CE100 Algorithms and Programming II

Week-4 (Heap/Heap Sort)

Spring Semester, 2021-2022

<iframe width=700, height=500 frameBorder=0 src="../ce100-week-4-heap.md_slide.html"></iframe>

Heap/Heap Sort

Outline (1)

- Heaps
    - Max / Min Heap
- Heap Data Structure
    - Heapify
        - Iterative
        - Recursive

Outline (2)

- Extract-Max
- Build Heap

Outline (3)

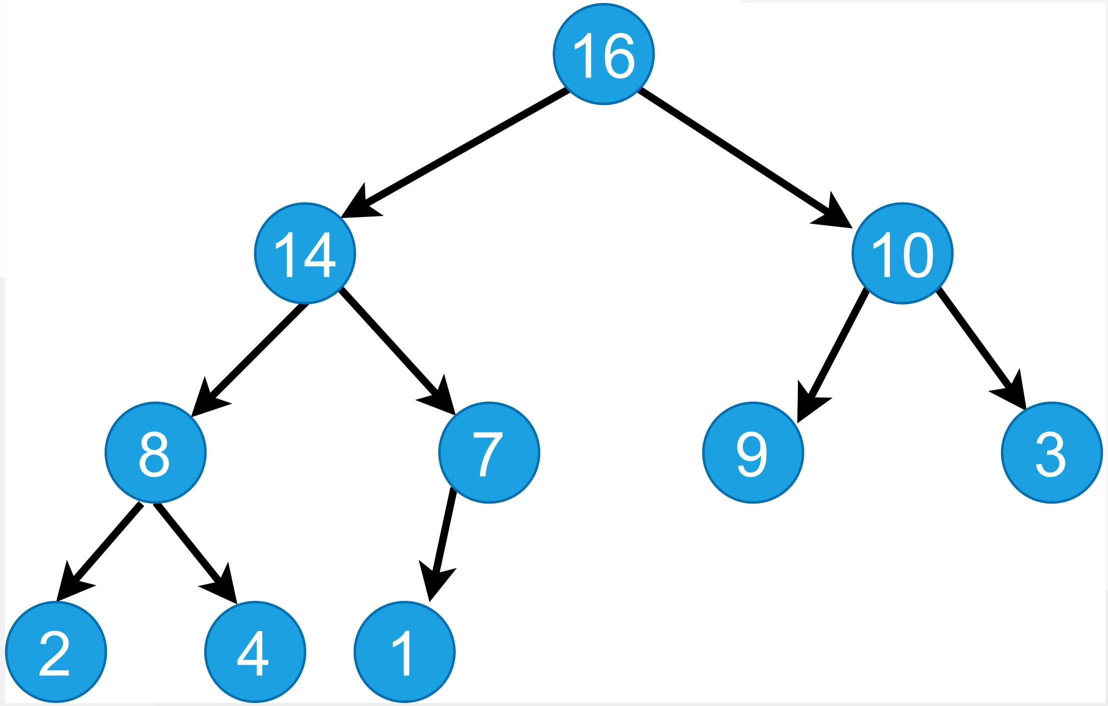
- Heap Sort
- Priority Queues
- Linked Lists
- Radix Sort
- Counting Sort

Heapsort

- Worst-case runtime: $O(nlgn)$
- Sorts in-place
- Uses a special data structure (heap) to manage information during execution of the algorithm
    - Another design paradigm

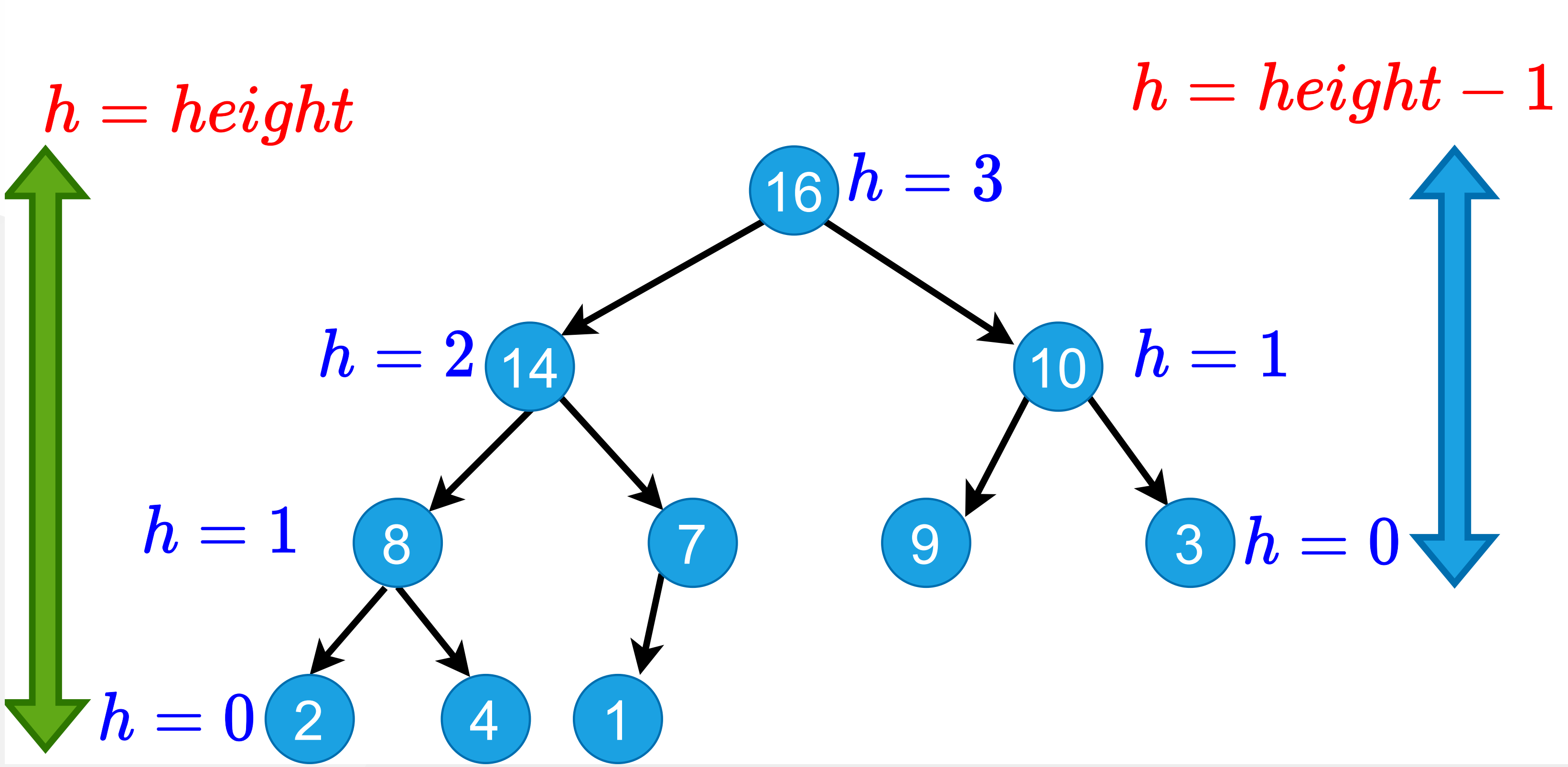
Heap Data Structure (1)

- Nearly complete binary tree
    - Completely filled on all levels except possibly the lowest level



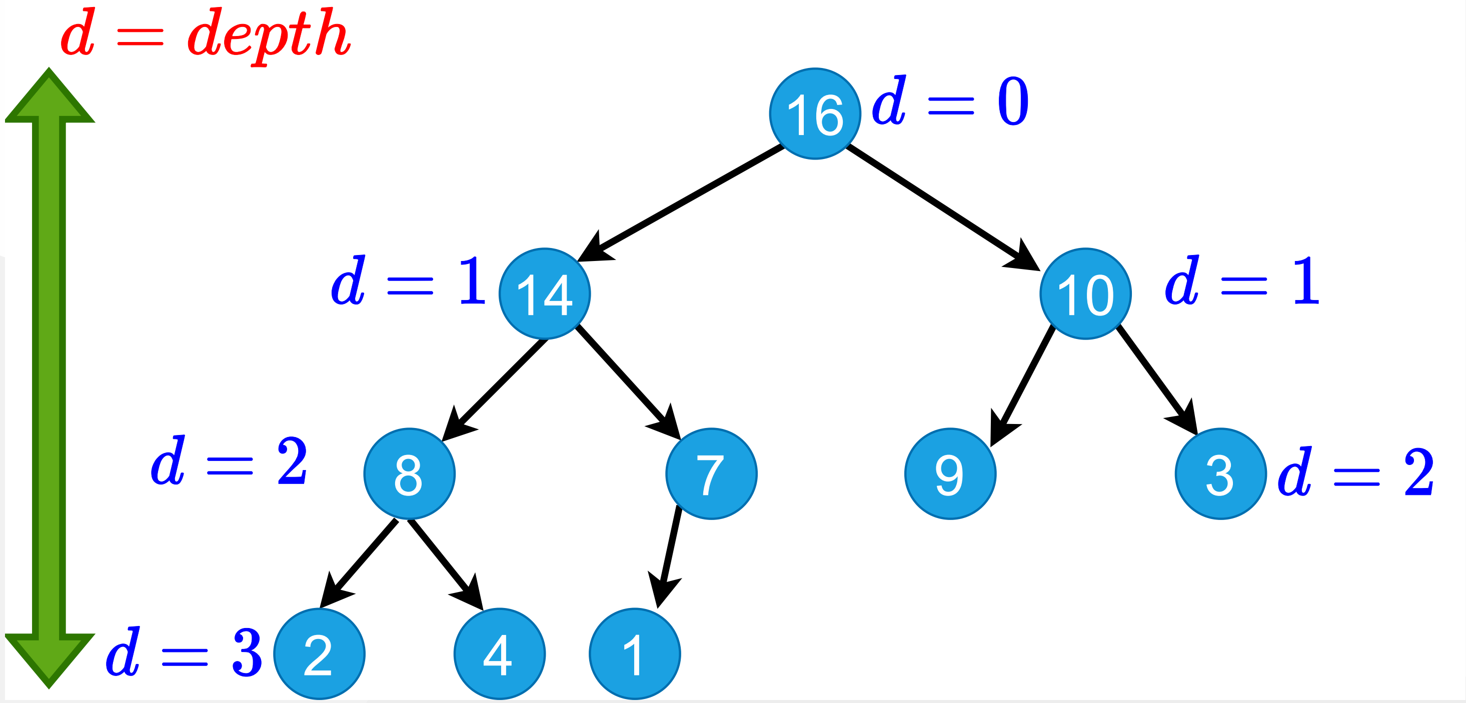
Heap Data Structure (2)

- **Height of node i:** Length of the longest simple downward path from **i** to a **leaf**
- **Height of the tree:** height of the **root**



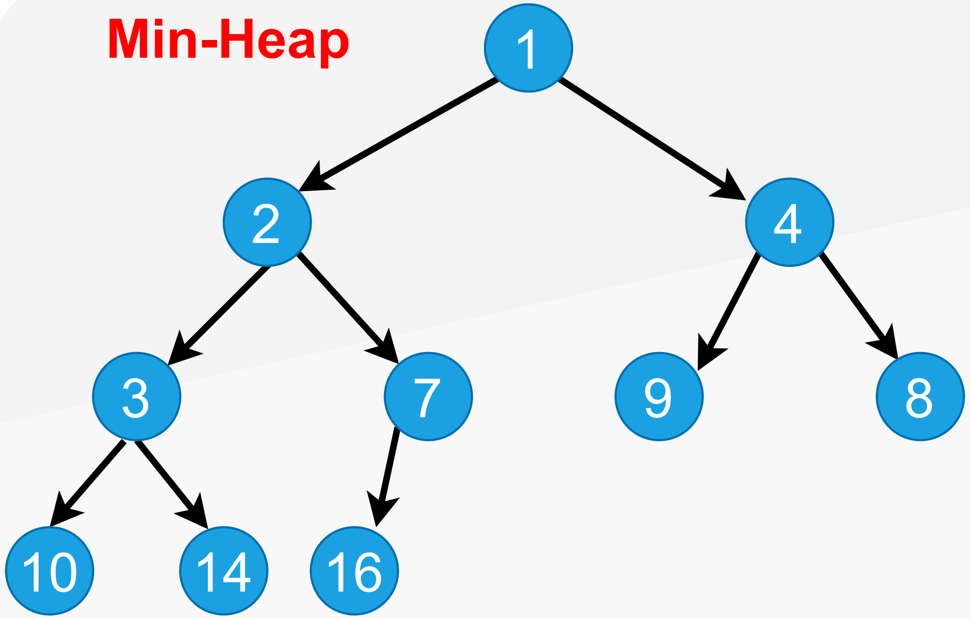
Heap Data Structures (3)

- **Depth of node i:** Length of the simple downward path from the **root** to node **i**



Heap Property: Min-Heap

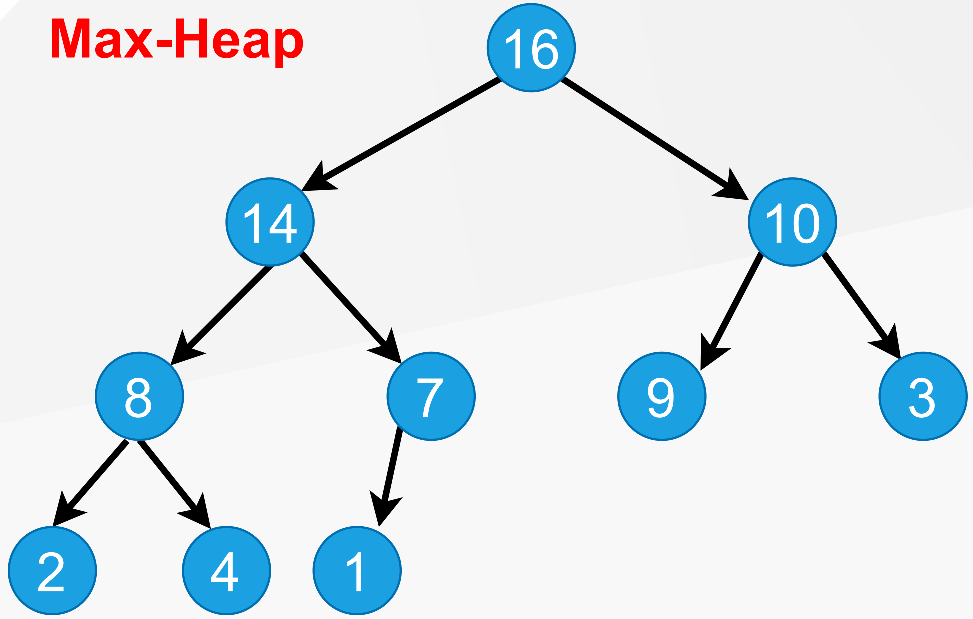
- The **smallest** element in any subtree is the **root** element in a **min-heap**
- **Min heap:** For every node **i** other than **root**, $A[parent(i)] \leq A[i]$
    - Parent node is always smaller than the child nodes



Heap Property: Max-Heap

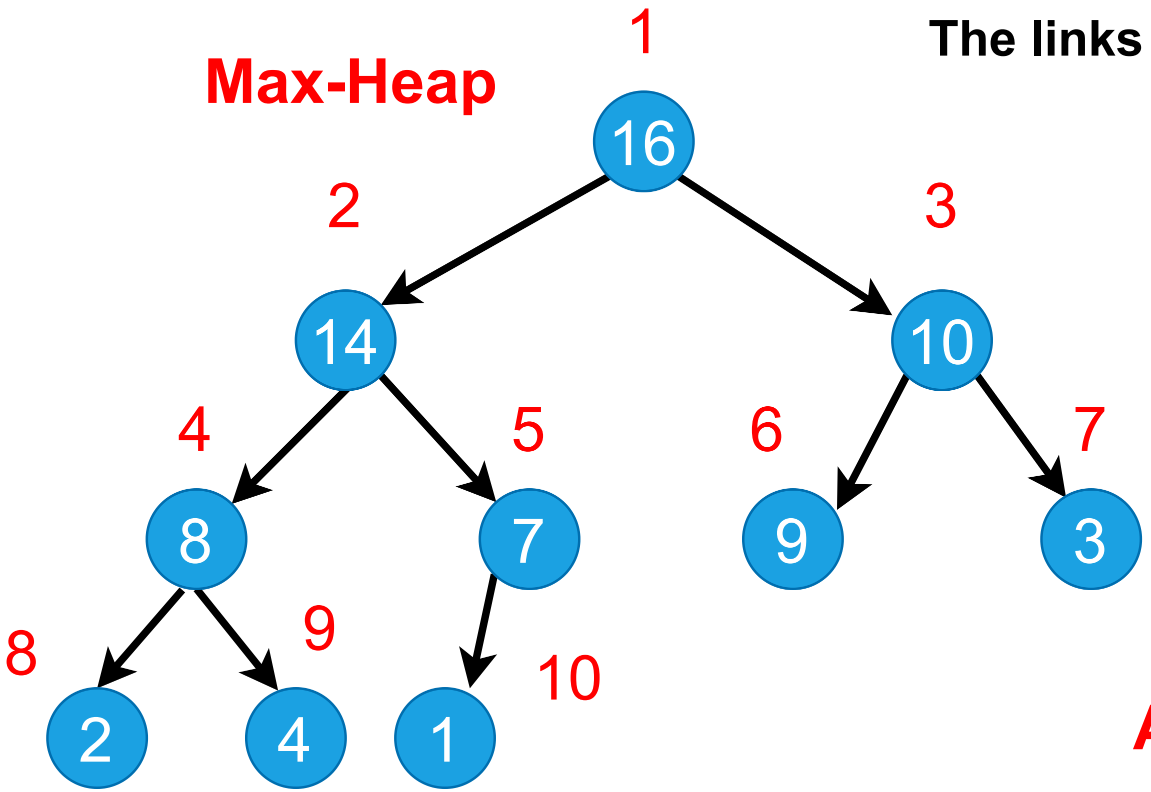
- The **largest** element in any subtree is the **root** element in a **max-heap**
  - We will focus on max-heaps
- **Max heap:** For every node **i** other than **root**, $A[parent(i)] \geq A[i]$
  - Parent node is always larger than the child nodes

Max-Heap



Heap Data Structures (4)

Max-Heap



The links in the heap are implicit

$$left(i) = 2i$$

e.g.Left child of node 4 has index 8

$$right(i) = 2i + 1$$

e.g. Right child of node 2 has index 5

$$parent(i) = \lfloor i/2 \rfloor$$

e.g.Parent of node 7 has index 3

Array Storage

Heap can be stored in a linear array

1	2	3	4	5	6	7	8	9	10
16	14	10	8	7	9	3	2	4	1

Heap Data Structures (5)

- Computing left child, right child, and parent indices very fast
  - **left(i) = 2i** $\Longrightarrow$ binary left shift
  - **right(i) = 2i+1** $\Longrightarrow$ binary left shift, then set the lowest bit to 1
  - **parent(i) = floor(i/2)** $\Longrightarrow$ right shift in binary
- $A[1]$ is always the **root** element
- Array A has two attributes:
  - **length(A):** The number of elements in A
  - **n = heap-size(A):** The number elements in $heap$
    - $n \leq length(A)$

Heap Operations : EXTRACT-MAX (1)

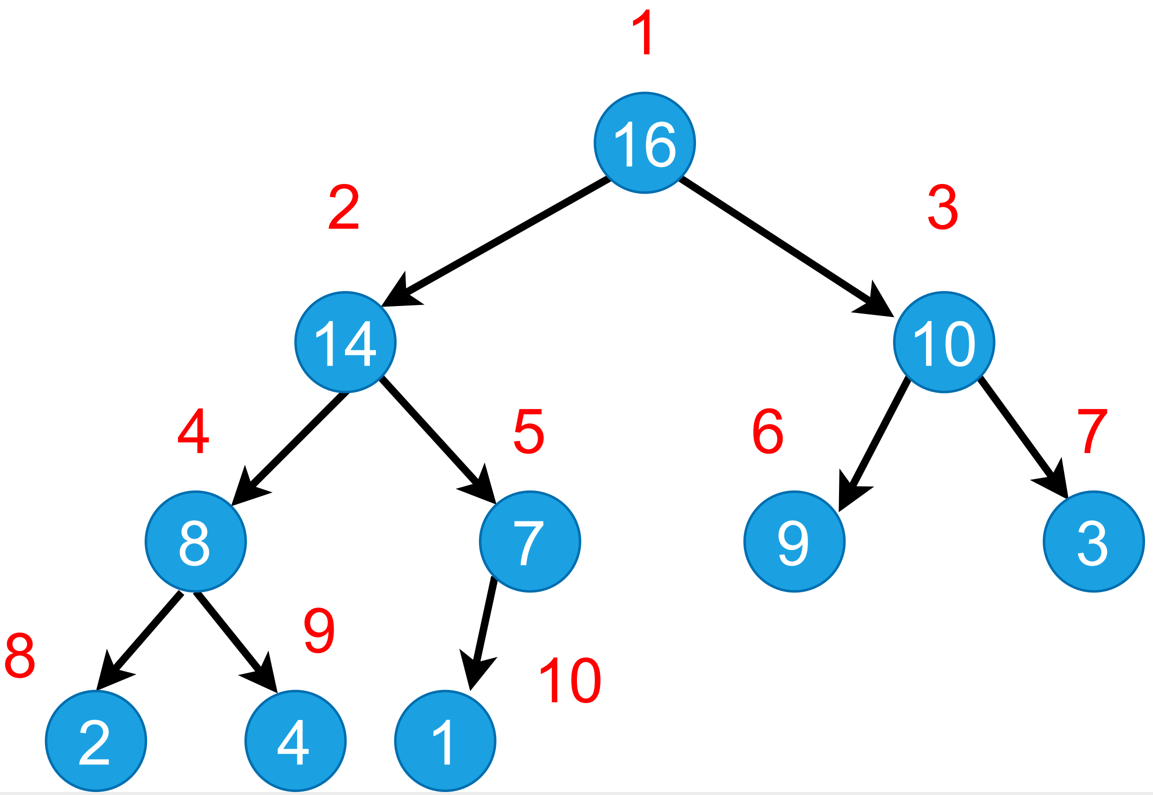
```
EXTRACT-MAX(A, n)
  max = A[1]
  A[1] = A[n]
  n = n - 1
  HEAPIFY(A, 1,n)
  return max
```

Heap Operations : EXTRACT-MAX (2)

- Return the max element,and reorganize the heap to maintain heap property

```
EXTRACT-MAX(A, n)
   max    = A[1]
   A[1]   = A[n]
   n = n - 1
   HEAPIFY(A, 1,n)
   return max
```

max=?

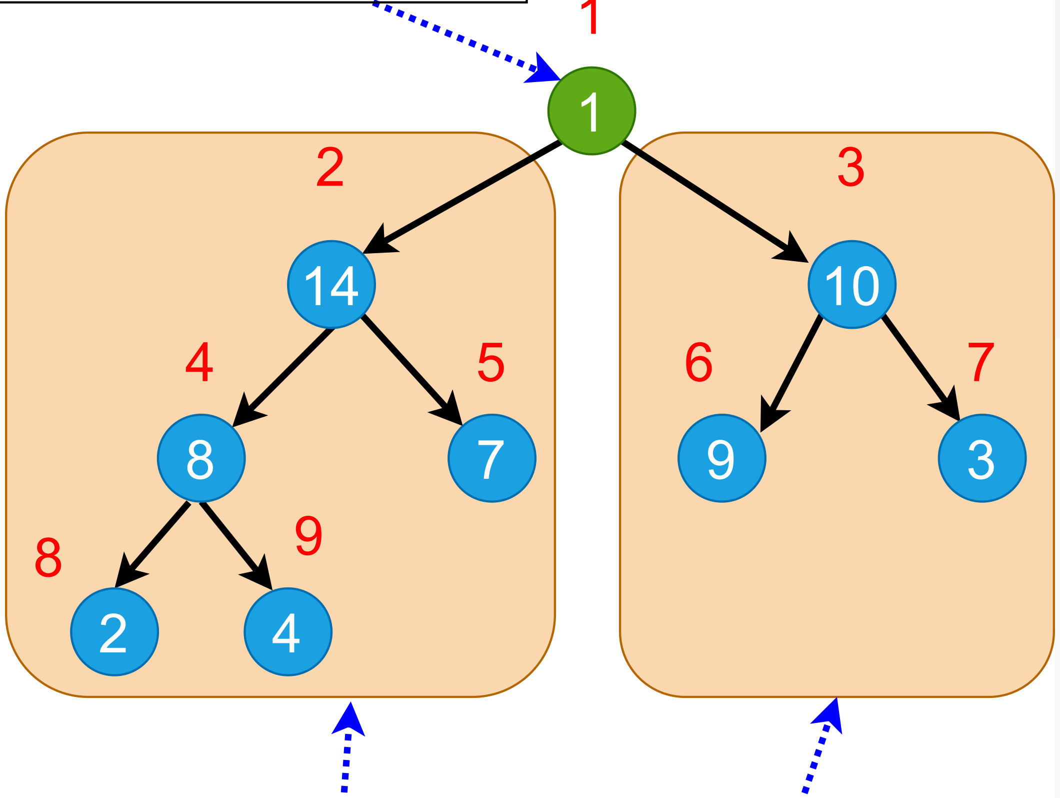


Heap Operations: HEAPIFY (1)

```
EXTRACT-MAX(A, n)
   max    = A[1]
   A[1]   = A[n]
   n = n - 1
   HEAPIFY(A, 1,n)
   return max
```

max= 16

Heap property violated at the root



Heap property satisfied for left and right subtrees

Heap Operations: HEAPIFY (2)

- Maintaining heap property:
    - Subtrees rooted at $left[i]$ and $right[i]$ are already heaps.
    - But, $A[i]$ may violate the heap property (i.e., may be smaller than its children)
- **Idea:** Float down the value at $A[i]$ in the heap so that subtree rooted at i becomes a heap.

Heap Operations: HEAPIFY (2)

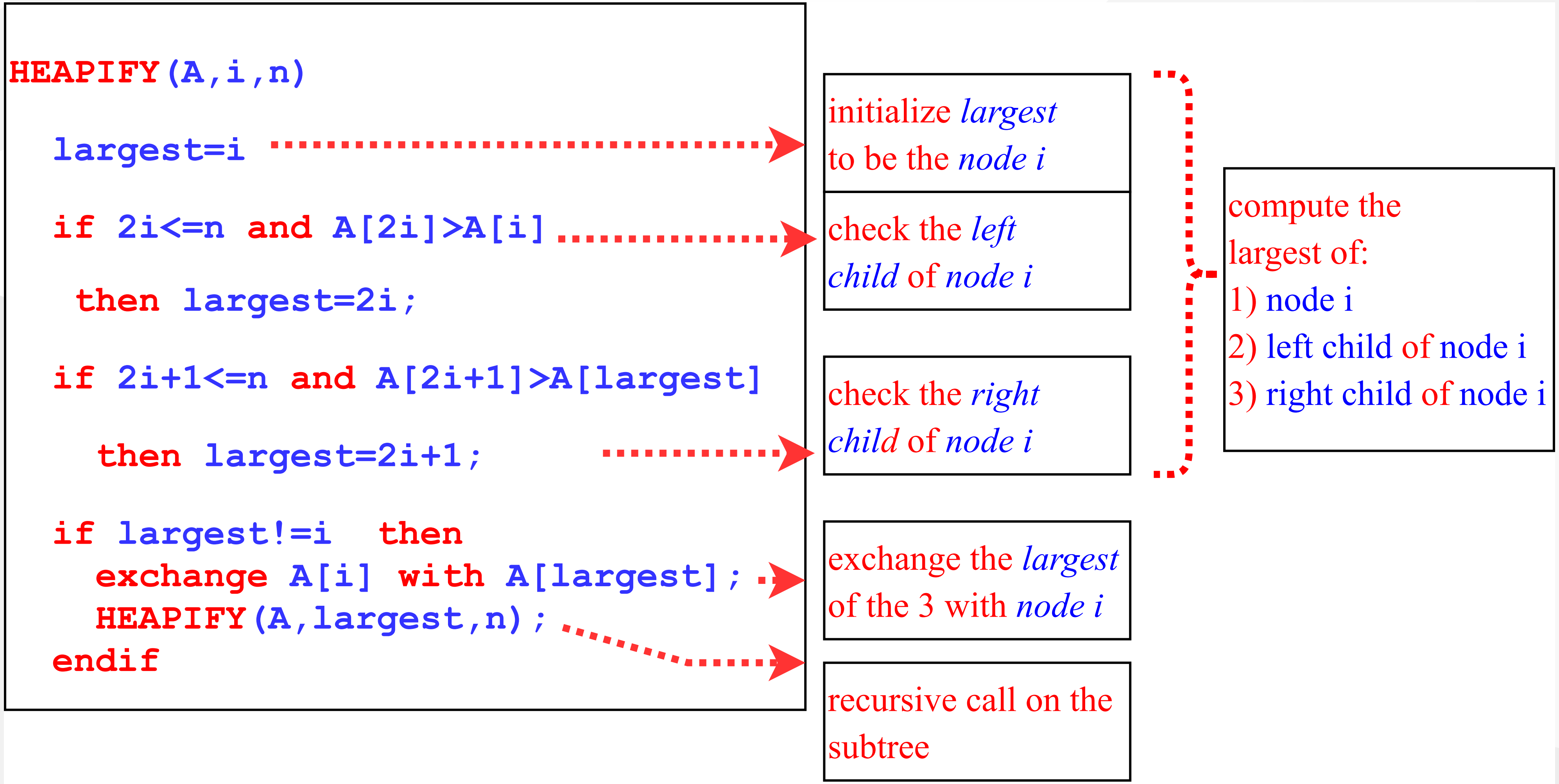
```
HEAPIFY(A, i, n)
  largest = i

  if 2i <= n and A[2i] > A[i] then
    largest = 2i;
  endif

  if 2i+1 <= n and A[2i+1] > A[largest] then
    largest = 2i+1;
  endif

  if largest != i  then
    exchange A[i] with A[largest];
    HEAPIFY(A, largest, n);
  endif
```

Heap Operations: HEAPIFY (3)



```
HEAPIFY(A,i,n)
  largest=i
  if 2i<=n and A[2i]>A[i]
   then largest=2i;
  if 2i+1<=n and A[2i+1]>A[largest]
    then largest=2i+1;
  if largest!=i  then
    exchange A[i] with A[largest];
    HEAPIFY(A,largest,n);
  endif
```

initialize *largest* to be the *node i*

check the *left child* of *node i*

check the *right child* of *node i*

exchange the *largest* of the 3 with *node i*

recursive call on the subtree

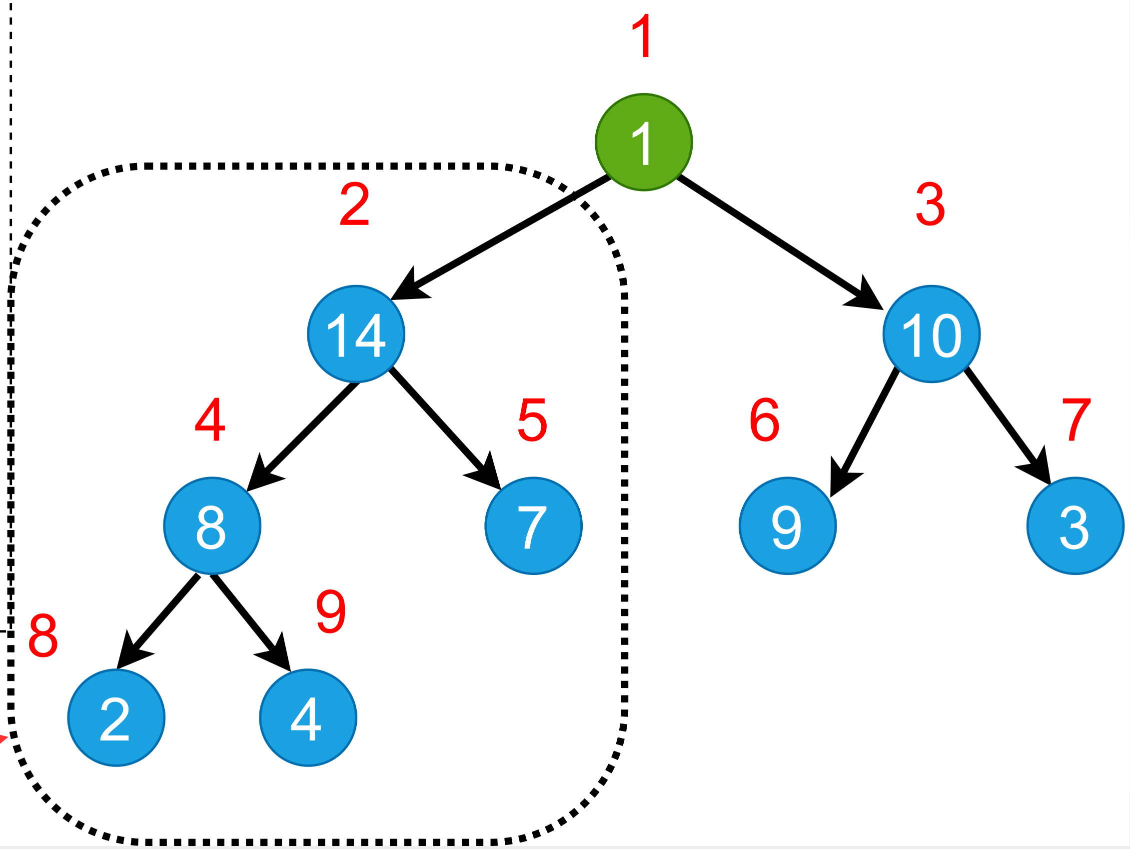
compute the largest of:

1) node i
2) left child of node i
3) right child of node i

Heap Operations: HEAPIFY (4)

```
HEAPIFY(A,i,n)
  largest=i
  if 2i<=n and A[2i]>A[i]
   then largest=2i;
  if 2i+1<=n and A[2i+1]>A[largest]
    then largest=2i+1;
  if largest!=i  then
    exchange A[i] with A[largest];
    HEAPIFY(A,largest,n);
  endif
```

$$HEAPIFY(A,1,9)$$

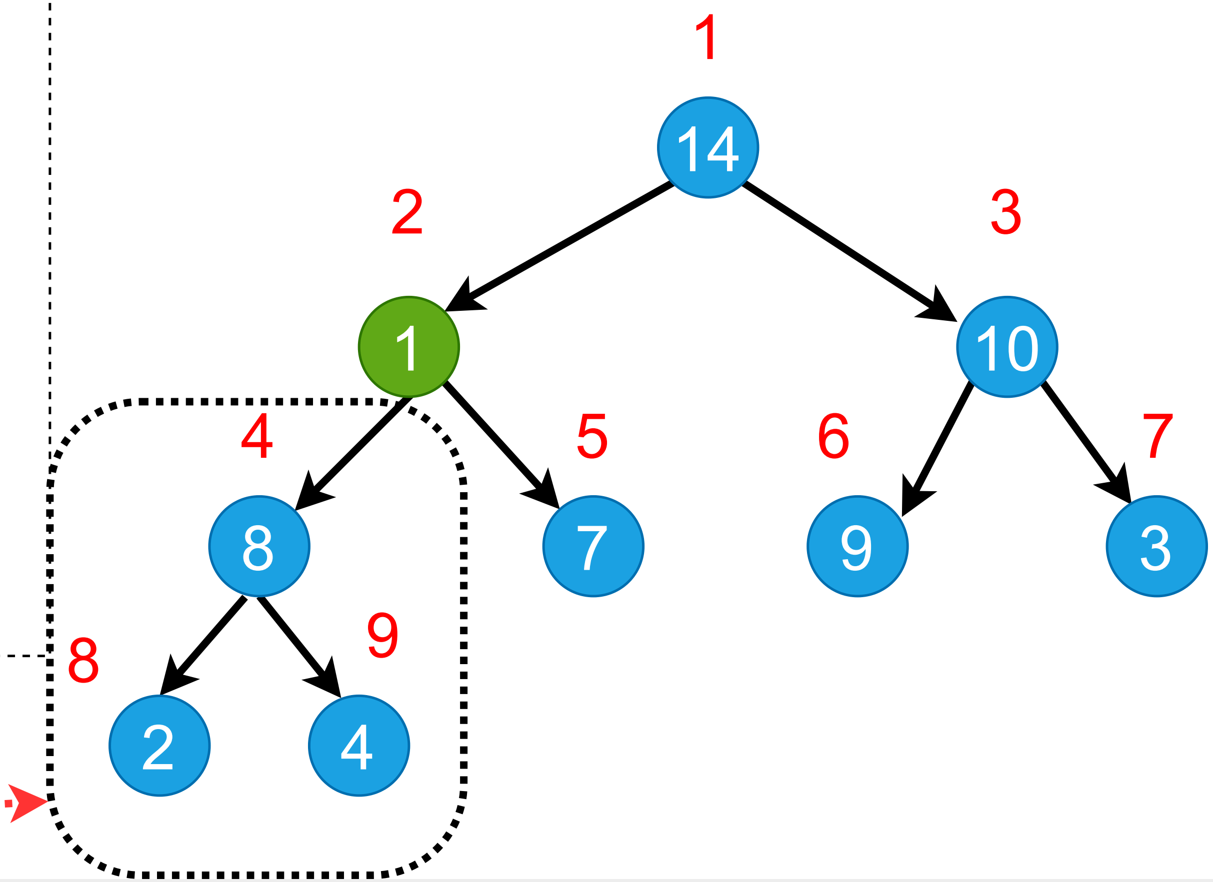


Recursive Call

Heap Operations: HEAPIFY (5)

```
HEAPIFY(A,i,n)
  largest=i
  if 2i<=n and A[2i]>A[i]
   then largest=2i;
  if 2i+1<=n and A[2i+1]>A[largest]
    then largest=2i+1;
  if largest!=i  then
    exchange A[i] with A[largest];
    HEAPIFY(A,largest,n);
  endif
```

$HEAPIFY(A, 2, 9)$

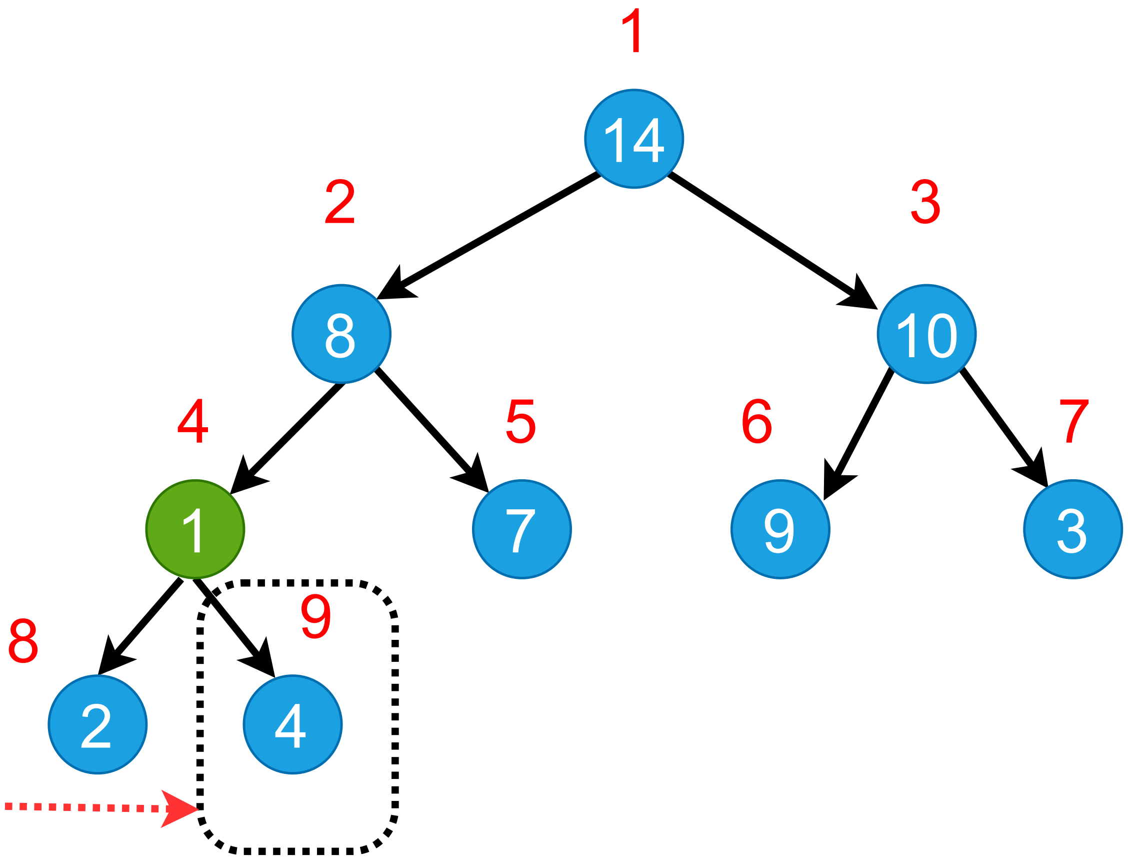


Recursive Call

Heap Operations: HEAPIFY (6)

```
HEAPIFY(A,i,n)
  largest=i
  if 2i<=n and A[2i]>A[i]
   then largest=2i;
  if 2i+1<=n and A[2i+1]>A[largest]
    then largest=2i+1;
  if largest!=i  then
    exchange A[i] with A[largest];
    HEAPIFY(A,largest,n);
  endif
```

$HEAPIFY(A, 4, 9)$

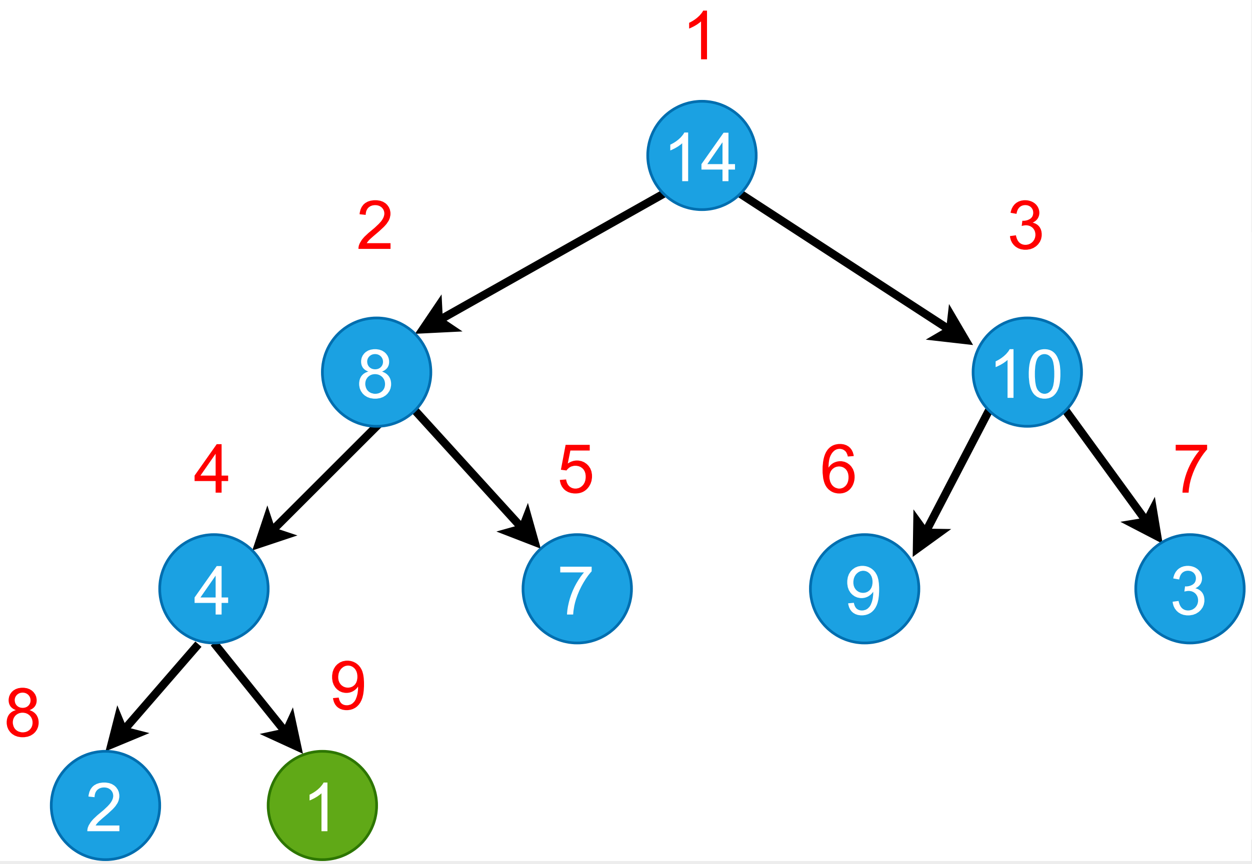


Recursive Call (Base Case)

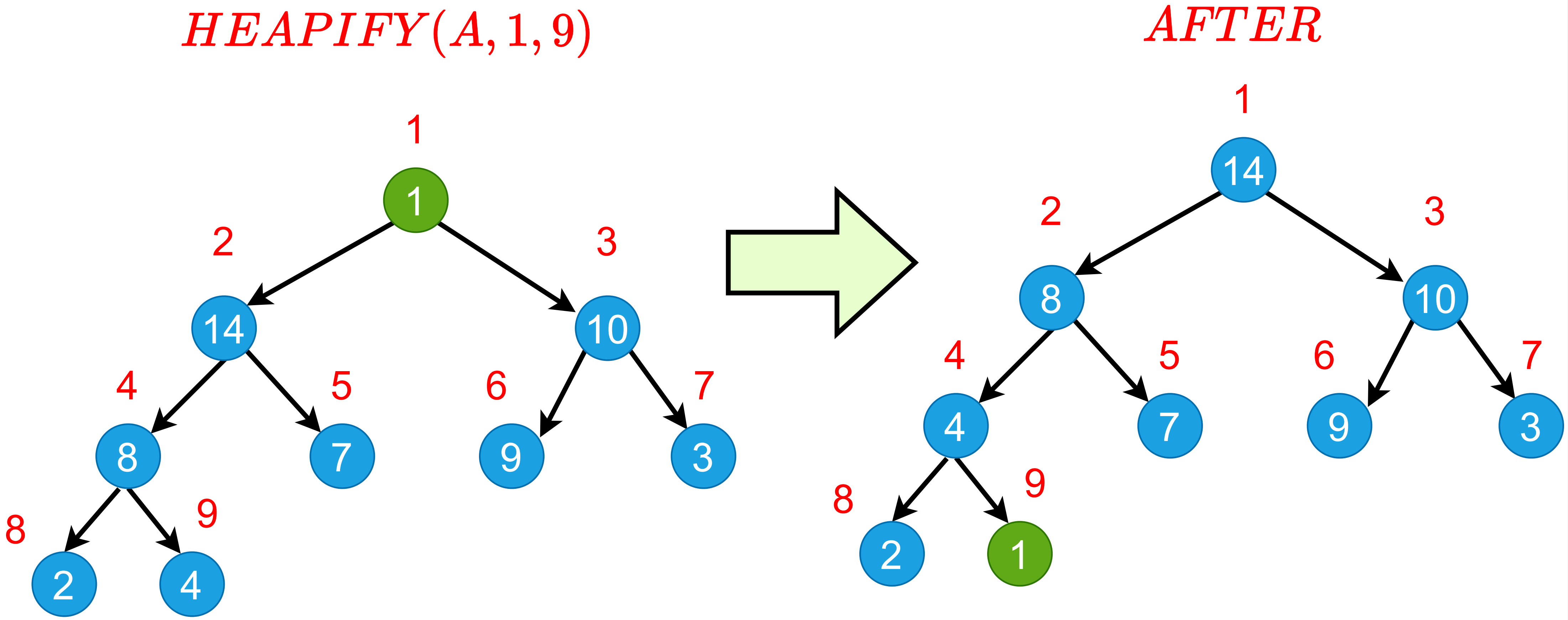
Heap Operations: HEAPIFY (7)

```
HEAPIFY(A,i,n)
  largest=i
  if 2i<=n and A[2i]>A[i]
   then largest=2i;
  if 2i+1<=n and A[2i+1]>A[largest]
    then largest=2i+1;
  if largest!=i  then
    exchange A[i] with A[largest];
    HEAPIFY(A,largest,n);
  endif
```

AFTER HEAPIFY



Heap Operations: HEAPIFY (8)



Intuitive Analysis of HEAPIFY

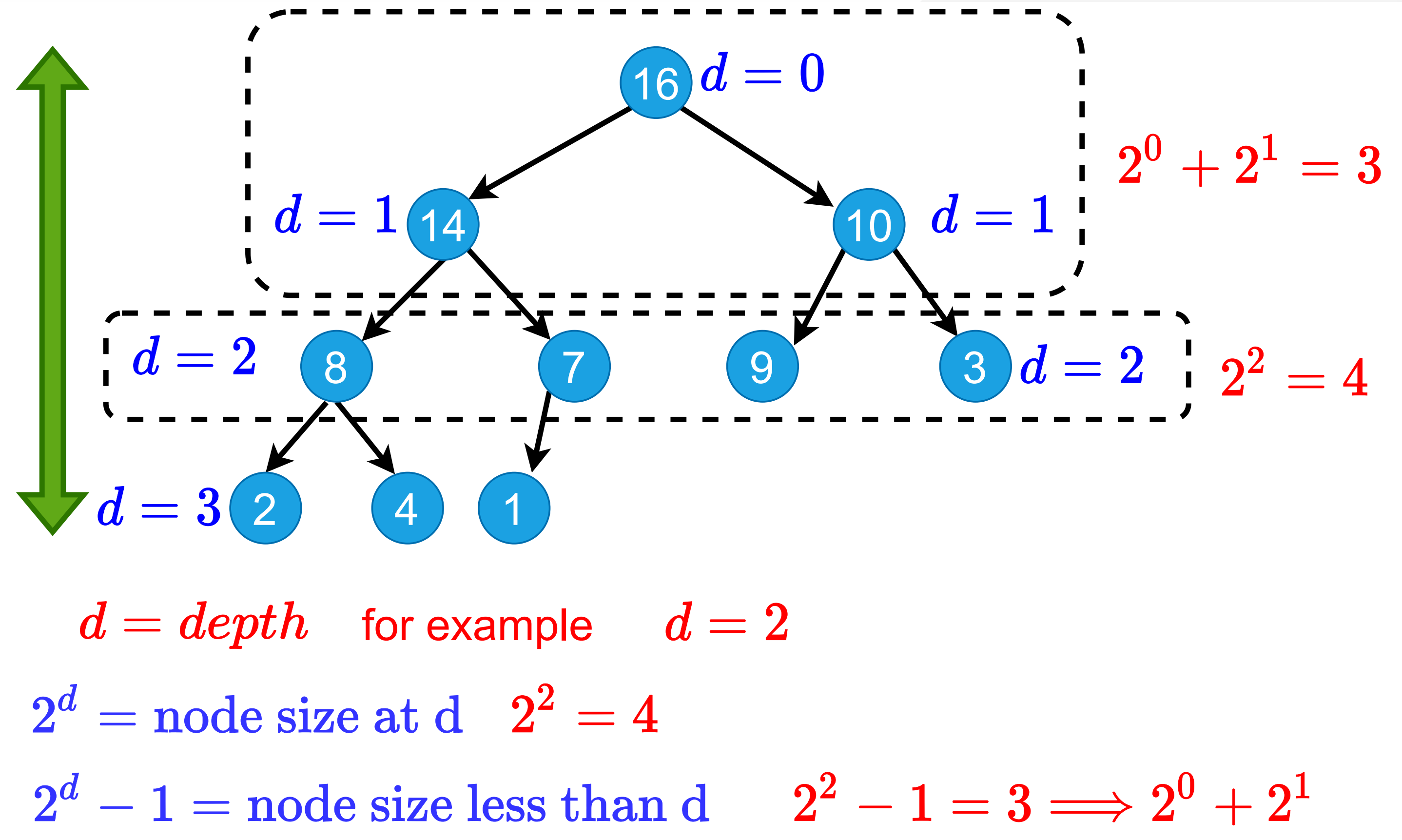
- Consider $HEAPIFY(A, i, n)$
    - let $h(i)$ be the height of node i
    - at most $h(i)$ recursion levels
        - Constant work at each level: $\Theta(1)$
    - Therefore $T(i) = O(h(i))$
- Heap is almost-complete binary tree
    - $h(n) = O(lgn)$
- Thus $T(n) = O(lgn)$

Formal Analysis of HEAPIFY

- **What is the recurrence?**
    - Depends on the size of the **subtree** on which recursive call is made
        - In the next, we try to compute an **upper bound** for this **subtree**.

Reminder: Binary trees

- For a complete binary tree:
  - $\#$ of nodes at depth d: 2^d
  - $\#$ of nodes with depths less than d: $2^d - 1$



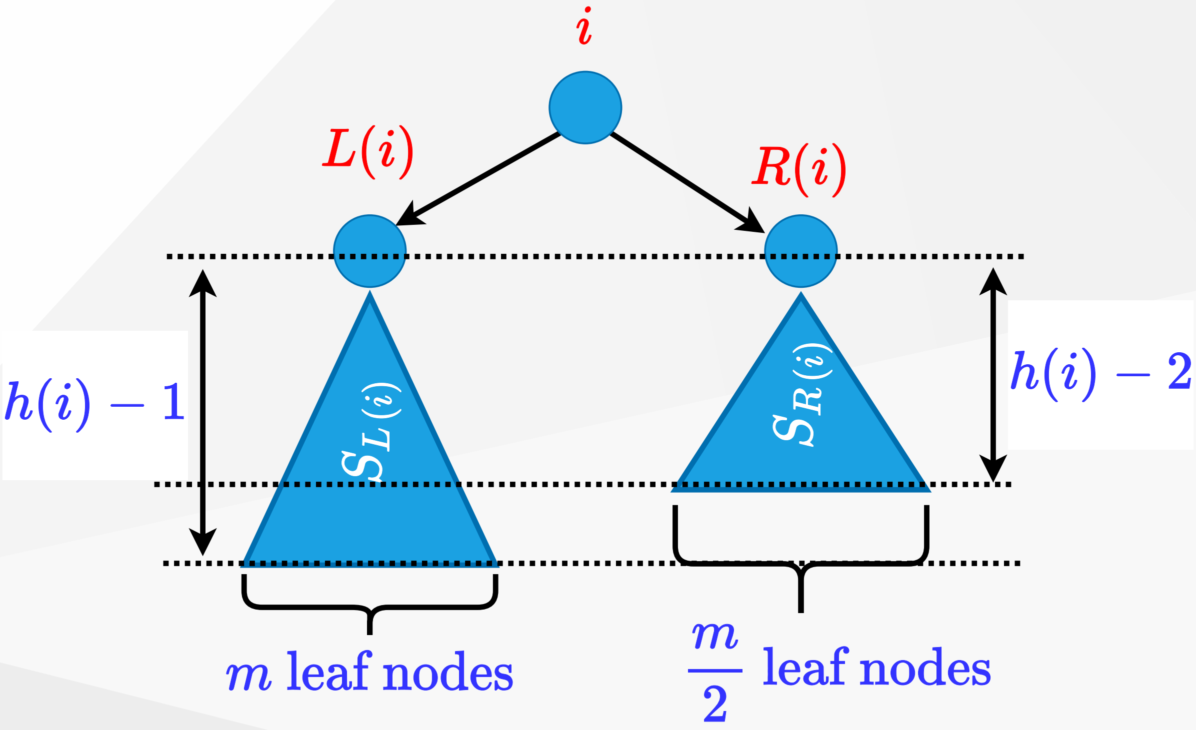
$d = depth$ for example $d = 2$

$2^d = \text{node size at d}$ $\quad 2^2 = 4$

$2^d - 1 = \text{node size less than d}$ $\quad 2^2 - 1 = 3 \Longrightarrow 2^0 + 2^1$

Formal Analysis of HEAPIFY (1)

- Worst case occurs when last row of the subtree S_i rooted at node i is **half full**
- $T(n) \leq T(|S_{L(i)}|) + \Theta(1)$
- $S_{L(i)}$ and $S_{R(i)}$ are complete binary trees of heights $h(i) - 1$ and $h(i) - 2$, respectively



Formal Analysis of HEAPIFY (2)

- Let m be the number of **leaf nodes** in $S_{L(i)}$
    - $|S_{L(i)}| = \overbrace{m}^{ext.} + \overbrace{(m–1)}^{int.} = 2m–1$
    - $|S_{R(i)}| = \overbrace{\frac{m}{2}}^{ext.} + \overbrace{(\frac{m}{2}–1)}^{int.} = m–1$
    - $|S_{L(i)}| + |S_{R(i)}| + 1 = n$

Formal Analysis of HEAPIFY (2)

$$
\begin{aligned}
(2m–1) + (m–1) + 1 &= n \\
m &= (n+1)/3 \\
|S_{L(i)}| &= 2m–1 \\
&= 2(n+1)/3–1 \\
&= (2n/3 + 2/3)–1 \\
&= \frac{2n}{3} - \frac{1}{3} \leq \frac{2n}{3} \\
T(n) &\leq T(2n/3) + \Theta(1) \\
T(n) &= O(lgn)
\end{aligned}
$$

- By CASE-2 of Master Theorem $\Longrightarrow T(n) = \Theta(n^{log_b^a} lgn)$

Formal Analysis of HEAPIFY (2)

- Recurrence: $T(n) = aT(n/b) + f(n)$
- *Case 2:* $\frac{f(n)}{n^{log_b^a}} = \Theta(1)$
- i.e., $f(n)$ and $n^{log_b^a}$ grow at similar rates
- **Solution:** $T(n) = \Theta(n^{log_b^a} lgn)$
    - $T(n) \leq T(2n/3) + \Theta(1)$ (drop constants.)
    - $T(n) \leq \Theta(n^{log_3^1} lgn)$
    - $T(n) \leq \Theta(n^0 lgn)$
    - $T(n) = O(lgn)$

HEAPIFY: Efficiency Issues

- **Recursion vs Iteration:**
    - In the absence of tail recursion, **iterative version** is in general **more efficient** because of the **pop/push** operations **to/from** stack at each **level of recursion**.

Heap Operations: HEAPIFY (1)

Recursive

```
HEAPIFY(A, i, n)
largest = i

if 2i <= n and A[2i] > A[i] then
  largest = 2i

if 2i+1 <= n and A[2i+1] > A[largest] then
  largest = 2i+1

if largest != i  then
  exchange A[i] with A[largest]
  HEAPIFY(A, largest, n)
```

Heap Operations: HEAPIFY (2)

Iterative

```
HEAPIFY(A, i, n)
  j = i
  while(true) do
    largest = j

  if 2j <= n and A[2j] > A[j] then
    largest = 2j

  if 2j+1 <= n and A[2j+1] > A[largest] then
    largest = 2j+1

  if largest != j  then
    exchange A[j] with A[largest]
    j = largest
  else return
```

Heap Operations: HEAPIFY (3)

Recursive

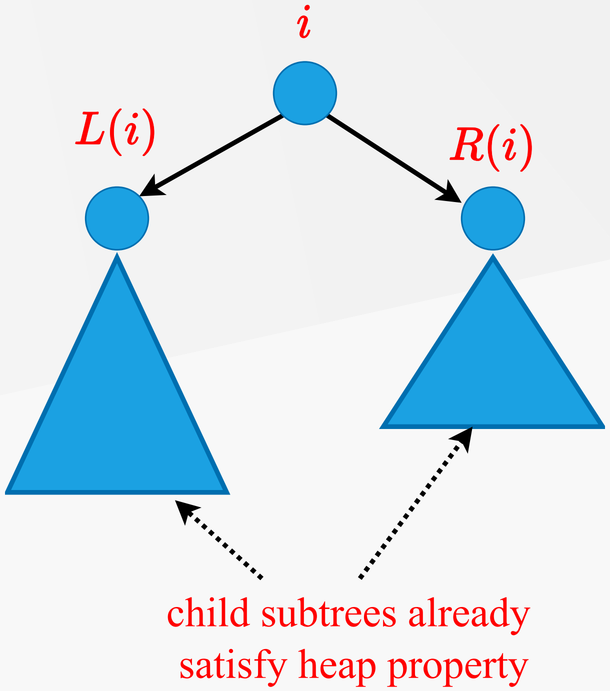
```
HEAPIFY(A,i, n)
 largest ← i
 if 2i ≤ n and A[2i] > A[i] then largest ← 2i
 if 2i +1 ≤ n and A[2i+1] > A[largest] then largest ← 2i +1
 if largest ≠ i  then
     exchange A[i]↔ A[largest]
     HEAPIFY(A, largest, n)
```

Iterative

```
HEAPIFY(A,i, n)
 j ← i
 while (true) do
    largest ← j
 if 2j ≤ n and A[2j] > A[j] then largest ← 2j
 if 2j +1 ≤ n and A[2j+1] > A[largest] then largest ← 2j +1
 if largest ≠ j  then
    exchange A[j]↔ A[largest]
    j ← largest
 else return
```

Heap Operations: Building Heap

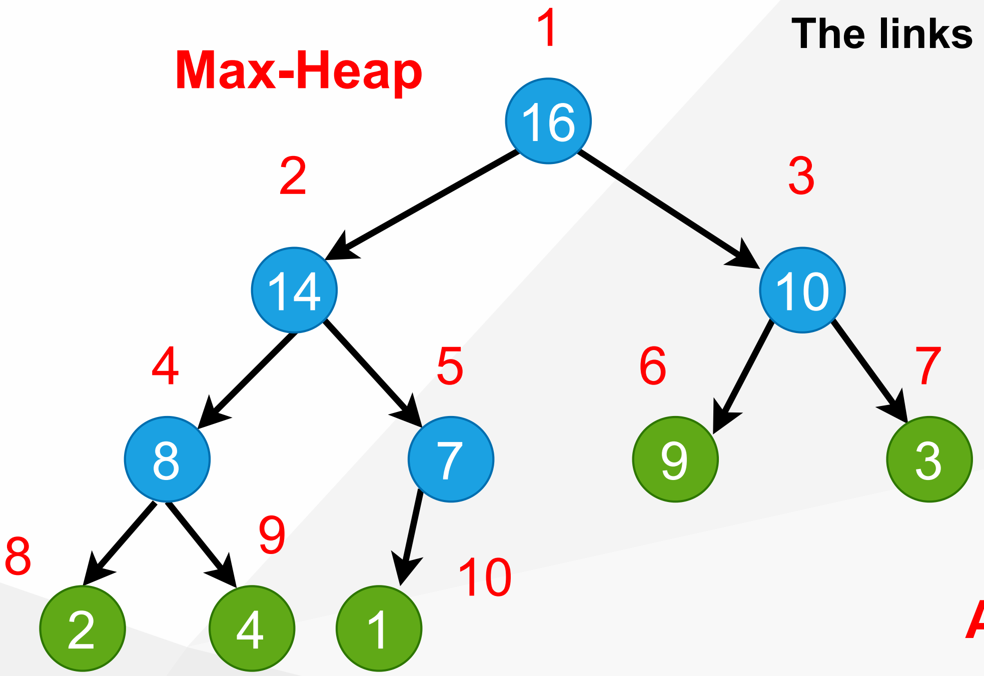
- Given an arbitrary array, how to build a heap from scratch?
- **Basic idea:** Call $HEAPIFY$ on each node bottom up
    - Start from the leaves (which trivially satisfy the heap property)
    - Process nodes in bottom up order.
    - When $HEAPIFY$ is called on node i, the subtrees connected to the $left$ and $right$ subtrees already satisfy the heap property.



Storage of the leaves (Lemma)

- **Lemma:** The last $\lceil \frac{n}{2} \rceil$ nodes of a heap are all leaves.

Max-Heap



The links in the heap are implicit

$$left(i) = 2i$$

e.g.Left child of node 4 has index 8

$$right(i) = 2i + 1$$

e.g. Right child of node 2 has index 5

$$parent(i) = \lfloor i/2 \rfloor$$

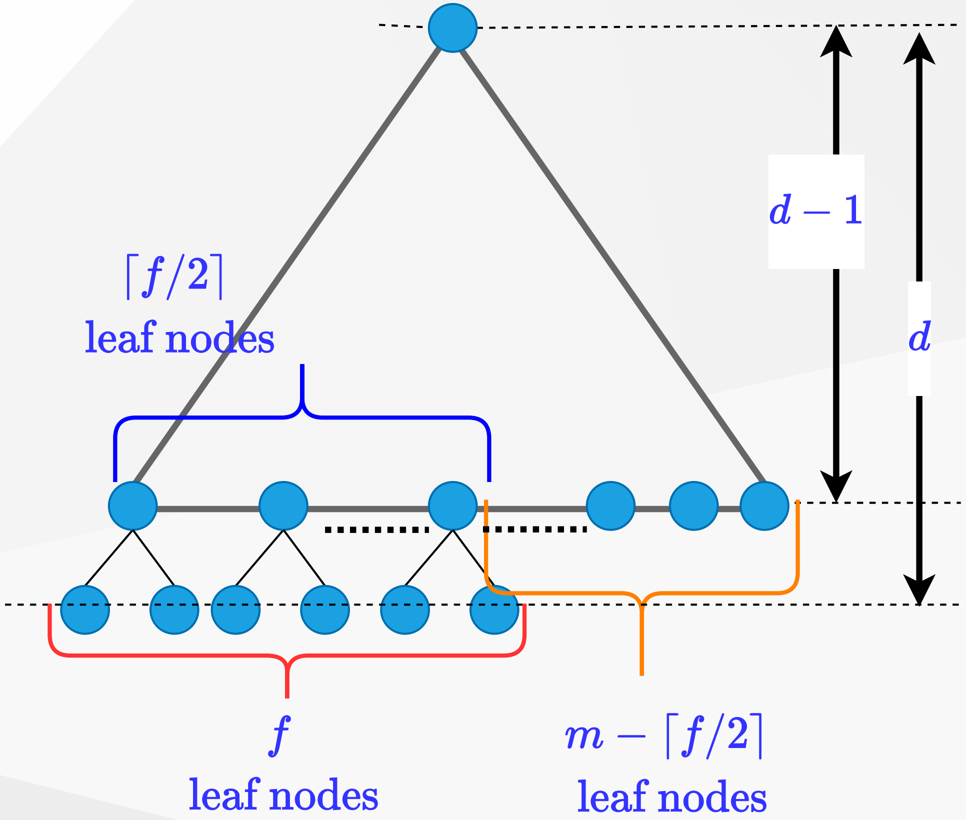
e.g.Parent of node 7 has index 3

Array Storage

1	2	3	4	5	6	7	8	9	10
16	14	10	8	7	9	3	2	4	1

Storage of the leaves (Proof of Lemma) (1)

- **Lemma:** last $\lceil n/2 \rceil$ nodes of a heap are all leaves
- Proof :
  - $m = 2^{d-1}$: $\#$ nodes at level $d-1$
  - f: $\#$ nodes at level d (last level)
- $\#$ of nodes with depth $d-1 : m$
- $\#$ of nodes with depth $< d-1 : m-1$
- $\#$ of nodes with depth $d : f$
- **Total** $\#$ of nodes :$n = f + 2m - 1$

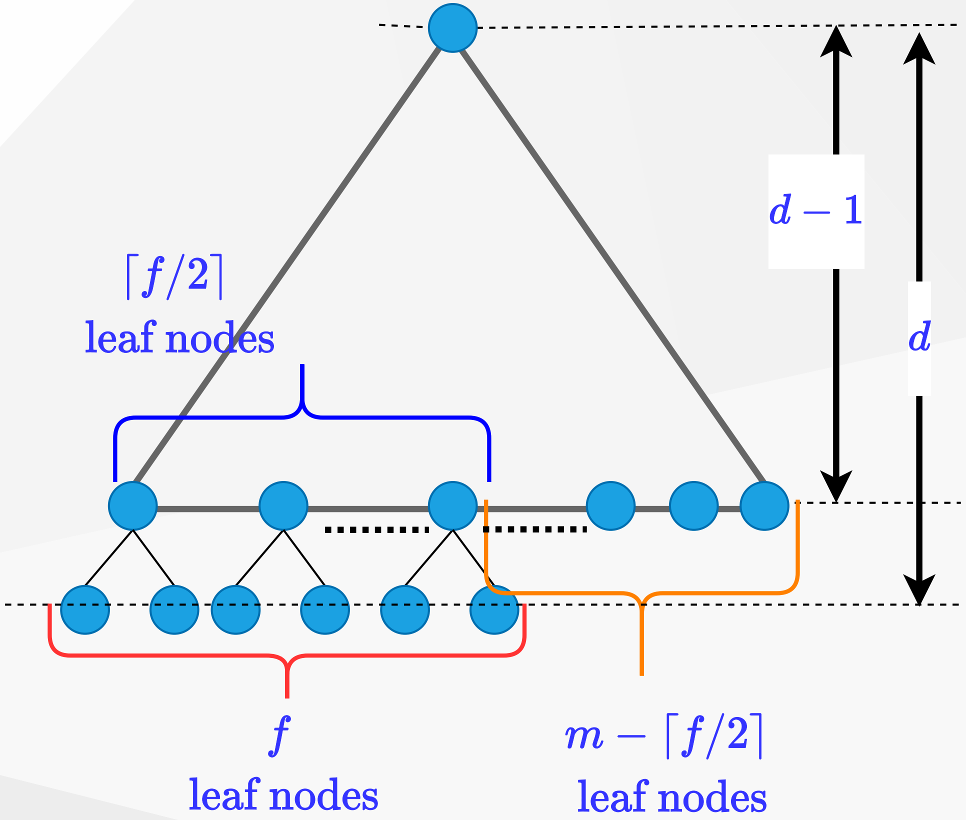


Storage of the leaves (Proof of Lemma) (2)

- **Total** $\#$ of nodes : $f = n - 2m + 1$

$$
\begin{aligned}
\#\text{ of leaves: } &= f + m - \lceil f/2 \rceil \\
&= m + \lfloor f/2 \rfloor \\
&= m + \lfloor (n - 2m + 1)/2 \rfloor \\
&= \lfloor (n+1)/2 \rfloor \\
&= \lceil n/2 \rceil
\end{aligned}
$$

Proof is Completed



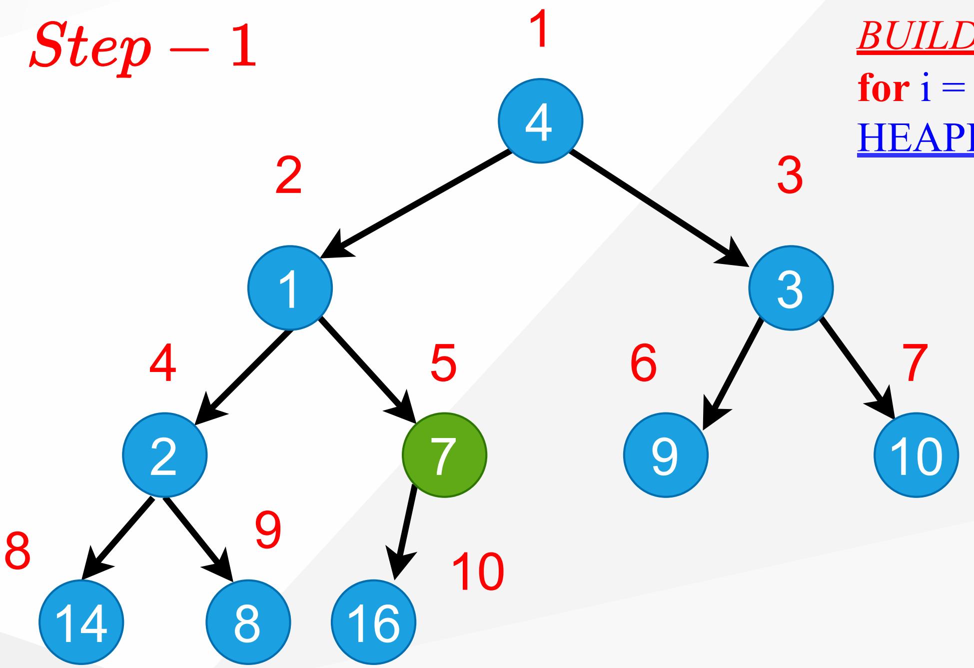
Heap Operations: Building Heap

```
BUILD-HEAP (A, n)
  for i = ceil(n/2) downto 1 do
    HEAPIFY(A, i, n)
```

- **Reminder:** The last $\lceil n/2 \rceil$ nodes of a heap are **all leaves**, which trivially satisfy the heap property

Build-Heap Example (Step-1)

$Step - 1$



```
BUILD-HEAP(A, n)
for i = ⌊n/2⌋ downto 1 do
    HEAPIFY(A, i, n)
```

$$i = \lfloor n/2 \rfloor - 0$$

$$i = \lfloor 10/2 \rfloor - 0$$

$$i = 5$$

$$HEAPIFY(A, 5, 10)$$

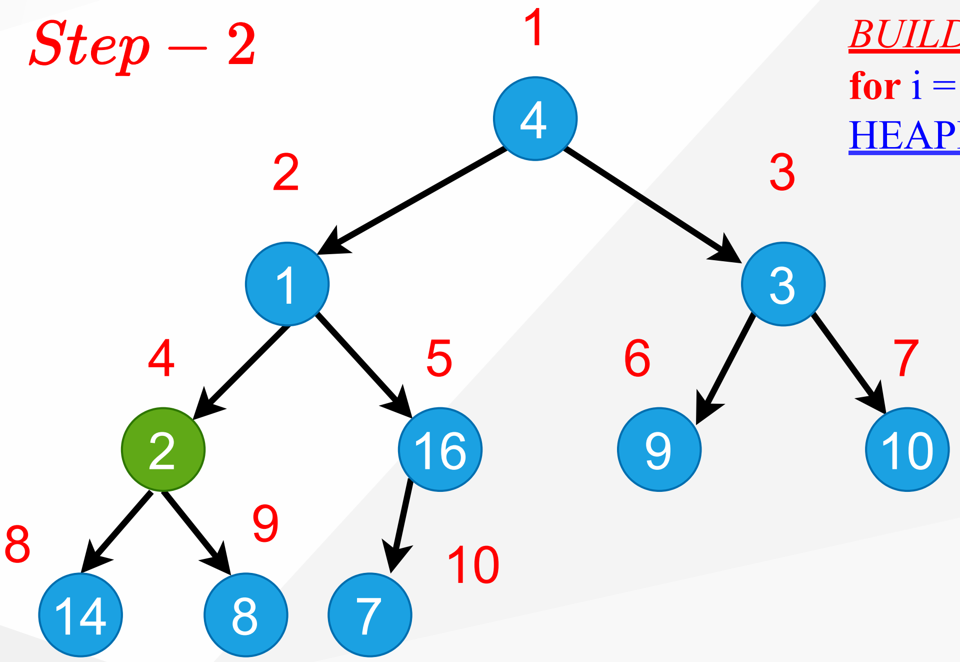
Array Storage

	1	2	3	4	5	6	7	8	9	10
$A =$	4	1	3	2	7	9	10	14	8	16

Exchange

Build-Heap Example (Step-2)

$Step - 2$



```
BUILD-HEAP(A, n)
for i = ⌊n/2⌋ downto 1 do
    HEAPIFY(A, i, n)
```

$$i = \lfloor n/2 \rfloor - 1$$

$$i = \lfloor 10/2 \rfloor - 1$$

$$i = 4$$

$$HEAPIFY(A, 4, 10)$$

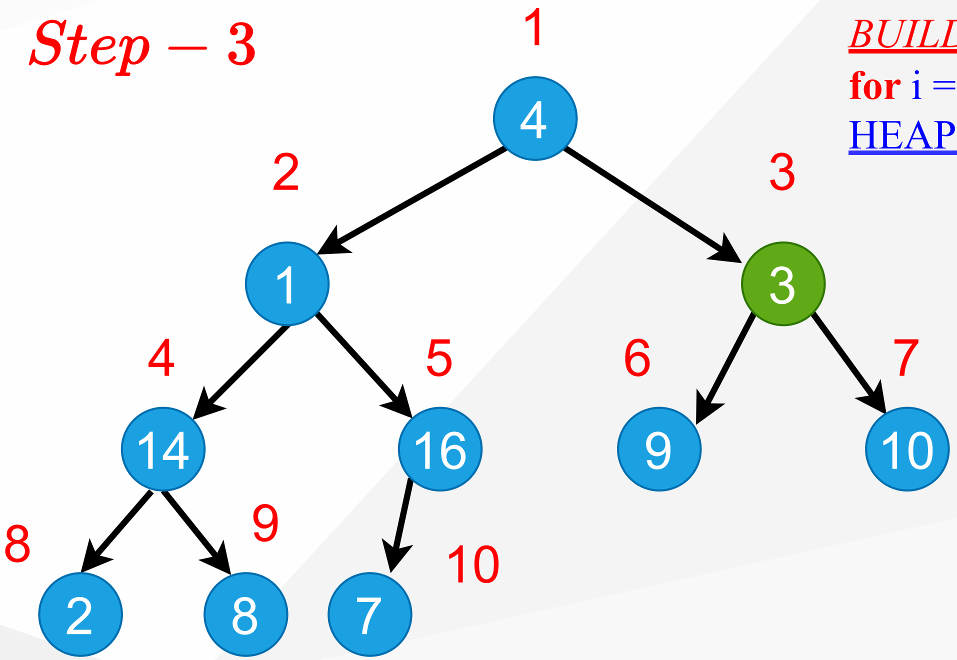
Array Storage

	1	2	3	4	5	6	7	8	9	10
$A =$	4	1	3	2	16	9	10	14	8	7

Exchange

Build-Heap Example (Step-3)

$Step - 3$



```
BUILD-HEAP(A, n)
for i = ⌊n/2⌋ downto 1 do
    HEAPIFY(A, i, n)
```

$i = \lfloor n/2 \rfloor - 2$

$i = \lfloor 10/2 \rfloor - 2$

$i = 3$

$HEAPIFY(A, 3, 10)$

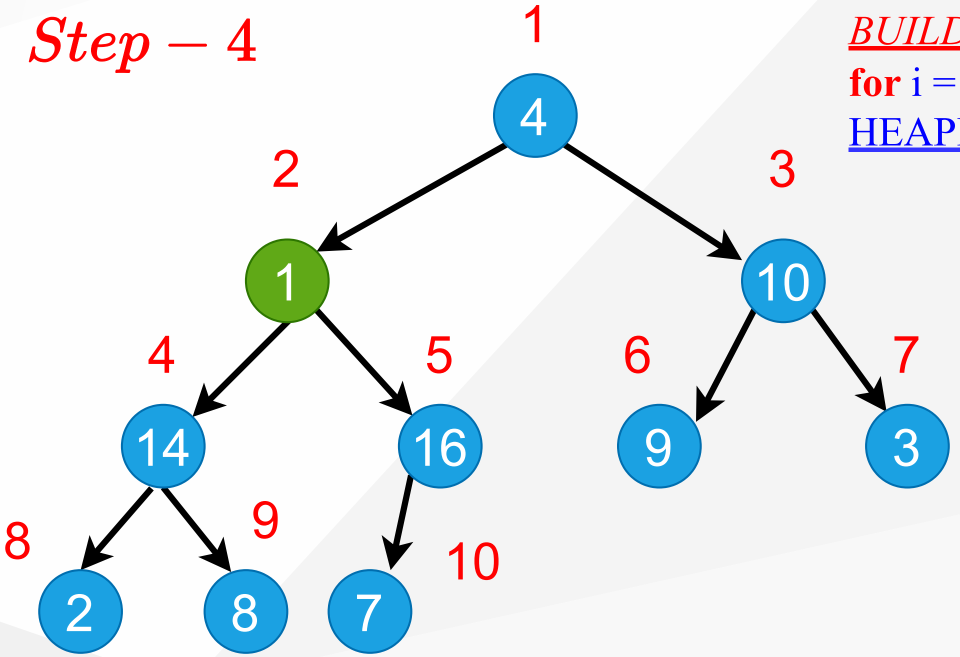
Array Storage

	1	2	3	4	5	6	7	8	9	10
$A =$	4	1	3	14	16	9	10	2	8	7

$Exchange$

Build-Heap Example (Step-4)

Step – 4



```
BUILD-HEAP(A, n)
for i = ⌊n/2⌋ downto 1 do
    HEAPIFY(A, i, n)
```

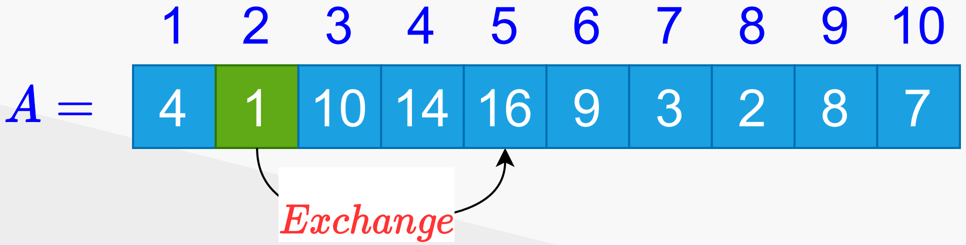
$$i = |n/2| - 3$$

$$i = \lfloor 10/2 \rfloor - 3$$

$$i = 2$$

$$HEAPIFY(A, 2, 10)$$

Array Storage



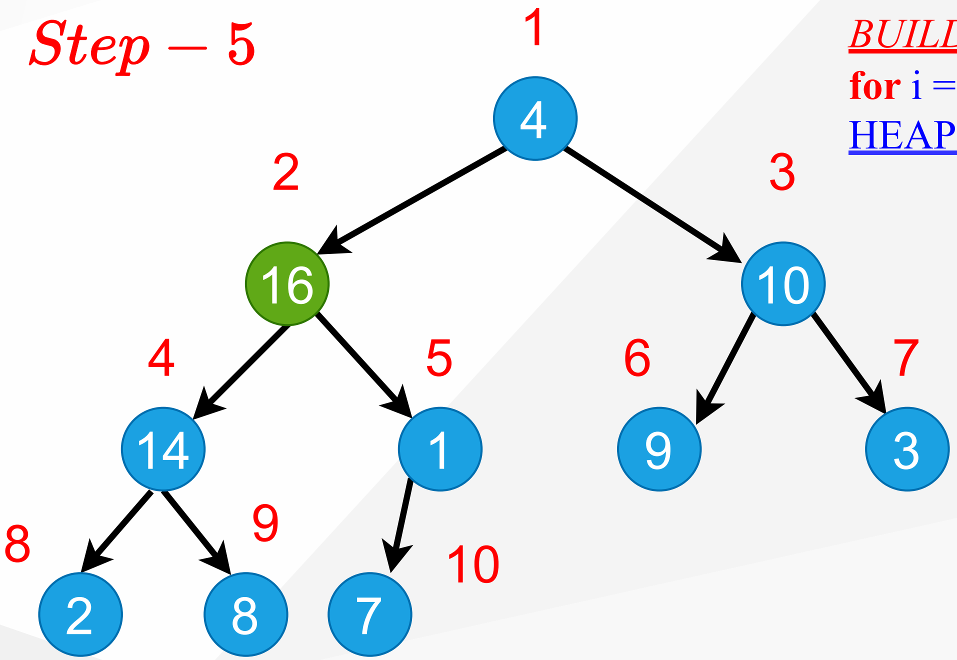
$A =$

1	2	3	4	5	6	7	8	9	10
4	1	10	14	16	9	3	2	8	7

Exchange

Build-Heap Example (Step-5)

$Step - 5$



```
BUILD-HEAP(A, n)
for i = ⌊n/2⌋ downto 1 do
    HEAPIFY(A, i, n)
```

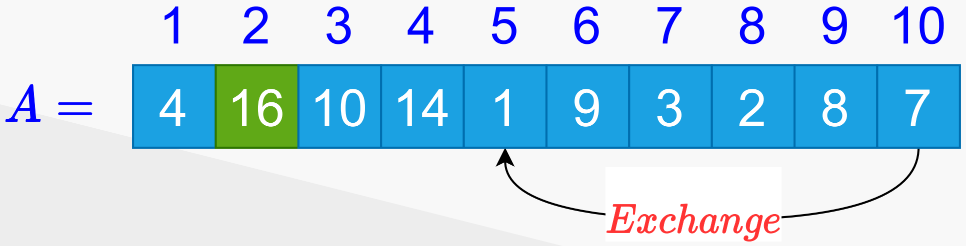
$$i = |n/2| - 3$$

$$i = \lfloor 10/2 \rfloor - 3$$

$$i = 2$$

$$HEAPIFY(A, 2, 10)$$

Array Storage

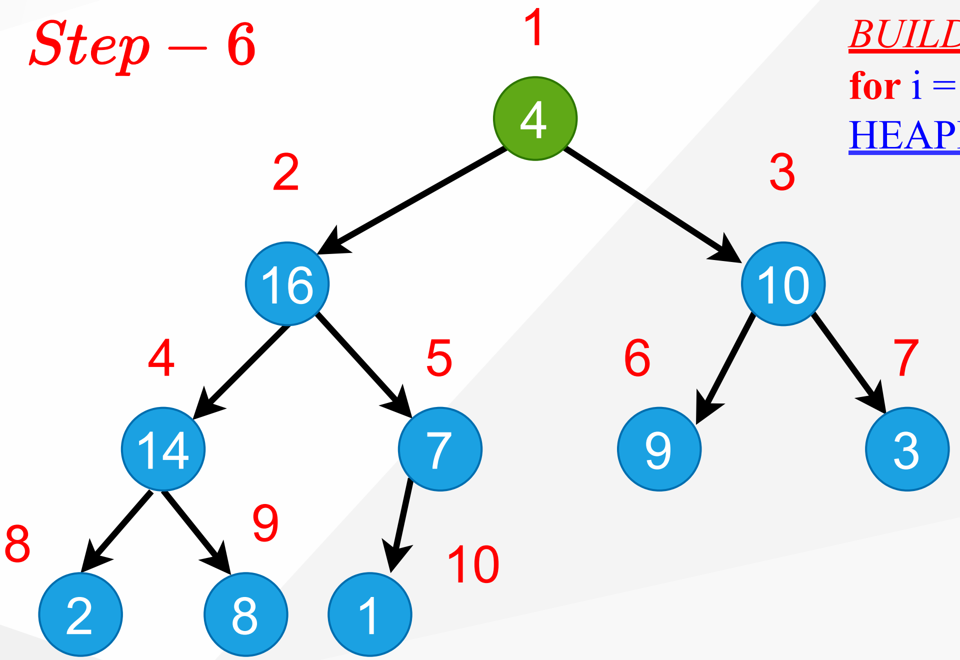


$A =$

1	2	3	4	5	6	7	8	9	10
4	16	10	14	1	9	3	2	8	7

Build-Heap Example (Step-6)

$Step - 6$



```
BUILD-HEAP(A, n)
for i = ⌊n/2⌋ downto 1 do
    HEAPIFY(A, i, n)
```

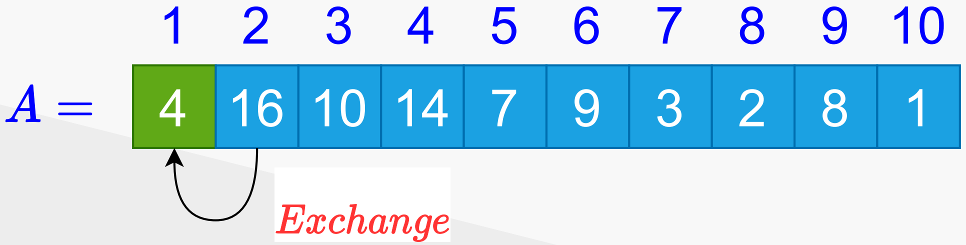
$$i = \lfloor n/2 \rfloor - 4$$

$$i = \lfloor 10/2 \rfloor - 4$$

$$i = 1$$

$$HEAPIFY(A, 1, 10)$$

Array Storage

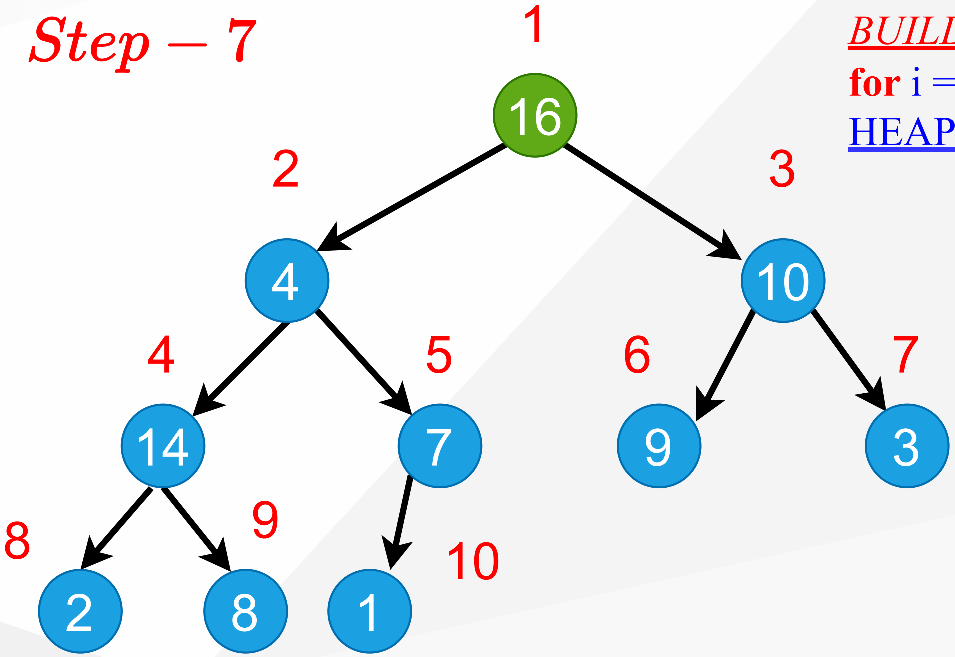


$A =$

1	2	3	4	5	6	7	8	9	10
4	16	10	14	7	9	3	2	8	1

Build-Heap Example (Step-7)

$Step - 7$



```
BUILD-HEAP(A, n)
for i = ⌊n/2⌋ downto 1 do
    HEAPIFY(A, i, n)
```

$i = \lfloor n/2 \rfloor - 4$

$i = \lfloor 10/2 \rfloor - 4$

$i = 1$

$HEAPIFY(A, 1, 10)$

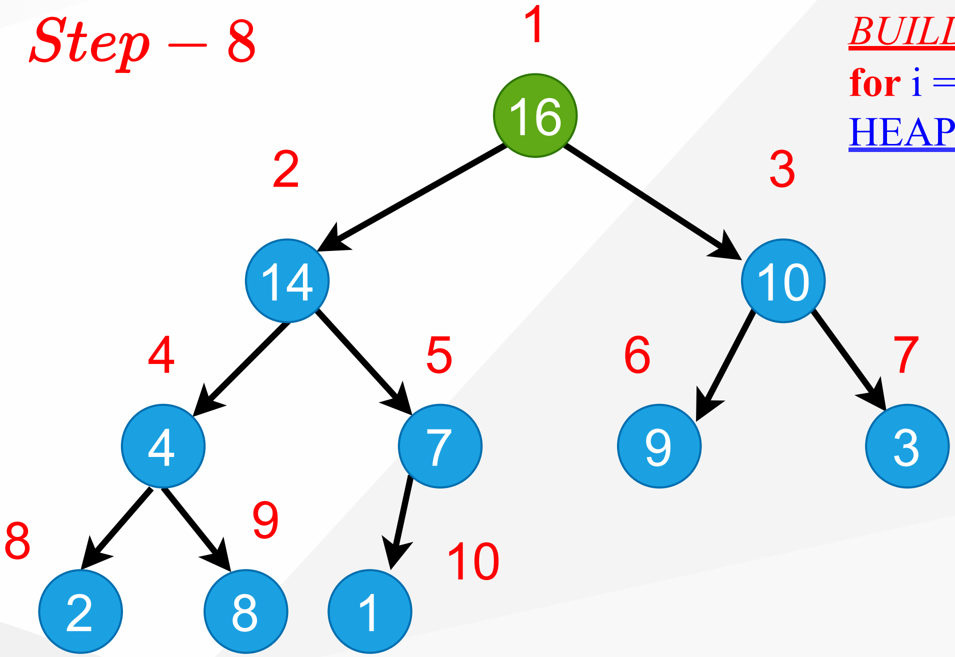
Array Storage

	1	2	3	4	5	6	7	8	9	10
$A =$	16	4	10	14	7	9	3	2	8	1

Exchange

Build-Heap Example (Step-8)

$Step - 8$



```
BUILD-HEAP(A, n)
for i = ⌊n/2⌋ downto 1 do
    HEAPIFY(A, i, n)
```

$i = |n/2| - 4$

$i = \lfloor 10/2 \rfloor - 4$

$i = 1$

$HEAPIFY(A, 1, 10)$

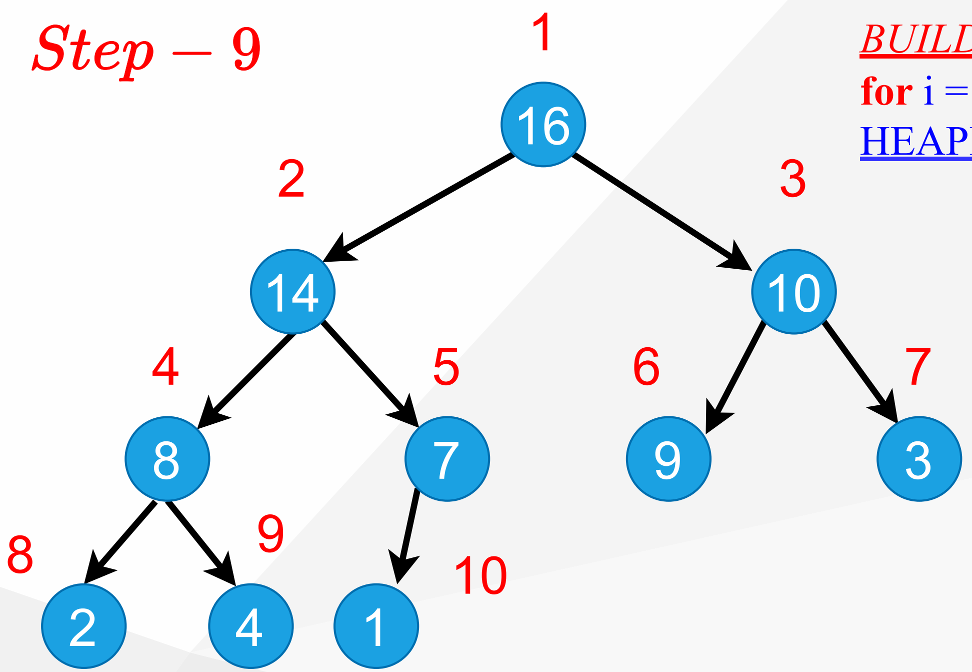
Array Storage

	1	2	3	4	5	6	7	8	9	10
$A =$	16	14	10	4	7	9	3	2	8	1

$Exchange$

Build-Heap Example (Step-9)

Step – 9



```
BUILD-HEAP(A, n)
for i = ⌊n/2⌋ downto 1 do
    HEAPIFY(A, i, n)
```

After Build-Heap

Array Storage

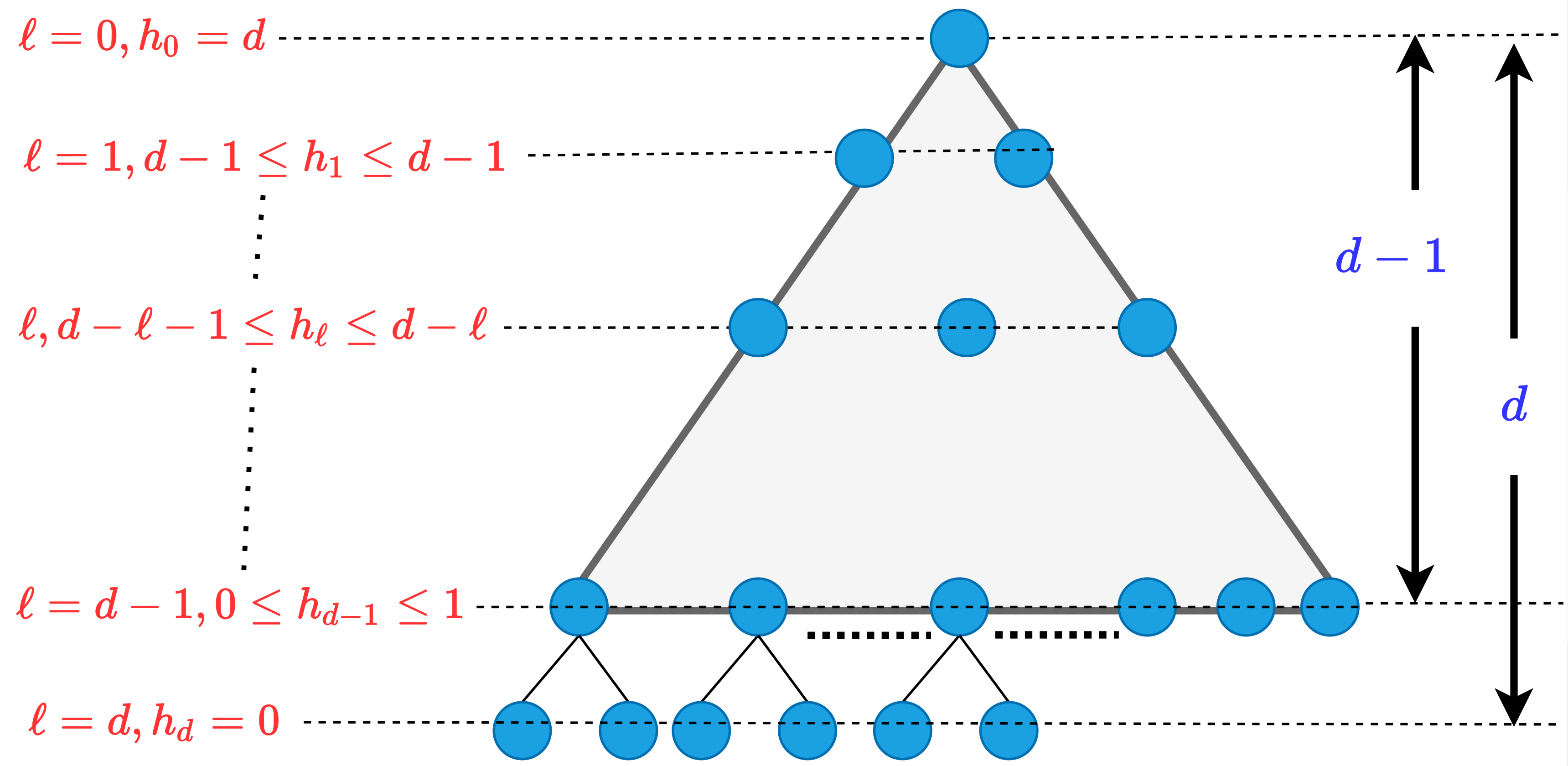
	1	2	3	4	5	6	7	8	9	10
$A =$	16	14	10	8	7	9	3	2	4	1

Build-Heap: Runtime Analysis

- Simple analysis:
    - $O(n)$ calls to $HEAPIFY$, each of which takes $O(lgn)$ time
    - $O(nlgn) \implies$ loose bound
- In general, a good approach:
    - Start by proving an easy bound
    - Then, try to tighten it
- Is there a tighter bound?

Build-Heap: Tighter Running Time Analysis

- If the heap is complete binary tree then $h_\ell = d–\ell$
- Otherwise, nodes at a given level do not all have the same height, But we have $d–\ell–1 \leq h_\ell \leq d–\ell$



Build-Heap: Tighter Running Time Analysis

- Assume that all nodes at level $\ell = d–1$ are processed

$$T(n) = \sum_{\ell=0}^{d-1} n_\ell O(h_\ell) = O(\sum_{\ell=0}^{d-1} n_\ell h_\ell) \begin{cases} n_\ell = 2^\ell = \# \text{ of nodes at level } \ell \\ h_\ell = \text{height of nodes at level } \ell \end{cases}$$

$$\therefore T(n) = O\left(\sum_{\ell=0}^{d-1} 2^\ell (d - \ell)\right)$$

$$\text{Let } h = d - \ell \implies \ell = d - h \text{ change of variables}$$

$$T(n) = O\left(\sum_{h=1}^{d} h 2^{d-h}\right) = O\left(\sum_{h=1}^{d} h \frac{2^d}{2^h}\right) = O\left(2^d \sum_{h=1}^{d} h (1/2)^h\right)$$

$$\text{but } 2^d = \Theta(n) \implies O\left(n \sum_{h=1}^{d} h (1/2)^h\right)$$

Build-Heap: Tighter Running Time Analysis

$$\sum_{h=1}^{d} h(1/2)^h \leq \sum_{h=0}^{d} h(1/2)^h \leq \sum_{h=0}^{\infty} h(1/2)^h$$

- recall infinite decreasing geometric series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \text{ where } |x| < 1$$

- differentiate both sides

$$\sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

Build-Heap: Tighter Running Time Analysis

$$\sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

- then, multiply both sides by x

$$\sum_{k=0}^{\infty} kx^{k} = \frac{x}{(1-x)^2}$$

- in our case: $x = 1/2$ and $k = h$

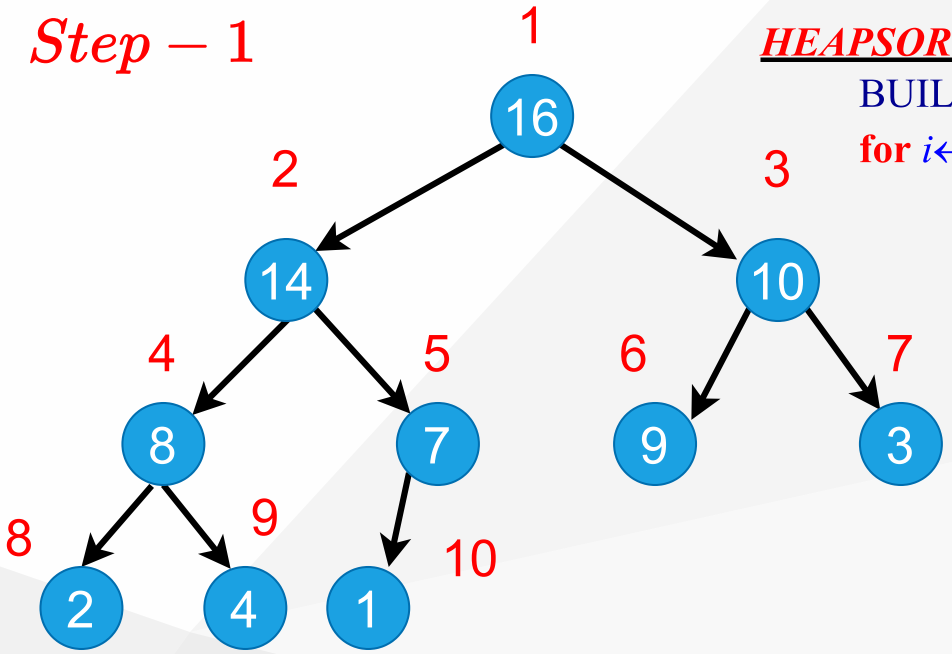
$$\therefore \sum_{h=0}^{\infty} h(1/2)^h = \frac{1/2}{(1-(1/2))^2} = 2 = O(1)$$

$$\therefore T(n) = O(n \sum_{h=1}^{d} h(1/2)^h) = O(n)$$

Heapsort Algorithm Steps

- **(1)** Build a heap on array $A[1 \dots n]$ by calling $BUILD-HEAP(A, n)$
- **(2)** The largest element is stored at the root $A[1]$
    - Put it into its correct final position $A[n]$ by $A[1] \longleftrightarrow A[n]$
- **(3)** Discard node n from the heap
- **(4)** Subtrees $(S2\&S3)$ rooted at children of root remain as heaps, but the new root element may violate the heap property.
    - Make $A[1 \dots n-1]$ a heap by calling $HEAPIFY(A, 1, n-1)$
- **(5)** $n \leftarrow n-1$
- **(6)** Repeat steps **(2-4)** until $n=2$

Heapsort Algorithm Example (Step-1)



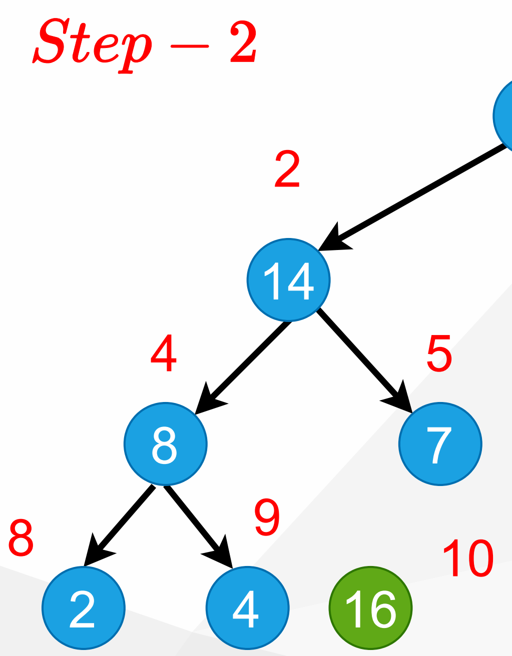
```
HEAPSORT(A, n)
    BUILD-HEAP(A, n)
    for i← n downto 2 do
        exchange A[1] ↔A[i]
        HEAPIFY(A,1, i –1)
```

Array Storage

	1	2	3	4	5	6	7	8	9	10
$A =$	16	14	10	8	7	9	3	2	4	1

Heapsort Algorithm Example (Step-2)

$Step - 2$



```
HEAPSORT(A, n)
    BUILD-HEAP(A, n)
    for i← n downto 2 do
        exchange A[1] ↔A[i]
        HEAPIFY(A,1, i –1)
```

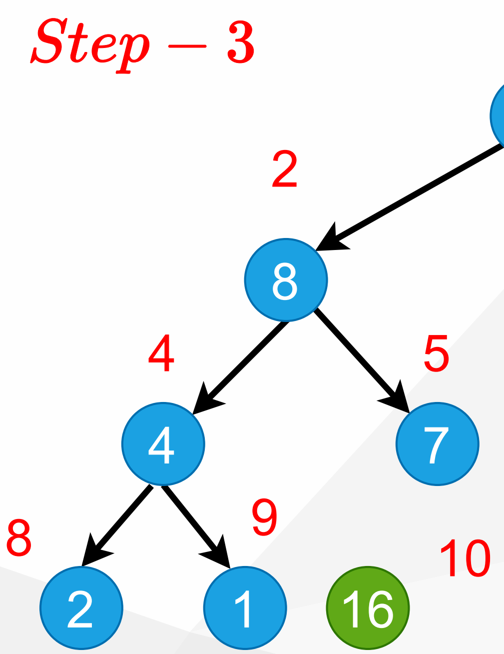
Array Storage

$A =$

1	2	3	4	5	6	7	8	9	10
1	14	10	8	7	9	3	2	4	16

Heapsort Algorithm Example (Step-3)

$Step - 3$



```
HEAPSORT(A, n)
    BUILD-HEAP(A, n)
    for i← n downto 2 do
        exchange A[1] ↔A[i]
        HEAPIFY(A,1, i –1)
```

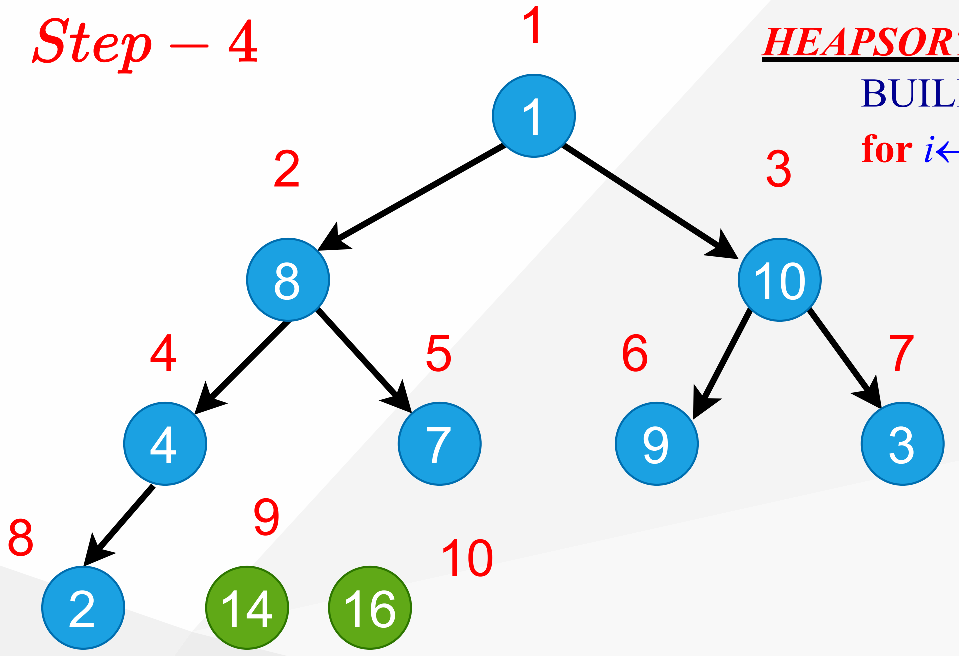
Array Storage

$A =$

1	2	3	4	5	6	7	8	9	10
14	8	10	4	7	9	3	2	1	16

Heapsort Algorithm Example (Step-4)

$Step - 4$



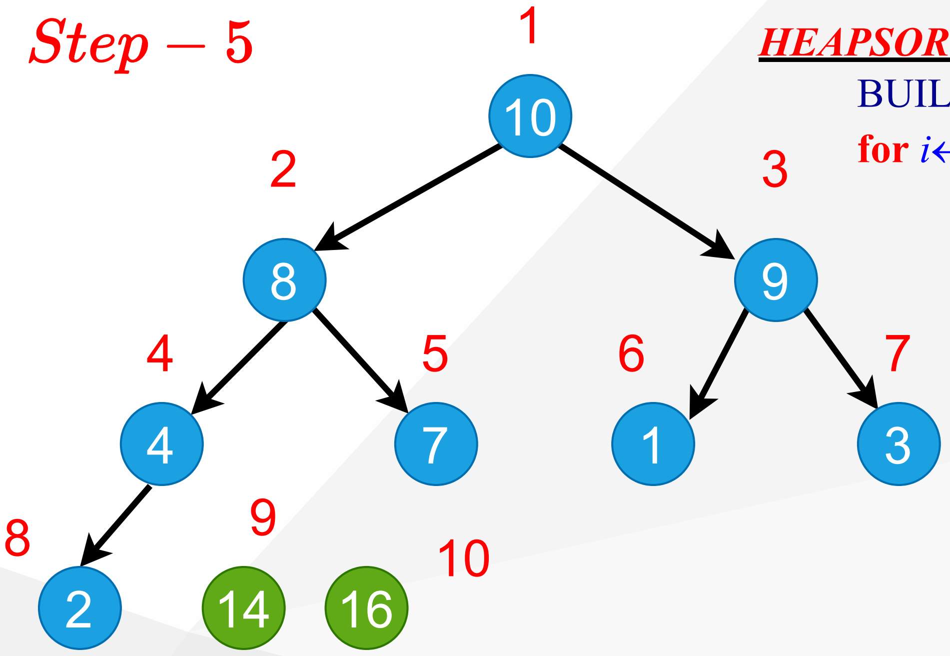
```
HEAPSORT(A, n)
    BUILD-HEAP(A, n)
    for i← n downto 2 do
        exchange A[1] ↔A[i]
        HEAPIFY(A,1, i –1)
```

Array Storage

	1	2	3	4	5	6	7	8	9	10
$A =$	1	8	10	4	7	9	3	2	14	16

Heapsort Algorithm Example (Step-5)

$Step - 5$



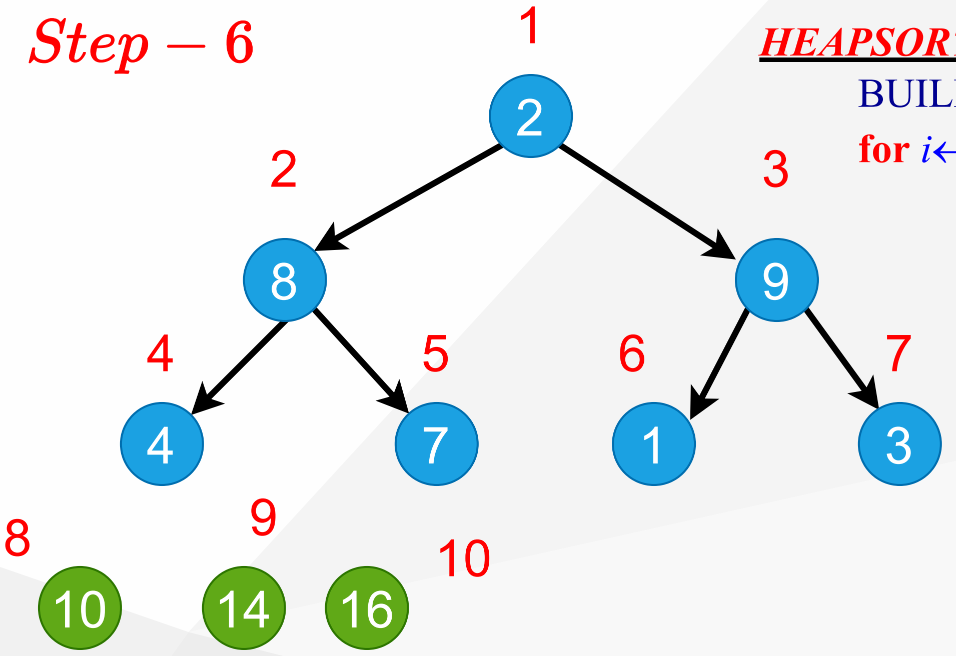
```
HEAPSORT(A, n)
    BUILD-HEAP(A, n)
    for i ← n downto 2 do
        exchange A[1] ↔ A[i]
        HEAPIFY(A, 1, i − 1)
```

Array Storage

$A =$

1	2	3	4	5	6	7	8	9	10
10	8	9	4	7	1	3	2	14	16

Heapsort Algorithm Example (Step-6)



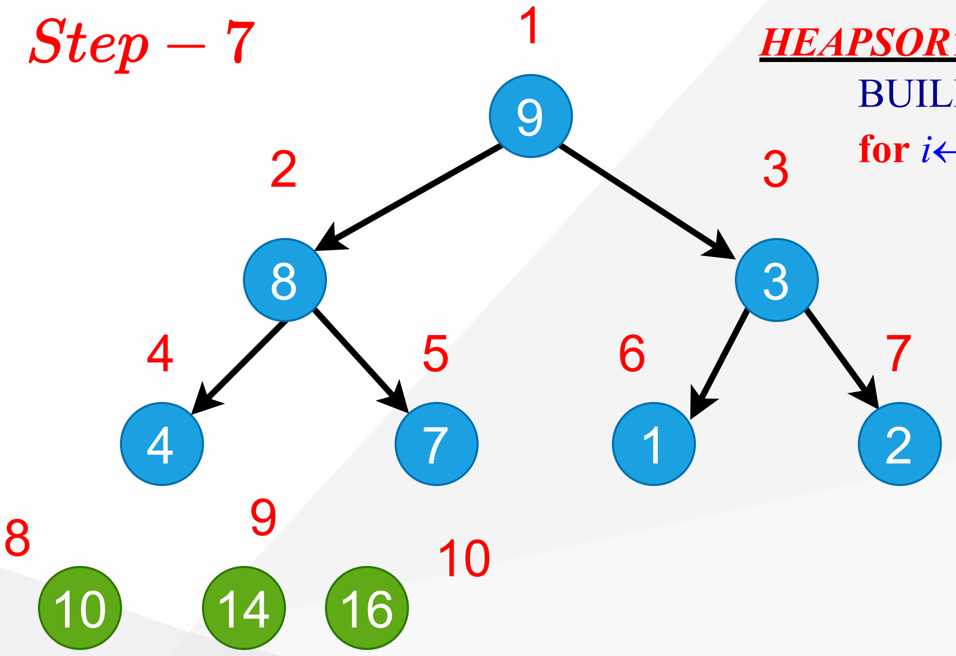
```
HEAPSORT(A, n)
    BUILD-HEAP(A, n)
    for i← n downto 2 do
        exchange A[1] ↔A[i]
        HEAPIFY(A,1, i –1)
```

Array Storage

	1	2	3	4	5	6	7	8	9	10
$A =$	2	8	9	4	7	1	3	10	14	16

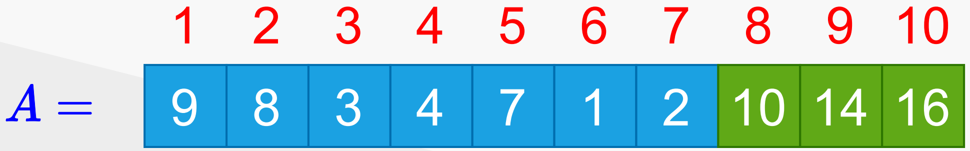
Heapsort Algorithm Example (Step-7)

$Step - 7$



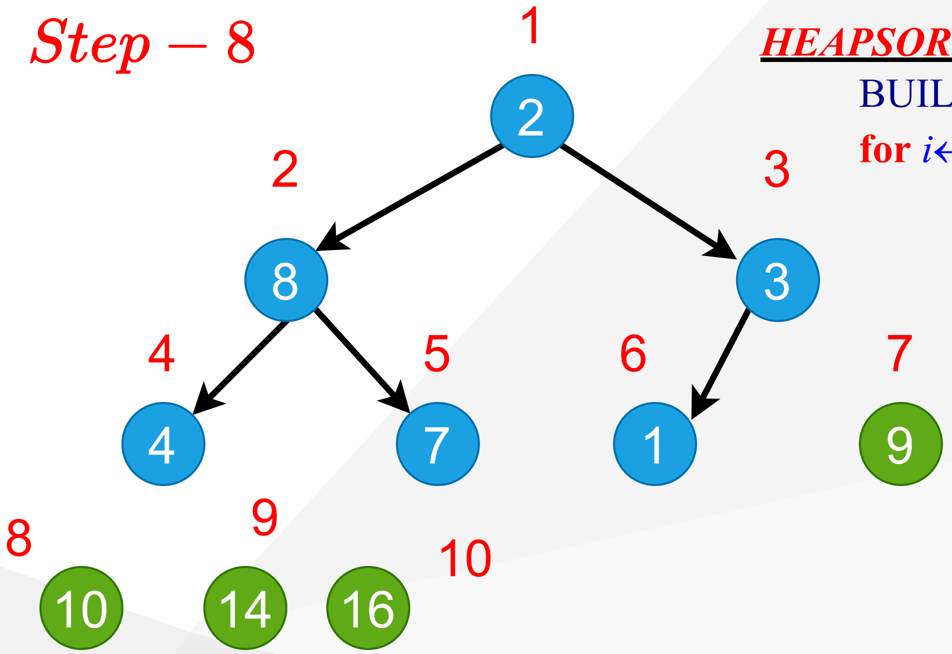
```
HEAPSORT(A, n)
    BUILD-HEAP(A, n)
    for i ← n downto 2 do
        exchange A[1] ↔ A[i]
        HEAPIFY(A,1, i −1)
```

Array Storage



	1	2	3	4	5	6	7	8	9	10
$A =$	9	8	3	4	7	1	2	10	14	16

Heapsort Algorithm Example (Step-8)

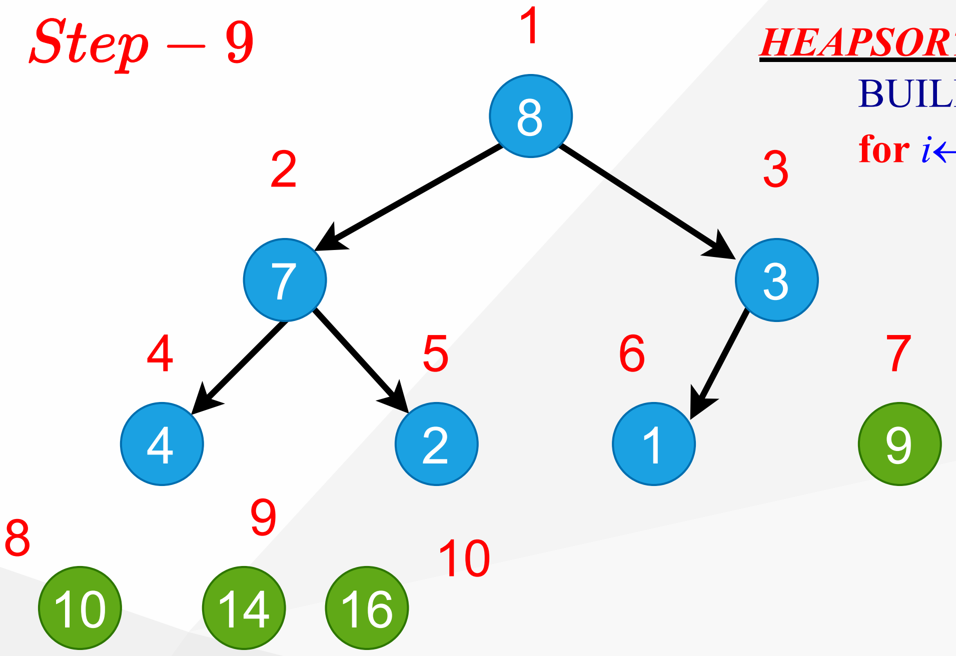


```
HEAPSORT(A, n)
    BUILD-HEAP(A, n)
    for i ← n downto 2 do
        exchange A[1] ↔ A[i]
        HEAPIFY(A,1, i −1)
```

Array Storage

	1	2	3	4	5	6	7	8	9	10
$A =$	2	8	3	4	7	1	9	10	14	16

Heapsort Algorithm Example (Step-9)

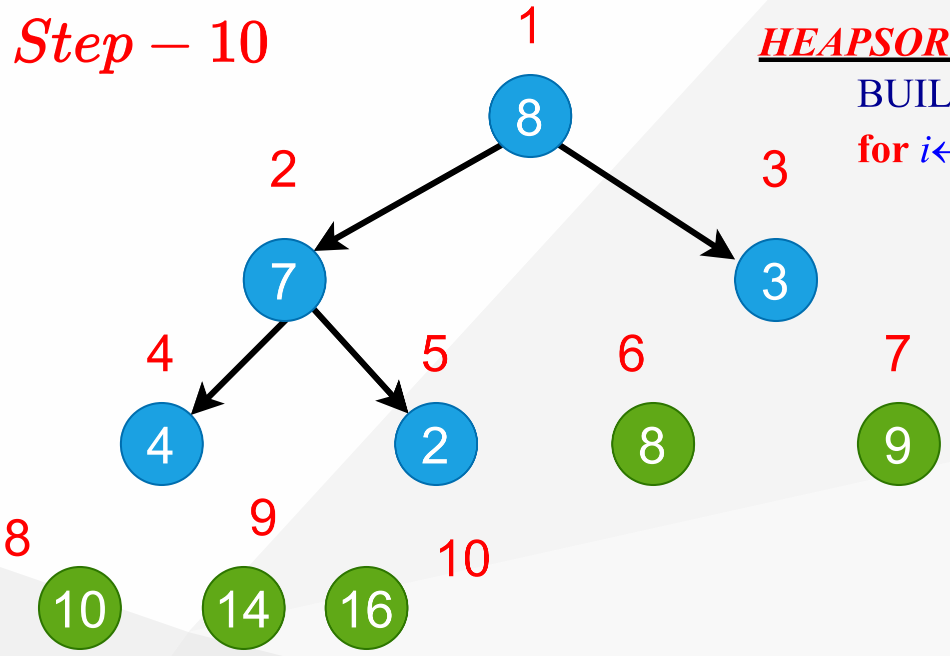


```
HEAPSORT(A, n)
    BUILD-HEAP(A, n)
    for i← n downto 2 do
        exchange A[1] ↔A[i]
        HEAPIFY(A,1, i –1)
```

Array Storage

	1	2	3	4	5	6	7	8	9	10
$A =$	8	7	3	4	2	1	9	10	14	16

Heapsort Algorithm Example (Step-10)



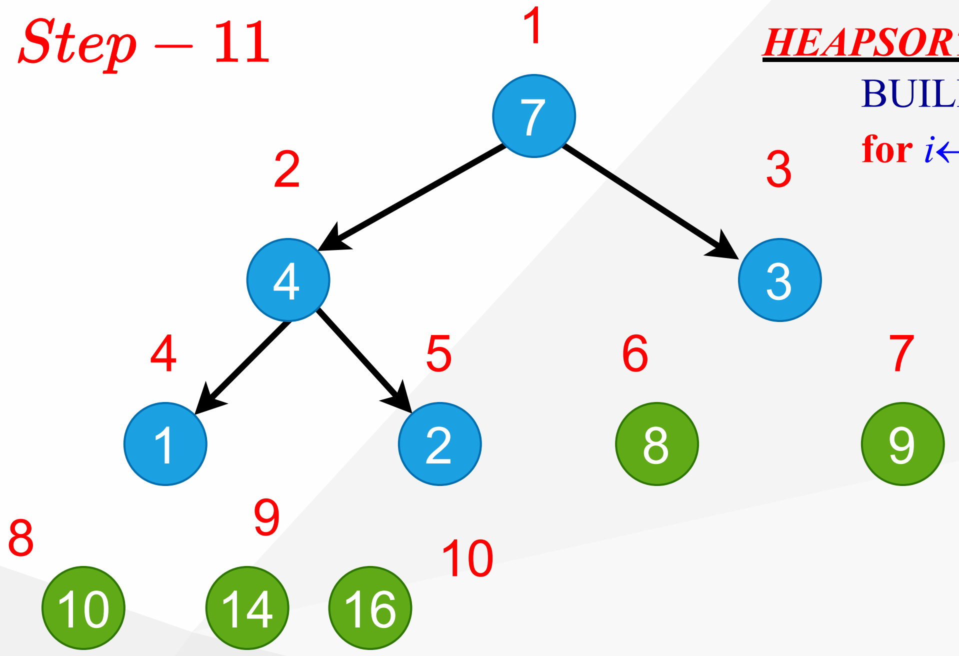
```
HEAPSORT(A, n)
    BUILD-HEAP(A, n)
    for i← n downto 2 do
        exchange A[1] ↔A[i]
        HEAPIFY(A,1, i –1)
```

Array Storage

	1	2	3	4	5	6	7	8	9	10
$A =$	1	7	3	4	2	8	9	10	14	16

Heapsort Algorithm Example (Step-11)

$Step - 11$



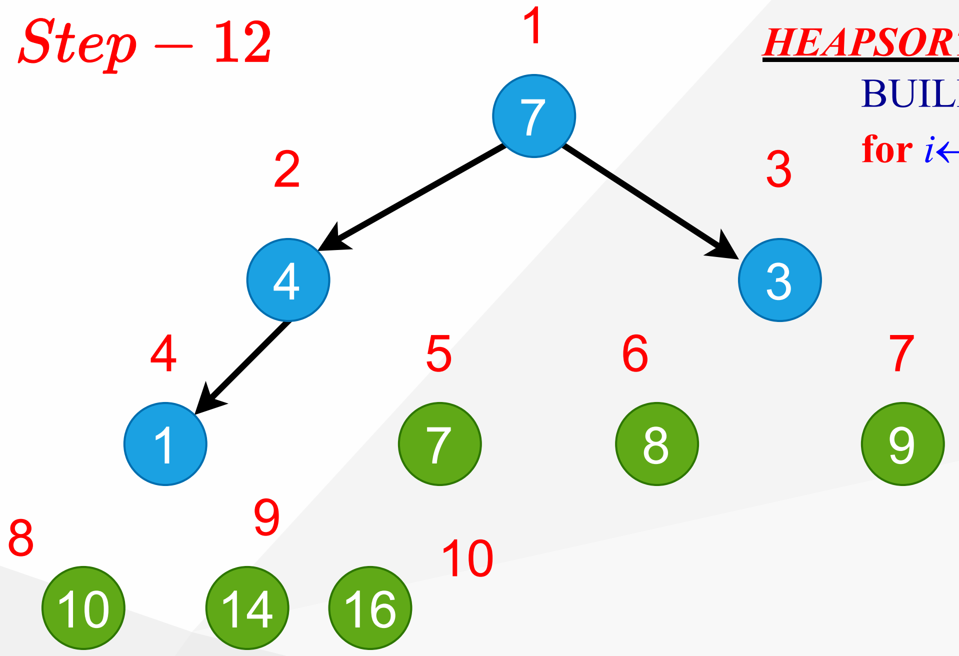
```
HEAPSORT(A, n)
    BUILD-HEAP(A, n)
    for i ← n downto 2 do
        exchange A[1] ↔ A[i]
        HEAPIFY(A,1, i −1)
```

Array Storage

	1	2	3	4	5	6	7	8	9	10
$A =$	7	4	3	1	2	8	9	10	14	16

Heapsort Algorithm Example (Step-12)

$Step - 12$

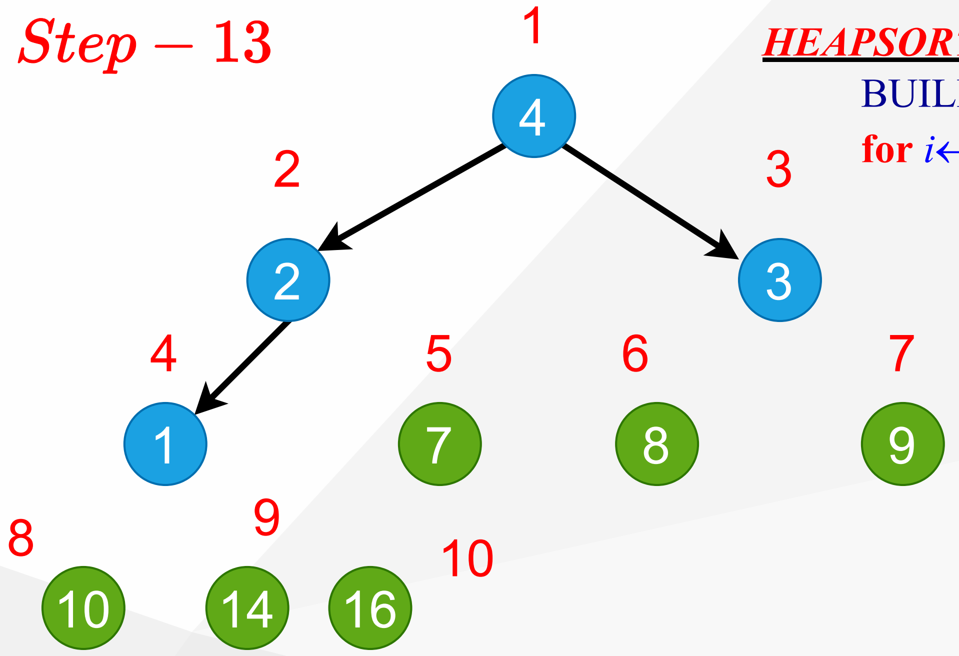


```
HEAPSORT(A, n)
    BUILD-HEAP(A, n)
    for i← n downto 2 do
        exchange A[1] ↔A[i]
        HEAPIFY(A,1, i –1)
```

Array Storage

	1	2	3	4	5	6	7	8	9	10
$A =$	2	4	3	1	7	8	9	10	14	16

Heapsort Algorithm Example (Step-13)



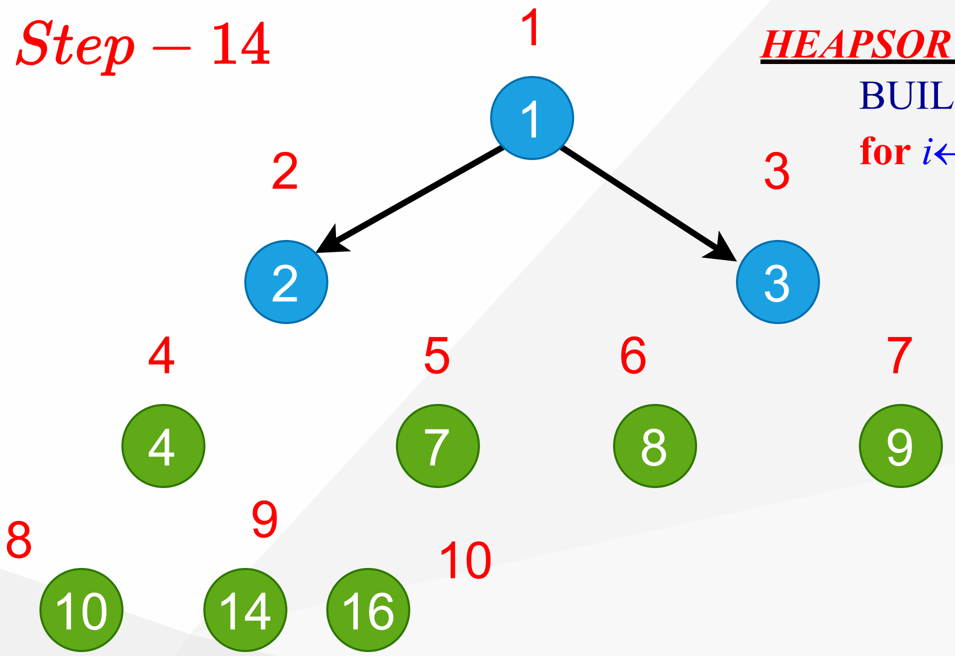
```
HEAPSORT(A, n)
    BUILD-HEAP(A, n)
    for i← n downto 2 do
        exchange A[1] ↔A[i]
        HEAPIFY(A,1, i –1)
```

Array Storage

$A =$

1	2	3	4	5	6	7	8	9	10
4	2	3	1	7	8	9	10	14	16

Heapsort Algorithm Example (Step-14)



```
HEAPSORT(A, n)
    BUILD-HEAP(A, n)
    for i← n downto 2 do
        exchange A[1] ↔A[i]
        HEAPIFY(A,1, i –1)
```

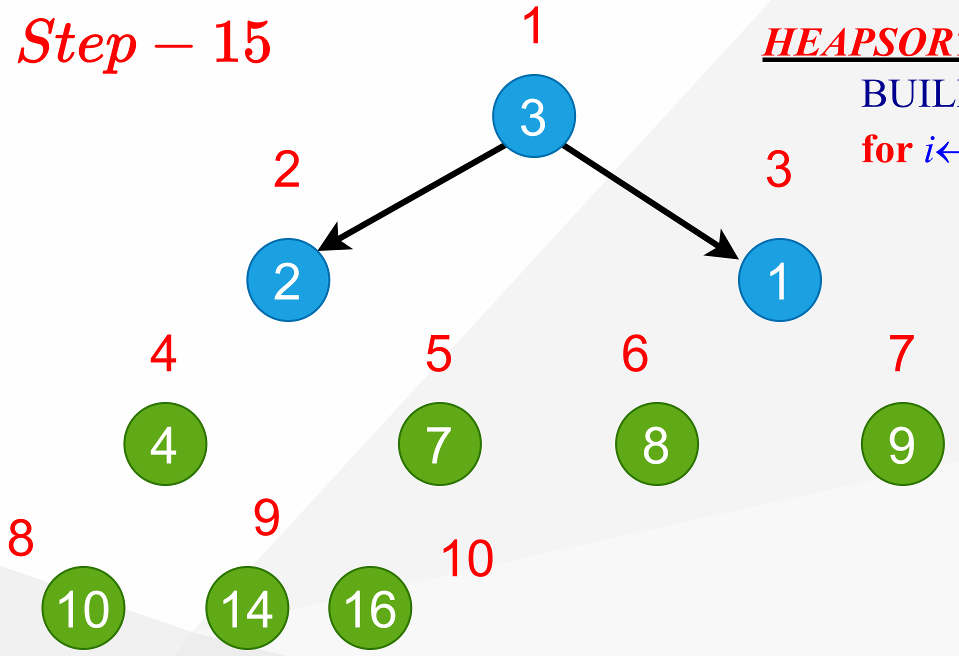
Array Storage

$A =$

1	2	3	4	5	6	7	8	9	10
1	2	3	4	7	8	9	10	14	16

Heapsort Algorithm Example (Step-15)

$Step - 15$

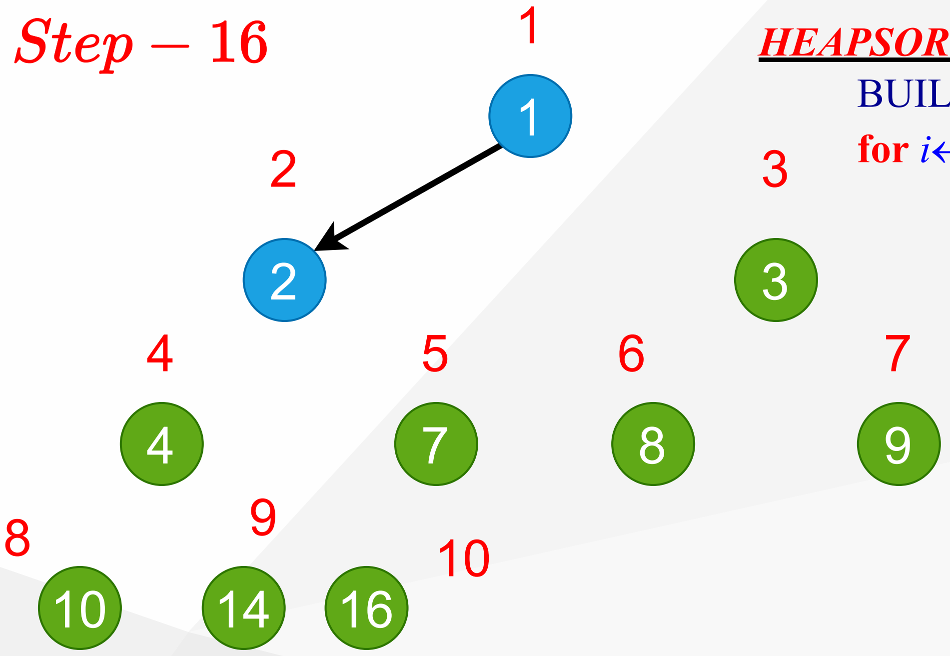


```
HEAPSORT(A, n)
    BUILD-HEAP(A, n)
    for i← n downto 2 do
        exchange A[1] ↔A[i]
        HEAPIFY(A,1, i –1)
```

Array Storage

	1	2	3	4	5	6	7	8	9	10
$A =$	3	2	1	4	7	8	9	10	14	16

Heapsort Algorithm Example (Step-16)



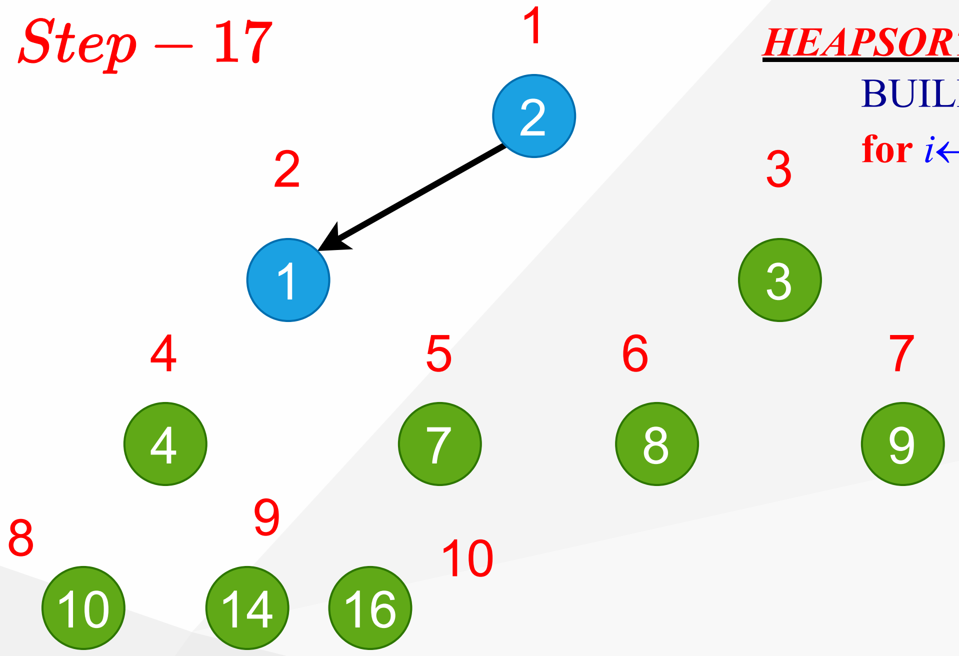
```
HEAPSORT(A, n)
    BUILD-HEAP(A, n)
    for i← n downto 2 do
        exchange A[1] ↔A[i]
        HEAPIFY(A,1, i –1)
```

Array Storage

1	2	3	4	5	6	7	8	9	10
1	2	3	4	7	8	9	10	14	16

$A =$

Heapsort Algorithm Example (Step-17)



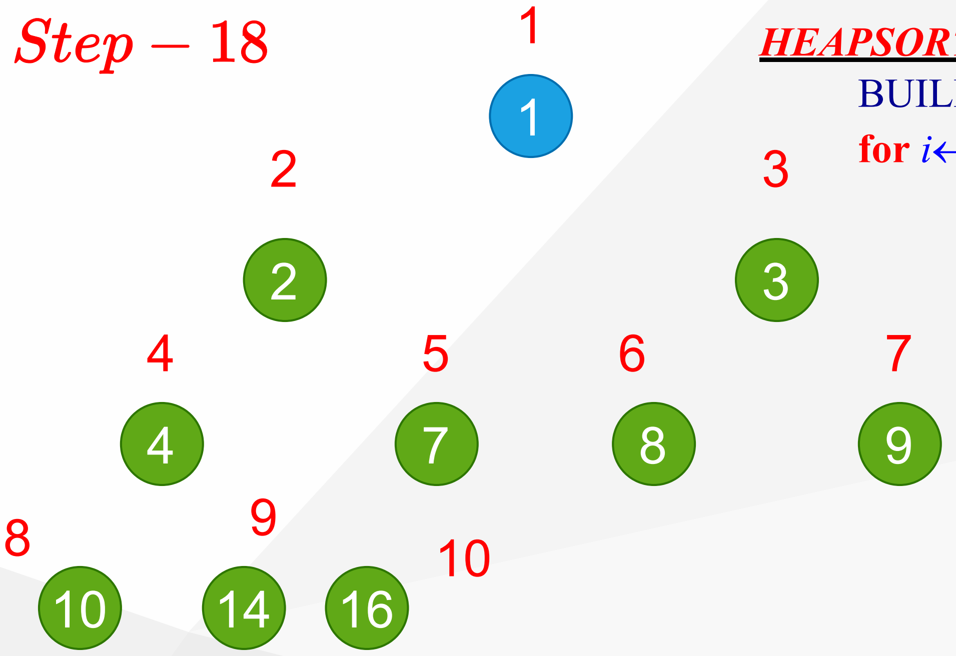
```
HEAPSORT(A, n)
    BUILD-HEAP(A, n)
    for i← n downto 2 do
        exchange A[1] ↔A[i]
        HEAPIFY(A,1, i –1)
```

Array Storage

	1	2	3	4	5	6	7	8	9	10
$A =$	2	1	3	4	7	8	9	10	14	16

Heapsort Algorithm Example (Step-18)

$Step - 18$



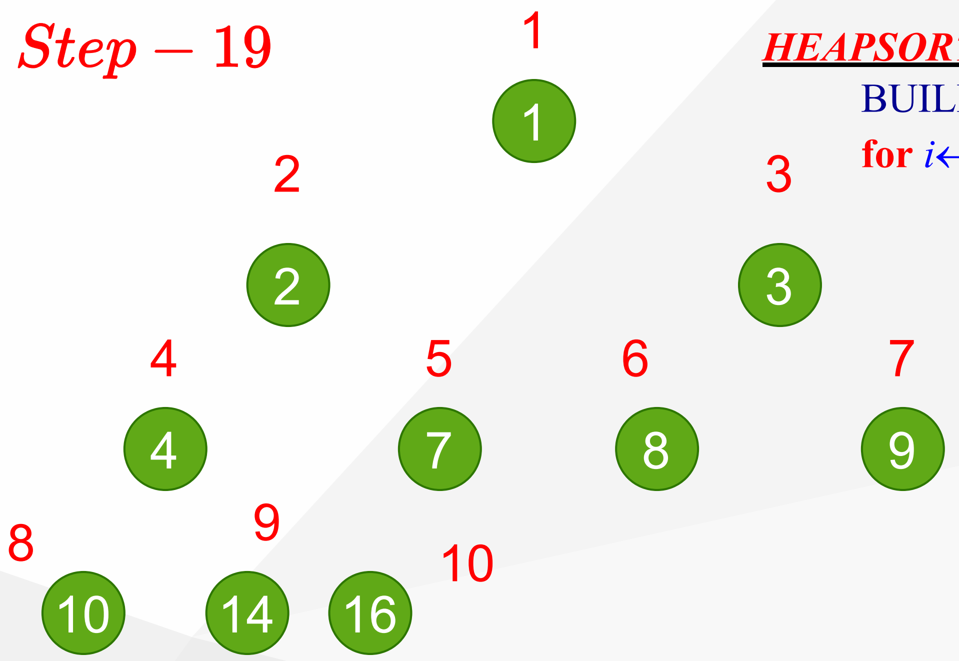
```
HEAPSORT(A, n)
    BUILD-HEAP(A, n)
    for i ← n downto 2 do
        exchange A[1] ↔ A[i]
        HEAPIFY(A,1, i −1)
```

Array Storage

	1	2	3	4	5	6	7	8	9	10
$A =$	1	2	3	4	7	8	9	10	14	16

Heapsort Algorithm Example (Step-19)

Step – 19



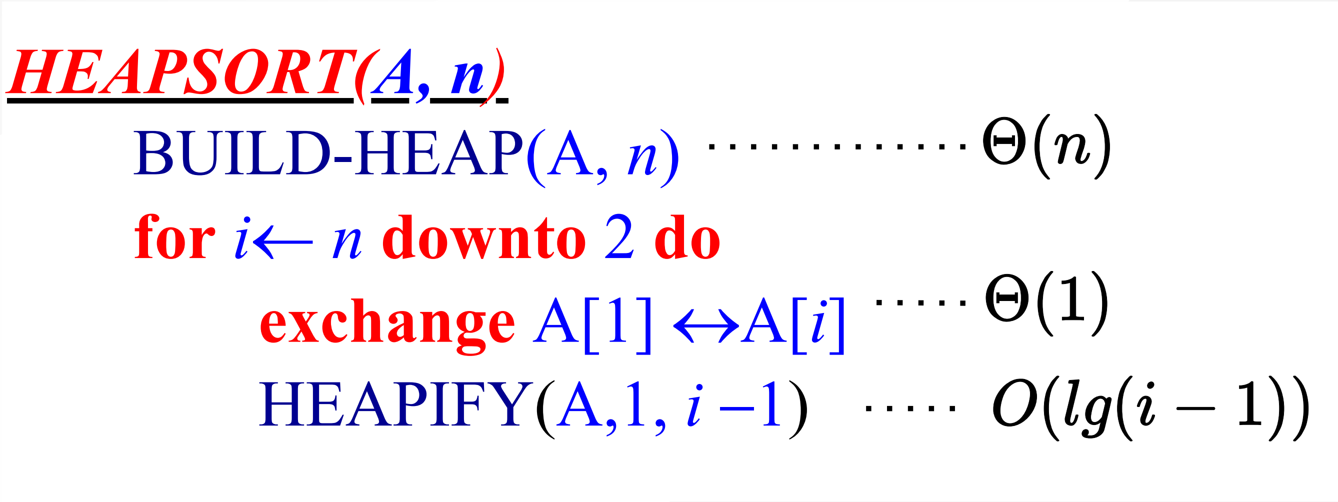
```
HEAPSORT(A, n)
    BUILD-HEAP(A, n)
    for i ← n downto 2 do
        exchange A[1] ↔ A[i]
        HEAPIFY(A,1, i –1)
```

Array Storage

$A =$

1	2	3	4	5	6	7	8	9	10
1	2	3	4	7	8	9	10	14	16

Heapsort Algorithm: Runtime Analysis



```
HEAPSORT(A, n)
    BUILD-HEAP(A, n) ............. $\Theta(n)$
    for $i \leftarrow n$ downto 2 do
        exchange A[1] ↔ A[i] ..... $\Theta(1)$
        HEAPIFY(A,1, i −1) ..... $O(lg(i-1))$
```

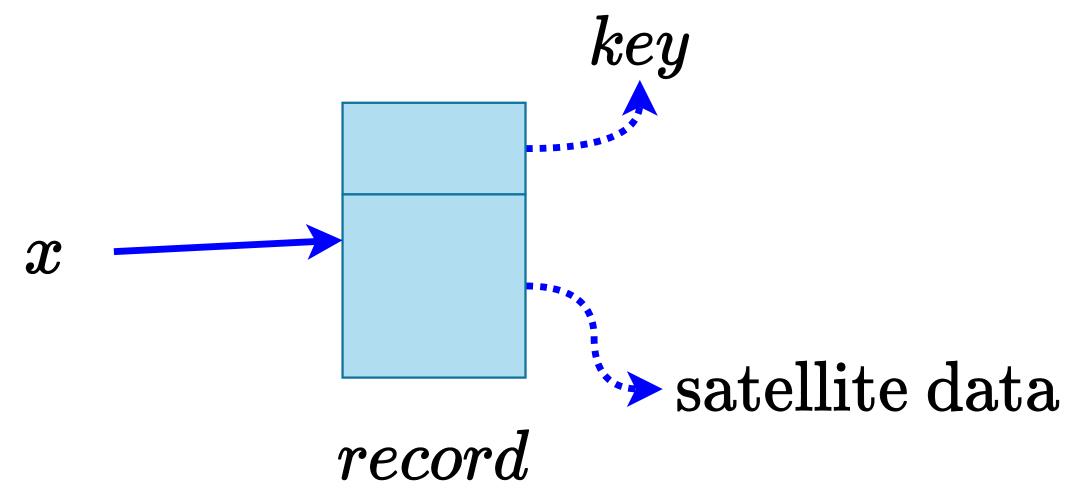
$$
\begin{aligned}
T(n) &= \Theta(n) + \sum_{i=2}^{n} O(lgi) \\
&= \Theta(n) + O\left(\sum_{i=2}^{n} O(lgn)\right) \\
&= O(nlgn)
\end{aligned}
$$

Heapsort - Notes

- **Heapsort** is a very good algorithm but, a good implementation of **quicksort** always **beats** heapsort **in practice**
- However, **heap data structure** has many popular applications, and it can be efficiently used for implementing **priority queues**

Data structures for Dynamic Sets

- Consider sets of records having **key** and **satellite** data



Operations on Dynamic Sets

- **Queries:** Simply return info;
  - $MAX(S)/MIN(S)$: (Query) return $x \in S$ with the **largest/smallest** key
  - $SEARCH(S, k)$: (Query) return $x \in S$ with $key[x] = k$
  - $SUCCESSOR(S, x)/PREDECESSOR(S, x)$: (Query) return $y \in S$ which is the next **larger/smaller** element after x
- **Modifying operations:** Change the set
  - $INSERT(S, x)$: (Modifying) $S \leftarrow S \cup \{x\}$
  - $DELETE(S, x)$: (Modifying) $S \leftarrow S - \{x\}$
  - $\text{EXTRACT-MAX}(S)/\text{EXTRACT-MIN}(S)$: (Modifying) return and delete $x \in S$ with the largest/smallest key
- Different data structures support/optimize different operations

Priority Queues (PQ)

- Supports
    - $INSERT$
    - MAX/MIN
    - $\text{EXTRACT-MAX/EXTRACT-MIN}$

Priority Queues (PQ)

- **One application:** Schedule jobs on a shared resource
    - **PQ** keeps track of jobs and their relative priorities
    - When a job is finished or interrupted, highest priority job is selected from those pending using EXTRACT-MAX
    - A new job can be added at any time using $INSERT$

Priority Queues (PQ)

- **Another application:** Event-driven simulation
    - Events to be simulated are the items in the **PQ**
    - Each event is associated with a time of occurrence which serves as a key
    - Simulation of an event can cause other events to be simulated in the future
    - Use EXTRACT-MIN at each step to choose the next event to simulate
    - As new events are produced insert them into the **PQ** using $INSERT$

Implementation of Priority Queue

- **Sorted linked list:** Simplest implementation
    - $INSERT$
        - $O(n)$ time
        - Scan the list to find place and splice in the new item
    - EXTRACT-MAX
        - $O(1)$ time
        - Take the first element
    - **Fast** extraction but **slow** insertion.

Implementation of Priority Queue

- **Unsorted linked list:** Simplest implementation
    - $INSERT$
        - $O(1)$ time
        - Put the new item at front
    - EXTRACT-MAX
        - $O(n)$ time
        - Scan the whole list
    - **Fast** insertion but **slow** extraction.
- Sorted linked list is better on the average
    - **Sorted list:** on the average, scans $n/2$ element per insertion
    - **Unsorted list:** always scans n element at each extraction

Heap Implementation of PQ

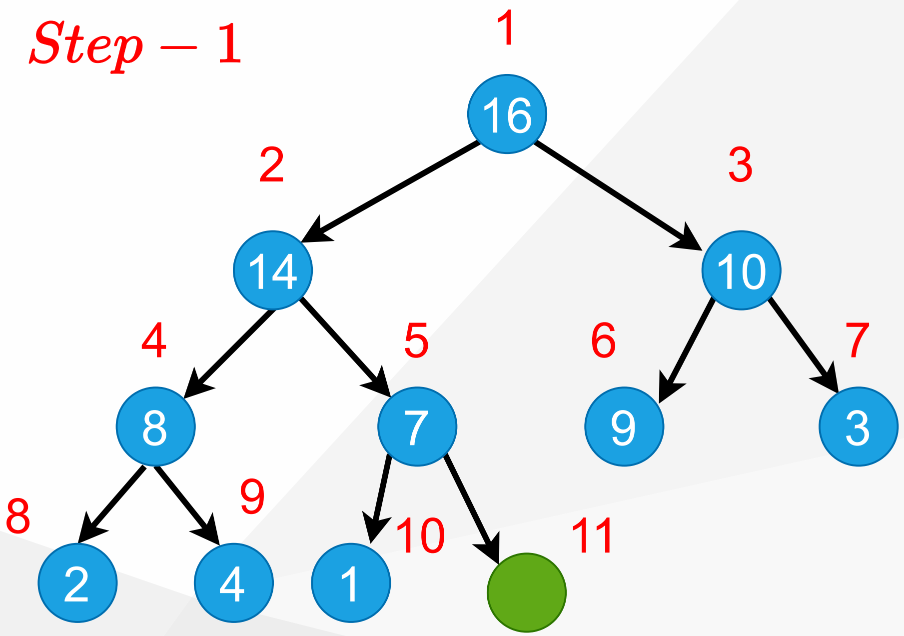
- $INSERT$ and EXTRACT-MAX are both $O(lgn)$
  - good compromise between fast insertion but slow extraction and vice versa
- EXTRACT-MAX: already discussed HEAP-EXTRACT-MAX
- $INSERT$: Insertion is like that of Insertion-Sort.

```
HEAP-INSERT(A, key, n)
  n = n+1
  i=n
  while i>1 and A[floor(i/2)] < key do
    A[i]=A[floor(i/2)]
    i= floor(i/2)
  A[i]=key
```

Heap Implementation of PQ

- Traverses $O(lgn)$ nodes, as $HEAPIFY$ does but makes fewer comparisons and assignments
    - $HEAPIFY$: compares parent with both children
    - $HEAP-INSERT$: with only one

HEAP-INSERT Example (Step-1)



```
HEAP-INSERT(A, key, n)
n ← n+1
i ← n
while i > 1 and A[⌊i/2⌋] < key do
        A[i] ← A⌊i/2⌋
        i ← ⌊i/2⌋
        A[i] ← key
```

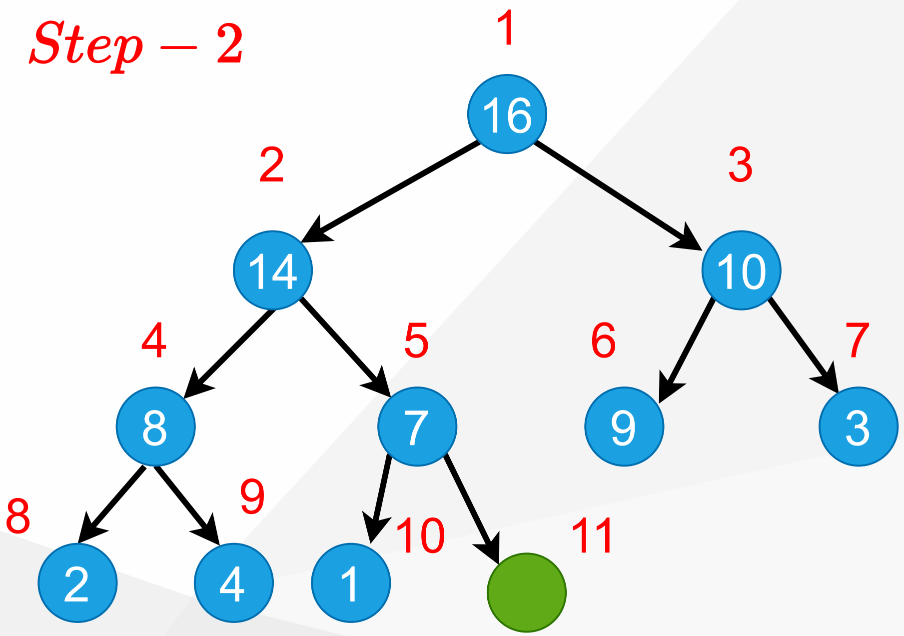
HEAP-INSERT(A, 15)

key=15

Array Storage

	1	2	3	4	5	6	7	8	9	10	11
$A =$	16	14	10	8	7	9	3	2	4	1	🟩

HEAP-INSERT Example (Step-2)


$Step - 2$

```
HEAP-INSERT(A, key, n)

$n \leftarrow n+1$
$i \leftarrow n$
while $i > 1$ and $A[\lfloor i/2 \rfloor] <$ key do
    $A[i] \leftarrow A[\lfloor i/2 \rfloor]$
    $i \leftarrow \lfloor i/2 \rfloor$
$A[i] \leftarrow$ key
```

HEAP-INSERT(A, 15)

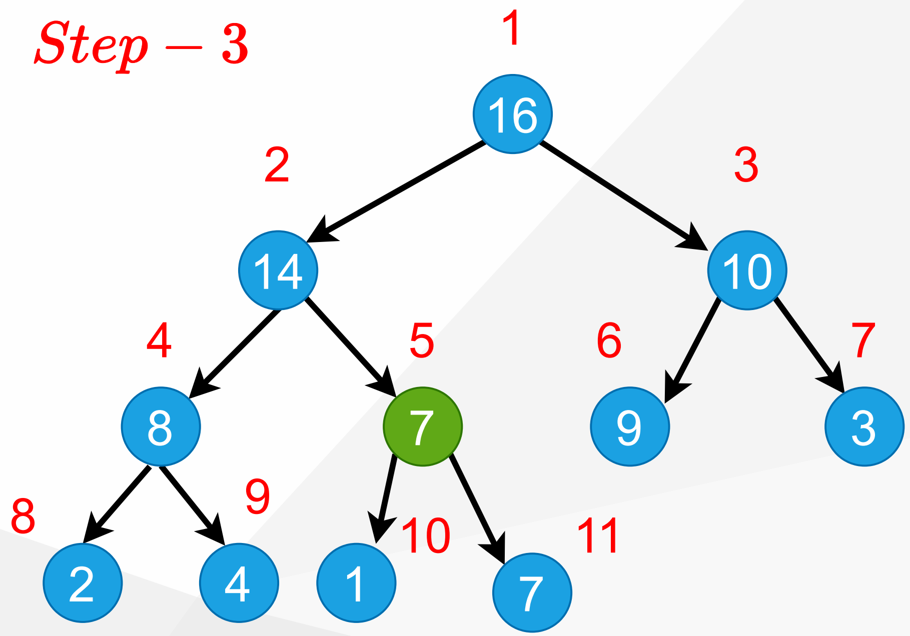
key=15

Array Storage

	1	2	3	4	5	6	7	8	9	10	11
$A =$	16	14	10	8	7	9	3	2	4	1	🟩

HEAP-INSERT Example (Step-3)

$Step - 3$



```
HEAP-INSERT(A, key, n)

n ← n+1
i ← n
while i >1 and A[⌊i/2⌋] < key do
        A[i] ← A⌊i/2⌋
        i ← ⌊i/2⌋
A[i] ← key
```

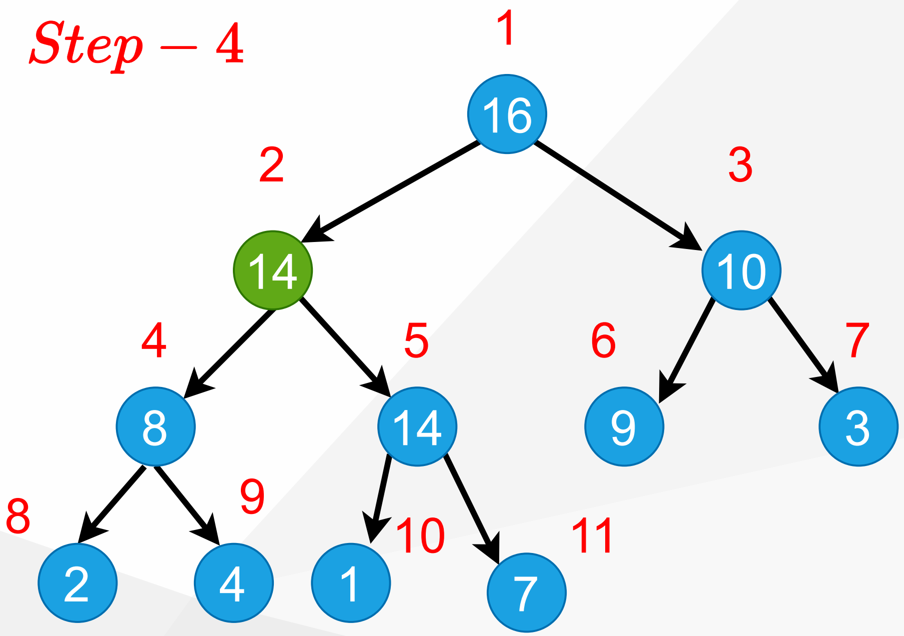
HEAP-INSERT(A, 15)

key=15

Array Storage

	1	2	3	4	5	6	7	8	9	10	11
$A =$	16	14	10	8	7	9	3	2	4	1	7

HEAP-INSERT Example (Step-4)



```
HEAP-INSERT(A, key, n)

$n \leftarrow n+1$
$i \leftarrow n$
while $i > 1$ and $A[\lfloor i/2 \rfloor] <$ key do
    $A[i] \leftarrow A\lfloor i/2 \rfloor$
    $i \leftarrow \lfloor i/2 \rfloor$
$A[i] \leftarrow$ key
```

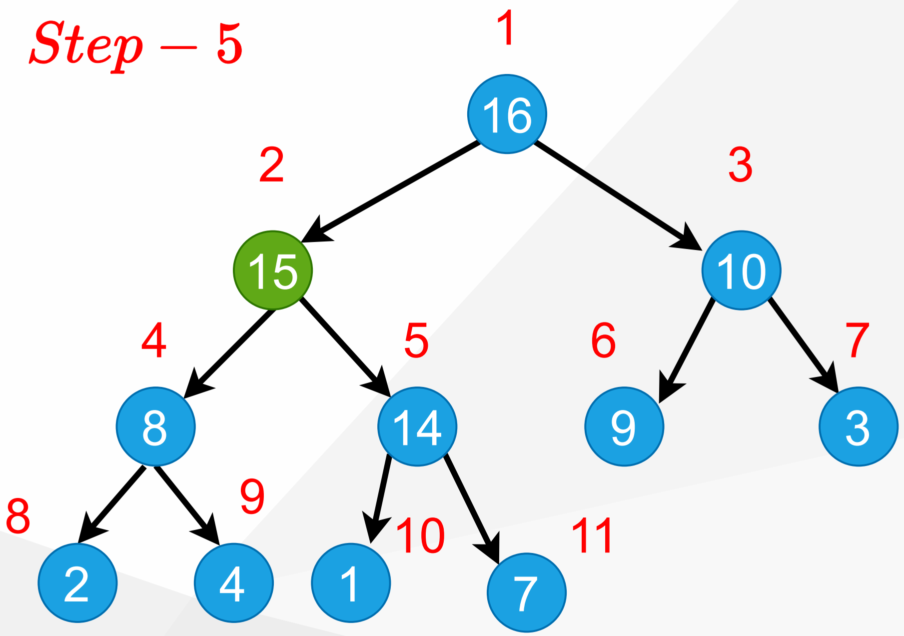
HEAP-INSERT(A, 15)

key=15

Array Storage

	1	2	3	4	5	6	7	8	9	10	11
$A =$	16	14	10	8	14	9	3	2	4	1	7

HEAP-INSERT Example (Step-5)



```
HEAP-INSERT(A, key, n)

n ← n+1
i ← n
while i >1 and A[⌊i/2⌋] < key do
        A[i] ← A⌊i/2⌋
        i ← ⌊i/2⌋
⇒ A[i] ← key
```

HEAP-INSERT(A, 15)

key=15

Array Storage

	1	2	3	4	5	6	7	8	9	10	11
$A =$	16	15	10	8	14	9	3	2	4	1	7

Heap Increase Key

- Key value of i^{th} element of heap is increased from $A[i]$ to key

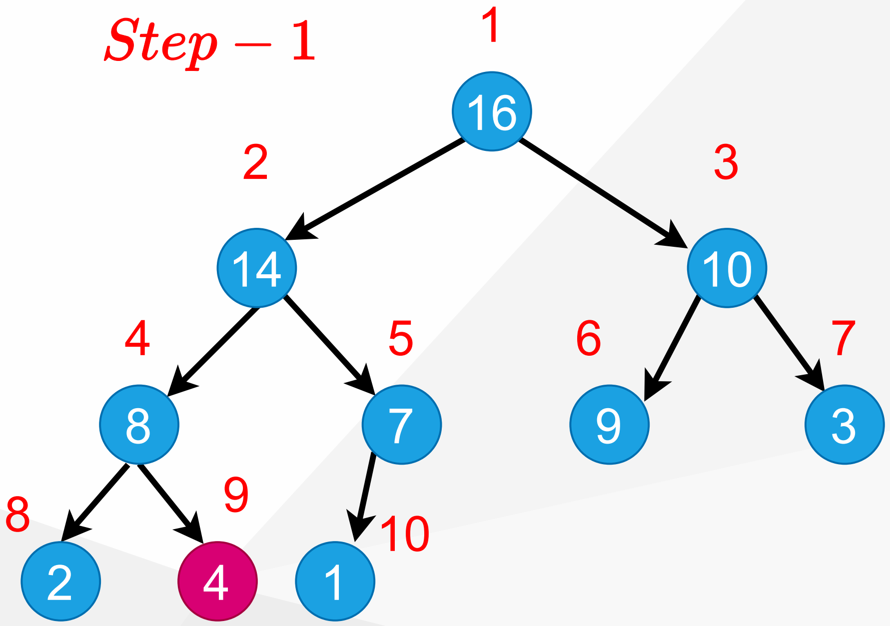
```
HEAP-INCREASE-KEY(A, i, key)

  if key < A[i] then
    return error

  while i > 1 and A[floor(i/2)] < key do
    A[i] = A[floor(i/2)]
    i = floor(i/2)

  A[i] = key
```

HEAP-INCREASE-KEY Example (Step-1)



```
HEAP-INCREASE-KEY(A, i, key)
  if key < A[i] then
  return error
  while i > 1 and A[⌊i/2⌋]<key do
    A[i] ← A[⌊i/2⌋]
    i ← ⌊i/2⌋

  A[i] ← key
```

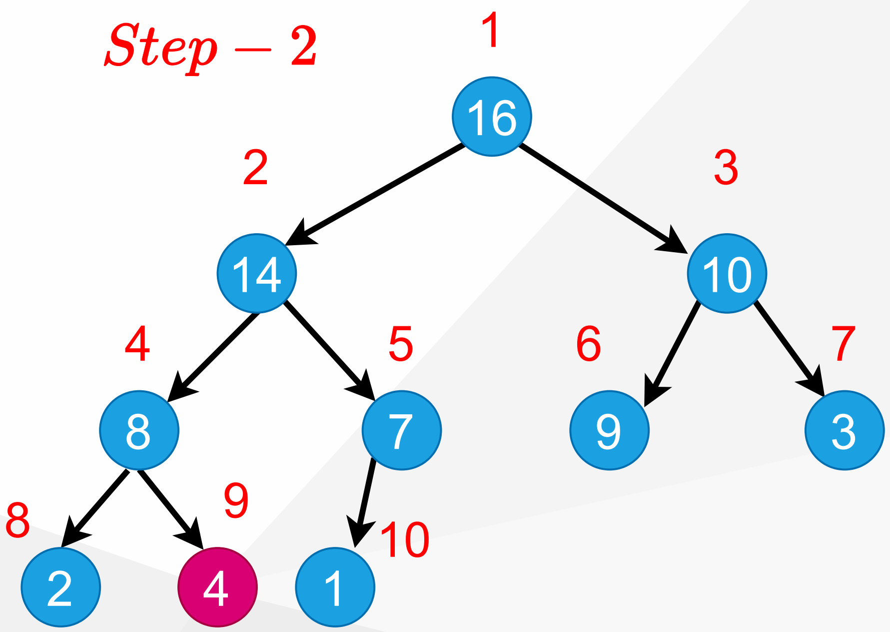
HEAP-INCREASE-KEY(A, 9, 15)

key=15

Array Storage

	1	2	3	4	5	6	7	8	9	10
$A =$	16	14	10	8	7	9	3	2	4	1

HEAP-INCREASE-KEY Example (Step-2)



```
HEAP-INCREASE-KEY(A, i, key)
  if key < A[i] then
  return error
  while $i > 1$ and $A[\lfloor i/2 \rfloor] < key$ do
    $A[i] \leftarrow A[\lfloor i/2 \rfloor]$
    $i \leftarrow \lfloor i/2 \rfloor$

  $A[i] \leftarrow key$
```

HEAP-INCREASE-KEY(A, 9, 15)

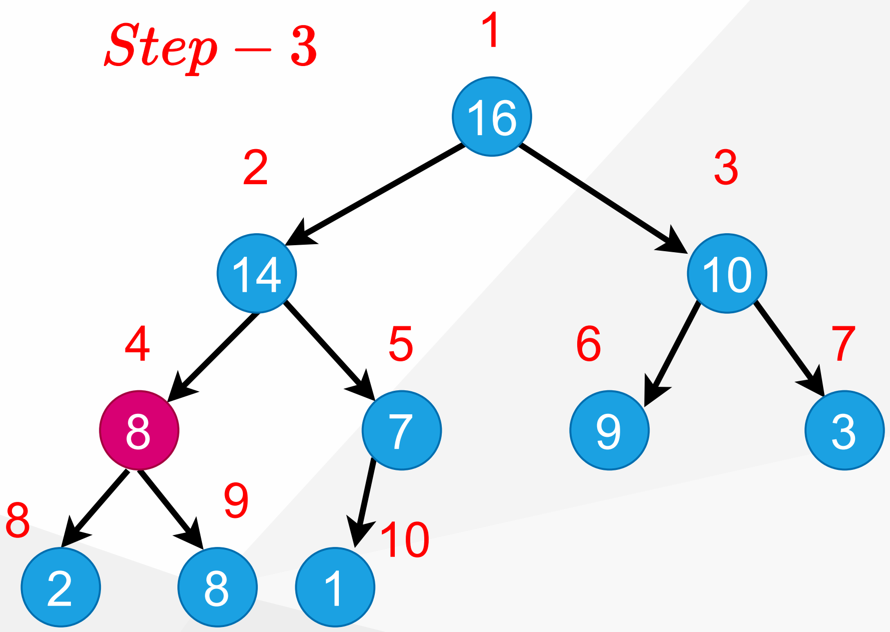
key=15

Array Storage

	1	2	3	4	5	6	7	8	9	10
$A =$	16	14	10	8	7	9	3	2	4	1

HEAP-INCREASE-KEY Example (Step-3)

$Step - 3$



```
HEAP-INCREASE-KEY(A, i, key)
  if key < A[i] then
  return error
  while i > 1 and A[⌊i/2⌋]<key do
    A[i] ← A[⌊i/2⌋]
    i ← ⌊i/2⌋

  A[i] ← key
```

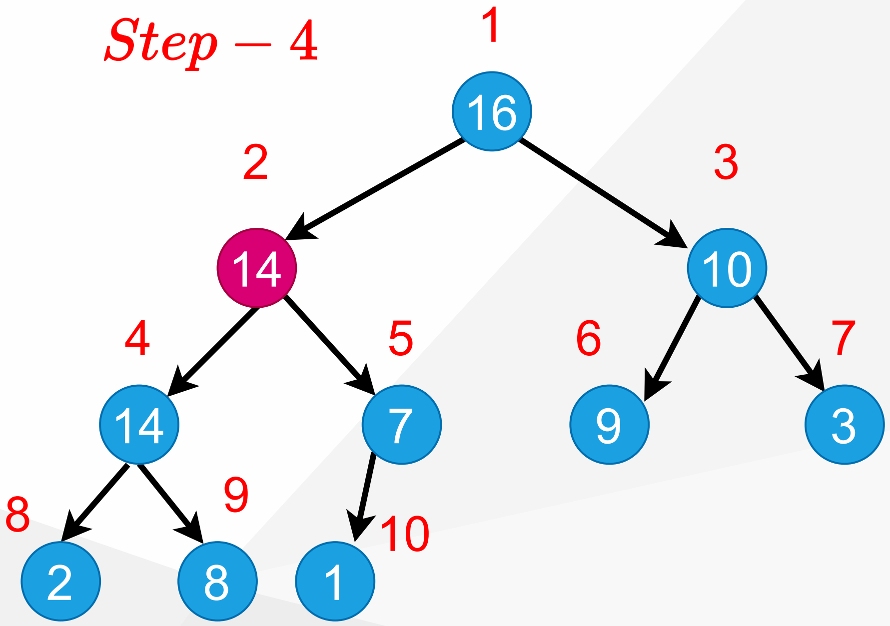
HEAP-INCREASE-KEY(A, 9, 15)

key=15

Array Storage

	1	2	3	4	5	6	7	8	9	10
$A =$	16	14	10	8	7	9	3	2	8	1

HEAP-INCREASE-KEY Example (Step-4)



```
HEAP-INCREASE-KEY(A, i, key)
  if key < A[i] then
  return error
  while i >1 and A[⌊i/2⌋]<key do
    A[i] ← A[⌊i/2⌋]
    i ← ⌊i/2⌋
  A[i] ← key
```

HEAP-INCREASE-KEY(A, 9, 15)

key=15

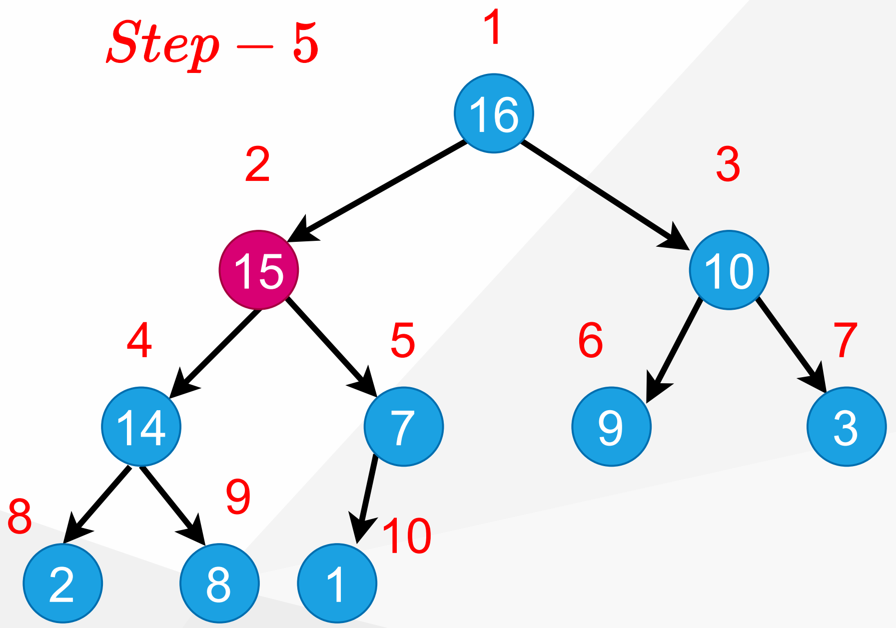
Array Storage

$A =$

1	2	3	4	5	6	7	8	9	10
16	14	10	14	7	9	3	2	8	1

HEAP-INCREASE-KEY Example (Step-5)

$Step - 5$



```
HEAP-INCREASE-KEY(A, i, key)
    if key < A[i] then
    return error
    while i >1 and A[⌊i/2⌋]<key do
        A[i] ← A[⌊i/2⌋]
        i ← ⌊i/2⌋
    A[i] ← key
```

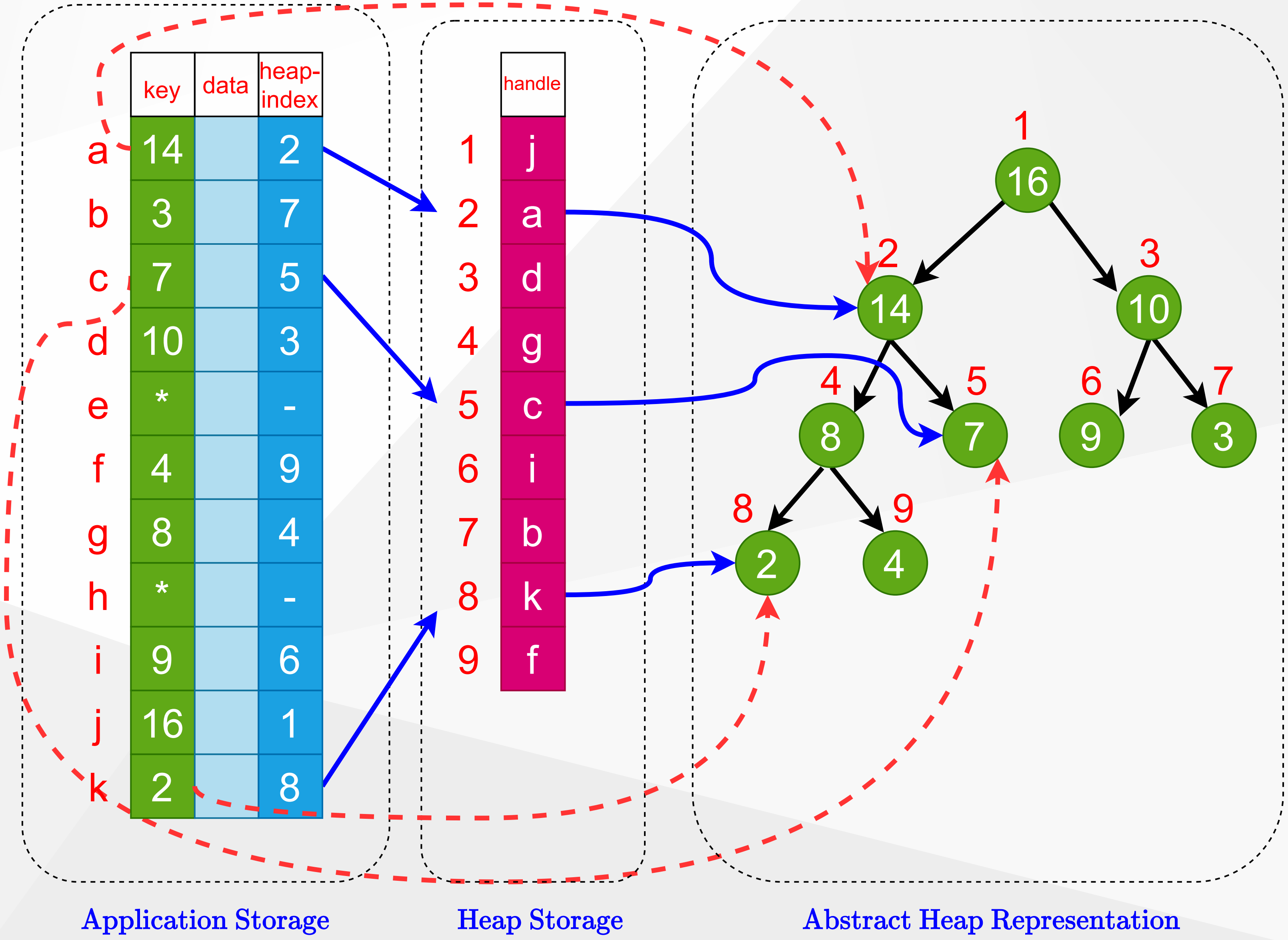
HEAP-INCREASE-KEY(A, 9, 15)

key=15

Array Storage

	1	2	3	4	5	6	7	8	9	10
$A =$	16	15	10	14	7	9	3	2	8	1

Heap Implementation of Priority Queue (PQ)


Application Storage

	key	data	heap-index
a	14		2
b	3		7
c	7		5
d	10		3
e	*		-
f	4		9
g	8		4
h	*		-
i	9		6
j	16		1
k	2		8

Heap Storage

	handle
1	j
2	a
3	d
4	g
5	c
6	i
7	b
8	k
9	f

Abstract Heap Representation

Summary: Max Heap

- **Heapify(A, i)**
  - Works when both child subtrees of node i are heaps
  - "*Floats down*" node i to satisfy the heap property
  - Runtime: $O(lgn)$
- **Max(A, n)**
  - Returns the max element of the heap (no modification)
  - Runtime: $O(1)$
- **Extract-Max(A, n)**
  - Returns and removes the max element of the heap
  - Fills the gap in $A[1]$ with $A[n]$, then calls **Heapify(A,1)**
  - Runtime: $O(lgn)$

Summary: Max Heap

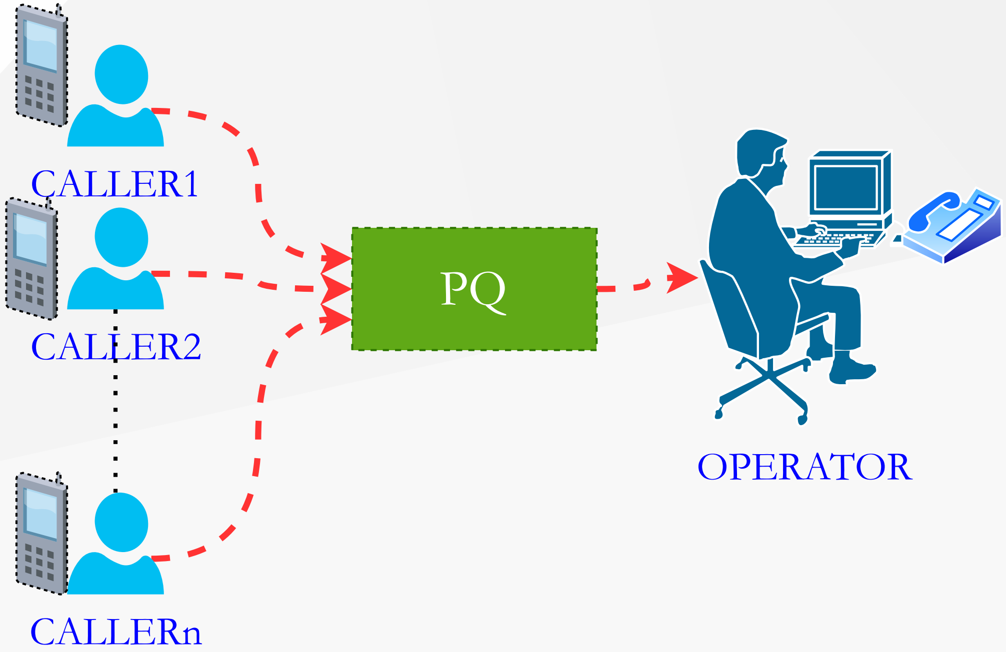
- **Build-Heap(A, n)**
  - Given an arbitrary array, builds a heap from scratch
  - Runtime: $O(n)$
- **Min(A, n)**
  - How to return the min element in a max-heap?
  - Worst case runtime: $O(n)$
    - because ~half of the heap elements are leaf nodes
  - Instead, use a min-heap for efficient min operations
- **Search(A, x)**
  - For an arbitrary x value, the worst-case runtime: $O(n)$
  - Use a sorted array instead for efficient search operations

Summary: Max Heap

- **Increase-Key(A, i, x)**
    - Increase the key of node i (from $A[i]$ to x)
    - *"Float up"* x until heap property is satisfied
    - Runtime: $O(lgn)$
- **Decrease-Key(A, i, x)**
    - Decrease the key of node i (from $A[i]$ to x)
    - Call **Heapify(A, i)**
    - Runtime: $O(lgn)$

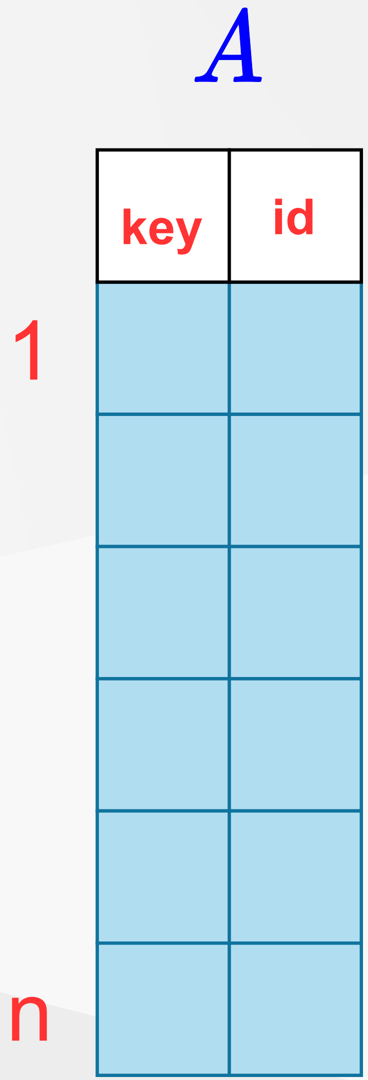
Phone Operator Problem

- A phone operator answering n **phones**
- Each phone i has x_i **people waiting** in line for their calls to be answered.
- Phone operator needs to answer the phone with the largest number of people waiting in line.
- New calls come continuously, and some people hang up after waiting.



Phone Operator Solution

- **Step 1:** Define the following array:
- $A[i]$: the ith element in heap
- $A[i].id$: the index of the corresponding phone
- $A[i].key$: $\#$ of people waiting in line for phone with index $A[i].id$

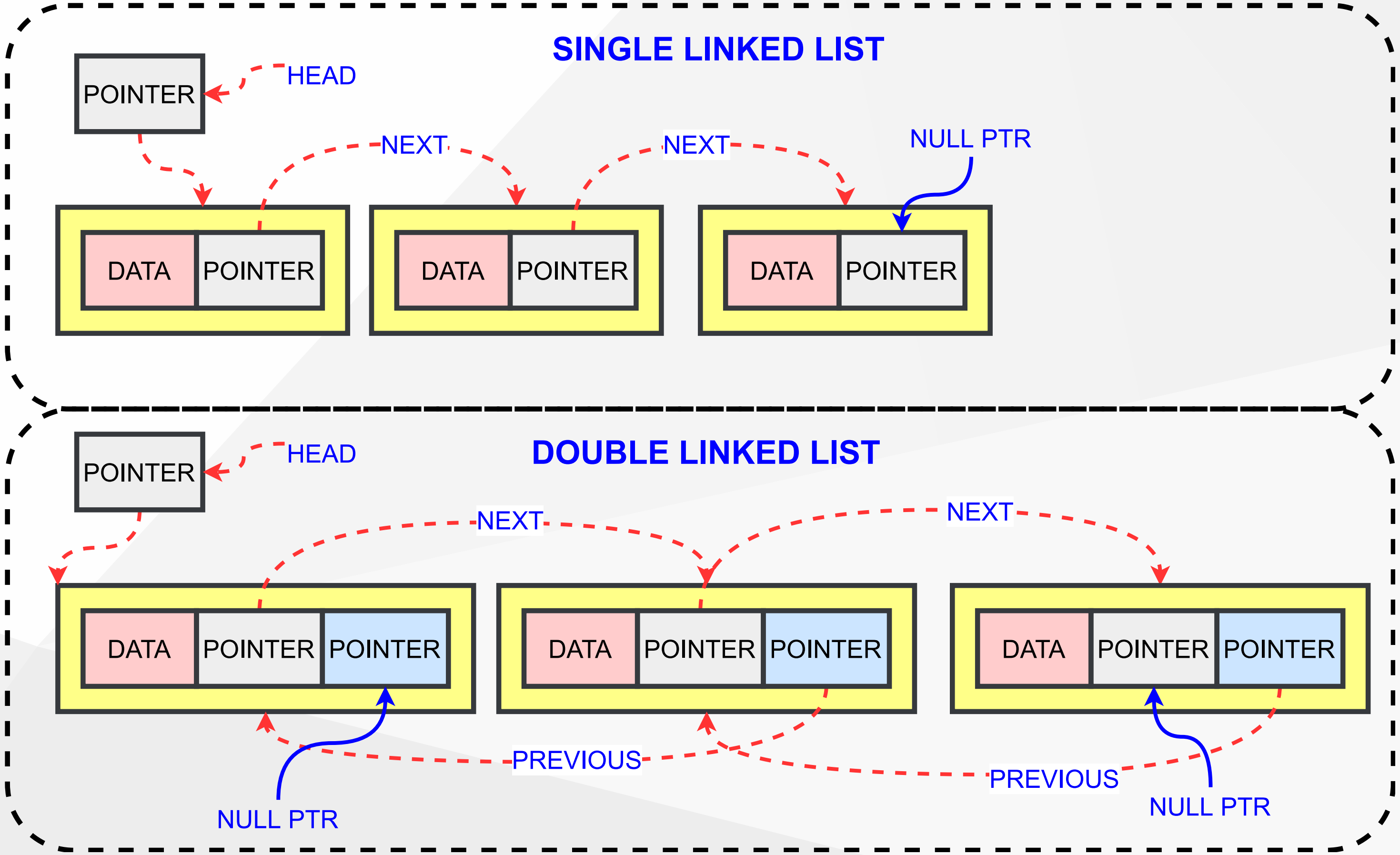


Phone Operator Solution

- **Step 2:** $\text{Build-Max-Heap}(A, n)$
    - **Execution:**
        - When the operator wants to answer a phone:
            - $id = A[1].id$
                - $\text{Decrease-Key}(A, 1, A[1].key - 1)$
                - answer phone with index id
            - When a new call comes in to phone i:
                - $\text{Increase-Key}(A, i, A[i].key + 1)$
            - When a call drops from phone i:
                - $\text{Decrease-Key}(A, i, A[i].key - 1)$

Linked Lists

- Like arrays, Linked List is a linear data structure.
- Unlike arrays, linked list elements are not stored at a contiguous location; the elements are linked using pointers.



Linked Lists - C Definition

- C

```
// A linked list node
struct Node {
  int data;
  struct Node* next;
};
```

Linked Lists - Cpp Definition

- Cpp

```cpp
class Node {
public:
  int data;
  Node* next;
};
```

Linked Lists - Java Definition

- Java

```
class LinkedList {
  Node head; // head of the list

  /* Linked list Node*/
  class Node {
      int data;
      Node next;

      // Constructor to create a new node
      // Next is by default initialized
      // as null
      Node(int d) { data = d; }
  }
}
```

Linked Lists - Csharp Definition

- Csharp

```
class LinkedList {
  // The first node(head) of the linked list
  // Will be an object of type Node (null by default)
  Node head;

  class Node {
      int data;
      Node next;

      // Constructor to create a new node
      Node(int d) { data = d; }
  }
}
```

Priority Queue using Linked List Methods

- Implement Priority Queue using Linked Lists.
    - **push():** This function is used to insert a new data into the queue.
    - **pop():** This function removes the element with the highest priority from the queue.
    - **peek()/top():** This function is used to get the highest priority element in the queue without removing it from the queue.

Priority Queue using Linked List Algorithm

```
PUSH(HEAD, DATA, PRIORITY)
  Create NEW.Data = DATA & NEW.Priority = PRIORITY
  If HEAD.priority < NEW.Priority
    NEW -> NEXT = HEAD
    HEAD = NEW
  Else
    Set TEMP to head of the list
  Endif

  WHILE TEMP -> NEXT != NULL and TEMP -> NEXT ->PRIORITY > PRIORITY THEN
    TEMP = TEMP -> NEXT
  ENDWHILE

  NEW -> NEXT = TEMP -> NEXT
  TEMP -> NEXT = NEW
```

Priority Queue using Linked List Algorithm

```
POP(HEAD)
//Set the head of the list to the next node in the list.
HEAD = HEAD -> NEXT.
Free the node at the head of the list
```

```
PEEK(HEAD):
Return HEAD -> DATA
```

Priority Queue using Linked List Notes

- LinkedList is already sorted.
- Time Complexities and Comparison with Binary Heap

	peek()	**push()**	**pop()**
Linked List	$O(1)$	$O(n)$	$O(1)$
Binary Heap	$O(1)$	$O(lgn)$	$O(lgn)$

Sorting in Linear Time

How Fast Can We Sort?

- The algorithms we have seen so far:
  - Based on comparison of elements
  - We only care about the relative ordering between the elements (not the actual values)
  - The smallest worst-case runtime we have seen so far: $O(nlgn)$
  - Is $O(nlgn)$ the best we can do?
- **Comparison sorts:** Only use comparisons to determine the relative order of elements.

Decision Trees for Comparison Sorts

- Represent a sorting algorithm abstractly in terms of a **decision tree**
    - A **binary tree** that represents the **comparisons between** elements in the sorting algorithm
    - Control, data movement, and other aspects are ignored
- One decision tree corresponds to one sorting algorithm and one value of n (*input size*)

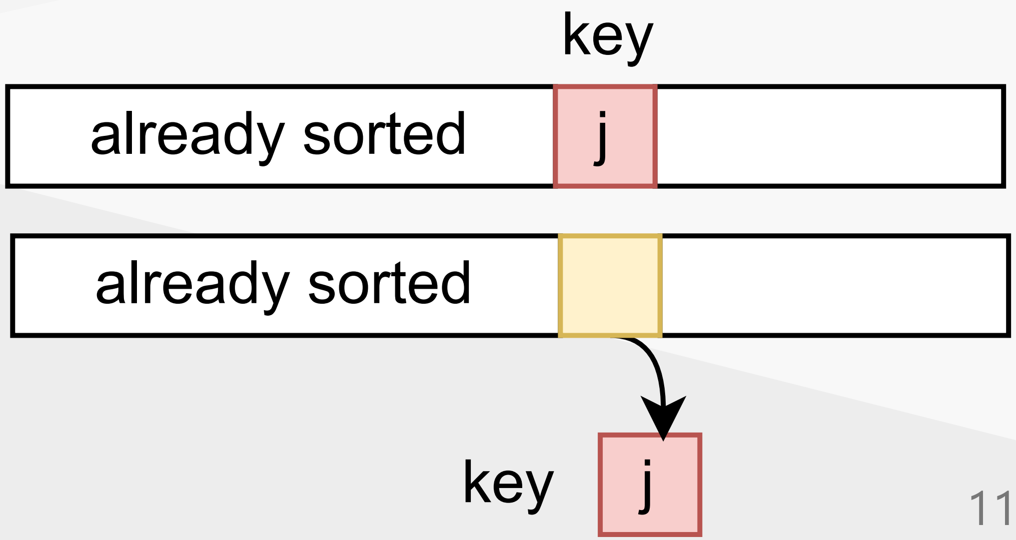
Reminder: Insertion Sort Step-By-Step Description (1)

```
Insertion-Sort(A)
for j = 2 to n do
    key = A[j];
    i = j-1;
    while i>0 and A[i]>key do
        A[i+1]=A[i];
        i = i-1;
    endwhile
    A[i+1]=key;
endfor
```

Iterate over array

Loop invariant:

The subarray $A[1..j-1]$ is always sorted



key

already sorted j

already sorted

key j

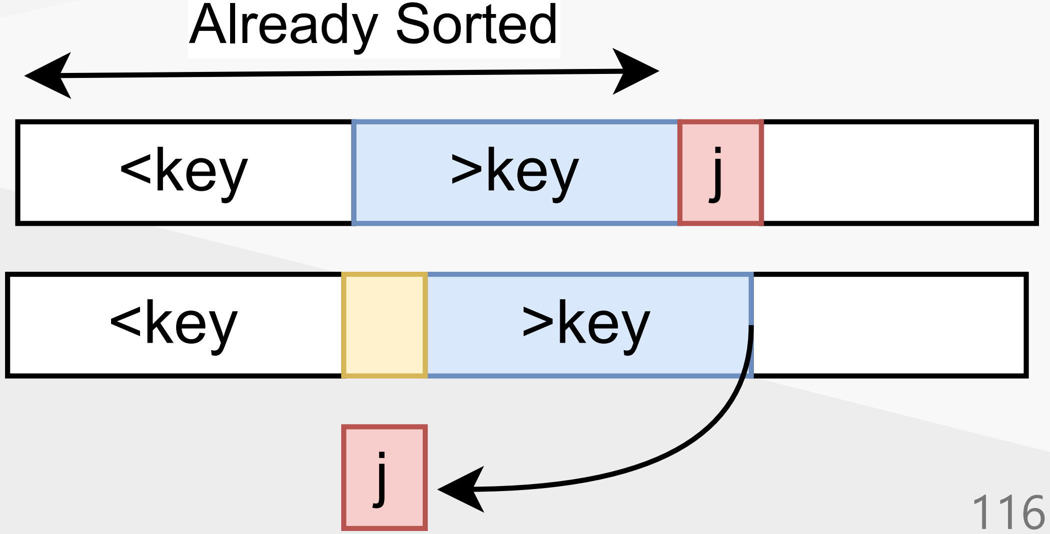
Reminder: Insertion Sort Step-By-Step Description (2)

```
Insertion-Sort(A)
for j = 2 to n do
    key = A[j];
    i = j-1;
    while i>0 and A[i]>key do
        A[i+1]=A[i];
        i = i-1;
    endwhile
    A[i+1]=key;
endfor
```

Shift right the entries in

$A[1..j-1]$

that are bigger than key = j



Already Sorted

<key

>key

j

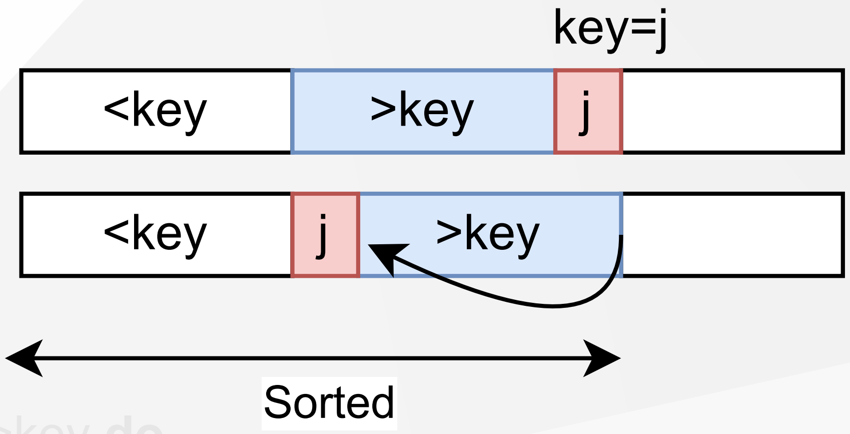
<key

>key

j

Reminder: Insertion Sort Step-By-Step Description (3)

```
Insertion-Sort(A)
for j = 2 to n do
    key = A[j];
    i = j-1;
    while i>0 and A[i]>key do
        A[i+1]=A[i];
        i = i-1;
    endwhile
    A[i+1]=key;
endfor
```

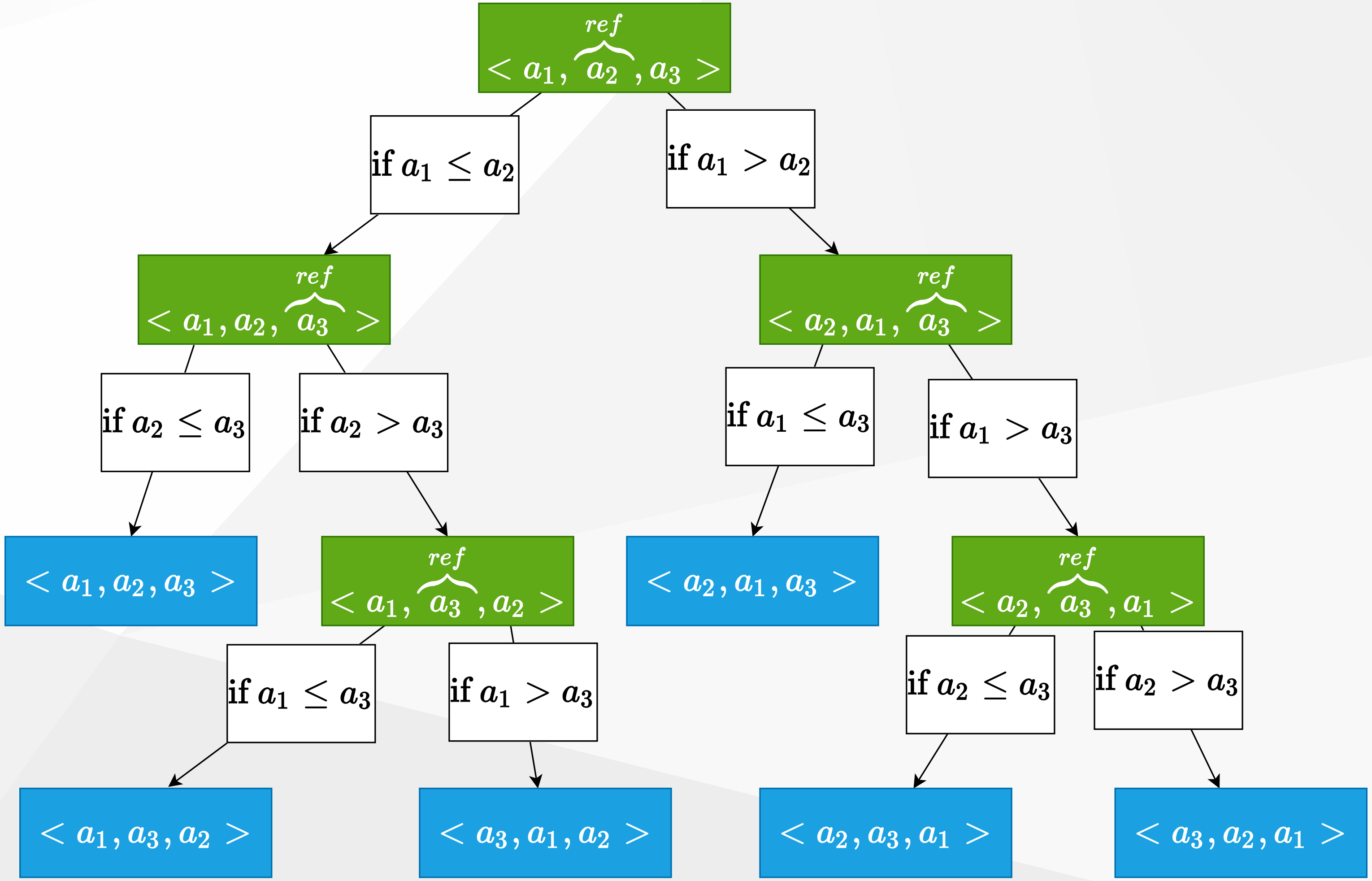


Insert key to the correct location

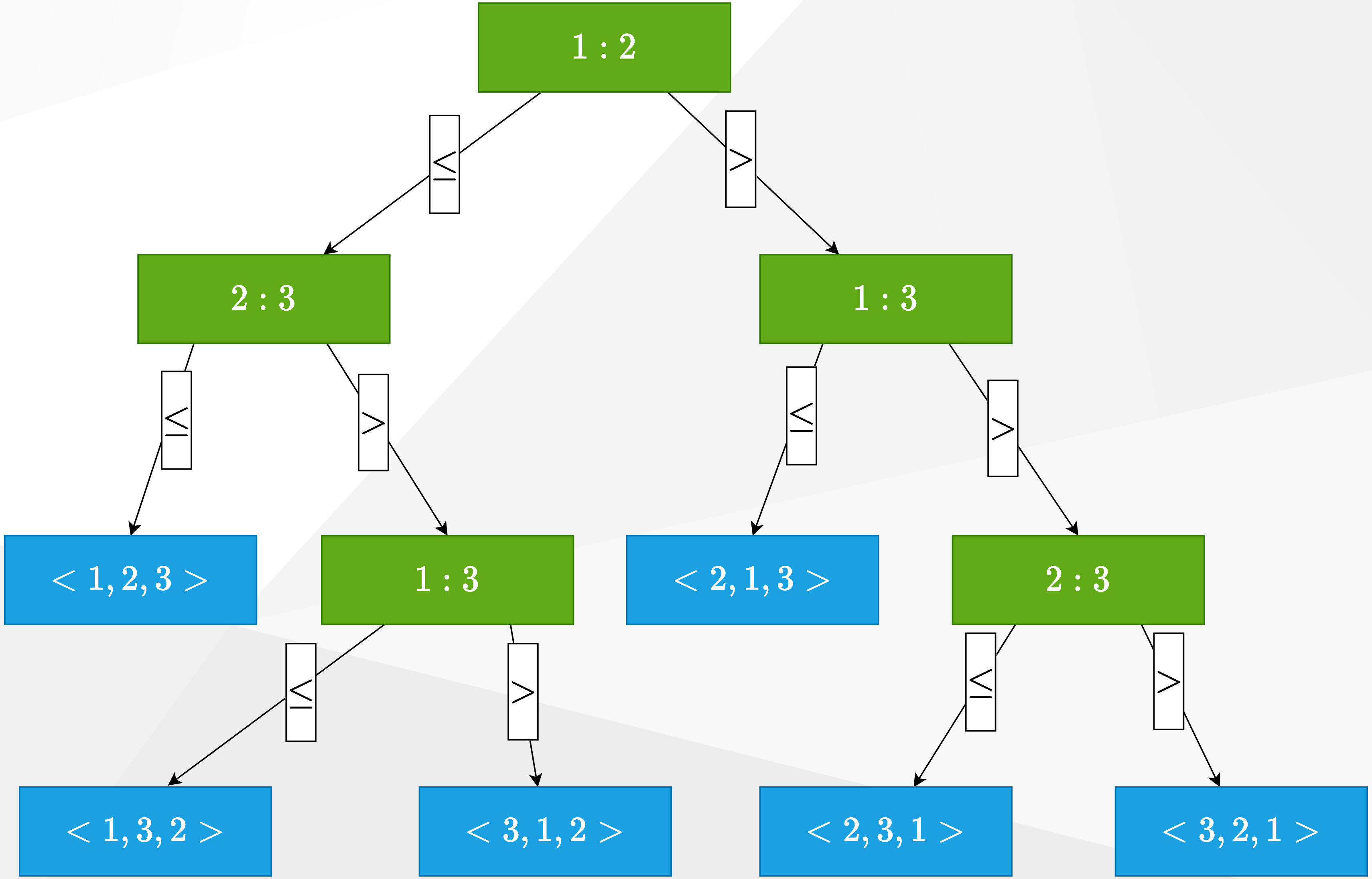
End of iteration $j : A[1..j]$ is sorted

Different Outcomes for Insertion Sort and n=3

- Input : $< a_1, a_2, a_3 >$



Decision Tree for Insertion Sort and n=3

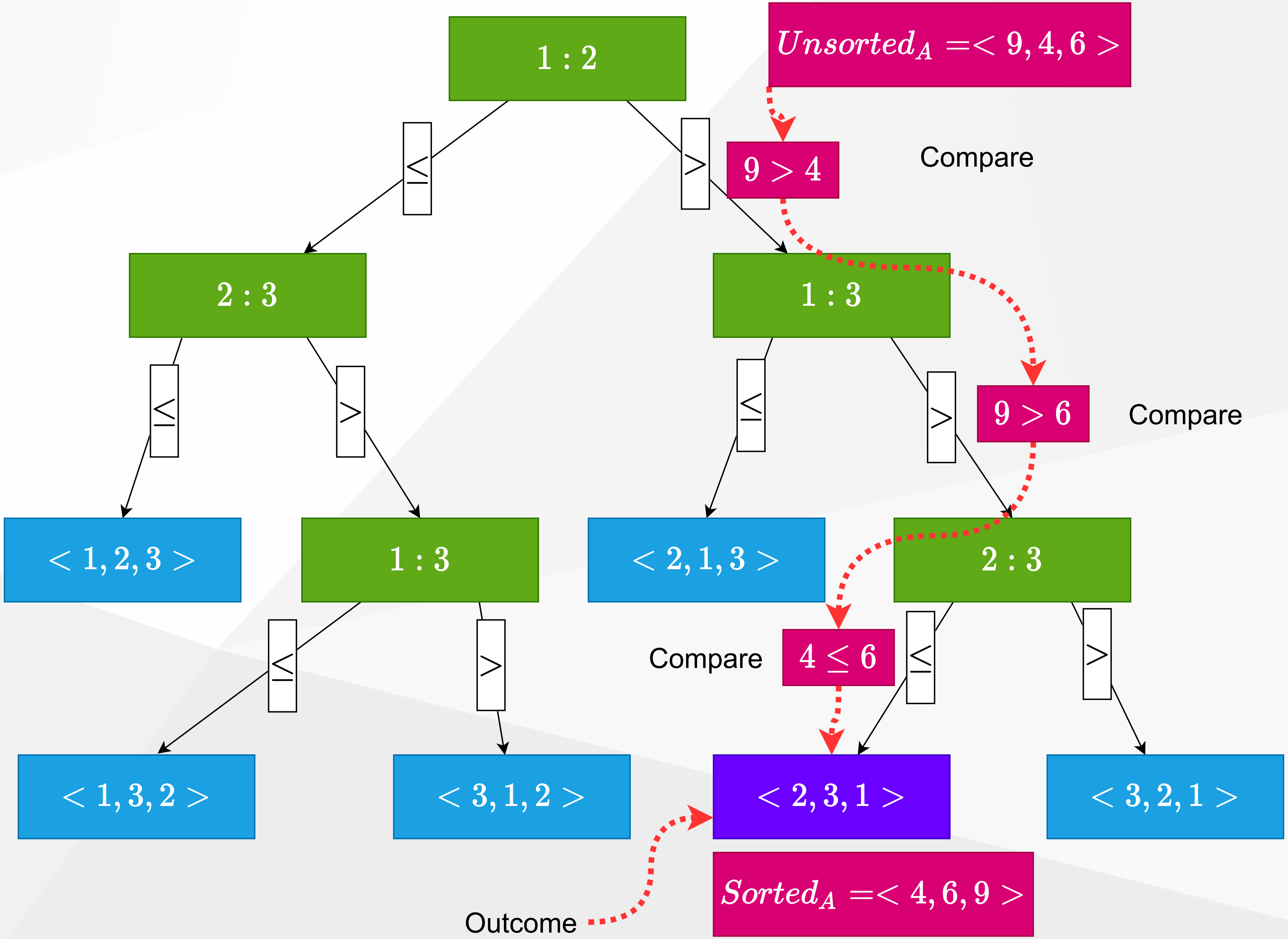


Decision Tree Model for Comparison Sorts

- **Internal node** $(i : j)$: Comparison between elements a_i and a_j
- **Leaf node:** An output of the sorting algorithm
- **Path from root to a leaf:** The execution of the sorting algorithm for a given input
- **All possible executions** are captured by the decision tree
- **All possible outcomes (permutations)** are in the leaf nodes

Decision Tree for Insertion Sort and n=3

- Input:
  $< 9, 4, 6 >$



Decision Tree Model

- A decision tree can model the execution of any comparison sort:
    - One tree for each input size n
    - View the algorithm as **splitting** whenever it compares two elements
    - The tree contains the **comparisons along all possible** instruction traces
- **The running time of the algorithm** $=$ *the length of the path taken*
- **Worst case running time** $=$ *height of the tree*

Counting Sort

Lower Bound for Comparison Sorts

- Let n be the number of elements in the input array.
- What is the min number of leaves in the decision tree?
    - $n!$ **(because there are n! permutations of the input array, and all possible outputs must be captured in the leaves)**
- What is the max number of leaves in a binary tree of height h? $\Longrightarrow 2^h$
- So, we must have:

$$2^h \geq n!$$

Lower Bound for Decision Tree Sorting

- **Theorem:** Any comparison sort algorithm requires $\Omega(nlgn)$ comparisons in the worst case.
- **Proof:** We'll prove that any decision tree corresponding to a comparison sort algorithm must have height $\Omega(nlgn)$

$$
\begin{aligned}
2^h &\geq n! \\
h &\geq lg(n!) \\
&\geq lg((n/e)^n)(Stirling Approximation) \\
&= nlgn - nlge \\
&= \Omega(nlgn)
\end{aligned}
$$

Lower Bound for Decision Tree Sorting

Corollary: Heapsort and merge sort are asymptotically optimal comparison sorts.

Proof: The $O(nlgn)$ upper bounds on the runtimes for heapsort and merge sort match the $\Omega(nlgn)$ **worst-case** lower bound from the previous theorem.

Sorting in Linear Time

- **Counting sort:** No comparisons between elements
  - **Input:** $A[1 \dots n]$, where $A[j] \in \{1, 2, \dots, k\}$
  - **Output:** $B[1 \dots n]$, sorted
  - **Auxiliary storage:** $C[1 \dots k]$

Counting Sort-1

```
for i ← 1 to k do
    C[i] ← 0
for j ← 1 to n do
    C[A[j]] ← C[A[j]] + 1
// C[i] = |{key = i}|
for i ← 2 to k do
    C[i] ← C[i] + C[i-1]
// C[i] = |{key ≤ i}|
for j ← n downto 1 do
    B[C[A[j]]] ← A[j]
    C[A[j]] ← C[A[j]] – 1
```

$A =$

4	1	3	4	3

$B =$

$C =$

1	2	3	4

Counting Sort-2

- **Step 1:** Initialize all counts to 0

```
for i ← 1 to k do
    C[i] ← 0
for j ← 1 to n do
    C[A[j]] ← C[A[j]] + 1
  // C[i] = |{key = i}|
for i ← 2 to k do
    C[i] ← C[i] + C[i-1]
// C[i] = |{key ≤ i}|
for j ← n downto 1 do
    B[C[A[j]]] ← A[j]
    C[A[j]] ← C[A[j]] – 1
```

$A =$

4	1	3	4	3

$B =$

$C =$

1	2	3	4
0	0	0	0

Counting Sort-3

- **Step 2:** Count the number of occurrences of each value in the input array

```
for $i \leftarrow 1$ to $k$ do
    $C[i] \leftarrow 0$
for $j \leftarrow 1$ to $n$ do
    $C[A[j]] \leftarrow C[A[j]] + 1$
// $C[i] = |\{key = i\}|$
for $i \leftarrow 2$ to $k$ do
    $C[i] \leftarrow C[i] + C[i-1]$
// $C[i] = |\{key \leq i\}|$
for $j \leftarrow n$ downto 1 do
    $B[C[A[j]]] \leftarrow A[j]$
    $C[A[j]] \leftarrow C[A[j]] - 1$
```

$A =$

4	1	3	4	3

$B =$

$C =$

1	2	3	4
0	0	0	0

Counting Sort-4

- **Step 3:** Compute the number of elements less than or equal to each value

```
for i ← 1 to k do
    C[i] ← 0
for j ← 1 to n do
    C[A[j]] ← C[A[j]] + 1
// C[i] = |{key = i}|
for i ← 2 to k do
    C[i] ← C[i] + C[i-1]
// C[i] = |{key ≤ i}|
for j ← n downto 1 do
    B[C[A[j]]] ← A[j]
    C[A[j]] ← C[A[j]] – 1
```

$A =$

4	1	3	4	3

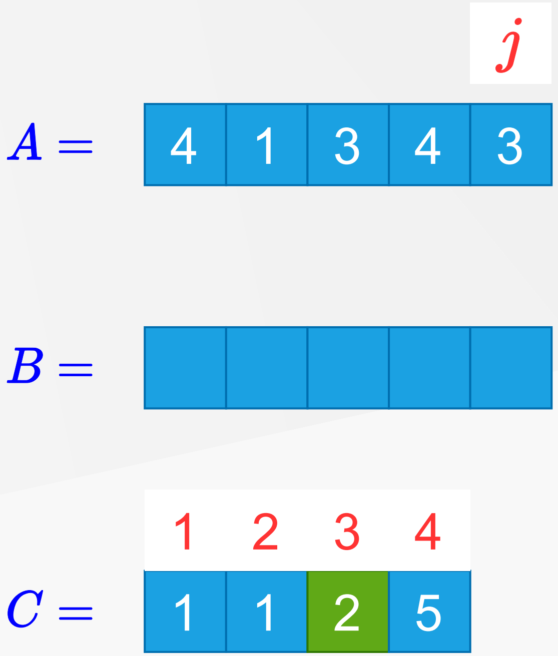
$B =$

i				
	1	2	3	4
$C =$	1	1	3	5

Counting Sort-5

- **Step 4:** Populate the output array
    - There are $C[3] = 3$ elements that are ≤ 3

```
for i ← 1 to k do
    C[i] ← 0
for j ← 1 to n do
    C[A[j]] ← C[A[j]] + 1
// C[i] = |{key = i}|
for i ← 2 to k do
    C[i] ← C[i] + C[i-1]
// C[i] = |{key ≤ i}|
for j ← n downto 1 do
    B[C[A[j]]] ← A[j]
    C[A[j]] ← C[A[j]] – 1
```



$A =$

				j
4	1	3	4	3

$B =$

$C =$

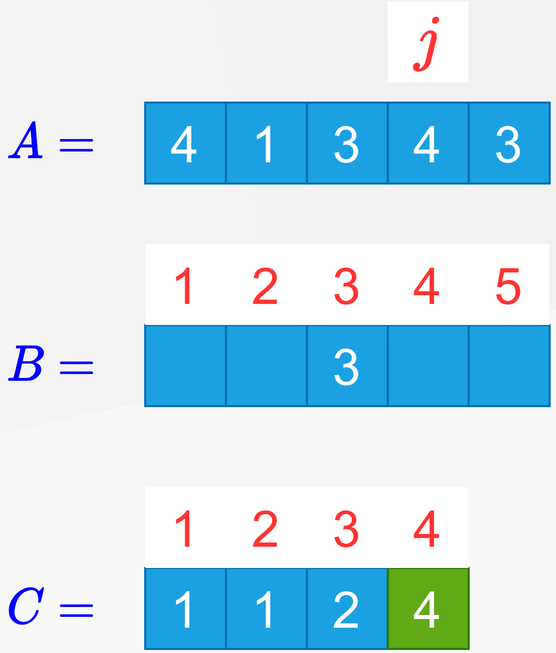
1	2	3	4
1	1	2	5

Counting Sort-6

- **Step 4:** Populate the output array
    - There are $C[4] = 5$ elements that are ≤ 4

Step-5: **Populate the output array**

```
for i ← 1 to k do
    C[i] ← 0
for j ← 1 to n do
    C[A[j]] ← C[A[j]] + 1
// C[i] = |{key = i}|
for i ← 2 to k do
    C[i] ← C[i] + C[i-1]
// C[i] = |{key ≤ i}|
for j ← n downto 1 do
    B[C[A[j]]] ← A[j]
    C[A[j]] ← C[A[j]] – 1
```



				j	
$A =$	4	1	3	4	3

	1	2	3	4	5
$B =$			3		

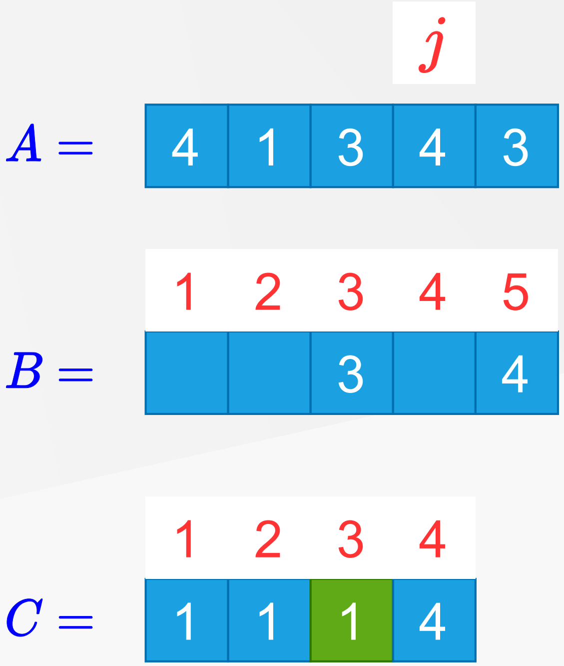
	1	2	3	4
$C =$	1	1	2	4

There are C[4] = 5 elts that are ≤ 4

Counting Sort-7

- **Step 4:** Populate the output array
  - There are $C[3] = 2$ elements that are ≤ 3

```
for i ← 1 to k do
    C[i] ← 0
for j ← 1 to n do
    C[A[j]] ← C[A[j]] + 1
// C[i] = |{key = i}|
for i ← 2 to k do
    C[i] ← C[i] + C[i-1]
// C[i] = |{key ≤ i}|
for j ← n downto 1 do
    B[C[A[j]]] ← A[j]
    C[A[j]] ← C[A[j]] – 1
```



$A =$

			j	
4	1	3	4	3

$B =$

1	2	3	4	5
		3		4

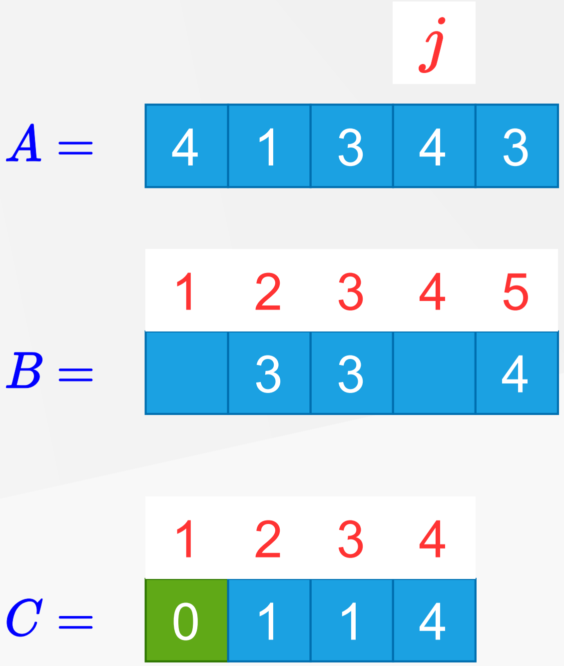
$C =$

1	2	3	4
1	1	1	4

Counting Sort-8

- **Step 4:** Populate the output array
    - There are $C[1] = 1$ elements that are ≤ 1

```
for i ← 1 to k do
    C[i] ← 0
for j ← 1 to n do
    C[A[j]] ← C[A[j]] + 1
// C[i] = |{key = i}|
for i ← 2 to k do
    C[i] ← C[i] + C[i-1]
// C[i] = |{key ≤ i}|
for j ← n downto 1 do
    B[C[A[j]]] ← A[j]
    C[A[j]] ← C[A[j]] – 1
```



				j	
$A =$	4	1	3	4	3

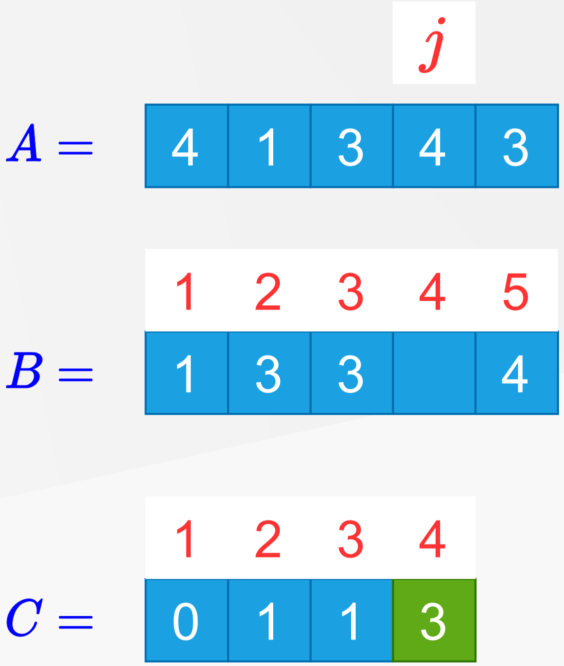
	1	2	3	4	5
$B =$		3	3		4

	1	2	3	4
$C =$	0	1	1	4

Counting Sort-9

- **Step 4:** Populate the output array
    - There are $C[4] =$ 4 elements that are ≤ 4

```
for i ← 1 to k do
    C[i] ← 0
for j ← 1 to n do
    C[A[j]] ← C[A[j]] + 1
// C[i] = |{key = i}|
for i ← 2 to k do
    C[i] ← C[i] + C[i-1]
// C[i] = |{key ≤ i}|
for j ← n downto 1 do
    B[C[A[j]]] ← A[j]
    C[A[j]] ← C[A[j]] – 1
```



				j	
$A =$	4	1	3	4	3

	1	2	3	4	5
$B =$	1	3	3		4

	1	2	3	4
$C =$	0	1	1	3

Counting Sort: Runtime Analysis

- **Total Runtime:** $\Theta(n + k)$
    - n : size of the input array
    - k : the range of input values

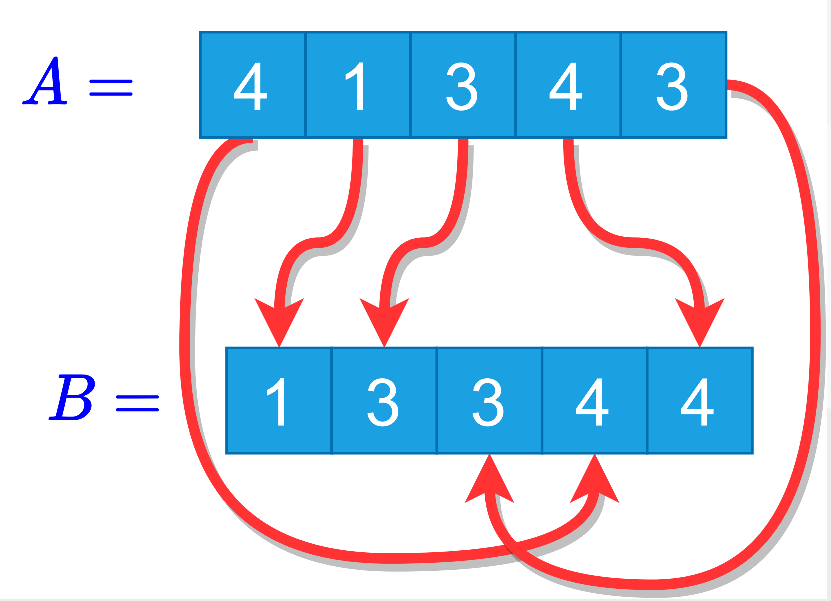
```
for i ← 1 to k do
    C[i] ← 0                            $\Theta(k)$
for j ← 1 to n do
    C[A[j]] ← C[A[j]] + 1               $\Theta(n)$
// C[i] = |{key = i}|
for i ← 2 to k do
    C[i] ← C[i] + C[i-1]                $\Theta(k)$
// C[i] = |{key ≤ i}|
for j ← n downto 1 do
    B[C[A[j]]] ← A[j]                   $\Theta(n)$
    C[A[j]] ← C[A[j]] – 1
```

Counting Sort: Runtime

- Runtime is $\Theta(n+k)$
    - If $k=O(n)$, then counting sort takes $\Theta(n)$
- **Question:** We proved a lower bound of $\Theta(nlgn)$ before! Where is the fallacy?
- **Answer:**
    - $\Theta(nlgn)$ lower bound is for comparison-based sorting
    - Counting sort is not a comparison sort
    - In fact, not a single comparison between elements occurs!

Stable Sorting

- Counting sort is a **stable sort:** It preserves the input order among equal elements.
  - i.e. The numbers with the same value appear in the output array in the same order as they do in the input array.
- **Note**: Which other sorting algorithms have this property?



Radix Sort

- **Origin:** Herman Hollerith's card-sorting machine for the 1890 US Census.
- **Basic idea:** Digit-by-digit sorting
- Two variations:
    - Sort from **MSD** to **LSD** (bad idea)
    - Sort from **LSD** to **MSD** (good idea)

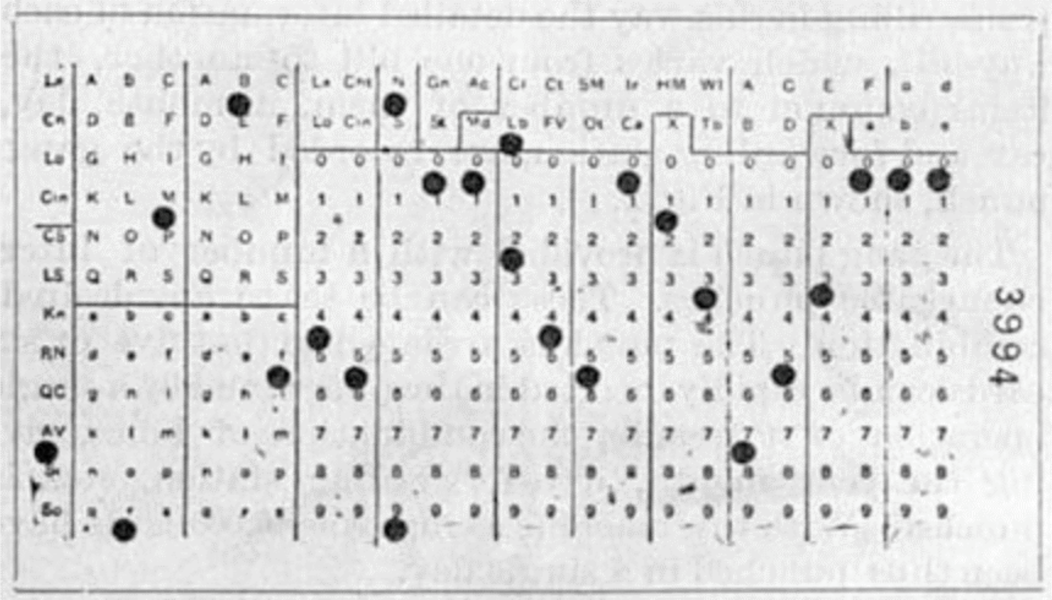
(*LSD/MSD: Least/most significant digit*)

Herman Hollerith (1860-1929)

- The 1880 U.S. Census took **almost 10 years** to process.
- While a lecturer at MIT, Hollerith prototyped **punched-card technology**.
- His machines, including a **card sorter**, allowed the 1890 census total to be reported in **6 weeks**.
- He founded the **Tabulating Machine Company** in 1911, which merged with other companies in 1924 to form **International Business Machines(IBM)**.

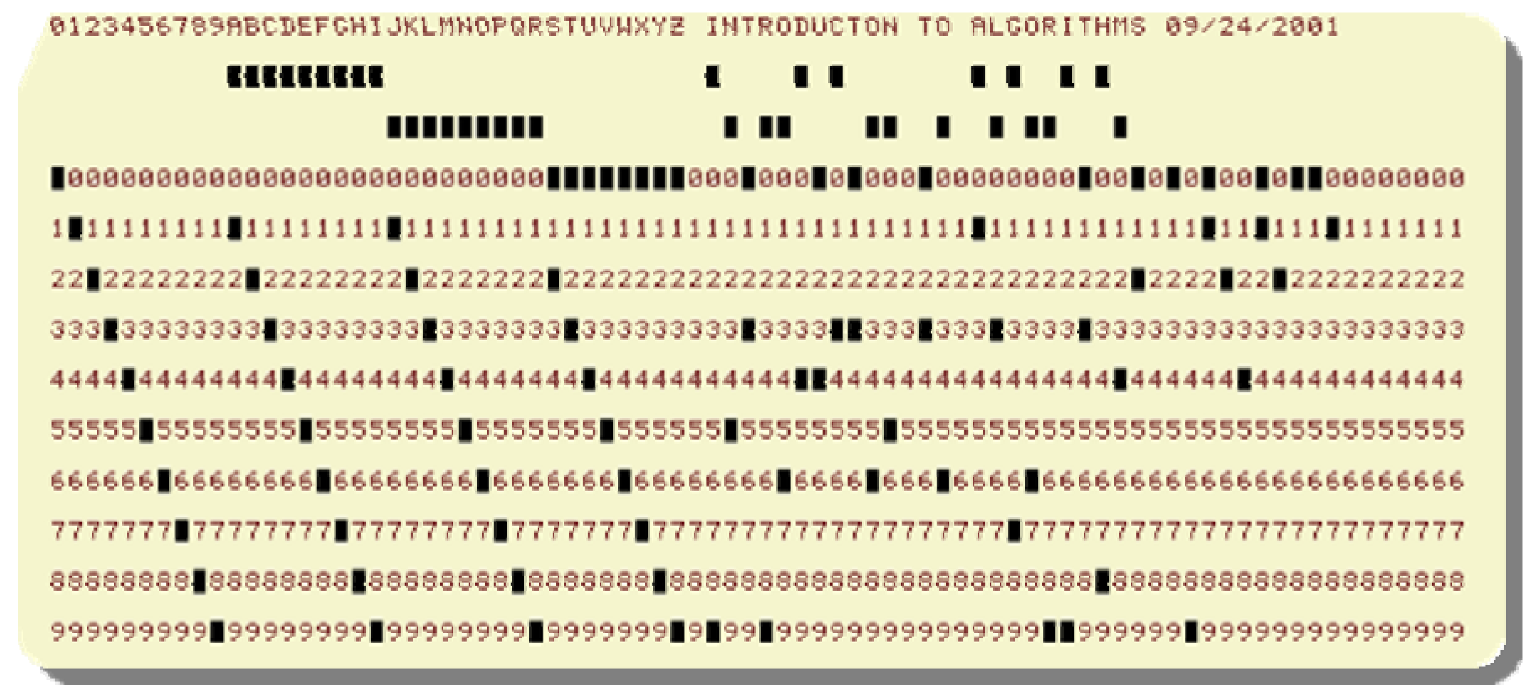
Hollerith Punched Card

- **Punched card:** A piece of stiff paper that contains digital information represented by the presence or absence of holes.
    - 12 rows and 24 columns
    - coded for age, state of residency, gender, etc.



Modern IBM card

- One character per column
    - So, that's why text windows have 80 columns!



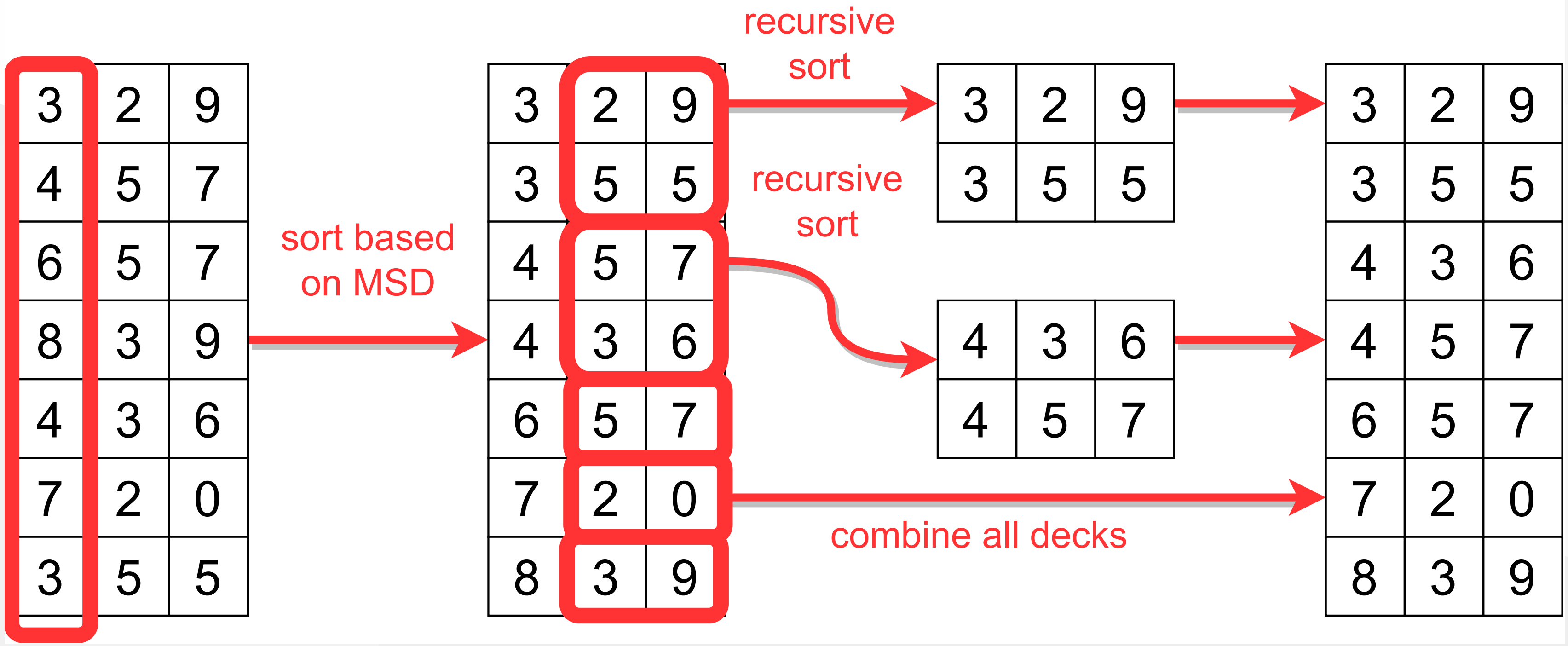
- for more samples visit https://en.wikipedia.org/wiki/Punched_card

Hollerith Tabulating Machine and Sorter

- Mechanically sorts the cards based on the hole locations.
- Sorting performed for one column at a time
- Human operator needed to load/retrieve/move cards at each stage

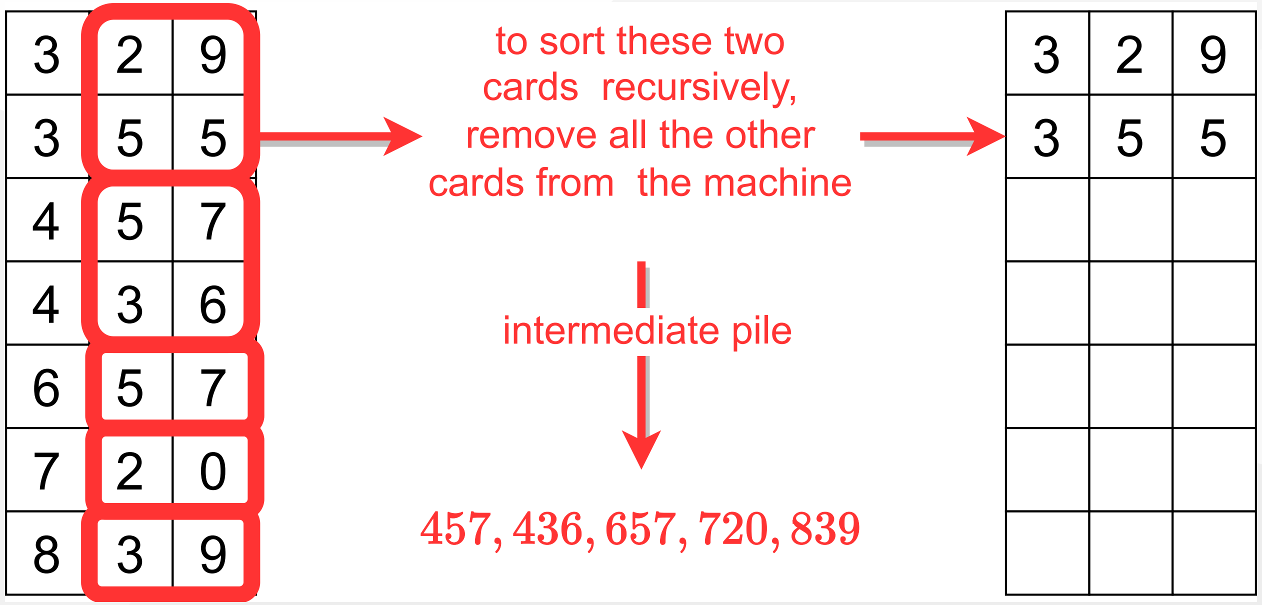
Hollerith's MSD-First Radix Sort

- Sort starting from the most significant digit (MSD)
- Then, sort each of the resulting bins recursively
- At the end, combine the decks in order



Hollerith’s MSD-First Radix Sort

- To sort a subset of cards recursively:
    - All the other cards need to be removed from the machine, because the machine can handle only one sorting problem at a time.
    - The human operator needs to keep track of the intermediate card piles



3	2	9
3	5	5
4	5	7
4	3	6
6	5	7
7	2	0
8	3	9

to sort these two cards recursively, remove all the other cards from the machine

intermediate pile

$457, 436, 657, 720, 839$

3	2	9
3	5	5

Hollerith’s MSD-First Radix Sort

- MSD-first sorting may require:
    - very large number of sorting passes
    - very large number of intermediate card piles to maintain
- **S(d):**
    - $\#$ of passes needed to sort d-digit numbers (worst-case)
- **Recurrence:**
    - $S(d) = 10S(d-1) + 1$ with $S(1) = 1$
        - **Reminder:** Recursive call made to each subset with the same most significant digit(MSD)

Hollerith's MSD-First Radix Sort

- **Recurrence:** $S(d) = 10S(d-1) + 1$

$$
\begin{aligned}
S(d) &= 10S(d-1) + 1 \\
&= 10\Big(10S(d-2) + 1\Big) + 1 \\
&= 10\Big(10\Big(10S(d-3) + 1\Big) + 1\Big) + 1 \\
&= 10iS(d-i) + 10i - 1 + 10i - 2 + \cdots + 101 + 100 \\
&= \sum_{i=0}^{d-1} 10^{i}
\end{aligned}
$$

- Iteration terminates when $i = d - 1$ with $S(d - (d-1)) = S(1) = 1$

Hollerith’s MSD-First Radix Sort

- **Recurrence:** $S(d) = 10S(d-1) + 1$

$$
\begin{aligned}
S(d) &= \sum_{i=0}^{d-1} 10^i \\
&= \frac{10^d - 1}{10 - 1} \\
&= \frac{1}{9}(10^d - 1)
\end{aligned}
$$

$$\Downarrow$$

$$S(d) = \frac{1}{9}(10^d - 1)$$

Hollerith’s MSD-First Radix Sort

- $P(d)$: $\#$ of intermediate card piles maintained (worst-case)
- **Reminder:** Each routing pass generates 9 intermediate piles except the sorting passes on least significant digits (LSDs)
    - There are 10^{d-1} sorting calls to LSDs

$$
\begin{aligned}
P(d) &= 9(S(d) - 10^{d-1}) \\
&= 9\frac{(10^{d-1})}{9-10^{d-1}} \\
&= (10^{d-1} - 9 * 10^{d-1}) \\
&= 10^{d-1} - 1
\end{aligned}
$$

Hollerith’s MSD-First Radix Sort

$$P(d) = 10^{d-1} - 1$$

Alternative solution: Solve the recurrence

$$P(d) = 10P(d-1) + 9$$
$$P(1) = 0$$

Hollerith's MSD-First Radix Sort

- **Example:** To sort 3 digit numbers, in the worst case:
  - $S(d) = (1/9)(103 - 1) = 111$ sorting passes needed
  - $P(d) = 10d - 1 - 1 = 99$ intermediate card piles generated
- MSD-first approach has more recursive calls and intermediate storage requirement
  - Expensive for a **tabulating machine** to sort punched cards
  - Overhead of recursive calls in a modern computer

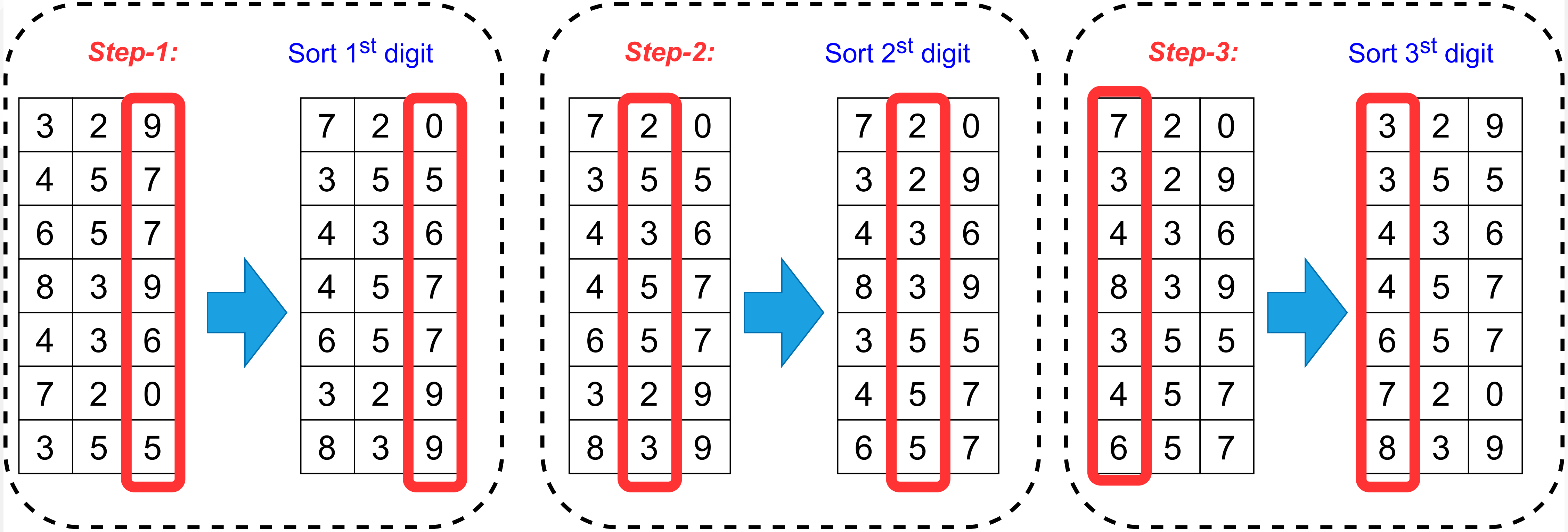
LSD-First Radix Sort

- Least significant digit (**LSD**)-first radix sort seems to be a folk invention originated by machine operators.
- It is the counter-intuitive, but the better algorithm.
- **Basic Algorithm:**

```
Sort numbers on their LSD first   (Stable Sorting Needed)
Combine the cards into a single deck in order
Continue this sorting process for the other digits
  from the LSD to MSD
```

- Requires only d sorting passes
- No intermediate card pile generated

LSD-first Radix Sort Example



Step-1:

3	2	9
4	5	7
6	5	7
8	3	9
4	3	6
7	2	0
3	5	5

Sort 1st digit

7	2	0
3	5	5
4	3	6
4	5	7
6	5	7
3	2	9
8	3	9

Step-2:

7	2	0
3	5	5
4	3	6
4	5	7
6	5	7
3	2	9
8	3	9

Sort 2st digit

7	2	0
3	2	9
4	3	6
8	3	9
3	5	5
4	5	7
6	5	7

Step-3:

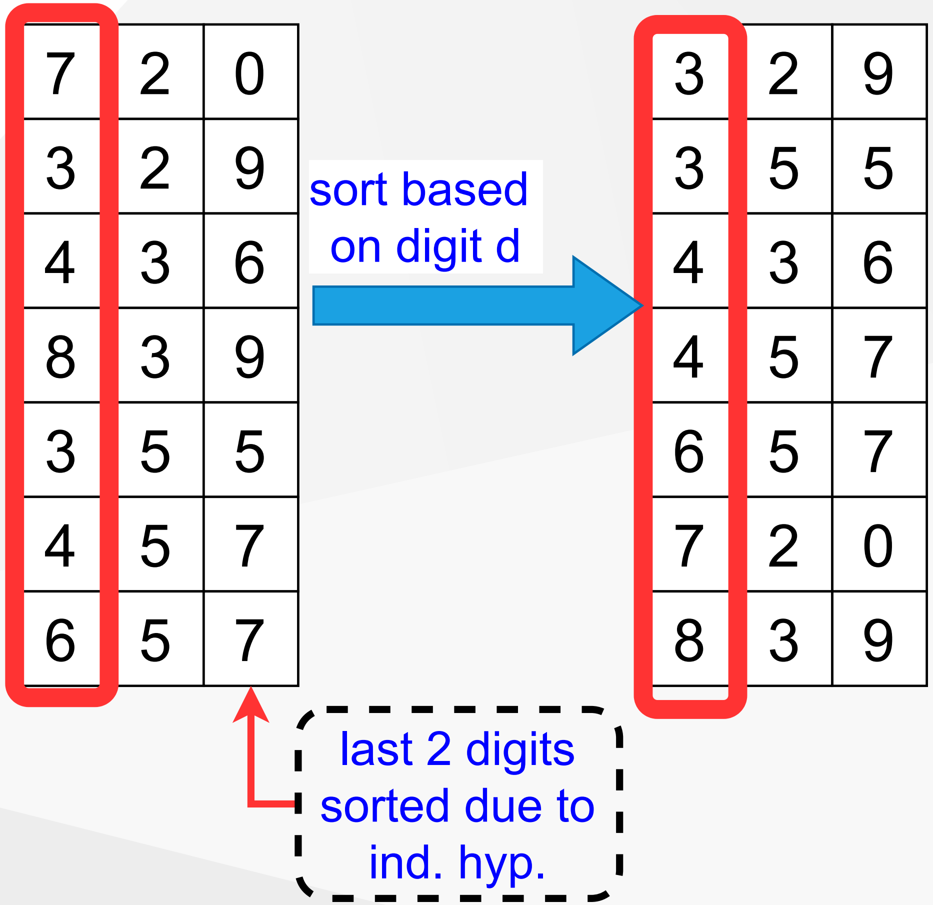
7	2	0
3	2	9
4	3	6
8	3	9
3	5	5
4	5	7
6	5	7

Sort 3st digit

3	2	9
3	5	5
4	3	6
4	5	7
6	5	7
7	2	0
8	3	9

Correctness of Radix Sort (LSD-first)

- **Proof by induction:**
  - **Base case:** $d = 1$ is correct (**trivial**)
  - **Inductive hyp:** Assume the first $d - 1$ digits are sorted correctly
- Prove that all d digits are sorted correctly after sorting digit d
- Two numbers that differ in digit d are correctly sorted (**e.g. 355 and 657**)
- Two numbers equal in digit d are put in the same order as the input
  - (**correct order**)



7	2	0
3	2	9
4	3	6
8	3	9
3	5	5
4	5	7
6	5	7

sort based on digit d

3	2	9
3	5	5
4	3	6
4	5	7
6	5	7
7	2	0
8	3	9

last 2 digits sorted due to ind. hyp.

Radix Sort Runtime

- Use counting-sort to sort each digit
- **Reminder:** Counting sort complexity: $\Theta(n+k)$
  - n: size of input array
  - k: the range of the values
- Radix sort runtime: $\Theta(d(n+k))$
  - d: $\#$ of digits

How to choose the d and k?

Radix Sort: Runtime – Example 1

- We have flexibility in choosing d and k
- Assume we are trying to sort **32-bit words**
  - We can define each digit to be **4 bits**
  - Then, the range for each digit $k = 2^4 = 16$
    - So, counting sort will take $\Theta(n + 16)$
  - The number of digits $d = 32/4 = 8$
  - Radix sort runtime: $\Theta(8(n + 16)) = \Theta(n)$
- $\overbrace{[4bits|4bits|4bits|4bits|4bits|4bits|4bits|4bits]}^{\text{32-bits}}$

Radix Sort: Runtime – Example 2

- We have flexibility in choosing d and k
- Assume we are trying to sort **32-bit words**
    - Or, we can define each digit to be **8 bits**
    - Then, the range for each digit $k = 2^8 = 256$
        - So, counting sort will take $\Theta(n + 256)$
    - The number of digits $d = 32/8 = 4$
    - Radix sort runtime: $\Theta(4(n + 256)) = \Theta(n)$
- $\overbrace{[8bits|8bits|8bits|8bits]}^{\text{32-bits}}$

Radix Sort: Runtime

- Assume we are trying to sort b-**bit** words
    - Define each digit to be r **bits**
    - Then, the range for each digit $k = 2^r$
        - So, **counting sort will take** $\Theta(n + 2^r)$
    - The number of digits $d = b/r$
        - **Radix sort runtime:**

$$T(n,b) = \Theta\left(\frac{b}{r}(n + 2^r)\right)$$

- $\overbrace{[rbits|rbits|rbits|rbits]}^{b/r \text{ bits}}$

Radix Sort: Runtime Analysis

$$T(n,b) = \Theta\left(\frac{b}{r}(n+2^r)\right)$$

- Minimize $T(n,b)$ by differentiating and setting to 0
- Or, intuitively:
  - We want to balance the terms (b/r) and $(n+2^r)$
  - **Choose** $r \approx lgn$
    - If we choose $r << lgn \implies (n+2^r)$ term **doesn't improve**
    - If we choose $r >> lgn \implies (n+2^r)$ increases **exponentially**

Radix Sort: Runtime Analysis

$$T(n, b) = \Theta\left(\frac{b}{r}(n + 2^r)\right)$$

$$\text{Choose } r = lgn \implies T(n, b) = \Theta(bn/lgn)$$

- For numbers in the range from 0 to $n^d – 1$, we have:
    - The number of bits $b = lg(nd) = dlgn$
        - Radix sort runs in $\Theta(dn)$

Radix Sort: Conclusions

$$\text{Choose } r = lgn \implies T(n,b) = \Theta(bn/lgn)$$

- **Example:** Compare radix sort with merge sort/heapsort
  - 1 million (2^{20}), 32-bit numbers $(n = 2^{20}, b = 32)$
    - **Radix sort:** $\lfloor 32/20 \rfloor = 2$ passes
    - **Merge sort/heap sort:** $lgn = 20$ passes
- **Downsides:**
  - Radix sort has **little locality of reference** (more cache misses)
  - The version that uses counting sort is not in-place
- On modern processors, a well-tuned quicksort implementation typically runs faster.

References

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$$-End - Of - Week - 4 - Course - Module-$$