# CE100 Algorithms and Programming II

# Week-6 (Matrix Chain Order / LCS)

## Spring Semester, 2021-2022

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# Matrix Chain Order / Longest Common Subsequence

## Outline

- Elements of Dynamic Programming
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  - Overlapping Subproblems

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- Constructing and LCS
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- Most Common Dynamic Programming Interview Questions

## Elements of Dynamic Programming

- When should we look for a DP solution to an optimization problem?
- Two key ingredients for the problem
  - Optimal substructure
  - Overlapping subproblems

## DP Hallmark #1

- **Optimal Substructure**
  - A problem exhibits optimal substructure
    - if an optimal solution to a problem contains within it optimal solutions to subproblems
  - **Example:** *matrix-chain-multiplication*
    - Optimal parenthesization of $A_1 A_2 \dots A_n$ that splits the product between $A_k$ and $A_{k+1}$, contains within it **optimal soln's** to the problems of parenthesizing $A_1 A_2 \dots A_k$ and $A_{k+1} A_{k+2} \dots A_n$

## Optimal Substructure

- Finding a suitable space of subproblems
    - Iterate on subproblem instances
    - **Example:** *matrix-chain-multiplication*
        - Iterate and look at the structure of optimal soln's to subproblems, sub-subproblems, and so forth
        - Discover that all subproblems consists of subchains of $\langle A_1, A_2, \dots, A_n \rangle$
        - Thus, the set of chains of the form $\langle A_i, A_{i+1}, \dots, A_j \rangle$ for $1 \leq i \leq j \leq n$
        - Makes a natural and reasonable space of subproblems

## DP Hallmark #2

- **Overlapping Subproblems**
  - Total number of distinct subproblems should be **polynomial** in the input size
  - When a **recursive** algorithm revisits the same problem **over and over again**,
    - We say that the optimization problem has **overlapping subproblems**

## Overlapping Subproblems

- **DP** algorithms typically take advantage of overlapping subproblems
  - by solving each problem once
  - then storing the solutions in a table
    - where it can be looked up when needed
  - using constant time per lookup

## Overlapping Subproblems

- Recursive matrix-chain order

$$\text{RMC}(p, i, j)\{$$

$$\quad \text{if } i = j \text{ then}$$

$$\qquad \text{return } 0$$

$$\quad m[i, j] \leftarrow \infty$$

$$\quad \text{for } k \leftarrow i \text{to } j - 1 \text{ do}$$

$$\qquad q \leftarrow \text{RMC}(p, i, k) + \text{RMC}(p, k + 1, j) + p_{i-1}p_kp_j$$

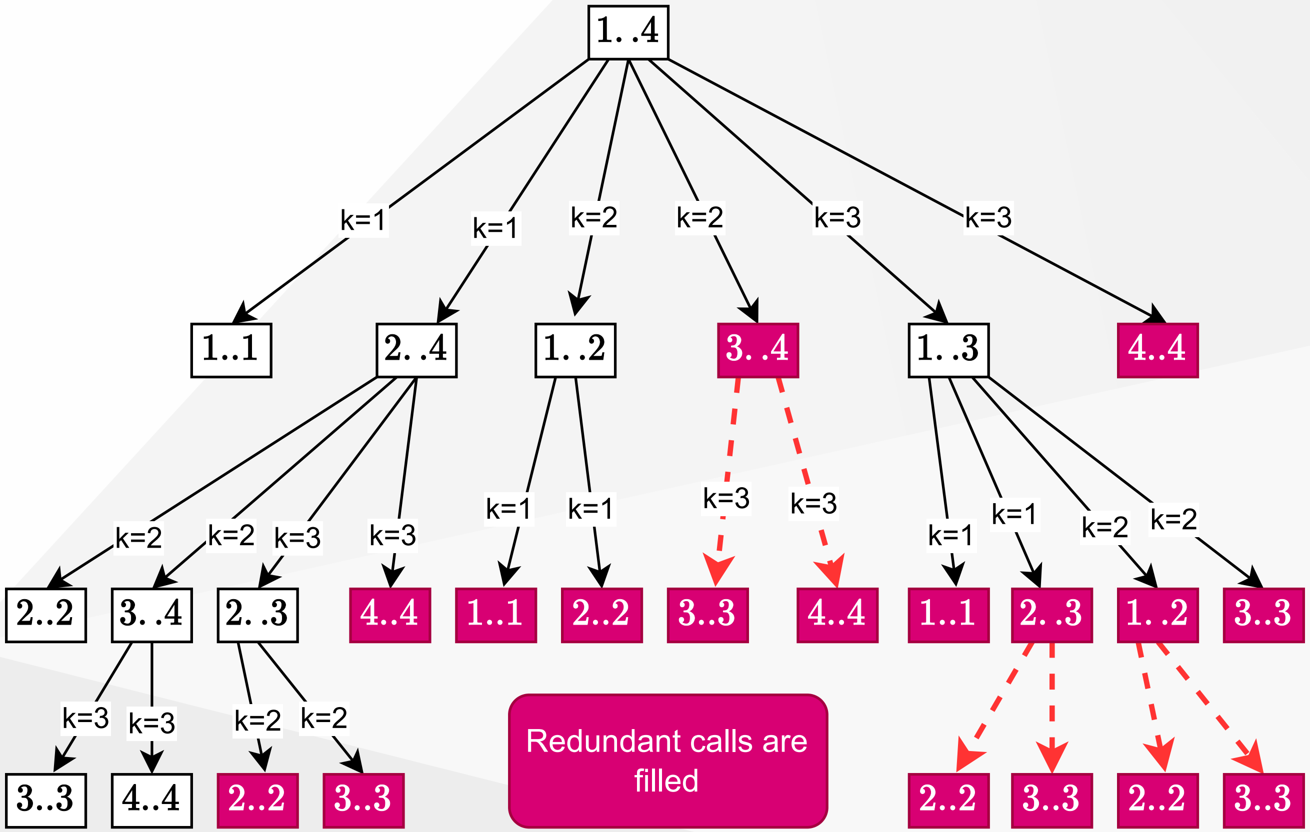
$$\quad if\ q < m[i, j] \text{ then}$$

$$\qquad m[i, j] \leftarrow q$$

$$\quad \text{return } m[i, j]\ \}$$

# Direct Recursion: Inefficient!

- Recursion tree for $RMC(p, 1, 4)$
- Nodes are labeled with $i$ and $j$ values



## Running Time of RMC

$T(1) \geq 1$

$T(n) \geq 1 + \sum_{k=1}^{n-1}(T(k) + T(n-k) + 1)$ for $n > 1$

- For $i = 1, 2, \dots, n$ each term $T(i)$ appears twice
  - Once as $T(k)$, and once as $T(n-k)$
- Collect $n-1$, $1$'s in the summation together with the front $1$

$$T(n) \geq 2\sum_{i=1}^{n-1} T(i) + n$$

- Prove that $T(n) = \Omega(2n)$ using the **substitution method**

# Running Time of RMC: Prove that $T(n) = \Omega(2n)$

- Try to show that $T(n) \geq 2^{n-1}$ (by substitution)
- Base case: $T(1) \geq 1 = 2^0 = 2^{1-1}$ for $n = 1$
- Ind. Hyp.:

$$T(i) \geq 2^{i-1} \text{ for all } i = 1, 2, \ldots, n-1 \text{ and } n \geq 2$$

$$T(n) \geq 2\sum_{i=1}^{n-1} 2^{i-1} + n$$

$$= 2\sum_{i=1}^{n-1} 2^{i-1} + n$$

$$= 2(2^{n-1} - 1) + n$$

$$= 2^{n-1} + (2^{n-1} - 2 + n)$$

$$\Rightarrow T(n) \geq 2^{n-1} \text{ Q.E.D.}$$

## Running Time of RMC: $T(n) \geq 2^{n-1}$

- **Whenever**
  - a recursion tree for the natural recursive solution to a problem contains the same subproblem repeatedly
  - the total number of different subproblems is small
    - it is a good idea to see if $DP(Dynamic\ Programming)$ can be applied

## Memoization

- Offers the efficiency of the usual $DP$ approach while maintaining **top-down** strategy
- Idea is to **memoize** the natural, but inefficient, **recursive algorithm**

## Memoized Recursive Algorithm

- Maintains an **entry** in a **table** for the soln to each subproblem
- Each table entry contains **a special value** to indicate that the entry has yet to be filled in
- When the subproblem is **first encountered** its solution is **computed** and then **stored** in the table
- Each **subsequent** time that the subproblem encountered the value stored in the table is simply **looked up** and **returned**

# Memoized Recursive Matrix-chain Order

- Shaded subtrees are looked-up rather than recomputing

$$
\begin{aligned}
&\text{MemoizedMatrixChain(p)} \\
&\quad n \leftarrow length[p] - 1 \\
&\quad \text{for } i \leftarrow 1 \text{ to } n \text{ do} \\
&\quad\quad \text{for } j \leftarrow 1 \text{ to } n \text{ do} \\
&\quad\quad\quad m[i,j] \leftarrow \infty \\
&\quad \text{return LookupC}(p, 1, n) \Longrightarrow
\end{aligned}
$$

$$
\begin{aligned}
&\Longrightarrow \text{LookupC}(p, i, j) \\
&\text{if } m[i,j] = \infty \text{ then} \\
&\quad \text{if } i = j \text{ then} \\
&\quad\quad m[i,j] \leftarrow 0 \\
&\quad \text{else} \\
&\quad\quad \text{for } k \leftarrow i \text{ to } j-1 \text{ do} \\
&\quad\quad\quad q \leftarrow \text{LookupC}(p, i, k) + \text{LookupC}(p, k+1, j) + p_{i-1}p_kp_j \\
&\quad\quad\quad \text{if } q < m[i,j] \text{ then} \\
&\quad\quad\quad\quad m[i,j] \leftarrow q \\
&\text{return } m[i,j]
\end{aligned}
$$

## Memoized Recursive Algorithm

- The approach assumes that
  - The set of **all possible subproblem parameters** are known
  - The relation between the **table positions** and **subproblems** is established
- Another approach is to memoize
  - by using **hashing** with subproblem parameters as key

## Dynamic Programming vs Memoization Summary (1)

- Matrix-chain multiplication can be solved in $O(n^3)$ time
  - by either a top-down memoized recursive algorithm
  - or a bottom-up dynamic programming algorithm
- Both methods exploit the **overlapping subproblems** property
  - There are only $\Theta(n^2)$ different subproblems in total
  - Both methods **compute** the soln to **each problem once**
- **Without memoization** the natural **recursive** algorithm runs in **exponential time** since subproblems are solved repeatedly

# Dynamic Programming vs Memoization Summary (2)

- **In general practice**
  - If all subproblems must be solved at once
    - a bottom-up **DP algorithm always outperforms** a top-down memoized algorithm by a constant factor
  - because, bottom-up **DP** algorithm
    - Has no overhead for recursion
    - Less overhead for maintaining the table
  - **DP: Regular** pattern of **table accesses** can be exploited to reduce the time and/or space requirements even further
  - **Memoized:** If some problems need not be solved at all, it has the advantage of avoiding solutions to those subproblems

# Problem 3: Longest Common Subsequence

## Definitions

- A **subsequence** of a given sequence is just the **given sequence** with **some elements** (possibly none) **left out**
- **Example:**
  - $X = \langle A, B, C, B, D, A, B \rangle$
  - $Z = \langle B, C, D, B \rangle$
    - $Z$ is a subsequence of $X$

## Problem 3: Longest Common Subsequence

### Definitions

- **Formal definition:** Given a sequence $X = \langle x_1, x_2, \ldots, x_m \rangle$, sequence $Z = \langle z_1, z_2, \ldots, z_k \rangle$ is a subsequence of $X$
  - if $\exists$ a **strictly increasing sequence** $\langle i_1, i_2, \ldots, i_k \rangle$ of indices of $X$ such that $x_{i_j} = z_j$ for all $j = 1, 2, \ldots, k$, where $1 \leq k \leq m$
- **Example:** $Z = \langle B, C, D, B \rangle$ is a subsequence of $X = \langle A, B, C, B, D, A, B \rangle$ with the **index sequence** $\langle i_1, i_2, i_3, i_4 \rangle = \langle 2, 3, 5, 7 \rangle$

## Problem 3: Longest Common Subsequence

### Definitions

- If $Z$ is a subsequence of both $X$ and $Y$, we denote $Z$ as a **common subsequence** of $X$ and $Y$.
- **Example:**

$$X = \langle A, B^*, C^*, B, D, A^*, B \rangle$$
$$Y = \langle B^*, D, C^*, A^*, B, A \rangle$$

- $Z_1 = \langle B^*, C^*, A^* \rangle$ is a common subsequence (**of length 3**) of $X$ and $Y$.
- **Two longest common subsequence (LCSs)** of $X$ and $Y$?
  - $Z2 = \langle B, C, B, A \rangle$ of length $4$
  - $Z3 = \langle B, D, A, B \rangle$ of length $4$
    - *The optimal solution value = 4*

## Longest Common Subsequence (LCS) Problem

- **LCS problem:** Given two sequences
  - $X = \langle x_1, x_2, \dots, x_m \rangle$ and
  - $Y = \langle y_1, y_2, \dots, y_n \rangle$, find the **LCS** of $X \& Y$
- **Brute force approach:**
  - Enumerate all subsequences of $X$
  - Check if each subsequence is also a subsequence of $Y$
  - Keep track of the **LCS**
  - What is the complexity?
  - There are $2^m$ subsequences of $X$
    - **Exponential runtime**

## Notation

- **Notation:** Let $X_i$ denote the $i^{th}$ prefix of $X$
  - i.e. $X_i = \langle x_1, x_2, \ldots, x_i \rangle$
- **Example:**

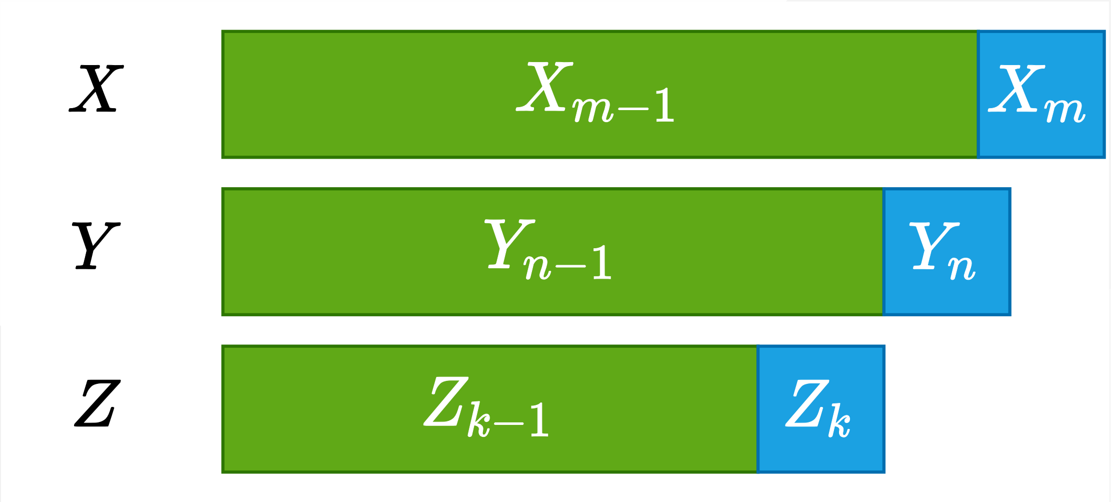
$$X = \langle A, B, C, B, D, A, B \rangle$$

$$X_4 = \langle A, B, C, B \rangle$$

$$X_0 = \langle \rangle$$

# Optimal Substructure of an LCS

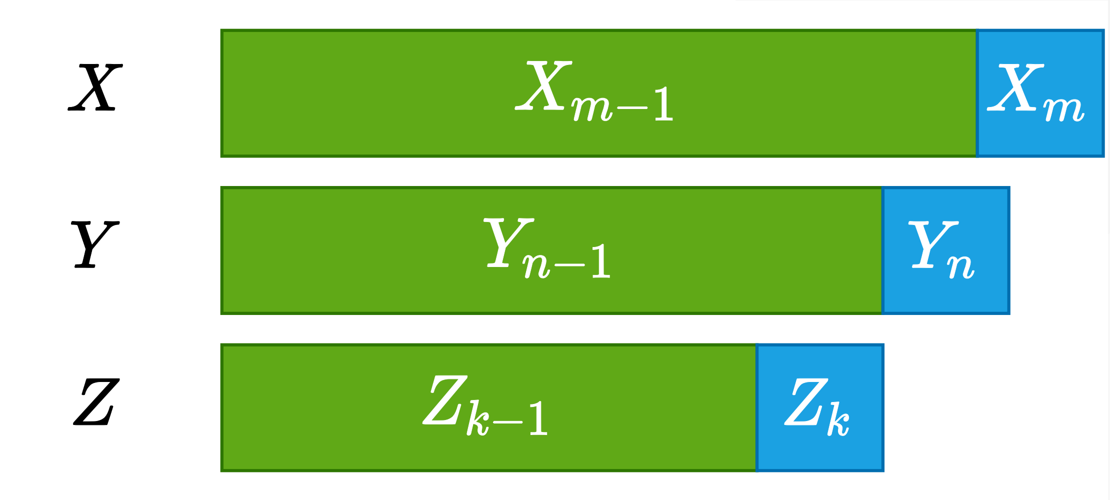
- Let $X =< x1, x2, \ldots, xm >$ and $Y = \langle y_1, y_2, \ldots, y_n \rangle$ are given
- Let $Z = \langle z_1, z_2, \ldots, z_k \rangle$ be an **LCS** of $X$ and $Y$




- **Question 1:** If $x_m = y_n$, how to define the optimal substructure?
  - We must have $z_k = x_m = y_n$ and
  - $Z_{k-1} = \text{LCS}(X_{m-1}, Y_{n-1})$

# Optimal Substructure of an LCS

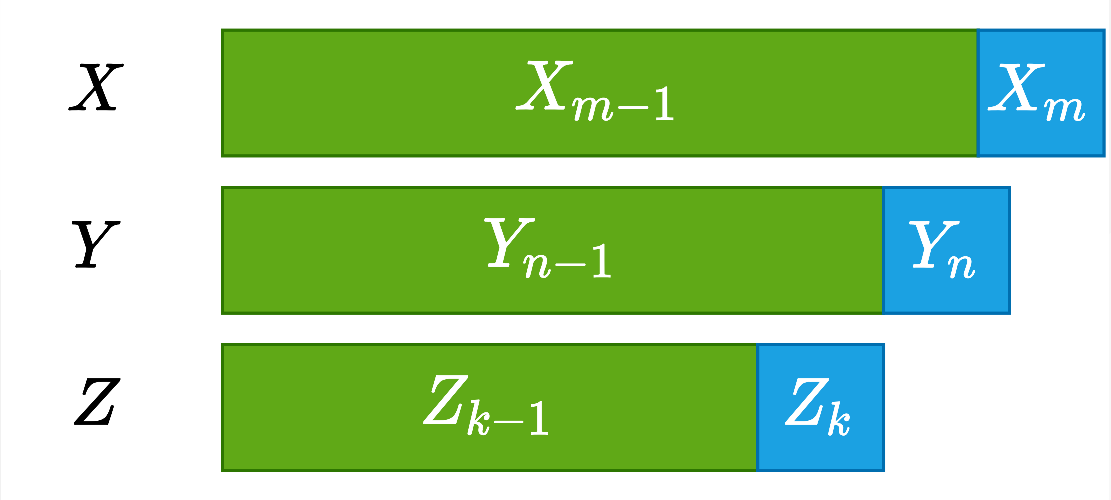
- Let $X =< x1, x2, \ldots, xm >$ and $Y = \langle y_1, y_2, \ldots, y_n \rangle$ are given
- Let $Z = \langle z_1, z_2, \ldots, z_k \rangle$ be an **LCS** of $X$ and $Y$




- **Question 2:** If $x_m \neq y_n$ and $z_k \neq x_m$, how to define the optimal substructure?
  - We must have $Z = \text{LCS}(X_{m-1}, Y)$

# Optimal Substructure of an LCS

- Let $X =< x1, x2, \ldots, xm >$ and $Y = \langle y_1, y_2, \ldots, y_n \rangle$ are given
- Let $Z = \langle z_1, z_2, \ldots, z_k \rangle$ be an **LCS** of $X$ and $Y$



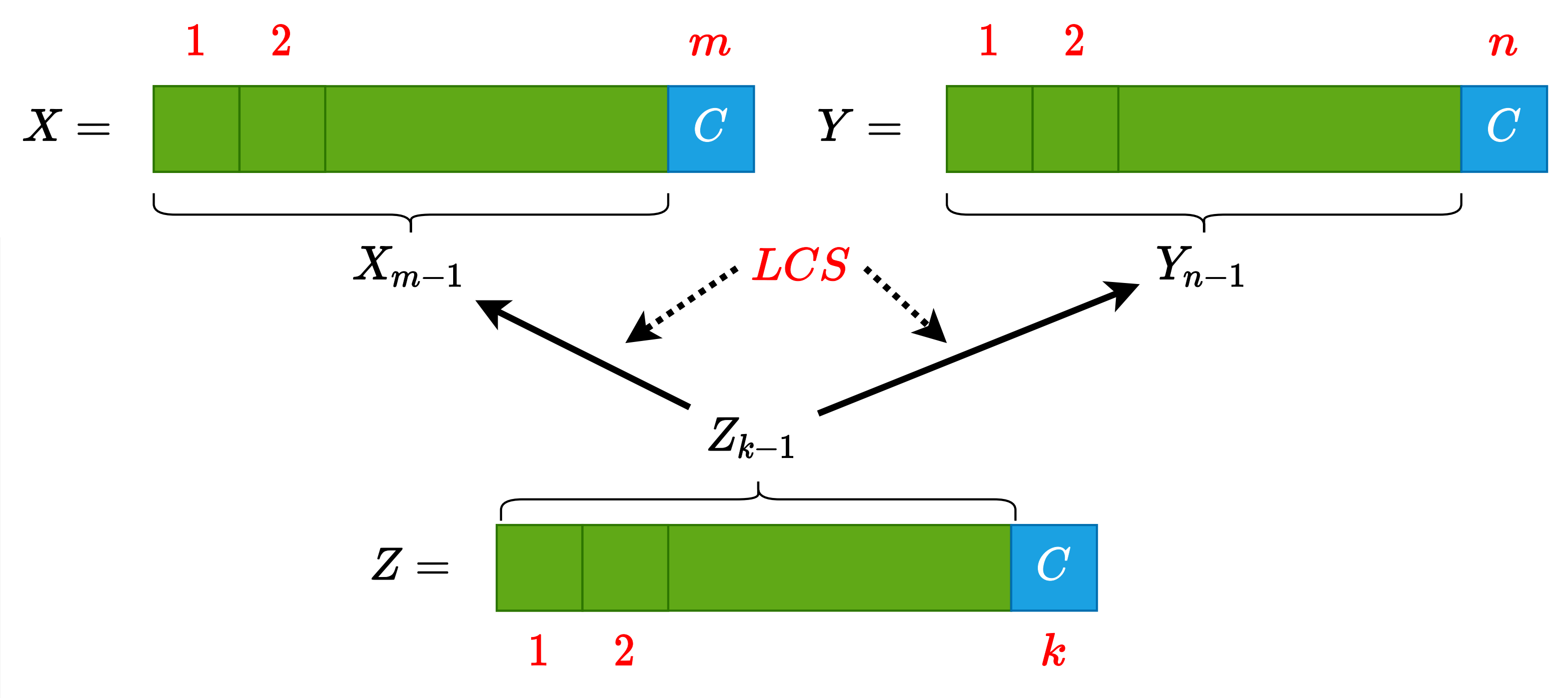

- **Question 3:** If $x_m \neq y_n$ and $z_k \neq y_n$, how to define the optimal substructure?
  - We must have $Z = \text{LCS}(X, Y_{n-1})$

# Theorem: Optimal Substructure of an LCS

- Let $X = \langle x_1, x_2, \ldots, x_m \rangle$ and Y = <y1, y2, ..., yn> are given
- Let $Z = \langle z_1, z_2, \ldots, z_k \rangle$ be an **LCS** of $X$ and $Y$
- **Theorem:** Optimal substructure of an LCS:
  - If $x_m = y_n$
    - then $z_k = x_m = y_n$ and $Z_{k-1}$ is an **LCS** of $X_{m-1}$ and $Y_{n-1}$
  - If $x_m \neq y_n$ and $z_k \neq x_m$
    - then $Z$ is an **LCS** of $X_{m-1}$ and $Y$
  - If $x_m \neq y_n$ and $z_k \neq y_n$
    - then $Z$ is an **LCS** of $X$ and $Y_{n-1}$

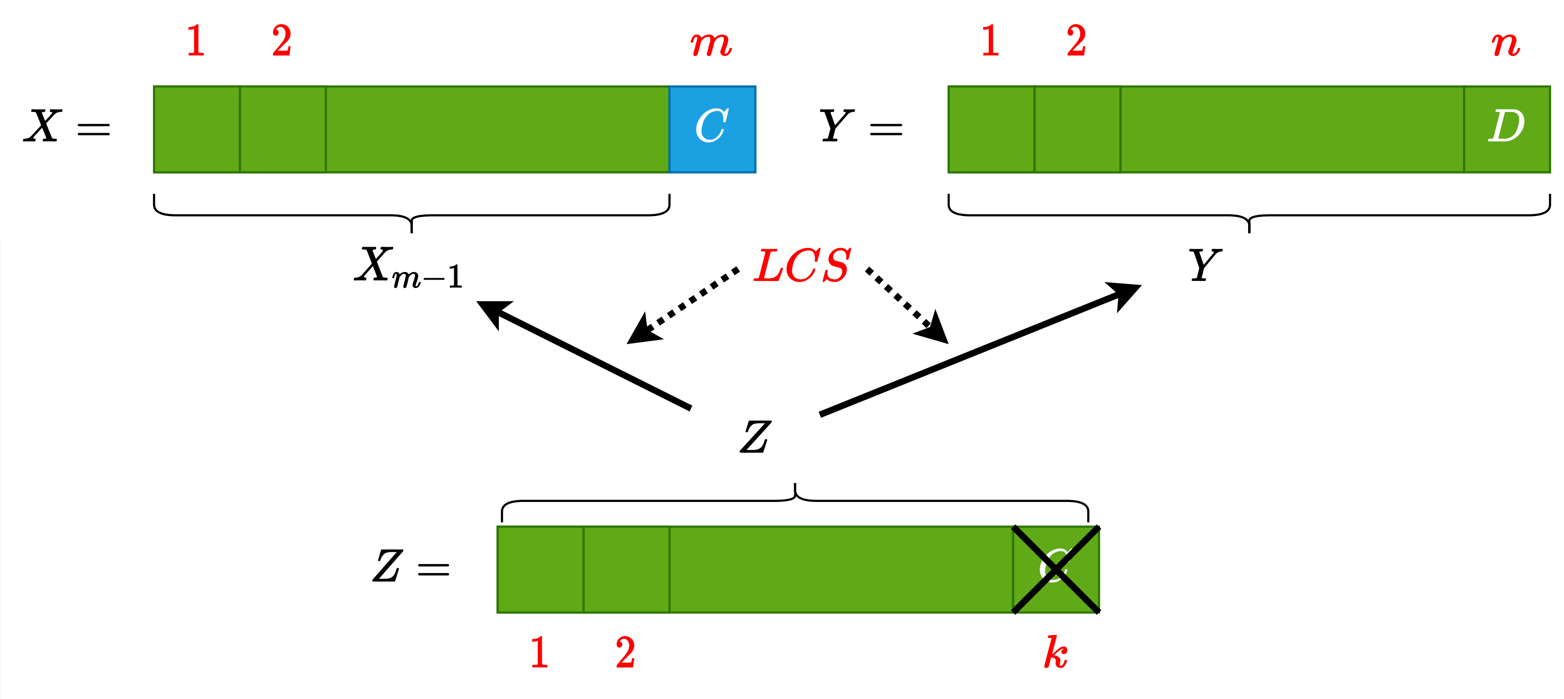
# Optimal Substructure Theorem (case 1)

- If $x_m = y_n$ then $z_k = x_m = y_n$ and $Z_{k-1}$ is an **LCS** of $X_{m-1}$ and $Y_{n-1}$



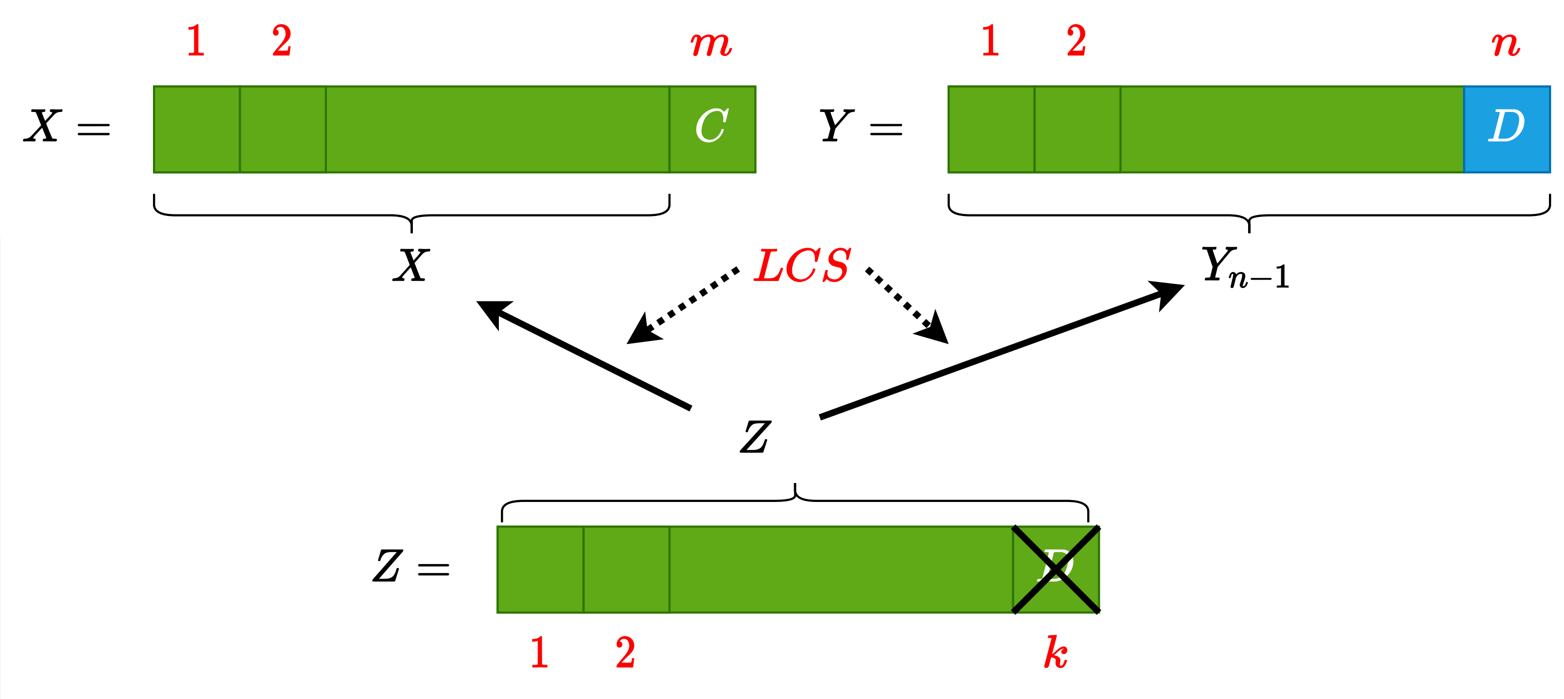
# Optimal Substructure Theorem (case 2)

- If $x_m \neq y_n$ and $z_k \neq x_m$ then $Z$ is an **LCS** of $X_{m-1}$ and $Y$



# Optimal Substructure Theorem (case 3)

- If $x_m \neq y_n$ and $z_k \neq y_n$ then $Z$ is an **LCS** of $X$ and $Y_{n-1}$



## Proof of Optimal Substructure Theorem (case 1)

- If $x_m = y_n$ then $z_k = x_m = y_n$ and $Z_{k-1}$ is an **LCS** of $X_{m-1}$ and $Y_{n-1}$
- **Proof:** If $z_k \neq x_m = y_n$ then
  - we can append $x_m = y_n$ to $Z$ to obtain a common subsequence of length $k+1 \implies$ **contradiction**
  - Thus, we must have $z_k = x_m = y_n$
  - Hence, the prefix $Z_{k-1}$ is a **length-($k-1$) CS** of $X_{m-1}$ and $Y_{n-1}$
- **We have to show that $Z_{k-1}$** is in fact an **LCS** of $X_{m-1}$ and $Y_{n-1}$
- **Proof by contradiction:**
  - **Assume that** $\exists$ a CS $W$ of $X_{m-1}$ and $Y_{n-1}$ with $|W| = k$
  - Then appending $x_m = y_n$ to $W$ produces a **CS** of length $k+1$

# Proof of Optimal Substructure Theorem (case 2)

- If $x_m \neq y_n$ and $z_k \neq x_m$ then $Z$ is an **LCS** of $X_{m-1}$ and $Y$
- **Proof :** If $z_k \neq x_m$ then $Z$ is a CS of $X_{m-1}$ and $Y_n$
  - **We have to show that** $Z$ is in fact an **LCS** of $X_{m-1}$ and $Y_n$
- **(Proof by contradiction)**
  - Assume that $\exists$ a CS $W$ of $X_{m-1}$ and $Y_n$ with $|W| > k$
  - Then $W$ would also be a CS of $X$ and $Y$
  - Contradiction to the assumption that
    - $Z$ is an LCS of $X$ and $Y$ with $|Z| = k$
- **Case 3:** Dual of the proof for (case 2)

# A Recursive Solution to Subproblems

- Theorem implies that there are one or two subproblems to examine
- **if** $x_m = y_n$ **then**
  - we must solve the subproblem of finding an **LCS** of $X_{m-1} \& Y_{n-1}$
  - appending $x_m = y_n$ to this **LCS** yields an **LCS** of $X \& Y$
- **else**
  - we must solve **two subproblems**
    - finding an **LCS** of $X_{m-1} \& Y$
    - finding an **LCS** of $X \& Y_{n-1}$
  - longer of these two **LCS** s is an **LCS** of $X \& Y$
- **endif**

## Recursive Algorithm (Inefficient)

$$
\begin{aligned}
&\text{LCS}(X, Y)\ \{\\
&\quad m \leftarrow length[X]\\
&\quad n \leftarrow length[Y]\\
&\quad \text{if } x_m = y_n \text{ then}\\
&\qquad Z \leftarrow \text{LCS}(X_{m-1}, Y_{n-1}) \triangleright \text{solve one subproblem}\\
&\qquad \text{return } \langle Z, x_m = y_n \rangle \triangleright \text{append } x_m = y_n \text{ to } Z\\
&\quad else\\
&\qquad Z^{'} \leftarrow \text{LCS}(X_{m-1}, Y) \triangleright \text{solve two subproblems}\\
&\qquad Z^{''} \leftarrow \text{LCS}(X, Y_{n-1})\\
&\qquad \text{return longer of } Z^{'} \text{ and } Z^{''}\\
&\}
\end{aligned}
$$

## A Recursive Solution

- $c[i, j]$ : length of an **LCS** of $X_i$ and $Y_j$

$$c[i, j] = \begin{cases} 0 & \text{if} \quad i = 0 \text{ or } j = 0 \\ c[i-1, j-1] + 1 & \text{if} \quad i, j > 0 \text{ and } x_i = y_j \\ \max\{c[i, j-1], c[i-1, j]\} & \text{if} \quad i, j > 0 \text{ and } x_i \neq y_j \end{cases}$$

## Computing the Length of an LCS

- We can easily write an **exponential-time recursive algorithm** based on the given recurrence. $\Longrightarrow$ **Inefficient!**
- How many distinct subproblems to solve?
  - $\Theta(mn)$
- **Overlapping subproblems property:** Many subproblems share the same sub-subproblems.
  - **e.g.** Finding an **LCS** to $X_{m-1}\&Y$ and an **LCS** to $X\&Y_{n-1}$
  - has the sub-subproblem of finding an **LCS** to $X_{m-1}\&Y_{n-1}$
- Therefore, we can use **dynamic programming**.

## Data Structures

- Let:
  - $c[i, j]$ : length of an **LCS** of $X_i$ and $Y_j$
  - $b[i, j]$ : direction towards the table entry corresponding to the optimal subproblem solution chosen when computing $c[i, j]$.
  - Used to simplify the construction of an optimal solution at the end.
- Maintain the following tables:
  - $c[0 \dots m, 0 \dots n]$
  - $b[1 \dots m, 1 \dots n]$

## Bottom-up Computation

- **Reminder:**

$$c[i,j] = \begin{cases} 0 & \text{if} \quad i = 0 \text{ or } j = 0 \\ c[i-1,j-1]+1 & \text{if} \quad i,j > 0 \text{ and } x_i = y_j \\ \max\{c[i,j-1], c[i-1,j]\} & \text{if} \quad i,j > 0 \text{ and } x_i \neq y_j \end{cases}$$

- How to choose the order in which we process $c[i,j]$ values?
- The values for $c[i-1,j-1]$, $c[i,j-1]$, and $c[i-1,j]$ must be computed before computing $c[i,j]$.

## Bottom-up Computation

$$c[i,j] = \begin{cases} 0 & \text{if} \quad i = 0 \text{ or } j = 0 \\ c[i-1,j-1]+1 & \text{if} \quad i,j > 0 \text{ and } x_i = y_j \\ \max\{c[i,j-1], c[i-1,j]\} & \text{if} \quad i,j > 0 \text{ and } x_i \neq y_j \end{cases}$$

**Need to process:**

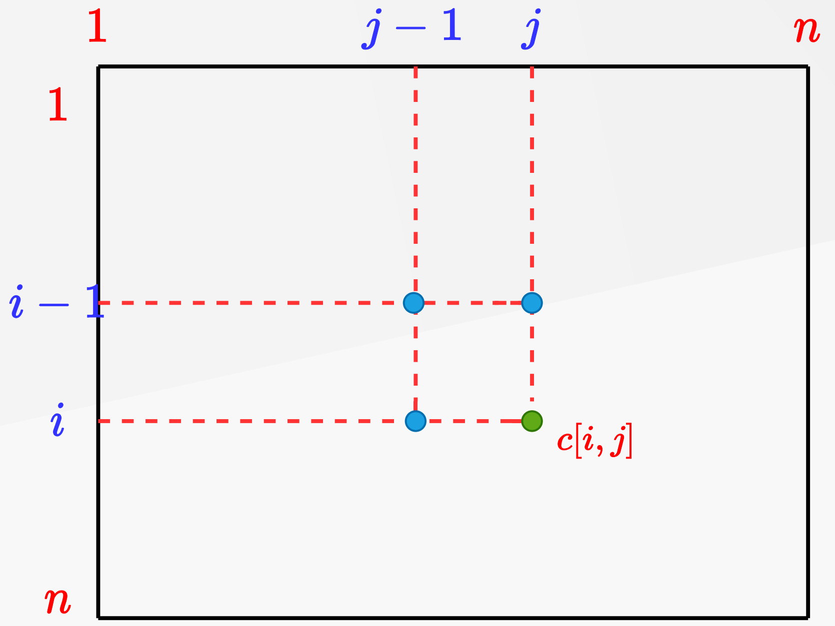
$c[i,j]$

**after computing:**

$c[i-1,j-1]$,

$c[i,j-1]$,

$c[i-1,j]$



## Bottom-up Computation

$$c[i,j] = \begin{cases} 0 & \text{if} \quad i = 0 \text{ or } j = 0 \\ c[i-1, j-1] + 1 & \text{if} \quad i, j > 0 \text{ and } x_i = y_j \\ \max\{c[i, j-1], c[i-1, j]\} & \text{if} \quad i, j > 0 \text{ and } x_i \neq y_j \end{cases}$$

$$\Downarrow$$

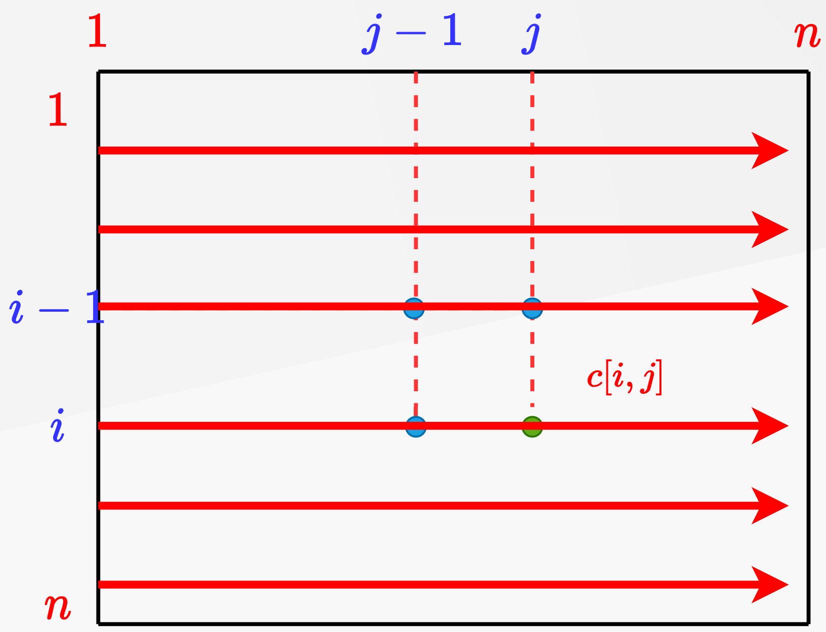
for $i \leftarrow 1$ to $m$

  for $j \leftarrow 1$ to $n$

  $\cdots$

  $\cdots$

  $c[i,j] = \cdots$



# Computing the Length of an LCS

$$\frac{\text{Total Runtime} = \Theta(mn)}{\text{Total Space} = \Theta(mn)}$$

$LCS - LENGTH(X, Y)$

$m \leftarrow length[X]; n \leftarrow length[Y]$

for $i \leftarrow 0$ to $m$ do $c[i, 0] \leftarrow 0$

for $j \leftarrow 0$ to $n$ do $c[0, j] \leftarrow 0$

for $i \leftarrow 1$ to $m$ do

  for $j \leftarrow 1$ to $n$ do

    if $x_i = y_j$ then

      $c[i, j] \leftarrow c[i-1, j-1] + 1$

      $b[i, j] \leftarrow " \nwarrow "$

    else if $c[i-1, j] \geq c[i, j-1]$

      $c[i, j] \leftarrow c[i-1, j]$

      $b[i, j] \leftarrow " \uparrow "$

    else

      $c[i, j] \leftarrow c[i, j-1]$

      $b[i, j] \leftarrow " \leftarrow "$

# Computing the Length of an LCS-1

Operation of LCS-LENGTH on the sequences

$$X = \langle \overset{1}{A}, \overset{2}{B}, \overset{3}{C}, \overset{4}{B}, \overset{5}{D}, \overset{6}{A}, \overset{7}{B} \rangle$$

$$Y = \langle \overset{1}{B}, \overset{2}{D}, \overset{3}{C}, \overset{4}{A}, \overset{5}{B}, \overset{6}{A} \rangle$$

| $\downarrow i/j \rightarrow$ | $0 y_j$ | 1 $B$ | 2 $D$ | 3 $C$ | 4 $A$ | 5 $B$ | 6 $A$ |
|---|---|---|---|---|---|---|---|
| $0 x_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 $A$ | 0 | | | | | | |
| 2 $B$ | 0 | | | | | | |
| 3 $C$ | 0 | | | | | | |
| 4 $B$ | 0 | | | | | | |
| 5 $D$ | 0 | | | | | | |
| 6 $A$ | 0 | | | | | | |
| 7 $B$ | 0 | | | | | | |

# Computing the Length of an LCS-2

Operation of LCS-LENGTH on the sequences

$$X = \langle \overset{1}{A}, \overset{2}{B}, \overset{3}{C}, \overset{4}{B}, \overset{5}{D}, \overset{6}{A}, \overset{7}{B} \rangle$$

$$Y = \langle \overset{1}{B}, \overset{2}{D}, \overset{3}{C}, \overset{4}{A}, \overset{5}{B}, \overset{6}{A} \rangle$$

| $\downarrow i/j \rightarrow$ | $0 y_j$ | 1 $B$ | 2 $D$ | 3 $C$ | 4 $A$ | 5 $B$ | 6 $A$ |
|---|---|---|---|---|---|---|---|
| $0 x_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 $A$ | 0 | $\uparrow$ 0 | $\uparrow$ 0 | $\uparrow$ 0 | $\nwarrow$ 1 | $\leftarrow$ 1 | $\nwarrow$ 1 |
| 2 $B$ | 0 | | | | | | |
| 3 $C$ | 0 | | | | | | |
| 4 $B$ | 0 | | | | | | |
| 5 $D$ | 0 | | | | | | |
| 6 $A$ | 0 | | | | | | |
| 7 $B$ | 0 | | | | | | |

# Computing the Length of an LCS-3

Operation of LCS-LENGTH on the sequences

$$X = \langle \overset{1}{A}, \overset{2}{B}, \overset{3}{C}, \overset{4}{B}, \overset{5}{D}, \overset{6}{A}, \overset{7}{B} \rangle$$

$$Y = \langle \overset{1}{B}, \overset{2}{D}, \overset{3}{C}, \overset{4}{A}, \overset{5}{B}, \overset{6}{A} \rangle$$

| $\downarrow i/j \rightarrow$ | $0 y_j$ | 1 $B$ | 2 $D$ | 3 $C$ | 4 $A$ | 5 $B$ | 6 $A$ |
|---|---|---|---|---|---|---|---|
| $0 x_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 $A$ | 0 | $\uparrow$ 0 | $\uparrow$ 0 | $\uparrow$ 0 | $\nwarrow$ 1 | $\leftarrow$ 1 | $\nwarrow$ 1 |
| 2 $B$ | 0 | $\nwarrow$ 1 | $\leftarrow$ 1 | $\leftarrow$ 1 | $\uparrow$ 1 | $\nwarrow$ 2 | $\leftarrow$ 2 |
| 3 $C$ | 0 | | | | | | |
| 4 $B$ | 0 | | | | | | |
| 5 $D$ | 0 | | | | | | |
| 6 $A$ | 0 | | | | | | |
| 7 $B$ | 0 | | | | | | |

## Computing the Length of an LCS-4

Operation of LCS-LENGTH on the sequences

$$X = \langle \overset{1}{A}, \overset{2}{B}, \overset{3}{C}, \overset{4}{B}, \overset{5}{D}, \overset{6}{A}, \overset{7}{B} \rangle$$

$$Y = \langle \overset{1}{B}, \overset{2}{D}, \overset{3}{C}, \overset{4}{A}, \overset{5}{B}, \overset{6}{A} \rangle$$

| $\downarrow i/j \rightarrow$ | $0\, y_j$ | $1\ B$ | $2\ D$ | $3\ C$ | $4\ A$ | $5\ B$ | $6\ A$ |
|---|---|---|---|---|---|---|---|
| $0\, x_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $1\ A$ | 0 | $\uparrow$ 0 | $\uparrow$ 0 | $\uparrow$ 0 | $\nwarrow$ 1 | $\leftarrow$ 1 | $\nwarrow$ 1 |
| $2\ B$ | 0 | $\nwarrow$ 1 | $\leftarrow$ 1 | $\leftarrow$ 1 | $\uparrow$ 1 | $\nwarrow$ 2 | $\leftarrow$ 2 |
| $3\ C$ | 0 | $\uparrow$ 1 | $\uparrow$ 1 | $\nwarrow$ 2 | $\leftarrow$ 2 | $\uparrow$ 2 | $\uparrow$ 2 |
| $4\ B$ | 0 | | | | | | |
| $5\ D$ | 0 | | | | | | |
| $6\ A$ | 0 | | | | | | |
| $7\ B$ | 0 | | | | | | |

# Computing the Length of an LCS-5

Operation of LCS-LENGTH on the sequences

$$X = \langle \overset{1}{A}, \overset{2}{B}, \overset{3}{C}, \overset{4}{B}, \overset{5}{D}, \overset{6}{A}, \overset{7}{B} \rangle$$

$$Y = \langle \overset{1}{B}, \overset{2}{D}, \overset{3}{C}, \overset{4}{A}, \overset{5}{B}, \overset{6}{A} \rangle$$

| $\downarrow i/j \rightarrow$ | $0 y_j$ | $1$ $B$ | $2$ $D$ | $3$ $C$ | $4$ $A$ | $5$ $B$ | $6$ $A$ |
|---|---|---|---|---|---|---|---|
| $0 x_i$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $1$ $A$ | $0$ | $\uparrow$ $0$ | $\uparrow$ $0$ | $\uparrow$ $0$ | $\nwarrow$ $1$ | $\leftarrow$ $1$ | $\nwarrow$ $1$ |
| $2$ $B$ | $0$ | $\nwarrow$ $1$ | $\leftarrow$ $1$ | $\leftarrow$ $1$ | $\uparrow$ $1$ | $\nwarrow$ $2$ | $\leftarrow$ $2$ |
| $3$ $C$ | $0$ | $\uparrow$ $1$ | $\uparrow$ $1$ | $\nwarrow$ $2$ | $\leftarrow$ $2$ | $\uparrow$ $2$ | $\uparrow$ $2$ |
| $4$ $B$ | $0$ | $\nwarrow$ $1$ | | | | | |
| $5$ $D$ | $0$ | | | | | | |
| $6$ $A$ | $0$ | | | | | | |
| $7$ $B$ | $0$ | | | | | | |

## Computing the Length of an LCS-6

Operation of LCS-LENGTH on the sequences

$$X = \langle \overset{1}{A}, \overset{2}{B}, \overset{3}{C}, \overset{4}{B}, \overset{5}{D}, \overset{6}{A}, \overset{7}{B} \rangle$$

$$Y = \langle \overset{1}{B}, \overset{2}{D}, \overset{3}{C}, \overset{4}{A}, \overset{5}{B}, \overset{6}{A} \rangle$$

| $\downarrow i/j \rightarrow$ | $0 y_j$ | 1 $B$ | 2 $D$ | 3 $C$ | 4 $A$ | 5 $B$ | 6 $A$ |
|---|---|---|---|---|---|---|---|
| $0 x_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 $A$ | 0 | $\uparrow$ 0 | $\uparrow$ 0 | $\uparrow$ 0 | $\nwarrow$ 1 | $\leftarrow$ 1 | $\nwarrow$ 1 |
| 2 $B$ | 0 | $\nwarrow$ 1 | $\leftarrow$ 1 | $\leftarrow$ 1 | $\uparrow$ 1 | $\nwarrow$ 2 | $\leftarrow$ 2 |
| 3 $C$ | 0 | $\uparrow$ 1 | $\uparrow$ 1 | $\nwarrow$ 2 | $\leftarrow$ 2 | $\uparrow$ 2 | $\uparrow$ 2 |
| 4 $B$ | 0 | $\nwarrow$ 1 | $\uparrow$ 1 | | | | |
| 5 $D$ | 0 | | | | | | |
| 6 $A$ | 0 | | | | | | |
| 7 $B$ | 0 | | | | | | |

## Computing the Length of an LCS-7

Operation of LCS-LENGTH on the sequences

$$X = \langle \overset{1}{A}, \overset{2}{B}, \overset{3}{C}, \overset{4}{B}, \overset{5}{D}, \overset{6}{A}, \overset{7}{B} \rangle$$

$$Y = \langle \overset{1}{B}, \overset{2}{D}, \overset{3}{C}, \overset{4}{A}, \overset{5}{B}, \overset{6}{A} \rangle$$

| $\downarrow i/j \rightarrow$ | $0 y_j$ | 1 $B$ | 2 $D$ | 3 $C$ | 4 $A$ | 5 $B$ | 6 $A$ |
|---|---|---|---|---|---|---|---|
| $0 x_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 $A$ | 0 | $\uparrow$ 0 | $\uparrow$ 0 | $\uparrow$ 0 | $\nwarrow$ 1 | $\leftarrow$ 1 | $\nwarrow$ 1 |
| 2 $B$ | 0 | $\nwarrow$ 1 | $\leftarrow$ 1 | $\leftarrow$ 1 | $\uparrow$ 1 | $\nwarrow$ 2 | $\leftarrow$ 2 |
| 3 $C$ | 0 | $\uparrow$ 1 | $\uparrow$ 1 | $\nwarrow$ 2 | $\leftarrow$ 2 | $\uparrow$ 2 | $\uparrow$ 2 |
| 4 $B$ | 0 | $\nwarrow$ 1 | $\uparrow$ 1 | $\uparrow$ 2 | | | |
| 5 $D$ | 0 | | | | | | |
| 6 $A$ | 0 | | | | | | |
| 7 $B$ | 0 | | | | | | |

## Computing the Length of an LCS-8

Operation of LCS-LENGTH on the sequences

$$X = \langle \overset{1}{A}, \overset{2}{B}, \overset{3}{C}, \overset{4}{B}, \overset{5}{D}, \overset{6}{A}, \overset{7}{B} \rangle$$

$$Y = \langle \overset{1}{B}, \overset{2}{D}, \overset{3}{C}, \overset{4}{A}, \overset{5}{B}, \overset{6}{A} \rangle$$

| $\downarrow i/j \rightarrow$ $0 y_j$ | | 1 $B$ | 2 $D$ | 3 $C$ | 4 $A$ | 5 $B$ | 6 $A$ |
|---|---|---|---|---|---|---|---|
| $0 x_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 $A$ | 0 | $\uparrow$ 0 | $\uparrow$ 0 | $\uparrow$ 0 | $\nwarrow$ 1 | $\leftarrow$ 1 | $\nwarrow$ 1 |
| 2 $B$ | 0 | $\nwarrow$ 1 | $\leftarrow$ 1 | $\leftarrow$ 1 | $\uparrow$ 1 | $\nwarrow$ 2 | $\leftarrow$ 2 |
| 3 $C$ | 0 | $\uparrow$ 1 | $\uparrow$ 1 | $\nwarrow$ 2 | $\leftarrow$ 2 | $\uparrow$ 2 | $\uparrow$ 2 |
| 4 $B$ | 0 | $\nwarrow$ 1 | $\uparrow$ 1 | $\uparrow$ 2 | $\uparrow$ 2 | | |
| 5 $D$ | 0 | | | | | | |
| 6 $A$ | 0 | | | | | | |
| 7 $B$ | 0 | | | | | | |

# Computing the Length of an LCS-9

Operation of LCS-LENGTH on the sequences

$$X = \langle \overset{1}{A}, \overset{2}{B}, \overset{3}{C}, \overset{4}{B}, \overset{5}{D}, \overset{6}{A}, \overset{7}{B} \rangle$$

$$Y = \langle \overset{1}{B}, \overset{2}{D}, \overset{3}{C}, \overset{4}{A}, \overset{5}{B}, \overset{6}{A} \rangle$$

| $\downarrow i/j \rightarrow$ | $0 y_j$ | 1 $B$ | 2 $D$ | 3 $C$ | 4 $A$ | 5 $B$ | 6 $A$ |
|---|---|---|---|---|---|---|---|
| $0 x_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 $A$ | 0 | $\uparrow$ 0 | $\uparrow$ 0 | $\uparrow$ 0 | $\nwarrow$ 1 | $\leftarrow$ 1 | $\nwarrow$ 1 |
| 2 $B$ | 0 | $\nwarrow$ 1 | $\leftarrow$ 1 | $\leftarrow$ 1 | $\uparrow$ 1 | $\nwarrow$ 2 | $\leftarrow$ 2 |
| 3 $C$ | 0 | $\uparrow$ 1 | $\uparrow$ 1 | $\nwarrow$ 2 | $\leftarrow$ 2 | $\uparrow$ 2 | $\uparrow$ 2 |
| 4 $B$ | 0 | $\nwarrow$ 1 | $\uparrow$ 1 | $\uparrow$ 2 | $\uparrow$ 2 | $\nwarrow$ 3 | |
| 5 $D$ | 0 | | | | | | |
| 6 $A$ | 0 | | | | | | |
| 7 $B$ | 0 | | | | | | |

# Computing the Length of an LCS-10

Operation of LCS-LENGTH on the sequences

$$X = \langle \overset{1}{A}, \overset{2}{B}, \overset{3}{C}, \overset{4}{B}, \overset{5}{D}, \overset{6}{A}, \overset{7}{B} \rangle$$

$$Y = \langle \overset{1}{B}, \overset{2}{D}, \overset{3}{C}, \overset{4}{A}, \overset{5}{B}, \overset{6}{A} \rangle$$

| $\downarrow i/j \rightarrow$ | $0 y_j$ | 1 $B$ | 2 $D$ | 3 $C$ | 4 $A$ | 5 $B$ | 6 $A$ |
|---|---|---|---|---|---|---|---|
| $0 x_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 $A$ | 0 | $\uparrow$ 0 | $\uparrow$ 0 | $\uparrow$ 0 | $\nwarrow$ 1 | $\leftarrow$ 1 | $\nwarrow$ 1 |
| 2 $B$ | 0 | $\nwarrow$ 1 | $\leftarrow$ 1 | $\leftarrow$ 1 | $\uparrow$ 1 | $\nwarrow$ 2 | $\leftarrow$ 2 |
| 3 $C$ | 0 | $\uparrow$ 1 | $\uparrow$ 1 | $\nwarrow$ 2 | $\leftarrow$ 2 | $\uparrow$ 2 | $\uparrow$ 2 |
| 4 $B$ | 0 | $\nwarrow$ 1 | $\uparrow$ 1 | $\uparrow$ 2 | $\uparrow$ 2 | $\nwarrow$ 3 | $\leftarrow$ 3 |
| 5 $D$ | 0 | | | | | | |
| 6 $A$ | 0 | | | | | | |
| 7 $B$ | 0 | | | | | | |

## Computing the Length of an LCS-11

Operation of LCS-LENGTH on the sequences

$$X = \langle \overset{1}{A}, \overset{2}{B}, \overset{3}{C}, \overset{4}{B}, \overset{5}{D}, \overset{6}{A}, \overset{7}{B} \rangle$$

$$Y = \langle \overset{1}{B}, \overset{2}{D}, \overset{3}{C}, \overset{4}{A}, \overset{5}{B}, \overset{6}{A} \rangle$$

| $\downarrow i/j \rightarrow$ | $0 y_j$ | 1 $B$ | 2 $D$ | 3 $C$ | 4 $A$ | 5 $B$ | 6 $A$ |
|---|---|---|---|---|---|---|---|
| $0 x_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 $A$ | 0 | $\uparrow$ 0 | $\uparrow$ 0 | $\uparrow$ 0 | $\nwarrow$ 1 | $\leftarrow$ 1 | $\nwarrow$ 1 |
| 2 $B$ | 0 | $\nwarrow$ 1 | $\leftarrow$ 1 | $\leftarrow$ 1 | $\uparrow$ 1 | $\nwarrow$ 2 | $\leftarrow$ 2 |
| 3 $C$ | 0 | $\uparrow$ 1 | $\uparrow$ 1 | $\nwarrow$ 2 | $\leftarrow$ 2 | $\uparrow$ 2 | $\uparrow$ 2 |
| 4 $B$ | 0 | $\nwarrow$ 1 | $\uparrow$ 1 | $\uparrow$ 2 | $\uparrow$ 2 | $\nwarrow$ 3 | $\leftarrow$ 3 |
| 5 $D$ | 0 | $\uparrow$ 1 | $\nwarrow$ 2 | $\uparrow$ 2 | $\uparrow$ 2 | $\uparrow$ 3 | $\uparrow$ 3 |
| 6 $A$ | 0 | | | | | | |
| 7 $B$ | 0 | | | | | | |

# Computing the Length of an LCS-12

Operation of LCS-LENGTH on the sequences

$$X = \langle \overset{1}{A}, \overset{2}{B}, \overset{3}{C}, \overset{4}{B}, \overset{5}{D}, \overset{6}{A}, \overset{7}{B} \rangle$$

$$Y = \langle \overset{1}{B}, \overset{2}{D}, \overset{3}{C}, \overset{4}{A}, \overset{5}{B}, \overset{6}{A} \rangle$$

| $\downarrow i/j \rightarrow$ | $0 y_j$ | 1 $B$ | 2 $D$ | 3 $C$ | 4 $A$ | 5 $B$ | 6 $A$ |
|---|---|---|---|---|---|---|---|
| $0 x_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 $A$ | 0 | $\uparrow$ 0 | $\uparrow$ 0 | $\uparrow$ 0 | $\nwarrow$ 1 | $\leftarrow$ 1 | $\nwarrow$ 1 |
| 2 $B$ | 0 | $\nwarrow$ 1 | $\leftarrow$ 1 | $\leftarrow$ 1 | $\uparrow$ 1 | $\nwarrow$ 2 | $\leftarrow$ 2 |
| 3 $C$ | 0 | $\uparrow$ 1 | $\uparrow$ 1 | $\nwarrow$ 2 | $\leftarrow$ 2 | $\uparrow$ 2 | $\uparrow$ 2 |
| 4 $B$ | 0 | $\nwarrow$ 1 | $\uparrow$ 1 | $\uparrow$ 2 | $\uparrow$ 2 | $\nwarrow$ 3 | $\leftarrow$ 3 |
| 5 $D$ | 0 | $\uparrow$ 1 | $\nwarrow$ 2 | $\uparrow$ 2 | $\uparrow$ 2 | $\uparrow$ 3 | $\uparrow$ 3 |
| 6 $A$ | 0 | $\uparrow$ 1 | $\uparrow$ 2 | $\uparrow$ 2 | $\nwarrow$ 3 | $\uparrow$ 3 | $\nwarrow$ 4 |
| 7 $B$ | 0 | | | | | | |

## Computing the Length of an LCS-13

Operation of LCS-LENGTH on the sequences

$$X = \langle \overset{1}{A}, \overset{2}{B}, \overset{3}{C}, \overset{4}{B}, \overset{5}{D}, \overset{6}{A}, \overset{7}{B} \rangle$$

$$Y = \langle \overset{1}{B}, \overset{2}{D}, \overset{3}{C}, \overset{4}{A}, \overset{5}{B}, \overset{6}{A} \rangle$$

- Running-time = $O(mn)$ since each table entry takes $O(1)$ time to compute

| $\downarrow i/j \rightarrow$ | $0 y_j$ | 1 $B$ | 2 $D$ | 3 $C$ | 4 $A$ | 5 $B$ | 6 $A$ |
|---|---|---|---|---|---|---|---|
| $0 x_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 $A$ | 0 | $\uparrow$ 0 | $\uparrow$ 0 | $\uparrow$ 0 | $\nwarrow$ 1 | $\leftarrow$ 1 | $\nwarrow$ 1 |
| 2 $B$ | 0 | $\nwarrow$ 1 | $\leftarrow$ 1 | $\leftarrow$ 1 | $\uparrow$ 1 | $\nwarrow$ 2 | $\leftarrow$ 2 |
| 3 $C$ | 0 | $\uparrow$ 1 | $\uparrow$ 1 | $\nwarrow$ 2 | $\leftarrow$ 2 | $\uparrow$ 2 | $\uparrow$ 2 |
| 4 $B$ | 0 | $\nwarrow$ 1 | $\uparrow$ 1 | $\uparrow$ 2 | $\uparrow$ 2 | $\nwarrow$ 3 | $\leftarrow$ 3 |
| 5 $D$ | 0 | $\uparrow$ 1 | $\nwarrow$ 2 | $\uparrow$ 2 | $\uparrow$ 2 | $\uparrow$ 3 | $\uparrow$ 3 |
| 6 $A$ | 0 | $\uparrow$ 1 | $\uparrow$ 2 | $\uparrow$ 2 | $\nwarrow$ 3 | $\uparrow$ 3 | $\nwarrow$ 4 |
| 7 $B$ | 0 | $\nwarrow$ 1 | $\uparrow$ 2 | $\uparrow$ 3 | $\uparrow$ 3 | $\nwarrow$ 4 | $\uparrow$ 4 |

# Computing the Length of an LCS-14

Operation of LCS-LENGTH on the sequences

$$X = \langle \overset{1}{A}, \overset{2}{B}, \overset{3}{C}, \overset{4}{B}, \overset{5}{D}, \overset{6}{A}, \overset{7}{B} \rangle$$

$$Y = \langle \overset{1}{B}, \overset{2}{D}, \overset{3}{C}, \overset{4}{A}, \overset{5}{B}, \overset{6}{A} \rangle$$

- Running-time = $O(mn)$ since each table entry takes $O(1)$ time to compute
- LCS of $X \& Y = \langle B, C, B, A \rangle$

| $\downarrow i/j \rightarrow$ | $0 y_j$ | 1 $B$ | 2 $D$ | 3 $C$ | 4 $A$ | 5 $B$ | 6 $A$ |
|---|---|---|---|---|---|---|---|
| $0 x_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 $A$ | 0 | $\uparrow$ 0 | $\uparrow$ 0 | $\uparrow$ 0 | $\nwarrow$ 1 | $\leftarrow$ 1 | $\nwarrow$ 1 |
| 2 $B$ | 0 | $\nwarrow$ 1 | $\leftarrow$ 1 | $\leftarrow$ 1 | $\uparrow$ 1 | $\nwarrow$ 2 | $\leftarrow$ 2 |
| 3 $C$ | 0 | $\uparrow$ 1 | $\uparrow$ 1 | $\nwarrow$ 2 | $\leftarrow$ 2 | $\uparrow$ 2 | $\uparrow$ 2 |
| 4 $B$ | 0 | $\nwarrow$ 1 | $\uparrow$ 1 | $\uparrow$ 2 | $\uparrow$ 2 | $\nwarrow$ 3 | $\leftarrow$ 3 |
| 5 $D$ | 0 | $\uparrow$ 1 | $\nwarrow$ 2 | $\uparrow$ 2 | $\uparrow$ 2 | $\uparrow$ 3 | $\uparrow$ 3 |
| 6 $A$ | 0 | $\uparrow$ 1 | $\uparrow$ 2 | $\uparrow$ 2 | $\nwarrow$ 3 | $\uparrow$ 3 | $\nwarrow$ 4 |
| 7 $B$ | 0 | $\nwarrow$ 1 | $\uparrow$ 2 | $\uparrow$ 3 | $\uparrow$ 3 | $\nwarrow$ 4 | $\uparrow$ 4 |

# Constructing an LCS

- The $b$ table returned by **LCS-LENGTH** can be used to quickly construct an **LCS** of $X\&Y$
- Begin at $b[m, n]$ and trace through the table following arrows
- Whenever you encounter a "$\nwarrow$" in entry $b[i, j]$ it implies that $x_i = y_j$ is an element of **LCS**
- The elements of **LCS** are encountered in **reverse order**

## Constructing an LCS

- The recursive procedure $\text{PRINT-LCS}$ prints out $\text{LCS}$ in proper order
- This procedure takes $O(m+n)$ time since at least one of $i$ and $j$ is decremented in each stage of the recursion

$$
\begin{aligned}
&\text{PRINT-LCS}(b, X, i, j) \\
&\quad \text{if } i = 0 \text{ or} j = 0 \text{ then} \\
&\quad \text{return} \\
&\quad \text{if } b[i,j] = \text{"} \nwarrow \text{" then} \\
&\quad\quad \text{PRINT-LCS}(b, X, i-1, j-1) \\
&\quad\quad \text{print } x_i \\
&\quad \text{else if } b[i,j] = \text{"} \uparrow \text{" then} \\
&\quad\quad \text{PRINT-LCS}(b, X, i-1, j) \\
&\quad \text{else} \\
&\quad\quad \text{PRINT-LCS}(b, X, i, j-1)
\end{aligned}
$$

- **The initial invocation:** $\text{PRINT-LCS}(b, X, length[X], length[Y])$

## Do we really need the b table (back-pointers)?

- **Question:** From which neighbor did we expand to the highlighted cell?
- **Answer:** Upper-left neighbor,because $X[i] = Y[j]$.

| $\downarrow i/j \rightarrow$ | $0 y_j$ | $1$ $B$ | $2$ $D$ | $3$ $C$ | $4$ $A$ | $5$ $B$ | $6$ $A$ |
|---|---|---|---|---|---|---|---|
| $0 x_i$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $1$ $A$ | $0$ | $\uparrow$ $0$ | $\uparrow$ $0$ | $\uparrow$ $0$ | $\nwarrow$ $1$ | $\leftarrow$ $1$ | $\nwarrow$ $1$ |
| $2$ $B$ | $0$ | $\nwarrow$ $1$ | $\leftarrow$ $1$ | $\leftarrow$ $1$ | $\uparrow$ $1$ | $\nwarrow$ $2$ | $\leftarrow$ $2$ |
| $3$ $C$ | $0$ | $\uparrow$ $1$ | $\uparrow$ $1$ | $\nwarrow$ $2$ | $\leftarrow$ $2$ | $\uparrow$ $2$ | $\uparrow$ $2$ |
| $4$ $B$ | $0$ | $\nwarrow$ $1$ | $\uparrow$ $1$ | $\uparrow$ $2$ | $\uparrow$ $2$ | $\nwarrow$ $3$ | $\leftarrow$ $3$ |
| $5$ $D$ | $0$ | $\uparrow$ $1$ | $\nwarrow$ $2$ | $\uparrow$ $2$ | $\uparrow$ $2$ | $\uparrow$ $3$ | $\uparrow$ $3$ |
| $6$ $A$ | $0$ | $\uparrow$ $1$ | $\uparrow$ $2$ | $\uparrow$ $2$ | $\nwarrow$ $3$ | $\uparrow$ $3$ | $\nwarrow$ $4$ |
| $7$ $B$ | $0$ | $\nwarrow$ $1$ | $\uparrow$ $2$ | $\uparrow$ $3$ | $\uparrow$ $3$ | $\nwarrow$ $4$ | $\uparrow$ $4$ |

## Do we really need the b table (back-pointers)?

- **Question:** From which neighbor did we expand to the highlighted cell?
- **Answer:** Left neighbor, because $X[i] \neq Y[j]$ and $LCS[i, j-1] > LCS[i-1, j]$.

| $\downarrow i/j \rightarrow$ | $0 y_j$ | 1 $B$ | 2 $D$ | 3 $C$ | 4 $A$ | 5 $B$ | 6 $A$ |
|---|---|---|---|---|---|---|---|
| $0 x_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 $A$ | 0 | $\uparrow$ 0 | $\uparrow$ 0 | $\uparrow$ 0 | $\nwarrow$ 1 | $\leftarrow$ 1 | $\nwarrow$ 1 |
| 2 $B$ | 0 | $\nwarrow$ 1 | $\leftarrow$ 1 | $\leftarrow$ 1 | $\uparrow$ 1 | $\nwarrow$ 2 | $\leftarrow$ 2 |
| 3 $C$ | 0 | $\uparrow$ 1 | $\uparrow$ 1 | $\nwarrow$ 2 | $\leftarrow$ 2 | $\uparrow$ 2 | $\uparrow$ 2 |
| 4 $B$ | 0 | $\nwarrow$ 1 | $\uparrow$ 1 | $\uparrow$ 2 | $\uparrow$ 2 | $\nwarrow$ 3 | $\leftarrow$ 3 |
| 5 $D$ | 0 | $\uparrow$ 1 | $\nwarrow$ 2 | $\uparrow$ 2 | $\uparrow$ 2 | $\uparrow$ 3 | $\uparrow$ 3 |
| 6 $A$ | 0 | $\uparrow$ 1 | $\uparrow$ 2 | $\uparrow$ 2 | $\nwarrow$ 3 | $\uparrow$ 3 | $\nwarrow$ 4 |
| 7 $B$ | 0 | $\nwarrow$ 1 | $\uparrow$ 2 | $\uparrow$ 3 | $\uparrow$ 3 | $\nwarrow$ 4 | $\uparrow$ 4 |

# Do we really need the b table (back-pointers)?

- **Question:** From which neighbor did we expand to the highlighted cell?
- **Answer:** Upper neighbor,because $X[i] \neq Y[j]$ and $LCS[i, j-1] = LCS[i-1, j]$. *(See pseudo-code to see how ties are handled.)*

| $\downarrow i/j \rightarrow$ | $0 y_j$ | 1 $B$ | 2 $D$ | 3 $C$ | 4 $A$ | 5 $B$ | 6 $A$ |
|---|---|---|---|---|---|---|---|
| $0 x_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 $A$ | 0 | $\uparrow$ 0 | $\uparrow$ 0 | $\uparrow$ 0 | $\nwarrow$ 1 | $\leftarrow$ 1 | $\nwarrow$ 1 |
| 2 $B$ | 0 | $\nwarrow$ 1 | $\leftarrow$ 1 | $\leftarrow$ 1 | $\uparrow$ 1 | $\nwarrow$ 2 | $\leftarrow$ 2 |
| 3 $C$ | 0 | $\uparrow$ 1 | $\uparrow$ 1 | $\nwarrow$ 2 | $\leftarrow$ 2 | $\uparrow$ 2 | $\uparrow$ 2 |
| 4 $B$ | 0 | $\nwarrow$ 1 | $\uparrow$ 1 | $\uparrow$ 2 | $\uparrow$ 2 | $\nwarrow$ 3 | $\leftarrow$ 3 |
| 5 $D$ | 0 | $\uparrow$ 1 | $\nwarrow$ 2 | $\uparrow$ 2 | $\uparrow$ 2 | $\uparrow$ 3 | $\uparrow$ 3 |
| 6 $A$ | 0 | $\uparrow$ 1 | $\uparrow$ 2 | $\uparrow$ 2 | $\nwarrow$ 3 | $\uparrow$ 3 | $\nwarrow$ 4 |
| 7 $B$ | 0 | $\nwarrow$ 1 | $\uparrow$ 2 | $\uparrow$ 3 | $\uparrow$ 3 | $\nwarrow$ 4 | $\uparrow$ 4 |

## Improving the Space Requirements

- We can eliminate the b table altogether
  - each $c[i, j]$ entry depends only on $3$ other $c$ table entries: $c[i-1, j-1]$, $c[i-1, j]$ and $c[i, j-1]$
- Given the value of $c[i, j]$:
  - We can determine in $O(1)$ time which of these $3$ values was used to compute $c[i, j]$ without inspecting table $b$
  - We save $\Theta(mn)$ space by this method
  - However, space requirement is still $\Theta(mn)$ since we need $\Theta(mn)$ space for the $c$ table anyway

## What if we store the last 2 rows only?

- To compute $c[i, j]$, we only need $c[i-1, j-1]$, $c[i-1, j]$,and $c[i-1, j-1]$
- So, we can store only the last two rows.

| $\downarrow i/j \rightarrow$ | $0 y_j$ | 1 $B$ | 2 $D$ | 3 $C$ | 4 $A$ | 5 $B$ | 6 $A$ |
|---|---|---|---|---|---|---|---|
| $0 x_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 $A$ | 0 | $\uparrow$ 0 | $\uparrow$ 0 | $\uparrow$ 0 | $\nwarrow$ 1 | $\leftarrow$ 1 | $\nwarrow$ 1 |
| 2 $B$ | 0 | $\nwarrow$ 1 | $\leftarrow$ 1 | $\leftarrow$ 1 | $\uparrow$ 1 | $\nwarrow$ 2 | $\leftarrow$ 2 |
| 3 $C$ | 0 | $\uparrow$ 1 | $\uparrow$ 1 | $\nwarrow$ 2 | $\leftarrow$ 2 | $\uparrow$ 2 | $\uparrow$ 2 |
| 4 $B$ | 0 | $\nwarrow$ 1 | $\uparrow$ 1 | $\uparrow$ 2 | $\uparrow$ 2 | $\nwarrow$ 3 | $\leftarrow$ 3 |
| 5 $D$ | 0 | $\uparrow$ 1 | $\nwarrow$ 2 | $\uparrow$ 2 | $\uparrow$ 2 | $\uparrow$ 3 | $\uparrow$ 3 |
| 6 $A$ | 0 | $\uparrow$ 1 | $\uparrow$ 2 | $\uparrow$ 2 | $\nwarrow$ 3 | $\uparrow$ 3 | $\nwarrow$ 4 |
| 7 $B$ | 0 | $\nwarrow$ 1 | $\uparrow$ 2 | $\uparrow$ 3 | $\uparrow$ 3 | $\nwarrow$ 4 | $\uparrow$ 4 |

## What if we store the last 2 rows only?

- To compute $c[i, j]$, we only need $c[i-1, j-1]$, $c[i-1, j]$, and $c[i-1, j-1]$
- So, we can store only the last two rows.

| $\downarrow i/j \rightarrow$ | $0 y_j$ | $1\ B$ | $2\ D$ | $3\ C$ | $4\ A$ | $5\ B$ | $6\ A$ |
|---|---|---|---|---|---|---|---|
| $0 x_i$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $1\ A$ | $0$ | $\uparrow 0$ | $\uparrow 0$ | $\uparrow 0$ | $\nwarrow 1$ | $\leftarrow 1$ | $\nwarrow 1$ |
| $2\ B$ | $0$ | $\nwarrow 1$ | $\leftarrow 1$ | $\leftarrow 1$ | $\uparrow 1$ | $\nwarrow 2$ | $\leftarrow 2$ |
| $3\ C$ | $0$ | $\uparrow 1$ | $\uparrow 1$ | $\nwarrow 2$ | $\leftarrow 2$ | $\uparrow 2$ | $\uparrow 2$ |
| $4\ B$ | $0$ | $\nwarrow 1$ | $\uparrow 1$ | $\uparrow 2$ | $\uparrow 2$ | $\nwarrow 3$ | $\leftarrow 3$ |
| $5\ D$ | $0$ | $\uparrow 1$ | $\nwarrow 2$ | $\uparrow 2$ | $\uparrow 2$ | $\uparrow 3$ | $\uparrow 3$ |
| $6\ A$ | $0$ | $\uparrow 1$ | $\uparrow 2$ | $\uparrow 2$ | $\nwarrow 3$ | $\uparrow 3$ | $\nwarrow 4$ |
| $7\ B$ | $0$ | $\nwarrow 1$ | $\uparrow 2$ | $\uparrow 3$ | $\uparrow 3$ | $\nwarrow 4$ | $\uparrow 4$ |

## What if we store the last 2 rows only?

- To compute $c[i,j]$, we only need $c[i-1,j-1]$, $c[i-1,j]$, and $c[i-1,j-1]$
- So, we can store only the last two rows.
- This reduces space complexity from $\Theta(mn)$ to $\Theta(n)$.
- Is there a problem with this approach?

| $\downarrow i/j \rightarrow$ | $0\ y_j$ | $1\ B$ | $2\ D$ | $3\ C$ | $4\ A$ | $5\ B$ | $6\ A$ |
|---|---|---|---|---|---|---|---|
| $0\ x_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $1\ A$ | 0 | $\uparrow 0$ | $\uparrow 0$ | $\uparrow 0$ | $\nwarrow 1$ | $\leftarrow 1$ | $\nwarrow 1$ |
| $2\ B$ | 0 | $\nwarrow 1$ | $\leftarrow 1$ | $\leftarrow 1$ | $\uparrow 1$ | $\nwarrow 2$ | $\leftarrow 2$ |
| $3\ C$ | 0 | $\uparrow 1$ | $\uparrow 1$ | $\nwarrow 2$ | $\leftarrow 2$ | $\uparrow 2$ | $\uparrow 2$ |
| $4\ B$ | 0 | $\nwarrow 1$ | $\uparrow 1$ | $\uparrow 2$ | $\uparrow 2$ | $\nwarrow 3$ | $\leftarrow 3$ |
| $5\ D$ | 0 | $\uparrow 1$ | $\nwarrow 2$ | $\uparrow 2$ | $\uparrow 2$ | $\uparrow 3$ | $\uparrow 3$ |
| $6\ A$ | 0 | $\uparrow 1$ | $\uparrow 2$ | $\uparrow 2$ | $\nwarrow 3$ | $\uparrow 3$ | $\nwarrow 4$ |
| $7\ B$ | 0 | $\nwarrow 1$ | $\uparrow 2$ | $\uparrow 3$ | $\uparrow 3$ | $\nwarrow 4$ | $\uparrow 4$ |

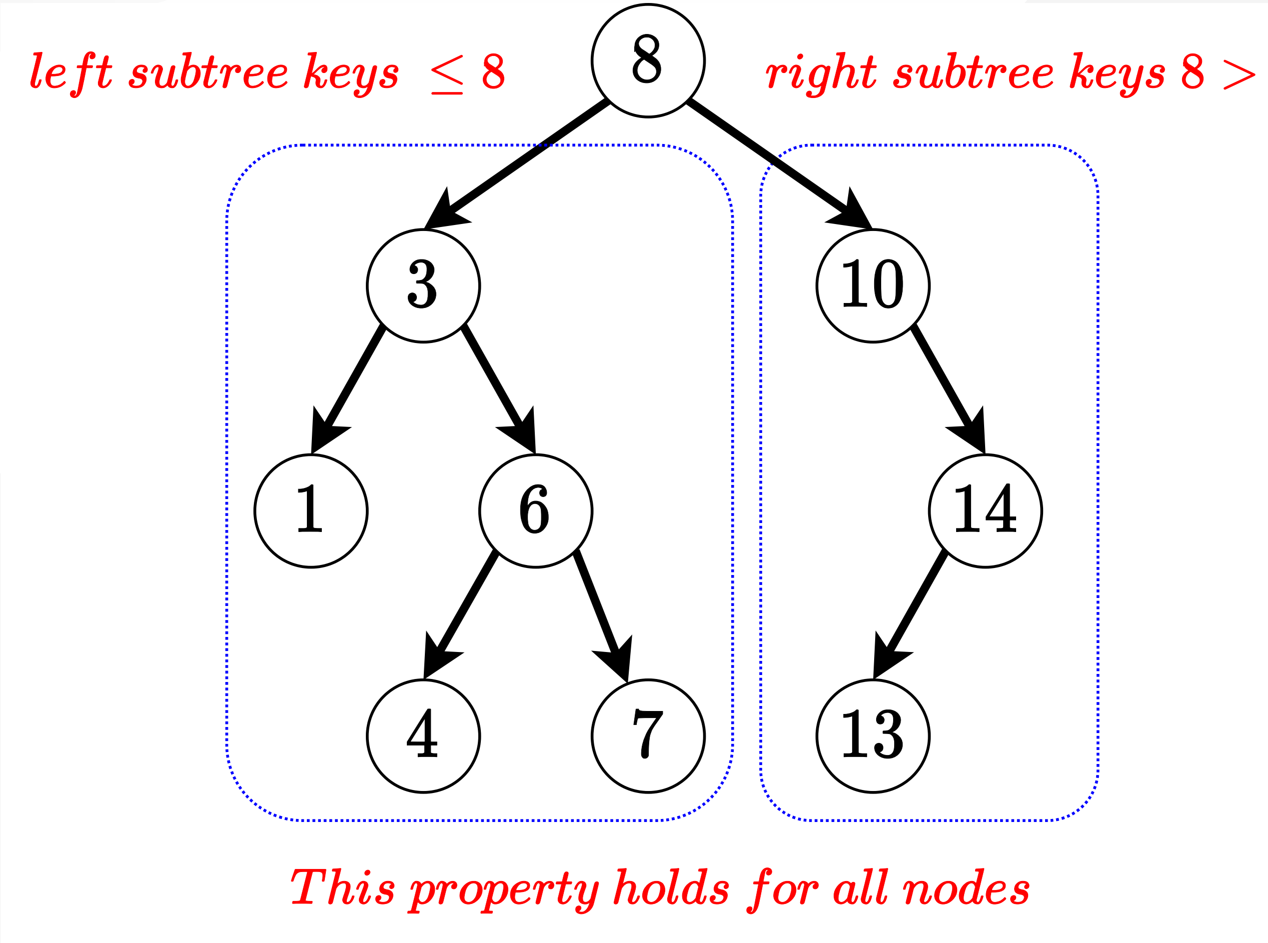
## What if we store the last 2 rows only?

- **Is there a problem with this approach?**
  - We cannot construct the optimal solution because we cannot backtrace anymore.
  - This approach works if we only need the length of an LCS, not the actual LCS.

| $\downarrow i/j \rightarrow$ | $0 y_j$ | 1 $B$ | 2 $D$ | 3 $C$ | 4 $A$ | 5 $B$ | 6 $A$ |
|---|---|---|---|---|---|---|---|
| $0 x_i$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 $A$ | 0 | $\uparrow$ 0 | $\uparrow$ 0 | $\uparrow$ 0 | $\nwarrow$ 1 | $\leftarrow$ 1 | $\nwarrow$ 1 |
| 2 $B$ | 0 | $\nwarrow$ 1 | $\leftarrow$ 1 | $\leftarrow$ 1 | $\uparrow$ 1 | $\nwarrow$ 2 | $\leftarrow$ 2 |
| 3 $C$ | 0 | $\uparrow$ 1 | $\uparrow$ 1 | $\nwarrow$ 2 | $\leftarrow$ 2 | $\uparrow$ 2 | $\uparrow$ 2 |
| 4 $B$ | 0 | $\nwarrow$ 1 | $\uparrow$ 1 | $\uparrow$ 2 | $\uparrow$ 2 | $\nwarrow$ 3 | $\leftarrow$ 3 |
| 5 $D$ | 0 | $\uparrow$ 1 | $\nwarrow$ 2 | $\uparrow$ 2 | $\uparrow$ 2 | $\uparrow$ 3 | $\uparrow$ 3 |
| 6 $A$ | 0 | $\uparrow$ 1 | $\uparrow$ 2 | $\uparrow$ 2 | $\nwarrow$ 3 | $\uparrow$ 3 | $\nwarrow$ 4 |
| 7 $B$ | 0 | $\nwarrow$ 1 | $\uparrow$ 2 | $\uparrow$ 3 | $\uparrow$ 3 | $\nwarrow$ 4 | $\uparrow$ 4 |

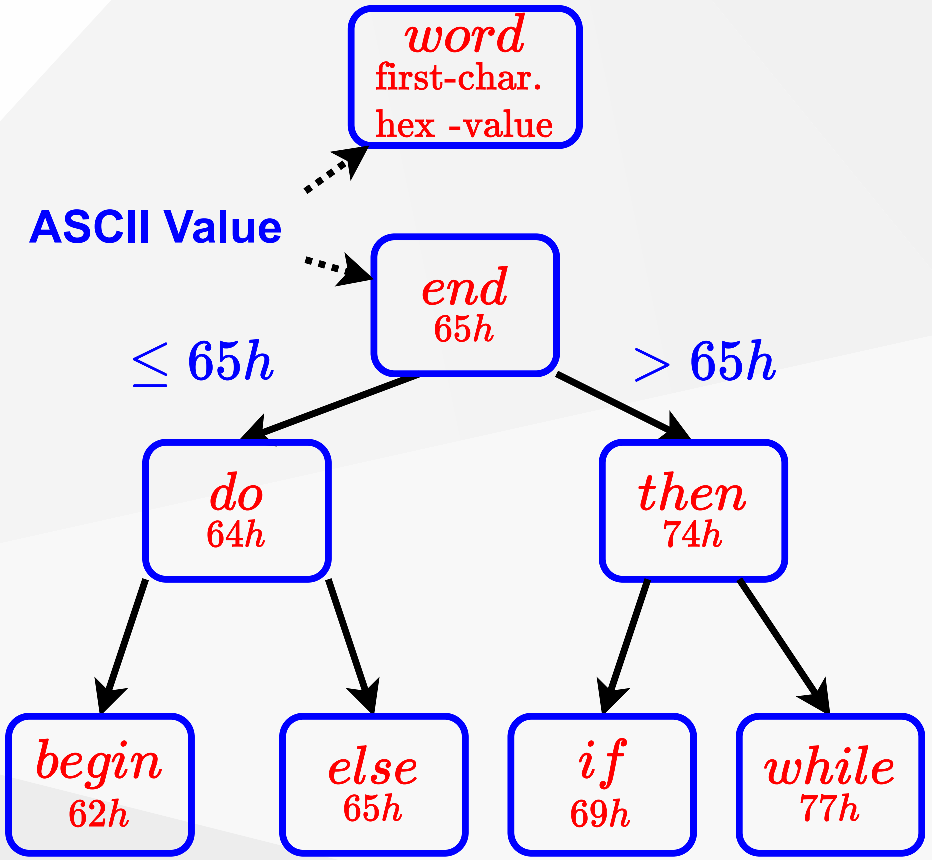
# Problem 4 Optimal Binary Search Tree

# Reminder: Binary Search Tree (BST)



# Binary Search Tree Example

- **Example:** English-to-French translation
    - Organize (English, French) word pairs in a BST
        - **Keyword:** English word
        - **Satellite Data:** French word
- We can search for an English word (node key) efficiently, and return the corresponding French word (satellite data).



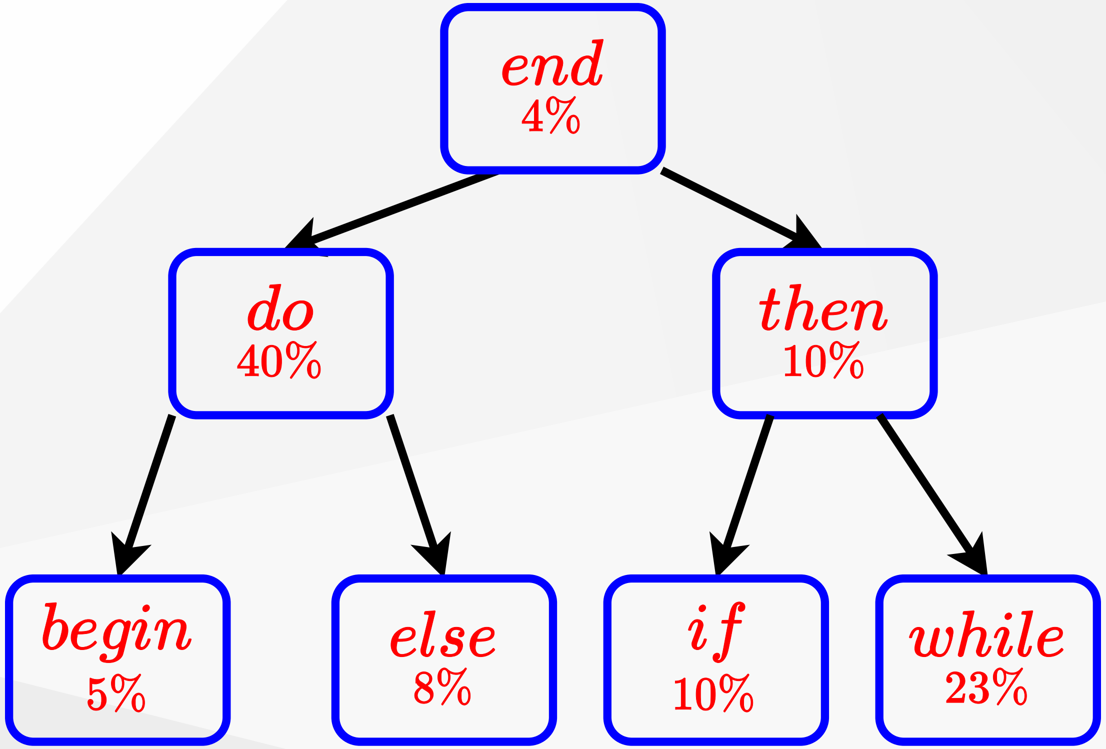
# ASCII Table

| Dec | Hex | Oct | Char | Dec | Hex | Oct | Char | Dec | Hex | Oct | Char | Dec | Hex | Oct | Char |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | | 32 | 20 | 40 | [space] | 64 | 40 | 100 | @ | 96 | 60 | 140 | ` |
| 1 | 1 | 1 | | 33 | 21 | 41 | ! | 65 | 41 | 101 | A | 97 | 61 | 141 | a |
| 2 | 2 | 2 | | 34 | 22 | 42 | " | 66 | 42 | 102 | B | 98 | 62 | 142 | b |
| 3 | 3 | 3 | | 35 | 23 | 43 | # | 67 | 43 | 103 | C | 99 | 63 | 143 | c |
| 4 | 4 | 4 | | 36 | 24 | 44 | $ | 68 | 44 | 104 | D | 100 | 64 | 144 | d |
| 5 | 5 | 5 | | 37 | 25 | 45 | % | 69 | 45 | 105 | E | 101 | 65 | 145 | e |
| 6 | 6 | 6 | | 38 | 26 | 46 | & | 70 | 46 | 106 | F | 102 | 66 | 146 | f |
| 7 | 7 | 7 | | 39 | 27 | 47 | ' | 71 | 47 | 107 | G | 103 | 67 | 147 | g |
| 8 | 8 | 10 | | 40 | 28 | 50 | ( | 72 | 48 | 110 | H | 104 | 68 | 150 | h |
| 9 | 9 | 11 | | 41 | 29 | 51 | ) | 73 | 49 | 111 | I | 105 | 69 | 151 | i |
| 10 | A | 12 | | 42 | 2A | 52 | * | 74 | 4A | 112 | J | 106 | 6A | 152 | j |
| 11 | B | 13 | | 43 | 2B | 53 | + | 75 | 4B | 113 | K | 107 | 6B | 153 | k |
| 12 | C | 14 | | 44 | 2C | 54 | , | 76 | 4C | 114 | L | 108 | 6C | 154 | l |
| 13 | D | 15 | | 45 | 2D | 55 | - | 77 | 4D | 115 | M | 109 | 6D | 155 | m |
| 14 | E | 16 | | 46 | 2E | 56 | . | 78 | 4E | 116 | N | 110 | 6E | 156 | n |
| 15 | F | 17 | | 47 | 2F | 57 | / | 79 | 4F | 117 | O | 111 | 6F | 157 | o |
| 16 | 10 | 20 | | 48 | 30 | 60 | 0 | 80 | 50 | 120 | P | 112 | 70 | 160 | p |
| 17 | 11 | 21 | | 49 | 31 | 61 | 1 | 81 | 51 | 121 | Q | 113 | 71 | 161 | q |
| 18 | 12 | 22 | | 50 | 32 | 62 | 2 | 82 | 52 | 122 | R | 114 | 72 | 162 | r |
| 19 | 13 | 23 | | 51 | 33 | 63 | 3 | 83 | 53 | 123 | S | 115 | 73 | 163 | s |
| 20 | 14 | 24 | | 52 | 34 | 64 | 4 | 84 | 54 | 124 | T | 116 | 74 | 164 | t |
| 21 | 15 | 25 | | 53 | 35 | 65 | 5 | 85 | 55 | 125 | U | 117 | 75 | 165 | u |
| 22 | 16 | 26 | | 54 | 36 | 66 | 6 | 86 | 56 | 126 | V | 118 | 76 | 166 | v |
| 23 | 17 | 27 | | 55 | 37 | 67 | 7 | 87 | 57 | 127 | W | 119 | 77 | 167 | w |
| 24 | 18 | 30 | | 56 | 38 | 70 | 8 | 88 | 58 | 130 | X | 120 | 78 | 170 | x |
| 25 | 19 | 31 | | 57 | 39 | 71 | 9 | 89 | 59 | 131 | Y | 121 | 79 | 171 | y |
| 26 | 1A | 32 | | 58 | 3A | 72 | : | 90 | 5A | 132 | Z | 122 | 7A | 172 | z |
| 27 | 1B | 33 | | 59 | 3B | 73 | ; | 91 | 5B | 133 | [ | 123 | 7B | 173 | { |
| 28 | 1C | 34 | | 60 | 3C | 74 | < | 92 | 5C | 134 | \ | 124 | 7C | 174 | \| |
| 29 | 1D | 35 | | 61 | 3D | 75 | = | 93 | 5D | 135 | ] | 125 | 7D | 175 | } |
| 30 | 1E | 36 | | 62 | 3E | 76 | > | 94 | 5E | 136 | ^ | 126 | 7E | 176 | ~ |
| 31 | 1F | 37 | | 63 | 3F | 77 | ? | 95 | 5F | 137 | _ | 127 | 7F | 177 | |

## Binary Search Tree Example

Suppose we know the frequency of each keyword in texts:

$$\underbrace{begin}_{5\%}, \underbrace{do}_{40\%}, \underbrace{else}_{8\%}, \underbrace{end}_{4\%}, \underbrace{if}_{10\%}, \underbrace{then}_{10\%}, \underbrace{while}_{23\%},$$

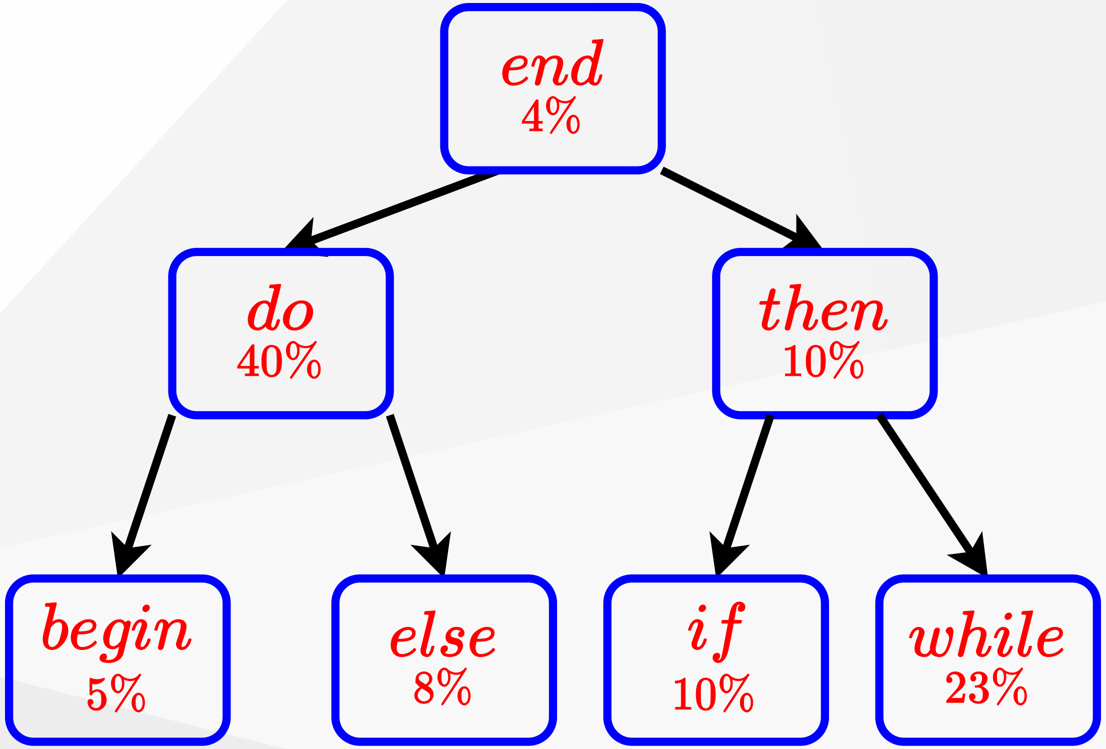


## Cost of a Binary Search Tree

**Example:** If we search for keyword "**while**", we need
to access $3$ nodes. So, $23$ of the queries will have cost of $3$.

$$\text{Total Cost} = \sum_i (\text{depth}(i) + 1)\text{freq}(i)$$

$$= 1 \times 0.04 + 2 \times 0.4 +$$
$$2 \times 0.1 + 3 \times 0.05 +$$
$$3 \times 0.08 + 3 \times 0.1 +$$
$$3 \times 0.23$$
$$= 2.42$$

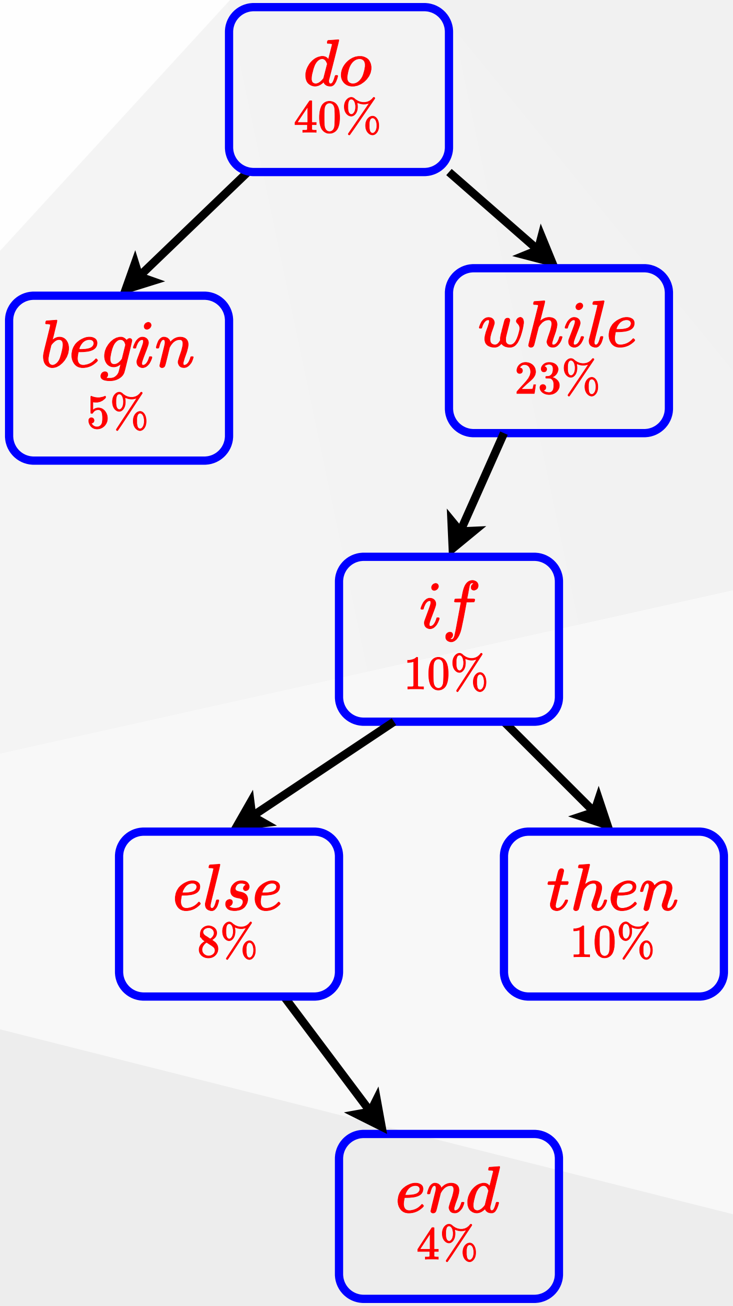


## Cost of a Binary Search Tree

**Example:** If we search for keyword **"while"**, we need to access $3$ nodes. So, $23$ of the queries will have cost of $3$.

$$
\begin{aligned}
\text{Total Cost} &= \sum_i (\text{depth}(i) + 1)\text{freq}(i) \\
&= 1 \times 0.4 + 2 \times 0.05 + 2 \times 0.23 + \\
&\quad 3 \times 0.1 + 4 \times 0.08 + \\
&\quad 4 \times 0.1 + 5 \times 0.04 \\
&= 2.18
\end{aligned}
$$

- This is in fact an optimal BST.



## Optimal Binary Search Tree Problem

- **Given:**
  - A collection of $n$ keys $K_1 < K_2 < \dots K_n$ to be stored in a **BST**.
  - The corresponding $p_i$ values for $1 \leq i \leq n$
    - $p_i$: probability of searching for key $K_i$
- **Find:**
  - An **optimal BST** with minimum total cost:

$$\text{Total Cost} = \sum_i (\text{depth}(i) + 1)\text{freq}(i)$$

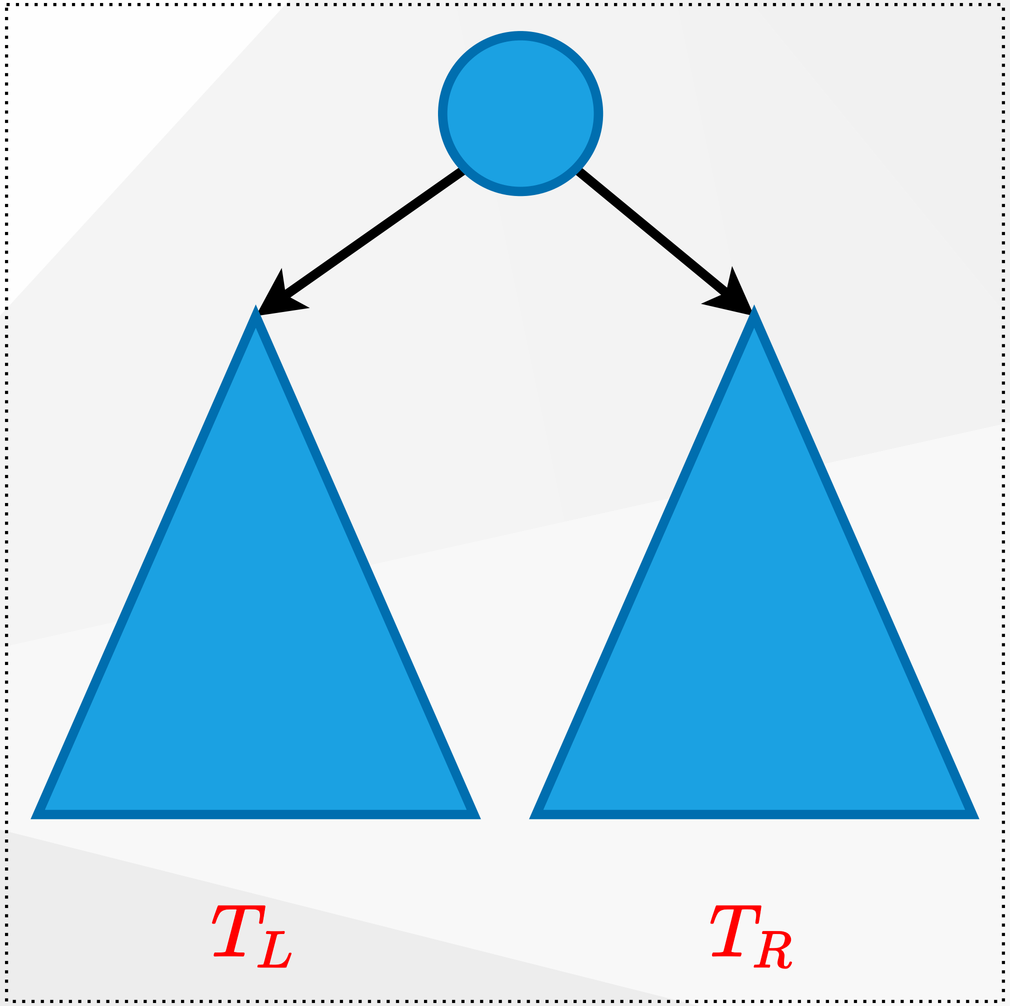
- **Note:** The BST will be static. Only search operations will be performed. No insert, no delete, etc.

## Cost of a Binary Search Tree

- **Lemma 1**: Let $Tij$ be a BST containing keys $K_i < K_{i+1} < \cdots < K_j$. Let $T_L$ and $T_R$ be the left and right subtrees of $T$. Then we have:

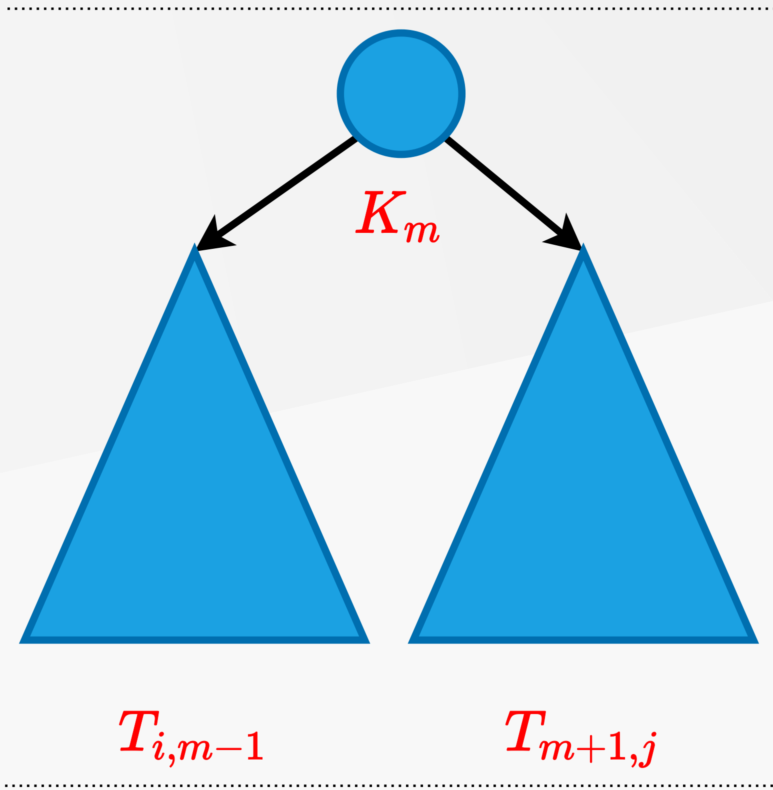
$$\text{cost}(T_{ij}) = \text{cost}(T_L) + \text{cost}(T_R) + \sum_{h=i}^{j} p_h$$

**Intuition:** When we add the root node, the depth of each node in $T_L$ and $T_R$ increases by $1$. So, the cost of node $h$ increases by $p_h$. In addition, the cost of root node $r$ is $p_r$. That's why, we have the last term at the end of the formula above.



# Optimal Substructure Property

- **Lemma 2:** Optimal substructure property
  - Consider an optimal **BST** $T_{ij}$ for keys $K_i < K_{i+1} < \cdots < K_j$
  - Let $K_m$ be the key at the root of $T_{ij}$
- **Then:**
  - $T_{i,m-1}$ is an **optimal BST** for subproblem containing keys:
    - $K_i < \cdots < K_{m-1}$
  - $T_{m+1,j}$ is an **optimal BST** for subproblem containing keys:
    - $K_{m+1} < \cdots < K_j$




$$\text{cost}(T_{ij}) = \text{cost}(T_{i,m-1}) + \text{cost}(T_{m+1,j}) + \sum_{h=i}^{j} p_h$$

## Recursive Formulation

- **Note:** *We don't know which root vertex leads to the minimum total cost. So, we need to try each vertex $m$, and choose the one with minimum total cost.*
- $c[i,j]$: cost of an optimal BST $T_{ij}$ for the subproblem $K_i < \cdots < K_j$

$$c[i,j] = \begin{cases} 0 & \text{if } i > j \\ \min\limits_{i \le r \le j} \{c[i, r-1] + c[r+1, j] + P_{ij}\} & \text{otherwise} \end{cases}$$

$$\text{where } P_{ij} = \sum_{h=i}^{j} p_h$$

## Bottom-up computation

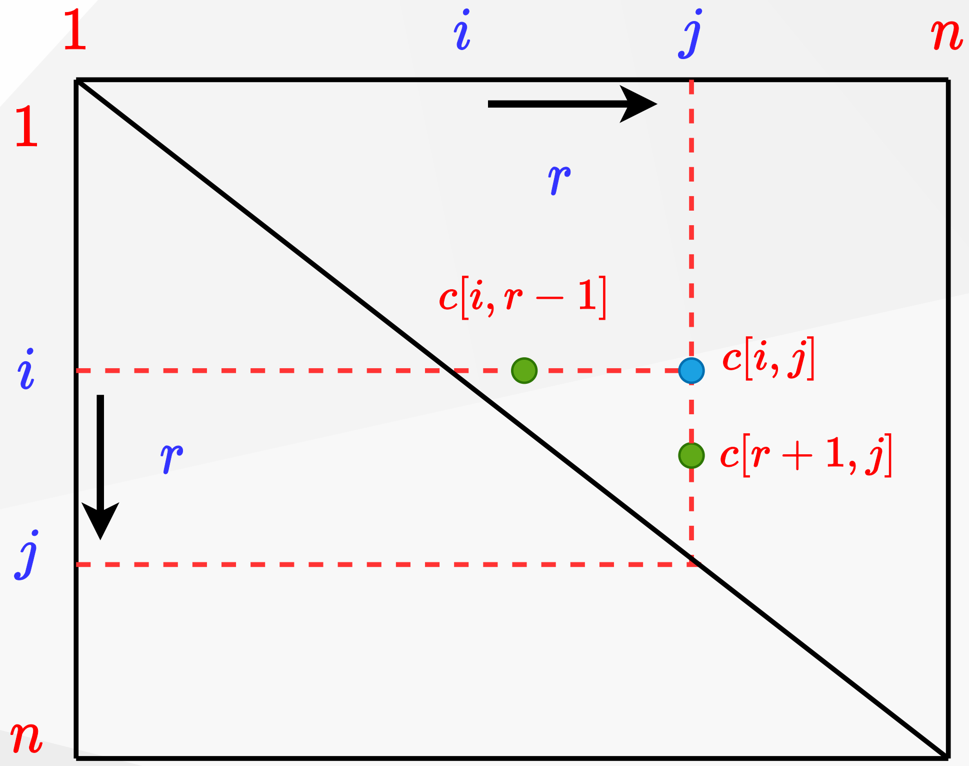
$$c[i,j] = \begin{cases} 0 & \text{if } i > j \\ \min\limits_{i \le r \le j} \{c[i,r-1] + c[r+1,j] + P_{ij}\} & \text{otherwise} \end{cases}$$

- How to choose the order in which we process $c[i,j]$ values?
- Before computing $c[i,j]$, we have to make sure that the values for $c[i,r-1]$ and $c[r+1,j]$ have been computed for all $r$.

## Bottom-up computation

$$c[i,j] = \begin{cases} 0 & \text{if } i > j \\ \min\limits_{i \le r \le j} \{c[i, r-1] + c[r+1, j] + P_{ij}\} & \text{otherwise} \end{cases}$$

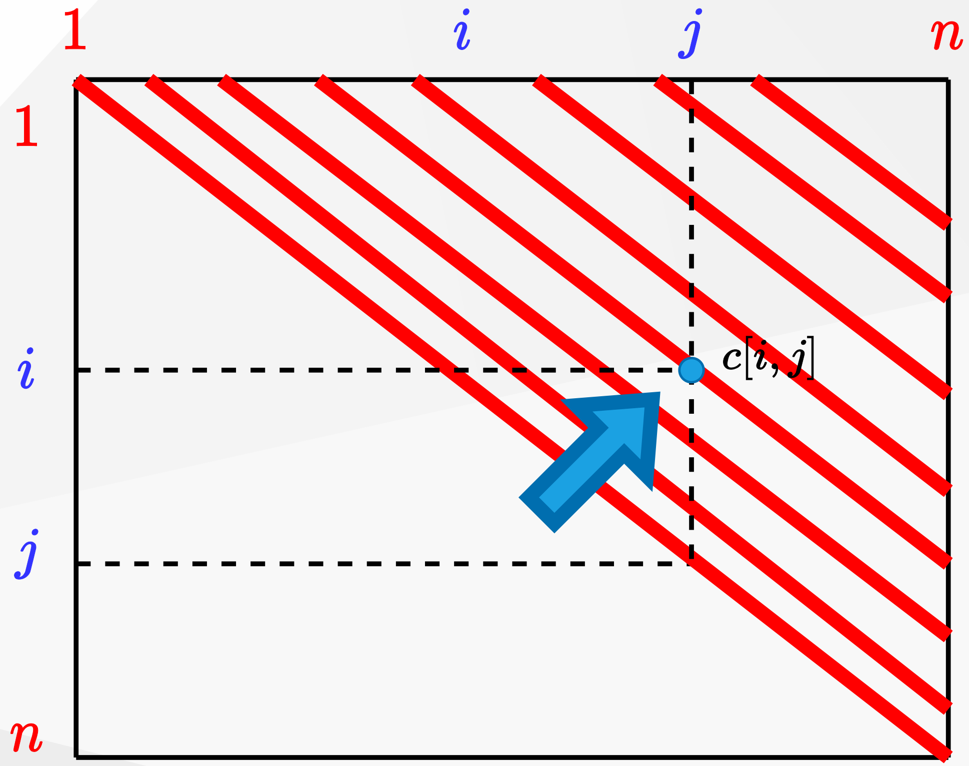
- $c[i,j]$ must be processed after $c[i, r-1]$ and $c[r+1, j]$



## Bottom-up computation

$$c[i,j] = \begin{cases} 0 & \text{if } i > j \\ \min\limits_{i \le r \le j} \{c[i,r-1] + c[r+1,j] + P_{ij}\} & \text{otherwise} \end{cases}$$

- If the entries $c[i,j]$ are computed in the shown order, then $c[i,r-1]$ and $c[r+1,j]$ values are guaranteed to be computed before $c[i,j]$.



# Computing the Optimal BST Cost

OPTIMAL-BST-COST$(p, n)$

for $i \leftarrow 1$ to $n$ do
  $c[i, i-1] \leftarrow 0$
  $c[i, i] \leftarrow p[i]$
  $R[i, j] \leftarrow i$
$PS[1] \leftarrow p[1] \Longleftarrow PS[i] \rightarrow$ prefix-sum $(i)$ : Sum of all $p[j]$ values for $j \leq i$
for $i \leftarrow 2$ to $n$ do
  $PS[i] \leftarrow p[i] + PS[i-1] \Longleftarrow$ compute the prefix sum
for $d \leftarrow 1$ to $n-1$ do $\Longleftarrow$ BSTs with $d+1$ consecutive keys
  for $i \leftarrow 1$ to $n$–$d$ do
    $j \leftarrow i + d$
    $c[i, j] \leftarrow \infty$
    for $r \leftarrow i$ to $j$ do
      $q \leftarrow min\{c[i, r-1] + c[r+1, j]\} + PS[j]–PS[i-1]\}$
      if $q < c[i, j]$ then
        $c[i, j] \leftarrow q$
        $R[i, j] \leftarrow r$
return $c[1, n], R$

## Note on Prefix Sum

- We need $P_{ij}$ values for each $i, j(1 \leq i \leq n \text{ and } 1 \leq j \leq n)$, where:

$$P_{ij} = \sum_{h=i}^{j} p_h$$

- If we compute the summation directly for every $(i, j)$ pair, the runtime would be $\Theta(n^3)$.
- Instead, we spend $O(n)$ time in preprocessing to compute the prefix sum array **PS**. Then we can compute each $P_{ij}$ in $O(1)$ time using **PS**.

## Note on Prefix Sum

- In preprocessing, compute for each $i$:
  - $PS[i]$: the sum of $p[j]$ values for $1 \leq j \leq i$
- Then, we can compute $P_{ij}$ in $O(1)$ time as follows:
  - $P_{ij} = PS[i] – PS[j-1]$
- **Example:**

$$
\begin{array}{rcccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
p: & 0.05 & 0.02 & 0.06 & 0.07 & 0.20 & 0.05 & 0.08 & 0.02 \\
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
PS: & 0.05 & 0.07 & 0.13 & 0.20 & 0.40 & 0.45 & 0.53 & 0.55
\end{array}
$$

$$P_{27} = PS[7] – PS[1] = 0.53 – 0.05 = 0.48$$

$$P_{36} = PS[6] – PS[2] = 0.45 – 0.07 = 0.38$$

# REVIEW

## Overlapping Subproblems Property in Dynamic Programming

Dynamic Programming is an algorithmic paradigm that solves a given complex problem by breaking it into subproblems and stores the results of subproblems to avoid computing the same results again.

## Overlapping Subproblems Property in Dynamic Programming

Following are the two main properties of a problem that suggests that the given problem can be solved using Dynamic programming.

1. Overlapping Subproblems
2. Optimal Substructure

## Overlapping Subproblems

- Like Divide and Conquer, Dynamic Programming combines solutions to sub-problems.
- Dynamic Programming is mainly used when solutions of the same subproblems are needed again and again.
- In dynamic programming, computed solutions to subproblems are stored in a table so that these don't have to be recomputed.
- So Dynamic Programming is not useful when there are no common (overlapping) subproblems because there is no point storing the solutions if they are not needed again.

## Overlapping Subproblems

- For example, Binary Search doesn’t have common subproblems.
- If we take an example of following recursive program for Fibonacci Numbers, there are many subproblems that are solved again and again.

# Simple Recursion

- $f(n) = f(n-1) + f(n-2)$
- C sample code:

```c
#include <stdio.h>
// a simple recursive program to compute fibonacci numbers
int fib(int n)
{
    if (n <= 1)
        return n;
    else
        return fib(n-1) + fib(n-2);
}

int main()
{
    int n = 5;
    printf("Fibonacci number is %d ", fib(n));
    return 0;
}
```

## Simple Recursion

- Output

```
Fibonacci number is 5
```

## Simple Recursion

- $f(n) = f(n-1) + f(n-2)$

```java
/* a simple recursive program for Fibonacci numbers */
public class Fibonacci {
    public static void main(String[] args) {
        int n = Integer.parseInt(args[0]);
        System.out.println(fib(n));
    }

    public static int fib(int n) {
        if (n <= 1)
            return n;

        return fib(n - 1) + fib(n - 2);
    }
}
```

## Simple Recursion

- $f(n) = f(n-1) + f(n-2)$

```
public class Fibonacci {
    public static void Main(string[] args) {
        int n = int.Parse(args[0]);
        Console.WriteLine(fib(n));
    }

    public static int fib(int n) {
        if (n <= 1)
            return n;

        return fib(n - 1) + fib(n - 2);
    }
}
```

## Recursion tree for execution of fib(5)

```
                      fib(5)
                  /              \
            fib(4)                fib(3)
          /       \              /      \
      fib(3)      fib(2)        fib(2)    fib(1)
     /     \        /    \      /    \
  fib(2)   fib(1)  fib(1) fib(0) fib(1) fib(0)
  /    \
fib(1) fib(0)
```

- We can see that the function `fib(3)` is being called 2 times.
- If we would have stored the value of `fib(3)`, then instead of computing it again, we could have reused the old stored value.

## Recursion tree for execution of fib(5)

There are following two different ways to store the values so that these values can be reused:

1. **Memoization (Top Down)**
2. **Tabulation (Bottom Up)**

## Memoization (Top Down)

- The memoized program for a problem is similar to the recursive version with a small modification that looks into a lookup table before computing solutions.
- We initialize a lookup array with all initial values as `NIL`. Whenever we need the solution to a subproblem, we first look into the lookup table.
- If the precomputed value is there then we return that value, otherwise, we calculate the value and put the result in the lookup table so that it can be reused later.

## Memoization (Top Down)

- Following is the memoized version for the nth Fibonacci Number.
- C++ Version:

```
/* C++ program for Memoized version
for nth Fibonacci number */
#include <bits/stdc++.h>
using namespace std;
#define NIL -1
#define MAX 100

int lookup[MAX];
```

## Memoization (Top Down)

- C++ Version:

```
/* Function to initialize NIL
values in lookup table */
void _initialize()
{
    int i;
    for (i = 0; i < MAX; i++)
        lookup[i] = NIL;
}
```

## Memoization (Top Down)

- C++ Version:

```cpp
/* function for nth Fibonacci number */
int fib(int n)
{
    if (lookup[n] == NIL) {
        if (n <= 1)
            lookup[n] = n;
        else
            lookup[n] = fib(n - 1) + fib(n - 2);
    }

    return lookup[n];
}
```

## Memoization (Top Down)

- C++ Version:

```cpp
// Driver code
int main()
{
    int n = 40;
    _initialize();
    cout << "Fibonacci number is " << fib(n);
    return 0;
}
```

## Memoization (Top Down)

- Java Version:

```java
/* Java program for Memoized version */
public class Fibonacci {
    final int MAX = 100;
    final int NIL = -1;

    int lookup[] = new int[MAX];

    /* Function to initialize NIL values in lookup table */
    void _initialize()
    {
        for (int i = 0; i < MAX; i++)
            lookup[i] = NIL;
    }
```

## Memoization (Top Down)

- Java Version:

```
/* function for nth Fibonacci number */
int fib(int n)
{
    if (lookup[n] == NIL) {
        if (n <= 1)
            lookup[n] = n;
        else
            lookup[n] = fib(n - 1) + fib(n - 2);
    }
    return lookup[n];
}
```

## Memoization (Top Down)

- Java Version:

```java
    public static void main(String[] args)
    {
        Fibonacci f = new Fibonacci();
        int n = 40;
        f._initialize();
        System.out.println("Fibonacci number is"
                        + " " + f.fib(n));
    }
}
```

# Memoization (Top Down)

- C# Version:

```
// C# program for Memoized versionof nth Fibonacci number
using System;

class FiboCalcMemoized {

    static int MAX = 100;
    static int NIL = -1;
    static int[] lookup = new int[MAX];

    /* Function to initialize NIL
    values in lookup table */
    static void initialize()
    {
        for (int i = 0; i < MAX; i++)
            lookup[i] = NIL;
    }
```

## Memoization (Top Down)

- C# Version:

```
/* function for nth Fibonacci number */
static int fib(int n)
{
    if (lookup[n] == NIL) {
        if (n <= 1)
            lookup[n] = n;
        else
            lookup[n] = fib(n - 1) + fib(n - 2);
    }
    return lookup[n];
}
```

## Memoization (Top Down)

- C# Version:

```
    // Driver code
    public static void Main()
    {

        int n = 40;
        initialize();
        Console.Write("Fibonacci number is"
                    + " " + fib(n));
    }
}
```

## Tabulation (Bottom Up)

- The tabulated program for a given problem builds a table in bottom-up fashion and returns the last entry from the table.
- For example, for the same Fibonacci number,
  - we first calculate `fib(0)` then `fib(1)` then `fib(2)` then `fib(3)`, and so on. So literally, we are building the solutions of subproblems bottom-up.

## Tabulation (Bottom Up)

- C++ Version:

```
/* C program for Tabulated version */
#include <stdio.h>
int fib(int n)
{
    int f[n + 1];
    int i;
    f[0] = 0;
    f[1] = 1;
    for (i = 2; i <= n; i++)
        f[i] = f[i - 1] + f[i - 2];

    return f[n];
}
```

## Tabulation (Bottom Up)

- C++ Version:

```cpp
...
int main()
{
    int n = 9;
    printf("Fibonacci number is %d ", fib(n));
    return 0;
}
```

Output:

```
Fibonacci number is 34
```

## Tabulation (Bottom Up)

- Java Version:

```java
/* Java program for Tabulated version */
public class Fibonacci {
  public static void main(String[] args)
  {
    int n = 9;
    System.out.println("Fibonacci number is " + fib(n));
  }
```

## Tabulation (Bottom Up)

- Java Version:

```java
  /* Function to calculate nth Fibonacci number */
  static int fib(int n)
  {
    int f[] = new int[n + 1];
    f[0] = 0;
    f[1] = 1;
    for (int i = 2; i <= n; i++)
      f[i] = f[i - 1] + f[i - 2];

    return f[n];
  }
}
```

# Tabulation (Bottom Up)

- C# Version:

```
// C# program for Tabulated version
using System;

class Fibonacci {
    static int fib(int n)
    {
        int[] f = new int[n + 1];
        f[0] = 0;
        f[1] = 1;
        for (int i = 2; i <= n; i++)
            f[i] = f[i - 1] + f[i - 2];
        return f[n];
    }

    public static void Main()
    {
        int n = 9;
        Console.Write("Fibonacci number is"
                      + " " + fib(n));
    }
}
```

- Both Tabulated and Memoized store the solutions of subproblems.
- In Memoized version, the table is filled on demand while in the Tabulated version, starting from the first entry, all entries are filled one by one.
- Unlike the Tabulated version, all entries of the lookup table are not necessarily filled in Memoized version.

- To see the optimization achieved by Memoized and Tabulated solutions over the basic Recursive solution, see the time taken by following runs for calculating the 40th Fibonacci number:
- Recursive Solution:
    - https://ide.geeksforgeeks.org/vHt6ly
- Memoized Solution:
    - https://ide.geeksforgeeks.org/Z94jYR
- Tabulated Solution:
    - https://ide.geeksforgeeks.org/12C5bP

## Optimal Substructure Property in Dynamic Programming

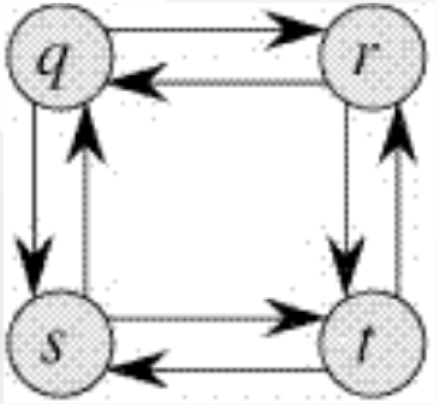
- A given problems has Optimal Substructure Property if optimal solution of the given problem can be obtained by using optimal solutions of its subproblems.
- For example, the Shortest Path problem has following optimal substructure property:
    - If a node x lies in the shortest path from a source node u to destination node v then the shortest path from u to v is combination of shortest path from u to x and shortest path from x to v. The standard All Pair Shortest Path algorithm like Floyd–Warshall and Single Source Shortest path algorithm for negative weight edges like Bellman–Ford are typical examples of Dynamic Programming.

## Optimal Substructure Property in Dynamic Programming

- On the other hand, the Longest Path problem doesn't have the Optimal Substructure property. Here by Longest Path we mean longest simple path (path without cycle) between two nodes

## Optimal Substructure Property in Dynamic Programming

- There are two longest paths from q to t: q→r→t and q→s→t. Unlike shortest paths, these longest paths do not have the optimal substructure property. For example, the longest path q→r→t is not a combination of longest path from q to r and longest path from r to t, because the longest path from q to r is q→s→t→r and the longest path from r to t is r→q→s→t.



# Most Common Dynamic Programming Interview Questions

## Problem-1: Longest Increasing Subsequence

- Problem-1: Longest Increasing Subsequence

## Problem-1: Longest Increasing Subsequence

---

## Problem-2: Edit Distance

- Problem-2: Edit Distance

# Problem-2: Edit Distance (Recursive)

## Problem-2: Edit Distance (DP)

https://www.coursera.org/learn/dna-sequencing

# Problem-2: Edit Distance (DP)

# Problem-2: Edit Distance (Other)

## Problem-3: Partition a set into two subsets such that the difference of subset sums is minimum

- Problem-3: Partition a set into two subsets such that the difference of subset sums is minimum

## Problem-4: Count number of ways to cover a distance

- Problem-4: Count number of ways to cover a distance

## Problem-5: Find the longest path in a matrix with given constraints

- Problem-5: Find the longest path in a matrix with given constraints

## Problem-6: Subset Sum Problem

- Problem-6: Subset Sum Problem

## Problem-7: Optimal Strategy for a Game

- Problem-7: Optimal Strategy for a Game

## Problem-8: 0-1 Knapsack Problem

- Problem-8: 0-1 Knapsack Problem

## Problem-9: Boolean Parenthesization Problem

- Problem-9: Boolean Parenthesization Problem

## Problem-10: Shortest Common Supersequence

- Problem-10: Shortest Common Supersequence

## Problem-11: Partition Problem

- Problem-11: Partition Problem

## Problem-12: Cutting a Rod

- Problem-12: Cutting a Rod

## Problem-13: Coin Change

- Problem-13: Coin Change

## Problem-14: Word Break Problem

- Problem-14: Word Break Problem

## Problem-15: Maximum Product Cutting

- Problem-15: Maximum Product Cutting

## Problem-16: Dice Throw

- Problem-16: Dice Throw

# Problem-16: Dice Throw

## Problem-17: Box Stacking Problem

- Problem-17: Box Stacking Problem

## Problem-18: Egg Dropping Puzzle

- Problem-18: Egg Dropping Puzzle

## References


- Introduction to Algorithms, Third Edition | The MIT Press
    - CLRS
- Bilkent CS473 Course Notes (new)
- Bilkent CS473 Course Notes (old)

$$-End - Of - Week - 6 - Course - Module-$$