CE100 Algorithms and Programming II

Heap/Heap Sort

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## CE100 Algorithms and Programming II

## Week-4 (Heap/Heap Sort)

#### Spring Semester, 2021-2022

Download [DOC](ce100-week-4-heap.md_doc.pdf), [SLIDE](ce100-week-4-heap.md_slide.pdf), [PPTX](ce100-week-4-heap.md_slide.pptx)

## Heap/Heap Sort

## Outline (1)

* Heaps
  + Max / Min Heap
* Heap Data Structure
  + Heapify
    - Iterative
    - Recursive

## Outline (2)

* Extract-Max
* Build Heap

## Outline (3)

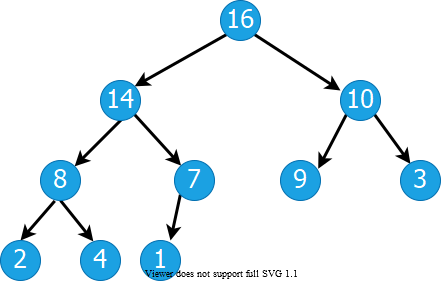
* Heap Sort
* Priority Queues
* Linked Lists
* Radix Sort
* Counting Sort

## Heapsort

* Worst-case runtime:
* Sorts in-place
* Uses a special data structure (heap) to manage information during execution of the algorithm
  + Another design paradigm

## Heap Data Structure (1)

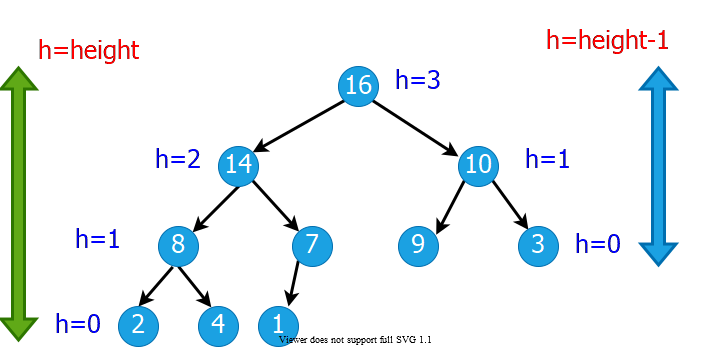
* Nearly complete binary tree
  + Completely filled on all levels except possibly the lowest level



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## Heap Data Structure (2)

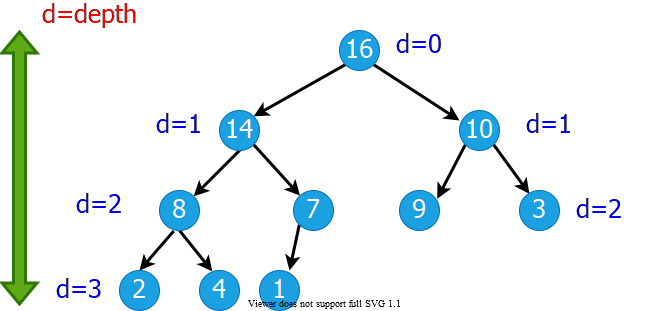
* **Height of node i:** Length of the longest simple downward path from **i** to a **leaf**
* **Height of the tree:** height of the **root**



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## Heap Data Structures (3)

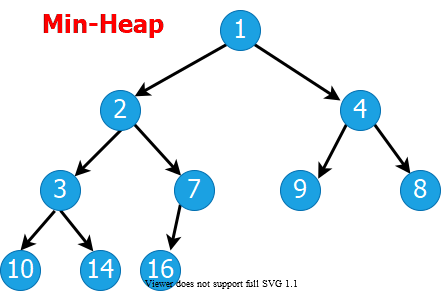
* **Depth of node i:** Length of the simple downward path from the **root** to node **i**



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## Heap Property: Min-Heap

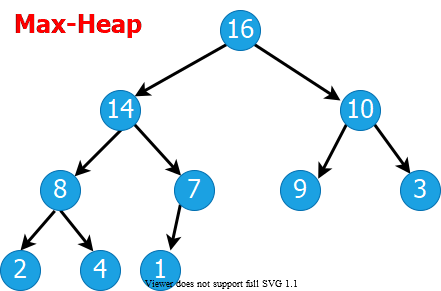
* The **smallest** element in any subtree is the **root** element in a **min-heap**
* **Min heap:** For every node **i** other than **root**,
  + Parent node is always smaller than the child nodes



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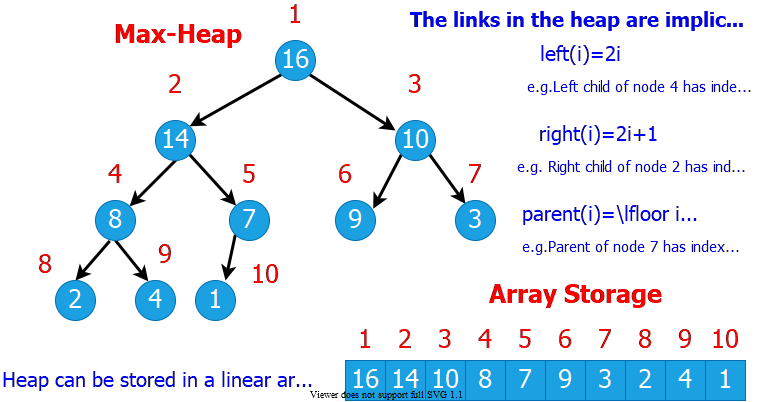
## Heap Property: Max-Heap

* The **largest** element in any subtree is the **root** element in a **max-heap**
  + We will focus on max-heaps
* **Max heap:** For every node **i** other than **root**,
  + Parent node is always larger than the child nodes



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## Heap Data Structures (4)



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## Heap Data Structures (5)

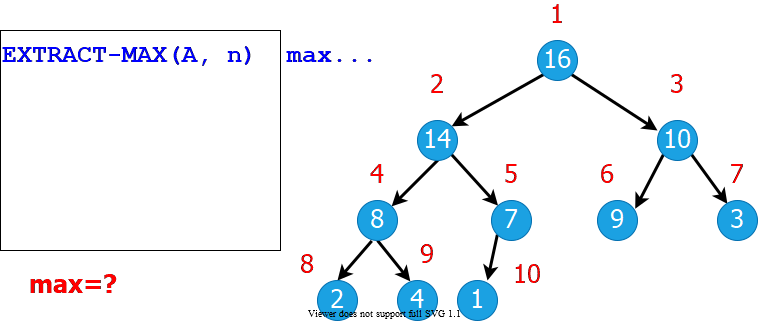
* Computing left child, right child, and parent indices very fast
  + **left(i) = 2i** binary left shift
  + **right(i) = 2i+1** binary left shift, then set the lowest bit to 1
  + **parent(i) = floor(i/2)** right shift in binary
* is always the **root** element
* Array has two attributes:
  + **length(A):** The number of elements in
  + **n = heap-size(A):** The number elements in

## Heap Operations : EXTRACT-MAX (1)

EXTRACT-MAX(A, n)  
 max = A[1]  
 A[1] = A[n]  
 n = n - 1  
 HEAPIFY(A, 1,n)  
 return max

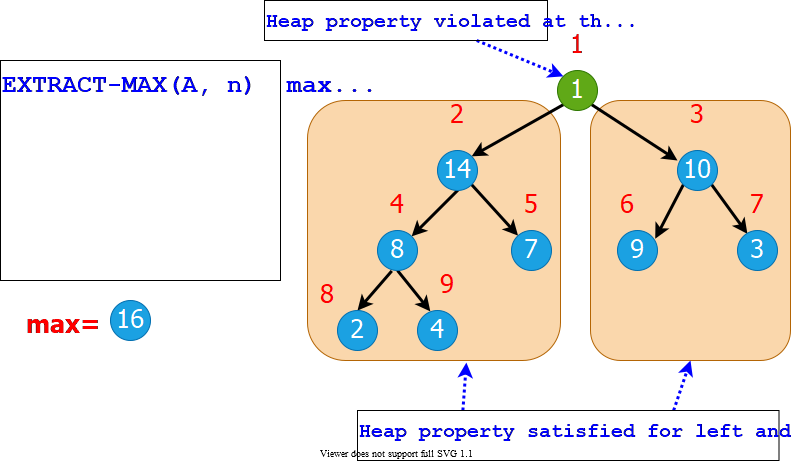
## Heap Operations : EXTRACT-MAX (2)

* Return the max element,and reorganize the heap to maintain heap property



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## Heap Operations: HEAPIFY (1)



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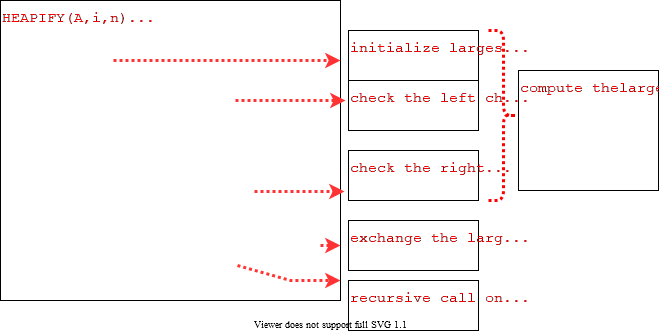
## Heap Operations: HEAPIFY (2)

* Maintaining heap property:
  + Subtrees rooted at and are already heaps.
  + But, may violate the heap property (i.e., may be smaller than its children)
* **Idea:** Float down the value at in the heap so that subtree rooted at becomes a heap.

## Heap Operations: HEAPIFY (2)

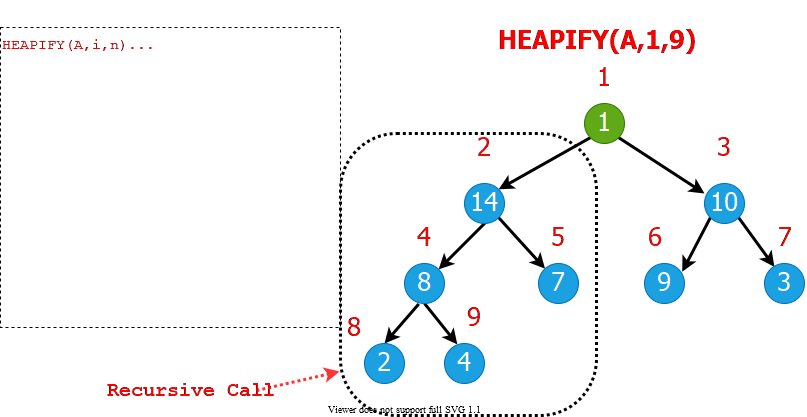
HEAPIFY(A, i, n)  
 largest = i   
  
 if 2i <= n and A[2i] > A[i] then   
 largest = 2i;  
 endif  
  
 if 2i+1 <= n and A[2i+1] > A[largest] then   
 largest = 2i+1;  
 endif  
  
 if largest != i then  
 exchange A[i] with A[largest];  
 HEAPIFY(A, largest, n);  
 endif

## Heap Operations: HEAPIFY (3)



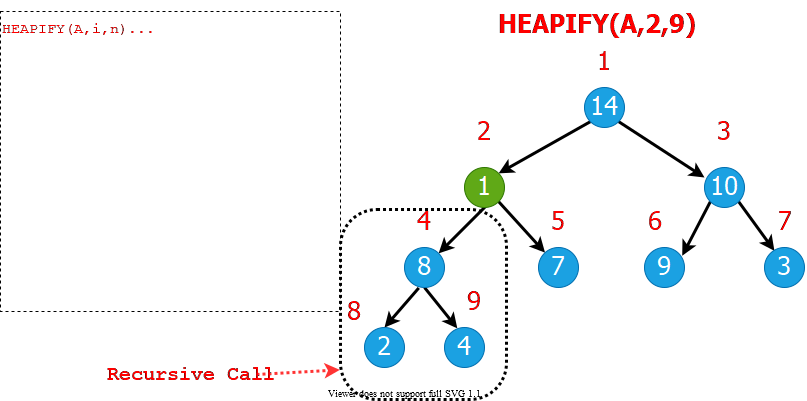
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## Heap Operations: HEAPIFY (4)



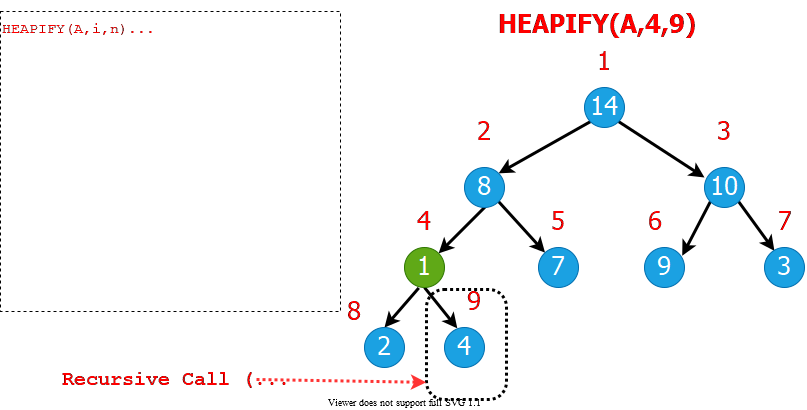
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## Heap Operations: HEAPIFY (5)



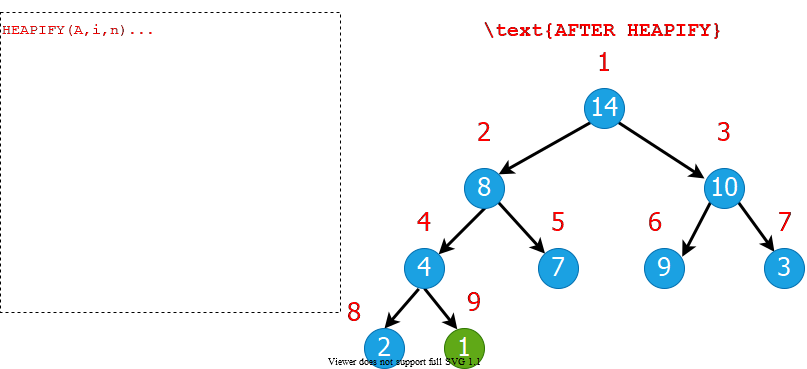
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## Heap Operations: HEAPIFY (6)



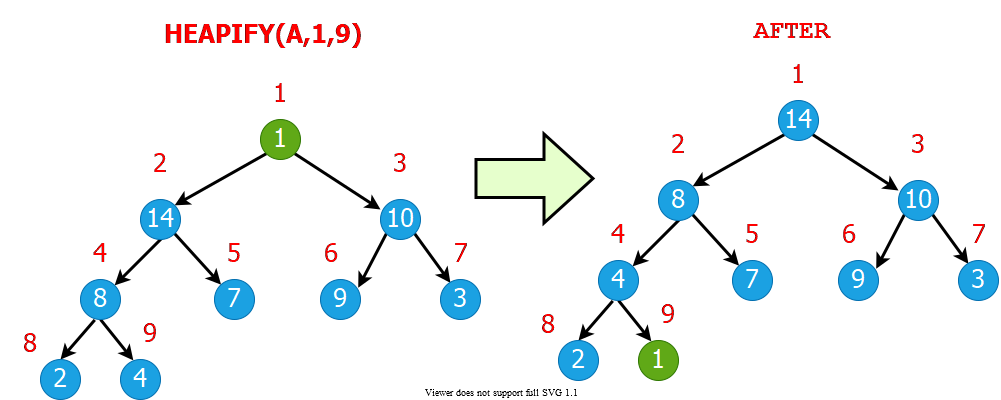
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## Heap Operations: HEAPIFY (7)



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## Heap Operations: HEAPIFY (8)



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## Intuitive Analysis of HEAPIFY

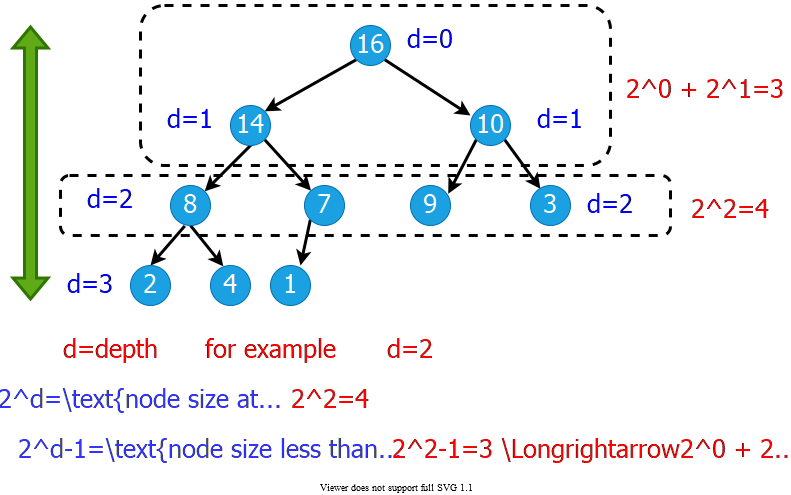
* Consider
  + let be the height of node
  + at most recursion levels
    - Constant work at each level:
  + Therefore
* Heap is almost-complete binary tree
* Thus

## Formal Analysis of HEAPIFY

* **What is the recurrence?**
  + Depends on the size of the **subtree** on which recursive call is made
    - In the next, we try to compute an **upper bound** for this **subtree**.

## Reminder: Binary trees

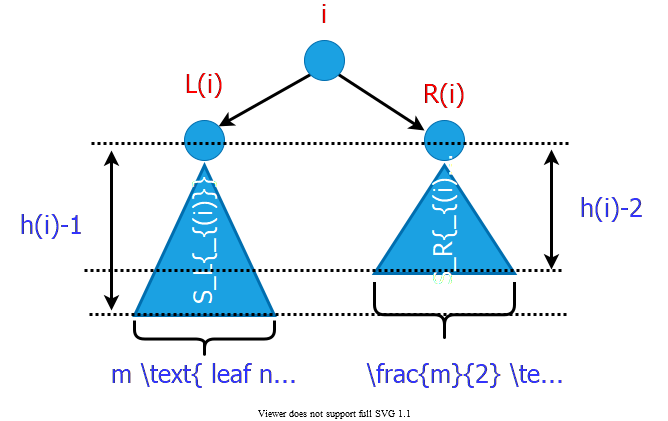
* For a complete binary tree:
  + of nodes at depth :
  + of nodes with depths less than :



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## Formal Analysis of HEAPIFY (1)

* Worst case occurs when last row of the subtree rooted at node is **half full**
* and are complete binary trees of heights and , respectively



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## Formal Analysis of HEAPIFY (2)

* Let be the number of **leaf nodes** in

## Formal Analysis of HEAPIFY (2)

* **By CASE-2 of Master Theorem**

## Formal Analysis of HEAPIFY (2)

* Recurrence:
* *Case 2:*
* i.e., and grow at similar rates
* **Solution:**
  + (drop constants.)

## HEAPIFY: Efficiency Issues

* **Recursion vs Iteration:**
  + In the absence of tail recursion, **iterative version** is in general **more efficient** because of the **pop/push** operations **to/from** stack at each **level of recursion**.

## Heap Operations: HEAPIFY (1)

**Recursive**

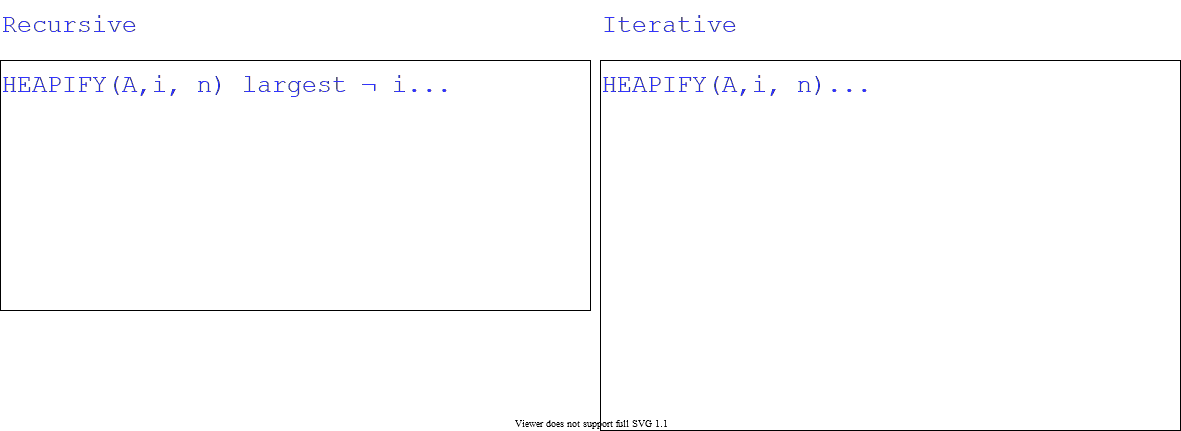
HEAPIFY(A, i, n)  
 largest = i   
  
 if 2i <= n and A[2i] > A[i] then   
 largest = 2i  
  
 if 2i+1 <= n and A[2i+1] > A[largest] then   
 largest = 2i+1  
  
 if largest != i then  
 exchange A[i] with A[largest]  
 HEAPIFY(A, largest, n)

## Heap Operations: HEAPIFY (2)

**Iterative**

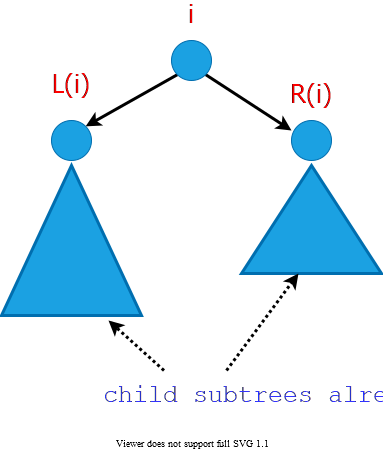
HEAPIFY(A, i, n)  
 j = i  
 while(true) do  
 largest = j   
  
 if 2j <= n and A[2j] > A[j] then   
 largest = 2j  
  
 if 2j+1 <= n and A[2j+1] > A[largest] then   
 largest = 2j+1  
  
 if largest != j then  
 exchange A[j] with A[largest]  
 j = largest  
 else return

## Heap Operations: HEAPIFY (3)



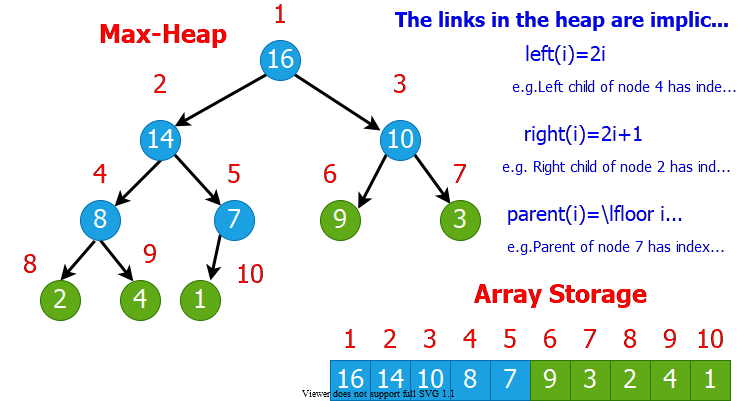
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## Heap Operations: Building Heap

* Given an arbitrary array, how to build a heap from scratch?
* **Basic idea:** Call on each node bottom up
  + Start from the leaves (which trivially satisfy the heap property)
  + Process nodes in bottom up order.
  + When is called on node , the subtrees connected to the and subtrees already satisfy the heap property.
* 
* bg right:25% w:300px

## Storage of the leaves (Lemma)

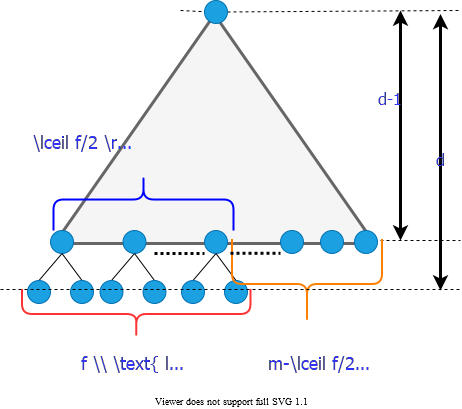
* **Lemma:** The last nodes of a heap are all leaves.



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## Storage of the leaves (Proof of Lemma) (1)

* **Lemma:** last nodes of a heap are all leaves
* Proof :
  + : nodes at level
  + : nodes at level (last level)
* of nodes with depth :
* of nodes with depth :
* of nodes with depth :
* **Total** of nodes :

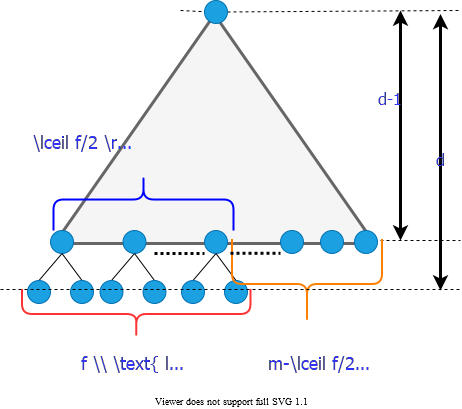


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## Storage of the leaves (Proof of Lemma) (2)

* **Total** of nodes :

Proof is Completed



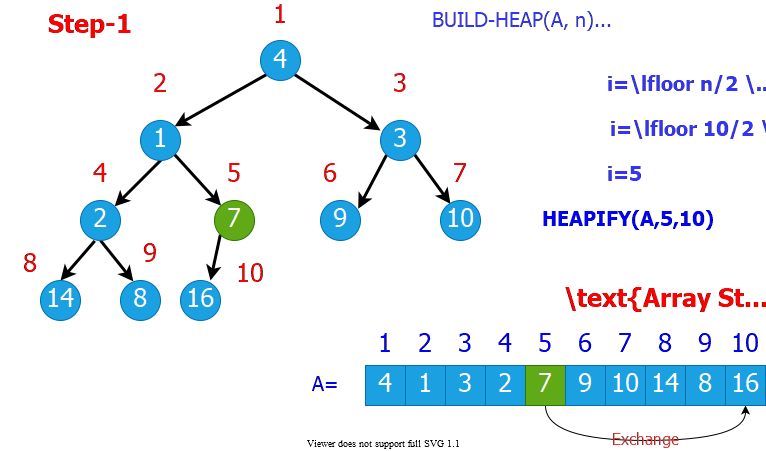
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## Heap Operations: Building Heap

BUILD-HEAP (A, n)  
 for i = ceil(n/2) downto 1 do  
 HEAPIFY(A, i, n)

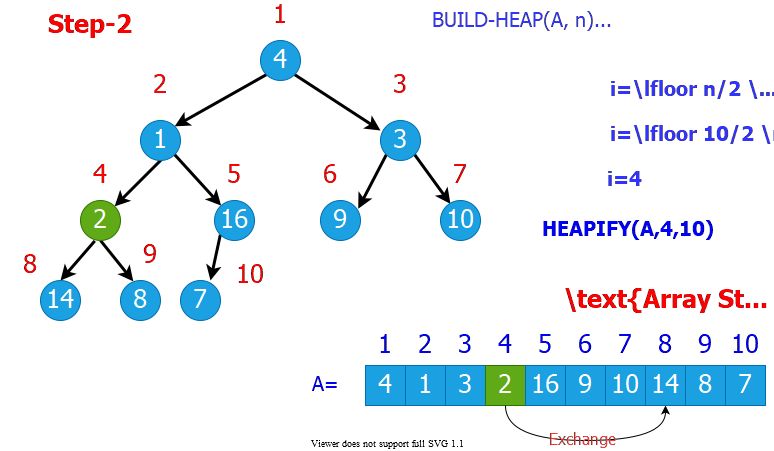
* **Reminder:** The last nodes of a heap are **all leaves**, which trivially satisfy the heap property

## Build-Heap Example (Step-1)



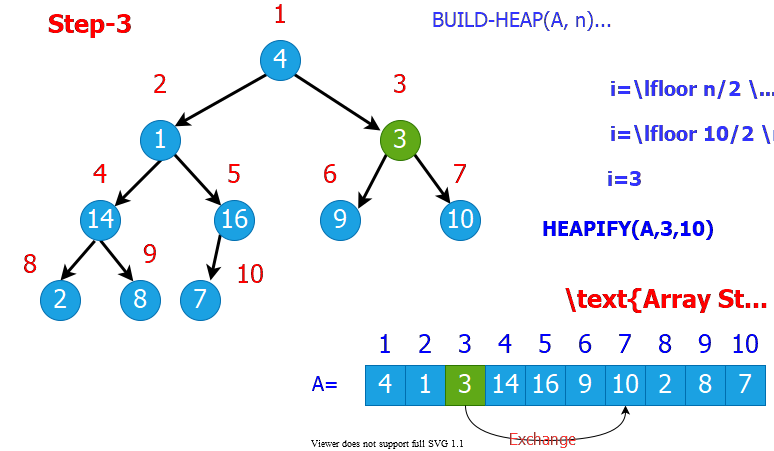
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## Build-Heap Example (Step-2)



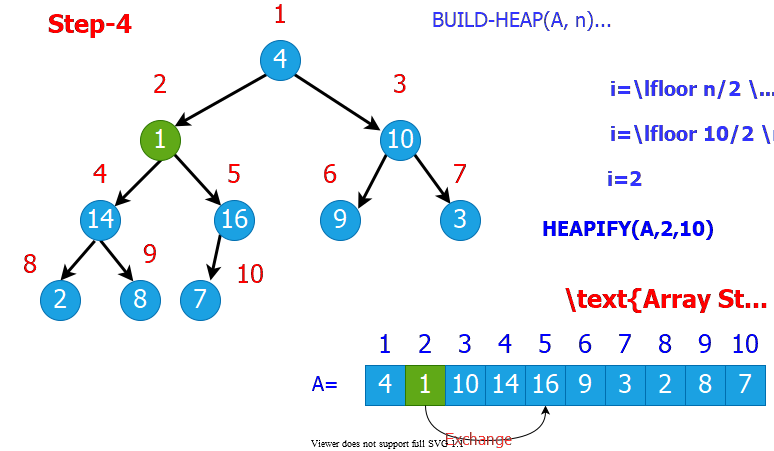
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## Build-Heap Example (Step-3)



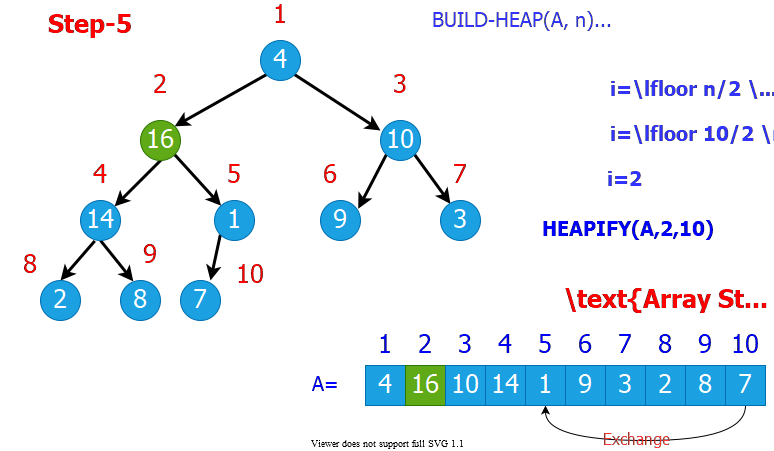
bg right:70% w:800px

## Build-Heap Example (Step-4)



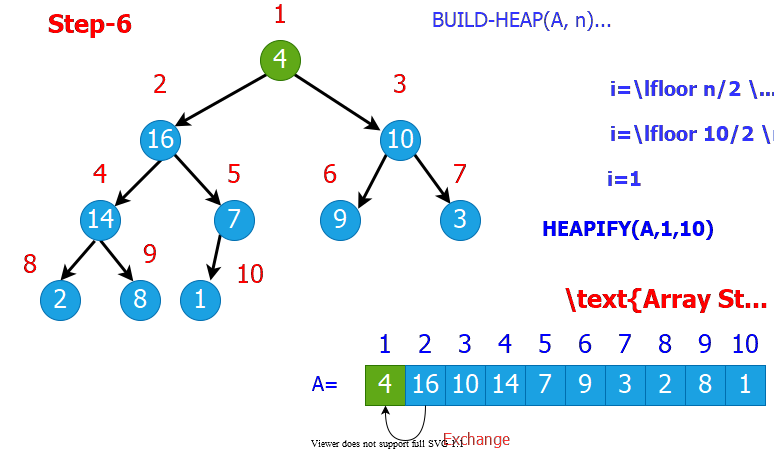
bg right:70% w:800px

## Build-Heap Example (Step-5)



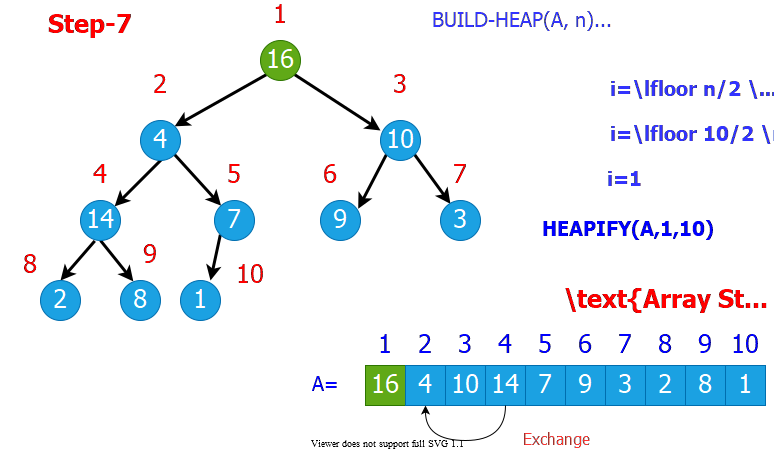
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## Build-Heap Example (Step-6)



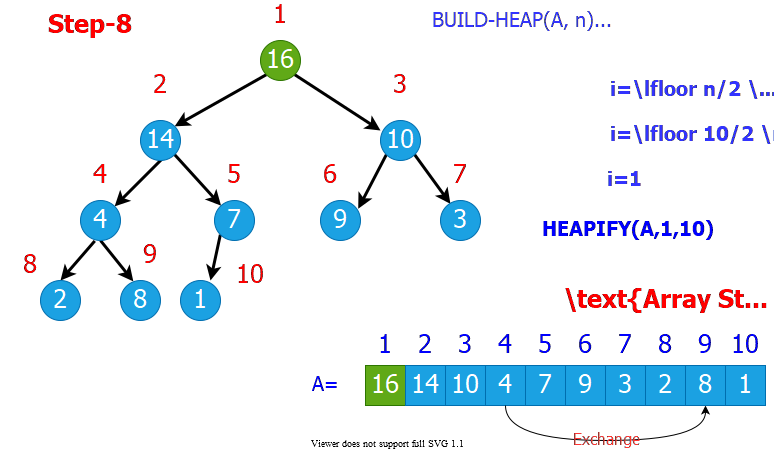
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## Build-Heap Example (Step-7)



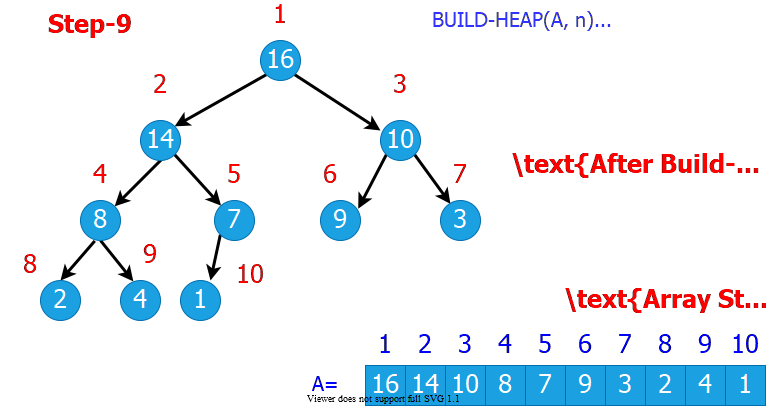
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## Build-Heap Example (Step-8)



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## Build-Heap Example (Step-9)



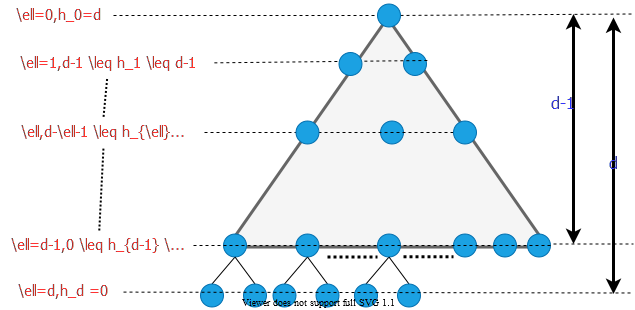
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## Build-Heap: Runtime Analysis

* Simple analysis:
  + calls to , each of which takes time
  + loose bound
* In general, a good approach:
  + Start by proving an easy bound
  + Then, try to tighten it
* Is there a tighter bound?

## Build-Heap: **Tighter** Running Time Analysis

* If the heap is complete binary tree then
* Otherwise, nodes at a given level do not all have the same height, But we have



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## Build-Heap: **Tighter** Running Time Analysis

* Assume that all nodes at level are processed

## Build-Heap: **Tighter** Running Time Analysis

* recall infinite decreasing geometric series
* differentiate both sides

## Build-Heap: **Tighter** Running Time Analysis

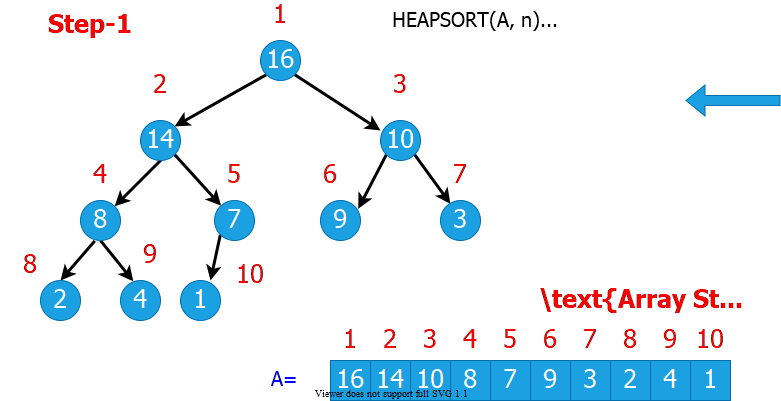
* then, multiply both sides by
* in our case: and

$$
\therefore \sum \limits\_{h=0}^{\infty}h(1/2)^h = \frac{1/2}{(1-(1/2))^2}=2=O(1) \\
\therefore T(n)=O(n\sum \limits\_{h=1}^{d}h(1/2)^h)=O(n)
$$

## Heapsort Algorithm Steps

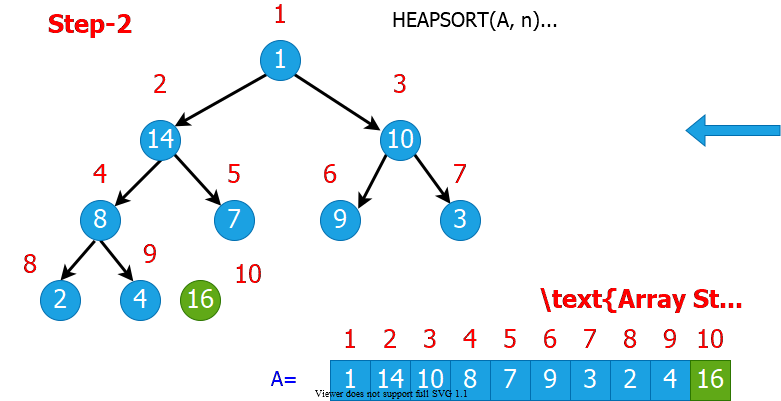
* **(1)** Build a heap on array by calling
* **(2)** The largest element is stored at the root
  + Put it into its correct final position by
* **(3)** Discard node from the heap
* **(4)** Subtrees rooted at children of root remain as heaps, but the new root element may violate the heap property.
  + Make a heap by calling
* **(5)**
* **(6)** Repeat steps **(2-4)** until

## Heapsort Algorithm Example (Step-1)



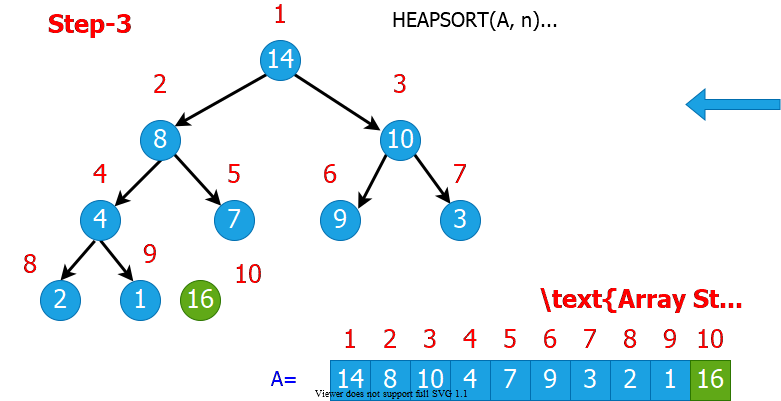
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## Heapsort Algorithm Example (Step-2)



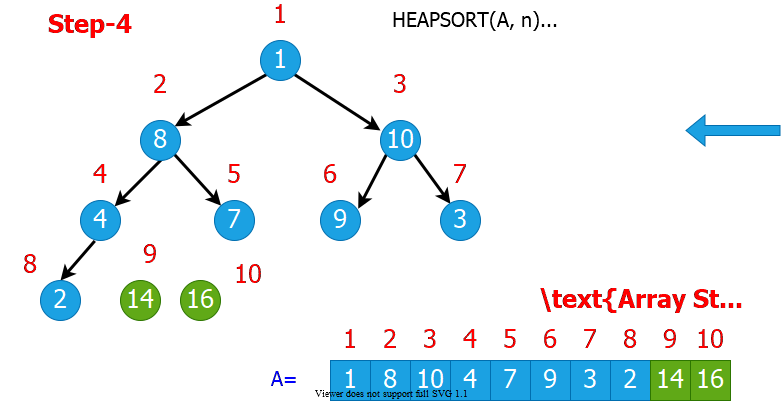
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## Heapsort Algorithm Example (Step-3)



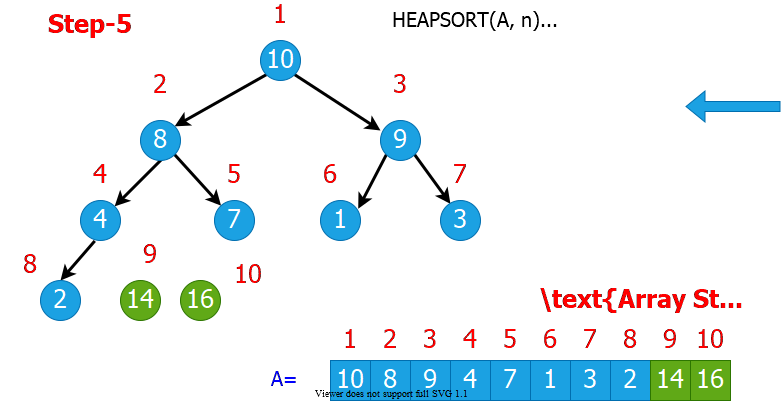
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## Heapsort Algorithm Example (Step-4)



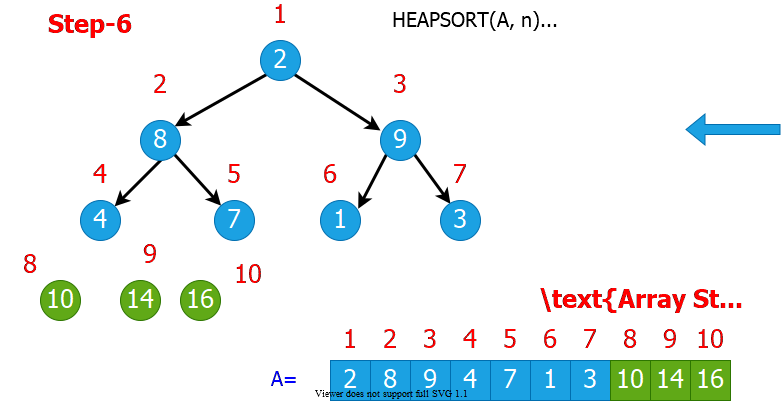
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## Heapsort Algorithm Example (Step-5)



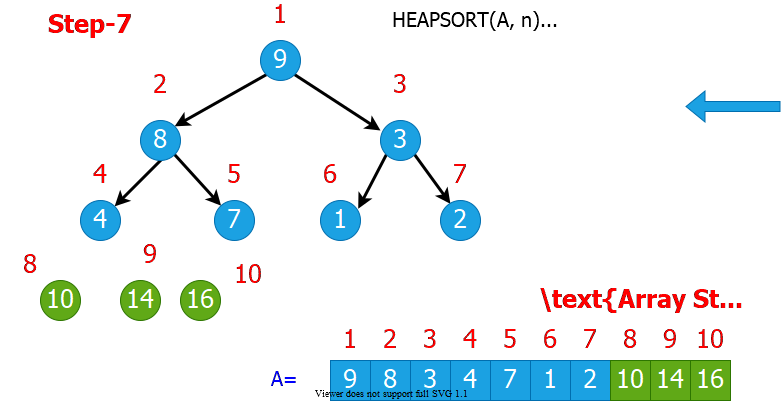
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## Heapsort Algorithm Example (Step-6)



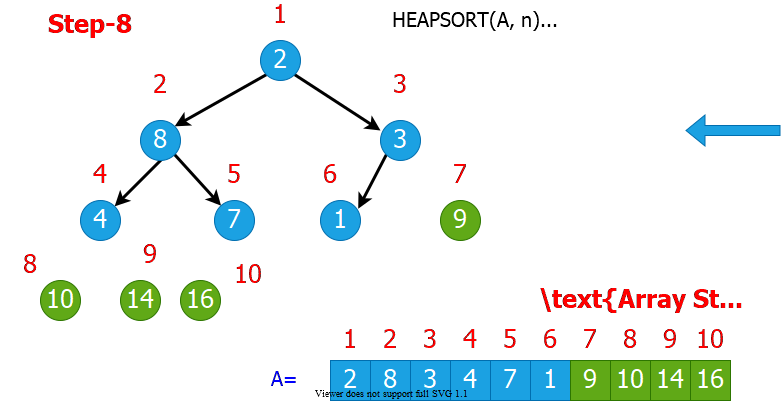
bg right:70% w:800px

## Heapsort Algorithm Example (Step-7)



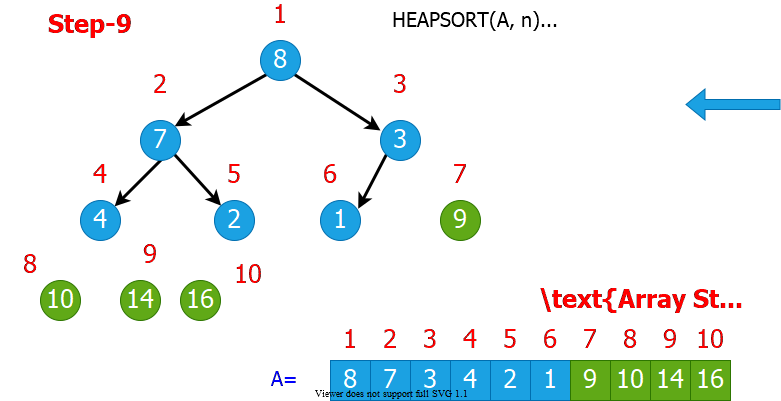
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## Heapsort Algorithm Example (Step-8)



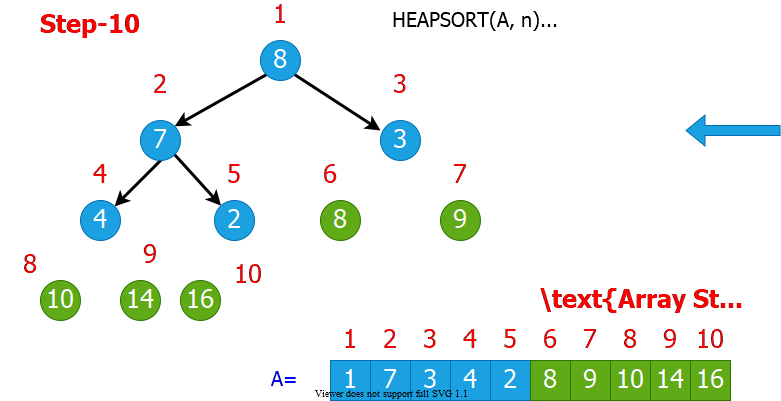
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## Heapsort Algorithm Example (Step-9)



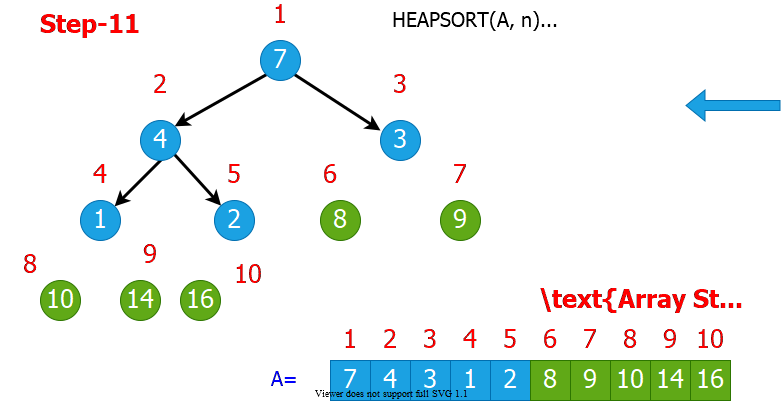
bg right:70% w:800px

## Heapsort Algorithm Example (Step-10)



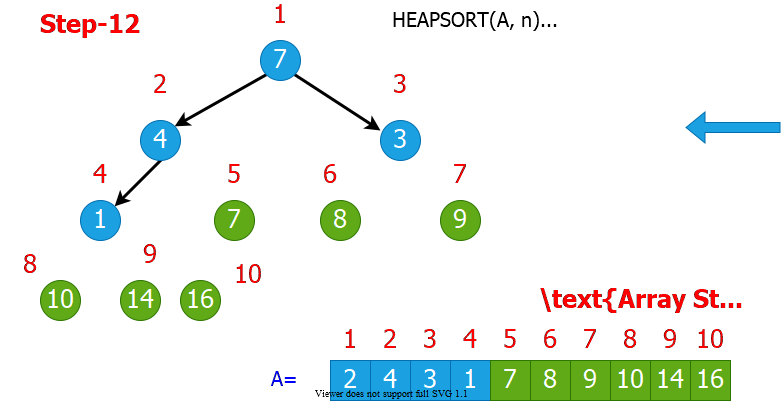
bg right:70% w:800px

## Heapsort Algorithm Example (Step-11)



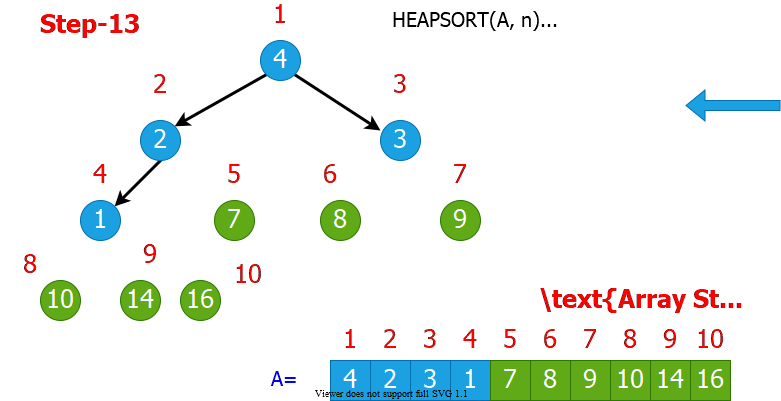
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## Heapsort Algorithm Example (Step-12)



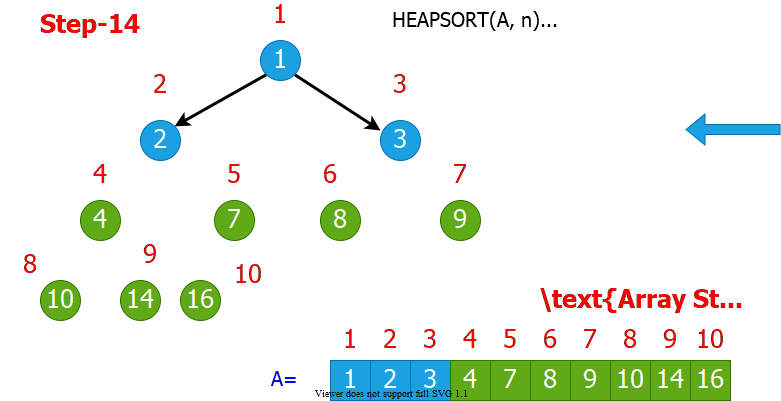
bg right:70% w:800px

## Heapsort Algorithm Example (Step-13)



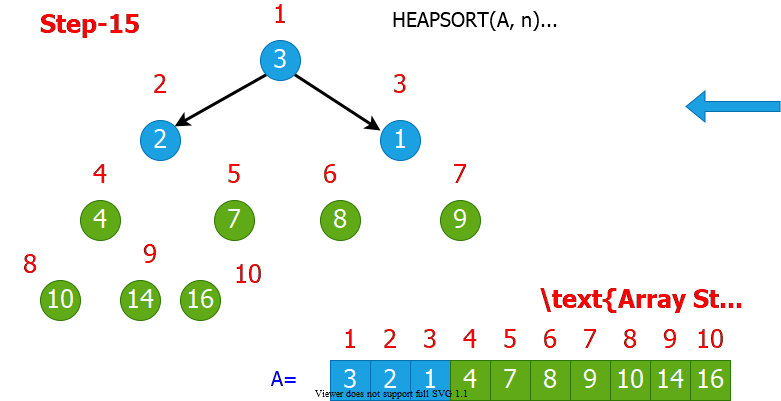
bg right:70% w:800px

## Heapsort Algorithm Example (Step-14)



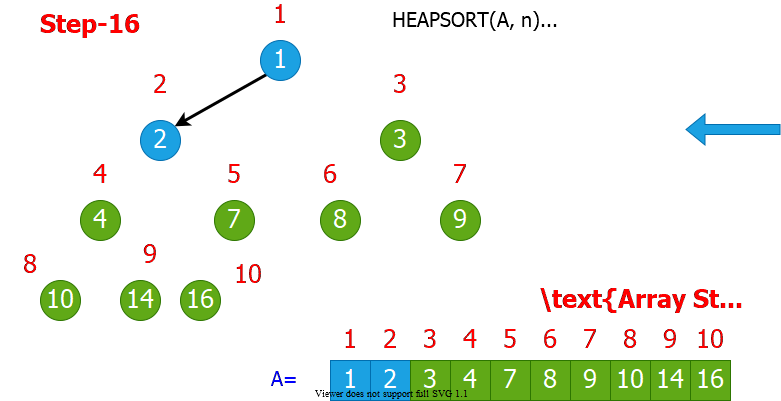
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## Heapsort Algorithm Example (Step-15)



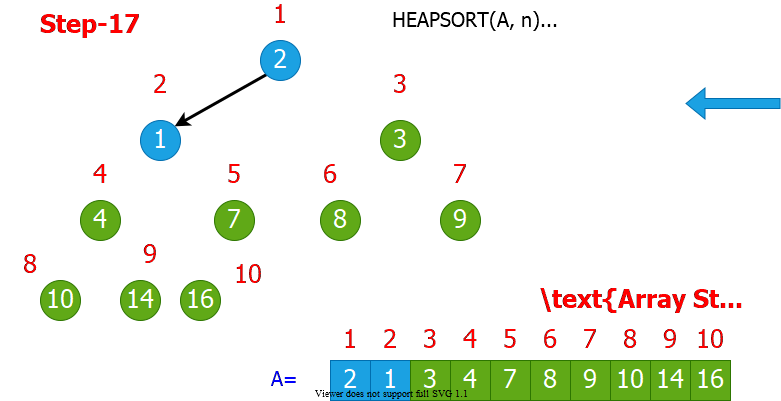
bg right:70% w:800px

## Heapsort Algorithm Example (Step-16)



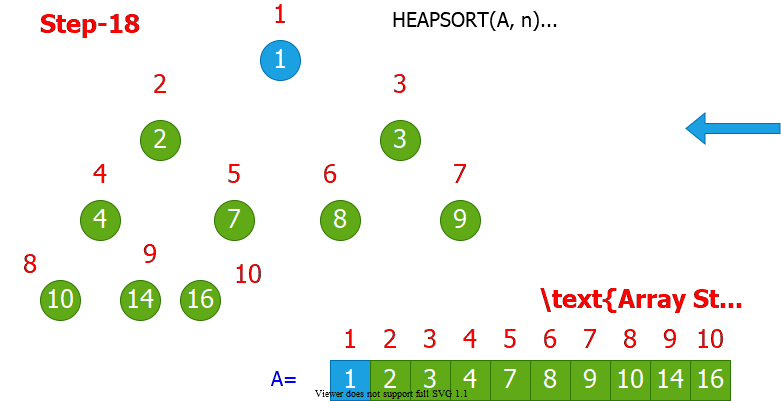
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## Heapsort Algorithm Example (Step-17)



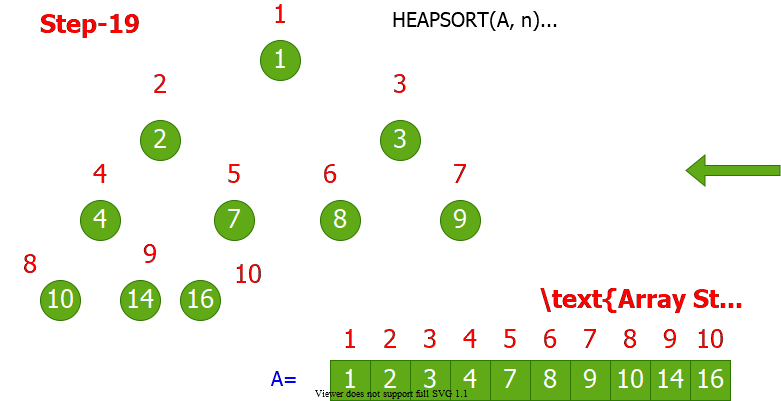
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## Heapsort Algorithm Example (Step-18)



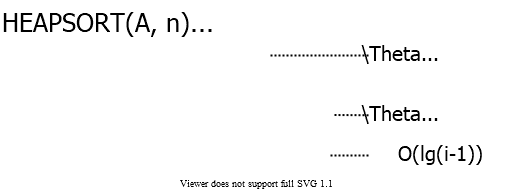
bg right:70% w:800px

## Heapsort Algorithm Example (Step-19)



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## Heapsort Algorithm: Runtime Analysis



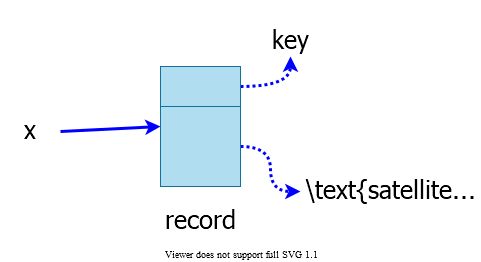
center height:250px

## Heapsort - Notes

* **Heapsort** is a very good algorithm but, a good implementation of **quicksort** always **beats** heapsort **in practice**
* However, **heap data structure** has many popular applications, and it can be efficiently used for implementing **priority queues**

## Data structures for **Dynamic Sets**

* Consider sets of records having **key** and **satellite** data



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## Operations on **Dynamic Sets**

* **Queries:** Simply return info;
  + (Query) return with the **largest/smallest**
  + (Query) return with
  + (Query) return which is the next **larger/smaller** element after
* **Modifying operations:** Change the set
  + (Modifying)
  + (Modifying)
  + (Modifying) return and delete with the largest/smallest
* Different data structures support/optimize different operations

## Priority Queues (PQ)

* Supports

## Priority Queues (PQ)

* **One application:** Schedule jobs on a shared resource
  + **PQ** keeps track of jobs and their relative priorities
  + When a job is finished or interrupted, highest priority job is selected from those pending using
  + A new job can be added at any time using

## Priority Queues (PQ)

* **Another application:** Event-driven simulation
  + Events to be simulated are the items in the **PQ**
  + Each event is associated with a time of occurrence which serves as a
  + Simulation of an event can cause other events to be simulated in the future
  + Use at each step to choose the next event to simulate
  + As new events are produced insert them into the **PQ** using

## Implementation of **Priority Queue**

* **Sorted linked list:** Simplest implementation
  + - time
    - Scan the list to find place and splice in the new item
    - time
    - Take the first element
  + **Fast** extraction but **slow** insertion.

## Implementation of **Priority Queue**

* **Unsorted linked list:** Simplest implementation
  + - time
    - Put the new item at front
    - time
    - Scan the whole list
  + **Fast** insertion but **slow** extraction.
* Sorted linked list is better on the average
  + **Sorted list:** on the average, scans element per insertion
  + **Unsorted list:** always scans element at each extraction

## Heap Implementation of **PQ**

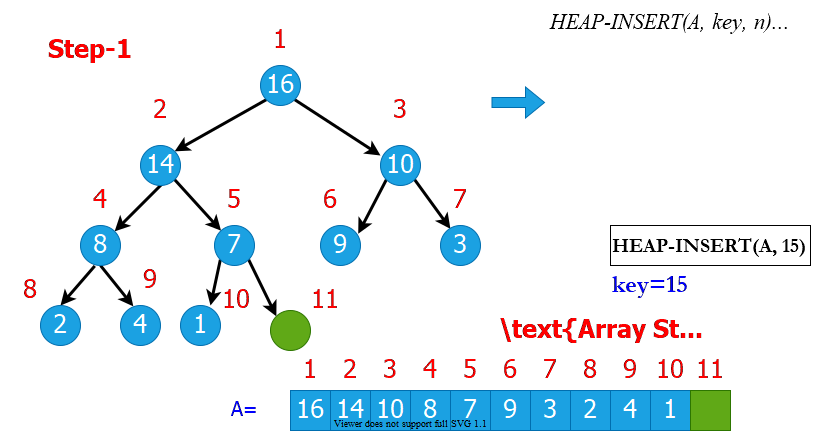
* and are both
  + good compromise between fast insertion but slow extraction and vice versa
* : already discussed
* : Insertion is like that of Insertion-Sort.

HEAP-INSERT(A, key, n)  
 n = n+1  
 i=n   
 while i>1 and A[floor(i/2)] < key do  
 A[i]=A[floor(i/2)]   
 i= floor(i/2)  
 A[i]=key

## Heap Implementation of **PQ**

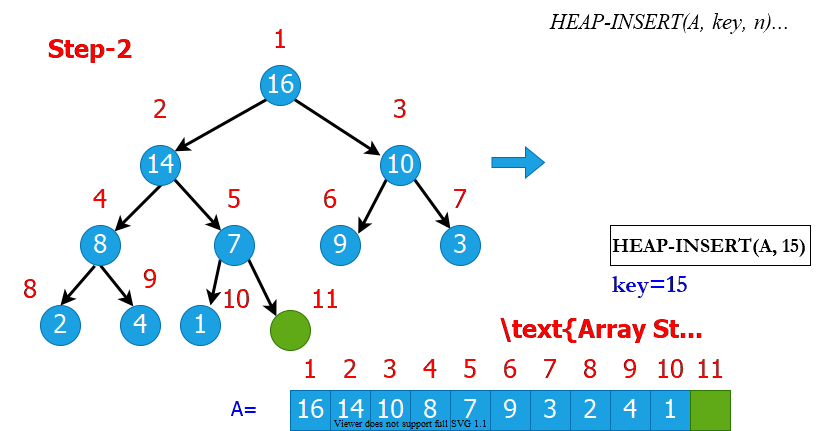
* Traverses nodes, as does but makes fewer comparisons and assignments
  + : compares parent with both children
  + : with only one

## HEAP-INSERT Example (Step-1)



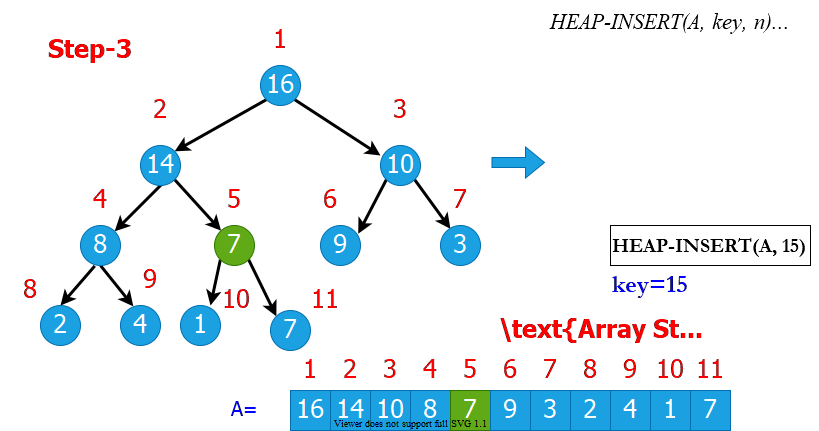
bg right:70% w:800px

## HEAP-INSERT Example (Step-2)



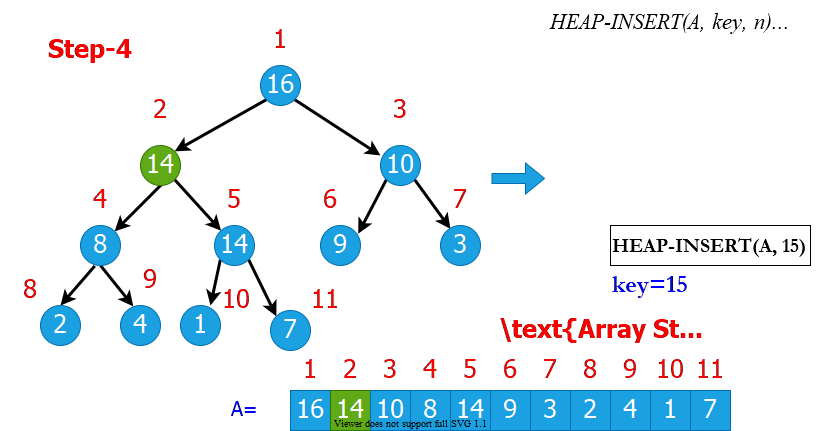
bg right:70% w:800px

## HEAP-INSERT Example (Step-3)



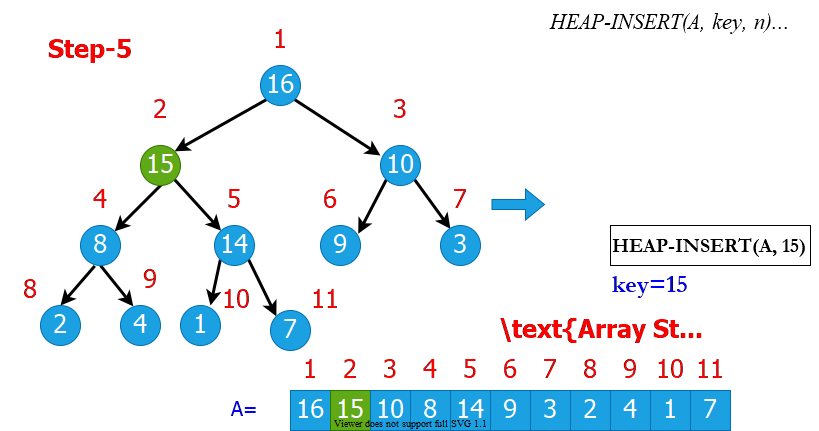
bg right:70% w:800px

## HEAP-INSERT Example (Step-4)



bg right:70% w:800px

## HEAP-INSERT Example (Step-5)



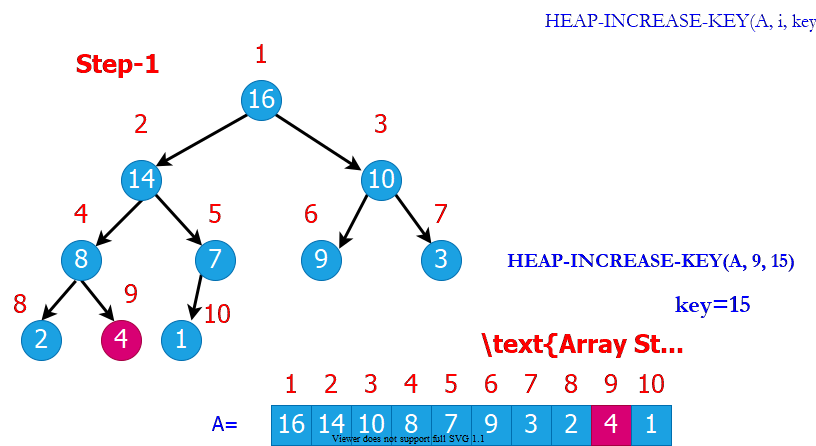
bg right:70% w:800px

## Heap Increase Key

* Key value of element of heap is increased from to

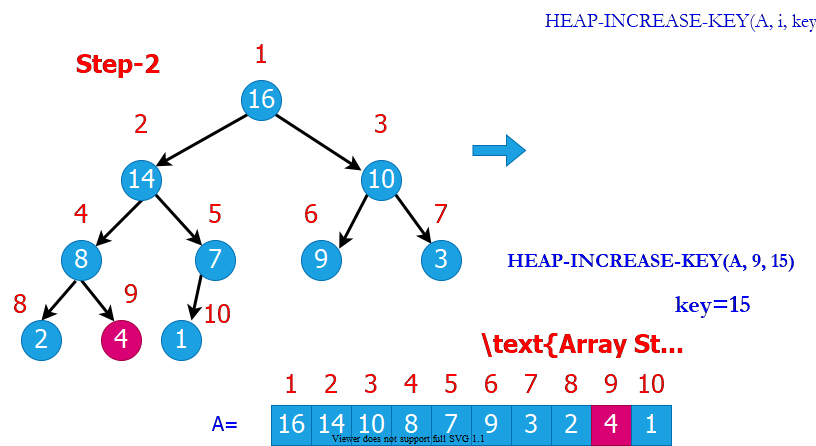
HEAP-INCREASE-KEY(A, i, key)  
  
 if key < A[i] then  
 return error  
  
 while i > 1 and A[floor(i/2)] < key do  
 A[i] = A[floor(i/2)]   
 i = floor(i/2)  
  
 A[i] = key

## HEAP-INCREASE-KEY Example (Step-1)



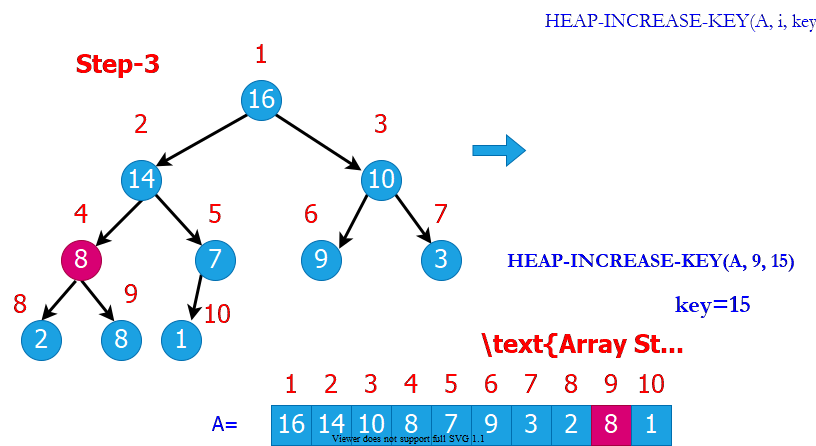
bg right:70% w:800px

## HEAP-INCREASE-KEY Example (Step-2)



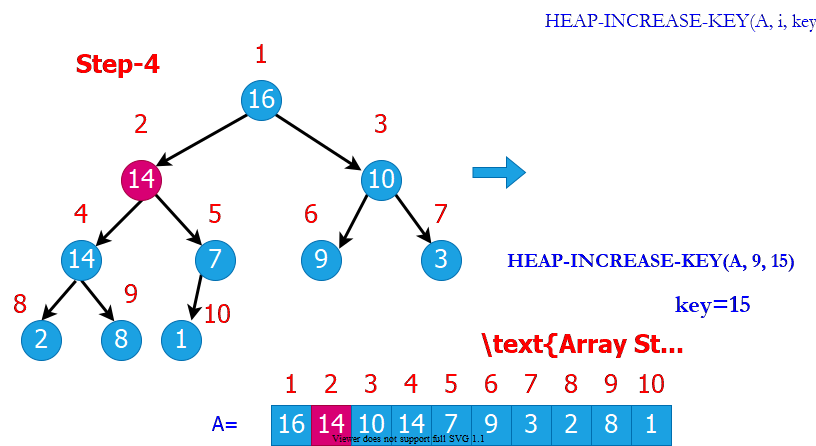
bg right:70% w:800px

## HEAP-INCREASE-KEY Example (Step-3)



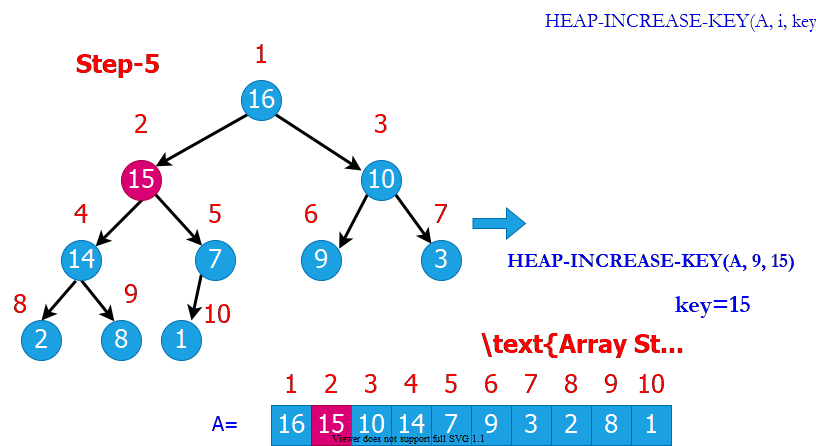
bg right:70% w:800px

## HEAP-INCREASE-KEY Example (Step-4)



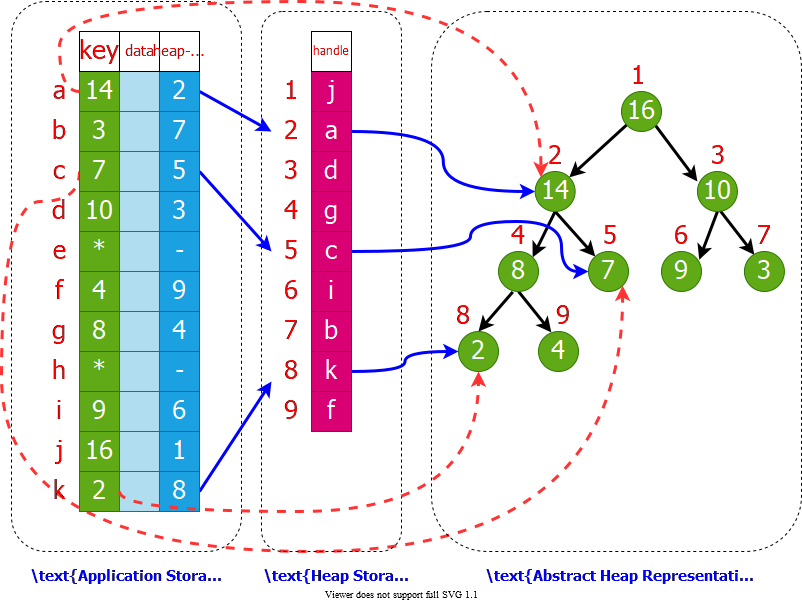
bg right:70% w:800px

## HEAP-INCREASE-KEY Example (Step-5)



bg right:70% w:800px

## Heap Implementation of Priority Queue (PQ)



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## Summary: **Max Heap**

* **Heapify(A, i)**
  + Works when both child subtrees of node i are heaps
  + “*Floats down*” node i to satisfy the heap property
  + Runtime:
* **Max(A, n)**
  + Returns the max element of the heap (no modification)
  + Runtime:
* **Extract-Max(A, n)**
  + Returns and removes the max element of the heap
  + Fills the gap in with , then calls **Heapify(A,1)**
  + Runtime:

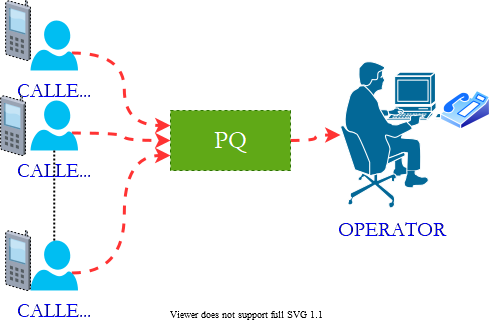
## Summary: **Max Heap**

* **Build-Heap(A, n)**
  + Given an arbitrary array, builds a heap from scratch
  + Runtime:
* **Min(A, n)**
  + How to return the min element in a max-heap?
  + Worst case runtime:
    - because ~half of the heap elements are leaf nodes
  + Instead, use a min-heap for efficient min operations
* **Search(A, x)**
  + For an arbitrary value, the worst-case runtime:
  + Use a sorted array instead for efficient search operations

## Summary: **Max Heap**

* **Increase-Key(A, i, x)**
  + Increase the key of node (from to )
  + “*Float up*” until heap property is satisfied
  + Runtime:
* **Decrease-Key(A, i, x)**
  + Decrease the key of node (from to )
  + Call **Heapify(A, i)**
  + Runtime:

## **Phone Operator** Problem

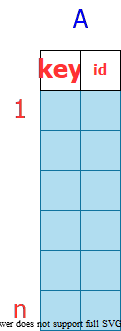


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* A phone operator answering  **phones**
* Each phone has  **people waiting** in line for their calls to be answered.
* Phone operator needs to answer the phone with the largest number of people waiting in line.
* New calls come continuously, and some people hang up after waiting.

## **Phone Operator** Solution

* **Step 1:** Define the following array:
* : the ith element in heap
* : the index of the corresponding phone
* : of people waiting in line for phone with index



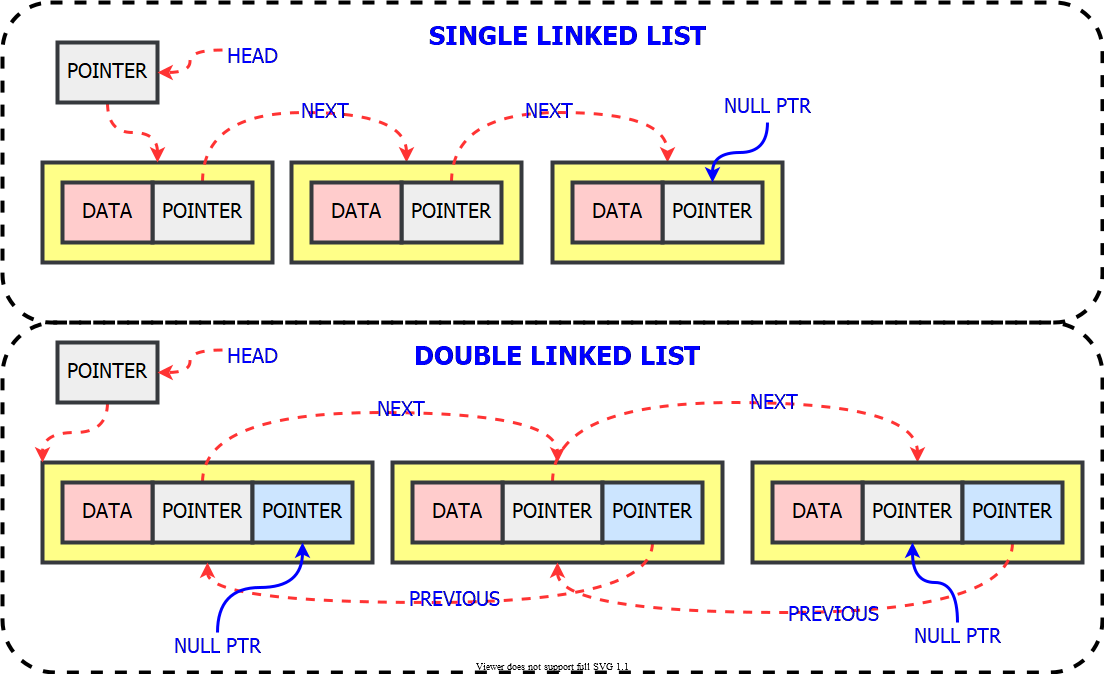
bg right:30% w:200px

## **Phone Operator** Solution

* **Step 2:**
  + **Execution:**
    - When the operator wants to answer a phone:
      * + answer phone with index
      * When a new call comes in to phone i:
      * When a call drops from phone i:

## Linked Lists

* Like arrays, Linked List is a linear data structure.
* Unlike arrays, linked list elements are not stored at a contiguous location; the elements are linked using pointers.



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## Linked Lists - C Definition

* C
* // A linked list node  
  struct Node {  
   int data;  
   struct Node\* next;  
  };

## Linked Lists - Cpp Definition

* Cpp
* class Node {  
  public:  
   int data;  
   Node\* next;  
  };

## Linked Lists - Java Definition

* Java
* class LinkedList {  
   Node head; // head of the list  
    
   /\* Linked list Node\*/  
   class Node {  
   int data;  
   Node next;  
    
   // Constructor to create a new node  
   // Next is by default initialized  
   // as null  
   Node(int d) { data = d; }  
   }  
  }

## Linked Lists - Csharp Definition

* Csharp
* class LinkedList {  
   // The first node(head) of the linked list  
   // Will be an object of type Node (null by default)  
   Node head;  
    
   class Node {  
   int data;  
   Node next;  
    
   // Constructor to create a new node  
   Node(int d) { data = d; }  
   }  
  }

## Priority Queue using **Linked List** Methods

* Implement Priority Queue using Linked Lists.
  + **push():** This function is used to insert a new data into the queue.
  + **pop():** This function removes the element with the highest priority from the queue.
  + **peek()/top():** This function is used to get the highest priority element in the queue without removing it from the queue.

## Priority Queue using **Linked List** Algorithm

PUSH(HEAD, DATA, PRIORITY)  
 Create NEW.Data = DATA & NEW.Priority = PRIORITY  
 If HEAD.priority < NEW.Priority   
 NEW -> NEXT = HEAD  
 HEAD = NEW   
 Else  
 Set TEMP to head of the list   
 Endif  
  
 WHILE TEMP -> NEXT != NULL and TEMP -> NEXT ->PRIORITY > PRIORITY THEN  
 TEMP = TEMP -> NEXT   
 ENDWHILE  
  
 NEW -> NEXT = TEMP -> NEXT   
 TEMP -> NEXT = NEW

## Priority Queue using **Linked List** Algorithm

POP(HEAD)  
//Set the head of the list to the next node in the list.  
HEAD = HEAD -> NEXT.  
Free the node at the head of the list

PEEK(HEAD):   
Return HEAD -> DATA

## Priority Queue using **Linked List** Notes

* LinkedList is already sorted.
* Time Complexities and Comparison with Binary Heap

|  | peek() | push() | pop() |
| --- | --- | --- | --- |
| Linked List |  |  |  |
| Binary Heap |  |  |  |

## Sorting in Linear Time

## How Fast Can We Sort?

* The algorithms we have seen so far:
  + Based on comparison of elements
  + We only care about the relative ordering between the elements (not the actual values)
  + The smallest worst-case runtime we have seen so far:
  + Is the best we can do?
* **Comparison sorts:** Only use comparisons to determine the relative order of elements.

## Decision Trees for Comparison Sorts

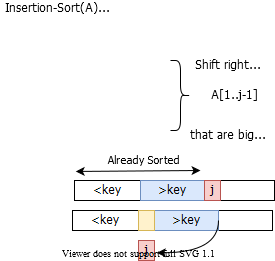
* Represent a sorting algorithm abstractly in terms of a **decision tree**
  + A **binary tree** that represents the **comparisons between** elements in the sorting algorithm
  + Control, data movement, and other aspects are ignored
* One decision tree corresponds to one sorting algorithm and one value of (*input size*)

## Reminder: Insertion Sort Step-By-Step Description (1)



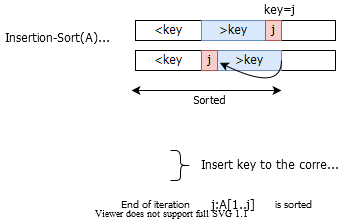
bg right:60% w:700px

## Reminder: Insertion Sort Step-By-Step Description (2)



bg right:60% w:700px

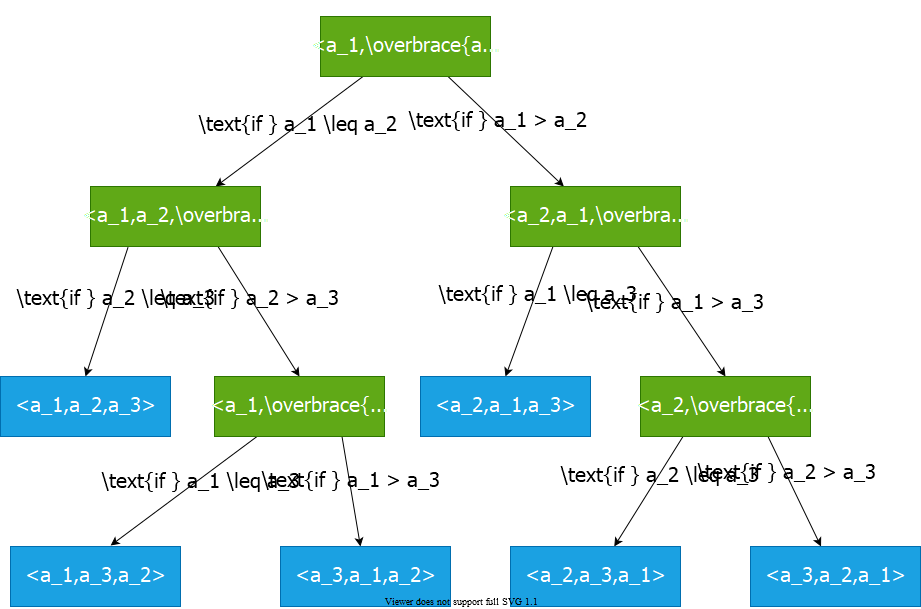
## Reminder: Insertion Sort Step-By-Step Description (3)



bg right:60% w:700px

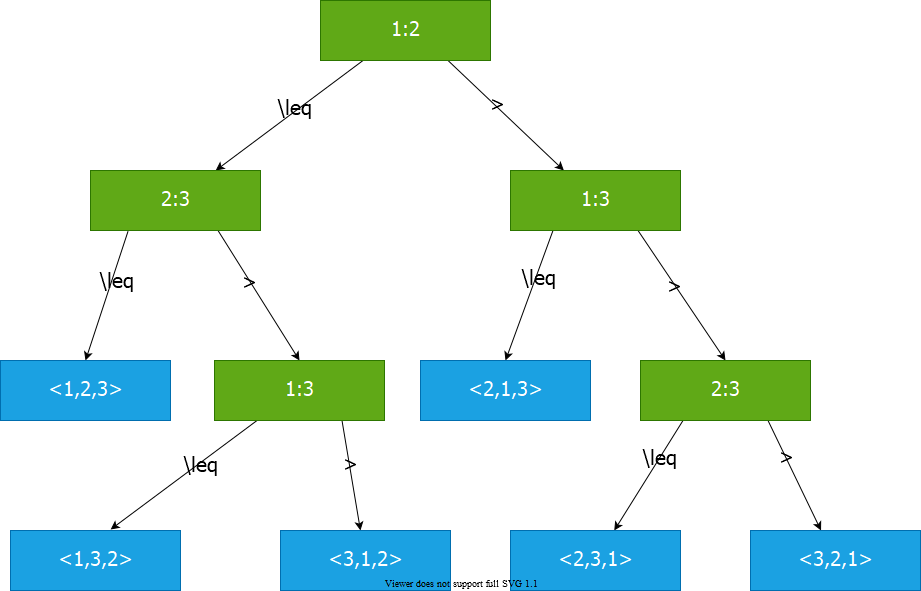
## Different Outcomes for Insertion Sort and n=3

* Input :



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## Decision Tree for Insertion Sort and n=3



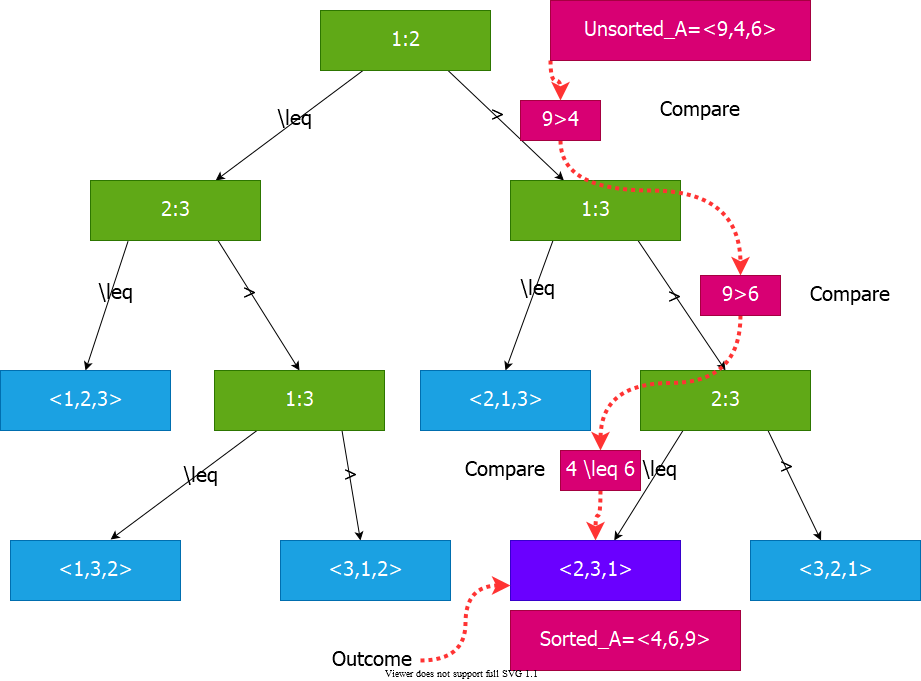
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## Decision Tree Model for Comparison Sorts

* **Internal node :** Comparison between elements and
* **Leaf node:** An output of the sorting algorithm
* **Path from root to a leaf:** The execution of the sorting algorithm for a given input
* **All possible executions** are captured by the decision tree
* **All possible outcomes (permutations)** are in the leaf nodes

## Decision Tree for Insertion Sort and n=3

* Input:



bg right:70% w:850px

## Decision Tree Model

* A decision tree can model the execution of any comparison sort:
  + One tree for each input size
  + View the algorithm as **splitting** whenever it compares two elements
  + The tree contains the **comparisons along all possible** instruction traces
* **The running time of the algorithm** *the length of the path taken*
* **Worst case running time** *height of the tree*

## Counting Sort

## Lower Bound for Comparison Sorts

* Let be the number of elements in the input array.
* What is the number of leaves in the decision tree?
  + **(because there are n! permutations of the input array, and all possible outputs must be captured in the leaves)**
* What is the max number of leaves in a binary tree of height ?
* So, we must have:

## Lower Bound for Decision Tree Sorting

* **Theorem:** Any comparison sort algorithm requires comparisons in the worst case.
* **Proof:** We’ll prove that any decision tree corresponding to a comparison sort algorithm must have height

## Lower Bound for Decision Tree Sorting

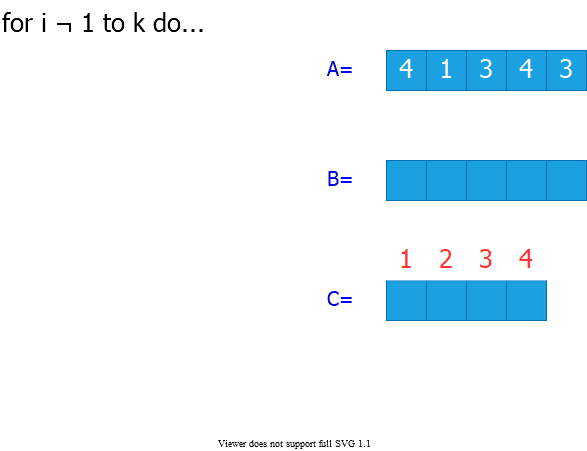
**Corollary:** Heapsort and merge sort are asymptotically optimal comparison sorts.

**Proof:** The upper bounds on the runtimes for heapsort and merge sort match the **worst-case** lower bound from the previous theorem.

## Sorting in Linear Time

* **Counting sort:** No comparisons between elements
  + **Input:** , where
  + **Output:** , sorted
  + **Auxiliary storage:**

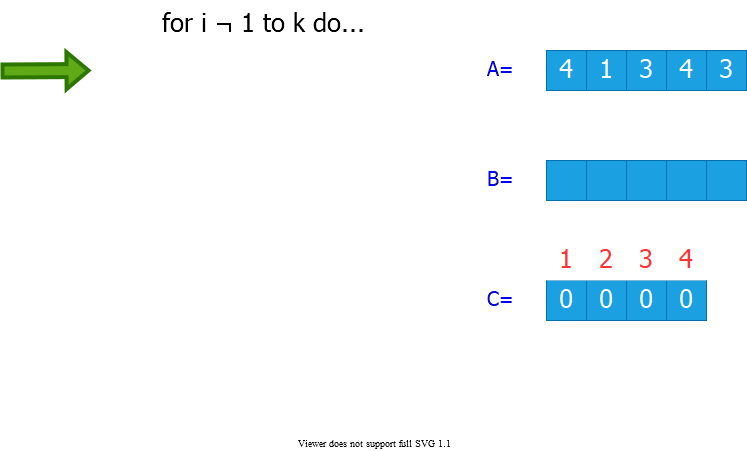
## Counting Sort-1



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## Counting Sort-2

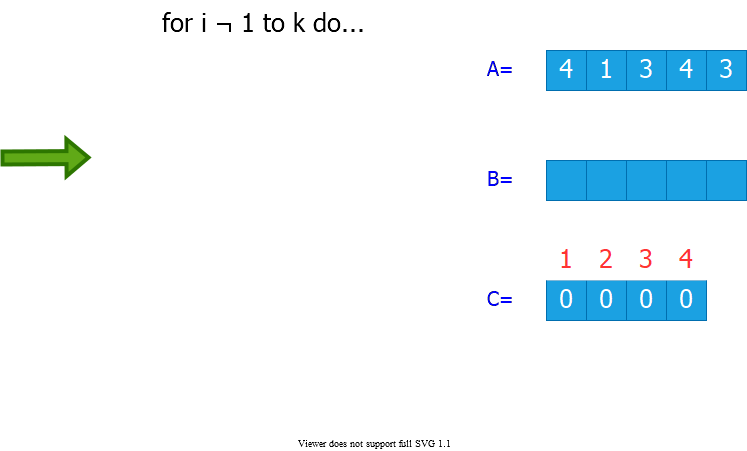
* **Step 1:** Initialize all counts to 0



bg right:60% w:750px

## Counting Sort-3

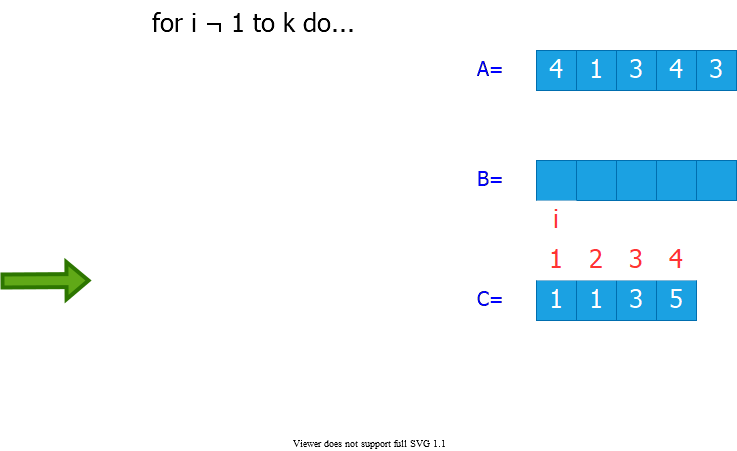
* **Step 2:** Count the number of occurrences of each value in the input array



bg right:60% w:750px

## Counting Sort-4

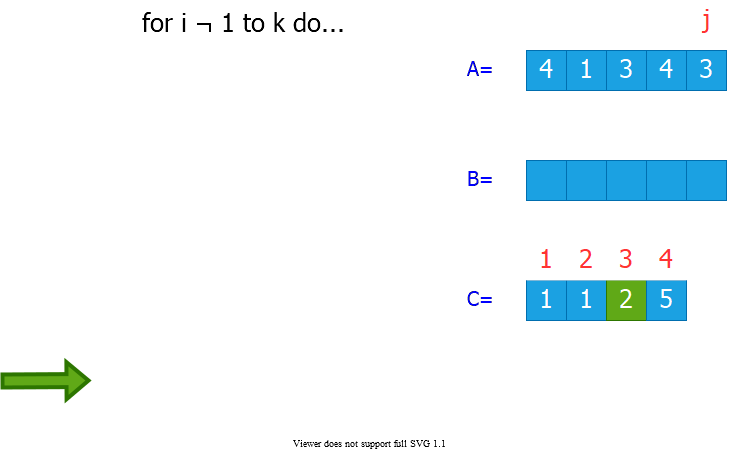
* **Step 3:** Compute the number of elements less than or equal to each value



bg right:60% w:750px

## Counting Sort-5

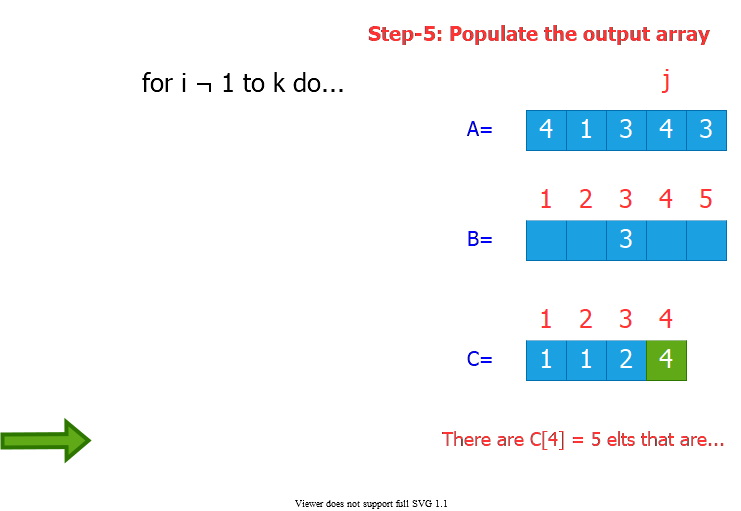
* **Step 4:** Populate the output array
  + There are elements that are



bg right:60% w:750px

## Counting Sort-6

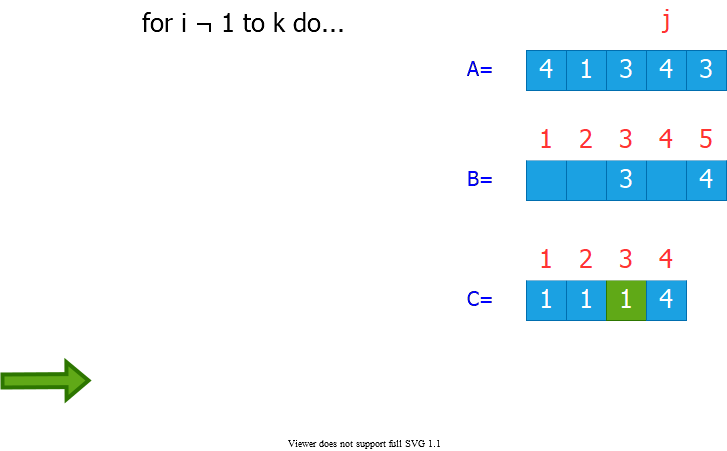
* **Step 4:** Populate the output array
  + There are elements that are



bg right:60% w:750px

## Counting Sort-7

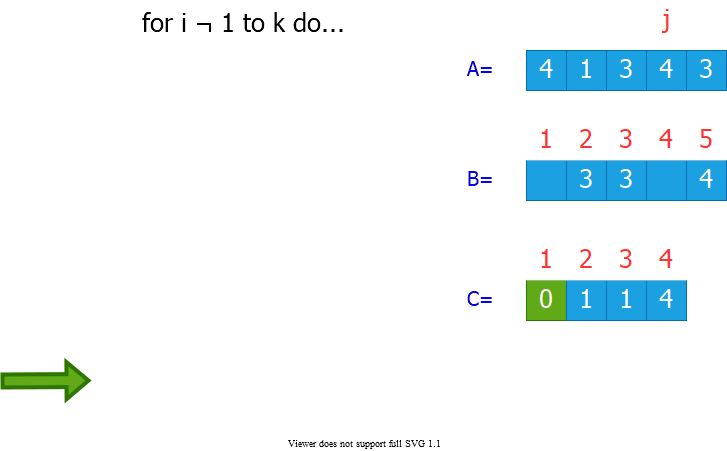
* **Step 4:** Populate the output array
  + There are elements that are



bg right:60% w:750px

## Counting Sort-8

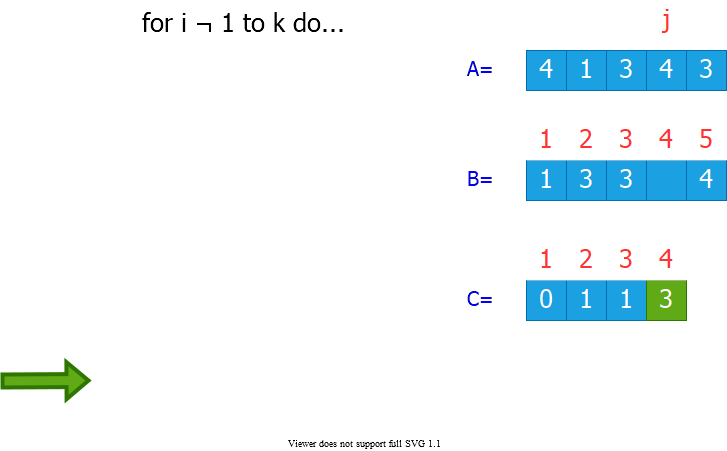
* **Step 4:** Populate the output array
  + There are elements that are



bg right:60% w:750px

## Counting Sort-9

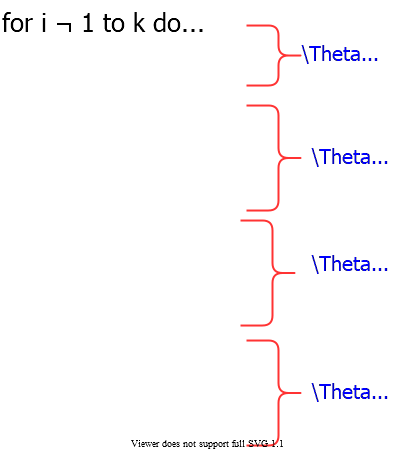
* **Step 4:** Populate the output array
  + There are elements that are



bg right:60% w:750px

## Counting Sort: Runtime Analysis

* **Total Runtime:**
  + : size of the input array
  + : the range of input values



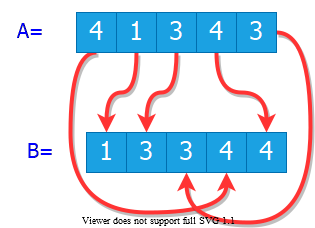
bg right:60% w:550px

## Counting Sort: Runtime

* Runtime is
  + If , then counting sort takes
* **Question:** We proved a lower bound of before! Where is the fallacy?
* **Answer:**
  + lower bound is for comparison-based sorting
  + Counting sort is not a comparison sort
  + In fact, not a single comparison between elements occurs!

## Stable Sorting

* Counting sort is a **stable sort:** It preserves the input order among equal elements.
  + i.e. The numbers with the same value appear in the output array in the same order as they do in the input array.
* **Note**: Which other sorting algorithms have this property?



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## Radix Sort

* **Origin:** Herman Hollerith’s card-sorting machine for the 1890 US Census.
* **Basic idea:** Digit-by-digit sorting
* Two variations:
  + Sort from **MSD** to **LSD** (bad idea)
  + Sort from **LSD** to **MSD** (good idea)

(*LSD/MSD: Least/most significant digit*)

## Herman Hollerith (1860-1929)

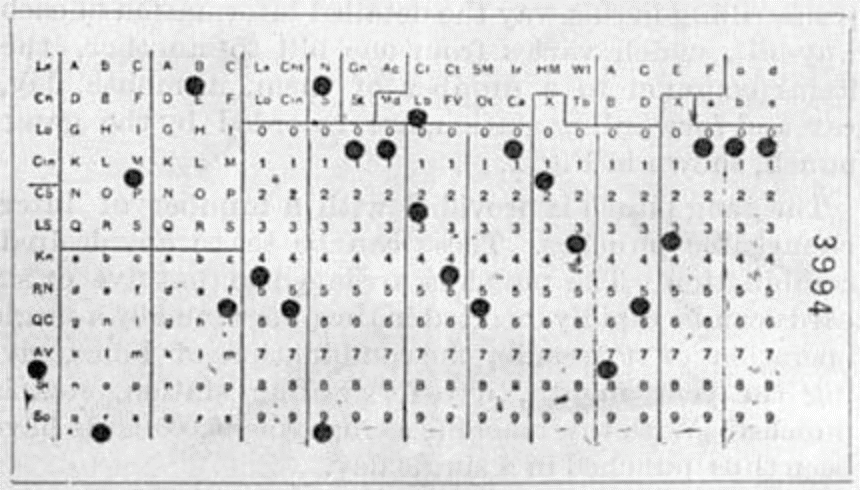
* The 1880 U.S. Census took **almost 10 years** to process.
* While a lecturer at MIT, Hollerith prototyped **punched-card technology**.
* His machines, including a **card sorter**, allowed the 1890 census total to be reported in **6 weeks**.
* He founded the **Tabulating Machine Company** in 1911, which merged with other companies in 1924 to form **International Business Machines(IBM)**.



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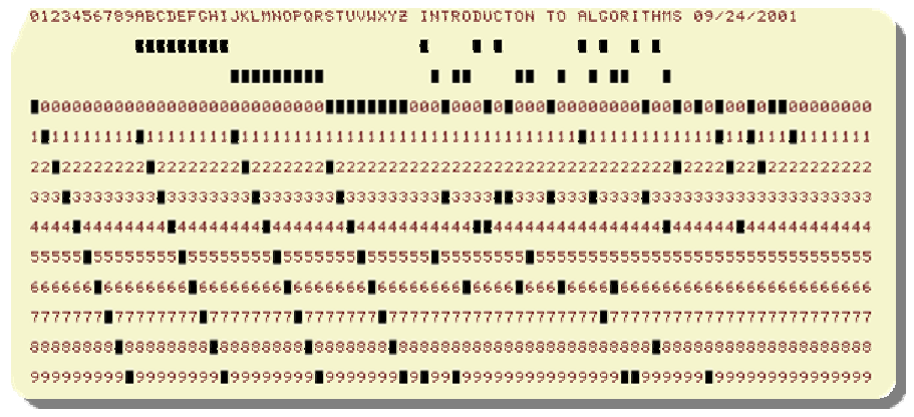
## Hollerith Punched Card

* **Punched card:** A piece of stiff paper that contains digital information represented by the presence or absence of holes.
  + 12 rows and 24 columns
  + coded for age, state of residency, gender, etc.



center h:300px

## **Modern** IBM card

* One character per column
  + So, that’s why text windows have 80 columns!
* 
* center h:350px
* for more samples visit https://en.wikipedia.org/wiki/Punched\_card

## Hollerith Tabulating Machine and Sorter

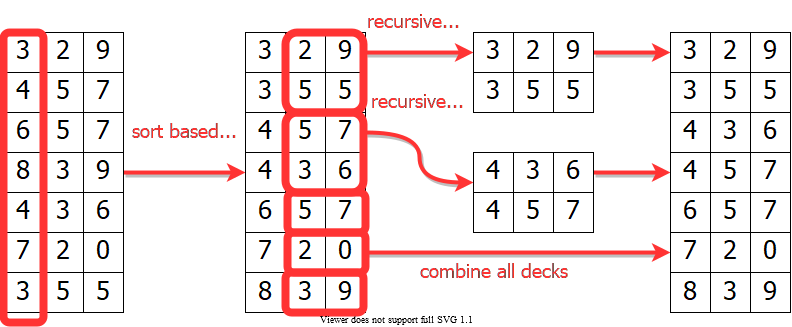
* Mechanically sorts the cards based on the hole locations.
* Sorting performed for one column at a time
* Human operator needed to load/retrieve/move cards at each stage



center h:350px

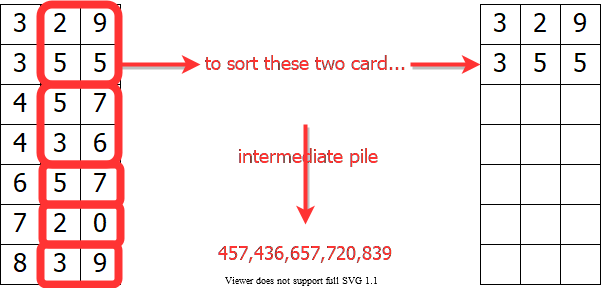
## Hollerith’s MSD-First Radix Sort

* Sort starting from the most significant digit (MSD)
* Then, sort each of the resulting bins recursively
* At the end, combine the decks in order



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## Hollerith’s MSD-First Radix Sort

* To sort a subset of cards recursively:
  + All the other cards need to be removed from the machine, because the machine can handle only one sorting problem at a time.
  + The human operator needs to keep track of the intermediate card piles
* 
* center h:300px

## Hollerith’s MSD-First Radix Sort

* MSD-first sorting may require:
  + very large number of sorting passes
  + very large number of intermediate card piles to maintain
* **S(d):**
  + of passes needed to sort d-digit numbers (worst-case)
* **Recurrence:**
  + with
    - **Reminder:** Recursive call made to each subset with the same most significant digit(MSD)

## Hollerith’s MSD-First Radix Sort

* **Recurrence:**
* Iteration terminates when with

## Hollerith’s MSD-First Radix Sort

* **Recurrence:**

## Hollerith’s MSD-First Radix Sort

* : of intermediate card piles maintained (worst-case)
* **Reminder:** Each routing pass generates 9 intermediate piles except the sorting passes on least significant digits (LSDs)
  + There are sorting calls to LSDs

## Hollerith’s MSD-First Radix Sort

**Alternative solution:** Solve the recurrence

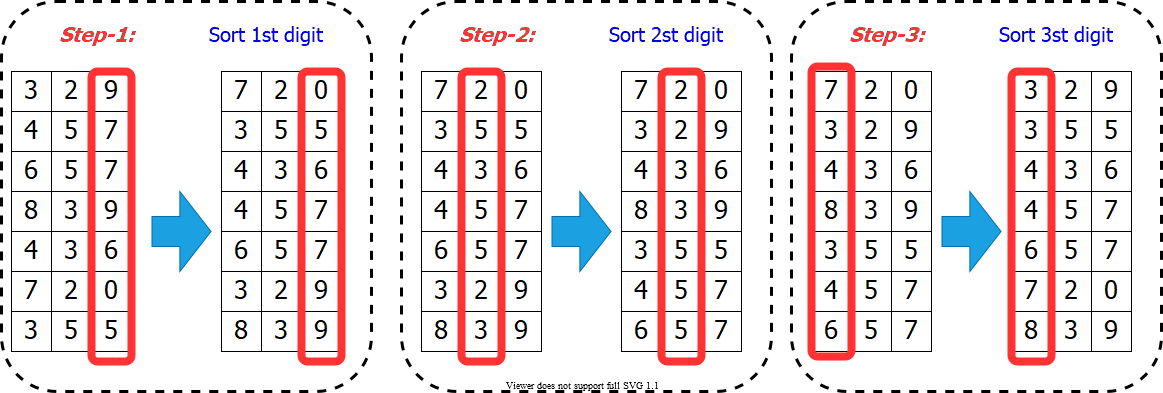
## Hollerith’s MSD-First Radix Sort

* **Example:** To sort digit numbers, in the worst case:
  + sorting passes needed
  + intermediate card piles generated
* MSD-first approach has more recursive calls and intermediate storage requirement
  + Expensive for a **tabulating machine** to sort punched cards
  + Overhead of recursive calls in a modern computer

## LSD-First Radix Sort

* Least significant digit (**LSD**)-first radix sort seems to be a folk invention originated by machine operators.
* It is the counter-intuitive, but the better algorithm.
* **Basic Algorithm:**
* Sort numbers on their LSD first (Stable Sorting Needed)  
  Combine the cards into a single deck in order   
  Continue this sorting process for the other digits  
   from the LSD to MSD
* Requires only sorting passes
* No intermediate card pile generated

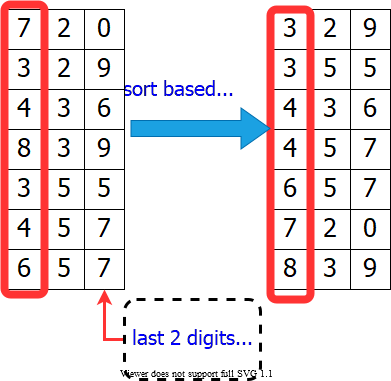
## LSD-first Radix Sort **Example**



center h:500px

## Correctness of Radix Sort **(LSD-first)**

* **Proof by induction:**
  + **Base case:** is correct (**trivial**)
  + **Inductive hyp:** Assume the first digits are sorted correctly
* Prove that all digits are sorted correctly after sorting digit
* Two numbers that differ in digit are correctly sorted (**e.g. 355 and 657**)
* Two numbers equal in digit d are put in the same order as the input
  + (**correct order**)



bg right:39% h:450px

## Radix Sort **Runtime**

* Use counting-sort to sort each digit
* **Reminder:** Counting sort complexity:
  + : size of input array
  + : the range of the values
* Radix sort runtime:
  + : of digits

**How to choose the and ?**

## Radix Sort: Runtime – **Example 1**

* We have flexibility in choosing and
* Assume we are trying to sort **32-bit words**
  + We can define each digit to be **4 bits**
  + Then, the range for each digit
    - So, counting sort will take
  + The number of digits
  + Radix sort runtime:

## Radix Sort: Runtime – **Example 2**

* We have flexibility in choosing and
* Assume we are trying to sort **32-bit words**
  + Or, we can define each digit to be **8 bits**
  + Then, the range for each digit
    - So, counting sort will take
  + The number of digits
  + Radix sort runtime:

## Radix Sort: **Runtime**

* Assume we are trying to sort **-bit** words
  + Define each digit to be  **bits**
  + Then, the range for each digit
    - So, **counting sort will take**
  + The number of digits
    - **Radix sort runtime:**

## Radix Sort: **Runtime Analysis**

* Minimize by differentiating and setting to
* Or, intuitively:
  + We want to balance the terms and
  + **Choose** 
    - If we choose term **doesn’t improve**
    - If we choose increases **exponentially**

## Radix Sort: **Runtime Analysis**

* For numbers in the range from to , we have:
  + The number of bits
    - Radix sort runs in

## Radix Sort: **Conclusions**

* **Example:** Compare radix sort with merge sort/heapsort
  + million (), -bit numbers
    - **Radix sort:** passes
    - **Merge sort/heap sort:** passes
* **Downsides:**
  + Radix sort has **little locality of reference** (more cache misses)
  + The version that uses counting sort is not in-place
* On modern processors, a well-tuned quicksort implementation typically runs faster.

## References

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