frac{d\theta}{dr}\right)^2 + \frac{1}{r^2}\sin^2\theta\cos^2\varphi\sin^2\varphi - \frac{2}{r}\sin\theta\cos\theta\cos^2\varphi\sin^2\varphi\left(\frac{d\theta}{dr}\right)$$

$$\left(\frac{\partial m_y}{\partial y}\right)^2 = \cos^2\theta\sin^4\varphi\left(\frac{d\theta}{dr}\right)^2 + \frac{1}{r^2}\sin^2\theta\cos^4\varphi + \frac{2}{r}\cos\theta\sin\theta\sin^2\varphi\cos^2\varphi\left(\frac{d\theta}{dr}\right)$$

$$\left(\frac{\partial m_y}{\partial z}\right)^2 = 0$$

$$\left(\frac{\partial m_z}{\partial x}\right)^2 = \sin^2\theta\cos^2\varphi\left(\frac{d\theta}{dr}\right)^2$$

$$\left(\frac{\partial m_z}{\partial y}\right)^2 = \sin^2\theta\sin^2\varphi\left(\frac{d\theta}{dr}\right)^2$$

$$\left(\frac{\partial m_z}{\partial z}\right)^2 = 0$$

$$w_{ex} = A\Big[\sin^2\theta\left(\frac{d\theta}{dr}\right)^2 + \frac{1}{r^2}\sin^2\theta\left(\sin^4\varphi + 2\sin^2\varphi\cos^2\varphi + \cos^2\varphi\right)$$

$$+\cos^2\theta\left(\frac{d\theta}{dr}\right)^2\left(\sin^4\varphi + 2\sin^2\varphi\cos^2\varphi + \cos^4\varphi\right)\Big]$$

$$= A\left[\left(\frac{d\theta}{dr}\right)^2\left(\sin^2\varphi + \cos^2\varphi\right) + \frac{1}{r^2}\sin^2\theta\right]$$

$$= A\left[\left(\frac{d\theta}{dr}\right)^2 + \frac{1}{r^2}\sin^2\theta\right]$$

DMI ENERGY (interfacial DMI)

$$w_{DMI} = D\left[m_x\frac{\partial m_z}{\partial x} - m_z\frac{\partial m_x}{\partial x} + m_y\frac{\partial m_z}{\partial y} - m_z\frac{\partial m_y}{\partial y}\right]$$

$$w_{DMI} = D\Big[-\sin^2\theta\cos^2\varphi\left(\frac{d\theta}{dr}\right) - \cos^2\theta\cos^2\varphi\left(\frac{d\theta}{dr}\right) - \frac{1}{r}\sin\theta\cos\theta\sin^2\varphi$$

$$-\sin^2\theta\sin^2\varphi\left(\frac{d\theta}{dr}\right) - \cos^2\theta\sin^2\varphi\left(\frac{d\theta}{dr}\right) - \frac{1}{r}\sin\theta\cos\theta\cos^2\varphi\Big]$$

$$= -D\Big[\sin^2\theta\frac{d\theta}{dr}\left(\cos^2\varphi + \sin^2\varphi\right) + \cos^2\theta\left(\frac{d\theta}{dr}\right)\left(\cos^2\varphi + \sin^2\varphi\right)$$

$$+\frac{1}{r}\sin\theta\cos\theta\left(\cos^2\varphi + \sin^2\varphi\right)\Big]$$

$$= -D\left[\frac{d\theta}{dr}\left(\sin^2\theta + \cos^2\theta\right) + \frac{1}{r}\sin\theta\cos\theta\right]$$

$$= -D\left[\left(\frac{d\theta}{dr}\right) + \frac{1}{r}\sin\theta\cos\theta\right]$$

ANISOTROPY ENERGY (uniaxial)

$$w_u = -K\left(\hat{z}\cdot\vec{m}\right)^2 = -K\cos^2\theta$$

$$-K\cos^2\theta = -K(1-\sin^2\theta) = K\sin^2\theta \underbrace{-K}_{\text{const.}}$$

In PRB 88, 184422 (2013) eq. (15) the anisotropy energy density is $+K\sin^2\theta$.

TOTAL ENERGY

$$W = \int_0^R w\,dA = \int_0^R \left(w_{ex} + w_{DMI} + w_A\right) dA$$

$$A = r^2\pi \;/\; d$$

$$dA = 2r\pi\, dr$$

$$W = 2\pi(t)\int_0^R \left\{A\left[\left(\frac{d\theta}{dr}\right)^2 + \frac{\sin^2\theta}{r^2}\right] - D\left[\frac{d\theta}{dr} + \frac{1}{r}\sin\theta\cos\theta\right] - K\cos^2\theta\right\} r\,dr$$

This energy expression is dimenionally wrong and should be multiplied by layer thickness.

COMMENTS ON DIFFERENT ANISOTROPY (Aharoni, Introduction to the theory of Ferromagnetism, Oxford 1996):

$$w_\mu = -K_1\cos^2\theta + K_2\cos^4\theta = -K_1 m_z^2 + K_2 m_z^2 \qquad (1)$$

where $\hat{z}$ is parallel to the crystalographic c-axis. "Most workers prefer to rewrite eq. (1) as

$$w_u = K_1\sin^2\theta + K_2\sin^4\theta = K_1(1-m_z^2) + K_2(1-m_z^2)^2 \qquad (2)$$

in which case the coefficient K_1 has a different value than in the case of eq. (1), unless $K_2 = 0$, or is negligibly small. Once K_1 is properly redefined, the difference between (1) and (2) is a constant, and a constant energy term does not have any physical meaning: it only means a shift in the definition of the zero energy, which is never important... Therefore the choice between (1) and (2) is completely arbitrary, as long as the definition is not swithed in the middle of calculation."

VARIATIONAL CALCULATION

$$F[\theta] = 2\pi A\left[r\left(\frac{d\theta}{dr}\right)^2 + \frac{\sin^2\theta}{r}\right] - 2\pi D\left[r\frac{d\theta}{dr} + \sin\theta\cos\theta\right] - 2\pi K r\cos^2\theta$$

$$\frac{\delta F[\theta]}{\delta\theta(r')} = 2\pi A r\cdot 2\frac{d\theta}{dr}\frac{\delta}{\delta\theta(r')}\frac{d\theta}{dr} + 2\pi A\frac{1}{r}\cdot 2\sin\theta\cos\theta\frac{\delta\theta}{\delta\theta(r')}$$

$$-2\pi D r\frac{\delta}{\delta\theta(r')}\frac{d\theta}{dr} - 2\pi D\left[\cos^2\theta - \sin^2\theta\right]\frac{\delta\theta}{\delta\theta(r')}$$

$$-2\pi K r\cdot 2\cos\theta(-\sin\theta)\frac{\delta\theta}{\delta\theta(r')} =$$

$$= 4\pi A r\frac{d\theta}{dr}\frac{d}{dr}\delta(r-r') + \frac{4\pi A}{r}\sin\theta\cos\theta\,\delta(r-r')$$

$$-2\pi D r\frac{d}{dr}\delta(r-r') - 2\pi D\cos 2\theta\,\delta(r-r') + 4\pi K r\cos\theta\sin\theta\,\delta(r-r')$$

$$\frac{\delta W[\theta]}{\delta\theta(r')} = \int_0^R \frac{\delta F[\theta]}{\delta\theta(r')}dr = 0$$

$$\frac{\delta W[\theta]}{\delta\theta(r')} = \underbrace{\int_0^R 4\pi A r\frac{d\theta}{dr}\frac{d}{dr}\delta(r-r')dr}_{I_1} + \underbrace{\int_0^R \frac{4\pi A}{r}\sin\theta\cos\theta\,\delta(r-r')dr}_{I_2}$$

$$-\underbrace{\int_0^R 2\pi D r\frac{d}{dr}\delta(r-r')dr}_{I_3} - \underbrace{\int_0^R 2\pi D\cos 2\theta\,\delta(r-r')dr}_{I_4}$$

$$+\underbrace{\int_0^R 4\pi K r\cos\theta\sin\theta\,\delta(r-r')dr}_{I_5}$$

$$I_1 = \int_0^R 4\pi A r\frac{d\theta}{dr}\frac{d}{dr}\delta(r-r')dr = 4\pi A\int_0^R r\frac{d\theta}{dr}\frac{d}{dr}\delta(r-r')dr$$

By calculating:

$$\frac{d}{dr}\left[r\frac{d\theta}{dr}\delta(r-r')\right] = \frac{d\theta}{dr}\delta(r-r') + r\left[\frac{d^2\theta}{dr^2}\delta(r-r') + \frac{d\theta}{dr}\frac{d}{dr}\delta(r-r')\right]$$

$$\Downarrow$$

$$r\frac{d\theta}{dr}\frac{d}{dr}\delta(r-r') = \frac{d}{dr}\left[r\frac{d\theta}{dr}\delta(r-r')\right] - \frac{d\theta}{dr}\delta(r-r') - r\frac{d^2\theta}{dr^2}\delta(r-r')$$

$$I_1 = 4\pi A\left[\int_0^R \frac{d}{dr}\left[r\frac{d\theta}{dr}\delta(r-r')\right]dr - \int_0^R \frac{d\theta}{dr}\delta(r-r')dr - \int_0^R r\frac{d^2\theta}{dr^2}\delta(r-r')dr\right] =$$

$$= 4\pi A\left[r\frac{d\theta}{dr}\delta(r-r')\Big|_0^R - \frac{d\theta(r')}{dr} - r'\frac{d^2\theta(r')}{dr^2}\right]$$

$$= 4\pi A r\frac{d\theta}{dr}\delta(r-r')\Big|_0^R - 4\pi A\frac{d\theta(r')}{dr} - 4\pi A r'\frac{d^2\theta(r')}{dr^2}$$

$$I_2 = \int_0^R \frac{4\pi A}{r}\sin\theta\cos\theta\,\delta(r-r')dr = 4\pi A\frac{1}{r'}\sin\theta(r')\cos\theta(r')$$

$$I_3 = -\int_0^R 2\pi D r\frac{d}{dr}\delta(r-r')dr = \begin{vmatrix} u=r & dv=\frac{d}{dr}\delta(r-r')dr \\ du=dr & v=\delta(r-r') \end{vmatrix} =$$

$$= -2\pi D\left[r\delta(r-r')\Big|_0^R + \int_0^R \delta(r-r')dr\right] =$$

$$= -2\pi D r\delta(r-r')\Big|_0^R + 2\pi D$$

$$I_4 = -2\pi D\int_0^R \cos 2\theta\,\delta(r-r')dr = -2\pi D\cos 2\theta(r') = -2\pi D\left(1 - 2\sin^2\theta(r')\right)$$

$$I_5 = 4\pi K\int_0^R r\cos\theta\sin\theta\,\delta(r-r')dr = 4\pi K r'\cos\theta(r')\sin\theta(r')$$

$$\frac{\delta W[\theta]}{\delta\theta(r')} = 4\pi A r\frac{d\theta}{dr}\delta(r-r')\Big|_0^R - 4\pi A\frac{d\theta(r')}{dr} - 4\pi A r'\frac{d^2\theta(r')}{dr^2}$$

$$+4\pi A\frac{1}{r'}\sin\theta(r')\cos\theta(r') - 2\pi D r\delta(r-r')\Big|_0^R + \cancel{2\pi D}$$

$$-\cancel{2\pi D} + 4\pi D\sin^2\theta(r') + 4\pi K r'\cos\theta(r')\sin\theta(r') = 0$$

Skyrmion profile ODE:

$$-4\pi A\frac{d\theta(r')}{dr} - 4\pi A r'\frac{d^2\theta(r')}{dr^2} + 4\pi A\frac{1}{r'}\sin\theta(r')\cos\theta(r') +$$

$$4\pi D\sin^2\theta(r') + 4\pi K r'\cos\theta(r')\sin\theta(r') = 0 \qquad /: 4\pi \quad r' \to r$$

$$-A\frac{d\theta}{dr} - A r\boxed{\frac{d^2\theta}{dr^2}} + A\frac{1}{r}\sin\theta\cos\theta + D\sin^2\theta + K r\cos\theta\sin\theta = 0$$

$$\frac{d^2\theta}{dr^2} = -\frac{1}{r}\frac{d\theta}{dr} + \frac{1}{r^2}\sin\theta\cos\theta + \frac{D}{A}\frac{1}{r}\sin^2\theta + \frac{K}{A}\cos\theta\sin\theta$$

$\Delta = \sqrt{\frac{A}{K}}$ $\xi = \frac{2A}{D}$ (substitutions)

$$\frac{d^2\Theta}{dr^2} = -\frac{1}{r}\frac{d\Theta}{dr} + \frac{\sin 2\Theta}{2}\left[\frac{1}{r^2} + \frac{1}{\Delta^2}\right] + \frac{2\sin^2\Theta}{\xi r}$$

Skyrmion profile boundary conditions (BC):

$$4\pi A r^2 \frac{d\Theta}{dr}\delta(r-r')\Big|_0^R - 2\pi D r \delta(r-r')\Big|_0^R = 0 \quad /: 2\pi r\delta(r-r')$$

$$2A\frac{d\Theta}{dr} - D = 0 \quad \text{for } r=R$$

$$\frac{d\Theta}{dr} = \frac{1}{\frac{2A}{D}} = \frac{1}{\xi} \quad \text{for } r=R$$

Skyrmion profile ODE + BC

$$\frac{d^2\Theta}{dr^2} = -\frac{1}{r}\frac{d\Theta}{dr} + \frac{\sin 2\Theta}{2}\left[\frac{1}{r^2} + \frac{1}{\Delta^2}\right] + \frac{2\sin^2\Theta}{\xi r}$$

$$\frac{d\Theta}{dr} = \frac{1}{\xi}, \quad \text{for } r=R$$

NUMERICAL INTEGRATION (SHOOTING METHOD)

The second order ODE is written as a system of two first order ODEs:

$$\frac{d\Theta}{dr} = \Theta' \qquad (1)$$

$$\frac{d\Theta'}{dr} = -\frac{1}{r}\Theta' + \frac{\sin 2\Theta}{2}\left[\frac{1}{r^2} + \frac{1}{\Delta^2}\right] + \frac{2\sin^2\Theta}{\xi r} \qquad (2)$$

The initial value for equation (1) is $\Theta(r=0)=0$, because the skyrmionic state profile solution is assumed. In this case, the magnetisation at the centre of the sample points in the positive z direction ($\Theta(r=0)=0$). Another known boundary condition is specified at the boundary ($r=R$) of the sample, where the derivative $\frac{d\Theta(R)}{dr} = \frac{1}{\xi}$. The initial condition $\frac{d\Theta'(r=0)}{dr}$ is chosen (changed) until the $\frac{d\Theta(R)}{dr} = \frac{1}{\xi}$

boundary condition is satisfied in the solution of initial value problem. Therefore, the initial values for the system of two first order ODEs are:

$$\Theta(0) = 0$$

$$\frac{d\Theta(0)}{dr} = S, \quad S \in \mathbb{R}$$

In an array form this system can be written as:

$$\frac{d\Theta[0]}{dr} = \Theta[1]$$

$$\frac{d\Theta[1]}{dr} = -\frac{1}{r}\Theta[1] + \frac{\sin 2\Theta[0]}{2}\left[\frac{1}{r^2} + \frac{1}{\Delta^2}\right] + \frac{2\sin^2\Theta[0]}{\xi r}$$

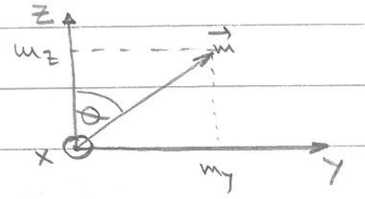
$$\Theta_0[0] = 0$$

$$\Theta_0[1] = S$$

2 CONFINED, BULK DMI $\left(D\vec{m}\cdot(\nabla\times\vec{m})\right)$

2.1 ONE-DIMENSIONAL CASE

Bulk DMI $\rightarrow$ Bloch type $\vec{m}\in(\hat{y},\hat{z})$ plane



$$\left.\begin{aligned} m_x &= 0 \\ m_y &= \sin\theta \\ m_z &= \cos\theta \end{aligned}\right\} \begin{aligned} |\vec{m}| &= 1 \\ \theta &= \theta(x) \end{aligned}$$

$$\vec{m} = \sin\theta\,\hat{y} + \cos\theta\,\hat{z}$$

- EXCHANGE

$$w_{ex} = A\left[(\nabla m_x)^2 + (\nabla m_y)^2 + (\nabla m_z)^2\right]$$

$$\nabla m_x = \frac{\partial m_x}{\partial x}\hat{x} + \frac{\partial m_x}{\partial y}\hat{y} + \frac{\partial m_x}{\partial z}\hat{z} = 0$$

$$\nabla m_y = \cos\theta\frac{d\theta}{dx}\hat{x}$$

$$\nabla m_z = -\sin\theta\frac{d\theta}{dx}\hat{x}$$

$$w_{ex} = A\left[\cos^2\theta\left(\frac{d\theta}{dx}\right)^2 + \sin^2\theta\left(\frac{d\theta}{dx}\right)^2\right] = A\left(\frac{d\theta}{dx}\right)^2$$

- DMI

$$w_{dmi} = D\,\vec{m}\cdot(\nabla\times\vec{m})$$
$$= D(m_x, m_y, m_z)\cdot\left[\cancelto{0}{\frac{\partial m_z}{\partial y}} - \cancelto{0}{\frac{\partial m_y}{\partial z}}, \cancelto{0}{\frac{\partial m_x}{\partial z}} - \frac{\partial m_z}{\partial x}, \frac{\partial m_y}{\partial x} - \cancelto{0}{\frac{\partial m_x}{\partial y}}\right]$$

$$= D(0, \sin\theta, \cos\theta)\cdot\left(0, \sin\theta\frac{d\theta}{dx}, \cos\theta\frac{d\theta}{dx}\right)$$

$$w_{dmi} = D\left[\sin^2\theta\frac{d\theta}{dx} + \cos^2\theta\frac{d\theta}{dx}\right] = D\frac{d\theta}{dx}$$

- ANISOTROPY

$$w_u = -K(\vec{m}\cdot\hat{z})^2 = -K\cos^2\theta$$

- ZEEMAN $(\vec{H}\parallel\hat{z})$

$$w_z = -M_s\,\vec{m}\cdot\vec{B} = -M_sB\cos\theta$$

- TOTAL ENERGY

$$W = (S)\int_{x_A}^{x_B}\underbrace{\left[A\left(\frac{d\theta}{dx}\right)^2 + D\left(\frac{d\theta}{dx}\right) - K\cos^2\theta - M_sB\cos\theta\right]}_{w}dx$$

cross section S

- VARIATIONAL COMPUTATION

$$\frac{\delta w}{\delta\theta(x')} = 2A\left(\frac{d\theta}{dx}\right)\frac{\delta}{\delta\theta(x')}\frac{d\theta}{dx} + D\frac{\delta}{\delta\theta(x')}\frac{d\theta}{dx} + 2K\cos\theta\sin\theta\frac{\delta\theta}{\delta\theta(x')}$$
$$+ M_sB\sin\theta\frac{\delta\theta}{\delta\theta(x')}$$

$$= 2A\frac{d\theta}{dx}\frac{d}{dx}\delta(x-x') + D\frac{d}{dx}\delta(x-x') + 2K\cos\theta\sin\theta\,\delta(x-x')$$
$$+ M_sB\sin\theta\,\delta(x-x')$$

$$\frac{\delta W}{\delta\theta(x')} = \underbrace{\int_{x_A}^{x_B}2A\frac{d\theta}{dx}\frac{d}{dx}\delta(x-x')dx}_{I_1} + \overbrace{\int_{x_A}^{x_B}D\frac{d}{dx}\delta(x-x')dx}^{I_2}$$
$$+ \underbrace{\int_{x_A}^{x_B}2K\cos\theta\sin\theta\,\delta(x-x')dx}_{I_3} + \underbrace{\int_{x_A}^{x_B}M_sB\sin\theta\,\delta(x-x')dx}_{I_4}$$

$$I_1 = \int_{x_A}^{x_B}2A\frac{d\theta}{dx}\frac{d}{dx}\delta(x-x')dx = \left|\begin{aligned} u &= \frac{d\theta}{dx} & dv &= \frac{d}{dx}\delta(x-x') \\ du &= \frac{d^2\theta}{dx^2}dx & v &= \delta(x-x') \end{aligned}\right|$$

$$= 2A\frac{d\theta}{dx}\delta(x-x')\Big|_{x_A}^{x_B} - 2A\int_{x_A}^{x_B}\frac{d^2\theta}{dx^2}\delta(x-x')dx =$$

$$= 2A\frac{d\theta}{dx}\delta(x-x')\Big|_{x_A}^{x_B} - 2A\frac{d^2\theta(x')}{dx'^2}$$

$$I_2 = \int_{x_A}^{x_B}D\frac{d}{dx}\delta(x-x')dx = D\delta(x-x')\Big|_{x_A}^{x_B}$$

$$I_3 = \int_{x_A}^{x_B}2K\cos\theta\sin\theta\,\delta(x-x')dx = 2K\cos\theta(x')\sin\theta(x')$$

$$I_4 = \int_{x_A}^{x_B} M_s B \sin\theta\, \delta(x-x')\, dx = M_s B \sin\theta(x')$$

$$\frac{\delta W}{\delta\theta(x')} = 2A \frac{d\theta}{dx}\delta(x-x')\Big|_{x_A}^{x_B} - 2A\frac{d^2\theta(x')}{dx^2} + D\delta(x-x')\Big|_{x_A}^{x_B} + 2K\cos\theta(x')\sin\theta(x')$$

$$+ M_s B \sin\theta(x') = 0$$

ODE

$$-2A\frac{d^2\theta(x')}{dx'^2} + 2K\cos\theta(x')\sin\theta(x') + M_s B\sin\theta(x') = 0 \qquad /:(-2A)$$

$$\frac{d^2\theta}{dx^2} - \frac{K}{A}\cos\theta\sin\theta - \frac{M_s B}{2A}\sin\theta = 0$$

$$\frac{d^2\theta}{dx^2} = \frac{\cos\theta\sin\theta}{\Delta^2} + \frac{M_s B}{2A}\sin\theta \qquad \Delta = \sqrt{\frac{A}{K}}$$

BOUNDARY CONDITIONS

$$2A\frac{d\theta}{dx}\delta(x-x')\Big|_{x_A}^{x_B} + D\delta(x-x')\Big|_{x_A}^{x_B} = 0 \qquad /:\delta(x-x')$$

$$2A\frac{d\theta}{dx} = -D$$

$$\frac{d\theta}{dx} = -\frac{1}{\frac{2A}{D}} = -\frac{1}{\xi} \qquad \xi = \frac{2A}{D}$$

The second order ODE can be written as a system of two first order ODEs.

$$\frac{d\theta}{dx} = \theta'$$

$$\frac{d\theta'}{dx} = \frac{\cos\theta\sin\theta}{\Delta^2} + \frac{M_s B}{2A}\sin\theta$$

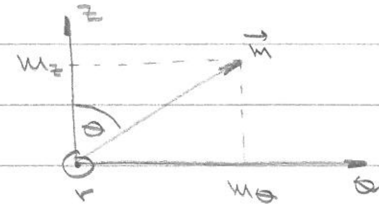
with boundary conditions:

$$\frac{d\theta}{dx} = -\frac{1}{\xi}$$

This ODE is solved similar to the case of an interfacial DMI.

2.2 TWO-DIMENSIONAL CASE (skyrmion)

Block type skyrmion $\vec{m} \in (\hat{\theta}, \hat{z})$ - chiral skyrmion



$$\left.\begin{aligned} m_r &= 0 \\ m_\theta &= \sin\theta \\ m_z &= \cos\theta \end{aligned}\right\} \quad \begin{aligned} &|\vec{m}| = 1 \\ &\vec{m} = \sin\theta\,\hat{\theta} + \cos\theta\,\hat{z} \\ &\theta = \theta(r) \end{aligned}$$

Conversion from cylindrical to Cartesian coordinate system.

$$\begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} \qquad \hat{\theta} = -\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y}$$

$$\vec{m} = \sin\theta\left[-\sin\varphi\,\hat{x} + \cos\varphi\,\hat{y}\right] + \cos\theta\,\hat{z}$$

$$\vec{m} = \underbrace{-\sin\theta\sin\varphi}_{m_x}\,\hat{x} + \underbrace{\sin\theta\cos\varphi}_{m_y}\,\hat{y} + \underbrace{\cos\theta}_{m_z}\,\hat{z}$$

$$m_x = -\sin\theta\sin\varphi$$
$$m_y = \sin\theta\cos\varphi$$
$$m_z = \cos\theta$$

$$r = \sqrt{x^2+y^2}$$

$$\frac{\partial r}{\partial x} = \frac{1}{\cancel{2}\sqrt{x^2+y^2}}\cdot \cancel{2}x = \frac{x}{r} = \cos\varphi \qquad \frac{\partial r}{\partial y} = \frac{1}{\cancel{2}\sqrt{x^2+y^2}}\cdot\cancel{2}y = \frac{y}{r} = \sin\varphi$$

$$\varphi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial\varphi}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(-\frac{y}{x^2}\right) = \frac{1}{\frac{x^2+y^2}{x^2}}\cdot\left(-\frac{y}{x^2}\right) = -\frac{y}{r^2} = -\frac{1}{r}\sin\varphi$$

$$\frac{\partial\varphi}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x} = \frac{1}{\frac{x^2+y^2}{x^2}}\cdot\frac{1}{x} = \frac{x}{r^2} = \frac{1}{r}\cos\varphi$$

$$\frac{\partial m_x}{\partial x} = -\left[\cos\theta\,\theta'\cos\varphi\sin\varphi - \cos\varphi\,\frac{1}{r}\sin\varphi\sin\theta\right]$$

$$= -\cos\theta\sin\varphi\cos\varphi\,\theta' + \frac{1}{r}\sin\theta\sin\varphi\cos\varphi$$

$$\frac{\partial m_x}{\partial y} = -\left[\cos\theta\,\theta'\sin^2\varphi + \cos\varphi\,\frac{1}{r}\cos\varphi\sin\theta\right]$$

$$= -\cos\theta\sin^2\varphi\,\theta' - \frac{1}{r}\sin\theta\cos^2\varphi$$

$$\frac{\partial m_x}{\partial z} = 0$$

$$\frac{\partial m_y}{\partial x} = \cos\theta\,\theta'\cos^2\varphi + (-\sin\varphi)\left(-\frac{1}{r}\right)\sin\varphi\sin\theta$$

$$= \cos\theta\cos^2\varphi\,\theta' + \frac{1}{r}\sin\theta\sin^2\varphi$$

$$\frac{\partial m_y}{\partial y} = \cos\theta\,\theta'\sin\varphi\cos\varphi + (-\sin\varphi)\frac{1}{r}\cos\varphi\sin\theta$$

$$= \cos\theta\sin\varphi\cos\varphi\,\theta' - \frac{1}{r}\sin\theta\sin\varphi\cos\varphi$$

$$\frac{\partial m_y}{\partial z} = 0$$

$$\frac{\partial m_z}{\partial x} = -\sin\theta\,\theta'\cos\varphi = -\sin\theta\cos\varphi\,\theta'$$

$$\frac{\partial m_z}{\partial y} = -\sin\theta\,\theta'\sin\varphi = -\sin\theta\sin\varphi\,\theta'$$

$$\frac{\partial m_z}{\partial z} = 0$$

- EXCHANGE ENERGY

$$w_{ex} = A\left[(\nabla m_x)^2 + (\nabla m_y)^2 + (\nabla m_z)^2\right]$$

$$\left(\frac{\partial m_x}{\partial x}\right)^2 = \cos^2\theta\sin^2\varphi\cos^2\varphi(\theta')^2 + \frac{1}{r^2}\sin^2\theta\sin^2\varphi\cos^2\varphi - \frac{2}{r}\sin\theta\cos\theta\sin^2\varphi\cos^2\varphi\,\theta'$$

$$\left(\frac{\partial m_x}{\partial y}\right)^2 = \cos^2\theta\sin^4\varphi(\theta')^2 + \frac{1}{r^2}\sin^2\theta\cos^4\varphi + \frac{2}{r}\sin\theta\cos\theta\sin^2\varphi\cos^2\varphi\,\theta'$$

$$\left(\frac{\partial m_x}{\partial z}\right)^2 = 0$$

$$\left(\frac{\partial m_y}{\partial x}\right)^2 = \cos^2\theta\cos^4\varphi(\theta')^2 + \frac{1}{r^2}\sin^2\theta\sin^4\varphi + \frac{2}{r}\sin\theta\cos\theta\sin^2\varphi\cos^2\varphi\,\theta'$$

$$\left(\frac{\partial m_y}{\partial y}\right)^2 = \cos^2\theta\sin^2\varphi\cos^2\varphi(\theta')^2 + \frac{1}{r^2}\sin^2\theta\sin^2\varphi\cos^2\varphi - \frac{2}{r}\sin\theta\cos\theta\sin^2\varphi\cos^2\varphi\,\theta'$$

$$\left(\frac{\partial m_y}{\partial z}\right)^2 = 0$$

$$\left(\frac{\partial m_z}{\partial x}\right)^2 = \sin^2\theta\cos^2\varphi(\theta')^2$$

$$\left(\frac{\partial m_z}{\partial y}\right)^2 = \sin^2\theta\sin^2\varphi(\theta')^2$$

$$w_{ex} = A\Big[\sin^2\theta(\theta')^2 + \cos^2\theta(\theta')^2\overbrace{\left(\sin^4\varphi + 2\sin^2\varphi\cos^2\varphi + \cos^4\varphi\right)}^{(\sin^2\varphi+\cos^2\varphi)^2}$$

$$+ \frac{1}{r^2}\sin^2\theta\underbrace{\left(\cos^4\varphi + 2\sin^2\varphi\cos^2\varphi + \sin^4\varphi\right)}_{(\sin^2\varphi+\cos^2\varphi)^2}\Big]$$

$$= A\left[\sin^2\theta(\theta')^2 + \cos^2\theta(\theta')^2 + \frac{1}{r^2}\sin^2\theta\right]$$

$$= A\left[\left(\frac{d\theta}{dr}\right)^2 + \frac{1}{r^2}\sin^2\theta\right]$$

- DMI ENERGY

$$w_{dmi} = D\vec{m}\cdot(\nabla\times\vec{m})$$

$$= D(m_x, m_y, m_z)\cdot\left(\frac{\partial m_z}{\partial y} - \overset{\to 0}{\frac{\partial m_y}{\partial z}},\ \overset{\to 0}{\frac{\partial m_x}{\partial z}} - \frac{\partial m_z}{\partial x},\ \frac{\partial m_y}{\partial x} - \frac{\partial m_x}{\partial y}\right)$$

$$m_x \frac{\partial m_z}{\partial y} = +\sin^2\theta \sin^2\varphi\, \theta'$$

$$-m_y \frac{\partial m_z}{\partial x} = +\sin^2\theta \cos^2\varphi\, \theta'$$

$$m_z\left(\frac{\partial m_y}{\partial x} - \frac{\partial m_x}{\partial y}\right) = \cos\theta\left[\cos\theta\cos^2\varphi\,\theta' + \frac{1}{r}\sin\theta\sin^2\varphi + \cos\theta\sin^2\varphi\,\theta'\right.$$

$$\left. + \frac{1}{r}\sin\theta\cos^2\varphi\right] = \cos^2\theta\,\theta' + \frac{1}{r}\sin\theta\cos\theta$$

$$w_{dmi} = D\left[\sin^2\theta\sin^2\varphi\,\theta' + \sin^2\theta\cos^2\varphi\,\theta' + \cos^2\theta\,\theta' + \frac{1}{r}\sin\theta\cos\theta\right]$$

$$= D\left[\frac{d\theta}{dr} + \frac{1}{r}\sin\theta\cos\theta\right]$$

- ANISOTROPY ENERGY

$$w_u = -K(\vec{m}\cdot\hat{z})^2 = -K\cos^2\theta$$

- ZEEMAN ENERGY ($\vec{H} \parallel \hat{z}$)

$$w_h = -M_S\,\vec{m}\cdot\vec{B} = -M_S B\cos\theta$$

- TOTAL ENERGY

$$W = \int_0^R w\, dV = \int_0^R (w_{ex} + w_{dmi} + w_u + w_h)\, dV$$

$$V = r^2\pi t$$

t ← thickness

$$dV = 2r\pi t\, dr$$

$$W = 2\pi(t)\int_0^R\left[A\left(\left(\frac{d\theta}{dr}\right)^2 + \frac{1}{r^2}\sin^2\theta\right) + D\left(\frac{d\theta}{dr} + \frac{1}{r}\sin\theta\cos\theta\right)\right.$$

$$\left. - K\cos^2\theta - M_S B\cos\theta\right] r\, dr$$

$$F[\theta] = A\left[r\left(\frac{d\theta}{dr}\right)^2 + \frac{1}{r}\sin^2\theta\right] + D\left[r\frac{d\theta}{dr} + \sin\theta\cos\theta\right]$$

$$-Kr\cos^2\theta - M_S B r\cos\theta$$

$$\frac{\delta F[\theta]}{\delta\theta(r')} = 2Ar\frac{d\theta}{dr}\frac{\delta}{\delta\theta(r')}\frac{d\theta}{dr} + \frac{2A}{r}\sin\theta\cos\theta\frac{\delta\theta}{\delta\theta(r')}$$

$$+ Dr\frac{\delta}{\delta\theta(r')}\frac{d\theta}{dr} + D\left(\cos^2\theta\frac{\delta\theta}{\delta\theta(r')} - \sin^2\theta\frac{\delta\theta}{\delta\theta(r')}\right)$$

$$+ 2Kr\cos\theta\sin\theta\frac{\delta\theta}{\delta\theta(r')} + M_S B r\sin\theta\frac{\delta\theta}{\delta\theta(r')} =$$

$$= 2Ar\frac{d\theta}{dr}\frac{d}{dr}\delta(r-r') + \frac{2A}{r}\sin\theta\cos\theta\,\delta(r-r')$$

$$+ Dr\frac{d}{dr}\delta(r-r') + D\cos 2\theta\,\delta(r-r')$$

$$+ 2Kr\cos\theta\sin\theta\,\delta(r-r') + M_S B r\sin\theta\,\delta(r-r')$$

$$\frac{\delta W[\theta]}{\delta\theta(r')} = \int_0^R \frac{\delta F[\theta]}{\delta\theta(r')}\, dr = 0$$

$$\frac{\delta W[\theta]}{\delta\theta(r')} = \underbrace{\int_0^R 2Ar\frac{d\theta}{dr}\frac{d}{dr}\delta(r-r')\,dr}_{I_1} + \underbrace{\int_0^R \frac{2A}{r}\sin\theta\cos\theta\,\delta(r-r')\,dr}_{I_2}$$

$$+ \underbrace{\int_0^R Dr\frac{d}{dr}\delta(r-r')\,dr}_{I_3} + \underbrace{\int_0^R D\cos 2\theta\,\delta(r-r')\,dr}_{I_4}$$

$$+ \underbrace{\int_0^R 2Kr\cos\theta\sin\theta\,\delta(r-r')\,dr}_{I_5} + \underbrace{\int_0^R M_S B r\sin\theta\,\delta(r-r')\,dr}_{I_6}$$

$$I_1 = \int_0^R 2Ar\frac{d\theta}{dr}\frac{d}{dr}\delta(r-r')\,dr = \ldots$$ By calculating:

$$\frac{d}{dr}\left[r\frac{d\theta}{dr}\delta(r-r')\right] = \frac{d\theta}{dr}\delta(r-r') + r\left(\frac{d^2\theta}{dr^2}\delta(r-r') + \frac{d\theta}{dr}\frac{d}{dr}\delta(r-r')\right)$$

$$\Rightarrow r\frac{d\theta}{dr}\frac{d}{dr}\delta(r-r') = \frac{d}{dr}\left[r\frac{d\theta}{dr}\delta(r-r')\right] - \frac{d\theta}{dr}\delta(r-r') - r\frac{d^2\theta}{dr^2}\delta(r-r')$$

$$I_1 = 2A\int_0^R \frac{d}{dr}\left[r\frac{d\theta}{dr}\delta(r-r')\right]dr - 2A\int_0^R \frac{d\theta}{dr}\delta(r-r')dr - 2A\int_0^R r\frac{d^2\theta}{dr^2}\delta(r-r')dr$$

$$= 2Ar\frac{d\theta}{dr}\delta(r-r')\Big|_0^R - 2A\frac{d\theta(r')}{dr} - 2Ar'\frac{d^2\theta(r')}{dr^2}$$

$$I_2 = \int_0^R \frac{2A}{r}\sin\theta\cos\theta\,\delta(r-r')dr = \frac{2A}{r'}\sin\theta(r')\cos\theta(r')$$

$$I_3 = \int_0^R Dr\frac{d}{dr}\delta(r-r')dr = \begin{vmatrix} u=r & dV=\frac{d}{dr}\delta(r-r')dr \\ du=dr & V=\delta(r-r') \end{vmatrix} =$$

$$= Dr\,\delta(r-r')\Big|_0^R - D\cancel{\int_0^R \delta(r-r')dr}^{\,1} = Dr\,\delta(r-r')\Big|_0^R - D$$

$$I_4 = \int_0^R D\cos2\theta\,\delta(r-r')dr = D\cos2\theta(r')$$

$$I_5 = \int_0^R 2Kr\cos\theta\sin\theta\,\delta(r-r')dr = 2Kr'\cos\theta(r')\sin\theta(r')$$

$$I_6 = \int_0^R M_sBr\sin\theta\,\delta(r-r')dr = M_sBr'\sin\theta(r')$$

$$I_1 + I_2 + I_3 + I_4 + I_5 + I_6 = 0$$

- ODE:

$$-2A\frac{d\theta}{dr} - 2Ar\frac{d^2\theta}{dr^2} + \frac{2A}{r}\sin\theta\cos\theta - D + D\cos2\theta$$

$$+ 2Kr\cos\theta\sin\theta + M_sBr\sin\theta = 0 \qquad /:(-2Ar)$$

$$\frac{d^2\theta}{dr^2} = -\frac{1}{r}\frac{d\theta}{dr} + \frac{\sin\theta\cos\theta}{r^2} - \frac{1}{\frac{2Ar}{D}} + \frac{\cos2\theta}{\frac{2Ar}{D}}$$

$$+ \frac{K}{A}\cos\theta\sin\theta + \frac{M_sB}{2A}\sin\theta$$

$$\frac{d^2\theta}{dr^2} = -\frac{1}{r}\frac{d\theta}{dr} + \left(\frac{1}{r^2} + \frac{1}{\Delta^2}\right)\sin\theta\cos\theta + \frac{1}{\xi r}(-1+\cos2\theta)$$

$$+ \frac{M_sB}{2A}\sin\theta$$

$$\frac{d^2\theta}{dr^2} = -\frac{1}{r}\frac{d\theta}{dr} + \left(\frac{1}{r^2} + \frac{1}{\Delta^2}\right)\frac{\sin2\theta}{2} - \frac{2\sin^2\theta}{\xi r} + \frac{M_sB}{2A}\sin\theta$$

- BOUNDARY CONDITIONS:

$$2Ar\frac{d\theta}{dr}\delta(r-r')\Big|_0^R + Dr\,\delta(r-r')\Big|_0^R = 0 \qquad /: r\delta(r-r')$$

$$2A\frac{d\theta}{dr} = -D$$

$$\frac{d\theta}{dr} = -\frac{1}{\frac{2A}{D}} = -\frac{1}{\xi}, \quad \text{for } r=R$$

ODE + BC:

$$\frac{d^2\theta}{dr^2} = -\frac{1}{r}\frac{d\theta}{dr} + \left(\frac{1}{r^2} + \frac{1}{\Delta^2}\right)\frac{\sin2\theta}{2} - \frac{2\sin^2\theta}{\xi r} + \frac{M_sB}{2A}\sin\theta$$

$$\frac{d\theta}{dr} = -\frac{1}{\xi}, \quad \text{for } r=R$$

Numerical integration of ODE+BC is performed similar to the case of interfacial DMI.

3 EFFECTIVE FIELD DERIVATION USING EULER-LAGRANGE

- used for zero-torque analytic model

3.1 EXCHANGE

$$w_{ex} = A\left[(\nabla m_x)^2 + (\nabla m_y)^2 + (\nabla m_z)^2\right]$$

$$\vec{H}_{eff} = -\frac{1}{\mu_0 M_s}\frac{\partial w_{ex}}{\partial \vec{m}} = -\frac{1}{\mu_0 M_s}\vec{h}_{eff}$$

$$\vec{h}_{eff} = h_x\hat{x} + h_y\hat{y} + h_z\hat{z}$$

$$w_{ex} = A\left[\left(\frac{\partial m_x}{\partial x}\right)^2 + \left(\frac{\partial m_x}{\partial y}\right)^2 + \left(\frac{\partial m_x}{\partial z}\right)^2 + \left(\frac{\partial m_y}{\partial x}\right)^2 + \left(\frac{\partial m_y}{\partial y}\right)^2 + \left(\frac{\partial m_y}{\partial z}\right)^2 + \left(\frac{\partial m_z}{\partial x}\right)^2 + \left(\frac{\partial m_z}{\partial y}\right)^2 + \left(\frac{\partial m_z}{\partial z}\right)^2\right]$$

$$h_x = \frac{\partial w_{ex}}{\partial m_x} - \frac{\partial}{\partial x}\frac{\partial w_{ex}}{\partial\left(\frac{\partial m_x}{\partial x}\right)} - \frac{\partial}{\partial y}\frac{\partial w_{ex}}{\partial\left(\frac{\partial m_x}{\partial y}\right)} - \frac{\partial}{\partial z}\frac{\partial w_{ex}}{\partial\left(\frac{\partial m_x}{\partial z}\right)}$$

$$h_x = A\left[0 - 2\frac{\partial}{\partial x}\frac{\partial m_x}{\partial x} - 2\frac{\partial}{\partial y}\frac{\partial m_x}{\partial y} - 2\frac{\partial}{\partial z}\frac{\partial m_x}{\partial z}\right]$$

$$h_x = -2A\left[\frac{\partial^2 m_x}{\partial x^2} + \frac{\partial^2 m_x}{\partial y^2} + \frac{\partial^2 m_x}{\partial z^2}\right]$$

$$h_x = -2A\nabla^2 m_x$$

$$h_y = \frac{\partial w_{ex}}{\partial m_y} - \frac{\partial}{\partial x}\frac{\partial w_{ex}}{\partial\left(\frac{\partial m_y}{\partial x}\right)} - \frac{\partial}{\partial y}\frac{\partial w_{ex}}{\partial\left(\frac{\partial m_y}{\partial y}\right)} - \frac{\partial}{\partial z}\frac{\partial w_{ex}}{\partial\left(\frac{\partial m_y}{\partial z}\right)}$$

$$h_y = A\left[0 - 2\frac{\partial}{\partial x}\frac{\partial m_y}{\partial x} - 2\frac{\partial}{\partial y}\frac{\partial m_y}{\partial y} - 2\frac{\partial}{\partial z}\frac{\partial m_y}{\partial z}\right]$$

$$h_y = -2A\left[\frac{\partial^2 m_y}{\partial x^2} + \frac{\partial^2 m_y}{\partial y^2} + \frac{\partial^2 m_y}{\partial z^2}\right]$$

$$h_y = -2A\nabla^2 m_y$$

$$h_z = \frac{\partial w_{ex}}{\partial m_z} - \frac{\partial}{\partial x}\frac{\partial w_{ex}}{\partial\left(\frac{\partial m_z}{\partial x}\right)} - \frac{\partial}{\partial y}\frac{\partial w_{ex}}{\partial\left(\frac{\partial m_z}{\partial y}\right)} - \frac{\partial}{\partial z}\frac{\partial w_{ex}}{\partial\left(\frac{\partial m_z}{\partial z}\right)}$$

$$h_z = A\left[0 - 2\frac{\partial}{\partial x}\frac{\partial m_z}{\partial x} - 2\frac{\partial}{\partial y}\frac{\partial m_z}{\partial y} - 2\frac{\partial}{\partial z}\frac{\partial m_z}{\partial z}\right]$$

$$h_z = -2A\left[\frac{\partial^2 m_z}{\partial x^2} + \frac{\partial^2 m_z}{\partial y^2} + \frac{\partial^2 m_z}{\partial z^2}\right]$$

$$h_z = -2A\nabla^2 m_z$$

$$\vec{h}_{eff} = -2A\left[\nabla^2 m_x\hat{x} + \nabla^2 m_y\hat{y} + \nabla^2 m_z\hat{z}\right] = -2A\nabla^2\vec{m}$$

$$\vec{H}_{eff} = \frac{2A}{\mu_0 M_s}\nabla^2\vec{m}$$

3.2 DMI

$$w_{dmi} = D\vec{m}\cdot(\nabla\times\vec{m})$$

$$w_{dmi} = D(m_x\hat{x} + m_y\hat{y} + m_z\hat{z})\cdot\left[\left(\frac{\partial m_z}{\partial y} - \frac{\partial m_y}{\partial z}\right)\hat{x} + \left(\frac{\partial m_x}{\partial z} - \frac{\partial m_z}{\partial x}\right)\hat{y} + \left(\frac{\partial m_y}{\partial x} - \frac{\partial m_x}{\partial y}\right)\hat{z}\right]$$

$$w_{dmi} = D\left[m_x\left(\frac{\partial m_z}{\partial y} - \frac{\partial m_y}{\partial z}\right) + m_y\left(\frac{\partial m_x}{\partial z} - \frac{\partial m_z}{\partial x}\right) + m_z\left(\frac{\partial m_y}{\partial x} - \frac{\partial m_x}{\partial y}\right)\right]$$

$$\vec{H}_{eff} = -\frac{1}{\mu_0 M_s}\frac{\partial w_{dmi}}{\partial \vec{m}} = -\frac{1}{\mu_0 M_s}\vec{h}_{eff}$$

$$\vec{h}_{eff} = h_x\hat{x} + h_y\hat{y} + h_z\hat{z}$$

$$h_x = \frac{\partial w_{dmi}}{\partial m_x} - \frac{\partial}{\partial x}\frac{\partial w_{dmi}}{\partial\left(\frac{\partial m_x}{\partial x}\right)} - \frac{\partial}{\partial y}\frac{\partial w_{dmi}}{\partial\left(\frac{\partial m_x}{\partial y}\right)} - \frac{\partial}{\partial z}\frac{\partial w_{dmi}}{\partial\left(\frac{\partial m_x}{\partial z}\right)}$$

$$h_x = D\left[\frac{\partial m_z}{\partial y} - \frac{\partial m_y}{\partial z} - 0 - \left(-\frac{\partial m_z}{\partial y}\right) - \frac{\partial m_y}{\partial z}\right]$$

$$h_x = 2D\left[\frac{\partial m_z}{\partial y} - \frac{\partial m_y}{\partial z}\right]$$

$$h_y = \frac{\partial w_{dmi}}{\partial m_y} - \frac{\partial}{\partial x}\frac{\partial w_{dmi}}{\partial\left(\frac{\partial m_y}{\partial x}\right)} - \frac{\partial}{\partial y}\frac{\partial w_{dmi}}{\partial\left(\frac{\partial m_y}{\partial y}\right)} - \frac{\partial}{\partial z}\frac{\partial w_{dmi}}{\partial\left(\frac{\partial m_y}{\partial z}\right)}$$

$$h_y = D\left[\frac{\partial m_x}{\partial z} - \frac{\partial m_z}{\partial x} - \frac{\partial m_z}{\partial x} - 0 - \left(-\frac{\partial m_x}{\partial z}\right)\right]$$

$$h_y = 2D\left[\frac{\partial m_x}{\partial z} - \frac{\partial m_z}{\partial x}\right]$$

$$h_z = \frac{\partial w_{dmi}}{\partial m_z} - \frac{\partial}{\partial x}\frac{\partial w_{dmi}}{\partial\left(\frac{\partial m_z}{\partial x}\right)} - \frac{\partial}{\partial y}\frac{\partial w_{dmi}}{\partial\left(\frac{\partial m_z}{\partial y}\right)} - \frac{\partial}{\partial z}\frac{\partial w_{dmi}}{\partial\left(\frac{\partial m_z}{\partial z}\right)}$$

$$h_z = D\left[\frac{\partial m_y}{\partial x} - \frac{\partial m_x}{\partial y} - \left(-\frac{\partial m_y}{\partial x}\right) - \frac{\partial m_x}{\partial y} - 0\right]$$

$$h_z = 2D\left[\frac{\partial m_y}{\partial x} - \frac{\partial m_x}{\partial y}\right]$$

$$\vec{h}_{eff} = 2D\left[\left(\frac{\partial m_z}{\partial y} - \frac{\partial m_y}{\partial z}\right)\hat{x} + \left(\frac{\partial m_x}{\partial z} - \frac{\partial m_z}{\partial x}\right)\hat{y} + \left(\frac{\partial m_y}{\partial x} - \frac{\partial m_x}{\partial y}\right)\hat{z}\right]$$

$$\vec{h}_{eff} = 2D(\nabla\times\vec{m})$$

$$\vec{H}_{eff} = -\frac{2D}{\mu_0 M_s}(\nabla\times\vec{m})$$

4. ZERO-TORQUE ANALYTIC MODEL [Beg et al. Sci. Rep. 5, 17137 (2015)]

A skyrmionic texture in two-dimensional disk sample can be approximated as (in cylindrical coordinates):

$$\left.\begin{aligned} m_r &= 0 \\ m_\phi &= \sin(kr) \\ m_z &= -\cos(kr) \end{aligned}\right\} \quad \begin{aligned} &|\vec{m}| = 1 \\ &\vec{m} = \sin(kr)\hat{\phi} - \cos(kr)\hat{z} \end{aligned}$$

This profile is valid for 'bulk-type' dmi and Bloch-type skyrmion (chiral skyrmion).

- EFFECTIVE FIELD

$$\vec{H}_{eff} = \frac{2A}{\mu_0 M_s}\nabla^2\vec{m} - \frac{2D}{\mu_0 M_s}(\nabla\times\vec{m}) = \frac{2}{\mu_0 M_s}\left[A\nabla^2\vec{m} - D(\nabla\times\vec{m})\right]$$

$$\nabla^2\vec{m} = \left(\cancel{\nabla^2 m_r}^{\,0} - \cancel{\frac{m_r}{r^2}}^{\,0} - \cancel{\frac{2}{r^2}\frac{\partial m_\phi}{\partial\phi}}^{\,0}\right)\hat{r} + \left(\nabla^2 m_\phi - \frac{m_\phi}{r^2} + \cancel{\frac{2}{r^2}\frac{\partial m_r}{\partial\phi}}^{\,0}\right)\hat{\phi}$$

$$+ \nabla^2 m_z\,\hat{z}$$

$$\nabla^2 m_\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial m_\phi}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 m_\phi}{\partial\phi^2}}^{\,0} + \cancel{\frac{\partial^2 m_\phi}{\partial z^2}}^{\,0}$$

$$\nabla^2 m_\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(rk\cos(kr)\right) = \frac{k}{r}\left[\cos(kr) - kr\sin(kr)\right]$$

$$\nabla^2 m_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial m_z}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 m_z}{\partial\phi^2}}^{\,0} + \cancel{\frac{\partial^2 m_z}{\partial z^2}}^{\,0}$$

$$\nabla^2 m_z = \frac{1}{r}\frac{\partial}{\partial r}\left(rk\sin(kr)\right) = \frac{k}{r}\left[\sin(kr) + kr\cos(kr)\right]$$

$$\nabla^2\vec{m} = \left[\frac{k}{r}(\cos(kr) - kr\sin(kr)) - \frac{\sin(kr)}{r^2}\right]\hat{\phi}$$

$$+ \left[\frac{k}{r}(\sin(kr) + kr\cos(kr))\right]\hat{z}$$

$$\nabla^2\vec{m} = \left[\frac{k}{r}\cos(kr) - \left(k^2 + \frac{1}{r^2}\right)\sin(kr)\right]\hat{\phi} + \left[\frac{k}{r}\sin(kr) + k^2\cos(kr)\right]\hat{z}$$

$$\nabla\times\vec{m} = \left(\cancelto{0}{\frac{1}{r}\frac{\partial m_z}{\partial\phi}} - \cancelto{0}{\frac{\partial m_\phi}{\partial z}}\right)\hat{r} + \left(\cancelto{0}{\frac{\partial m_r}{\partial z}} - \frac{\partial m_z}{\partial r}\right)\hat{\phi} + \frac{1}{r}\left(\frac{\partial}{\partial r}(r m_\phi) - \cancelto{0}{\frac{\partial m_r}{\partial\phi}}\right)\hat{z}$$

$$\nabla\times\vec{m} = -\frac{\partial m_z}{\partial r}\hat{\phi} + \frac{1}{r}\left(\frac{\partial}{\partial r}(r m_\phi)\right)\hat{z}$$

$$\nabla\times\vec{m} = -k\sin(kr)\,\hat{\phi} + \frac{1}{r}\left[\sin(kr) + kr\cos(kr)\right]\hat{z}$$

$$\nabla\times\vec{m} = -k\sin(kr)\,\hat{\phi} + \left[\frac{1}{r}\sin(kr) + k\cos(kr)\right]\hat{z}$$

$$\vec{H}_{eff} = \frac{2}{\mu_0 M_s}\left(A\nabla^2\vec{m} - D(\nabla\times\vec{m})\right)$$

$$\vec{H}_{eff} = \frac{2}{\mu_0 M_s}\left[\frac{Ak}{r}\cos(kr) - \left(Ak^2 + \frac{A}{r^2}\right)\sin(kr) + Dk\sin(kr)\right]\hat{\phi}$$

$$+ \frac{2}{\mu_0 M_s}\left[\frac{Ak}{r}\sin(kr) + Ak^2\cos(kr) - \frac{D}{r}\sin(kr) - Dk\cos(kr)\right]\hat{z}$$

$$\vec{H}_{eff} = \frac{2}{\mu_0 M_s}\left[\left(Dk - Ak^2 - \frac{A}{r^2}\right)\sin(kr) + \frac{Ak}{r}\cos(kr)\right]\hat{\phi}$$

$$+ \frac{2}{\mu_0 M_s}\left[\left(\frac{Ak}{r} - \frac{D}{r}\right)\sin(kr) + \left(Ak^2 - Dk\right)\cos(kr)\right]\hat{z}$$

$$\vec{m}\times\vec{H}_{eff} = 0$$

$$\vec{m}\times\vec{H}_{eff} = \left|\begin{array}{ccc|cc} \hat{r} & \hat{\phi} & \hat{z} & \hat{r} & \hat{\phi} \\ 0 & m_\phi & m_z & 0 & m_\phi \\ 0 & H_\phi & H_z & 0 & H_\phi \end{array}\right| = \left[m_\phi H_z - m_z H_\phi\right]\hat{r}$$

$$\vec{m}\times\vec{H}_{eff} = \frac{2}{\mu_0 M_s}\Bigg[\left(\frac{Ak}{r} - \frac{D}{r}\right)\sin^2(kr) + \left(Ak^2 - Dk\right)\sin(kr)\cos(kr)$$

$$+ \left(Dk - Ak^2 - \frac{A}{r^2}\right)\sin(kr)\cos(kr) + \frac{Ak}{r}\cos^2(kr)\Bigg]\hat{r}$$

$$|\vec{m}\times\vec{H}_{eff}| = \frac{2}{\mu_0 M_s}\left[\frac{Ak}{r} - \frac{D}{r}\sin^2(kr) - \frac{A}{r^2}\sin(kr)\cos(kr)\right] = 0$$

$$\frac{Ak}{r} - \frac{D}{r}\sin^2(kr) - \frac{A}{2r^2}\sin(2kr) = 0 \quad /: \frac{Ak}{r}$$

$$\underbrace{-\frac{D}{kA}\sin^2(kr) - \frac{1}{2kr}\sin(2kr) + 1 = 0}$$

Zero - torque equation

Existence of solution:

$$g(kr) = -\frac{D}{kA}\sin^2(kr) - \frac{1}{2kr}\sin(2kr) + 1$$

Taylor expansions:

$$\sin(2kr) = 2kr - \frac{(2kr)^3}{6}$$

$$\cos(2kr) = 1 - \frac{(2kr)^2}{2}$$

$$g(kr) = -\frac{D}{kA}\,\frac{1-\cos(2kr)}{2} - \frac{\sin(2kr)}{2kr} + 1$$

$$g(kr) = -\frac{D}{kA}\,\frac{\cancel{1} - \cancel{1} + \frac{(2kr)^2}{2}}{2} - \cancel{1} + \frac{(2kr)^2}{6} + \cancel{1}$$

$$g(kr) = -\frac{D}{kA}k^2r^2 + \frac{2k^2r^2}{3} \quad \boxed{<0} \rightarrow \text{by visually inspecting graphs}$$

$$-\frac{D}{kA} + \frac{2}{3} < 0$$

$$\frac{D}{kA} > \frac{2}{3}$$

This model was initially proposed and derived by Robert L. Stamps.

5. EXCHANGE AND BULK DMI

$$w = w_{ex} + w_{dmi}$$

$$w_{ex} = A\left[\left(\frac{d\theta}{dr}\right)^2 + \frac{1}{r^2}\sin^2\theta\right]$$

$$w_{dmi} = D\left[\frac{d\theta}{dr} + \frac{1}{r}\sin\theta\cos\theta\right]$$

$$W = \int_0^R (w_{ex} + w_{dmi})\, dV \qquad dV = 2r\pi t\, dr$$

$$W = 2\pi(h)\int_0^R \left[A\left(\left(\frac{d\theta}{dr}\right)^2 + \frac{1}{r^2}\sin^2\theta\right) + D\left(\frac{d\theta}{dr} + \frac{1}{r}\sin\theta\cos\theta\right)\right] r\, dr$$

$$F[\theta] = A\left[r\left(\frac{d\theta}{dr}\right)^2 + \frac{1}{r}\sin^2\theta\right] + D\left[r\frac{d\theta}{dr} + \frac{\sin 2\theta}{2}\right]$$

$$\frac{\delta F[\theta]}{\delta\theta(r')} = 2Ar\frac{d\theta}{dr}\frac{\delta}{\delta\theta(r')}\frac{d\theta}{dr} + A\frac{1}{r}2\sin\theta\cos\theta\frac{\delta\theta}{\delta\theta(r')} + Dr\frac{\delta}{\delta\theta(r')}\frac{d\theta}{dr} + \frac{D}{\cancel{2}}\cos 2\theta\cdot\cancel{2}\frac{\delta\theta}{\delta\theta(r')}$$

$$\frac{\delta F[\theta]}{\delta\theta(r')} = 2Ar\frac{d\theta}{dr}\frac{d}{dr}\delta(r-r') + \frac{A}{r}\sin 2\theta\,\delta(r-r') + Dr\frac{d}{dr}\delta(r-r') + D\cos 2\theta\,\delta(r-r')$$

$$\frac{\delta W[\theta]}{\delta\theta(r')} = \int_0^R \frac{\delta F[\theta]}{\delta\theta(r')}\, dr = 0$$

$$\frac{\delta W[\theta]}{\delta\theta(r')} = \underbrace{\int_0^R 2Ar\frac{d\theta}{dr}\frac{d}{dr}\delta(r-r')\,dr}_{I_1} + \underbrace{\int_0^R \frac{A}{r}\sin 2\theta\,\delta(r-r')\,dr}_{I_2} + \underbrace{\int_0^R Dr\frac{d}{dr}\delta(r-r')\,dr}_{I_3} + \underbrace{\int_0^R D\cos 2\theta\,\delta(r-r')\,dr}_{I_4}$$

$$I_1 = 2Ar\frac{d\theta}{dr}\delta(r-r')\Big|_0^R - 2A\frac{d\theta(r')}{dr} - 2Ar'\frac{d^2\theta(r')}{dr^2}$$

$$I_2 = \frac{A}{r'}\sin 2\theta(r')$$

These integrals were calculated in one of the previous sections.

$$I_3 = Dr\,\delta(r-r')\Big|_0^R - D$$

$$I_4 = D\cos 2\theta(r')$$

ODE:

$$-2A\frac{d\theta(r')}{dr} - 2Ar'\frac{d^2\theta(r')}{dr^2} + \frac{A}{r'}\sin 2\theta(r') - D + D\cos 2\theta(r') = 0 \qquad /:(-2Ar')$$

$(r = r')$ substitution

$$\frac{1}{r}\frac{d\theta(r)}{dr} + \frac{d^2\theta(r)}{dr^2} - \frac{1}{2r^2}\sin 2\theta(r) + \frac{D}{2Ar} - \frac{D}{2Ar}\cos 2\theta(r) = 0$$

$$\frac{d^2\theta}{dr^2} = -\frac{1}{r}\frac{d\theta}{dr} + \frac{1}{2r^2}\sin 2\theta - \frac{D}{2Ar}\underbrace{(-\cos 2\theta + 1)}_{2\sin^2\theta} = 0$$

$$\frac{d^2\theta}{dr^2} = -\frac{1}{r}\frac{d\theta}{dr} + \frac{1}{r^2}\frac{\sin 2\theta}{2} - \frac{2\sin^2\theta}{\xi r}$$

BC:

$$2Ar\frac{d\theta}{dr}\delta(r-r')\Big|_0^R + Dr\,\delta(r-r')\Big|_0^R = 0 \qquad /: r\delta(r-r')$$

$$2A\frac{d\theta}{dr}\Big|_0^R + D\Big|_0^R = 0$$

$$\frac{d\theta}{dr} = -\frac{D}{2A} = -\frac{1}{\xi} \quad \text{for } r = R$$

ODE + BC

$$\frac{d^2\theta}{dr^2} = -\frac{1}{r}\frac{d\theta}{dr} + \frac{1}{r^2}\frac{\sin 2\theta}{2} - \frac{2\sin^2\theta}{\xi r}$$

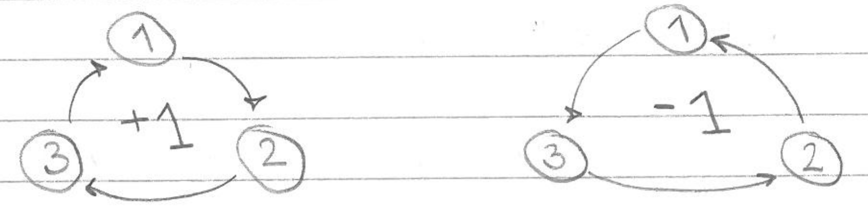
$$\frac{d\theta}{dr} = -\frac{1}{\xi}$$

6. ANGULAR MOMENTUM EIGENVALUES

From the fundamental commutation relations of angular momentum:

$$[J_i , J_j] = i\hbar\, \varepsilon_{ijk} J_k$$

$\varepsilon_{ijk} \to$ Levi-Civita symbol



$$\vec{J}^2 = J_x J_x + J_y J_y + J_z J_z$$

$J_\pm \equiv J_x \pm i J_y$ - Ladder operator

COMMUTATORS:

1) $$\begin{aligned}[\vec{J}^2, J_z] &= [J_xJ_x + J_yJ_y + J_zJ_z , J_z] \\ &= [J_xJ_x, J_z] + [J_yJ_y, J_z] + \cancelto{0}{[J_zJ_z, J_z]} \\ &= J_x[J_x, J_z] + [J_x, J_z]J_x + J_y[J_y, J_z] + [J_y, J_z]J_y \\ &= J_x(-i\hbar J_y) + (-i\hbar J_y)J_x + J_y i\hbar J_x + i\hbar J_x J_y \\ &= 0\end{aligned}$$

2) $$\begin{aligned}[J_z, J_\pm] &= [J_z, J_x \pm i J_y] \\ &= [J_z, J_x] \pm i[J_z, J_y] \\ &= i\hbar J_y \pm i(-i\hbar J_x) \\ &= i\hbar J_y \pm \hbar J_x \\ &= \pm\hbar(J_x \pm i J_y) \\ &= \pm\hbar J_\pm\end{aligned}$$

3) $$\begin{aligned}[\vec{J}^2, J_\pm] &= [\vec{J}^2, J_x \pm i J_y] \\ &= [\vec{J}^2, J_x] \pm i[\vec{J}^2, J_y] \\ &= 0\end{aligned}$$

If it is assumed that eigenket of both $\vec{J}^2$ and J_z is $|a,b\rangle$ and eigenvalues are a and b:

$$\vec{J}^2|a,b\rangle = a|a,b\rangle$$
$$J_z|a,b\rangle = b|a,b\rangle$$

$$\underbrace{\vec{J}^2(J_\pm|a,b\rangle) = J_\pm(\vec{J}^2|a,b\rangle)}_{[\vec{J}^2, J_\pm]=0} = J_\pm(a|a,b\rangle) = a J_\pm|a,b\rangle$$

$$\begin{aligned}J_z(J_\pm|a,b\rangle) &= [J_z, J_\pm]|a,b\rangle + J_\pm(J_z|a,b\rangle) \\ &= \pm\hbar J_\pm|a,b\rangle + J_\pm(b|a,b\rangle) \\ &= (b\pm\hbar)(J_\pm|a,b\rangle)\end{aligned}$$

From condition $J_z^2 \le \vec{J}^2 \Rightarrow b^2 \le a$

$J_+|a, b_{max}\rangle = 0$ - top rung
$J_-|a, b_{min}\rangle = 0$ - bottom rung

$$\begin{aligned}J_+J_- &= (J_x + iJ_y)(J_x - iJ_y) \\ &= J_x^2 + J_y^2 + iJ_yJ_x - iJ_xJ_y \\ &= \vec{J}^2 - J_z^2 + i[J_y, J_x] \\ &= \vec{J}^2 - J_z^2 + \hbar J_z\end{aligned}$$

$$\begin{aligned}J_-J_+ &= (J_x - iJ_y)(J_x + iJ_y) \\ &= J_x^2 + J_y^2 - iJ_yJ_x + iJ_xJ_y \\ &= \vec{J}^2 - J_z^2 - i[J_y, J_x] \\ &= \vec{J}^2 - J_z^2 - \hbar J_z\end{aligned}$$

$$J_-(\underbrace{J_+|a,b_{max}\rangle}_{0}) = J_-J_+|a,b_{max}\rangle$$
$$= (\vec{J}^2 - J_z^2 - \hbar J_z)|a,b_{max}\rangle$$
$$= (a^2 - b_{max}^2 - \hbar b_{max})|a,b_{max}\rangle$$

$$J_+(\underbrace{J_-|a,b_{min}\rangle}_{0}) = J_+J_-|a,b_{min}\rangle$$
$$= (\vec{J}^2 - J_z^2 + \hbar J_z)|a,b_{min}\rangle$$
$$= (a^2 - b_{min}^2 + \hbar b_{min})|a,b_{min}\rangle$$

$$a^2 = b_{max}^2 + \hbar b_{max} = b_{max}(b_{max} + \hbar)$$
$$a^2 = b_{min}^2 - \hbar b_{min} = b_{min}(b_{min} - \hbar)$$

$$b_{max}(b_{max} + \hbar) = b_{min}(b_{min} - \hbar)$$

This has two possible solutions:

a) $b_{min} = b_{max} + \hbar$ - physically impossible

b) $\boxed{b_{max} = -b_{min}}$

$$b_{max} = b_{min} + N\hbar$$
$$b_{max} = -b_{max} + N\hbar$$
$$b_{max} = \frac{N}{2}\hbar$$

$$\boxed{j \equiv \frac{b_{max}}{\hbar} = \frac{N}{2}}$$

Maximum value of J_z is $j\hbar$:

$$a = b_{max}(b_{max} + \hbar) = j\hbar(j\hbar + \hbar) = \hbar^2 j(j+1)$$
$$b = m\hbar$$

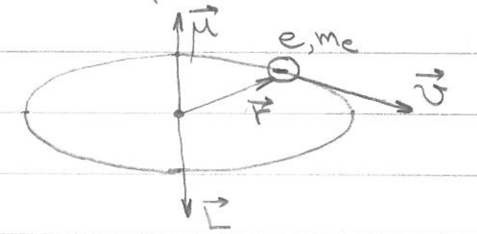
$$|a,b\rangle \rightarrow |j,m\rangle$$

$$\boxed{\begin{aligned}\vec{J}^2|j,m\rangle &= \hbar^2 j(j+1)|j,m\rangle \\ J_z|j,m\rangle &= \hbar m|j,m\rangle\end{aligned}}$$

$j = \frac{N}{2}$ $\quad N \in \mathbb{N}$

$m = \underbrace{-j, -j+1, \dots, j-1, j}_{2j+1 \text{ values}}$

7. GYROMAGNETIC RATIO



$$\vec{L} = \vec{r} \times \vec{p} = m_e \vec{r} \times \vec{v}$$

in orbital motion: $\vec{r} \perp \vec{v}$

$$L = m_e r v \quad \Rightarrow v = \frac{L}{m_e r}$$

$$\mu = I \cdot S = \frac{-e}{T} \cdot r^2\pi$$
$$T = \frac{2r\pi}{v}$$

$$\mu = -\frac{ev}{2\cancel{r\pi}} \cdot r^2\cancel{\pi} = -\frac{eL}{2m_e\cancel{r}} \cdot \cancel{r} = \underbrace{\boxed{-\frac{e}{2m_e}}}^{\gamma} L$$

$\boxed{\gamma = -\frac{e}{2m_e}}$ gyromagnetic ratio

IF $L = \hbar$: $\quad \mu = \gamma\hbar = \boxed{-\frac{e\hbar}{2m_e} \equiv \mu_B}$ Bohr magneton

8. MAGNETIC MOMENT PRECESSION (LLG PRECESSION TERM)

8.1. QUANTUM APPROACH

Heisenberg picture: $\frac{d}{dt}A(t) = \frac{i}{\hbar}[H, A(t)] + \left(\frac{\partial A}{\partial t}\right)_{H}$

$$\vec{J} = (J_x, J_y, J_z) \Rightarrow \frac{d\vec{J}}{dt} = \left(\frac{dJ_x}{dt}, \frac{dJ_y}{dt}, \frac{dJ_z}{dt}\right)$$

$$\frac{d}{dt}J_x = \frac{i}{\hbar}[H, J_x] = \frac{i}{\hbar}\sum_{j=x,y,z}\frac{\partial H}{\partial J_j}[J_j, J_x]$$

$$= \frac{i}{\hbar}\left[\frac{\partial H}{\partial J_x}\cancel{[J_x, J_x]}^{0} + \frac{\partial H}{\partial J_y}[J_y, J_x] + \frac{\partial H}{\partial J_z}[J_z, J_x]\right]$$

$$= \frac{i}{\hbar}\left[\frac{\partial H}{\partial J_y}(-i\hbar J_z) + \frac{\partial H}{\partial J_z} i\hbar J_y\right]$$

$$= \frac{\partial H}{\partial J_y}J_z - \frac{\partial H}{\partial J_z}J_y$$

$$\frac{d}{dt}J_y = \frac{i}{\hbar}[H, J_y] = \frac{i}{\hbar}\sum_{j=x,y,z}\frac{\partial H}{\partial J_j}[J_j, J_y]$$

$$= \frac{i}{\hbar}\left[\frac{\partial H}{\partial J_x}[J_x, J_y] + \frac{\partial H}{\partial J_y}\cancel{[J_y, J_y]}^{0} + \frac{\partial H}{\partial J_z}[J_z, J_y]\right]$$

$$= \frac{i}{\hbar}\left[\frac{\partial H}{\partial J_x} i\hbar J_z + \frac{\partial H}{\partial J_z}(-i\hbar J_x)\right]$$

$$= \frac{\partial H}{\partial J_z}J_x - \frac{\partial H}{\partial J_x}J_z$$

$$\frac{d}{dt}J_z = \frac{i}{\hbar}[H, J_z] = \frac{i}{\hbar}\sum_{j=x,y,z}\frac{\partial H}{\partial J_j}[J_j, J_z]$$

$$= \frac{i}{\hbar}\left[\frac{\partial H}{\partial J_x}[J_x, J_z] + \frac{\partial H}{\partial J_y}[J_y, J_z] + \frac{\partial H}{\partial J_z}\cancel{[J_z, J_z]}^{0}\right]$$

$$= \frac{i}{\hbar}\left[\frac{\partial H}{\partial J_x}(-i\hbar J_y) + \frac{\partial H}{\partial J_y} i\hbar J_x\right]$$

$$= \frac{\partial H}{\partial J_x}J_y - \frac{\partial H}{\partial J_y}J_x$$

$$\frac{d\vec{J}}{dt} = \left(\frac{\partial H}{\partial J_y}J_z - \frac{\partial H}{\partial J_z}J_y,\ \frac{\partial H}{\partial J_z}J_x - \frac{\partial H}{\partial J_x}J_z,\ \frac{\partial H}{\partial J_x}J_y - \frac{\partial H}{\partial J_y}J_x\right)$$

$$\vec{J} \times \frac{\partial H}{\partial \vec{J}} = \left|\begin{array}{ccc|cc} \hat{x} & \hat{y} & \hat{z} & \hat{x} & \hat{y} \\ J_x & J_y & J_z & J_x & J_y \\ \frac{\partial H}{\partial J_x} & \frac{\partial H}{\partial J_y} & \frac{\partial H}{\partial J_z} & \frac{\partial H}{\partial J_x} & \frac{\partial H}{\partial J_y} \end{array}\right| =$$

$$= \left(\frac{\partial H}{\partial J_z}J_y - \frac{\partial H}{\partial J_y}J_z,\ \frac{\partial H}{\partial J_x}J_z - \frac{\partial H}{\partial J_z}J_x,\ \frac{\partial H}{\partial J_y}J_x - \frac{\partial H}{\partial J_x}J_y\right)$$

$\gamma < 0$

$$\frac{d\vec{J}}{dt} = -\vec{J} \times \frac{\partial H}{\partial \vec{J}} \quad / \cdot \gamma \qquad \vec{\mu} = \gamma\vec{J}$$

$$\frac{d\vec{\mu}}{dt} = -\vec{\mu} \times \frac{\partial H}{\partial \vec{J}} \qquad \frac{\partial H}{\partial \vec{J}} = \gamma \cdot \frac{\partial H}{\partial \vec{\mu}}$$

$$\frac{d\vec{\mu}}{dt} = -\gamma\vec{\mu} \times \overbrace{\frac{\partial H}{\partial \vec{\mu}}}^{\vec{B}_{eff}} \qquad \frac{\partial H}{\partial \vec{\mu}} = \frac{\partial E}{\partial \vec{\mu}} = \vec{B}_{eff} = \mu_0 \vec{H}_{eff}$$

$$\frac{d\vec{\mu}}{dt} = -\gamma\vec{\mu} \times \mu_0 \vec{H}_{eff} \qquad \vec{H}_{eff} = \frac{1}{\mu_0}\frac{\partial E}{\partial \vec{\mu}}$$

$$\frac{d\vec{\mu}}{dt} = -\overset{\gamma_0}{\boxed{\gamma\mu_0}}\,\vec{\mu} \times \vec{H}_{eff}$$

$$\frac{d\vec{\mu}}{dt} = -\gamma_0\vec{\mu} \times \vec{H}_{eff}$$

If $\gamma > 0$: $\vec{\mu} = -\gamma\vec{J}$

$$\frac{d\vec{J}}{dt} = -\vec{J} \times \frac{\partial H}{\partial \vec{J}} \quad / \cdot (-\gamma) \qquad \frac{d\vec{\mu}}{dt} = -\vec{\mu} \times \frac{\partial H}{\partial \vec{J}}$$

$$\frac{\partial H}{\partial \vec{J}} = -\gamma\frac{\partial H}{\partial \vec{\mu}} \qquad \frac{d\vec{\mu}}{dt} = \gamma\vec{\mu} \times \frac{\partial H}{\partial \vec{\mu}} \qquad \frac{\partial H}{\partial \vec{\mu}} = \vec{B}_{eff} = \mu_0\vec{H}_{eff} = \frac{\partial E}{\partial \vec{\mu}}$$

$$\vec{H}_{eff} = \frac{1}{\mu_0}\frac{\partial E}{\partial \vec{\mu}}$$

$$\frac{d\vec{\mu}}{dt} = \gamma\mu_0\vec{\mu} \times \vec{H}_{eff} = \gamma_0\vec{\mu} \times \vec{H}_{eff}$$

If $\vec{H}_{eff}$ is defined as

$$\vec{H}_{eff} = -\frac{1}{\mu_0}\frac{\partial E}{\partial \vec{\mu}},$$

the precession term is:

$$\frac{d\vec{\mu}}{dt} = -\gamma_0 \vec{\mu} \times \vec{H}_{eff}$$

Now, if the magnetic moment $\vec{\mu}$ is replaced with a continuous magnetisation function $\vec{M} = M_s \vec{m}$

$$\frac{d\vec{M}}{dt} = -\gamma_0 \vec{M} \times \vec{H}_{eff} \qquad \vec{H}_{eff} = -\frac{1}{\mu_0}\frac{\delta E}{\delta \vec{M}}$$

or in $\vec{m}$

$$\frac{d\vec{m}}{dt} = -\gamma_0 \vec{m} \times \vec{H}_{eff} \qquad \vec{H}_{eff} = -\frac{1}{\mu_0 M_s}\frac{\delta E}{\delta \vec{m}}$$

8.2 CLASSICAL APPROACH

Torque $\vec{\tau} = \frac{d\vec{J}}{dt}$ & $\vec{\tau} = \mu_0 \vec{\mu} \times \vec{H}$

$$\frac{d\vec{J}}{dt} = \mu_0 \vec{\mu} \times \vec{H} \qquad / \cdot \gamma (<0)$$

$$\frac{d\vec{\mu}}{dt} = \gamma_0 \vec{\mu} \times \underset{\vec{H}_{eff}}{\vec{H}} \qquad \text{(Gilbert 2004)}$$

$$\vec{H}_{eff} = -\frac{1}{\mu_0}\frac{\partial E}{\partial \vec{\mu}} \qquad \left(\vec{H}_{eff} = -\frac{\partial E}{\partial \vec{\mu}} \text{ in Gilbert 2004 - cgs units}\right)$$

$$\frac{d\vec{m}}{dt} = \gamma_0 \vec{m} \times \vec{H}_{eff}, \quad \vec{H}_{eff} = -\frac{1}{\mu_0 M_s}\frac{\delta E}{\delta \vec{m}}$$

if $\gamma > 0$:

$$\frac{d\vec{m}}{dt} = -\gamma_0 \vec{m} \times \vec{H}_{eff}, \quad \vec{H}_{eff} = -\frac{1}{\mu_0 M_s}\frac{\delta E}{\delta \vec{m}}$$

9. LANDAU-LIFSHITZ (AND) GILBERT EQUATIONS

LL equation

$$\frac{d\vec{m}}{dt} = -\gamma_0'(\vec{m} \times \vec{H}_{eff}) - \lambda \vec{m} \times (\vec{m} \times \vec{H}_{eff}) \qquad \text{(LL)}$$

LLG equation

$$\frac{d\vec{m}}{dt} = -\gamma_0(\vec{m} \times \vec{H}_{eff}) + \alpha\left(\vec{m} \times \frac{d\vec{m}}{dt}\right) \qquad \text{(LLG)}$$

Equivalence of equations:

$$\text{(LLG)} \rightarrow \vec{m} \times \frac{d\vec{m}}{dt} = -\gamma_0 \vec{m} \times (\vec{m} \times \vec{H}) + \alpha \vec{m} \times \left(\vec{m} \times \frac{d\vec{m}}{dt}\right)$$

Grassman identity: $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

$$\vec{m} \times \frac{d\vec{m}}{dt} = -\gamma_0 \vec{m} \times (\vec{m} \times \vec{H}_{eff}) + \alpha\left[\underbrace{\cancel{\left(\vec{m} \cdot \frac{d\vec{m}}{dt}\right)}}_{\vec{m} \perp \frac{d\vec{m}}{dt}}\vec{m} - \underbrace{(\vec{m} \cdot \vec{m})}_{|\vec{m}|^2 = 1}\frac{d\vec{m}}{dt}\right]$$

$$\vec{m} \times \frac{d\vec{m}}{dt} = -\gamma_0 \vec{m} \times (\vec{m} \times \vec{H}_{eff}) - \alpha \frac{d\vec{m}}{dt} \xrightarrow[\text{in}]{\text{substitute}} \text{(LLG)}$$

$$\frac{d\vec{m}}{dt} = -\gamma_0(\vec{m} \times \vec{H}_{eff}) + \alpha\left(-\gamma_0 \vec{m} \times (\vec{m} \times \vec{H}_{eff}) - \alpha \frac{d\vec{m}}{dt}\right)$$

$$\frac{d\vec{m}}{dt} = -\gamma_0(\vec{m} \times \vec{H}_{eff}) - \gamma_0 \alpha \vec{m} \times (\vec{m} \times \vec{H}_{eff}) - \alpha^2 \frac{d\vec{m}}{dt}$$

$$(1+\alpha^2)\frac{d\vec{m}}{dt} = -\gamma_0(\vec{m} \times \vec{H}_{eff}) - \gamma_0 \alpha \vec{m} \times (\vec{m} \times \vec{H}_{eff})$$

$$\frac{d\vec{m}}{dt} = -\underbrace{\boxed{\frac{\gamma_0}{1+\alpha^2}}}_{\gamma_0'}(\vec{m} \times \vec{H}_{eff}) - \underbrace{\boxed{\frac{\gamma_0 \alpha}{1+\alpha^2}}}_{\lambda}\vec{m} \times (\vec{m} \times \vec{H}_{eff})$$

$$\gamma_0' = \frac{\gamma_0}{1+\alpha^2} \qquad \lambda = \frac{\gamma_0 \alpha}{1+\alpha^2}$$

More common notation for γ_0^{LL}, γ_0^{LLG}, λ, and α:

LL: $\frac{d\vec{m}}{dt} = -\gamma_0(\vec{m}\times\vec{H}_{eff}) - \lambda\vec{m}\times(\vec{m}\times\vec{H}_{eff})$, $\gamma_0>0$

LLG: $\frac{d\vec{m}}{dt} = -\gamma_0^*(\vec{m}\times\vec{H}_{eff}) + \alpha\left(\vec{m}\times\frac{d\vec{m}}{dt}\right)$, $\gamma_0>0$

$\gamma_0^* = \gamma_0(1+\alpha^2)$ $\quad \gamma_0^*\approx\gamma_0$, $\alpha<<1$

$\lambda = \frac{\gamma_0\alpha}{1+\alpha^2}$

Comment on γ_0 value:

$\gamma_0 = \gamma\mu_0 = \underbrace{g\frac{e}{2m_e}}_{\gamma}\mu_0 = 2.21\cdot10^5\ \frac{m}{As}$, $g\approx2$

10. EXCHANGE ENERGY

If two spins $\vec{S}_1$ and $\vec{S}_2$ are coupled via
$H = A\vec{S}_1\cdot\vec{S}_2$

$\vec{S}_{tot} = \vec{S}_1 + \vec{S}_2 \quad |^2$

$\vec{S}_{tot}^2 = (\vec{S}_1)^2 + (\vec{S}_2)^2 + 2\vec{S}_1\cdot\vec{S}_2$

$\vec{S}_1\cdot\vec{S}_2 = \frac{1}{2}(\vec{S}_{tot}^2 - \vec{S}_1^2 - \vec{S}_2^2)$

$\vec{S}_{tot}^2 = s(s+1)\hbar = \begin{cases} 0 & , s=0 \to \text{singlet state} \\ 2\hbar & , s=1 \to \text{triplet state}\end{cases}$

$\vec{S}_1^2 = \vec{S}_2^2 = s(s+1)\hbar = \frac{3}{4}\hbar$, $s=\frac{1}{2}$

$\vec{S}_1\cdot\vec{S}_2 = \begin{cases} -\frac{3}{4}\hbar & , s=0 \to \text{singlet state} \\ \frac{1}{4}\hbar & , s=1 \to \text{triplet state}\end{cases}$

$E_S = -\frac{3}{4}A\hbar \quad E_T = \frac{1}{4}A\hbar$, $\boxed{E_S - E_T = -A\hbar}$

For electrons (fermions), two particle wave function must be antisymmetric:

$\underset{\text{antisymmetric}}{\Psi_S(\vec{r}_1,\vec{r}_2)} = \frac{1}{\sqrt{2}}\underbrace{[\Psi_a(\vec{r}_1)\Psi_b(\vec{r}_2) + \Psi_a(\vec{r}_2)\Psi_b(\vec{r}_1)]}_{\text{symmetric}}\underset{\text{antisymmetric}}{\chi_S}$

$\Psi_T(\vec{r}_1,\vec{r}_2) = \frac{1}{\sqrt{2}}\underbrace{[\Psi_a(\vec{r}_1)\Psi_b(\vec{r}_2) - \Psi_a(\vec{r}_2)\Psi_b(\vec{r}_1)]}_{\text{antisymmetric}}\underset{\text{symmetric}}{\chi_T}$

$$E_S = \int\Psi_S^*(\vec{r}_1,\vec{r}_2)\,\hat{H}_{ex}\Psi_S(\vec{r}_1,\vec{r}_2) =$$
$$= \frac{1}{2}\int\Psi_a^*(\vec{r}_1)\Psi_b^*(\vec{r}_2)\chi_S^*\hat{H}_{ex}\Psi_a(\vec{r}_1)\Psi_b(\vec{r}_2)\chi_S\,d\vec{r}_1 d\vec{r}_2$$
$$+\frac{1}{2}\int\Psi_a^*(\vec{r}_2)\Psi_b^*(\vec{r}_1)\chi_S^*\hat{H}_{ex}\Psi_a(\vec{r}_1)\Psi_b(\vec{r}_2)\chi_S\,d\vec{r}_1 d\vec{r}_2$$
$$+\frac{1}{2}\int\Psi_a^*(\vec{r}_1)\Psi_b(\vec{r}_2)\chi_S^*\hat{H}_{ex}\Psi_a(\vec{r}_2)\Psi_b(\vec{r}_1)\chi_S\,d\vec{r}_1 d\vec{r}_2$$
$$+\frac{1}{2}\int\Psi_a^*(\vec{r}_2)\Psi_b(\vec{r}_1)\chi_S^*\hat{H}_{ex}\Psi_a(\vec{r}_2)\Psi_b(\vec{r}_1)\chi_S\,d\vec{r}_1 d\vec{r}_2$$

$$E_T = \int \psi_T^*(\vec r_1,\vec r_2)\hat H_{ex}\psi_T(\vec r_1,\vec r_2)\,d\vec r_1 d\vec r_2$$
$$= \frac{1}{2}\int \psi_a^*(\vec r_1)\psi_b(\vec r_2)\chi_T^*\hat H_{ex}\psi_a(\vec r_1)\psi_b(\vec r_2)\chi_T\,d\vec r_1 d\vec r_2$$
$$-\frac{1}{2}\int \psi_a^*(\vec r_2)\psi_b^*(\vec r_1)\chi_T^*\hat H_{ex}\psi_a(\vec r_1)\psi_b(\vec r_2)\chi_T\,d\vec r_1 d\vec r_2$$
$$-\frac{1}{2}\int \psi_a^*(\vec r_1)\psi_b^*(\vec r_2)\chi_T^*\hat H_{ex}\psi_a(\vec r_2)\psi_b(\vec r_1)\chi_T\,d\vec r_1 d\vec r_2$$
$$+\frac{1}{2}\int \psi_a^*(\vec r_2)\psi_b(\vec r_1)\chi_T^*\hat H_{ex}\psi_a(\vec r_2)\psi_b(\vec r_1)\chi_T\,d\vec r_1 d\vec r_2$$

$\chi_S^*\chi_S = 1$ and $\chi_T^*\chi_T = 1$

$$E_S - E_T = \int \psi_a^*(\vec r_2)\psi_b^*(\vec r_1)\hat H_{ex}\psi_a(\vec r_1)\psi_b(\vec r_2)\,d\vec r_1 d\vec r_2 + \int \psi_a^*(\vec r_1)\psi_b^*(\vec r_2)\hat H_{ex}\psi_a(\vec r_2)\psi_b(\vec r_1)\,d\vec r_1 d\vec r_2$$

Because wave functions are symmetric

$$E_S - E_T = 2\int \psi_a^*(\vec r_1)\psi_b(\vec r_2)\hat H_{ex}\psi_a(\vec r_2)\psi_b(\vec r_1)\,d\vec r_1 d\vec r_2 = J$$

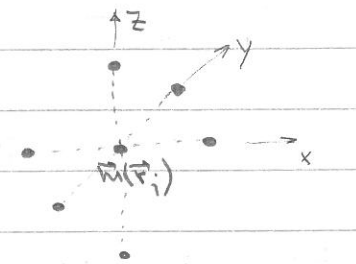
$$E_{ex} = -\underbrace{(E_S - E_T)}_{J}\vec S_1\cdot\vec S_2 = -J\vec S_1\cdot\vec S_2 \rightarrow \text{between two spins}$$

ATOMISTIC:

$$E_{ex} = -\sum_{i=1}^{N}\sum_{j\neq i}^{N} J_{ij}\vec S_i\cdot\vec S_j = -J\sum_{i=1}^{N}\sum_{j}^{nN}\vec S_i\cdot\vec S_j = -JS^2\sum_{i=1}^{N}\vec m(\vec r_i)\cdot\sum_j^{nN}\vec m(\vec r_j)$$

CONTINUOUS EXPRESSION:

$$E_{ex} = -JS^2\sum_{i=1}^{N}\vec m(\vec r_i)\cdot\underbrace{\sum_j^{nN}\vec m(\vec r_j)}$$



$$\sum_j^{nN}\vec m(\vec r_j) = \vec m(\vec r_i) + \frac{\partial \vec m(\vec r)}{\partial x}\Big|_{\vec r=\vec r_i} a + \frac{1}{2}\frac{\partial^2\vec m(\vec r)}{\partial x^2}\Big|_{\vec r=\vec r_i} a^2 + \dots$$
$$+\vec m(\vec r_i) + \frac{\partial \vec m(\vec r)}{\partial x}\Big|_{\vec r=\vec r_i}(-a) + \frac{1}{2}\frac{\partial^2\vec m(\vec r)}{\partial x^2}\Big|_{\vec r=\vec r_i} a^2 + \dots$$
$$+\vec m(\vec r_i) + \frac{\partial \vec m(\vec r)}{\partial y}\Big|_{\vec r=\vec r_i} a + \frac{1}{2}\frac{\partial^2\vec m(\vec r)}{\partial y^2}\Big|_{\vec r=\vec r_i} a^2 + \dots$$
$$+\vec m(\vec r_i) + \frac{\partial \vec m(\vec r)}{\partial y}\Big|_{\vec r=\vec r_i}(-a) + \frac{1}{2}\frac{\partial^2\vec m(\vec r)}{\partial y^2}\Big|_{\vec r=\vec r_i} a^2 + \dots$$

$$+\vec m(\vec r_i) + \frac{\partial \vec m(\vec r)}{\partial z}\Big|_{\vec r=\vec r_i} a + \frac{1}{2}\frac{\partial^2\vec m(\vec r)}{\partial z^2}\Big|_{\vec r=\vec r_i} a^2 + \dots$$
$$+\vec m(\vec r_i) + \frac{\partial \vec m(\vec r)}{\partial z}\Big|_{\vec r=\vec r_i}(-a) + \frac{1}{2}\frac{\partial^2\vec m(\vec r)}{\partial z^2}\Big|_{\vec r=\vec r_i} a^2$$

$$\sum_j^{nN}\vec m(\vec r_j) = 6\vec m(r_i) + \underbrace{\left[\frac{\partial^2\vec m(\vec r)}{\partial x^2} + \frac{\partial^2\vec m(\vec r)}{\partial y^2} + \frac{\partial^2\vec m(\vec r)}{\partial z^2}\right]}_{\nabla^2\vec m}\Bigg|_{\vec r=\vec r_i} a^2 + O(a^4)$$

6 = const.

$$E_{ex} = -JS^2\sum_{i=1}^{N}\left[\vec m(\vec r_i)\cdot 6\vec m(\vec r_i) + \vec m(\vec r_i)\cdot\nabla^2\vec m(\vec r)\Big|_{\vec r=\vec r_i}\cdot a^2\right]$$

$$E_{ex} = -JS^2a^2\sum_{i=1}^{N}\vec m(\vec r_i)\cdot\nabla^2\vec m(\vec r)\Big|_{\vec r=\vec r_i}$$

$$= -JS^2a^2\left(\frac{n}{a^3}\right)\int\vec m\cdot\nabla^2\vec m\,dV \qquad \left(\frac{n}{a^3}dV \text{ atoms in } dV\right)$$

$$= -\underbrace{\boxed{\frac{JS^2n}{a}}}_{A}\int\vec m\cdot\nabla^2\vec m\,dV = -A\int\vec m\cdot\nabla^2\vec m\,dV$$

Continuous expression is usually written as $E_{ex} = A\int(\nabla\vec m)^2 dV$

$$\vec m\cdot\nabla^2\vec m = m_x\nabla^2 m_x + m_y\nabla^2 m_y + m_z\nabla^2 m_z$$

by using: $\nabla\cdot(\psi\vec A) = \psi\nabla\cdot\vec A + \vec A\cdot\nabla\psi$

$$\left.\begin{aligned} m_x\nabla^2 m_x &= m_x\nabla\cdot(\nabla m_x) = \nabla\cdot(m_x\nabla m_x) - \nabla m_x\cdot\nabla m_x\\ m_y\nabla^2 m_y &= \nabla\cdot(m_y\nabla m_y) - (\nabla m_y)^2\\ m_z\nabla^2 m_z &= \nabla\cdot(m_z\nabla m_z) - (\nabla m_z)^2\end{aligned}\right\} +$$

$$\vec m\cdot\nabla^2\vec m = \nabla\cdot(m_x\nabla m_x + m_y\nabla m_y + m_z\nabla m_z) - (\nabla m_x)^2 - (\nabla m_y)^2 - (\nabla m_z)^2$$

From $\vec m\cdot\vec m = 1 \;/\; \frac{\partial}{\partial x}$

$$\frac{\partial\vec m}{\partial x}\cdot\vec m + \vec m\cdot\frac{\partial\vec m}{\partial x} = 0 \qquad 2\frac{\partial\vec m}{\partial x}\cdot\vec m = 0 \qquad \vec m\cdot\frac{\partial\vec m}{\partial x} = 0$$

$$\frac{\partial\vec m}{\partial y}\cdot\vec m + \vec m\cdot\frac{\partial\vec m}{\partial y} = 0 \qquad 2\frac{\partial\vec m}{\partial y}\cdot\vec m = 0 \qquad \vec m\cdot\frac{\partial\vec m}{\partial y} = 0$$

$$\frac{\partial\vec m}{\partial z}\cdot\vec m + \vec m\cdot\frac{\partial\vec m}{\partial y} = 0 \qquad 2\frac{\partial\vec m}{\partial z}\cdot\vec m = 0 \qquad \vec m\,\frac{\partial\vec m}{\partial z} = 0$$

$$\left.\begin{aligned} m_x \nabla m_x &= m_x\left(\frac{\partial m_x}{\partial x}, \frac{\partial m_x}{\partial y}, \frac{\partial m_x}{\partial z}\right) \\ m_y \nabla m_y &= m_y\left(\frac{\partial m_y}{\partial x}, \frac{\partial m_y}{\partial y}, \frac{\partial m_y}{\partial z}\right) \\ m_z \nabla m_z &= m_z\left(\frac{\partial m_z}{\partial x}, \frac{\partial m_z}{\partial y}, \frac{\partial m_z}{\partial z}\right) \end{aligned}\right\} +$$

$$m_x \nabla m_x + m_y \nabla m_y + m_z \nabla m_z = \left(\cancel{\vec{m}\cdot\frac{\partial \vec{m}}{\partial x}}^{\,0}, \cancel{\vec{m}\cdot\frac{\partial \vec{m}}{\partial y}}^{\,0}, \cancel{\vec{m}\cdot\frac{\partial \vec{m}}{\partial z}}^{\,0}\right) = 0$$

$$\vec{m}\cdot\nabla^2\vec{m} = -\left[(\nabla m_x)^2 + (\nabla m_y)^2 + (\nabla m_z)^2\right] = -(\nabla\vec{m})^2$$

$$E_{ex} = A\int(\nabla\vec{m})^2\, dV \quad \text{or} \quad E_{ex} = A\int\left[\left(\frac{\partial\vec{m}}{\partial x}\right)^2 + \left(\frac{\partial\vec{m}}{\partial y}\right)^2 + \left(\frac{\partial\vec{m}}{\partial z}\right)^2\right] dV$$

EFFECTIVE FIELD:

$$\vec{H}_{eff} = -\frac{1}{\mu_0 M_s}\vec{h}_{eff}, \quad \vec{h}_{eff} = \left(h^x_{eff}, h^y_{eff}, h^z_{eff}\right)$$

$$h^x_{eff} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[E_{ex}[\vec{m} + \varepsilon\hat{x}\delta(\vec{r}-\vec{r}_0)] - E_{ex}[\vec{m}]\right]$$

Using $E_{ex} = A\int\left[\left(\frac{\partial\vec{m}}{\partial x}\right)^2 + \left(\frac{\partial\vec{m}}{\partial y}\right)^2 + \left(\frac{\partial\vec{m}}{\partial z}\right)^2\right] dV$

$$h^x_{eff} = A\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int\Bigg[\left(\frac{\partial\vec{m}}{\partial x} + \frac{\varepsilon\hat{x}\,\partial\delta(\vec{r}-\vec{r}_0)}{\partial x}\right)^2 + \left(\frac{\partial\vec{m}}{\partial y} + \frac{\varepsilon\hat{x}\,\partial\delta(\vec{r}-\vec{r}_0)}{\partial y}\right)^2 + \left(\frac{\partial\vec{m}}{\partial z} + \frac{\varepsilon\hat{x}\,\partial\delta(\vec{r}-\vec{r}_0)}{\partial z}\right)$$
$$- \left(\frac{\partial\vec{m}}{\partial x}\right)^2 - \left(\frac{\partial\vec{m}}{\partial y}\right)^2 - \left(\frac{\partial\vec{m}}{\partial z}\right)^2\Bigg] dV$$

$$= A\lim_{\varepsilon\to 0}\int\Bigg[\left(\frac{\partial\vec{m}}{\partial x}\right)^2 + 2\frac{\partial\vec{m}}{\partial x}\frac{\varepsilon\hat{x}\,\partial\delta(\vec{r}-\vec{r}_0)}{\partial x} + \varepsilon^2\left(\frac{\partial\delta(\vec{r}-\vec{r}_0)}{\partial x}\right)^2 + \left(\frac{\partial\vec{m}}{\partial y}\right)^2 + 2\frac{\partial\vec{m}}{\partial y}\frac{\varepsilon\hat{x}\,\partial\delta(\vec{r}-\vec{r}_0)}{\partial y}$$
$$+ \varepsilon^2\left(\frac{\partial\delta(\vec{r}-\vec{r}_0)}{\partial y}\right)^2 + \left(\frac{\partial\vec{m}}{\partial z}\right)^2 + 2\frac{\partial\vec{m}}{\partial z}\frac{\varepsilon\hat{x}\,\partial\delta(\vec{r}-\vec{r}_0)}{\partial z} + \varepsilon^2\left(\frac{\partial\delta(\vec{r}-\vec{r}_0)}{\partial z}\right)^2$$
$$- \left(\frac{\partial\vec{m}}{\partial x}\right)^2 - \left(\frac{\partial\vec{m}}{\partial y}\right)^2 - \left(\frac{\partial\vec{m}}{\partial z}\right)^2\Bigg] dV$$

$$= 2A\hat{x}\int\Bigg[\underbrace{\frac{\partial\vec{m}}{\partial x}\frac{\partial\delta(\vec{r}-\vec{r}_0)}{\partial x}}_{I_1} + \underbrace{\frac{\partial\vec{m}}{\partial y}\frac{\partial\delta(\vec{r}-\vec{r}_0)}{\partial y}}_{I_2} + \underbrace{\frac{\partial\vec{m}}{\partial z}\frac{\partial\delta(\vec{r}-\vec{r}_0)}{\partial z}}_{I_3}\Bigg] dV$$

$$I_1 = \int\frac{\partial\vec{m}}{\partial x}\frac{\partial\delta(\vec{r}-\vec{r}_0)}{\partial x}\, dV = \left|\begin{array}{ll} u = \dfrac{\partial\vec{m}}{\partial x} & dv = \dfrac{\partial\delta(\vec{r}-\vec{r}_0)}{\partial x} \\ du = \dfrac{\partial^2\vec{m}}{\partial x^2}dx & v = \delta(\vec{r}-\vec{r}_0) \end{array}\right|$$

$$= \frac{\partial\vec{m}}{\partial x}\delta(\vec{r}-\vec{r}_0)\Big|_B - \int\frac{\partial^2\vec{m}}{\partial x^2}\delta(\vec{r}-\vec{r}_0)\, dV = -\frac{\partial^2\vec{m}}{\partial x^2}$$

$I_2 = -\dfrac{\partial^2\vec{m}}{\partial y^2} \qquad I_3 = -\dfrac{\partial^2\vec{m}}{\partial z^2}$ (similar to I_1)

$$h^x_{eff} = -2A\hat{x}\left(\frac{\partial^2\vec{m}}{\partial x^2} + \frac{\partial^2\vec{m}}{\partial y^2} + \frac{\partial^2\vec{m}}{\partial z^2}\right) = -2A\nabla^2 m_x$$

Similarly: $h^y_{eff} = -2A\nabla^2 m_y \qquad h^z_{eff} = -2A\nabla^2 m_z$

$$\vec{H}_{eff} = \frac{2A}{\mu_0 M_s}\left(\nabla^2 m_x, \nabla^2 m_y, \nabla^2 m_z\right) = \frac{2A}{\mu_0 M_s}\nabla^2\vec{m}$$

11. DMI ENERGY

- ATOMISTIC EXPRESSION

$$E_{dmi} = \vec{d}_{ij}\cdot(\vec{S}_i\times\vec{S}_j)$$

- CONTINUOUS EXPRESSION

In bulk (helimagnetic) materials

$$\vec{d}_{ij} = d\,\vec{r}_{ij}$$

$$E_{dmi} = \sum_{i=1}^{N}\sum_{j}^{nN}\vec{d}_{ij}\cdot(\vec{S}_i\times\vec{S}_j) = d\sum_{i=1}^{N}\sum_{j}^{nN}\vec{r}_{ij}\cdot(\vec{S}_i\times\vec{S}_j)$$

$$E_{dmi} = dS^2\sum_{i=1}^{N}\underbrace{\sum_{j}^{nN}\vec{r}_{ij}\cdot[\vec{m}(\vec{r}_i)\times\vec{m}(\vec{r}_j)]}$$

$$= \hat{x}\cdot\left\{\vec{m}(\vec{r}_i)\times\left[\vec{m}(\vec{r}_i)+\frac{\partial\vec{m}(\vec{r})}{\partial x}\Big|_{\vec{r}=\vec{r}_i}a+\frac{1}{2}\frac{\partial^2\vec{m}(\vec{r})}{\partial x^2}\Big|_{\vec{r}=\vec{r}_i}a^2+\dots\right]\right\}$$

$$+(-\hat{x})\cdot\left\{\vec{m}(\vec{r}_i)\times\left[\vec{m}(\vec{r}_i)+\frac{\partial\vec{m}(\vec{r})}{\partial x}\Big|_{\vec{r}=\vec{r}_i}(-a)+\frac{1}{2}\frac{\partial^2\vec{m}(\vec{r})}{\partial x^2}\Big|_{\vec{r}=\vec{r}_i}a^2+\dots\right]\right\}$$

$$+\hat{y}\cdot\left\{\vec{m}(\vec{r}_i)\times\left[\vec{m}(\vec{r}_i)+\frac{\partial\vec{m}(\vec{r})}{\partial y}\Big|_{\vec{r}=\vec{r}_i}a+\frac{1}{2}\frac{\partial^2\vec{m}(\vec{r})}{\partial y^2}\Big|_{\vec{r}=\vec{r}_i}a^2+\dots\right]\right\}$$

$$+(-\hat{y})\cdot\left\{\vec{m}(\vec{r}_i)\times\left[\vec{m}(\vec{r}_i)+\frac{\partial\vec{m}(\vec{r})}{\partial y}\Big|_{\vec{r}=\vec{r}_i}(-a)+\frac{1}{2}\frac{\partial^2\vec{m}(\vec{r})}{\partial y^2}a^2+\dots\right]\right\}$$

$$+\hat{z}\cdot\left\{\vec{m}(\vec{r}_i)\times\left[\vec{m}(\vec{r}_i)+\frac{\partial\vec{m}(\vec{r})}{\partial z}\Big|_{\vec{r}=\vec{r}_i}a+\frac{1}{2}\frac{\partial^2\vec{m}(\vec{r})}{\partial z^2}a^2+\dots\right]\right\}$$

$$+(-\hat{z})\cdot\left\{\vec{m}(\vec{r}_i)\times\left[\vec{m}(\vec{r}_i)+\frac{\partial\vec{m}(\vec{r})}{\partial z}\Big|_{\vec{r}=\vec{r}_i}(-a)+\frac{1}{2}\frac{\partial^2\vec{m}(\vec{r})}{\partial z^2}a^2+\dots\right]\right\} =$$

$$= 2\hat{x}\cdot\left[\vec{m}(\vec{r}_i)\times\frac{\partial\vec{m}(\vec{r})}{\partial x}a\right]+2\hat{y}\cdot\left[\vec{m}(\vec{r}_i)\times\frac{\partial\vec{m}(\vec{r})}{\partial y}a\right]+2\hat{z}\cdot\left[\vec{m}(\vec{r}_i)\times\frac{\partial\vec{m}(\vec{r})}{\partial z}a\right]$$

$$= 2a\left(m_y\frac{\partial m_z}{\partial x}-m_z\frac{\partial m_y}{\partial x}\right)+2a\left(m_z\frac{\partial m_x}{\partial y}-m_x\frac{\partial m_z}{\partial y}\right)+2a\left(m_x\frac{\partial m_y}{\partial z}-m_y\frac{\partial m_x}{\partial z}\right)$$

$$= 2a\left[m_x\left(\frac{\partial m_y}{\partial z}-\frac{\partial m_z}{\partial y}\right)+m_y\left(\frac{\partial m_z}{\partial x}-\frac{\partial m_x}{\partial z}\right)+m_z\left(\frac{\partial m_x}{\partial y}-\frac{\partial m_y}{\partial x}\right)\right] \quad (1)$$

$$= 2a\,\vec{m}\cdot(-\nabla\times\vec{m}) = -2a\,\vec{m}\cdot(\nabla\times\vec{m})$$

$$E_{dmi} = -2a\,d\,S^2\sum_{i=1}^{N}\vec{m}(\vec{r}_i)\cdot(\nabla\times\vec{m}(\vec{r}))\Big|_{\vec{r}=\vec{r}_i}\cdot\boxed{\frac{1}{2}}$$

→ to avoid double counting

$$= -a\,d\,S^2\sum_{i=1}^{N}\vec{m}(\vec{r}_i)\cdot(\nabla\times\vec{m}(\vec{r}))\Big|_{\vec{r}=\vec{r}_i}$$

$$= \underbrace{\boxed{-\frac{dS^2 n}{a^2}}}_{D}\int\vec{m}\cdot(\nabla\times\vec{m})\,dV \qquad \left(\frac{n}{a^3}dV \text{ atoms in } dV\right)$$

$$E_{dmi} = D\int\vec{m}\cdot(\nabla\times\vec{m})\,dV$$

From (1) using the Lifshitz invariant definition:

$$\mathcal{L}_{ij}^{(k)} = m_i\frac{\partial m_j}{\partial k}-m_j\frac{\partial m_i}{\partial k}$$

$$(1)\rightarrow 2a\left(\mathcal{L}_{yz}^{(x)}+\mathcal{L}_{zx}^{(y)}+\mathcal{L}_{xy}^{(z)}\right) = -2a\,\vec{m}\cdot(\nabla\times\vec{m})$$

EFFECTIVE FIELD

$$\vec{H}_{eff} = -\frac{1}{\mu_0 M_s}\vec{h}_{eff}, \qquad \vec{h}_{eff} = (h^x_{eff}, h^y_{eff}, h^z_{eff})$$

$$h^x_{eff} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[E_{dmi}[\vec{m}+\varepsilon\hat{x}\delta(\vec{r}-\vec{r}_0)]-E_{dmi}[\vec{m}]\right]$$

Using $E_{dmi} = D\int\left[m_x\frac{\partial m_z}{\partial y}-m_x\frac{\partial m_y}{\partial z}+m_y\frac{\partial m_x}{\partial z}-m_y\frac{\partial m_z}{\partial x}+m_z\frac{\partial m_y}{\partial x}-m_z\frac{\partial m_x}{\partial y}\right]dV$

$$h^x_{eff} = \lim_{\varepsilon\to 0}\frac{1}{\cancel{\varepsilon}}D\int \cancel{m_x\frac{\partial m_z}{\partial y}}+\cancel{\varepsilon}\delta(\vec{r}-\vec{r}_0)\frac{\partial m_z}{\partial y}-\cancel{m_x\frac{\partial m_y}{\partial z}}-\cancel{\varepsilon}\delta(\vec{r}-\vec{r}_0)\frac{\partial m_y}{\partial z}$$

$$+\cancel{m_y\frac{\partial m_x}{\partial z}}+m_y\frac{\cancel{\varepsilon}\,\partial\delta(\vec{r}-\vec{r}_0)}{\partial z}-\cancel{m_y\frac{\partial m_z}{\partial x}}+\cancel{m_z\frac{\partial m_y}{\partial x}}-\cancel{m_z\frac{\partial m_x}{\partial y}}-m_z\frac{\cancel{\varepsilon}\,\partial\delta(\vec{r}-\vec{r}_0)}{\partial y}$$

$$-\cancel{m_x\frac{\partial m_z}{\partial y}}+\cancel{m_x\frac{\partial m_y}{\partial z}}-\cancel{m_y\frac{\partial m_x}{\partial z}}+\cancel{m_y\frac{\partial m_z}{\partial x}}-\cancel{m_z\frac{\partial m_y}{\partial x}}+\cancel{m_z\frac{\partial m_x}{\partial y}}\Big]dV$$

$$= D\int\left[\delta(\vec{r}-\vec{r}_0)\frac{\partial m_z}{\partial y}-\delta(\vec{r}-\vec{r}_0)\frac{\partial m_y}{\partial z}+m_y\frac{\partial\delta(\vec{r}-\vec{r}_0)}{\partial z}-m_z\frac{\partial\delta(\vec{r}-\vec{r}_0)}{\partial y}\right]dV$$

$$= D\left[\frac{\partial m_z}{\partial y}-\frac{\partial m_y}{\partial z}\right]+D\left[-\frac{\partial m_y}{\partial z}+\frac{\partial m_z}{\partial y}\right] = 2D\left(\frac{\partial m_z}{\partial y}-\frac{\partial m_y}{\partial z}\right)$$

Similarly: $h^y_{eff} = 2D\left(\frac{\partial m_x}{\partial z}-\frac{\partial m_z}{\partial x}\right)$ and $h^z_{eff} = 2D\left(\frac{\partial m_y}{\partial x}-\frac{\partial m_x}{\partial y}\right)$

Accordingly:

$$\vec{H}_{eff} = -\frac{2D}{\mu_0 M_s}\left(\frac{\partial m_z}{\partial y} - \frac{\partial m_y}{\partial z}, \frac{\partial m_x}{\partial z} - \frac{\partial m_z}{\partial x}, \frac{\partial m_y}{\partial x} - \frac{\partial m_x}{\partial y}\right)$$

$$\vec{H}_{eff} = -\frac{2D}{\mu_0 M_s}(\nabla \times \vec{m})$$

UNIAXIAL ANISOTROPY ENERGY

- ATOMISTIC EXPRESSION

$$E_a = K(\vec{m}\cdot\hat{u})^2$$

$K>0$ hard-axis (easy-plane)

$K<0$ easy-axis

- CONTINUOUS EXPRESSION

$$E_a = K\int(\vec{m}\cdot\hat{u})^2\, dV$$

- EFFECTIVE FIELD

$$\vec{H}_{eff} = -\frac{1}{\mu_0 M_s}\vec{h}_{eff}, \quad \vec{h}_{eff} = (h^x_{eff}, h^y_{eff}, h^z_{eff})$$

$$h^x_{eff} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[E_a[\vec{m}+\hat{x}\,\varepsilon\delta(\vec{r}-\vec{r}_0)] - E_a[\vec{m}]\right]$$

$$= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int K\left[(\vec{m}\cdot\hat{u} + \hat{x}\,\varepsilon\delta(\vec{r}-\vec{r}_0)\cdot\hat{u})^2 - (\vec{m}\cdot\hat{u})^2\right] dV$$

$$= \lim_{\varepsilon\to 0}\frac{1}{\cancel{\varepsilon}}\int\left[\cancel{K(\vec{m}\cdot\hat{u})^2} + 2K(\vec{m}\cdot\hat{u})(\hat{x}\,\cancel{\varepsilon}\delta(\vec{r}-\vec{r}_0)\cdot\hat{u}) + K\cancel{\varepsilon^2}\delta^2(\vec{r}-\vec{r}_0) - \cancel{K(\vec{m}\cdot\hat{u})^2}\right] dV$$

$$= \int 2K(\vec{m}\cdot\hat{u})(\hat{x}\cdot\hat{u})\,\delta(\vec{r}-\vec{r}_0)\, dV = 2K u_x m_x$$

Similarly $h^y_{eff} = 2K u_y m_y$ and $h^z_{eff} = 2K u_z m_z$

$$\vec{H}_{eff} = \frac{2K}{\mu_0 M_s}(m_x u_x, m_y u_y, m_z u_z)$$

2. ZEEMAN ENERGY

- ATOMISTIC EXPRESSION

$$E_h = -\mu_0 M_s\, \vec{m}\cdot\vec{H}$$

- CONTINUOUS EXPRESSION

$$E_h = -\mu_0 M_s \int \vec{m}\cdot\vec{H}\, dV$$

- EFFECTIVE FIELD

$$\vec{H}_{eff} = -\frac{1}{\mu_0 M_s}, \quad \vec{h}_{eff} = (h^x_{eff}, h^y_{eff}, h^z_{eff})$$

$$h^x_{eff} = -\frac{1}{\mu_0 M_s}\left[E_h[\vec{m}+\hat{x}\,\varepsilon\delta(\vec{r}-\vec{r}_0)] - E_h[\vec{m}]\right]$$

$$= -\frac{1}{\cancel{\mu_0 M_s}}\lim_{\varepsilon\to 0}\frac{1}{\cancel{\varepsilon}}\left[-\cancel{\mu_0 M_s}\int\left[\cancel{\vec{m}\cdot\vec{H}} + \hat{x}\,\cancel{\varepsilon}\delta(\vec{r}-\vec{r}_0)\cdot\vec{H} - \cancel{\vec{m}\cdot\vec{H}}\right] dV\right] =$$

$$= \int \hat{x}\cdot\vec{H}\,\delta(\vec{r}-\vec{r}_0)\, dV = \vec{H}\cdot\hat{x} = H_x$$

Similarly $h^y_{eff} = H_y$ and $h^z_{eff} = H_z$

$$\vec{H}_{eff} = \vec{H}$$

13. EIGENVALUE METHOD

Undamped LLG equation (precession term) is:

$$\dot{\vec{m}} = \frac{\partial \vec{m}}{\partial t} = -\gamma_0^* (\vec{m} \times \vec{H}_{eff})$$

Any magnetisation perturbation from the equilibrium state $\vec{m}_0$ can be written as

$$\vec{m}(t) = \vec{m}_0 + \varepsilon \vec{v}(t) \quad , \quad \varepsilon > 0 \quad \text{and} \quad \vec{v}(t) \perp \vec{m}_0$$

$$\frac{\partial}{\partial t}(\vec{m}_0 + \varepsilon \vec{v}(t)) = -\gamma_0^* \left[(\vec{m}_0 + \varepsilon \vec{v}(t)) \times \vec{H}_{eff}(\vec{m}_0 + \varepsilon \vec{v}(t)) \right]$$

$$\vec{H}_{eff}(\vec{m}_0 + \varepsilon \vec{v}(t)) = \vec{H}_{eff}(\vec{m}_0) + \varepsilon \vec{H}'_{eff}(\vec{m}_0) \cdot \vec{v}(t) + \mathcal{O}(\varepsilon^2)$$

$$\varepsilon \frac{\partial v(t)}{\partial t} = \varepsilon \dot{v} = -\gamma_0^* \left[\vec{m}_0 \times \vec{H}_{eff}(\vec{m}_0) + \varepsilon \vec{v}(t) \times \vec{H}_{eff}(\vec{m}_0) + \vec{m}_0 \times \left(\varepsilon \vec{H}'_{eff}(\vec{m}_0) \cdot \vec{v}(t) \right) + \varepsilon^2 \vec{v}(t) \times \left(\vec{H}'_{eff}(\vec{m}_0) \cdot \vec{v}(t) \right) \right]$$

All term of ε^2 order or higher are neglected and in equilibrium $\vec{m}_0 \times \vec{H}_{eff}(\vec{m}_0) = 0$:

$$\dot{v}(t) = -\gamma_0^* \left[\vec{v}(t) \times \vec{H}_{eff}(\vec{m}_0) + \vec{m}_0 \times \left(\vec{H}'_{eff}(\vec{m}_0) \cdot \vec{v}(t) \right) \right]$$

If $\vec{H}_{eff}(\vec{m}_0) = \vec{H}_0 = h_0 \vec{m}_0$, because $\vec{H}_0 \parallel \vec{m}_0$ in equilibrium and $h_0 = |\vec{H}_0|$.

$$\dot{v}(t) = -\gamma_0^* \left[-\vec{m}_0 \times h_0 \vec{v}(t) + m_0 \times \left(\vec{H}'_{eff}(\vec{m}_0) \cdot \vec{v}(t) \right) \right]$$

$$= \gamma_0^* \vec{m}_0 \times \left[h_0 \vec{v}(t) - \left(\vec{H}'_{eff}(\vec{m}_0) \cdot \vec{v}(t) \right) \right]$$

Now, if the problem is discretised:

$$\dot{v}(t) = \gamma_0^* \vec{m}_0 \times \underbrace{\left[h_0 \cdot \mathbb{1} - \vec{H}'_{eff}(\vec{m}_0) \right]}_{\mathcal{A}_0} \cdot \vec{v}(t) \qquad (1)$$

The matrix with $\vec{w} \times \vec{v} = \Lambda(\vec{w}) \cdot \vec{v}$ property is:

$$\Lambda(\vec{w}) = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}$$

Now, Eq. (1) can be written as

$$\dot{v}(t) = A \cdot v(t) \quad , \text{ where } A = \gamma_0^* \Lambda(\vec{m}_0) \cdot \mathcal{A}_0$$

A linear differential equation of this form has a full set of solutions of the form $\vec{v}(t) = \vec{v}_0 e^{i\omega t}$. Using this ansatz, the differential equation becomes:

$$i\omega \cancel{e^{i\omega t}} \vec{v}_0 = A \vec{v}_0 \cancel{e^{i\omega t}}$$

So, the eigenvalue problem is:

$$i\omega \vec{v}_0 = A \vec{v}_0$$

The eigenvalues $\omega = 2\pi f$ are the resonant frequencies and from eigenvectors $\vec{v}_0$, the magnetisation dynamics of eigenmodes can be calculated:

$$\vec{m}(t) = \vec{m}_0 + \vec{v}_0 e^{i\omega t}$$

14. ANTIFERROMAGNETIC EXCHANGE (SUBLATTICES)

Atomistic model:

$$E_{ex} = -\sum_{i}^{N} \sum_{j\neq i}^{N} J_{ij} \vec{S}_i \times \vec{S}_j$$

If this expression is subdivided into two sublattices:

$$E_{ex} = \left[-J \sum_{i}^{N/2} \vec{S}_i^{A} \sum_{j}^{nN} \vec{S}_j^{B} \right] + \left[-J \sum_{i}^{N/2} \vec{S}_i^{B} \sum_{j}^{nN} \vec{S}_j^{A} \right]$$

$$= -2J \sum_{i}^{N/2} \vec{S}_i^{A} \sum_{j}^{nN} \vec{S}_j^{B} = -2JS^2 \sum_{i} \vec{m}^A(\vec{r}_i) \sum_{j} \vec{m}^B(\vec{r}_j)$$

$$\sum_{j}^{nN} \vec{m}^B(\vec{r}_j) = \vec{m}^B(\vec{r}_i) + \frac{\partial \vec{m}^B}{\partial x}(\vec{r}_i)a + \frac{1}{2}\frac{\partial^2 \vec{m}^B}{\partial x^2} a^2 + O(a^3)$$

$$+ \vec{m}^B(\vec{r}_i) + \frac{\partial \vec{m}^B}{\partial x}(\vec{r}_i)(-a) + \frac{1}{2}\frac{\partial^2 \vec{m}^B}{\partial x^2}(-a^2) + O(a^3)$$

$$+ \vec{m}^B(\vec{r}_i) + \frac{\partial \vec{m}^B}{\partial y}(\vec{r}_i)a + \frac{1}{2}\frac{\partial^2 \vec{m}^B}{\partial y^2} a^2 + O(a^3)$$

$$+ \vec{m}^B(\vec{r}_i) + \frac{\partial \vec{m}^B}{\partial y}(\vec{r}_i)(-a) + \frac{1}{2}\frac{\partial^2 \vec{m}^B}{\partial y^2}(-a)^2 + O(a^3)$$

$$+ \vec{m}^B(\vec{r}_i) + \frac{\partial \vec{m}^B}{\partial z}(\vec{r}_i)a + \frac{1}{2}\frac{\partial^2 \vec{m}^B}{\partial z^2} a^2 + O(a^3)$$

$$+ \vec{m}^B(\vec{r}_i) + \frac{\partial \vec{m}^B}{\partial z}(\vec{r}_i)(-a) + \frac{1}{2}\frac{\partial^2 \vec{m}^B}{\partial z^2}(-a)^2 + O(a^3)$$

$$= 6\vec{m}^B(\vec{r}_i) + \left(\frac{\partial^2 \vec{m}^B}{\partial x^2}(\vec{r}_i) + \frac{\partial \vec{m}^B}{\partial y^2}(\vec{r}_i) + \frac{\partial \vec{m}^B}{\partial z^2}(\vec{r}_i) \right) a^2 + O(a^4)$$

$$= 6\vec{m}^B(\vec{r}_i) + a^2 \nabla^2 \vec{m}^B(\vec{r}_i)$$

$$E_{ex} = -2JS^2 \sum_{i=1}^{N/2} \vec{m}^A(\vec{r}_i) \cdot \left[6\vec{m}^B(\vec{r}_i) + a^2 \nabla^2 \vec{m}^B(\vec{r}_i) \right]$$

$$= -6JS^2 \sum_{i=1}^{N} \vec{m}^A(\vec{r}_i) \cdot \vec{m}^B(\vec{r}_i) - JS^2 a^2 \sum_{i=1}^{N} \vec{m}^A(\vec{r}_i) \cdot \nabla^2 \vec{m}^B(\vec{r}_i)$$

In the continuous limit: $\sum \to \int$, $\frac{n}{a^3}dV$ atoms in dV

$$E_{ex} = -\frac{6JS^2 n}{a^3} \int \vec{m}^A \cdot \vec{m}^B \, dV - \frac{JS^2 n}{a} \int \vec{m}^A \cdot \nabla^2 \vec{m}^B \, dV$$

Using the 1st Green's identity:

$$\int_V \left(\vec{m}^A \nabla^2 \vec{m}^B + \nabla \vec{m}^A \cdot \nabla \vec{m}^B \right) dV = \cancelto{0}{\oint_{\partial V} \vec{m}^A \nabla \vec{m}^B \cdot d\vec{S}}$$

$$\int \vec{m}^A \nabla^2 \vec{m}^B \, dV = -\int \nabla \vec{m}^A \cdot \nabla \vec{m}^B \, dV$$

$$E_{ex} = -\frac{6JS^2 n}{a^3} \int \vec{m}^A \cdot \vec{m}^B \, dV + \frac{JS^2 n}{a} \int \nabla \vec{m}^A \cdot \nabla \vec{m}^B \, dV$$