

# CE100 Algorithms and Programming II

## Week-3 (Matrix Multiplication/ Quick Sort)

Spring Semester, 2021-2022

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# Matrix Multiplication / Quick Sort

## Outline (1)

- Matrix Multiplication
  - Traditional
  - Recursive
  - Strassen

## Outline (2)

- Quicksort
  - Hoare Partitioning
  - Lomuto Partitioning
  - Recursive Sorting

## Outline (3)

- Quicksort Analysis
  - Randomized Quicksort
  - Randomized Selection
    - Recursive
    - Medians

## Matrix Multiplication (1)

- Input:  $A = [a_{ij}]$ ,  $B = [b_{ij}]$
- Output:  $C = [c_{ij}] = A \cdot B \implies i, j = 1, 2, 3, \dots, n$

$$\begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \vdots & \ddots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \vdots & \ddots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{bmatrix}$$

# Matrix Multiplication (2)

$$\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ \boxed{c_{21}} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \boxed{a_{21} \quad a_{22} \quad \cdots \quad a_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ \boxed{b_{21}} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$$

- $c_{ij} = \sum_{1 \leq k \leq n} a_{ik} \cdot b_{kj}$

# Matrix Multiplication: Standard Algorithm

Running Time:  $\Theta(n^3)$

```
for i=1 to n do
  for j=1 to n do
    C[i,j] = 0
    for k=1 to n do
      C[i,j] = C[i,j] + A[i,k] + B[k,j]
    endfor
  endfor
endfor
```

# Matrix Multiplication: Divide & Conquer (1)

**IDEA:** Divide the  $n \times n$  matrix into  $2 \times 2$  matrix of  $(n/2) \times (n/2)$  submatrices.

$$\begin{pmatrix} \boxed{c_{11}} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} \boxed{a_{11}} & \boxed{a_{12}} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} \boxed{b_{11}} & b_{12} \\ \boxed{b_{21}} & b_{22} \end{pmatrix} \quad \begin{pmatrix} c_{11} & c_{12} \\ \boxed{c_{21}} & c_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ \boxed{a_{21}} & \boxed{a_{22}} \end{pmatrix} \cdot \begin{pmatrix} \boxed{b_{11}} & b_{12} \\ \boxed{b_{21}} & b_{22} \end{pmatrix}$$

$$c_{11} = a_{11}b_{11} + a_{12}b_{21}$$

$$c_{21} = a_{21}b_{11} + a_{22}b_{21}$$

$$\begin{pmatrix} c_{11} & \boxed{c_{12}} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} \boxed{a_{11}} & \boxed{a_{12}} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & \boxed{b_{12}} \\ b_{21} & \boxed{b_{22}} \end{pmatrix} \quad \begin{pmatrix} c_{11} & c_{12} \\ \boxed{c_{21}} & c_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ \boxed{a_{21}} & \boxed{a_{22}} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & \boxed{b_{12}} \\ b_{21} & \boxed{b_{22}} \end{pmatrix}$$

$$c_{12} = a_{11}b_{12} + a_{12}b_{22}$$

$$c_{22} = a_{21}b_{12} + a_{22}b_{22}$$



## Matrix Multiplication: Divide & Conquer (2)

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$8 \text{ mults and } 4 \text{ adds of } (n/2) \times (n/2) \text{ submatrices} = \begin{cases} c_{11} = a_{11}b_{11} + a_{12}b_{21} \\ c_{21} = a_{21}b_{11} + a_{22}b_{21} \\ c_{12} = a_{11}b_{12} + a_{12}b_{22} \\ c_{22} = a_{21}b_{12} + a_{22}b_{22} \end{cases}$$

# Matrix Multiplication: Divide & Conquer (3)

```
MATRIX-MULTIPLY(A, B)
// Assuming that both A and B are nxn matrices
if n == 1 then
    return A * B
else
    //partition A, B, and C as shown before
    C[1,1] = MATRIX-MULTIPLY (A[1,1], B[1,1]) +
             MATRIX-MULTIPLY (A[1,2], B[2,1]);

    C[1,2] = MATRIX-MULTIPLY (A[1,1], B[1,2]) +
             MATRIX-MULTIPLY (A[1,2], B[2,2]);

    C[2,1] = MATRIX-MULTIPLY (A[2,1], B[1,1]) +
             MATRIX-MULTIPLY (A[2,2], B[2,1]);

    C[2,2] = MATRIX-MULTIPLY (A[2,1], B[1,2]) +
             MATRIX-MULTIPLY (A[2,2], B[2,2]);
endif

return C
```

## Matrix Multiplication: Divide & Conquer Analysis

$$T(n) = 8T(n/2) + \Theta(n^2)$$

- 8 recursive calls  $\implies 8T(\dots)$
- each problem has size  $n/2 \implies \dots T(n/2)$
- Submatrix addition  $\implies \Theta(n^2)$

## Matrix Multiplication: Solving the Recurrence

- $T(n) = 8T(n/2) + \Theta(n^2)$ 
  - $a = 8, b = 2$
  - $f(n) = \Theta(n^2)$
  - $n^{\log_b^a} = n^3$
- Case 1:  $\frac{n^{\log_b^a}}{f(n)} = \Omega(n^\epsilon) \implies T(n) = \Theta(n^{\log_b^a})$

Similar with ordinary (iterative) algorithm.

## Matrix Multiplication: Strassen's Idea (1)

Compute  $c_{11}, c_{12}, c_{21}, c_{22}$  using 7 recursive multiplications.

In normal case we need 8 as below.

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$8 \text{ mults and } 4 \text{ adds of } (n/2) \times (n/2) \text{ submatrices} = \begin{cases} c_{11} = a_{11}b_{11} + a_{12}b_{21} \\ c_{21} = a_{21}b_{11} + a_{22}b_{21} \\ c_{12} = a_{11}b_{12} + a_{12}b_{22} \\ c_{22} = a_{21}b_{12} + a_{22}b_{22} \end{cases}$$

# Matrix Multiplication: Strassen's Idea (2)

- **Reminder:**
  - Each submatrix is of size  $(n/2) * (n/2)$
  - Each add/sub operation takes  $\Theta(n^2)$  time
- Compute  $P_1 \dots P_7$  using 7 recursive calls to matrix-multiply

$$P_1 = a_{11} * (b_{12} - b_{22})$$

$$P_2 = (a_{11} + a_{12}) * b_{22}$$

$$P_3 = (a_{21} + a_{22}) * b_{11}$$

$$P_4 = a_{22} * (b_{21} - b_{11})$$

$$P_5 = (a_{11} + a_{22}) * (b_{11} + b_{22})$$

$$P_6 = (a_{12} - a_{22}) * (b_{21} + b_{22})$$

$$P_7 = (a_{11} - a_{21}) * (b_{11} + b_{12})$$

# Matrix Multiplication: Strassen's Idea (3)

$$P_1 = a_{11} * (b_{12} - b_{22})$$

$$P_2 = (a_{11} + a_{12}) * b_{22}$$

$$P_3 = (a_{21} + a_{22}) * b_{11}$$

$$P_4 = a_{22} * (b_{21} - b_{11})$$

$$P_5 = (a_{11} + a_{22}) * (b_{11} + b_{22})$$

$$P_6 = (a_{12} - a_{22}) * (b_{21} + b_{22})$$

$$P_7 = (a_{11} - a_{21}) * (b_{11} + b_{12})$$

- How to compute  $c_{ij}$  using  $P_1 \dots P_7$  ?

$$c_{11} = P_5 + P_4 - P_2 + P_6$$

$$c_{12} = P_1 + P_2$$

$$c_{21} = P_3 + P_4$$

$$c_{22} = P_5 + P_1 - P_3 - P_7$$

## Matrix Multiplication: Strassen's Idea (4)

- 7 recursive multiply calls
- 18 add/sub operations



## Matrix Multiplication: Strassen's Idea (5)

e.g. Show that  $c_{12} = P_1 + P_2$  :

$$\begin{aligned}c_{12} &= P_1 + P_2 \\&= a_{11}(b_{12}-b_{22}) + (a_{11} + a_{12})b_{22} \\&= a_{11}b_{12} - a_{11}b_{22} + a_{11}b_{22} + a_{12}b_{22} \\&= a_{11}b_{12} + a_{12}b_{22}\end{aligned}$$

## Strassen's Algorithm

- **Divide:** Partition  $A$  and  $B$  into  $(n/2) * (n/2)$  submatrices. Form terms to be multiplied using  $+$  and  $-$ .
- **Conquer:** Perform 7 multiplications of  $(n/2) * (n/2)$  submatrices recursively.
- **Combine:** Form  $C$  using  $+$  and  $-$  on  $(n/2) * (n/2)$  submatrices.

Recurrence:  $T(n) = 7T(n/2) + \Theta(n^2)$

# Strassen's Algorithm: Solving the Recurrence (1)

- $T(n) = 7T(n/2) + \Theta(n^2)$ 
  - $a = 7, b = 2$
  - $f(n) = \Theta(n^2)$
  - $n^{\log_b^a} = n^{\lg 7}$
- Case 1:  $\frac{n^{\log_b^a}}{f(n)} = \Omega(n^\epsilon) \implies T(n) = \Theta(n^{\log_b^a})$

$$T(n) = \Theta(n^{\log_2^7})$$

$$2^3 = 8, 2^2 = 4 \text{ so } \implies \log_2^7 \approx 2.81$$

or use <https://www.omnicalculator.com/math/log>

## Strassen's Algorithm: Solving the Recurrence (2)

- The number 2.81 may not seem much smaller than 3
- But, it is significant because the difference is in the exponent.
- Strassen's algorithm beats the ordinary algorithm on today's machines for  $n \geq 30$  or so.
- Best to date:  $\Theta(n^{2.376\dots})$  (of theoretical interest only)

# Maximum Subarray Problem

**Input:** An array of values

**Output:** The contiguous subarray that has the largest sum of elements

- Input array:

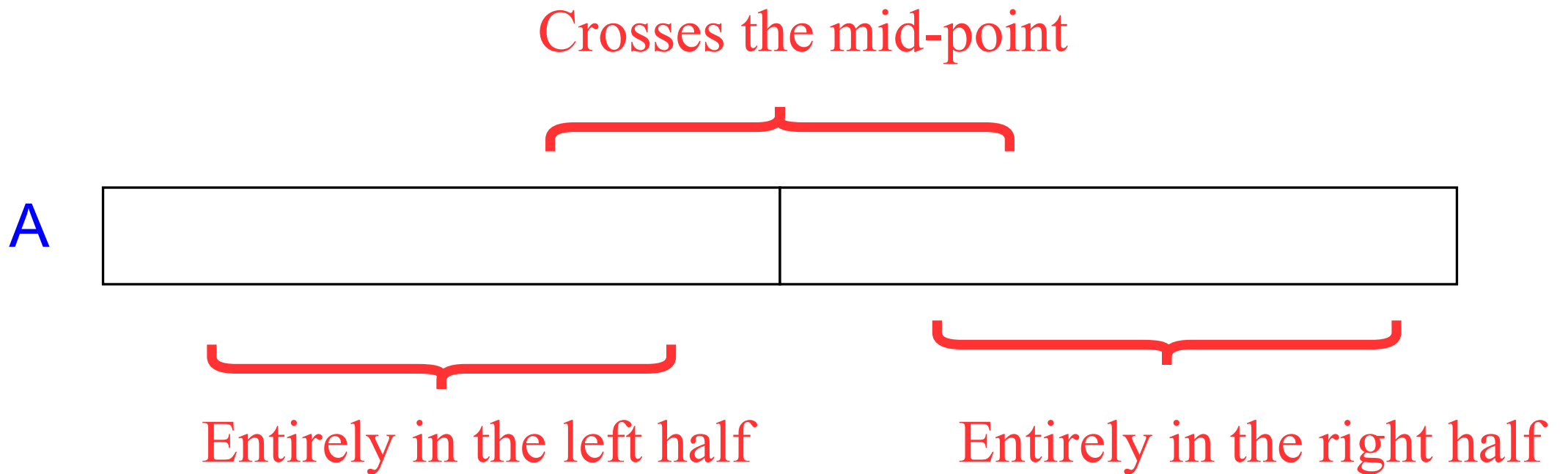
max. contiguous subarray

$[13] [-3] [-25] [20] [-3] [-16] [-23] \quad \overbrace{[18] [20] [-7] [12]} \quad [-22] [-4] [7]$

## Maximum Subarray Problem: Divide & Conquer (1)

- **Basic idea:**
- **Divide** the input array into 2 from the middle
- Pick the **best** solution among the following:
  - The max subarray of the **left half**
  - The max subarray of the **right half**
  - The max subarray **crossing the mid-point**

## Maximum Subarray Problem: Divide & Conquer (2)



## Maximum Subarray Problem: Divide & Conquer (3)

- **Divide:** Trivial (divide the array from the middle)
- **Conquer:** Recursively compute the max subarrays of the left and right halves
- **Combine:** Compute the max-subarray crossing the *mid – point*
  - (can be done in  $\Theta(n)$  time).
  - Return the max among the following:
    - the max subarray of the left-subarray
    - the max subarray of the rightsubarray
    - the max subarray crossing the mid-point

TODO : detailed solution in textbook...



## Conclusion : Divide & Conquer

- Divide and conquer is just one of several powerful techniques for algorithm design.
- Divide-and-conquer algorithms can be analyzed using recurrences and the master method (so practice this math).
- Can lead to more efficient algorithms

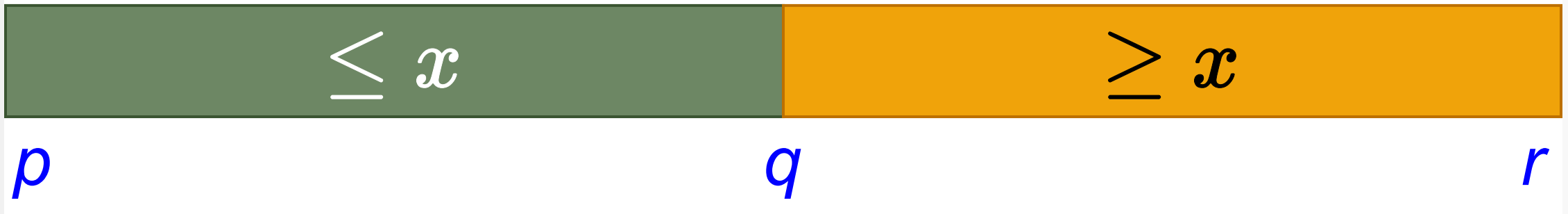
# Quicksort (1)

- One of the most-used algorithms in practice
- Proposed by **C.A.R. Hoare** in 1962.
- Divide-and-conquer algorithm
- In-place algorithm
  - The additional space needed is  $O(1)$
  - The sorted array is returned in the input array
  - *Reminder: Insertion-sort is also an in-place algorithm, but Merge-Sort is not in-place.*
- Very practical

## Quicksort (2)

- **Divide:** Partition the array into 2 subarrays such that elements in the lower part  $\leq$  elements in the higher part
- **Conquer:** Recursively sort 2 subarrays
- **Combine:** Trivial (because in-place)

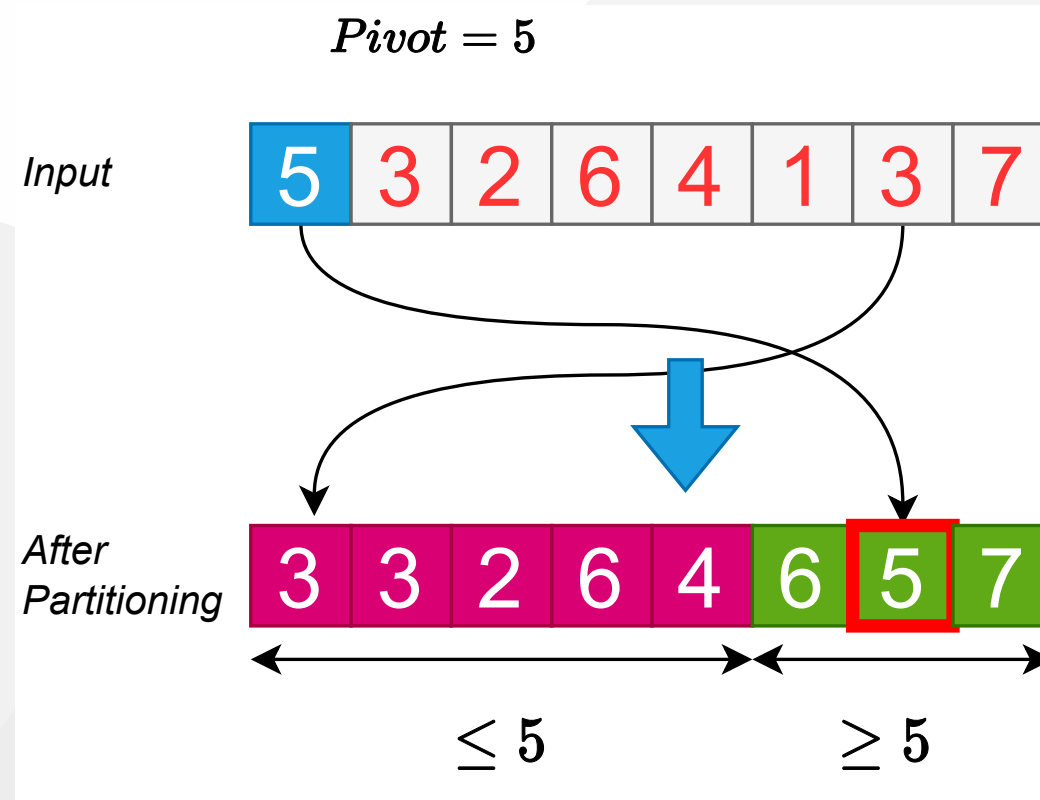
Key: Linear-time ( $\Theta(n)$ ) partitioning algorithm



# Divide: Partition the array around a pivot element

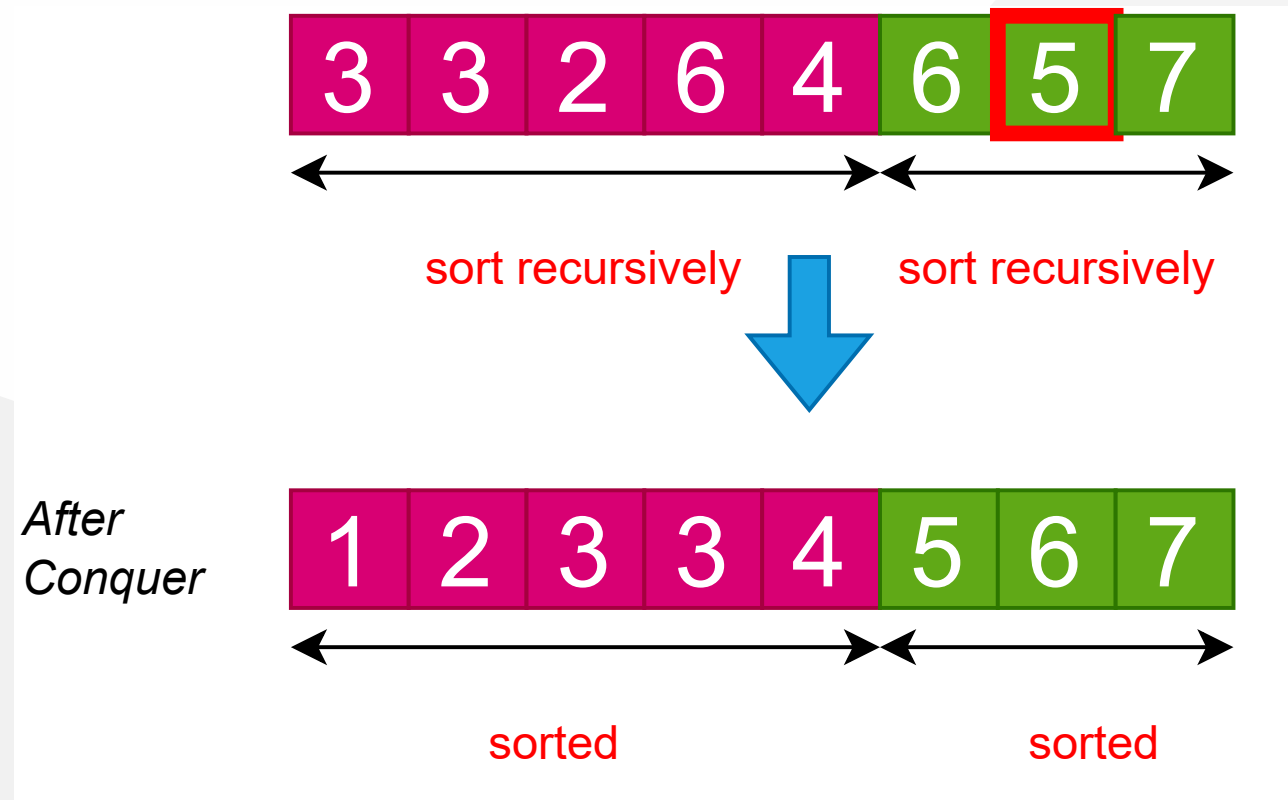
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- Choose a pivot element  $x$
- Rearrange the array such that:
  - Left subarray: All elements  $\leq x$
  - Right subarray: All elements  $\geq x$



## Conquer: Recursively Sort the Subarrays

Note: Everything in the left subarray  $\leq$  everything in the right subarray



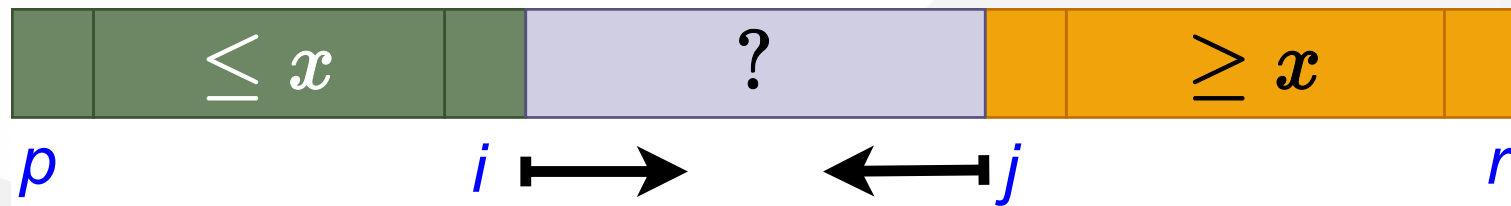
Note: Combine is trivial after conquer. Array already sorted.

## Two partitioning algorithms

- **Hoare's algorithm:**

Partitions around the first element of subarray

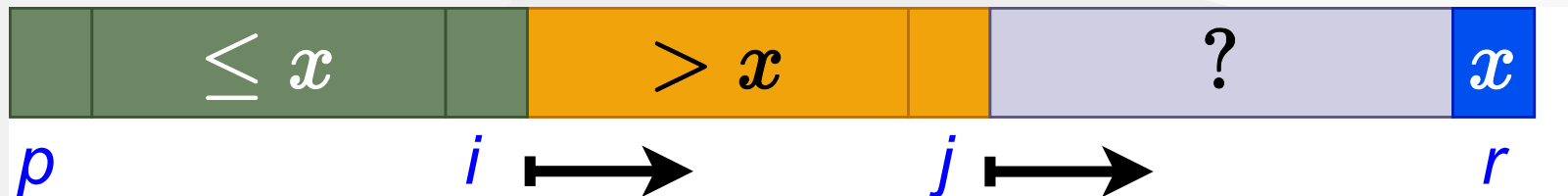
- ( $pivot = x = A[p]$ )



- **Lomuto's algorithm:**

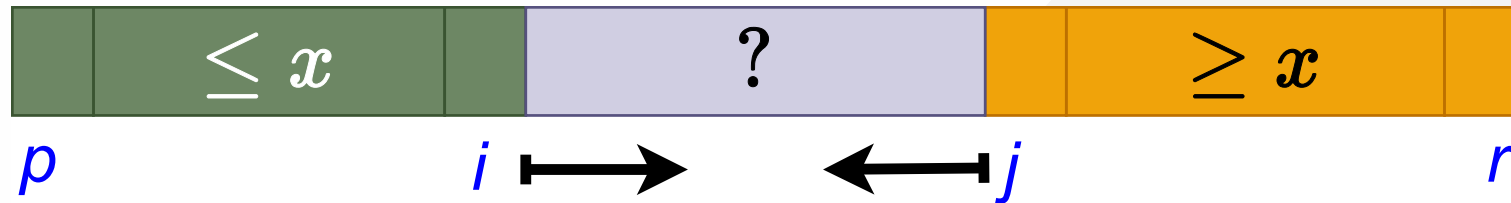
Partitions around the last element of subarray

- ( $pivot = x = A[r]$ )



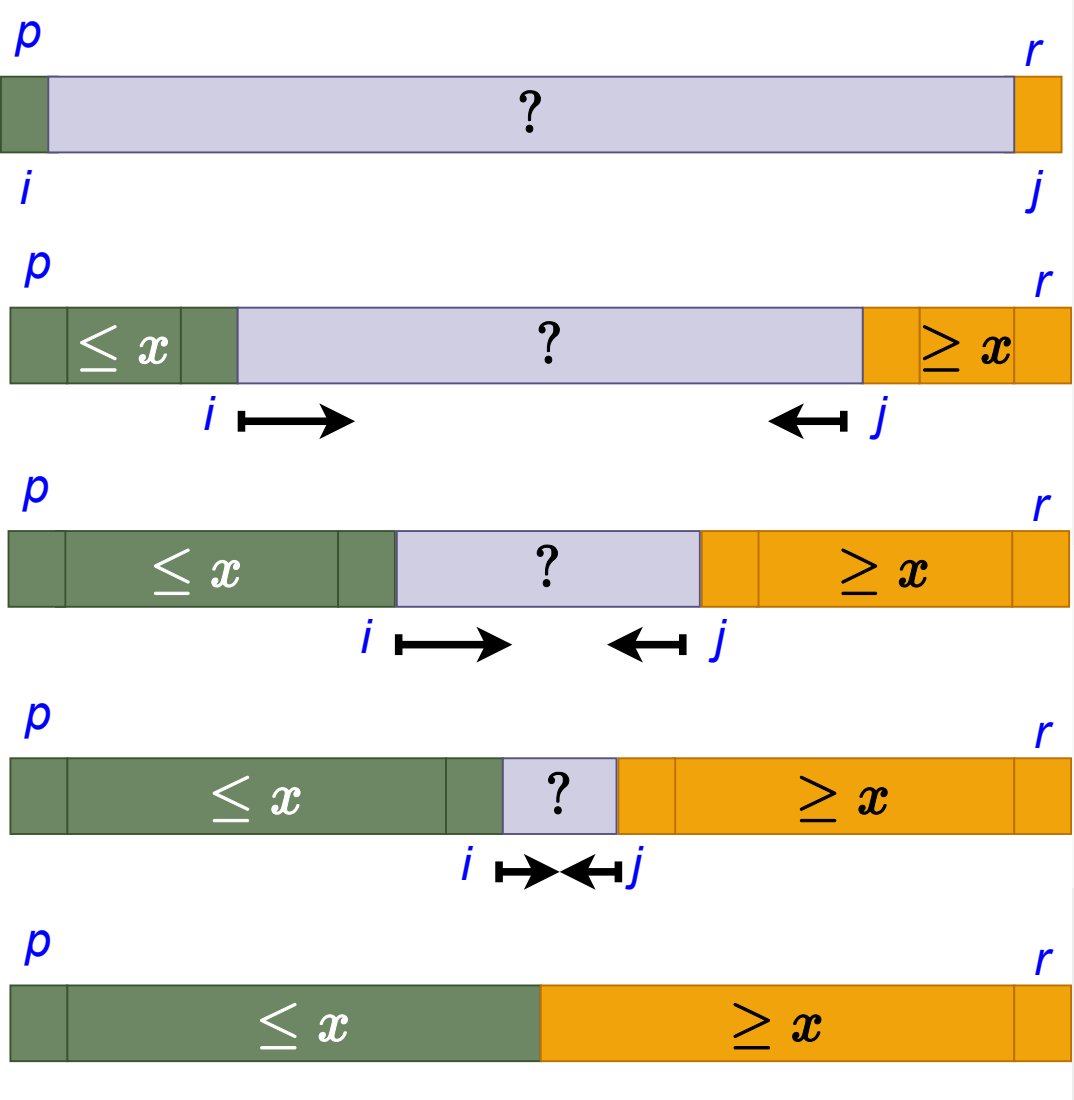
# Hoare's Partitioning Algorithm (1)

- Choose a pivot element:  $pivot = x = A[p]$



- Grow two regions:
  - from left to right:  $A[p \dots i]$
  - from right to left:  $A[j \dots r]$ 
    - such that:
      - every element in  $A[p \dots i] \leq pivot$
      - every element in  $A[j \dots r] \geq pivot$

# Hoare's Partitioning Algorithm (2)





# Hoare's Partitioning Algorithm (3)

- Elements are exchanged when
  - $A[i]$  is **too large** to belong to the **left** region
  - $A[j]$  is **too small** to belong to the **right** region
    - assuming that the inequality is strict
- The two regions  $A[p \dots i]$  and  $A[j \dots r]$  grow until  $A[i] \geq pivot \geq A[j]$

```
H-PARTITION(A, p, r)
    pivot = A[p]
    i = p - 1
    j = r - 1
    while true do
        repeat j = j - 1 until A[j] <= pivot
        repeat i = i - 1 until A[i] <= pivot
        if i < j then
            exchange A[i] with A[j]
        else
            return j
```

# Hoare's Partitioning Algorithm Example (Step-1)

*Pivot = 5*

*Input*

5	3	2	6	4	1	3	7
---	---	---	---	---	---	---	---

*STEP – 1*

**H-PARTITION** (A, p, r).

*pivot*  $\leftarrow$  A[p]

*i*  $\leftarrow$  p – 1

*j*  $\leftarrow$  r + 1

**while true do**

**repeat** *j*  $\leftarrow$  *j* – 1 **until** A[*j*]  $\leq$  *pivot*

**repeat** *i*  $\leftarrow$  *i* + 1 **until** A[*i*]  $\geq$  *pivot*

**if** *i* < *j* **then**

    exchange A[*i*]  $\leftrightarrow$  A[*j*]

**else**

**return** *j*

# Hoare's Partitioning Algorithm Example (Step-2)

**H-PARTITION** ( $A, p, r$ ).

$pivot \leftarrow A[p]$

$i \leftarrow p - 1$

$j \leftarrow r + 1$

**while true do**

**repeat**  $j \leftarrow j - 1$  **until**  $A[j] \leq pivot$

**repeat**  $i \leftarrow i + 1$  **until**  $A[i] \geq pivot$

**if**  $i < j$  **then**

    exchange  $A[i] \leftrightarrow A[j]$

**else**

**return**  $j$

$Pivot = 5$

Input

5	3	2	6	4	1	3	7
---	---	---	---	---	---	---	---

$i$

$j$

$STEP - 2$

# Hoare's Partitioning Algorithm Example (Step-3)

**H-PARTITION** (A, p, r).

$pivot \leftarrow A[p]$

$i \leftarrow p - 1$

$j \leftarrow r + 1$

**while true do**

**repeat**  $j \leftarrow j - 1$  **until**  $A[j] \leq pivot$

**repeat**  $i \leftarrow i + 1$  **until**  $A[i] \geq pivot$

**if**  $i < j$  **then**

    exchange  $A[i] \leftrightarrow A[j]$

**else**

**return**  $j$

*Pivot* = 5

Input

5	3	2	6	4	1	3	7
---	---	---	---	---	---	---	---

**i**

**j**

*STEP* – 3

# Hoare's Partitioning Algorithm Example (Step-4)

**H-PARTITION** (A, p, r).

$pivot \leftarrow A[p]$

$i \leftarrow p - 1$

$j \leftarrow r + 1$

**while true do**

**repeat**  $j \leftarrow j - 1$  **until**  $A[j] \leq pivot$

**repeat**  $i \leftarrow i + 1$  **until**  $A[i] \geq pivot$

**if**  $i < j$  **then**

    exchange  $A[i] \leftrightarrow A[j]$

**else**

**return**  $j$

*Pivot* = 5

*Input*

5	3	2	6	4	1	3	7
---	---	---	---	---	---	---	---

*i*

*j*

*STEP* – 4

# Hoare's Partitioning Algorithm Example (Step-5)

**H-PARTITION** (A, p, r).

$pivot \leftarrow A[p]$

$i \leftarrow p - 1$

$j \leftarrow r + 1$

**while true do**

**repeat**  $j \leftarrow j - 1$  **until**  $A[j] \leq pivot$

**repeat**  $i \leftarrow i + 1$  **until**  $A[i] \geq pivot$

**if**  $i < j$  **then**

    exchange  $A[i] \leftrightarrow A[j]$

**else**

**return**  $j$

*Pivot* = 5

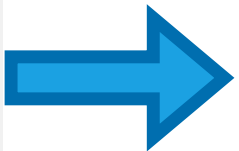
*Input*

5	3	2	6	4	1	3	7
---	---	---	---	---	---	---	---

*i*

*j*

*STEP* – 5



# Hoare's Partitioning Algorithm Example (Step-6)

**H-PARTITION** ( $A, p, r$ )

$pivot \leftarrow A[p]$

$i \leftarrow p - 1$

$j \leftarrow r + 1$

**while true do**

**repeat**  $j \leftarrow j - 1$  **until**  $A[j] \leq pivot$

**repeat**  $i \leftarrow i + 1$  **until**  $A[i] \geq pivot$

**if**  $i < j$  **then**

    exchange  $A[i] \leftrightarrow A[j]$

**else**

**return**  $j$

$Pivot = 5$

Input

3	3	2	6	4	1	5	7
						$i$	$j$

**STEP - 6**

# Hoare's Partitioning Algorithm Example (Step-7)

**H-PARTITION** ( $A, p, r$ ).

$pivot \leftarrow A[p]$

$i \leftarrow p - 1$

$j \leftarrow r + 1$

**while true do**

**repeat**  $j \leftarrow j - 1$  **until**  $A[j] \leq pivot$

**repeat**  $i \leftarrow i + 1$  **until**  $A[i] \geq pivot$

**if**  $i < j$  **then**

    exchange  $A[i] \leftrightarrow A[j]$

**else**

**return**  $j$

$Pivot = 5$

Input

3	3	2	6	4	1	5	7
---	---	---	---	---	---	---	---

$i$

$j$

$STEP - 7$



# Hoare's Partitioning Algorithm Example (Step-8)

**H-PARTITION** ( $A, p, r$ ).

$pivot \leftarrow A[p]$

$i \leftarrow p - 1$

$j \leftarrow r + 1$

**while true do**

**repeat**  $j \leftarrow j - 1$  **until**  $A[j] \leq pivot$

**repeat**  $i \leftarrow i + 1$  **until**  $A[i] \geq pivot$

**if**  $i < j$  **then**

    exchange  $A[i] \leftrightarrow A[j]$

**else**

**return**  $j$

*Pivot* = 5

Input

3	3	2	6	4	1	5	7
---	---	---	---	---	---	---	---

$i$   $j$

*STEP* – 8

# Hoare's Partitioning Algorithm Example (Step-9)

**H-PARTITION** ( $A, p, r$ ).

$pivot \leftarrow A[p]$

$i \leftarrow p - 1$

$j \leftarrow r + 1$

**while true do**

**repeat**  $j \leftarrow j - 1$  **until**  $A[j] \leq pivot$

**repeat**  $i \leftarrow i + 1$  **until**  $A[i] \geq pivot$

**if**  $i < j$  **then**

exchange  $A[i] \leftrightarrow A[j]$

**else**

**return**  $j$

*Pivot* = 5

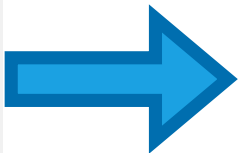
*Input*

3	3	2	6	4	1	5	7
---	---	---	---	---	---	---	---

$i$

$j$

*STEP* – 9



# Hoare's Partitioning Algorithm Example (Step-10)

**H-PARTITION** ( $A, p, r$ ).

$pivot \leftarrow A[p]$

$i \leftarrow p - 1$

$j \leftarrow r + 1$

**while true do**

**repeat**  $j \leftarrow j - 1$  **until**  $A[j] \leq pivot$

**repeat**  $i \leftarrow i + 1$  **until**  $A[i] \geq pivot$

**if**  $i < j$  **then**

    exchange  $A[i] \leftrightarrow A[j]$

**else**

**return**  $j$

$Pivot = 5$

Input

3	3	2	1	4	6	5	7
---	---	---	---	---	---	---	---

$i$

$j$

$STEP - 10$

# Hoare's Partitioning Algorithm Example (Step-11)

**H-PARTITION** ( $A, p, r$ ).

$pivot \leftarrow A[p]$

$i \leftarrow p - 1$

$j \leftarrow r + 1$

**while true do**

**repeat**  $j \leftarrow j - 1$  **until**  $A[j] \leq pivot$

**repeat**  $i \leftarrow i + 1$  **until**  $A[i] \geq pivot$

**if**  $i < j$  **then**

    exchange  $A[i] \leftrightarrow A[j]$

**else**

**return**  $j$

$Pivot = 5$

Input

3	3	2	1	4	6	5	7
---	---	---	---	---	---	---	---

$i$     $j$

$STEP - 11$

# Hoare's Partitioning Algorithm Example (Step-12)

**H-PARTITION** ( $A, p, r$ ).

$pivot \leftarrow A[p]$

$i \leftarrow p - 1$

$j \leftarrow r + 1$

**while true do**

**repeat**  $j \leftarrow j - 1$  **until**  $A[j] \leq pivot$

**repeat**  $i \leftarrow i + 1$  **until**  $A[i] \geq pivot$

**if**  $i < j$  **then**

    exchange  $A[i] \leftrightarrow A[j]$

**else**

**return**  $j$

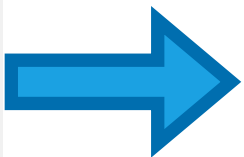
$Pivot = 5$

Input

3	3	2	1	4	6	5	7
---	---	---	---	---	---	---	---

$j$     $i$

$STEP - 12$



# Hoare's Partitioning Algorithm - Notes

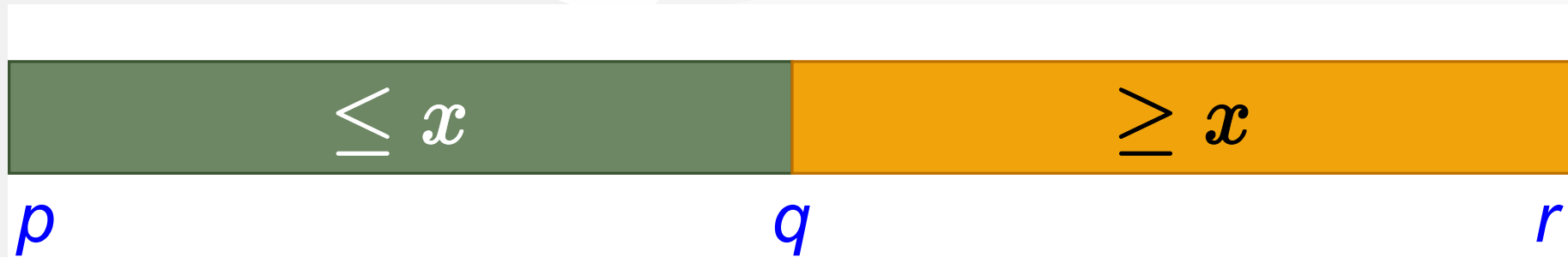
- Elements are exchanged when
  - $A[i]$  is **too large** to belong to the **left** region
  - $A[j]$  is **too small** to belong to the **right** region
    - assuming that the inequality is strict
- The two regions  $A[p \dots i]$  and  $A[j \dots r]$  grow until  $A[i] \geq pivot \geq A[j]$
- The asymptotic runtime of Hoare's partitioning algorithm  $\Theta(n)$

```
H-PARTITION(A, p, r)
    pivot = A[p]
    i = p - 1
    j = r - 1
    while true do
        repeat j = j - 1 until A[j] <= pivot
        repeat i = i - 1 until A[i] <= pivot
        if i < j then exchange A[i] with A[j]
        else return j
```

# Quicksort with Hoare's Partitioning Algorithm

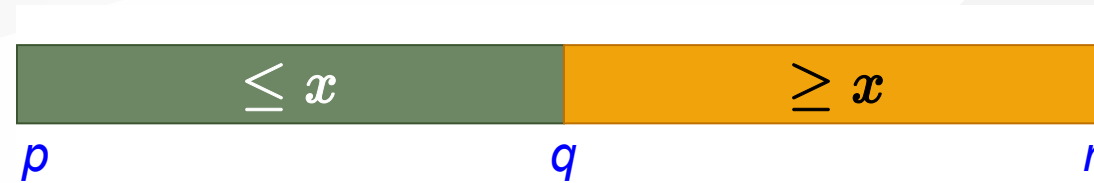
```
QUICKSORT (A, p, r)
  if p < r then
    q = H-PARTITION(A, p, r)
    QUICKSORT(A, p, q)
    QUICKSORT(A, q + 1, r)
  endif
```

Initial invocation: QUICKSORT(A, 1, n)



# Hoare's Partitioning Algorithm: Pivot Selection

- if we select pivot to be  $A[r]$  instead of  $A[p]$  in H-PARTITION



*Pivot = 7*



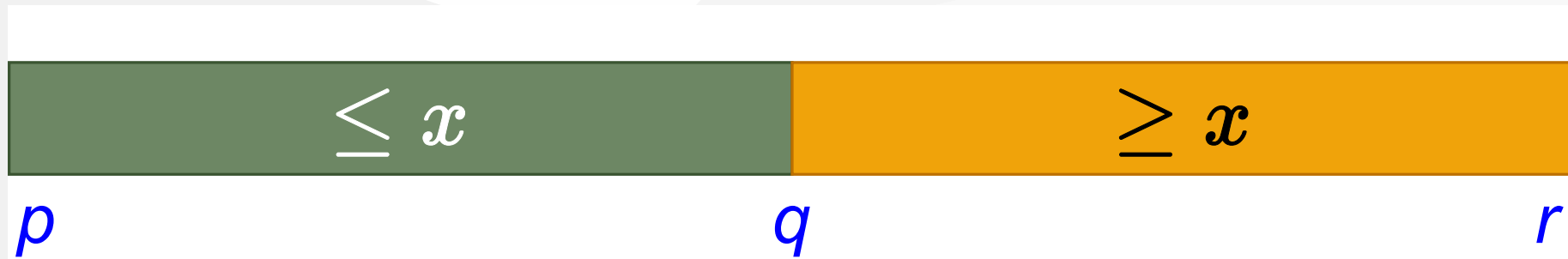
- Consider the example where  $A[r]$  is the largest element in the array:
  - End of H-PARTITION:  $i = j = r$
  - In QUICKSORT:  $q = r$ 
    - So, recursive call to:
      - QUICKSORT( $A$ ,  $p$ ,  $q=r$ )
        - infinite loop



## Correctness of Hoare's Algorithm (1)

We need to prove 3 claims to show correctness:

- Indices  $i$  and  $j$  never reference  $A$  outside the interval  $A[p \dots r]$
- Split is always non-trivial; i.e.,  $j \neq r$  at termination
- Every element in  $A[p \dots j] \leq$  every element in  $A[j + 1 \dots r]$  at termination



## Correctness of Hoare's Algorithm (2)

- Notations:
  - $k$ : # of times the while-loop iterates until termination
  - $i_m$ : the value of index  $i$  at the end of iteration  $m$
  - $j_m$ : the value of index  $j$  at the end of iteration  $m$
  - $x$ : the value of the pivot element
- **Note:** We always have  $i_1 = p$  and  $p \leq j_1 \leq r$   
because  $x = A[p]$

## Correctness of Hoare's Algorithm (3)

**Lemma 1:** Either  $i_k = j_k$  or  $i_k = j_k + 1$  at termination

**Proof of Lemma 1:**

- The algorithm terminates when  $i \geq j$  (the else condition).
- So, it is sufficient to prove that  $i_k - j_k \leq 1$
- There are 2 cases to consider:
  - Case 1:  $k = 1$ , i.e. the algorithm terminates in a single iteration
  - Case 2:  $k > 1$ , i.e. the alg. does not terminate in a single iter.

**By contradiction,** assume there is a run with  $i_k - j_k > 1$

## Correctness of Hoare's Algorithm (4)

Original correctness claims:

- Indices  $i$  and  $j$  never reference  $A$  outside the interval  $A[p \dots r]$
- Split is always non-trivial; i.e.,  $j \neq r$  at termination

Proof:

- For  $k = 1$ :
  - Trivial because  $i_1 = j_1 = p$  (see Case 1 in proof of Lemma 2)
- For  $k > 1$ :
  - $i_k > p$  and  $j_k < r$  (due to the repeat-until loops moving indices)
  - $i_k \leq r$  and  $j_k \geq p$  (due to Lemma 1 and the statement above)

The proof of claims (a) and (b) complete

## Correctness of Hoare's Algorithm (5)

**Lemma 2:** At the end of iteration  $m$ , where  $m < k$  (i.e.  $m$  is not the last iteration), we must have:

$$A[p \dots i_m] \leq x \text{ and } A[j_m \dots r] \geq x$$

**Proof of Lemma 2:**

- **Base case:**  $m = 1$  and  $k > 1$  (i.e. the alg. does not terminate in the first iter.)

**Ind. Hyp.:** At the end of iteration  $m - 1$ , where  $m < k$  (i.e.  $m$  is not the last iteration), we must have:

$$A[p \dots i_m - 1] \leq x \text{ and } A[j_m - 1 \dots r] \geq x$$

**General case:** The lemma holds for  $m$ , where  $m < k$

**Proof of base case complete!**

## Correctness of Hoare's Algorithm (6)

Original correctness claim:

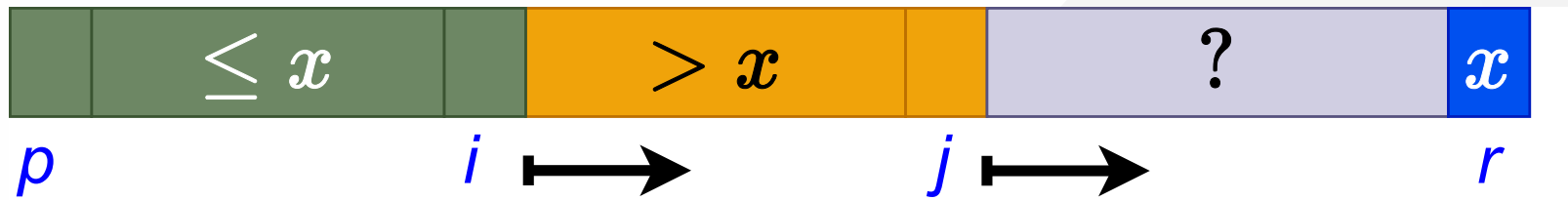
- (c) Every element in  $A[\dots j] \leq$  every element in  $A[j + \dots r]$  at termination

Proof of claim (c)

- There are 3 cases to consider:
  - **Case 1:**  $k = 1$ , i.e. the algorithm terminates in a single iteration
  - **Case 2:**  $k > 1$  and  $i_k = j_k$
  - **Case 3:**  $k > 1$  and  $i_k = j_k + 1$

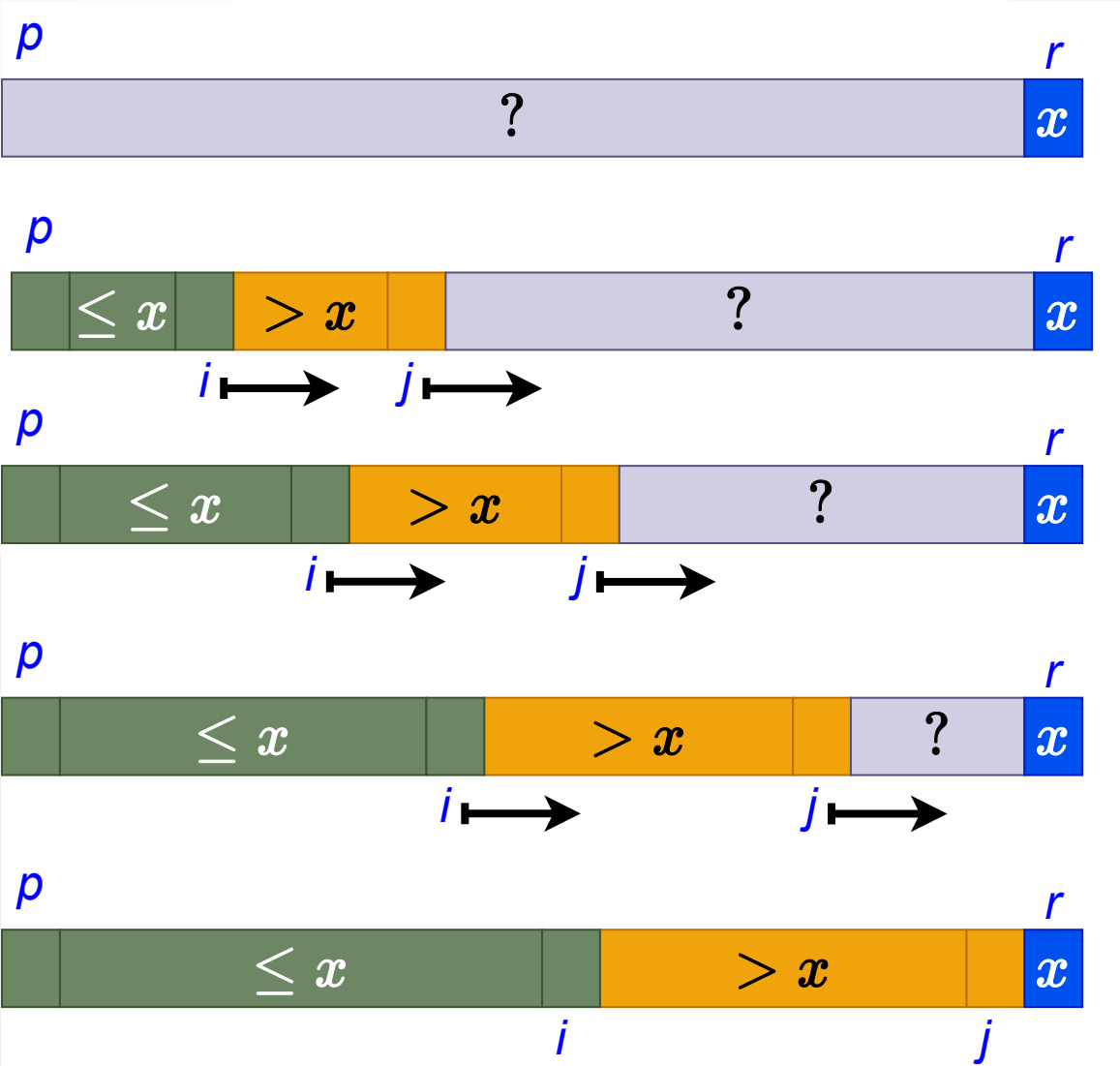
## Lomuto's Partitioning Algorithm (1)

- Choose a pivot element:  $pivot = x = A[r]$



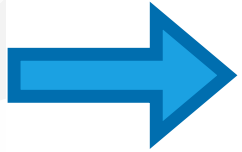
- Grow two regions:
  - from left to right:  $A[p \dots i]$
  - from left to right:  $A[i + 1 \dots j]$ 
    - such that:
      - every element in  $A[p \dots i] \leq pivot$
      - every element in  $A[i + 1 \dots j] > pivot$

# Lomuto's Partitioning Algorithm (2)





# Lomuto's Partitioning Algorithm Ex. (Step-1)



**L-PARTITION** ( $A, p, r$ ).

$pivot \leftarrow A[r]$

$i \leftarrow p - 1$

**for**  $j \leftarrow p$  **to**  $r - 1$  **do**

**if**  $A[j] \leq pivot$  **then**

$i \leftarrow i + 1$

**exchange**  $A[i] \leftrightarrow A[j]$

**exchange**  $A[i + 1] \leftrightarrow A[r]$

**return**  $i + 1$

Input

$p$	$Pivot = 4$						$r$
7	8	2	6	5	1	3	4

*STEP - 1*

# Lomuto's Partitioning Algorithm Ex. (Step-2)

**L-PARTITION** ( $A, p, r$ ).

$pivot \leftarrow A[r]$

$i \leftarrow p - 1$

**for**  $j \leftarrow p$  **to**  $r - 1$  **do**

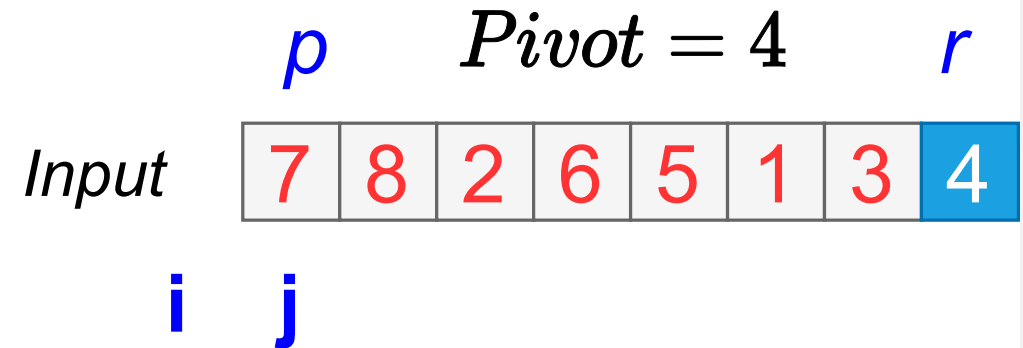
**if**  $A[j] \leq pivot$  **then**

$i \leftarrow i + 1$

**exchange**  $A[i] \leftrightarrow A[j]$

**exchange**  $A[i + 1] \leftrightarrow A[r]$

**return**  $i + 1$



**STEP - 2**

# Lomuto's Partitioning Algorithm Ex. (Step-3)

**L-PARTITION** ( $A, p, r$ ).

$pivot \leftarrow A[r]$

$i \leftarrow p - 1$

**for**  $j \leftarrow p$  **to**  $r - 1$  **do**

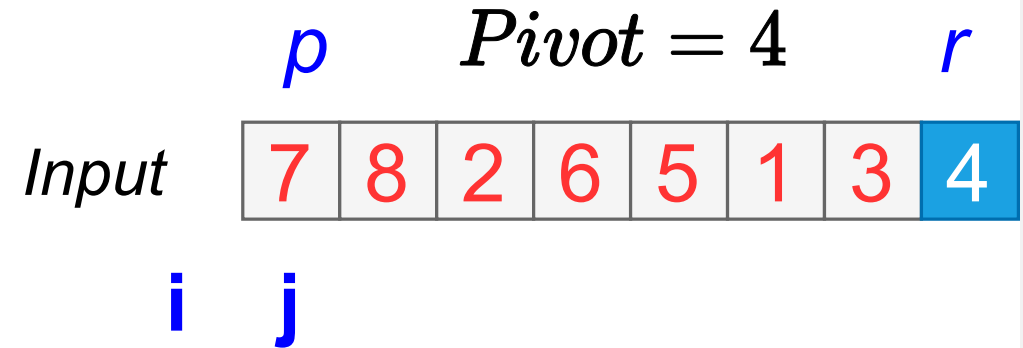
**if**  $A[j] \leq pivot$  **then**

$i \leftarrow i + 1$

**exchange**  $A[i] \leftrightarrow A[j]$

**exchange**  $A[i + 1] \leftrightarrow A[r]$

**return**  $i + 1$



**STEP - 3**

# Lomuto's Partitioning Algorithm Ex. (Step-4)

**L-PARTITION** ( $A, p, r$ ).

$pivot \leftarrow A[r]$

$i \leftarrow p - 1$

**for**  $j \leftarrow p$  **to**  $r - 1$  **do**

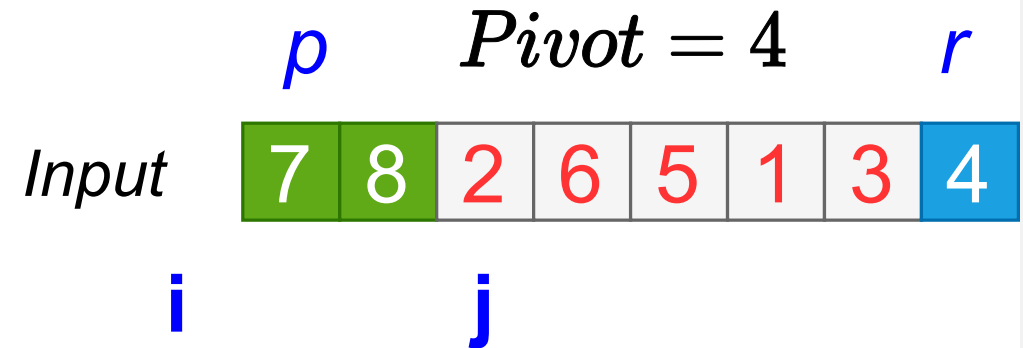
**if**  $A[j] \leq pivot$  **then**

$i \leftarrow i + 1$

**exchange**  $A[i] \leftrightarrow A[j]$

**exchange**  $A[i + 1] \leftrightarrow A[r]$

**return**  $i + 1$



**STEP - 4**

# Lomuto's Partitioning Algorithm Ex. (Step-5)

**L-PARTITION** ( $A, p, r$ ).

$pivot \leftarrow A[r]$

$i \leftarrow p - 1$

**for**  $j \leftarrow p$  **to**  $r - 1$  **do**

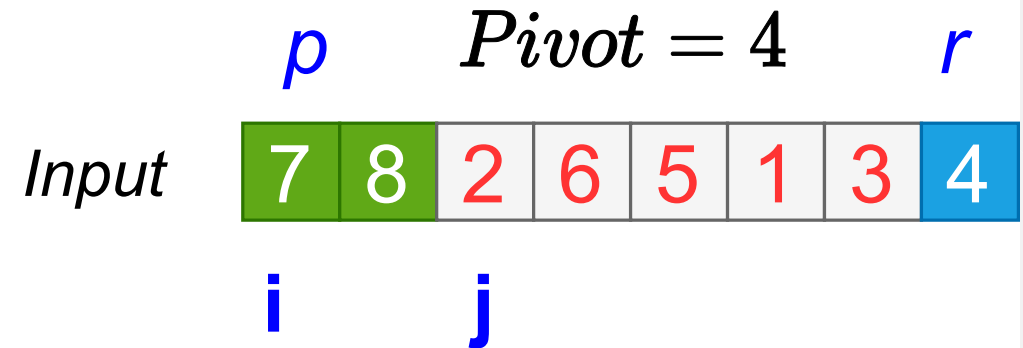
**if**  $A[j] \leq pivot$  **then**

$i \leftarrow i + 1$

**exchange**  $A[i] \leftrightarrow A[j]$

**exchange**  $A[i + 1] \leftrightarrow A[r]$

**return**  $i + 1$



**STEP - 5**

# Lomuto's Partitioning Algorithm Ex. (Step-6)

**L-PARTITION** ( $A, p, r$ ).

$pivot \leftarrow A[r]$

$i \leftarrow p - 1$

**for**  $j \leftarrow p$  **to**  $r - 1$  **do**

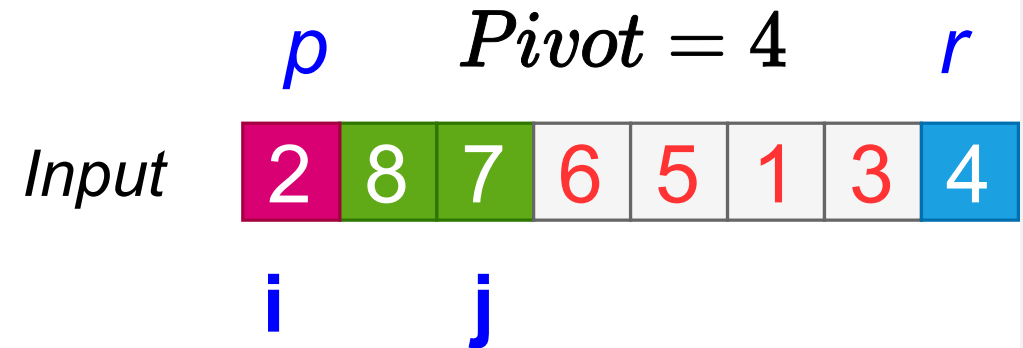
**if**  $A[j] \leq pivot$  **then**

$i \leftarrow i + 1$

**exchange**  $A[i] \leftrightarrow A[j]$

**exchange**  $A[i + 1] \leftrightarrow A[r]$

**return**  $i + 1$



**STEP - 6**



















# Lomuto's Partitioning Algorithm Ex. (Step-15)

**L-PARTITION** ( $A, p, r$ ).

$pivot \leftarrow A[r]$

$i \leftarrow p - 1$

**for**  $j \leftarrow p$  **to**  $r - 1$  **do**

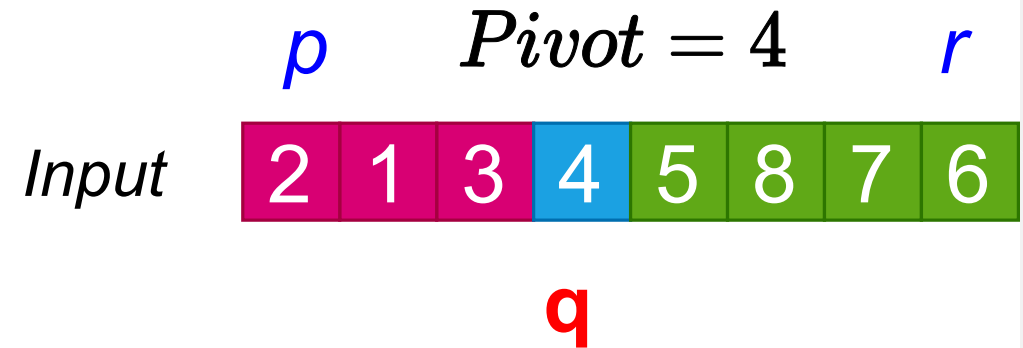
**if**  $A[j] \leq pivot$  **then**

$i \leftarrow i + 1$

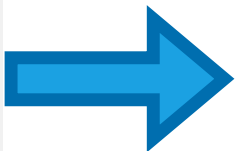
**exchange**  $A[i] \leftrightarrow A[j]$

**exchange**  $A[i + 1] \leftrightarrow A[r]$

**return**  $i + 1$



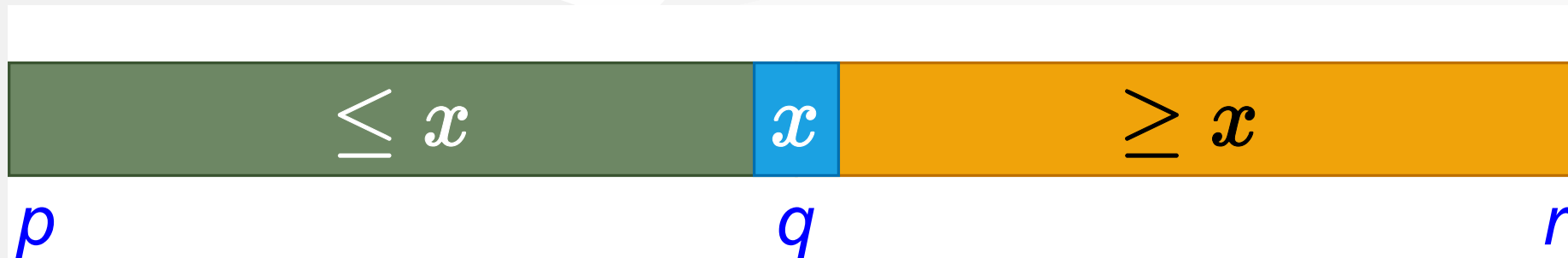
**STEP** – 15



# Quicksort with Lomuto's Partitioning Algorithm

```
QUICKSORT (A, p, r)
  if p < r then
    q = L-PARTITION(A, p, r)
    QUICKSORT(A, p, q - 1)
    QUICKSORT(A, q + 1, r)
  endif
```

Initial invocation: QUICKSORT(A, 1, n)





# Comparison of Hoare's & Lomuto's Algorithms (1)

- Notation:  $n = r - p + 1$ 
  - $pivot = A[p]$  (Hoare)
  - $pivot = A[r]$  (Lomuto)
- # of element exchanges:  $e(n)$ 
  - Hoare:  $0 \leq e(n) \leq \lfloor \frac{n}{2} \rfloor$ 
    - **Best:**  $k = 1$  with  $i_1 = j_1 = p$  (i.e.,  $A[p + 1 \dots r] > pivot$ )
    - **Worst:**  $A[p + 1 \dots p + \lfloor \frac{n}{2} \rfloor - 1] \geq pivot \geq A[p + \lceil \frac{n}{2} \rceil \dots r]$
  - Lomuto :  $1 \leq e(n) \leq n$ 
    - **Best:**  $A[p \dots r - 1] > pivot$
    - **Worst:**  $A[p \dots r - 1] \leq pivot$

## Comparison of Hoare's & Lomuto's Algorithms (2)

- # of element comparisons:  $c_e(n)$ 
  - Hoare:  $n + 1 \leq c_e(n) \leq n + 2$ 
    - Best:  $i_k = j_k$
    - Worst:  $i_k = j_k + 1$
  - Lomuto:  $c_e(n) = n - 1$
- # of index comparisons:  $c_i(n)$ 
  - Hoare:  $1 \leq c_i(n) \leq \lfloor \frac{n}{2} \rfloor + 1 \mid (c_i(n) = e(n) + 1)$
  - Lomuto:  $c_i(n) = n - 1$

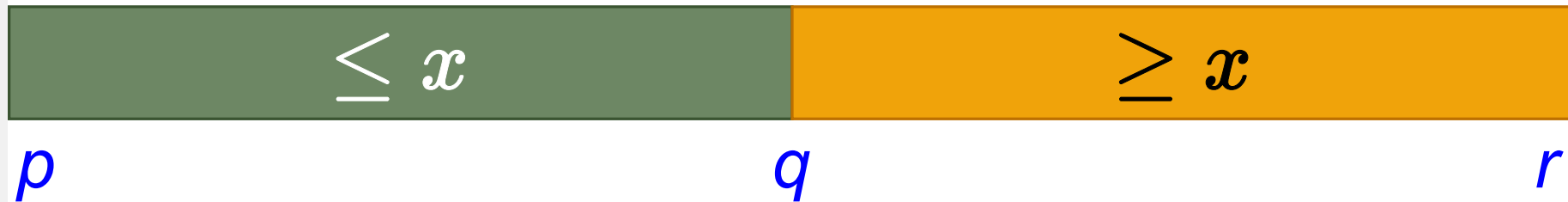
## Comparison of Hoare's & Lomuto's Algorithms (3)

- # of index increment/decrement operations:  $a(n)$ 
  - **Hoare:**  $n + 1 \leq a(n) \leq n + 2 \mid (a(n) = c_e(n))$
  - **Lomuto:**  $n \leq a(n) \leq 2n - 1 \mid (a(n) = e(n) + (n - 1))$
- Hoare's algorithm is in general faster
- Hoare behaves better when pivot is repeated in  $A[p \dots r]$ 
  - **Hoare:** Evenly distributes them between left & right regions
  - **Lomuto:** Puts all of them to the left region

# Analysis of Quicksort (1)

```
QUICKSORT (A, p, r)
  if p < r then
    q = H-PARTITION(A, p, r)
    QUICKSORT(A, p, q)
    QUICKSORT(A, q + 1, r)
  endif
```

Initial invocation: QUICKSORT(A, 1, n)



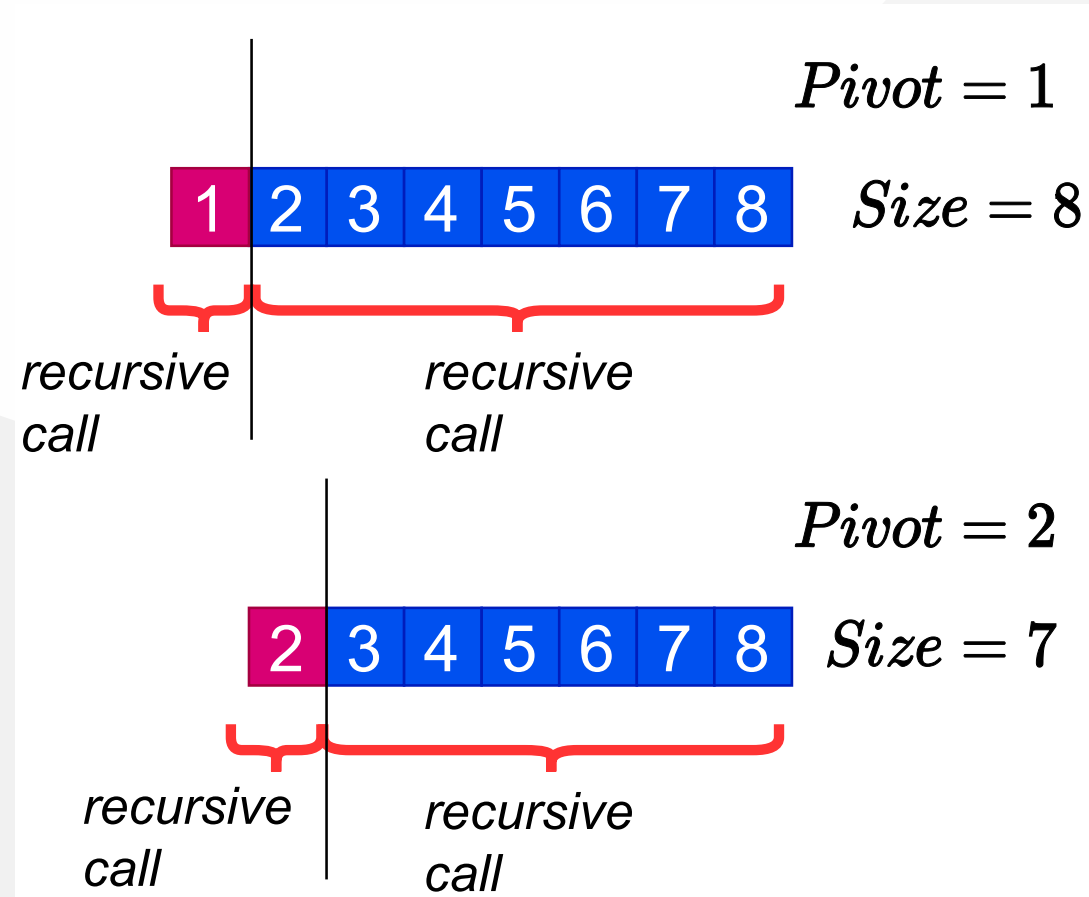
Assume **all** elements are **distinct** in the following analysis

## Analysis of Quicksort (2)

- **H-PARTITION** always chooses  $A[p]$  (the first element) as the pivot.
- The runtime of **QUICKSORT** on an already-sorted array is  $\Theta(n^2)$

## Example: An Already Sorted Array

Partitioning always leads to 2 parts of size 1 and  $n - 1$



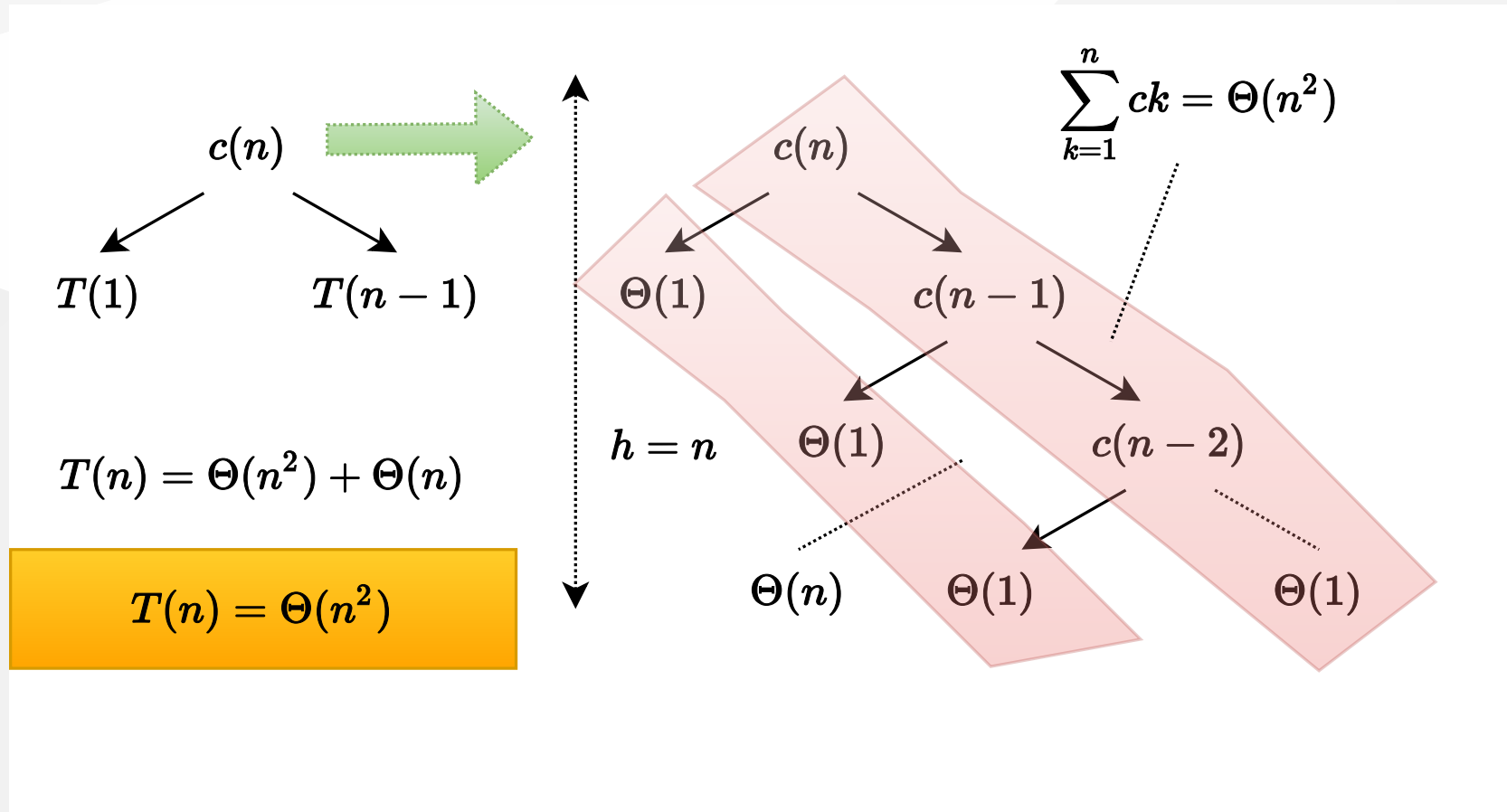
## Worst Case Analysis of Quicksort

- **Worst case** is when the **PARTITION** algorithm always returns **imbalanced partitions** (of size 1 and  $n - 1$ ) in every recursive call.
  - This happens when the pivot is selected to be either the min or **max** element.
  - This happens for **H-PARTITION** when the input array is already sorted or reverse sorted

$$\begin{aligned}T(n) &= T(1) + T(n - 1) + \Theta(n) \\&= T(n - 1) + \Theta(n) \\&= \Theta(n^2)\end{aligned}$$

# Worst Case Recursion Tree

$$T(n) = T(1) + T(n-1) + cn$$





## Best Case Analysis (for intuition only)

- If we're extremely lucky, **H-PARTITION** splits the array evenly at every recursive call

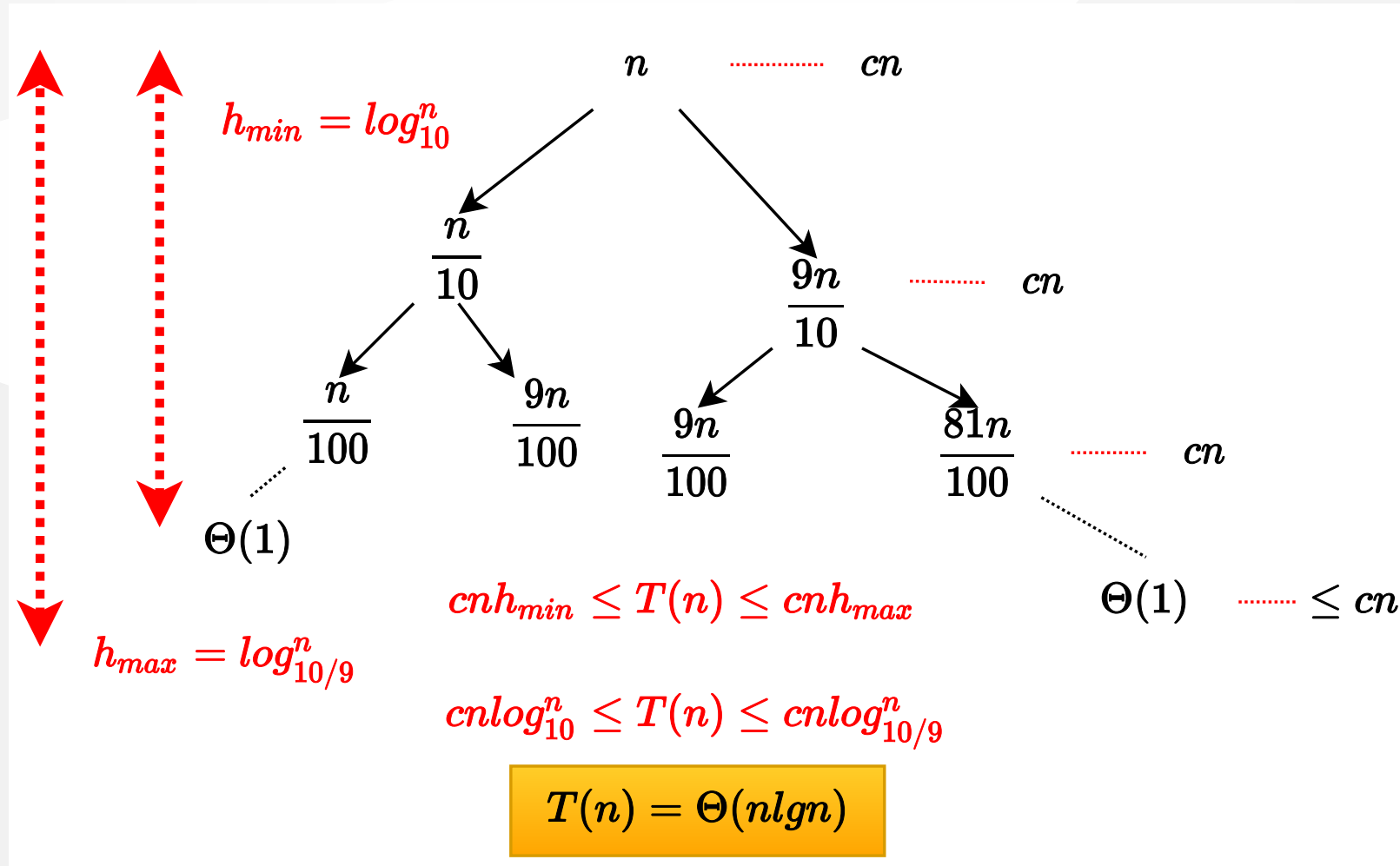
$$\begin{aligned}T(n) &= 2T(n/2) + \Theta(n) \\ &= \Theta(n \lg n)\end{aligned}$$

(same as merge sort)

- Instead of splitting 0.5 : 0.5, if we split 0.1 : 0.9 then we need solve following equation.

$$\begin{aligned}T(n) &= T(n/10) + T(9n/10) + \Theta(n) \\ &= \Theta(n \lg n)\end{aligned}$$

# "Almost-Best" Case Analysis



## Balanced Partitioning (1)

- We have seen that if **H-PARTITION** always splits the array with  $0.1 - to - 0.9$  ratio, the runtime will be  $\Theta(n \lg n)$ .
- Same is true with a split ratio of  $0.01 - to - 0.99$ , etc.
- Possible to show that if the split has always constant ( $\Theta(1)$ ) proportionality, then the runtime will be  $\Theta(n \lg n)$ .
- In other words, for a **constant**  $\alpha | (0 < \alpha \leq 0.5)$ :
  - $\alpha - to - (1 - \alpha)$  proportional split yields  $\Theta(n \lg n)$  total runtime

## Balanced Partitioning (2)

- In the rest of the analysis, assume that all input permutations are equally likely.
  - This is only to gain some intuition
  - We cannot make this assumption for average case analysis
  - We will revisit this assumption later
- Also, assume that all input elements are distinct.

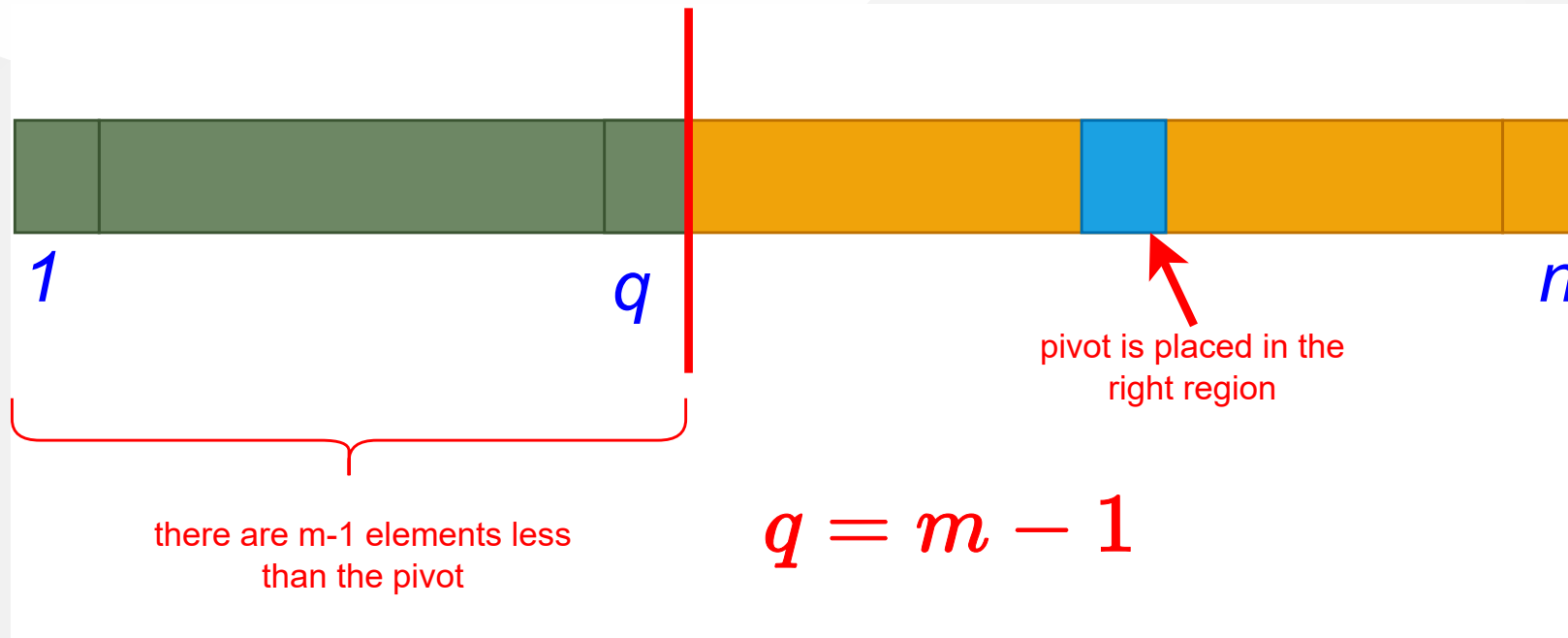
## Balanced Partitioning (3)

- **Question:** What is the probability that H-PARTITION returns a split that is more balanced than  $0.1 - t_0 - 0.9$ ?

## Balanced Partitioning (4)

**Reminder:** *H-PARTITION* will place the pivot in the right partition unless the pivot is the smallest element in the arrays.

**Question:** If the pivot selected is the  $m$ th smallest value ( $1 < m \leq n$ ) in the input array, what is the size of the left region after partitioning?

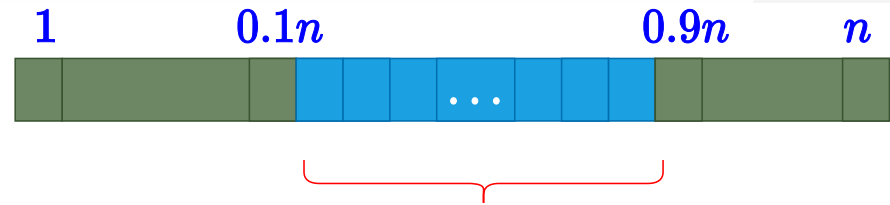


## Balanced Partitioning (5)

- **Question:** What is the probability that the **pivot** selected is the  $m^{th}$  smallest value in the array of size  $n$ ?
  - $1/n$  (*since all input permutations are equally likely*)
- **Question:** What is the probability that the left partition returned by **H-PARTITION** has size  $m$ , where  $1 < m < n$ ?
  - $1/n$  (*due to the answers to the previous 2 questions*)

# Balanced Partitioning (6)

- **Question:** What is the probability that H-PARTITION returns a split that is more balanced than  $0.1 - to - 0.9$ ?



The partition boundary will be in this region for a more balanced split than

$0.1 - to - 0.9$

$$Probability = \sum_{q=0.1n+1}^{0.9n-1} \frac{1}{n} = \frac{1}{n} (0.9n - 1 - 0.1n - 1 + 1)$$

$$= 0.8 - \frac{1}{n}$$

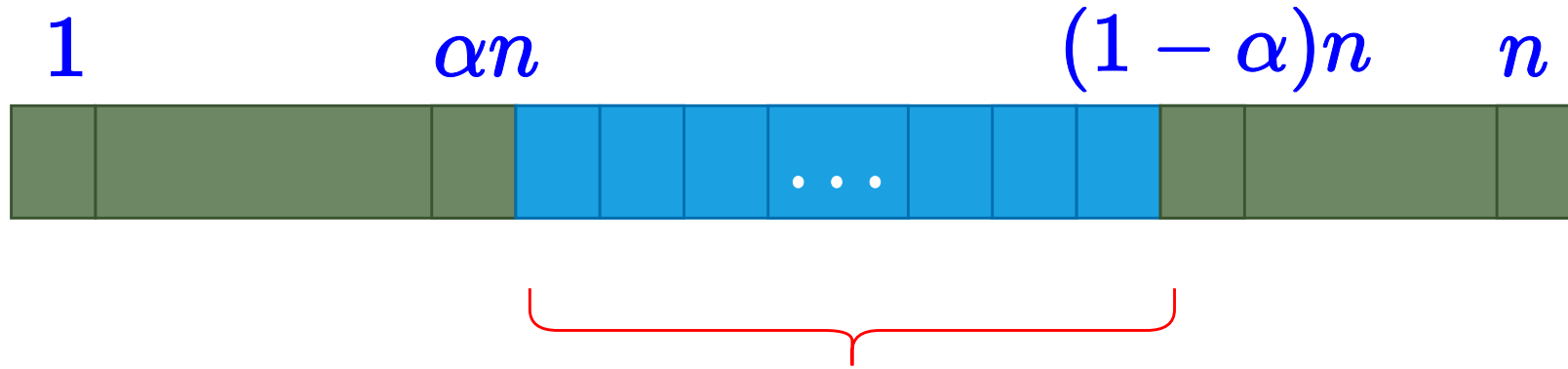
$\approx 0.8$  for large  $n$



## Balanced Partitioning (7)

- The probability that **H-PARTITION** yields a split that is more balanced than  $0.1 - to - 0.9$  is 80% on a random array.
- Let  $P_{\alpha>}$  be the probability that **H-PARTITION** yields a split more balanced than  $\alpha - to - (1 - \alpha)$ , where  $0 < \alpha \leq 0.5$
- Repeat the analysis to generalize the previous result

balanced than  $\alpha - to - (1 - \alpha)$ ?



The partition boundary will be in this region for a more balanced split than

$$\alpha n - to - (1 - \alpha)n$$

$$\begin{aligned}
 \text{Probability} &= \sum_{q=\alpha n+1}^{(1-\alpha)n-1} \frac{1}{n} \\
 &= \frac{1}{n} ((1 - \alpha)n - 1 - \alpha n - 1 + 1)
 \end{aligned}$$

## Balanced Partitioning (9)

- We found  $P_{\alpha>} = 1 - 2\alpha$ 
  - Ex:  $P_{0.1>} = 0.8$  and  $P_{0.01>} = 0.98$
- Hence, **H-PARTITION** produces a split
  - **more balanced than a**
    - $0.1 - to - 0.9$  split 80% of the time
    - $0.01 - to - 0.99$  split 98% of the time
  - **less balanced than a**
    - $0.1 - to - 0.9$  split 20% of the time
    - $0.01 - to - 0.99$  split 2% of the time

## Intuition for the Average Case (1)

- **Assumption:** All permutations are equally likely
  - Only for intuition; we'll revisit this assumption later
- **Unlikely:** Splits always the same way at every level
- **Expectation:**
  - Some splits will be reasonably balanced
  - Some splits will be fairly unbalanced
- **Average case:** A mix of good and bad splits
  - **Good** and **bad** splits distributed randomly thru the tree

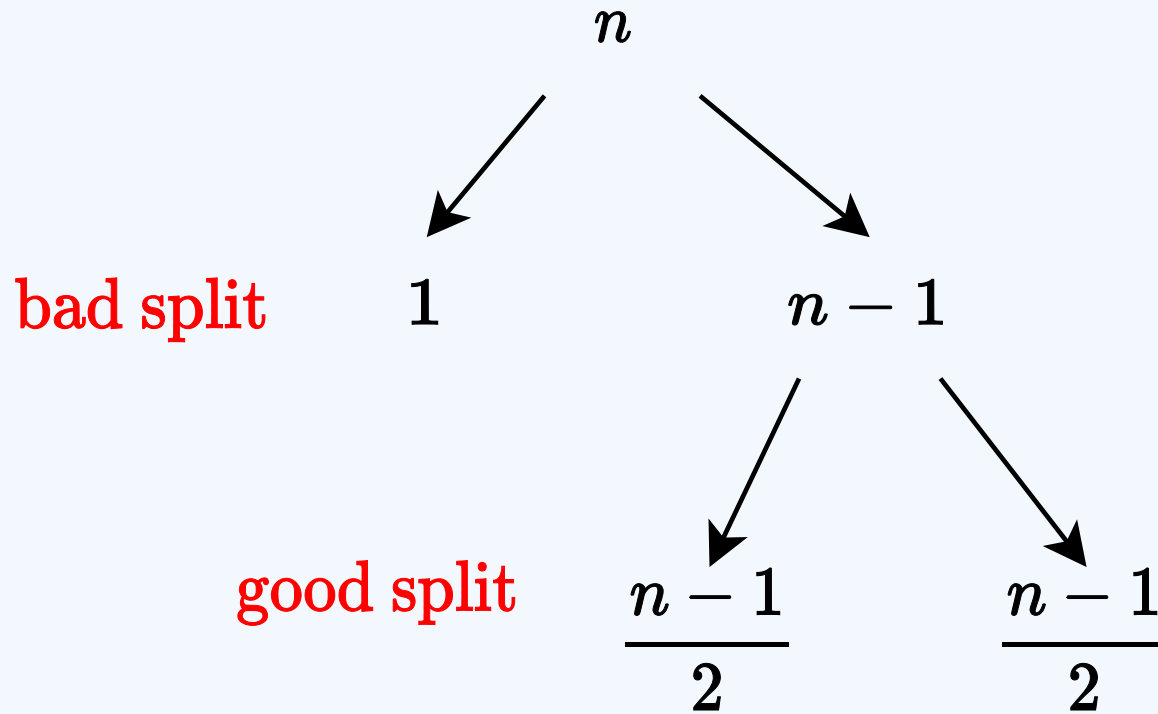
## Intuition for the Average Case (2)

- **Assume for intuition:** Good and bad splits occur in the alternate levels of the tree
  - **Good split:** Best case split
  - **Bad split:** Worst case split

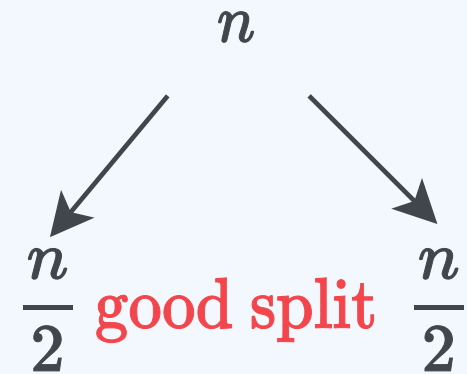
## Intuition for the Average Case (3)

Compare 2-successive levels of avg case vs. 1 level of best case

### AVERAGE CASE



### BEST CASE



## Intuition for the Average Case (4)

- In terms of the remaining subproblems, **two levels of avg case** is slightly better than the **single level of the best case**
- The avg case has **extra divide cost of  $\Theta(n)$**  at alternate levels
- The extra divide cost  $\Theta(n)$  of bad splits absorbed into the  $\Theta(n)$  of good splits.
- Running time is still  $\Theta(n \lg n)$ 
  - But, slightly larger hidden constants, because the height of the recursion tree is about twice of that of best case.

# Intuition for the Average Case (5)

- Another way of looking at it:
  - Suppose we alternate lucky, unlucky, lucky, unlucky, . . .
  - We can write the recurrence as:
    - $L(n) = 2U(n/2) + \Theta(n)$  lucky split (best)
    - $U(n) = L(n - 1) + \Theta(n)$  unlucky split (worst)
  - Solving:

$$\begin{aligned} L(n) &= 2(L(n/2 - 1) + \Theta(n/2)) + \Theta(n) \\ &= 2L(n/2 - 1) + \Theta(n) \\ &= \Theta(n \lg n) \end{aligned}$$

- How can we make sure we are usually lucky for all inputs?



## Summary: Quicksort Runtime Analysis (1)

- **Worst case:** Unbalanced split at every recursive call

$$T(n) = T(1) + T(n - 1) + \Theta(n)$$

$$T(n) = \Theta(n^2)$$

- **Best case:** Balanced split at every recursive call (*extremely lucky*)

$$T(n) = 2T(n/2) + \Theta(n)$$

$$T(n) = \Theta(n \lg n)$$

## Summary: Quicksort Runtime Analysis (2)

- **Almost-best case:** Almost-balanced split at every recursive call

$$T(n) = T(n/10) + T(9n/10) + \Theta(n)$$

$$\text{or } T(n) = T(n/100) + T(99n/100) + \Theta(n)$$

$$\text{or } T(n) = T(\alpha n) + T((1 - \alpha)n) + \Theta(n)$$

for any constant  $\alpha$ ,  $0 < \alpha \leq 0.5$

## Summary: Quicksort Runtime Analysis (3)

- For a random input array, the probability of having a split
  - more balanced than  $0.1$ – $to$ – $0.9$  :  $80\%$
  - more balanced than  $0.01$ – $to$ – $0.99$  :  $98\%$
  - more balanced than  $\alpha$ – $to$ – $(1 - \alpha)$  :  $1 - 2\alpha$   
for any constant  $\alpha$ ,  $0 < \alpha \leq 0.5$

## Summary: Quicksort Runtime Analysis (4)

- **Avg case intuition:** Different splits expected at different levels
  - some balanced (good), some unbalanced (bad)
- **Avg case intuition:** Assume the good and bad splits alternate
  - i.e. good split -> bad split -> good split -> ...
  - $T(n) = \Theta(n \lg n)$ 
    - (informal analysis for intuition)

## Randomized Quicksort

- In the avg-case analysis, we assumed that **all permutations** of the input array are **equally likely**.
  - But, this assumption **does not always hold**
  - e.g. What if **all** the input arrays are **reverse sorted**?
    - **Always worst-case behavior**
- Ideally, the avg-case runtime should be **independent of the input permutation**.
- **Randomness should be within the algorithm**, not based on the distribution of the inputs.
  - i.e. The avg case should hold for all possible inputs

## Randomized Algorithms (1)

- Alternative to assuming a uniform distribution:
  - **Impose a uniform distribution**
  - e.g. Choose a random pivot rather than the first element
- Typically useful when:
  - there are many ways that an algorithm can proceed
  - but, it's **difficult** to determine a way that is **always guaranteed to be good**.
  - If there are **many good alternatives**; simply **choose one randomly**.

## Randomized Algorithms (1)

- Ideally:
  - Runtime should be **independent of the specific inputs**
  - No specific input should cause worst-case behavior
  - Worst-case should be determined only by output of a random number generator.

# Randomized Quicksort (1)

- Using Hoare's partitioning algorithm:

```
R-QUICKSORT(A, p, r)
  if p < r then
    q = R-PARTITION(A, p, r)
    R-QUICKSORT(A, p, q)
    R-QUICKSORT(A, q+1, r)
```

```
R-PARTITION(A, p, r)
  s = RANDOM(p, r)
  exchange A[p] with A[s]
  return H-PARTITION(A, p, r)
```

- Alternatively, permuting the whole array would also work
  - but, would be more difficult to analyze



## Randomized Quicksort (2)

- Using Lomuto's partitioning algorithm:

```
R-QUICKSORT(A, p, r)
  if p < r then
    q = R-PARTITION(A, p, r)
    R-QUICKSORT(A, p, q-1)
    R-QUICKSORT(A, q+1, r)
```

```
R-PARTITION(A, p, r)
  s = RANDOM(p, r)
  exchange A[r] with A[s]
  return L-PARTITION(A, p, r)
```

- Alternatively, permuting the whole array would also work
  - but, would be more difficult to analyze

## Notations for Formal Analysis

- Assume all elements in  $A[p \dots r]$  are distinct
  - Let  $n = r - p + 1$
- Let  $rank(x) = |A[i] : p \leq i \leq r \text{ and } A[i] \leq x|$
- i.e.  $rank(x)$  is the number of array elements with value less than or equal to  $x$ 
  - $A = \{5, 9, 7, 6, 8, 1, 4\}$
  - $p = 5, r = 4$
  - $rank(5) = 3$ 
    - i.e. it is the 3<sup>rd</sup> smallest element in the array

## Formal Analysis for Average Case

- The following analysis will be for **Quicksort** using **Hoare's** partitioning algorithm.
- **Reminder:** The **pivot** is selected **randomly** and exchanged with  $A[p]$  before calling **H-PARTITION**
- Let  $x$  be the **random pivot** chosen.
- What is the probability that  $rank(x) = i$  for  $i = 1, 2, \dots, n$  ?
  - $P(rank(x) = i) = 1/n$

## Various Outcomes of H-PARTITION (1)

- Assume that  $rank(x) = 1$ 
  - i.e. the **random pivot** chosen is the **smallest** element
  - What will be the **size of the left partition** ( $|L|$ )?
  - **Reminder:** Only the elements less than or equal to  $x$  will be in the left partition.

$$A = \{ \overset{p=x=pivot}{\underbrace{2}}, \underbrace{9, 7, 6, 8, 5}_{\Rightarrow |L|=1}, \overset{r}{\underbrace{4}} \}$$

$$p = 2, r = 4$$

$$pivot = x = 2$$

# Various Outcomes of H-PARTITION (2)

- Assume that  $\text{rank}(x) > 1$ 
  - i.e. the random pivot chosen is not the smallest element
  - What will be the size of the left partition ( $|L|$ )?
  - **Reminder:** Only the elements less than or equal to  $x$  will be in the left partition.
  - **Reminder:** The pivot will stay in the right region after H-PARTITION if  $\text{rank}(x) > 1$

$$A = \{ \overbrace{2}^p, 4, \underbrace{7, 6, 8, \overbrace{5}^{\text{pivot}}, \overbrace{9}^r} \}$$

$$\implies |L| = \text{rank}(x) - 1$$


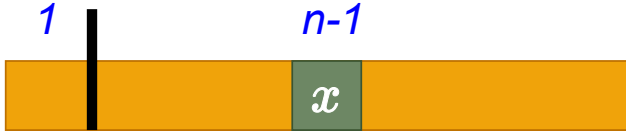
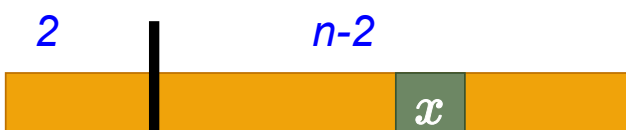


$$p = 2, r = 4$$

$$\text{pivot} = x = 5$$

## Various Outcomes of H-PARTITION - Summary (1)

- $x : pivot$
- $|L|$  : size of left region
- $P(rank(x) = i) = 1/n$  for  $1 \leq i \leq n$ 
  - if  $rank(x) = 1$  then  $|L| = 1$
  - if  $rank(x) > 1$  then  $|L| = rank(x) - 1$
- $P(|L| = 1) = P(rank(x) = 1) + P(rank(x) = 2)$ 
  - $P(|L| = 1) = 2/n$
- $P(|L| = i) = P(rank(x) = i + 1)$  for  $1 < i < n$ 
  - $P(|L| = i) = 1/n$  for  $1 < i < n$

## Various Outcomes of H-PARTITION - Summary (2)

<i>rank(x)</i>	<i>probability</i>	<i>T(n)</i>	
1	$\frac{1}{n}$	$T(1) + T(n-1) + \Theta(n)$	
2	$\frac{1}{n}$	$T(1) + T(n-1) + \Theta(n)$	
3	$\frac{1}{n}$	$T(2) + T(n-2) + \Theta(n)$	
$\vdots$	$\vdots$	$\vdots$	
$i+1$	$\frac{1}{n}$	$T(i) + T(n-i) + \Theta(n)$	
$\vdots$	$\vdots$	$\vdots$	
$n$	$\frac{1}{n}$	$T(n-1) + T(1) + \Theta(n)$	

# Average - Case Analysis: Recurrence (1)

$x = pivot$

$$\begin{aligned}
 T(n) &= \frac{1}{n}(T(1) + t(n-1)) & rank : 1 \\
 &+ \frac{1}{n}(T(1) + t(n-1)) & rank : 2 \\
 &+ \frac{1}{n}(T(2) + t(n-2)) & rank : 3 \\
 &\vdots & \vdots \\
 &+ \frac{1}{n}(T(i) + t(n-i)) & rank : i + 1 \\
 &\vdots & \vdots \\
 &+ \frac{1}{n}(T(n-1) + t(1)) & rank : n \\
 &+ \Theta(n)
 \end{aligned}$$



## Average - Case Analysis: Recurrence (2)

$$T(n) = \frac{1}{n} \sum_{q=1}^{n-1} (T(q) + T(n-q)) + \frac{1}{n} (T(1) + T(n-1)) + \Theta(n)$$

Note:  $\frac{1}{n} (T(1) + T(n-1)) = \frac{1}{n} (\Theta(1) + O(n^2)) = O(n)$

$$T(n) = \frac{1}{n} \sum_{q=1}^{n-1} (T(q) + T(n-q)) + \Theta(n)$$

for  $k = 1, 2, \dots, n-1$  each term  $T(k)$  appears twice once for  $q = k$  and once for  $q = n - k$

$$T(n) = \frac{2}{n} \sum_{k=1}^{n-1} T(k) + \Theta(n)$$

# Average - Case Analysis -Solving Recurrence: Substitution

- Guess:  $T(n) = O(n \lg n)$
- $T(k) \leq a k \lg k$  for  $k < n$ , for some constant  $a > 0$

$$\begin{aligned} T(n) &= \frac{2}{n} \sum_{k=1}^{n-1} T(k) + \Theta(n) \\ &\leq \frac{2}{n} \sum_{k=1}^{n-1} a k \lg k + \Theta(n) \\ &\leq \frac{2a}{n} \sum_{k=1}^{n-1} k \lg k + \Theta(n) \end{aligned}$$

- Need a tight bound for  $\sum k \lg k$

## Tight bound for $\sum klgk$ (1)

- Bounding the terms
  - $\sum_{k=1}^{n-1} klgk \leq \sum_{k=1}^{n-1} nlg n = n(n-1)lg n \leq n^2lg n$
  - This bound is **not strong** enough because
  - $T(n) \leq \frac{2a}{n} n^2lg n + \Theta(n)$
  - $= 2anlg n + \Theta(n) \implies$  couldn't prove  $T(n) \leq anlg n$

## Tight bound for $\sum klgk$ (2)

- **Splitting summations:** ignore ceilings for simplicity
- $\sum_{k=1}^{n-1} klgk \leq \sum_{k=1}^{n/2-1} klgk + \sum_{k=n/2}^{n-1} klgk$ 
  - **First summation:**  $lgk < lg(n/2) = lgn - 1$
  - **Second summation:**  $lgk < lgn$

**Splitting:**  $\sum_{k=1}^{n-1} k \lg k \leq \sum_{k=1}^{n/2-1} k \lg k + \sum_{k=n/2}^{n-1} k \lg k \quad (3)$

- $\sum_{k=1}^{n-1} k \lg k \leq (\lg(n-1)) \sum_{k=1}^{n/2-1} k + \lg n \sum_{k=n/2}^{n-1} k$ 
  - $= \lg n \sum_{k=1}^{n-1} k - \sum_{k=1}^{n/2-1} k$
  - $= \frac{1}{2}n(n-1)\lg n - \frac{1}{2} \frac{n}{2} \left( \frac{n}{2} - 1 \right)$
  - $= \frac{1}{2}n^2 \lg n - \frac{1}{8}n^2 - \frac{1}{2}n(\lg n - 1/2)$
- $\sum_{k=1}^{n-1} k \lg k \leq \frac{1}{2}n^2 \lg n - \frac{1}{8}n^2$  for  $\lg n \geq 1/2 \implies n \geq \sqrt{2}$

**Substituting:** -  $\sum_{k=1}^{n-1} k \lg k \leq \frac{1}{2}n^2 \lg n - \frac{1}{8}n^2$  (4)

$$\begin{aligned} T(n) &\leq \frac{2a}{n} \sum_{k=1}^{n-1} k \lg k + \Theta(n) \\ &\leq \frac{2a}{n} \left( \frac{1}{2}n^2 \lg n - \frac{1}{8}n^2 \right) + \Theta(n) \\ &= a n \lg n - \left( \frac{a}{4}n - \Theta(n) \right) \end{aligned}$$

- We can choose  $a$  large enough so that  $\frac{a}{4}n \geq \Theta(n)$

$$T(n) \leq a n \lg n$$

$$T(n) = O(n \lg n)$$

Q.E.D.

# Medians and Order Statistics

- **ith order statistic:**  $i^{th}$  smallest element of a set of  $n$  elements
- **minimum:** *first* order statistic
- **maximum:**  $n^{th}$  order statistic
- **median:** “halfway point” of the set

$$i = \left\lfloor \frac{(n + 1)}{2} \right\rfloor$$

or

$$i = \left\lceil \frac{(n + 1)}{2} \right\rceil$$

## Selection Problem

- **Selection problem:** Select the  $i^{th}$  smallest of  $n$  elements
- **Naïve algorithm:** Sort the input array  $A$ ; then return  $A[i]$ 
  - $T(n) = \theta(n \lg n)$ 
    - *using e.g. merge sort (but not quicksort)*
- Can we do any better?



## Selection in Expected Linear Time

- Randomized algorithm using divide and conquer
- Similar to randomized quicksort
  - **Like quicksort:** Partitions input array recursively
  - **Unlike quicksort:** Makes a single recursive call
    - **Reminder:** *Quicksort makes two recursive calls*
- Expected runtime:  $\Theta(n)$ 
  - **Reminder:** *Expected runtime of quicksort:  $\Theta(n \lg n)$*

## Selection in Expected Linear Time: Example 1

- Select the  $2^{nd}$  smallest element:

$$A = \{6, 10, 13, 5, 8, 3, 2, 11\}$$
$$i = 2$$

- Partition the input array:

$$A = \{ \underbrace{2, 3, 5}_{\text{left subarray}}, \underbrace{13, 8, 10, 6, 11}_{\text{right subarray}} \}$$

- make a recursive call to select the  $2^{nd}$  smallest element in **left subarray**

## Selection in Expected Linear Time: Example 2

- Select the 7<sup>th</sup> smallest element:

$$A = \{6, 10, 13, 5, 8, 3, 2, 11\}$$
$$i = 7$$

- Partition the input array:

$$A = \{ \underbrace{2, 3, 5}_{\text{left subarray}}, \underbrace{13, 8, 10, 6, 11}_{\text{right subarray}} \}$$

- make a recursive call to select the 4<sup>th</sup> smallest element in right subarray

## Selection in Expected Linear Time (1)

```

R-SELECT(A,p,r,i)
  if p == r then
    return A[p];
  q = R-PARTITION(A, p, r)
  k = q-p+1;
  if i <= k then
    return R-SELECT(A, p, q, i);
  else
    return R-SELECT(A, q+1, r, i-k);

```

$$A = \{ \underbrace{\quad}_p \cdots \leq x(\text{k smallest elements}) \cdots \underbrace{\quad}_q \cdots \geq x \cdots \underbrace{\quad}_r \}$$

$x = \text{pivot}$

## Selection in Expected Linear Time (2)

$$A = \left\{ \overbrace{\quad}^p \mid \underbrace{\cdots \leq x \cdots}_L \mid \underbrace{\cdots \geq x \cdots}_R \mid \overbrace{\quad}^r \right\}$$

$$x = \textit{pivot}$$

- All elements in  $L \leq$  all elements in  $R$
- $L$  contains:
- $|L| = q - p + 1 = k$  smallest elements of  $A[p \dots r]$
- if  $i \leq |L| = k$  then
  - **search**  $L$  recursively for its  $i^{\text{th}}$  smallest element
- else
  - **search**  $R$  recursively for its  $(i - k)^{\text{th}}$  smallest element

## Runtime Analysis (1)

- **Worst case:**
  - Imbalanced partitioning at every level and the recursive call always to the larger partition

$$= \{1, \underbrace{2, 3, 4, 5, 6, 7, 8}_{\text{recursive call}}\} \quad i = 8$$

$$= \{2, \underbrace{3, 4, 5, 6, 7, 8}_{\text{recursive call}}\} \quad i = 7$$

## Runtime Analysis (2)

- **Worst case:** Worse than the naïve method (based on sorting)

$$T(n) = T(n - 1) + \Theta(n)$$

$$T(n) = \Theta(n^2)$$

- **Best case:** Balanced partitioning at every recursive level

$$T(n) = T(n/2) + \Theta(n)$$

$$T(n) = \Theta(n)$$

- **Avg case:** Expected runtime – need analysis T.B.D.

## Reminder: Various Outcomes of H-PARTITION

- $x : \text{pivot}$
- $|L|$  : size of left region
- $P(\text{rank}(x) = i) = 1/n$  for  $1 \leq i \leq n$ 
  - if  $\text{rank}(x) = 1$  then  $|L| = 1$
  - if  $\text{rank}(x) > 1$  then  $|L| = \text{rank}(x) - 1$
- $P(|L| = 1) = P(\text{rank}(x) = 1) + P(\text{rank}(x) = 2)$ 
  - $P(|L| = 1) = 2/n$
- $P(|L| = i) = P(\text{rank}(x) = i + 1)$  for  $1 < i < n$ 
  - $P(|L| = i) = 1/n$  for  $1 < i < n$



## Average Case Analysis of Randomized Select


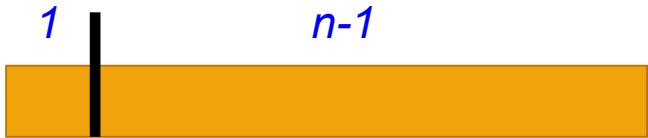
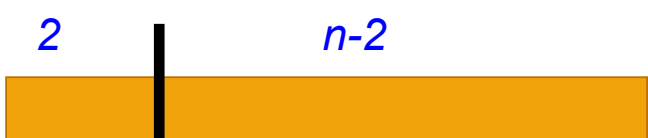


- To compute the **upper bound** for the **avg case**, assume that the  $i^{th}$  element always falls into the **larger partition**.

$$A = \left\{ \overbrace{\quad}^p \mid \underbrace{\dots \leq x \dots}_{LeftPartition} \mid \underbrace{\dots \geq x \dots}_{RightPartition} \mid \overbrace{\quad}^r \right\}$$

$$x = pivot$$

- We will analyze the case where the recursive call is always made to the larger partition
  - This will give us an upper bound for the avg case*

# Various Outcomes of H-PARTITION

<i>rank(x)</i>	<i>probability</i>	<i>T(n)</i>	
1	$\frac{1}{n}$	$\leq T(\max(1, n - 1)) + \Theta(n)$	
2	$\frac{1}{n}$	$\leq T(\max(1, n - 1)) + \Theta(n)$	
3	$\frac{1}{n}$	$\leq T(\max(2, n - 2)) + \Theta(n)$	
$\vdots$	$\vdots$	$\vdots$	
$i + 1$	$\frac{1}{n}$	$\leq T(\max(i, n - i)) + \Theta(n)$	
$\vdots$	$\vdots$	$\vdots$	
$n$	$\frac{1}{n}$	$\leq T(\max(n - 1, 1)) + \Theta(n)$	

## Average-Case Analysis of Randomized Select (1)

$$\text{Recall: } P(|L| = i) = \begin{cases} 2/n & \text{for } i = 1 \\ 1/n & \text{for } i = 2, 3, \dots, n-1 \end{cases}$$

**Upper bound:** Assume  $i^{\text{th}}$  element always falls into the larger part.

$$T(n) \leq \frac{1}{n}T(\max(1, n-1)) + \frac{1}{n} \sum_{q=1}^{n-1} T(\max(q, n-q)) + O(n)$$

$$\text{Note : } \frac{1}{n}T(\max(1, n-1)) = \frac{1}{n}T(n-1) = \frac{1}{n}O(n^2) = O(n)$$

$$\therefore \text{ (3 dot mean therefore) } T(n) \leq \frac{1}{n} \sum_{q=1}^{n-1} T(\max(q, n-q)) + O(n)$$

## Average-Case Analysis of Randomized Select (2)

$$\therefore T(n) \leq \frac{1}{n} \sum_{q=1}^{n-1} T(\max(q, n-q)) + O(n)$$

$$\max(q, n-q) = \begin{cases} q & \text{if } q \geq \lceil n/2 \rceil \\ n-q & \text{if } q < \lceil n/2 \rceil \end{cases}$$

- $n$  is odd:  $T(k)$  appears twice for  $k = \lceil n/2 \rceil + 1, \lceil n/2 \rceil + 2, \dots, n-1$
- $n$  is even:  $T(\lceil n/2 \rceil)$  appears once  $T(k)$  appears twice for  $k = \lceil n/2 \rceil + 1, \lceil n/2 \rceil + 2, \dots, n-1$

## Average-Case Analysis of Randomized Select (3)

- Hence, in both cases:

$$\sum_{q=1}^{n-1} T(\max(q, n - q)) + O(n) \leq 2 \sum_{q=\lceil n/2 \rceil}^{n-1} T(q) + O(n)$$

$$\therefore T(n) \leq \frac{2}{n} \sum_{q=\lceil n/2 \rceil}^{n-1} T(q) + O(n)$$

# Average-Case Analysis of Randomized Select (4)

$$T(n) \leq \frac{2}{n} \sum_{q=\lceil n/2 \rceil}^{n-1} T(q) + O(n)$$

- By substitution guess  $T(n) = O(n)$
- Inductive hypothesis:  $T(k) \leq ck, \forall k < n$

$$T(n) \leq \frac{2}{n} \sum_{q=\lceil n/2 \rceil}^{n-1} ck + O(n)$$

$$= \frac{2c}{n} \left( \sum_{k=1}^{n-1} k - \sum_{k=1}^{\lceil n/2 \rceil - 1} k \right) + O(n)$$

$$= \frac{2c}{n} \left( \frac{1}{2}n(n-1) - \frac{1}{2}\lceil \frac{n}{2} \rceil \left( \frac{n}{2} - 1 \right) \right) + O(n)$$

## Average-Case Analysis of Randomized Select (5)

$$\begin{aligned}
 T(n) &\leq \frac{2c}{n} \left( \frac{1}{2}n(n-1) - \frac{1}{2} \lceil \frac{n}{2} \rceil \left( \frac{n}{2} - 1 \right) \right) + O(n) \\
 &\leq c(n-1) - \frac{c}{4}n + \frac{c}{2} + O(n) \\
 &= cn - \frac{c}{4}n - \frac{c}{2} + O(n) \\
 &= cn - \left( \left( \frac{c}{4}n + \frac{c}{2} \right) + O(n) \right) \\
 &\leq cn
 \end{aligned}$$

- since we can choose  $c$  large enough so that  $(cn/4 + c/2)$  dominates  $O(n)$

## Summary of Randomized Order-Statistic Selection

- Works fast: linear expected time
- Excellent algorithm in practise
- But, the worst case is very bad:  $\Theta(n^2)$
- **Blum, Floyd, Pratt, Rivest & Tarjan[1973]** algorithms are runs in **linear time** in the **worst case**.
- Generate a **good pivot** recursively



# Selection in Worst Case Linear Time

```
//return i-th element in set S with n elements
SELECT(S, n, i)

    if n <= 5 then

        SORT S and return the i-th element

    DIVIDE S into ceil(n/5) groups
    //first ceil(n/5) groups are of size 5, last group is of size n mod 5

    FIND median set M={m , ..., m_ceil(n/5)}
    // m_j : median of j-th group

    x = SELECT(M,ceil(n/5),floor((ceil(n/5)+1)/2))

    PARTITION set S around the pivot x into L and R

    if i <= |L| then
        return SELECT(L, |L|, i)
    else
        return SELECT(R, n-|L|, i-|L|)
```

## Selection in Worst Case Linear Time - Example (1)

- **Input:** Array  $S$  and index  $i$
- **Output:** The  $i^{th}$  smallest value

25	9	16	8	11	27	39	42	15	63	2	14	36	20	33	22	31	4	17	3	30	41
2	13	19	7	21	10	34	1	37	23	40	5	29	18	24	12	38	28	26	35	43	

## Selection in Worst Case Linear Time - Example (2)

Step 1: Divide the input array into groups of size 5

group size=5

25	9	16	8	11
27	39	42	15	6
32	14	36	20	33
22	31	4	17	3
30	41	2	13	19
7	21	10	34	1
37	23	40	5	29
18	24	12	38	28
26	35	43		

# Selection in Worst Case Linear Time - Example (3)

Step 2: Compute the median of each group ( $\Theta(n)$ )

<i>Medians</i>				
25	16	11	8	9
39	42	27	6	15
36	33	32	20	14
22	31	17	3	4
41	30	19	13	2
21	34	10	1	7
37	40	29	23	5
38	28	24	12	18
	26	35	43	

- Let  $M$  be the set of the medians computed:
  - $M = \{11, 27, 32, 17, 19, 10, 29, 24, 35\}$

## Selection in Worst Case Linear Time - Example (4)

Step 3: Compute the median of the median group  $M$

$x \leftarrow SELECT(M, |M|, \lfloor (|M| + 1)/2 \rfloor)$  where  $|M| = \lceil n/5 \rceil$

- Let  $M$  be the set of the medians computed:

$$\circ M = \{11, 27, 32, 17, 19, 10, 29, \overset{\text{Median}}{\underbrace{24}}, 35\}$$

- $Median = 24$
- The runtime of the recursive call:  $T(|M|) = T(\lceil n/5 \rceil)$

## Selection in Worst Case Linear Time - Example (5)

**Step 4:** Partition the input array  $S$  around the median-of-medians  $x$

25	9	16	8	11	27	39	42	15	63	2	14	36	20	33	22	31	4	17	3	30	41
2	13	19	7	21	10	34	1	37	23	40	5	29	18	24	12	38	28	26	35	43	

Partition  $S$  around  $x = 24$

**Claim:** Partitioning around  $x$  is guaranteed to be **well-balanced**.

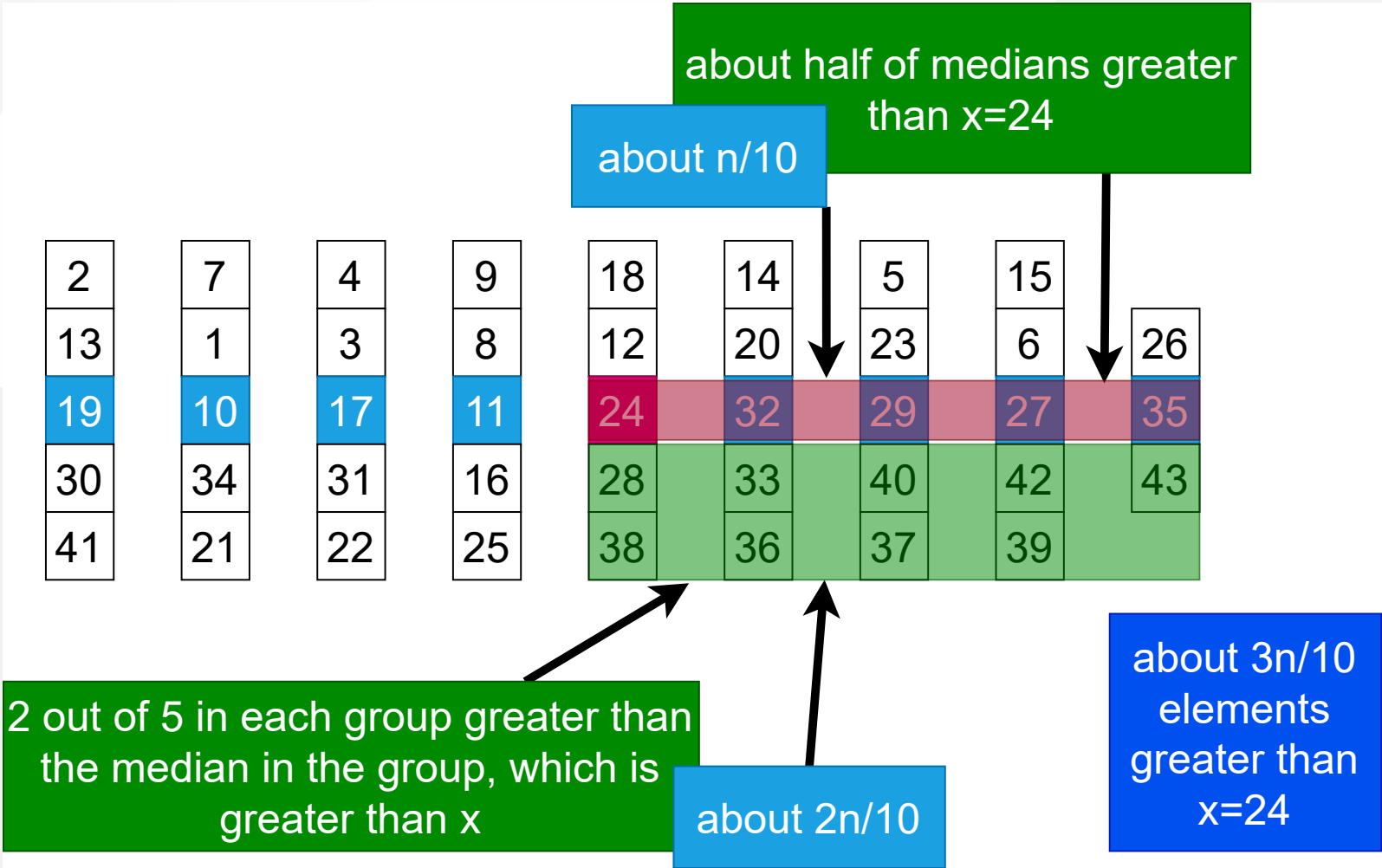
# Selection in Worst Case Linear Time - Example (6)

- $M$  : Median,  $M^*$  : Median of Medians

		$\overbrace{M}$		
41	30	19	13	2
21	34	10	1	7
22	31	17	3	4
25	16	11	8	9
		$\overbrace{M^*}$		
38	28	24	12	18
36	33	32	20	14
37	40	29	23	5
39	42	27	6	15
	26	35	43	

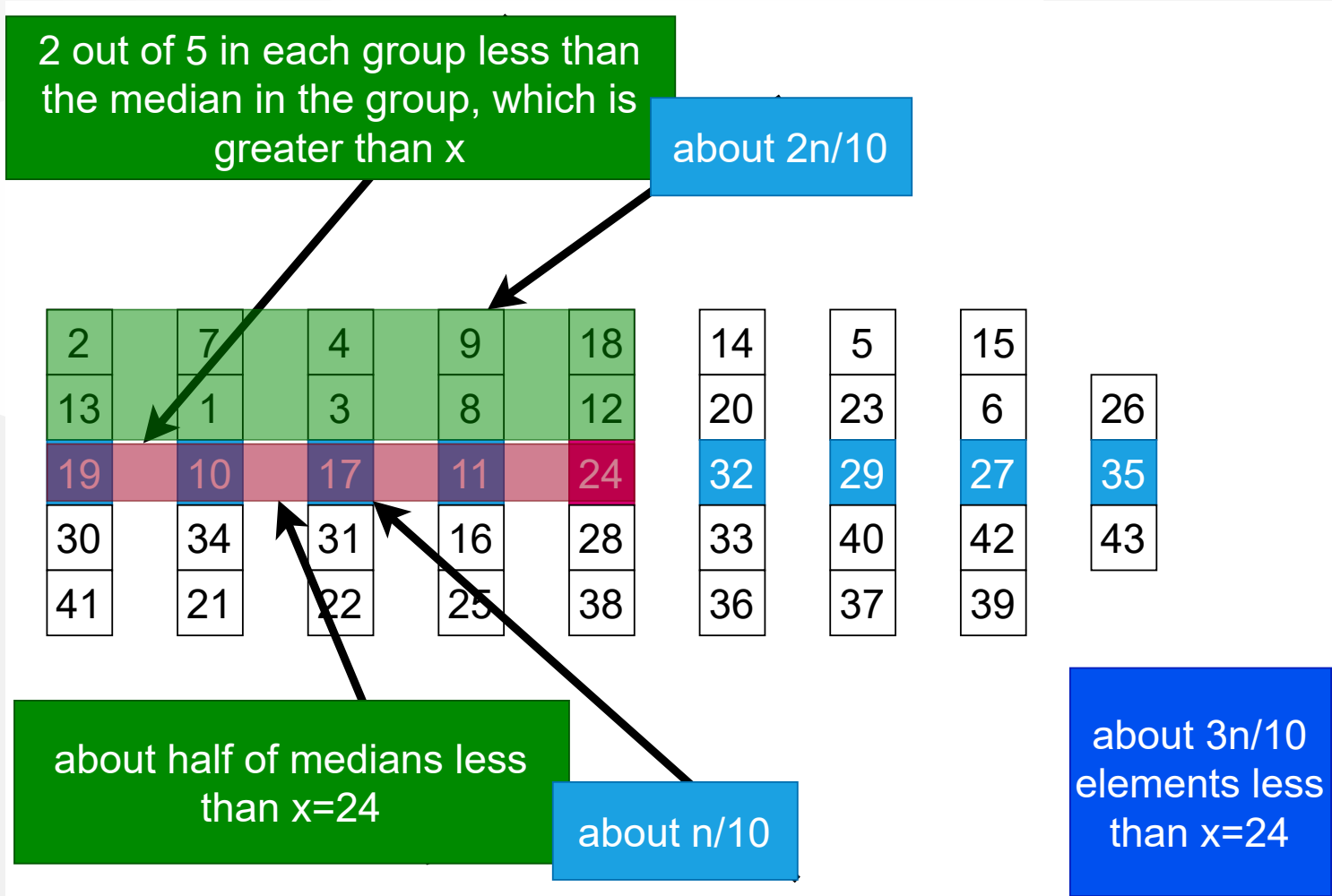
- About half of the medians greater than  $x = 24$  (about  $n/10$ )

# Selection in Worst Case Linear Time - Example (7)





# Selection in Worst Case Linear Time - Example (8)



## Selection in Worst Case Linear Time - Example (9)

$S = \begin{Bmatrix} 25 & 9 & 16 & 8 & 11 & 27 & 39 & 42 & 15 & 63 & 14 & 36 & 20 & 33 & 22 & 31 & 4 & 17 & 3 & 30 & 41 \\ 2 & 13 & 19 & 7 & 21 & 10 & 34 & 1 & 37 & 23 & 40 & 5 & 29 & 18 & 24 & 12 & 38 & 28 & 26 & 35 & 43 \end{Bmatrix}$

- Partitioning  $S$  around  $x = 24$  will lead to partitions of sizes  $\sim 3n/10$  and  $\sim 7n/10$  in the **worst case**.

**Step 5:** Make a recursive call to one of the partitions

```

if i <= |L| then
    return SELECT(L, |L|, i)
else
    return SELECT(R, n - |L|, i - |L|)
  
```

# Selection in Worst Case Linear Time

```
//return i-th element in set S with n elements
SELECT(S, n, i)

    if n <= 5 then

        SORT S and return the i-th element

    DIVIDE S into ceil(n/5) groups
    //first ceil(n/5) groups are of size 5, last group is of size n mod 5

    FIND median set M={m , ..., m_ceil(n/5)}
    // m_j : median of j-th group

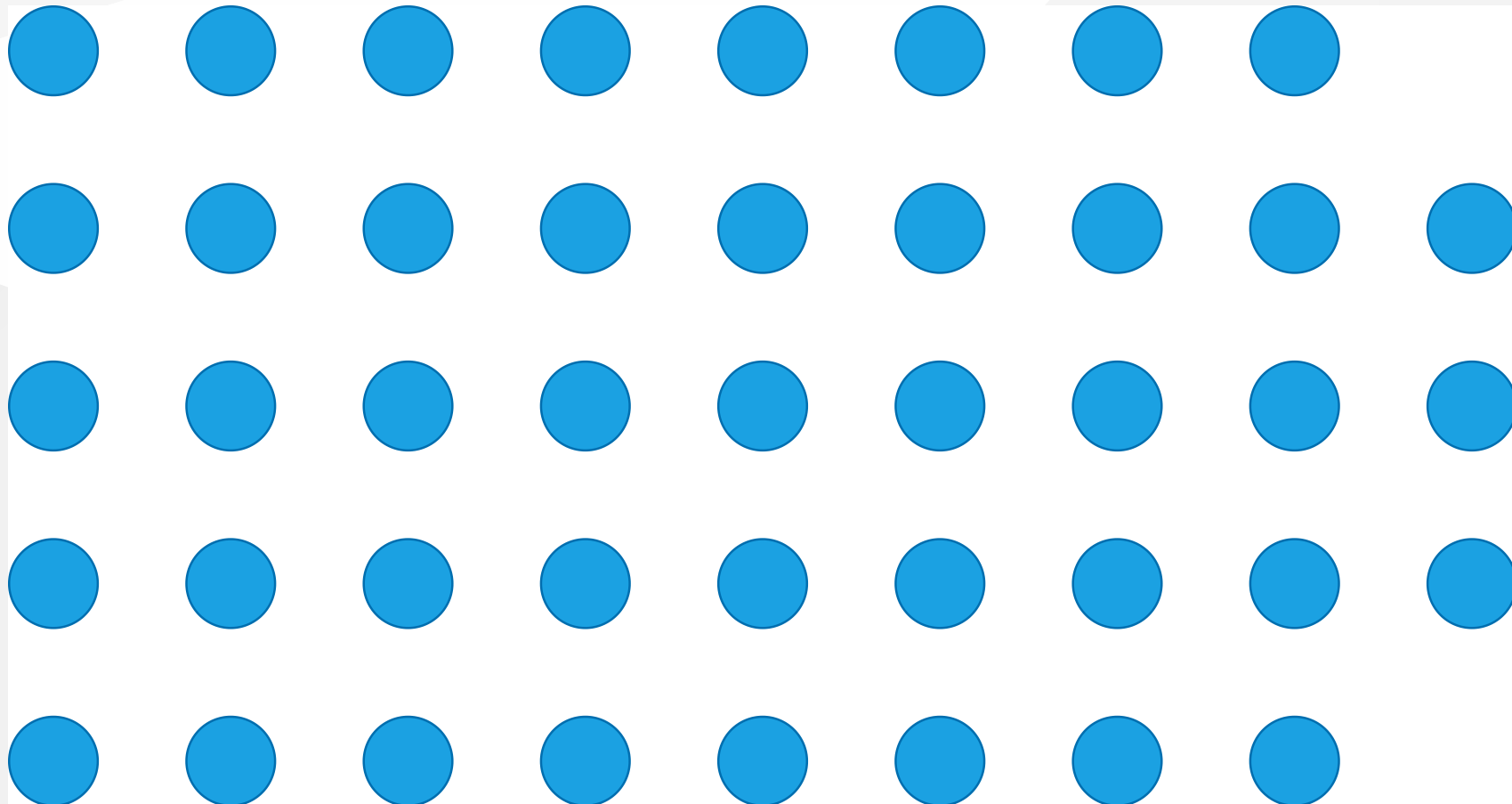
    x = SELECT(M,ceil(n/5),floor((ceil(n/5)+1)/2))

    PARTITION set S around the pivot x into L and R

    if i <= |L| then
        return SELECT(L, |L|, i)
    else
        return SELECT(R, n-|L|, i-|L|)
```

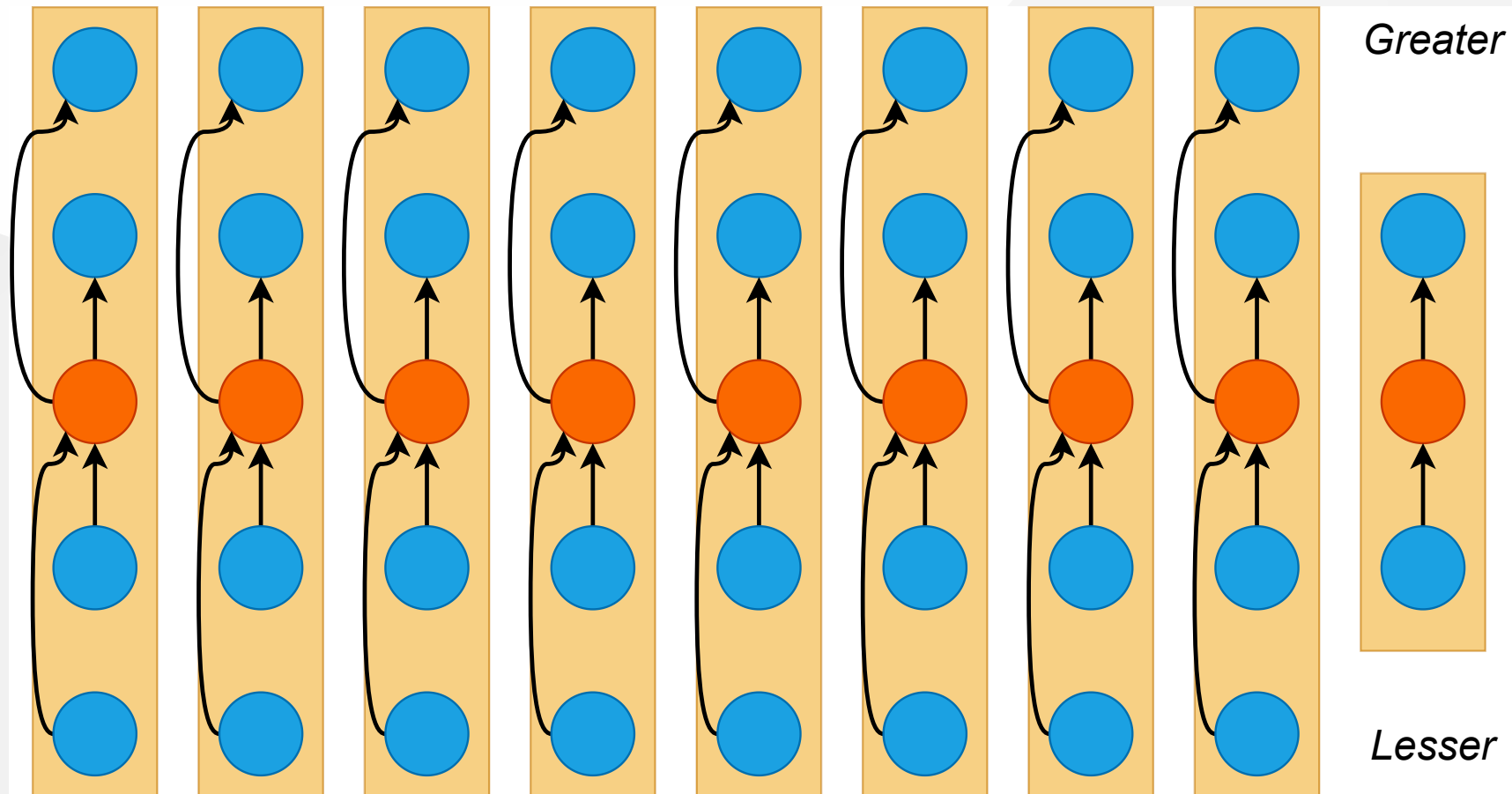
# Choosing the Pivot (1)

1. Divide S into groups of size 5



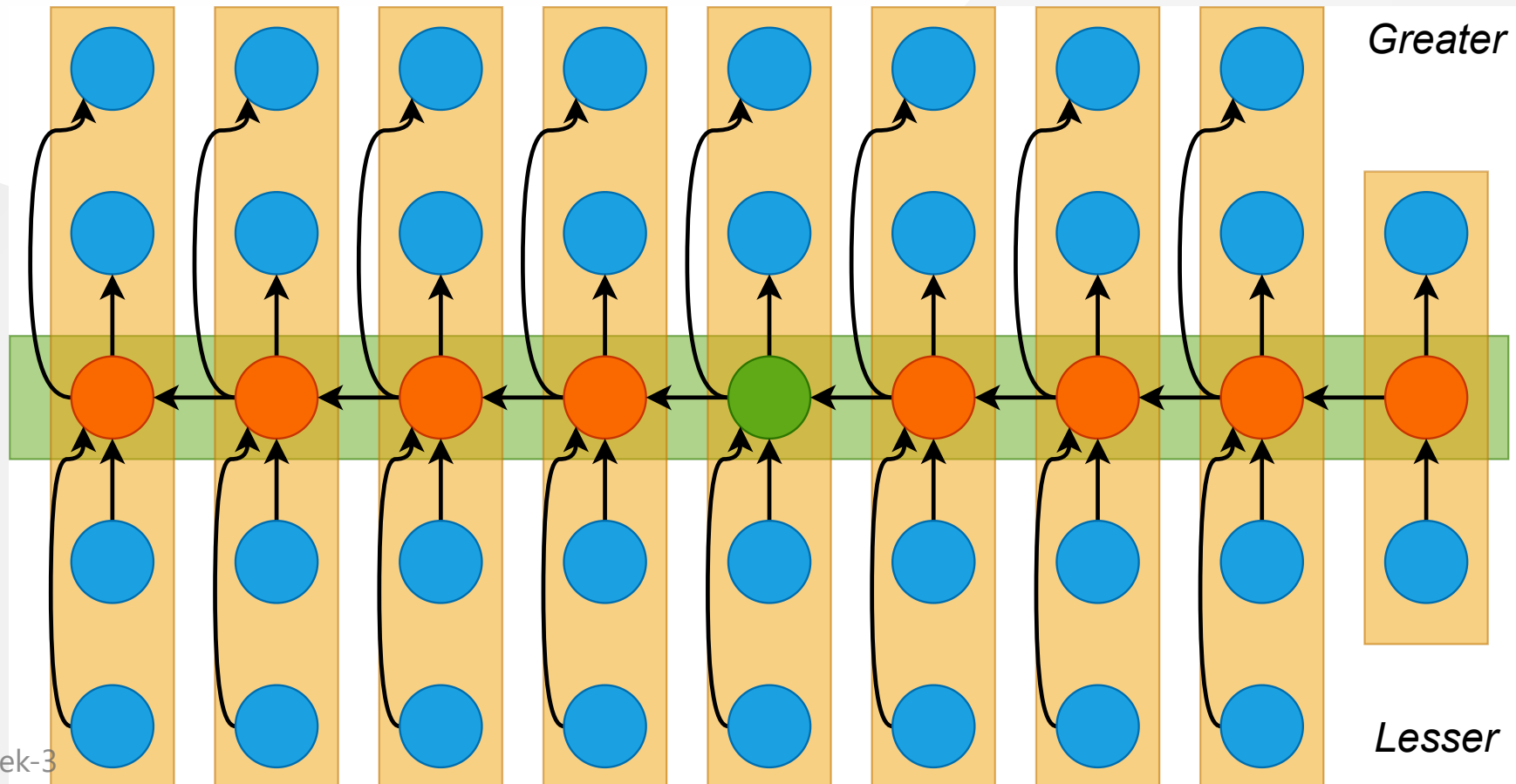
## Choosing the Pivot (2)

- Divide S into groups of size 5
- Find the median of each group



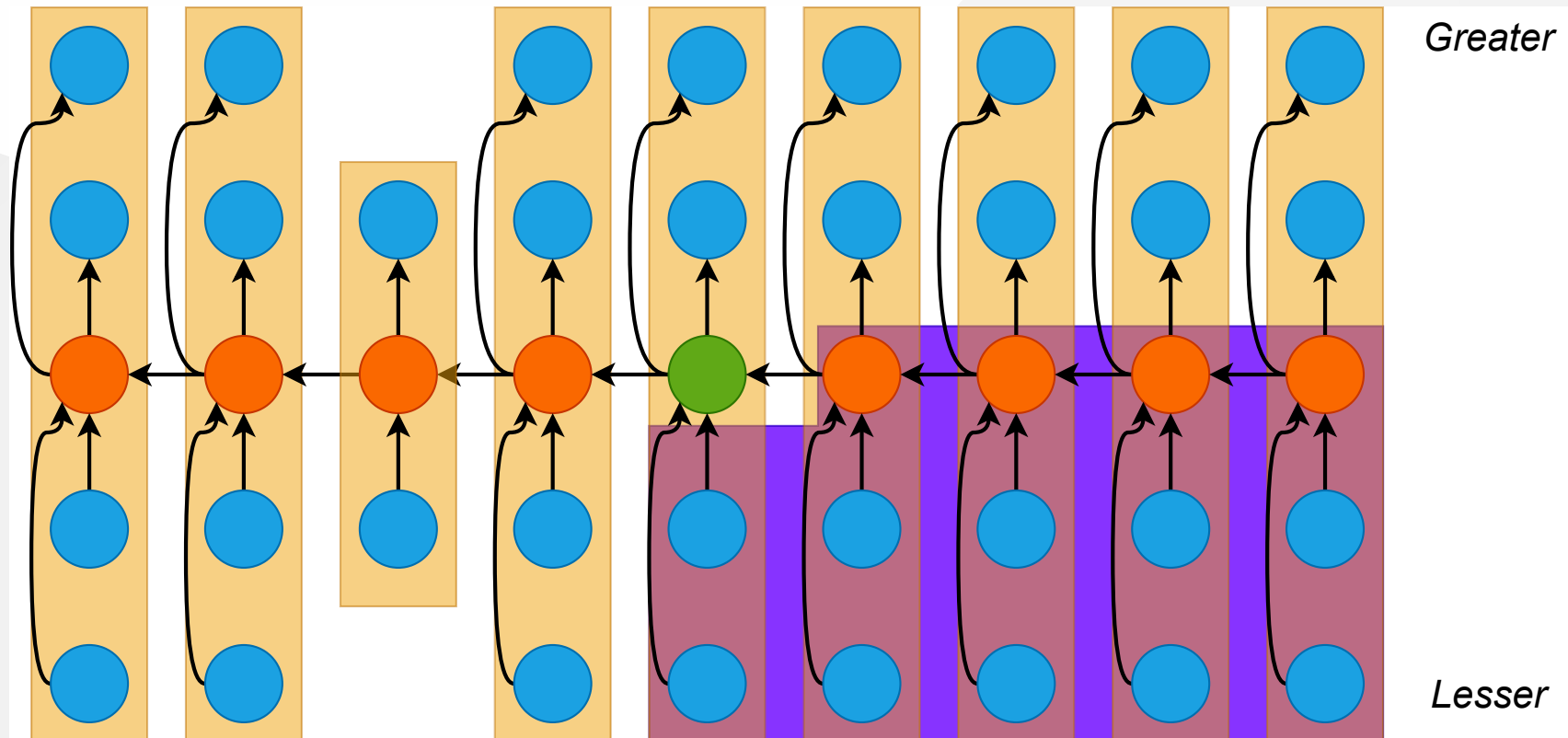
# Choosing the Pivot (3)

- Divide S into groups of size 5
- Find the median of each group
- Recursively select the median  $x$  of the medians



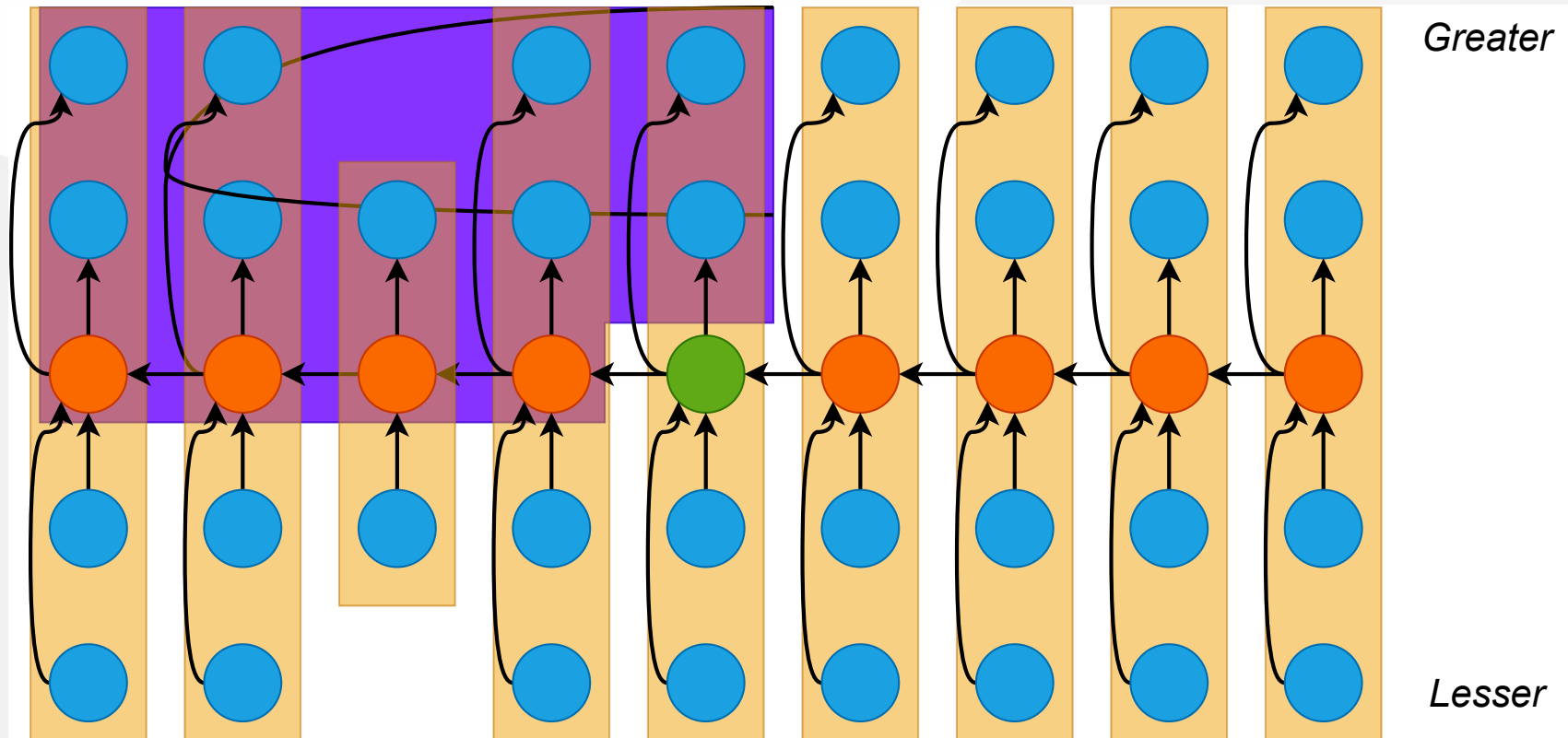
# Choosing the Pivot (4)

- At least half of the medians  $\geq x$
- Thus  $m = \lceil \lceil n/5 \rceil / 2 \rceil$  groups contribute 3 elements to  $R$  except possibly the last group and the group that contains  $x$ ,  $|R| \geq 3(m-2) \geq \frac{3n}{10} - 6$



## Choosing the Pivot (5)

- Similarly  $|L| \geq \frac{3n}{10} - 6$
- Therefore, **SELECT** is recursively called on at most  $n - (\frac{3n}{10} - 6) = \frac{7n}{10} + 6$  elements





# Selection in Worst Case Linear Time (1)

```

//return i-th element in set S with n elements
SELECT(S, n, i)
  if n <= 5 then
    SORT S and return the i-th element
   $\Theta(n)$  { DIVIDE S into  $\text{ceil}(n/5)$  groups
           //first  $\text{ceil}(n/5)$  groups are of size 5, last group is of size  $n \bmod 5$ 
   $\Theta(n)$  { FIND median set  $M = \{m_1, \dots, m_{\text{ceil}(n/5)}\}$ 
           //  $m_j$  : median of j-th group
   $T(\lceil n/5 \rceil)$  {  $x = \text{SELECT}(M, \text{ceil}(n/5), \text{floor}((\text{ceil}(n/5)+1)/2))$ 
   $\Theta(n)$  { PARTITION set S around the pivot x into L and R
           if  $i \leq |L|$  then
             return SELECT(L, |L|, i)
           else
             return SELECT(R, n-|L|, i-|L|)
   $T(\frac{7n}{10} + 6)$ 

```

## Selection in Worst Case Linear Time (2)

- Thus recurrence becomes
  - $T(n) \leq T(\lceil \frac{n}{5} \rceil) + T(\frac{7n}{10} + 6) + \Theta(n)$
- Guess  $T(n) = O(n)$  and prove by induction
- Inductive step:

$$\begin{aligned} T(n) &\leq c\lceil n/5 \rceil + c(7n/10 + 6) + \Theta(n) \\ &\leq cn/5 + c + 7cn/10 + 6c + \Theta(n) \\ &= 9cn/10 + 7c + \Theta(n) \\ &= cn - [c(n/10 - 7) - \Theta(n)] \leq cn \quad (\text{for large } c) \end{aligned}$$

- Work at each level of recursion is a constant factor (9/10) smaller

# References

- [Introduction to Algorithms, Third Edition | The MIT Press](#)
- [Bilkent CS473 Course Notes \(new\)](#)
- [Bilkent CS473 Course Notes \(old\)](#)
- [Insertion Sort - GeeksforGeeks](#)
- [NIST Dictionary of Algorithms and Data Structures](#)
- [NIST - Dictionary of Algorithms and Data Structures](#)
- [NIST - big-O notation](#)
- [NIST - big-Omega notation](#)

*–End – Of – Week – 3 – Course – Module–*