CE100 Algorithms and Programming II

Week-3 (Matrix Multiplication/ Quick Sort)

Spring Semester, 2021-2022

Download DOC, SLIDE, PPTX

<iframe width=700, height=500 frameBorder=0 src="../ce100-week-3-matrix.md_slide.html"></iframe>

Matrix Multiplication / Quick Sort

Outline

- Matrix Multiplication
    - Traditional
    - Recursive
    - Strassen

Outline

- Quicksort
  - Hoare Partitioning
  - Lomuto Partitioning
  - Recursive Sorting

Outline

- Quicksort Analysis
  - Randomized Quicksort
  - Randomized Selection
    - Recursive
    - Medians

Matrix Multiplication

- **Input:** $A = [a_{ij}], B = [b_{ij}]$
- **Output:** $C = [c_{ij}] = A \cdot B \Longrightarrow i, j = 1, 2, 3, \dots, n$

$$\begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \vdots & \ddots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \vdots & \ddots \\ b_{n1} & a_{n2} & \dots & b_{nn} \end{bmatrix}$$

Matrix Multiplication

$$\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ \boxed{c_{21}} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$$

- $c_{ij} = \sum_{1 \leq k \leq n} a_{ik} . b_{kj}$

Matrix Multiplication: Standard Algorithm

Running Time: $\Theta(n^3)$

```
for i=1 to n do
    for j=1 to n do
        C[i,j] = 0
        for k=1 to n do
            C[i,j] = C[i,j] + A[i,k] + B[k,j]
        endfor
    endfor
endfor
```

Matrix Multiplication: Divide & Conquer

IDEA: Divide the nxn matrix into $2x2$ matrix of $(n/2)x(n/2)$ submatrices.

$$\begin{pmatrix} \boxed{c_{11}} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$c_{11} = a_{11}b_{11} + a_{12}b_{21}$$

$$\begin{pmatrix} c_{11} & c_{12} \\ \boxed{c_{21}} & c_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$c_{21} = a_{21}b_{11} + a_{22}b_{21}$$

$$\begin{pmatrix} c_{11} & \boxed{c_{12}} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$c_{12} = a_{11}b_{12} + a_{12}b_{22}$$

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & \boxed{c_{22}} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$c_{22} = a_{21}b_{12} + a_{22}b_{22}$$

Matrix Multiplication: Divide & Conquer

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\text{8 mults and 4 adds of (n/2)*(n/2) submatrices} = \begin{cases} c_{11} &= a_{11}b_{11} + a_{12}b_{21} \\ c_{21} &= a_{21}b_{11} + a_{22}b_{21} \\ c_{12} &= a_{11}b_{12} + a_{12}b_{22} \\ c_{22} &= a_{21}b_{12} + a_{22}b_{22} \end{cases}$$

Matrix Multiplication: Divide & Conquer

```
MATRIX-MULTIPLY(A, B)
    // Assuming that both A and B are nxn matrices
    if n == 1 then
        return A * B
    else
        //partition A, B, and C as shown before
        C[1,1] = MATRIX-MULTIPLY (A[1,1], B[1,1]) +
                 MATRIX-MULTIPLY (A[1,2], B[2,1]);

        C[1,2] = MATRIX-MULTIPLY (A[1,1], B[1,2]) +
                MATRIX-MULTIPLY (A[1,2], B[2,2]);

        C[2,1] = MATRIX-MULTIPLY (A[2,1], B[1,1]) +
        MATRIX-MULTIPLY (A[2,2], B[2,1]);

        C[2,2] = MATRIX-MULTIPLY (A[2,1], B[1,2]) +
        MATRIX-MULTIPLY (A[2,2], B[2,2]);
    endif

    return C
```

Matrix Multiplication: Divide & Conquer Analysis

$$T(n) = 8T(n/2) + \Theta(n^2)$$

- 8 recursive calls $\Longrightarrow 8T(\cdots)$
- each problem has size $n/2 \Longrightarrow \cdots T(n/2)$
- Submatrix addition $\Longrightarrow \Theta(n^2)$

Matrix Multiplication: Solving the Recurrence

- $T(n) = 8T(n/2) + \Theta(n^2)$
  - $a = 8, b = 2$
  - $f(n) = \Theta(n^2)$
  - $n^{log_b^a} = n^3$
- Case 1: $\frac{n^{log_b^a}}{f(n)} = \Omega(n^{\varepsilon}) \implies T(n) = \Theta(n^{log_b^a})$

Similar with ordinary (iterative) algorithm.

Matrix Multiplication: Strassen's Idea

Compute $c_{11}, c_{12}, c_{21}, c_{22}$ using 7 recursive multiplications.

In normal case we need 8 as below.

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\text{8 mults and 4 adds of (n/2)*(n/2) submatrices} = \begin{cases} c_{11} = a_{11}b_{11} + a_{12}b_{21} \\ c_{21} = a_{21}b_{11} + a_{22}b_{21} \\ c_{12} = a_{11}b_{12} + a_{12}b_{22} \\ c_{22} = a_{21}b_{12} + a_{22}b_{22} \end{cases}$$

Matrix Multiplication: Strassen's Idea

- **Reminder:**
    - Each submatrix is of size $(n/2) * (n/2)$
    - Each add/sub operation takes $\Theta(n^2)$ time
- Compute $P1 \dots P7$ using 7 recursive calls to matrix-multiply

$$
\begin{aligned}
P_1 &= a_{11} * (b_{12} - b_{22}) \\
P_2 &= (a_{11} + a_{12}) * b_{22} \\
P_3 &= (a_{21} + a_{22}) * b_{11} \\
P_4 &= a_{22} * (b_{21} - b_{11}) \\
P_5 &= (a_{11} + a_{22}) * (b_{11} + b_{22}) \\
P_6 &= (a_{12} - a_{22}) * (b_{21} + b_{22}) \\
P_7 &= (a_{11} - a_{21}) * (b_{11} + b_{12})
\end{aligned}
$$

Matrix Multiplication: Strassen’s Idea

$$P_1 = a_{11} * (b_{12} - b_{22})$$
$$P_2 = (a_{11} + a_{12}) * b_{22}$$
$$P_3 = (a_{21} + a_{22}) * b_{11}$$
$$P_4 = a_{22} * (b_{21} - b_{11})$$
$$P_5 = (a_{11} + a_{22}) * (b_{11} + b_{22})$$
$$P_6 = (a_{12} - a_{22}) * (b_{21} + b_{22})$$
$$P_7 = (a_{11} - a_{21}) * (b_{11} + b_{12})$$

- How to compute c_{ij} using $P1 \dots P7$?

$$c_{11} = P_5 + P_4 - P_2 + P_6$$
$$c_{12} = P_1 + P_2$$
$$c_{21} = P_3 + P_4$$
$$c_{22} = P_5 + P_1 - P_3 - P_7$$

Matrix Multiplication: Strassen’s Idea

- 7 recursive multiply calls
- 18 add/sub operations

Matrix Multiplication: Strassen's Idea

e.g. Show that $c_{12} = P_1 + P_2$:

$$
\begin{aligned}
c_{12} &= P_1 + P_2 \\
&= a_{11}(b_{12} – b_{22}) + (a_{11} + a_{12})b_{22} \\
&= a_{11}b_{12} - a_{11}b_{22} + a_{11}b_{22} + a_{12}b_{22} \\
&= a_{11}b_{12} + a_{12}b_{22}
\end{aligned}
$$

Strassen's Algorithm

- **Divide:** Partition A and B into $(n/2) * (n/2)$ submatrices. Form terms to be multiplied using $+$ and $-$.
- **Conquer:** Perform 7 multiplications of $(n/2) * (n/2)$ submatrices recursively.
- **Combine:** Form C using $+$ and $-$ on $(n/2) * (n/2)$submatrices.

Recurrence: $T(n) = 7T(n/2) + \Theta(n^2)$

Strassen's Algorithm: Solving the Recurrence

- $T(n) = 7T(n/2) + \Theta(n^2)$
  - $a = 7, b = 2$
  - $f(n) = \Theta(n^2)$
  - $n^{log_b^a} = n^{lg7}$
- Case 1: $\frac{n^{log_b^a}}{f(n)} = \Omega(n^{\varepsilon}) \implies T(n) = \Theta(n^{log_b^a})$

$T(n) = \Theta(n^{log_2^7})$

$2^3 = 8, 2^2 = 4$ so $\implies log_2^7 \approx 2.81$

or use https://www.omnicalculator.com/math/log

Strassen's Algorithm

- The number 2.81 may not seem much smaller than 3
- But, it is significant because the difference is in the exponent.
- Strassen's algorithm beats the ordinary algorithm on today's machines for $n \geq 30$ or so.
- Best to date: $\Theta(n^{2.376\ldots})$ (of theoretical interest only)

Maximum Subarray Problem

Input: An array of values

Output: The contiguous subarray that has the largest sum of elements

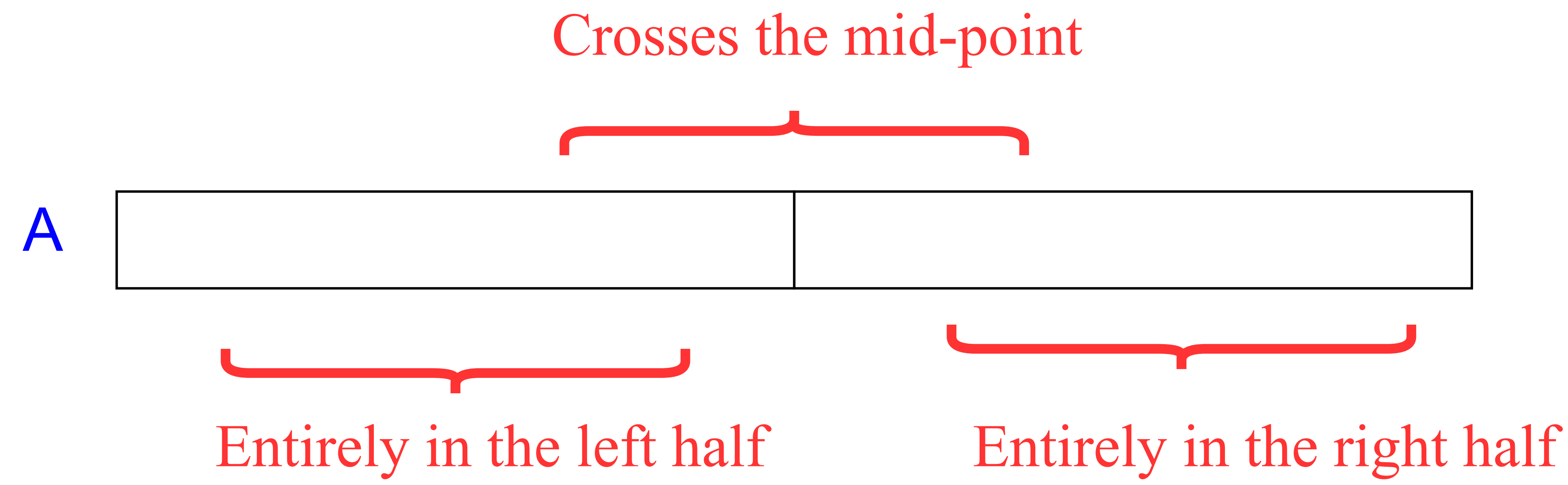
- Input array:

$$[13][-3][-25][20][-3][-16][-23] \overbrace{[18][20][-7][12]}^{\text{max. contiguous subarray}} [-22][-4][7]$$

Maximum Subarray Problem: Divide & Conquer

- **Basic idea:**
- **Divide** the input array into 2 from the middle
- Pick the **best** solution among the following:
  - The max subarray of the **left half**
  - The max subarray of the **right half**
  - The max subarray **crossing the mid-point**

Maximum Subarray Problem: Divide & Conquer



Maximum Subarray Problem: Divide & Conquer

- **Divide:** Trivial (divide the array from the middle)
- **Conquer:** Recursively compute the max subarrays of the left and right halves
- **Combine:** Compute the max-subarray crossing the $mid - point$
  - (can be done in $\Theta(n)$ time).
  - Return the max among the following:
    - the max subarray of the left-subarray
    - the max subarray of the rightsubarray
    - the max subarray crossing the mid-point

TODO : detailed solution in textbook...

Conclusion : Divide & Conquer

- Divide and conquer is just one of several powerful techniques for algorithm design.
- Divide-and-conquer algorithms can be analyzed using recurrences and the master method (so practice this math).
- Can lead to more efficient algorithms

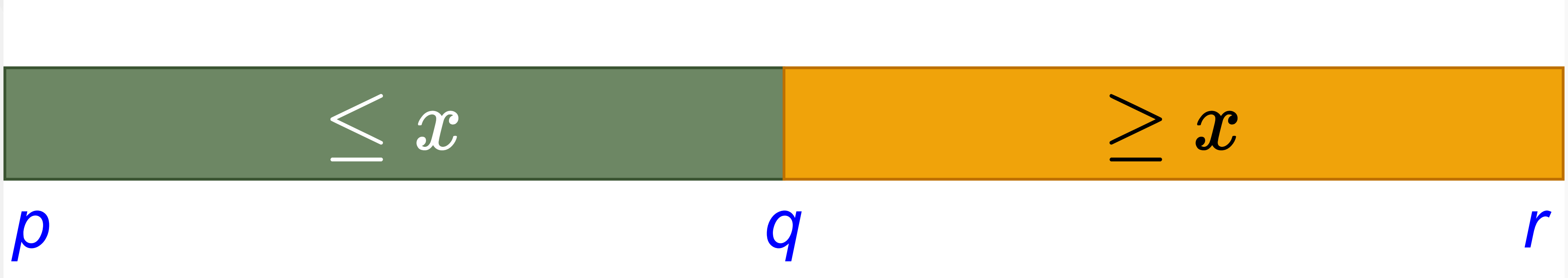
Quicksort

- One of the most-used algorithms in practice
- Proposed by **C.A.R.** *Hoare* in 1962.
- Divide-and-conquer algorithm
- In-place algorithm
    - The additional space needed is O(1)
    - The sorted array is returned in the input array
    - *Reminder: Insertion-sort is also an in-place algorithm, but Merge-Sort is not in-place.*
- Very practical

Quicksort

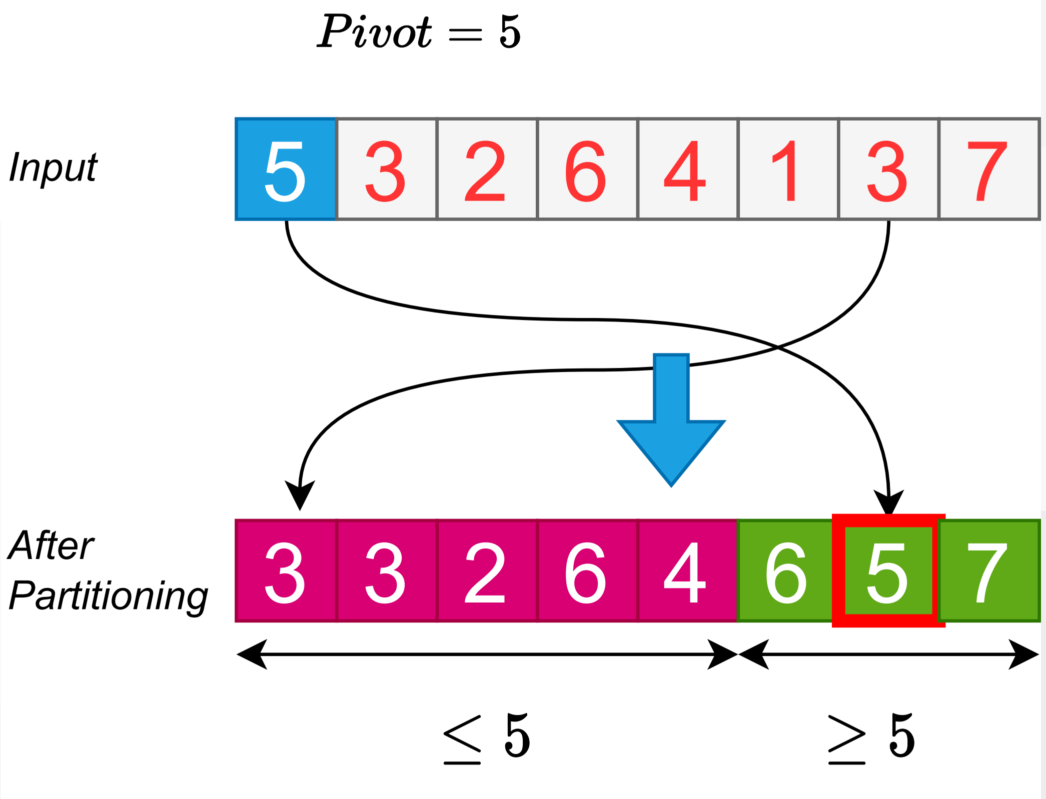
- **Divide:** Partition the array into 2 subarrays such that elements in the lower part $\leq$ elements in the higher part
- **Conquer:** Recursively sort 2 subarrays
- **Combine:** Trivial (because in-place)

Key: Linear-time $(\Theta(n))$ partitioning algorithm



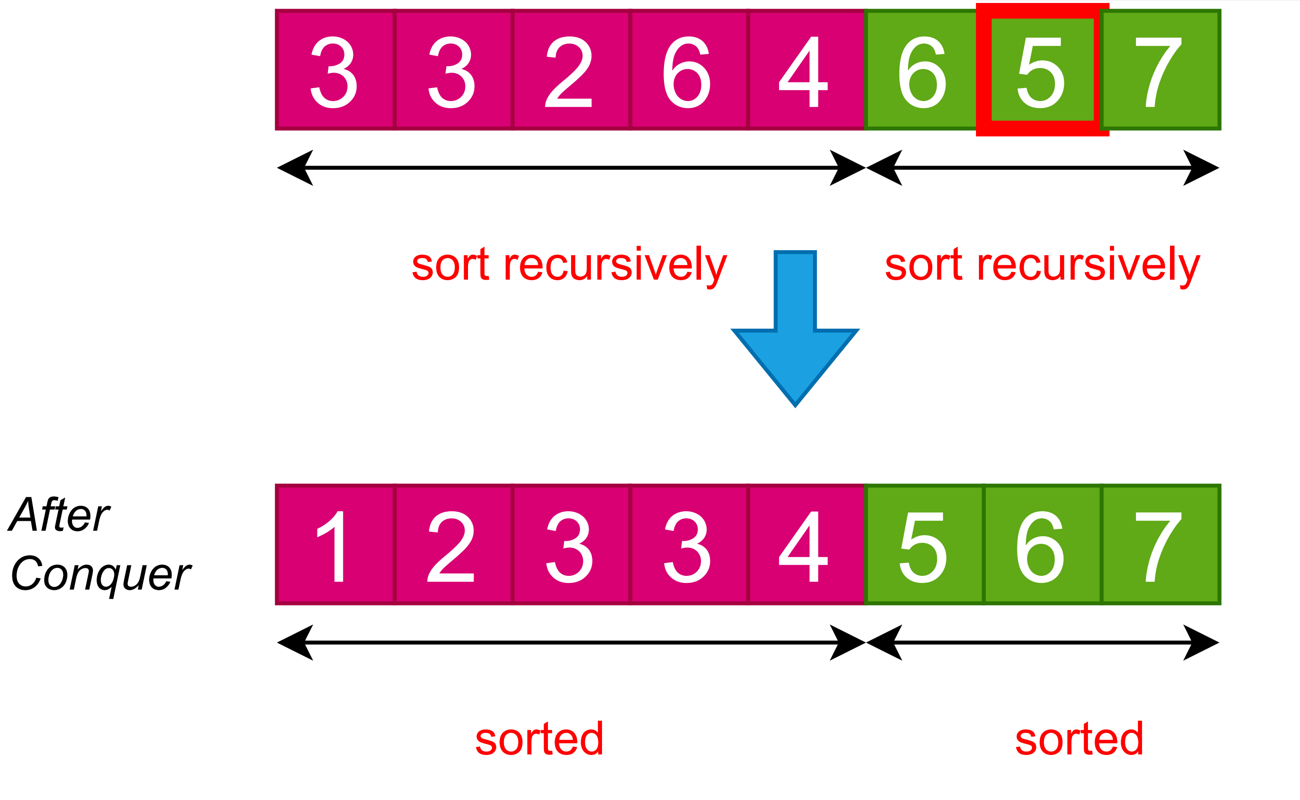
Divide: Partition the array around a pivot element

- Choose a pivot element x
- Rearrange the array such that:
  - Left subarray: All elements $\leq x$
  - Right subarray: All elements $\geq x$



Conquer: Recursively Sort the Subarrays

Note: Everything in the left subarray ≤ everything in the right subarray

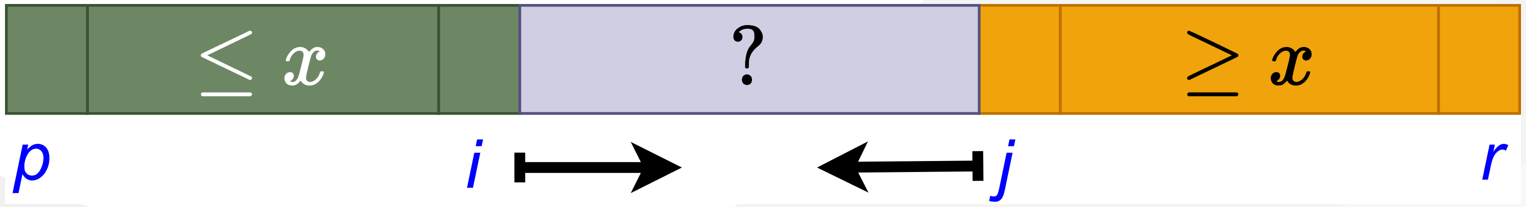


Note: Combine is trivial after conquer. Array already sorted.

Two partitioning algorithms

- **Hoare's algorithm:**

  Partitions around the first element of subarray
  - $(pivot = x = A[p])$

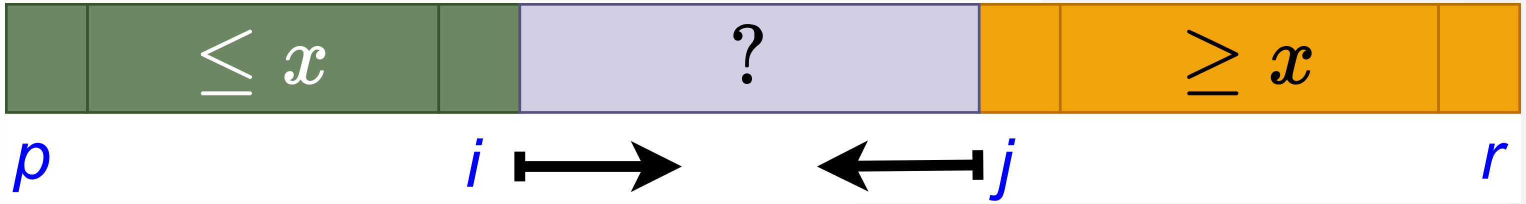


- **Lomuto's algorithm:**

  Partitions around the last element of subarray
  - $(pivot = x = A[r])$

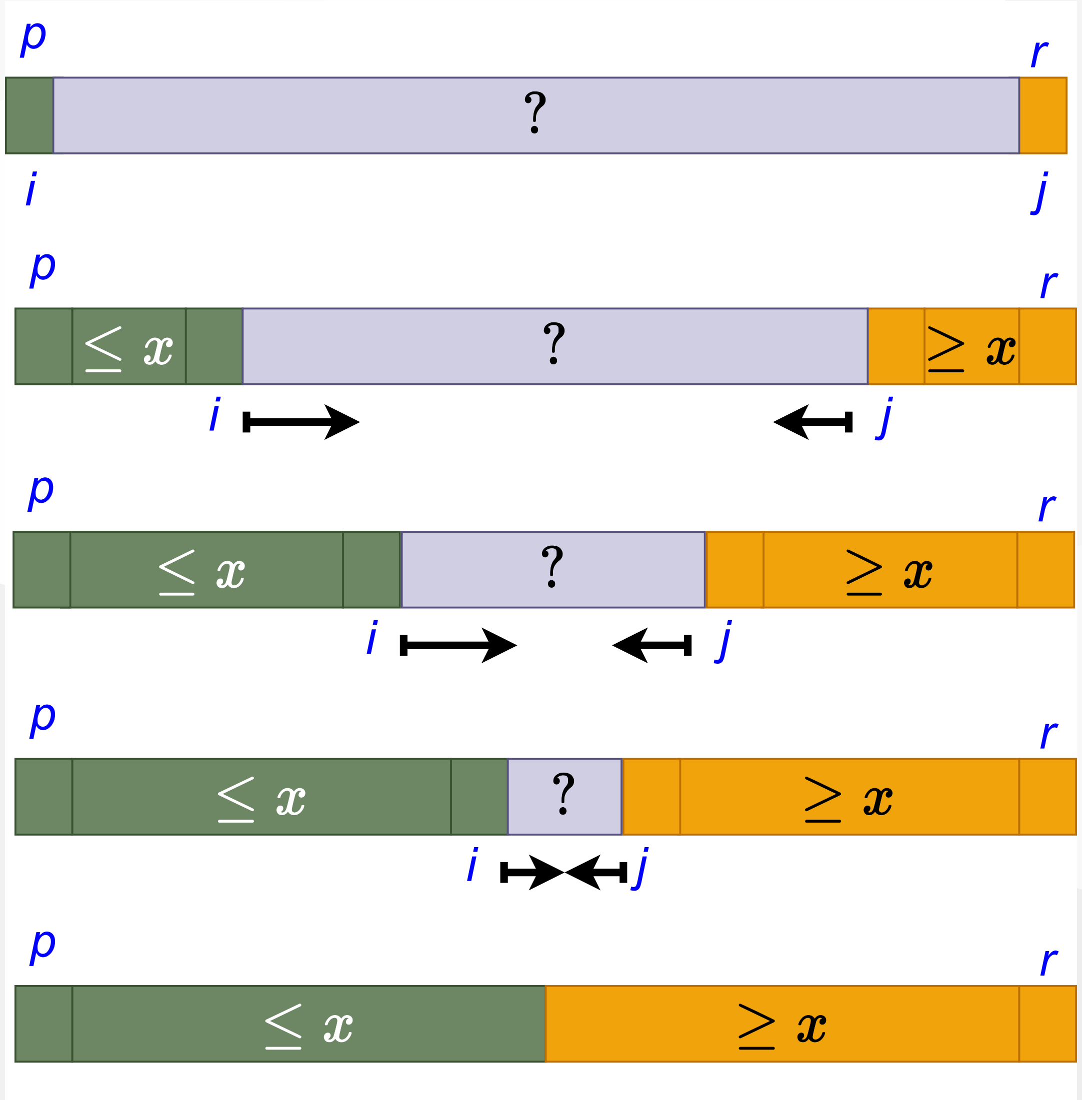
Hoare's Partitioning Algorithm

- Choose a pivot element: $pivot = x = A[p]$



- Grow two regions:
    - from left to right: $A[p \ldots i]$
    - from right to left: $A[j \ldots r]$
        - such that:
    - every element in $A[p \ldots i] \leq$ pivot
    - every element in $A[p \ldots i] \geq$ pivot

Hoare's Partitioning Algorithm



Hoare's Partitioning Algorithm

- Elements are exchanged when
  - $A[i]$ is **too large** to belong to the **left** region
  - $A[j]$ is **too small** to belong to the **right** region
    - assuming that the inequality is strict
- The two regions $A[p \dots i]$ and $A[j \dots r]$ grow until $A[i] \geq pivot \geq A[j]$

```
H-PARTITION(A, p, r)
      pivot = A[p]
        i = p - 1
        j = r - 1
        while true do
          repeat j = j - 1 until A[j] <= pivot
          repeat i = i - 1 until A[i] <= pivot
        if i < j then
          exchange A[i] with A[j]
        else
          return j
```

Hoare's Partitioning Algorithm Example (Step-1)

H-PARTITION (A, p, r)

$pivot \leftarrow A[p]$

$i \leftarrow p - 1$

$j \leftarrow r + 1$

while true do

repeat $j \leftarrow j - 1$ **until** $A[j] \leq pivot$

repeat $i \leftarrow i + 1$ **until** $A[i] \geq pivot$

if $i < j$ then

exchange $A[i] \leftrightarrow A[j]$

else

return j

$Pivot = 5$

Input	5	3	2	6	4	1	3	7

$STEP - 1$

Hoare's Partitioning Algorithm Example (Step-2)

H-PARTITION (A, *p*, *r*)

$pivot \leftarrow A[p]$

$i \leftarrow p - 1$

$j \leftarrow r + 1$

while true do

repeat $j \leftarrow j - 1$ **until** $A[j] \leq pivot$

repeat $i \leftarrow i + 1$ **until** $A[i] \geq pivot$

if $i < j$ then

exchange $A[i] \leftrightarrow A[j]$

else

return j

$Pivot = 5$

Input

5	3	2	6	4	1	3	7

i **j**

$STEP - 2$

Hoare's Partitioning Algorithm Example (Step-3)

H-PARTITION (A, p, r)

$pivot \leftarrow A[p]$
$i \leftarrow p - 1$
$j \leftarrow r + 1$
while true do
  repeat $j \leftarrow j - 1$ **until** $A[j] \le pivot$
  repeat $i \leftarrow i + 1$ **until** $A[i] \ge pivot$
  if $i < j$ then
    exchange $A[i] \leftrightarrow A[j]$
  else
    return j

$Pivot = 5$

Input

5	3	2	6	4	1	3	7

i j

$STEP - 3$

Hoare's Partitioning Algorithm Example (Step-4)

H-PARTITION (A, p, r)

$pivot \leftarrow A[p]$
$i \leftarrow p - 1$
$j \leftarrow r + 1$
while true do
  repeat $j \leftarrow j - 1$ **until** $A[j] \leq pivot$
  repeat $i \leftarrow i + 1$ **until** $A[i] \geq pivot$
  if $i < j$ then
    exchange $A[i] \leftrightarrow A[j]$
  else
    return j

$Pivot = 5$

Input

5	3	2	6	4	1	3	7

i (at 5), j (at 3)

$STEP-4$

Hoare's Partitioning Algorithm Example (Step-5)

H-PARTITION (A, *p*, *r*)

> $pivot \leftarrow A[p]$
> $i \leftarrow p - 1$
> $j \leftarrow r + 1$
> **while true do**
> **repeat** $j \leftarrow j - 1$ **until** $A[j] \leq pivot$
> **repeat** $i \leftarrow i + 1$ **until** $A[i] \geq pivot$
> **if** $i < j$ **then**
> exchange $A[i] \leftrightarrow A[j]$
> **else**
> **return** j

$Pivot = 5$

Input

5	3	2	6	4	1	3	7

i (at index 1), j (at index 7)

$STEP - 5$

Hoare's Partitioning Algorithm Example (Step-6)

H-PARTITION (A, *p*, *r*)

$pivot \leftarrow A[p]$
$i \leftarrow p - 1$
$j \leftarrow r + 1$
while true **do**
  repeat $j \leftarrow j - 1$ **until** $A[j] \leq pivot$
  repeat $i \leftarrow i + 1$ **until** $A[i] \geq pivot$
  if $i < j$ then
    exchange $A[i] \leftrightarrow A[j]$
  else
    return j

$$Pivot = 5$$

Input

3	3	2	6	4	1	5	7
i						j	

$$STEP - 6$$

Hoare's Partitioning Algorithm Example (Step-7)

H-PARTITION (A, p, r)

$pivot \leftarrow A[p]$

$i \leftarrow p - 1$

$j \leftarrow r + 1$

while true do

repeat $j \leftarrow j - 1$ until $A[j] \leq pivot$

repeat $i \leftarrow i + 1$ until $A[i] \geq pivot$

if $i < j$ then

exchange $A[i] \leftrightarrow A[j]$

else

return j

$Pivot = 5$

Input

3	3	2	6	4	1	5	7
i					j		

$STEP - 7$

Hoare’s Partitioning Algorithm Example (Step-8)

H-PARTITION (A, p, r)

```
pivot ← A[p]
i ← p − 1
j ← r + 1
while true do
  repeat j ← j − 1 until A[j] ≤ pivot
  repeat i ← i + 1 until A[i] ≥ pivot
  if i < j then
        exchange A[i] ↔ A[j]
  else
        return j
```

$$Pivot = 5$$

Input

3	3	2	6	4	1	5	7
i					j		

$$STEP - 8$$

Hoare’s Partitioning Algorithm Example (Step-9)

H-PARTITION (A, p, r)

$pivot \leftarrow A[p]$
$i \leftarrow p - 1$
$j \leftarrow r + 1$
while true **do**
  repeat $j \leftarrow j - 1$ **until** $A[j] \leq pivot$
  repeat $i \leftarrow i + 1$ **until** $A[i] \geq pivot$
  if $i < j$ then
    exchange $A[i] \leftrightarrow A[j]$
  else
    return j

$Pivot = 5$

Input

3	3	2	6	4	1	5	7
			i		j		

$STEP - 9$

Hoare's Partitioning Algorithm Example (Step-10)

H-PARTITION (A, p, r)

$pivot \leftarrow A[p]$

$i \leftarrow p - 1$

$j \leftarrow r + 1$

while true do

repeat $j \leftarrow j - 1$ **until** $A[j] \leq pivot$

repeat $i \leftarrow i + 1$ **until** $A[i] \geq pivot$

if $i < j$ then

exchange $A[i] \leftrightarrow A[j]$

else

return j

$Pivot = 5$

Input

3	3	2	1	4	6	5	7

i (at 1), j (at 6)

$STEP - 10$

Hoare's Partitioning Algorithm Example (Step-11)

H-PARTITION (A, p, r)

```
pivot ← A[p]
i ← p − 1
j ← r + 1
while true do
  repeat j ← j − 1 until A[j] ≤ pivot
  repeat i ← i + 1 until A[i] ≥ pivot
  if i < j then
      exchange A[i] ↔ A[j]
  else
      return j
```

$Pivot = 5$

Input

3	3	2	1	4	6	5	7
			i	j			

$STEP - 11$

Hoare's Partitioning Algorithm Example (Step-12)

H-PARTITION (A, *p*, *r*)

```
pivot ← A[p]
i ← p − 1
j ← r + 1
while true do
  repeat j ← j − 1 until A[j] ≤ pivot
  repeat i ← i + 1 until A[i] ≥ pivot
  if i < j then
        exchange A[i] ↔ A[j]
  else
        return j
```

$Pivot = 5$

Input

3	3	2	1	4	6	5	7
				j	i		

$STEP - 12$

Hoare's Partitioning Algorithm - Notes

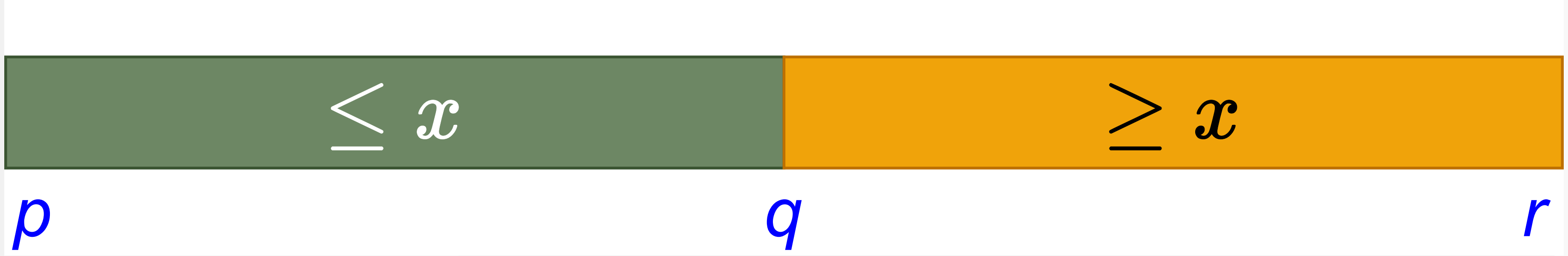
- Elements are exchanged when
  - $A[i]$ is **too large** to belong to the **left** region
  - $A[j]$ is **too small** to belong to the **right** region
    - assuming that the inequality is strict
- The two regions $A[p \dots i]$ and $A[j \dots r]$ grow until $A[i] \geq pivot \geq A[j]$
- The asymptotic runtime of Hoare's partitioning algorithm $\Theta(n)$

```
H-PARTITION(A, p, r)
      pivot = A[p]
        i = p - 1
        j = r - 1
        while true do
          repeat j = j - 1 until A[j] <= pivot
          repeat i = i - 1 until A[i] <= pivot
        if i < j then exchange A[i] with A[j]
        else return j
```

Quicksort with Hoare's Partitioning Algorithm

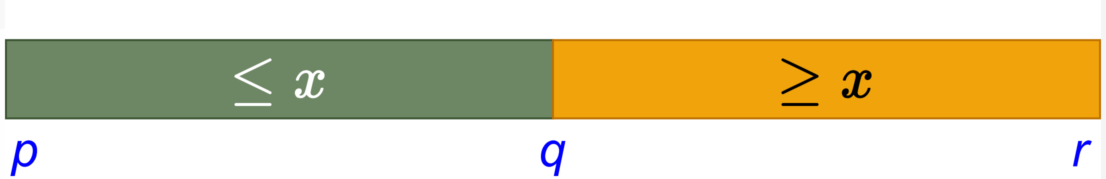
```
QUICKSORT (A, p, r)
  if p < r then
    q = H-PARTITION(A, p, r)
    QUICKSORT(A, p, q)
    QUICKSORT(A, q + 1, r)
  endif
```

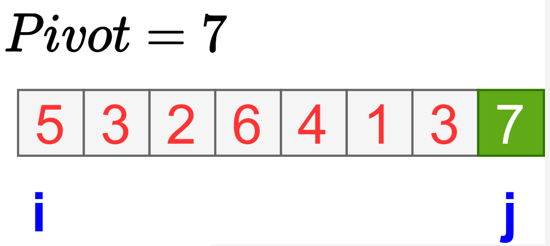
Initial invocation: `QUICKSORT(A,1,n)`



Hoare's Partitioning Algorithm: Pivot Selection

- if we select pivot to be $A[r]$ instead of $A[p]$ in **H-PARTITION**



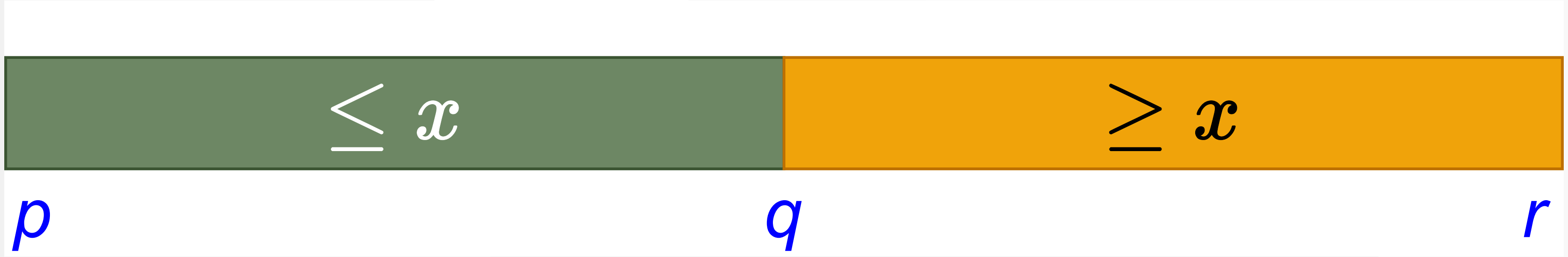


- Consider the example where $A[r]$ is the largest element in the array:
    - End of H-PARTITION: $i = j = r$
    - In QUICKSORT: $q = r$
        - So, recursive call to:
            - `QUICKSORT(A, p, q=r)`
                - **infinite loop**

Correctness of Hoare's Algorithm (1)

We need to prove 3 claims to show correctness:

- Indices i and j never reference A outside the interval $A[p \dots r]$
- Split is always non-trivial; i.e., $j \neq r$ at termination
- Every element in $A[p \dots j] \leq$ every element in $A[j+1 \dots r]$ at termination



Correctness of Hoare's Algorithm (2)

- Notations:
  - k: $\#$ of times the while-loop iterates until termination
  - i_m: the value of index i at the end of iteration m
  - j_m: the value of index j at the end of iteration m
  - x: the value of the pivot element
- **Note**: We always have $i_1 = p$ and $p \leq j_1 \leq r$ because $x = A[p]$

Correctness of Hoare's Algorithm (3)

Lemma 1: Either $i_k = j_k$ or $i_k = j_k + 1$ at termination

Proof of Lemma 1:

- The algorithm terminates when $i \geq j$ (the else condition).
- So, it is sufficient to prove that $i_k – j_k \leq 1$
- There are 2 cases to consider:
  - Case 1: $k = 1$, i.e. the algorithm terminates in a single iteration
  - Case 2: $k > 1$, i.e. the alg. does not terminate in a single iter.

By contradiction, assume there is a run with $i_k – j_k > 1$

Correctness of Hoare's Algorithm (4)

Original correctness claims:

- Indices i and j never reference A outside the interval $A[p \dots r]$
- Split is always non-trivial; i.e., $j \neq r$ at termination

Proof:

- **For $k = 1$:**
  - Trivial because $i_1 = j_1 = p$ (*see Case 1 in proof of Lemma 2*)
- **For $k > 1$:**
  - $i_k > p$ and $j_k < r$ (*due to the repeat-until loops moving indices*)
  - $i_k \leq r$ and $j_k \geq p$ (*due to Lemma 1 and the statement above*)

The proof of claims (a) and (b) complete

Correctness of Hoare's Algorithm (5)

Lemma 2: At the end of iteration m, where $m < k$ (*i.e. m is not the last iteration*), we must have:

$A[p \dots i_m] \leq x$ and $A[j_m \dots r] \geq x$

Proof of Lemma 2:

- **Base case:** $m = 1$ and $k > 1$ (*i.e. the alg. does not terminate in the first iter.*)

Ind. Hyp.: At the end of iteration $m - 1$, where $m < k$ (*i.e. m is not the last iteration*), we must have:

$A[p \dots i_m - 1] \leq x$ and $A[j_m - 1 \dots r] \geq x$

General case: The lemma holds for m, where $m < k$

Proof of base case complete!

Correctness of Hoare's Algorithm (6)

Original correctness claim:

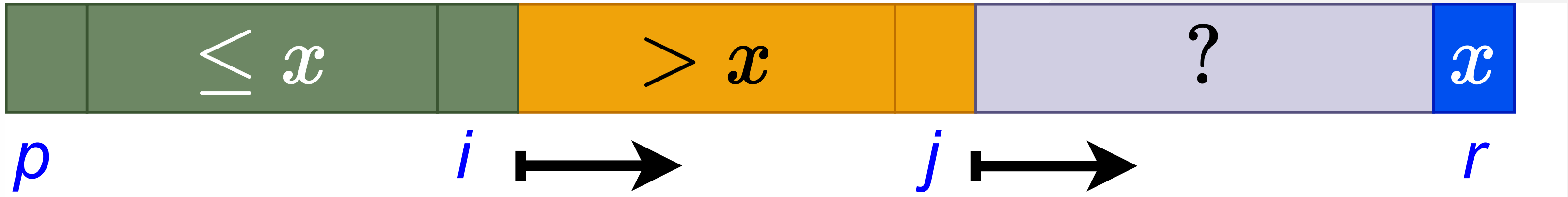
- (c) Every element in $A[\dots j] \leq$ every element in $A[j+\dots r]$ at termination

Proof of claim (c)

- There are 3 cases to consider:
  - **Case 1:** $k=1$, i.e. the algorithm terminates in a single iteration
  - **Case 2:** $k>1$ and $i_k=j_k$
  - **Case 3:** $k>1$ and $i_k=j_k+1$

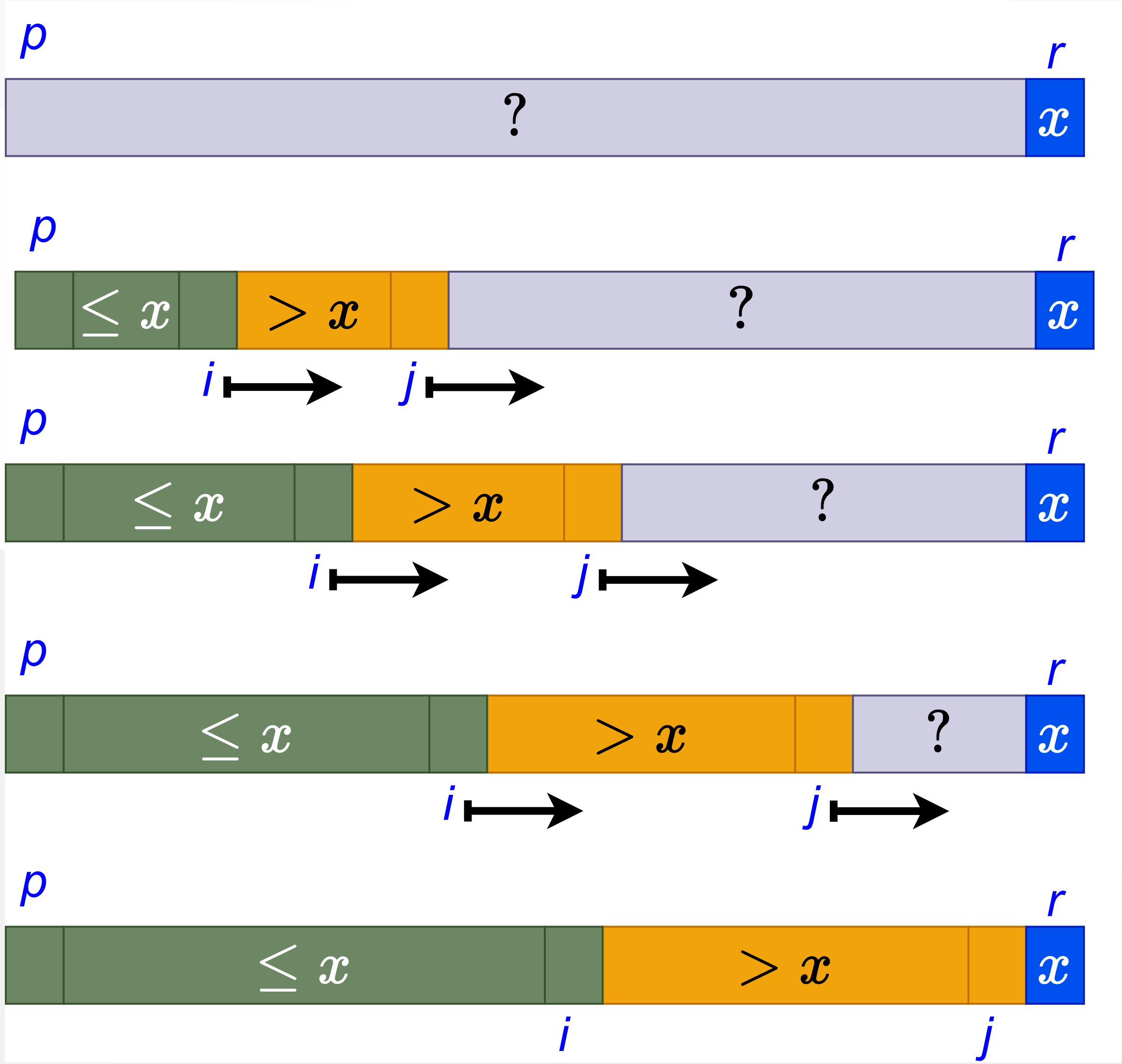
Lomuto's Partitioning Algorithm

- Choose a pivot element: $pivot = x = A[r]$



- Grow two regions:
  - from left to right: $A[p \dots i]$
  - from left to right: $A[i+1 \dots j]$
    - such that:
      - every element in $A[p \dots i] \leq pivot$
      - every element in $A[i+1 \dots j] > pivot$

Lomuto's Partitioning Algorithm



Lomuto's Partitioning Algorithm Ex. (Step-1)

L-PARTITION (A, p, r)

$pivot \leftarrow A[r]$

$i \leftarrow p - 1$

for $j \leftarrow p$ **to** $r - 1$ **do**

  if $A[j] \leq pivot$ **then**

    $i \leftarrow i + 1$

    exchange $A[i] \leftrightarrow A[j]$

exchange $A[i + 1] \leftrightarrow A[r]$

return $i + 1$

p $\quad Pivot = 4 \quad$ r

Input	7	8	2	6	5	1	3	4

$STEP - 1$

Lomuto's Partitioning Algorithm Ex. (Step-2)

L-PARTITION (A, *p*, *r*)

$pivot \leftarrow A[r]$

$i \leftarrow p - 1$

for $j \leftarrow p$ **to** $r - 1$ **do**

if $A[j] \leq pivot$ **then**

$i \leftarrow i + 1$

exchange $A[i] \leftrightarrow A[j]$

exchange $A[i + 1] \leftrightarrow A[r]$

return $i + 1$

p $\quad Pivot = 4 \quad$ r

Input	7	8	2	6	5	1	3	4

i j

$STEP - 2$

Lomuto's Partitioning Algorithm Ex. (Step-3)

L-PARTITION (A, p, r)

$pivot \leftarrow A[r]$

$i \leftarrow p - 1$

for $j \leftarrow p$ **to** $r - 1$ **do**

  if $A[j] \leq pivot$ **then**

    $i \leftarrow i + 1$

    exchange $A[i] \leftrightarrow A[j]$

exchange $A[i + 1] \leftrightarrow A[r]$

return $i + 1$

p $\quad Pivot = 4 \quad$ r

Input	7	8	2	6	5	1	3	4
	i j							

$STEP - 3$

Lomuto's Partitioning Algorithm Ex. (Step-4)

L-PARTITION (A, *p*, *r*)

$pivot \leftarrow A[r]$

$i \leftarrow p - 1$

for $j \leftarrow p$ **to** $r - 1$ **do**

  if $A[j] \leq pivot$ **then**

    $i \leftarrow i + 1$

    exchange $A[i] \leftrightarrow A[j]$

exchange $A[i+1] \leftrightarrow A[r]$

return $i + 1$

p $Pivot = 4$ r

Input	7	8	2	6	5	1	3	4

i j

$STEP - 4$

Lomuto's Partitioning Algorithm Ex. (Step-5)

L-PARTITION (A, p, r)

$pivot \leftarrow A[r]$
$i \leftarrow p - 1$
for $j \leftarrow p$ **to** $r - 1$ **do**
    if $A[j] \le pivot$ **then**
        $i \leftarrow i + 1$
        exchange $A[i] \leftrightarrow A[j]$
exchange $A[i+1] \leftrightarrow A[r]$
return $i + 1$

$Pivot = 4$

	p							r
Input	7	8	2	6	5	1	3	4
	i		j					

$STEP - 5$

Lomuto's Partitioning Algorithm Ex. (Step-6)

L-PARTITION (A, p, r)

$pivot \leftarrow A[r]$
$i \leftarrow p - 1$
for $j \leftarrow p$ **to** $r - 1$ **do**
  if $A[j] \leq pivot$ **then**
    $i \leftarrow i + 1$
    exchange $A[i] \leftrightarrow A[j]$
exchange $A[i + 1] \leftrightarrow A[r]$
return $i + 1$

p $\quad Pivot = 4 \quad$ r

Input	2	8	7	6	5	1	3	4
	i		j					

$STEP - 6$

Lomuto's Partitioning Algorithm Ex. (Step-7)

L-PARTITION (A, *p*, *r*)

pivot ← A[*r*]
i ← *p* – 1
for j ← *p* to *r* – 1 do
  if A[*j*] ≤ *pivot* then
    i ← *i* + 1
    exchange A[*i*] ↔ A[*j*]
exchange A[*i* + 1] ↔ A[*r*]
return *i* + 1

p $\quad Pivot = 4 \quad$ r

Input	2	8	7	6	5	1	3	4
	i					j		

$STEP - 7$

Lomuto's Partitioning Algorithm Ex. (Step-8)

L-PARTITION (A, *p*, *r*)

$pivot \leftarrow A[r]$

$i \leftarrow p - 1$

for $j \leftarrow p$ **to** $r - 1$ **do**

  if $A[j] \leq pivot$ **then**

    $i \leftarrow i + 1$

    exchange $A[i] \leftrightarrow A[j]$

exchange $A[i+1] \leftrightarrow A[r]$

return $i + 1$

p $\quad Pivot = 4 \quad$ r

Input	2	8	7	6	5	1	3	4
	i					j		

$STEP - 8$

Lomuto's Partitioning Algorithm Ex. (Step-9)

L-PARTITION (A, p, r)

$pivot \leftarrow A[r]$
$i \leftarrow p - 1$
for $j \leftarrow p$ **to** $r - 1$ **do**
  if $A[j] \leq pivot$ **then**
    $i \leftarrow i + 1$
    exchange $A[i] \leftrightarrow A[j]$
exchange $A[i + 1] \leftrightarrow A[r]$
return $i + 1$

p $\quad Pivot = 4 \quad$ r

Input	2	1	7	6	5	8	3	4
		i				j		

$STEP - 9$

Lomuto's Partitioning Algorithm Ex. (Step-10)

L-PARTITION (A, p, r)

$pivot \leftarrow A[r]$
$i \leftarrow p - 1$
for $j \leftarrow p$ **to** $r - 1$ **do**
  if $A[j] \leq pivot$ **then**
    $i \leftarrow i + 1$
    exchange $A[i] \leftrightarrow A[j]$
exchange $A[i + 1] \leftrightarrow A[r]$
return $i + 1$

p $\quad Pivot = 4 \quad$ r

Input	2	1	7	6	5	8	3	4
		i					j	

$STEP - 10$

Lomuto's Partitioning Algorithm Ex. (Step-11)

L-PARTITION (A, *p*, *r*)

$pivot \leftarrow A[r]$
$i \leftarrow p - 1$
for $j \leftarrow p$ **to** $r - 1$ **do**
  if $A[j] \leq pivot$ **then**
    $i \leftarrow i + 1$
    exchange $A[i] \leftrightarrow A[j]$
exchange $A[i + 1] \leftrightarrow A[r]$
return $i + 1$

p $Pivot = 4$ r

Input	2	1	7	6	5	8	3	4
			i				j	

$STEP - 11$

Lomuto's Partitioning Algorithm Ex. (Step-12)

L-PARTITION (A, *p*, *r*)

$pivot \leftarrow A[r]$

$i \leftarrow p - 1$

for $j \leftarrow p$ **to** $r - 1$ **do**

 if $A[j] \leq pivot$ **then**

  $i \leftarrow i + 1$

  exchange $A[i] \leftrightarrow A[j]$

exchange $A[i + 1] \leftrightarrow A[r]$

return $i + 1$

p $Pivot = 4$ r

Input	2	1	3	6	5	8	7	4
			i				j	

$STEP - 12$

Lomuto's Partitioning Algorithm Ex. (Step-13)

L-PARTITION (A, p, r)

$pivot \leftarrow A[r]$

$i \leftarrow p - 1$

for $j \leftarrow p$ **to** $r - 1$ **do**

  if $A[j] \leq pivot$ **then**

    $i \leftarrow i + 1$

    exchange $A[i] \leftrightarrow A[j]$

exchange $A[i+1] \leftrightarrow A[r]$

return $i + 1$

p $Pivot = 4$ r

Input

2	1	3	6	5	8	7	4
		i				j	

$STEP - 13$

Lomuto's Partitioning Algorithm Ex. (Step-14)

L-PARTITION (A, p, r)

$pivot \leftarrow A[r]$
$i \leftarrow p - 1$
for $j \leftarrow p$ **to** $r - 1$ **do**
  if $A[j] \leq pivot$ **then**
    $i \leftarrow i + 1$
    exchange $A[i] \leftrightarrow A[j]$
exchange $A[i + 1] \leftrightarrow A[r]$
return $i + 1$

p $\quad Pivot = 4 \quad$ r

Input	2	1	3	4	5	8	7	6
			i				j	

$STEP - 14$

Lomuto's Partitioning Algorithm Ex. (Step-15)

L-PARTITION (A, *p*, *r*)

$pivot \leftarrow A[r]$

$i \leftarrow p - 1$

for $j \leftarrow p$ **to** $r - 1$ **do**

  if $A[j] \leq pivot$ **then**

    $i \leftarrow i + 1$

    exchange $A[i] \leftrightarrow A[j]$

exchange $A[i+1] \leftrightarrow A[r]$

return $i + 1$

p $Pivot = 4$ r

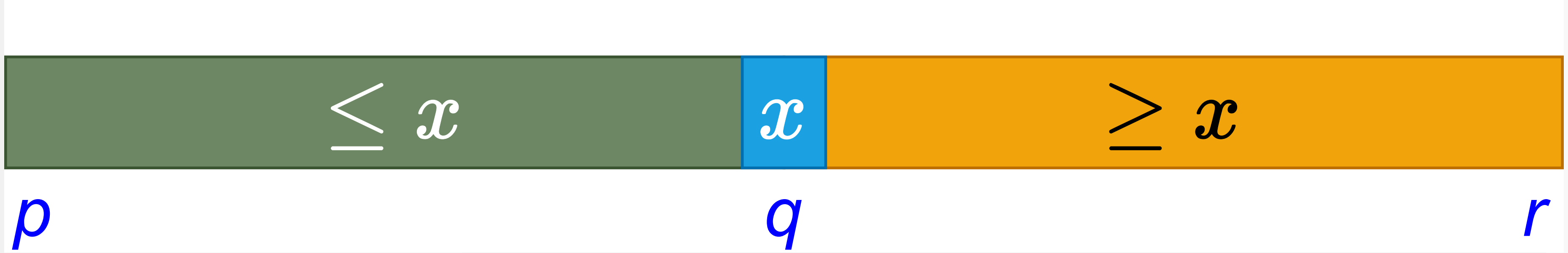
Input	2	1	3	4	5	8	7	6
				q				

$STEP - 15$

Quicksort with Lomuto’s Partitioning Algorithm

```
QUICKSORT (A, p, r)
  if p < r then
    q = L-PARTITION(A, p, r)
    QUICKSORT(A, p, q - 1)
    QUICKSORT(A, q + 1, r)
  endif
```

Initial invocation: `QUICKSORT(A,1,n)`



Comparison of Hoare's & Lomuto's Algorithms

- Notation: $n = r - p + 1$
  - $pivot = A[p]$ (*Hoare*)
  - $pivot = A[r]$ (*Lomuto*)
- $\#$ **of element exchanges:** $e(n)$
  - Hoare: $0 \geq e(n) \geq \lfloor \frac{n}{2} \rfloor$
    - **Best**: $k = 1$ with $i_1 = j_1 = p$ (i.e., $A[p+1 \dots r] > pivot$)
    - **Worst**: $A[p+1 \dots p + \lfloor \frac{n}{2} \rfloor - 1] \geq pivot \geq A[p + \lceil \frac{n}{2} \rceil \dots r]$
  - Lomuto : $1 \leq e(n) \leq n$
    - **Best**: $A[p \dots r-1] > pivot$
    - **Worst**: $A[p \dots r-1] \leq pivot$

Comparison of Hoare's & Lomuto's Algorithms

- $\#$ of element comparisons: $c_e(n)$
  - Hoare: $n + 1 \leq c_e(n) \leq n + 2$
    - Best: $i_k = j_k$
    - Worst: $i_k = j_k + 1$
  - Lomuto: $c_e(n) = n - 1$
- $\#$ of index comparisons: $c_i(n)$
  - Hoare: $1 \leq c_i(n) \leq \lfloor \frac{n}{2} \rfloor + 1 | (c_i(n) = e(n) + 1)$
  - Lomuto: $c_i(n) = n - 1$

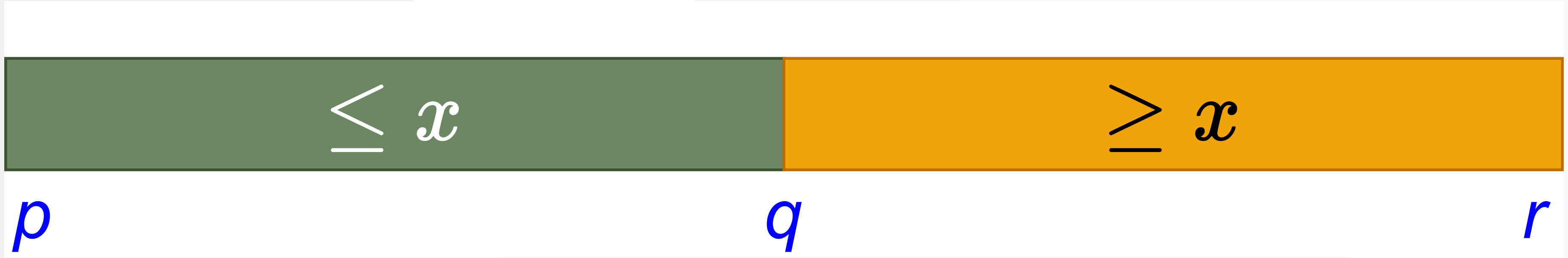
Comparison of Hoare's & Lomuto's Algorithms

- **$\#$ of index increment/decrement operations: $a(n)$**
  - **Hoare**: $n+1 \leq a(n) \leq n+2 | (a(n) = c_e(n))$
  - **Lomuto**: $n \leq a(n) \leq 2n-1 | (a(n) = e(n) + (n-1))$
- Hoare's algorithm is in general faster
- Hoare behaves better when pivot is repeated in $A[p \dots r]$
  - **Hoare**: Evenly distributes them between left & right regions
  - **Lomuto**: Puts all of them to the left region

Analysis of Quicksort

```
QUICKSORT (A, p, r)
  if p < r then
    q = H-PARTITION(A, p, r)
    QUICKSORT(A, p, q)
    QUICKSORT(A, q + 1, r)
  endif
```

Initial invocation: `QUICKSORT(A,1,n)`



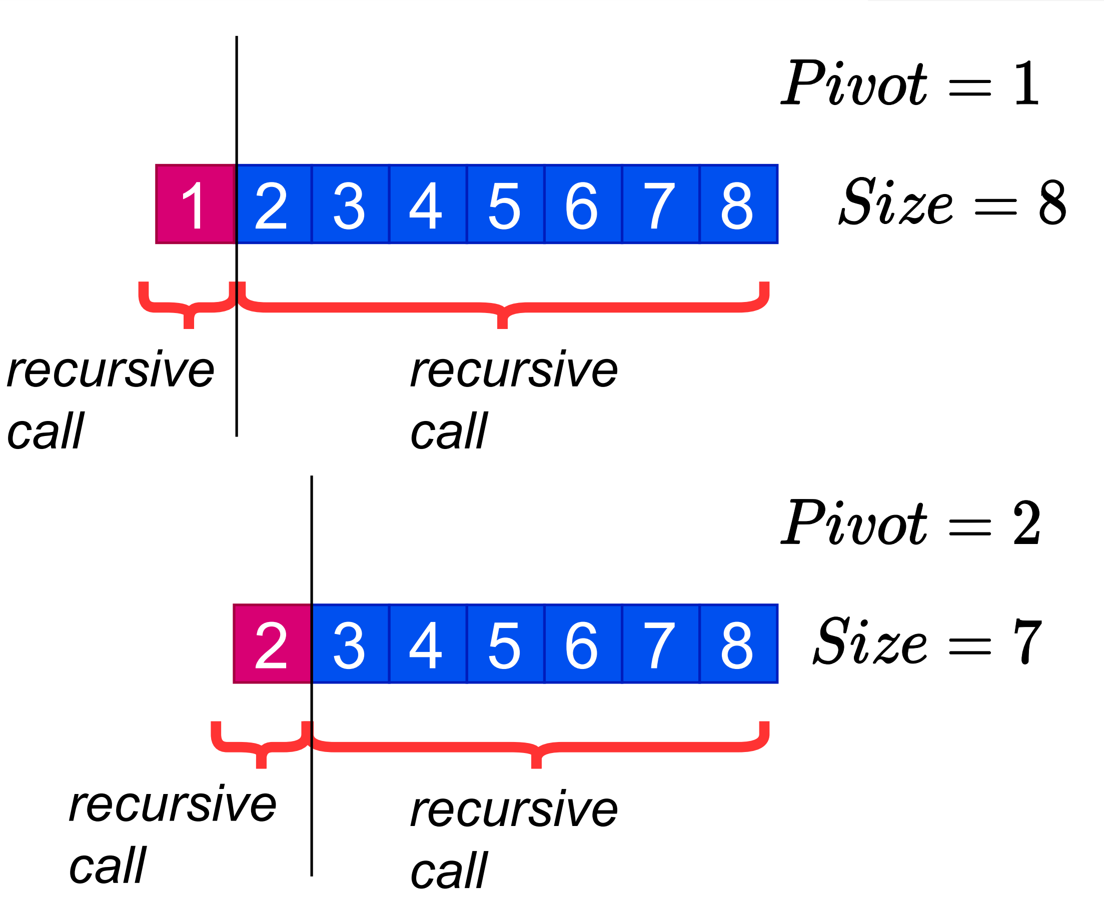
Assume **all elements are distinct** in the following analysis

Analysis of Quicksort

- **H-PARTITION** always chooses $A[p]$ (the first element) as the pivot.
- The runtime of **QUICKSORT** on an already-sorted array is $\Theta(n^2)$

Example: An Already Sorted Array

Partitioning always leads to 2 parts of size 1 and $n-1$



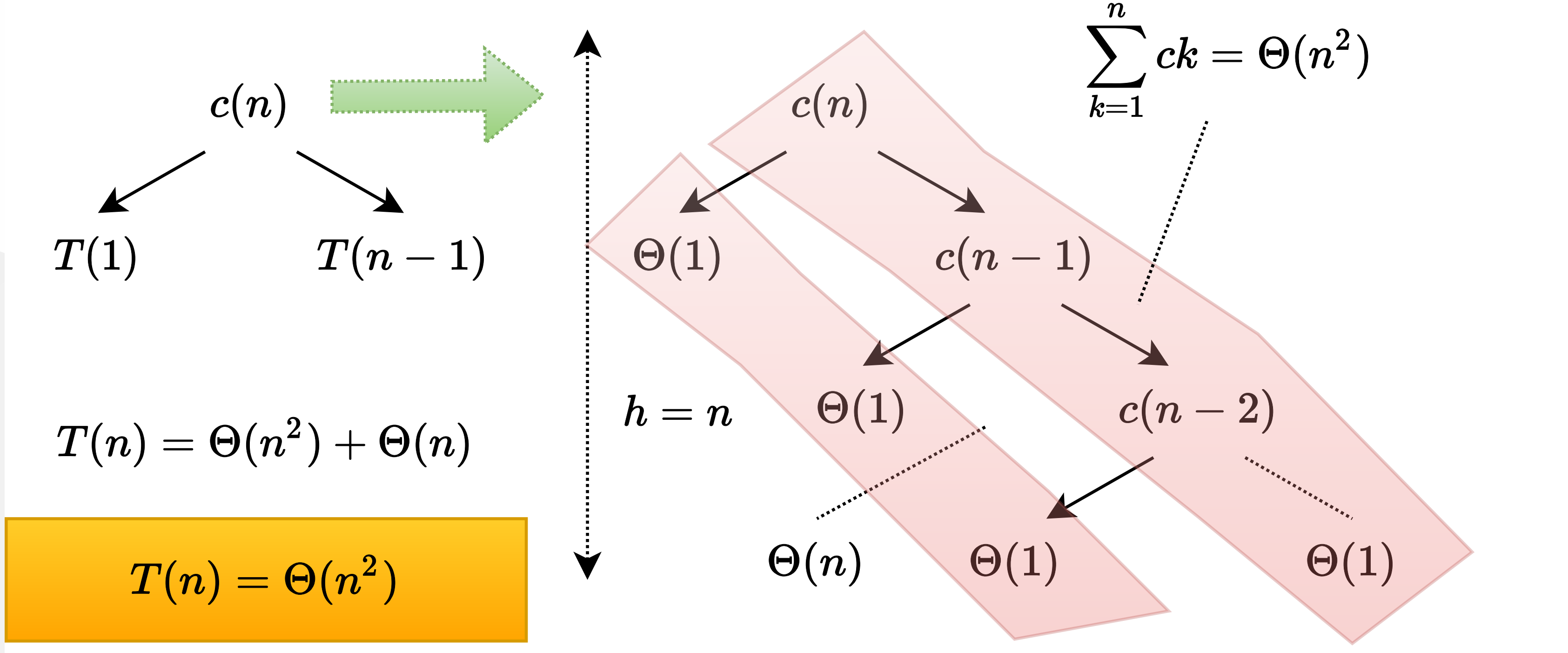
Worst Case Analysis of Quicksort

- **Worst case** is when the **PARTITION** algorithm always returns **imbalanced partitions** (of size 1 and $n-1$) in every recursive call.
  - This happens when the pivot is selected to be either the min or **max** element.
  - This happens for **H-PARTITION** when the input array is already sorted or reverse sorted

$$\begin{aligned} T(n) &= T(1) + T(n-1) + \Theta(n) \\ &= T(n-1) + \Theta(n) \\ &= \Theta(n2) \end{aligned}$$

Worst Case Recursion Tree

$$T(n) = T(1) + T(n-1) + cn$$



Best Case Analysis (for intuition only)

- If we're extremely lucky, **H-PARTITION** splits the array evenly at every recursive call

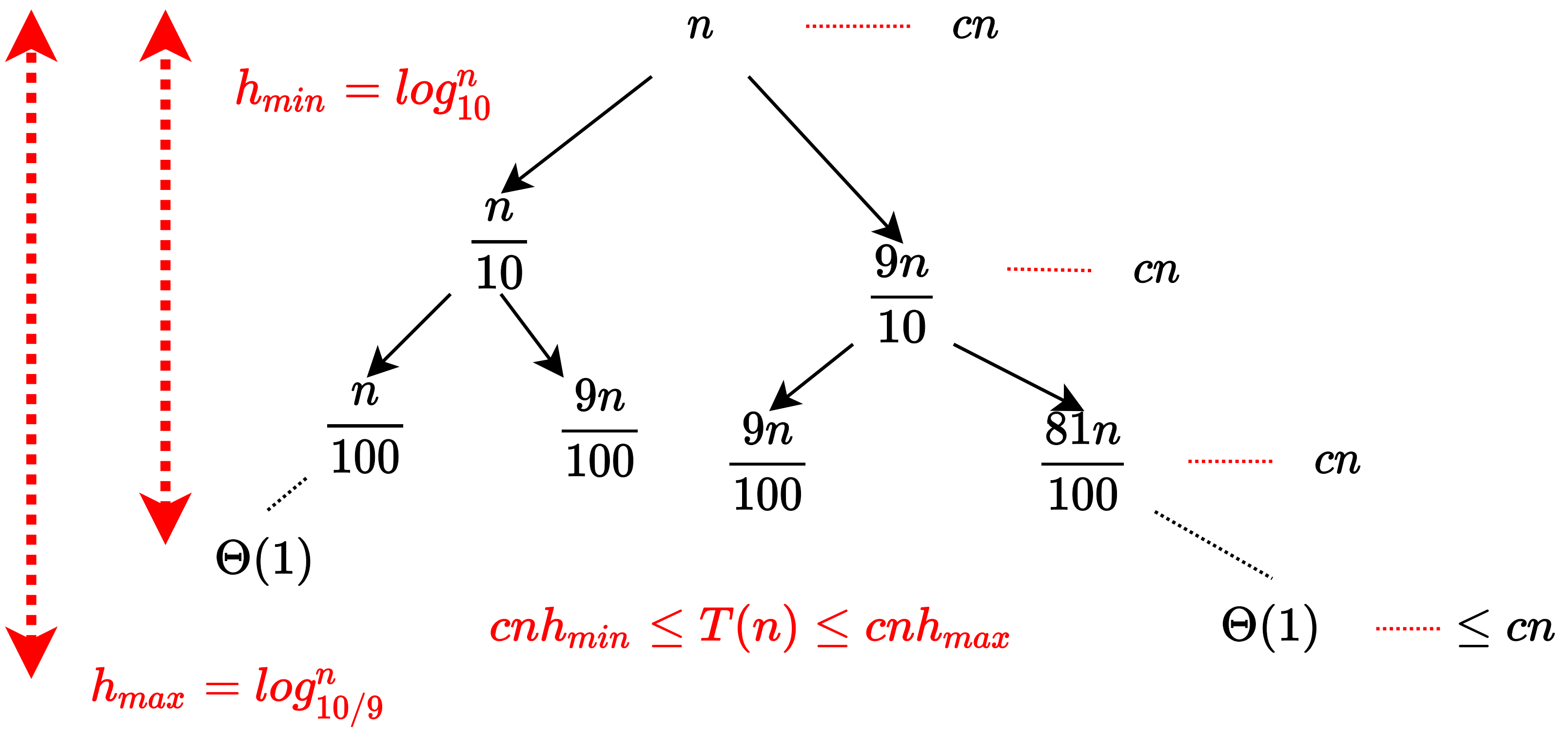
$$\begin{aligned} T(n) &= 2T(n/2) + \Theta(n) \\ &= \Theta(nlgn) \end{aligned}$$

(same as merge sort)

- Instead of splitting $0.5 : 0.5$, if we split $0.1 : 0.9$ then we need solve following equation.

$$\begin{aligned} T(n) &= T(n/10) + T(9n/10) + \Theta(n) \\ &= \Theta(nlgn) \end{aligned}$$

"Almost-Best" Case Analysis



$$cnlog_{10}^{n} \leq T(n) \leq cnlog_{10/9}^{n}$$

$$\boxed{T(n) = \Theta(nlgn)}$$

Balanced Partitioning

- We have seen that if **H-PARTITION** always splits the array with $0.1-to-0.9$ ratio, the runtime will be $\Theta(nlgn)$.
- Same is true with a split ratio of $0.01-to-0.99$, etc.
- Possible to show that if the split has always constant $(\Theta(1))$ proportionality, then the runtime will be $\Theta(nlgn)$.
- In other words, for a **constant** $\alpha|(0<\alpha\leq 0.5)$:
  - α–to–$(1-\alpha)$ proportional split yields $\Theta(nlgn)$ total runtime

Balanced Partitioning

- In the rest of the analysis, assume that all input permutations are equally likely.
  - This is only to gain some intuition
  - We cannot make this assumption for average case analysis
  - We will revisit this assumption later
- Also, assume that all input elements are distinct.

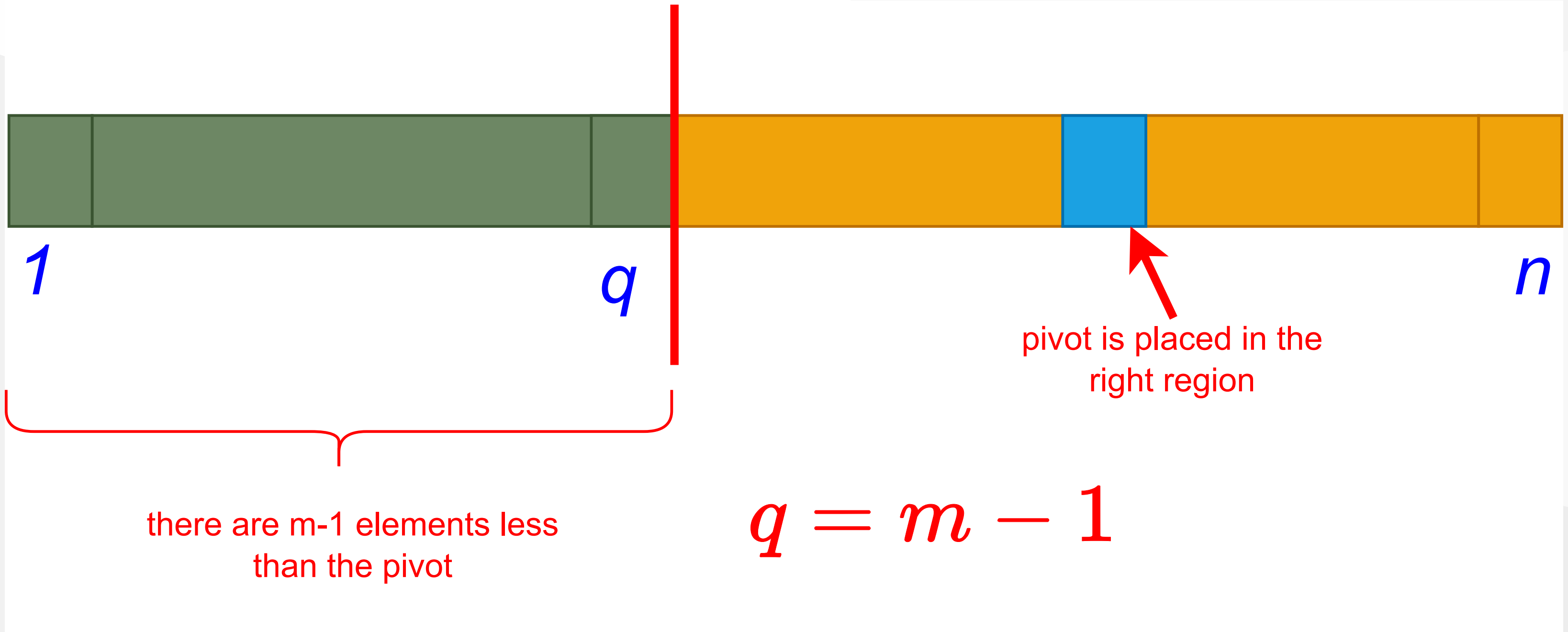
Balanced Partitioning

- **Question:** What is the probability that H-PARTITION returns a split that is more balanced than $0.1-to-0.9$?

Balanced Partitioning

Reminder: *H-PARTITION* will place the pivot in the right partition unless the pivot is the smallest element in the arrays.

Question: If the pivot selected is the mth smallest value $(1 < m \leq n)$ in the input array, what is the size of the left region after partitioning?

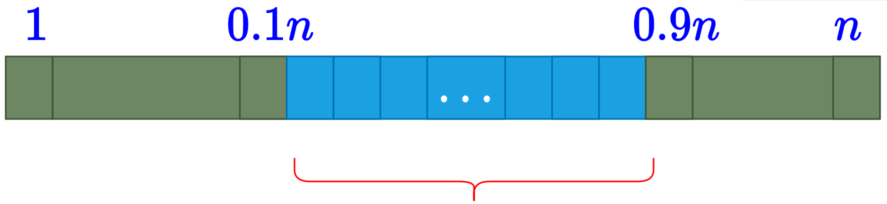


Balanced Partitioning

- **Question:** What is the probability that the **pivot** selected is the m^{th} smallest value in the array of size n?
  - $1/n$ (*since all input permutations are equally likely*)
- **Question:** What is the probability that the left partition returned by **H-PARTITION** has size m, where $1 < m < n$?
  - $1/n$ (*due to the answers to the previous 2 questions*)

Balanced Partitioning

- **Question:** What is the probability that H-PARTITION returns a split that is more balanced than $0.1 - to - 0.9$?



The partition boundary will be in
this region for a more balanced split than
$0.1 - to - 0.9$

$$Probability = \sum_{q=0.1n+1}^{0.9n-1} \frac{1}{n} = \frac{1}{n}(0.9n - 1 - 0.1n - 1 + 1)$$

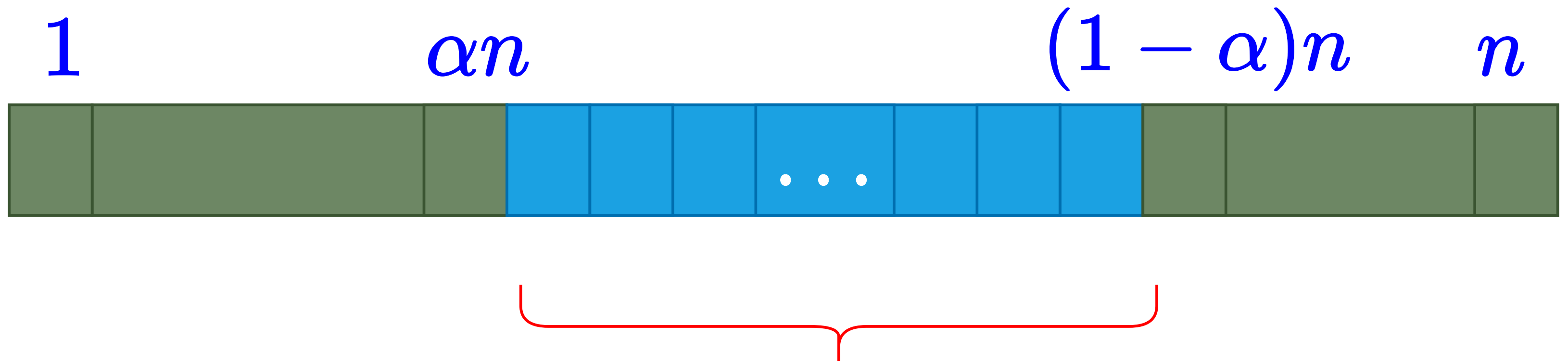
$$= 0.8 - \frac{1}{n}$$

$$\approx 0.8 \text{ for large n}$$

Balanced Partitioning

- The probability that **H-PARTITION** yields a split that is more balanced than $0.1-to-0.9$ is 80% on a random array.
- Let $P_{\alpha>}$ be the probability that **H-PARTITION** yields a split more balanced than $\alpha-to-(1-\alpha)$, where $0<\alpha\leq 0.5$
- Repeat the analysis to generalize the previous result

balanced than $\alpha - to - (1-\alpha)$?



The partition boundary will be in this region for a more balanced split than

$$\alpha n - to - (1-\alpha)n$$

$$Probability = \sum_{q=\alpha n+1}^{(1-\alpha)n-1} \frac{1}{n}$$

$$= \frac{1}{n}((1-\alpha)n - 1 - \alpha n - 1 + 1)$$

Balanced Partitioning

- We found $P_{\alpha>} = 1 - 2\alpha$
  - Ex: $P_{0.1>} = 0.8$ and $P_{0.01>} = 0.98$
- Hence, **H-PARTITION** produces a split
  - **more balanced than a**
    - $0.1 - to - 0.9$ split 80% of the time
    - $0.01 - to - 0.99$ split 98% of the time
  - **less balanced than a**
    - $0.1 - to - 0.9$ split 20% of the time
    - $0.01 - to - 0.99$ split 2% of the time

Intuition for the Average Case

- **Assumption:** All permutations are equally likely
    - Only for intuition; we’ll revisit this assumption later
- **Unlikely:** Splits always the same way at every level
- **Expectation:**
    - Some splits will be reasonably balanced
    - Some splits will be fairly unbalanced
- **Average case:** A mix of good and bad splits
    - **Good** and **bad** splits distributed randomly thru the tree

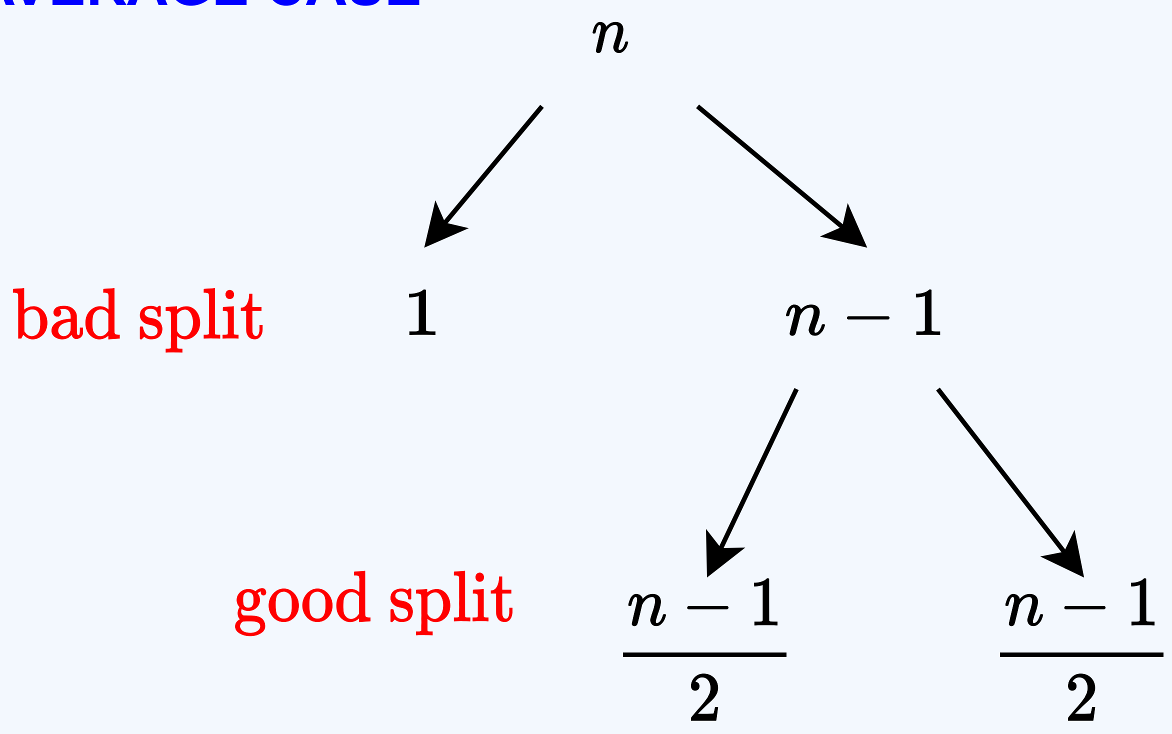
Intuition for the Average Case

- **Assume for intuition:** Good and bad splits occur in the alternate levels of the tree
  - **Good split:** Best case split
  - **Bad split:** Worst case split

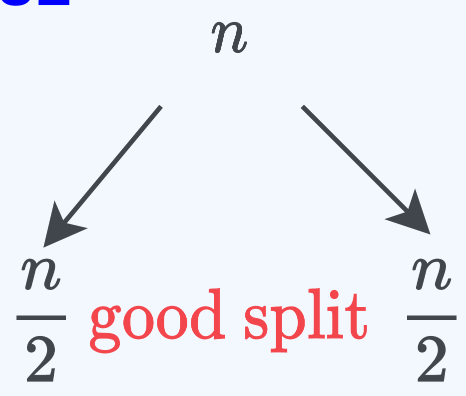
Intuition for the Average Case

Compare 2-successive levels of avg case vs. 1 level of best case

AVERAGE CASE



BEST CASE



Intuition for the Average Case

- In terms of the remaining subproblems, **two levels of avg case** is slightly better than the **single level of the best case**
- The avg case has **extra divide cost of** $\Theta(n)$ at alternate levels
- The extra divide cost $\Theta(n)$ of bad splits absorbed into the $\Theta(n)$ of good splits.
- Running time is still $\Theta(nlgn)$
  - But, slightly larger hidden constants, because the height of the recursion tree is about twice of that of best case.

Intuition for the Average Case

- Another way of looking at it:
  - Suppose we alternate lucky, unlucky, lucky, unlucky, ...
  - We can write the recurrence as:
    - $L(n) = 2U(n/2) + \Theta(n)$ lucky split (best)
    - $U(n) = L(n-1) + \Theta(n)$ unlucky split (worst)
  - Solving:

$$
\begin{aligned}
L(n) &= 2(L(n/2-1) + \Theta(n/2)) + \Theta(n) \\
&= 2L(n/2-1) + \Theta(n) \\
&= \Theta(nlgn)
\end{aligned}
$$

- How can we make sure we are usually lucky for all inputs?

Summary: Quicksort Runtime Analysis

- **Worst case:** Unbalanced split at every recursive call

$$T(n) = T(1) + T(n-1) + \Theta(n)$$
$$T(n) = \Theta(n2)$$

- **Best case:** Balanced split at every recursive call (*extremely lucky*)

$$T(n) = 2T(n/2) + \Theta(n)$$
$$T(n) = \Theta(nlgn)$$

Summary: Quicksort Runtime Analysis

- **Almost-best case:** Almost-balanced split at every recursive call

$$T(n) = T(n/10) + T(9n/10) + \Theta(n)$$

$$\text{or } T(n) = T(n/100) + T(99n/100) + \Theta(n)$$

$$\text{or } T(n) = T(\alpha n) + T((1 - \alpha n) + \Theta(n)$$

for any constant $\alpha, 0 < \alpha \leq 0.5$

Summary: Quicksort Runtime Analysis

- For a random input array, the probability of having a split
  - more balanced than $0.1\text{–}to\text{–}0.9$: 80%
  - more balanced than $0.01\text{–}to\text{–}0.99$: 98%
  - more balanced than $\alpha\text{–}to\text{–}(1-\alpha)$: $1\text{–}2\alpha$

    for any constant $\alpha, 0 < \alpha \leq 0.5$

Summary: Quicksort Runtime Analysis

- **Avg case intuition:** Different splits expected at different levels
  - some balanced (good), some unbalanced (bad)
- **Avg case intuition:** Assume the good and bad splits alternate
  - i.e. good split -> bad split -> good split -> …
  - $T(n) = \Theta(nlgn)$
    - (informal analysis for intuition)

Randomized Quicksort

- In the avg-case analysis, we assumed that **all permutations** of the input array are **equally likely.**
    - But, this assumption **does not always hold**
    - e.g. What if **all** the input arrays are **reverse sorted**?
        - **Always worst-case behavior**
- Ideally, the avg-case runtime should be **independent of the input permutation**.
- **Randomness should be within the algorithm**, not based on the distribution of the inputs.
    - i.e. The avg case should hold for all possible inputs

Randomized Algorithms

- Alternative to assuming a uniform distribution:
  - **Impose a uniform distribution**
  - e.g. Choose a random pivot rather than the first element
- Typically useful when:
  - there are many ways that an algorithm can proceed
  - but, it's **difficult** to determine a way that is **always guaranteed to be good**.
  - If there are **many good alternatives**; simply **choose one randomly**.

Randomized Algorithms

- Ideally:
  - Runtime should be **independent of the specific inputs**
  - No specific input should cause worst-case behavior
  - Worst-case should be determined only by output of a random number generator.

Randomized Quicksort

- Using Hoare's partitioning algorithm:

```
R-QUICKSORT(A, p, r)
  if p < r then
    q = R-PARTITION(A, p, r)
    R-QUICKSORT(A, p, q)
    R-QUICKSORT(A, q+1, r)
```

```
R-PARTITION(A, p, r)
  s = RANDOM(p, r)
  exchange A[p] with A[s]
  return H-PARTITION(A, p, r)
```

- Alternatively, permuting the whole array would also work
  - but, would be more difficult to analyze

Randomized Quicksort

- Using Lomuto's partitioning algorithm:

```
R-QUICKSORT(A, p, r)
  if p < r then
    q = R-PARTITION(A, p, r)
    R-QUICKSORT(A, p, q-1)
    R-QUICKSORT(A, q+1, r)
```

```
R-PARTITION(A, p, r)
  s = RANDOM(p, r)
  exchange A[r] with A[s]
  return L-PARTITION(A, p, r)
```

- Alternatively, permuting the whole array would also work
  - but, would be more difficult to analyze

Notations for Formal Analysis

- Assume all elements in $A[p \dots r]$ are distinct
  - Let $n = r – p + 1$
- Let $rank(x) = |A[i] : p \leq i \leq r \text{ and } A[i] \leq x|$
- i.e. $rank(x)$ is the number of array elements with value less than or equal to x
  - $A = \{5, 9, 7, 6, 8, 1, 4\}$
  - $p = 5, r = 4$
  - $rank(5) = 3$
    - i.e. it is the 3^{rd} smallest element in the array

Formal Analysis for Average Case

- The following analysis will be for **Quicksort** using **Hoare's** partitioning algorithm.
- **Reminder:** The **pivot** is selected **randomly** and exchanged with $A[p]$ before calling **H-PARTITION**
- Let x be the **random pivot** chosen.
- What is the probability that $rank(x) = i$ for $i = 1, 2, \dots n$?
  - $P(rank(x) = i) = 1/n$

Various Outcomes of H-PARTITION

- Assume that $rank(x) = 1$
  - i.e. the **random pivot** chosen is the **smallest** element
  - What will be the **size of the left partition** $(|L|)$?
  - **Reminder:** Only the elements less than or equal to x will be in the left partition.

$$A = \{ \overbrace{2}^{p=x=pivot} \underbrace{,}_{\implies |L|=1} 9, 7, 6, 8, 5, \overbrace{4}^{r} \}$$

$$p = 2, r = 4$$
$$pivot = x = 2$$

TODO: convert to image...S6_P9

Various Outcomes of H-PARTITION

- Assume that $rank(x) > 1$
  - i.e. the random pivot chosen is not the smallest element
  - What will be the size of the left partition $(|L|)$?
  - **Reminder:** Only the elements less than or equal to x will be in the left partition.
  - **Reminder:** The pivot will stay in the right region after **H-PARTITION** if $rank(x) > 1$

$$A = \{ \overbrace{2}^{p}, 4 \underbrace{,}_{\Longrightarrow |L| = rank(x) - 1} 7, 6, 8, \overbrace{5,}^{pivot} \overbrace{9}^{r} \}$$

$p = 2, r = 4$

$pivot = x = 5$

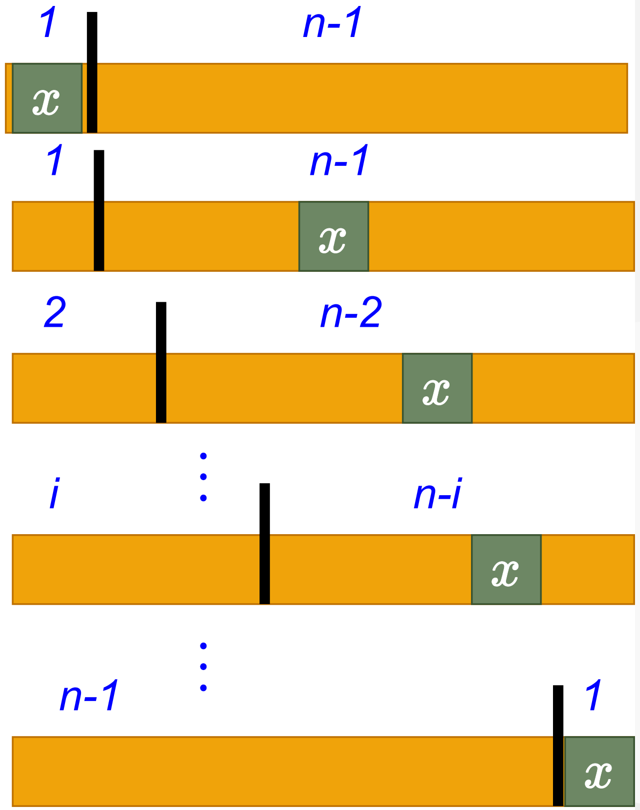
TODO: convert to image...S6_P10

Various Outcomes of H-PARTITION - Summary

- $x : pivot$
- $|L|$: size of left region
- $P(rank(x) = i) = 1/n$ for $1 \leq i \leq n$
  - if $rank(x) = 1$ then $|L| = 1$
  - if $rank(x) > 1$ then $|L| = rank(x) - 1$
- $P(|L| = 1) = P(rank(x) = 1) + P(rank(x) = 2)$
  - $P(|L| = 1) = 2/n$
- $P(|L| = i) = P(rank(x) = i + 1)$ for $1 < i < n$
  - $P(|L| = i) = 1/n$ for $1 < i < n$

Various Outcomes of H-PARTITION - Summary

$rank(x)$	$probability$	$T(n)$
1	$\frac{1}{n}$	$T(1) + T(n-1) + \Theta(n)$
2	$\frac{1}{n}$	$T(1) + T(n-1) + \Theta(n)$
3	$\frac{1}{n}$	$T(2) + T(n-2) + \Theta(n)$
$\vdots$	$\vdots$	$\vdots$
$i+1$	$\frac{1}{n}$	$T(i) + T(n-i) + \Theta(n)$
$\vdots$	$\vdots$	$\vdots$
n	$\frac{1}{n}$	$T(n-1) + T(1) + \Theta(n)$



Average - Case Analysis: Recurrence

$x = pivot$

$$\begin{aligned}
T(n) = & \frac{1}{n}(T(1) + t(n-1)) & rank: 1 \\
& + \frac{1}{n}(T(1) + t(n-1)) & rank: 2 \\
& + \frac{1}{n}(T(2) + t(n-2)) & rank: 3 \\
& \vdots & \vdots \\
& + \frac{1}{n}(T(i) + t(n-i)) & rank: i+1 \\
& \vdots & \vdots \\
& + \frac{1}{n}(T(n-1) + t(1)) & rank: n \\
& + \Theta(n)
\end{aligned}$$

Recurrence

$$T(n) = \frac{1}{n}\sum_{q=1}^{n-1}(T(q)+T(n-q)) + \frac{1}{n}(T(1)+T(n-1)) + \Theta(n)$$

$$\text{Note: } \frac{1}{n}(T(1)+T(n-1)) = \frac{1}{n}(\Theta(1)+O(n^2)) = O(n)$$

$$T(n) = \frac{1}{n}\sum_{q=1}^{n-1}(T(q)+T(n-q)) + \Theta(n)$$

for $k = 1, 2, \dots, n-1$ each term $T(k)$ appears twice once for $q = k$ and once for $q = n - k$

$$T(n) = \frac{2}{n}\sum_{k=1}^{n-1}T(k) + \Theta(n)$$

Solving Recurrence: Substitution

- Guess: $T(n) = O(nlgn)$
- $T(k) \leq aklgk$ for $k < n$, for some constant $a > 0$

$$
\begin{aligned}
T(n) &= \frac{2}{n}\sum_{k=1}^{n-1} T(k) + \Theta(n) \\
&\leq \frac{2}{n}\sum_{k=1}^{n-1} aklgk + \Theta(n) \\
&\leq \frac{2a}{n}\sum_{k=1}^{n-1} klgk + \Theta(n)
\end{aligned}
$$

- Need a tight bound for $\sum klgk$

Tight bound for $\sum klgk$

- Bounding the terms
  - $\sum_{k=1}^{n-1} klgk \leq \sum_{k=1}^{n-1} nlgn = n(n-1)lgn \leq n^2lgn$
  - This bound **is not strong** enough because
  - $T(n) \leq \frac{2a}{n}n^2lgn + \Theta(n)$
  - $= 2anlgn + \Theta(n) \implies$ couldn't prove $T(n) \leq anlgn$

Tight bound for $\sum klgk$

- **Splitting summations:** ignore ceilings for simplicity
- $\sum_{k=1}^{n-1} klgk \leq \sum_{k=1}^{n/2-1} klgk + \sum_{k=n/2}^{n-1} klgk$
  - **First summation:** $lgk < lg(n/2) = lgn - 1$
  - **Second summation:** $lgk < lgn$

Splitting: $\sum_{k=1}^{n-1} klgk \leq \sum_{k=1}^{n/2-1} klgk + \sum_{k=n/2}^{n-1} klgk$

- $\sum_{k=1}^{n-1} klgk \leq (lg(n-1)) \sum_{k=1}^{n/2-1} k + lgn \sum_{k=n/2}^{n-1} k$
  - $= lgn \sum_{k=1}^{n-1} k - \sum_{k=1}^{n/2-1} k$
  - $= \frac{1}{2}n(n-1)lgn - \frac{1}{2}\frac{n}{2}(\frac{n}{2}-1)$
  - $= \frac{1}{2}n^2lgn - \frac{1}{8}n^2 - \frac{1}{2}n(lgn - 1/2)$
- $\sum_{k=1}^{n-1} klgk \leq \frac{1}{2}n^2lgn - \frac{1}{8}n^2$ for $lgn \geq 1/2 \implies n \geq \sqrt{2}$

Substituting: - $\sum_{k=1}^{n-1} klgk \leq \frac{1}{2}n^2lgn - \frac{1}{8}n^2$

$$
\begin{aligned}
T(n) &\leq \frac{2a}{n}\sum_{k=1}^{n-1} klgk + \Theta(n) \\
&\leq \frac{2a}{n}(\frac{1}{2}n^2lgn - \frac{1}{8}n^2) + \Theta(n) \\
&= anlgn - (\frac{a}{4}n - \Theta(n))
\end{aligned}
$$

- We can choose a large enough so that $\frac{a}{4}n \geq \Theta(n)$

$$T(n) \leq anlgn$$

$$T(n) = O(nlgn)$$

Q.E.D.

Medians and Order Statistics

- **ith order statistic**: i^{th} smallest element of a set of n elements
- **minimum:** *first* order statistic
- **maximum:** n^{th} order statistic
- **median**: “halfway point” of the set

$$i = \lfloor \frac{(n+1)}{2} \rfloor$$

or

$$i = \lceil \frac{(n+1)}{2} \rceil$$

Selection Problem

- **Selection problem:** Select the i^{th} smallest of n elements
- **Naïve algorithm:** Sort the input array A; then return $A[i]$
  - $T(n) = \theta(nlgn)$
    - *using e.g. merge sort (but not quicksort)*
- Can we do any better?

Selection in Expected Linear Time

- Randomized algorithm using divide and conquer
- Similar to randomized quicksort
  - **Like quicksort:** Partitions input array recursively
  - **Unlike quicksort:** Makes a single recursive call
    - **Reminder:** *Quicksort makes two recursive calls*
- Expected runtime: $\Theta(n)$
  - **Reminder:** *Expected runtime of quicksort:* $\Theta(nlgn)$

Selection in Expected Linear Time: Example 1

- Select the 2^{nd} smallest element:

$$A = \{6, 10, 13, 5, 8, 3, 2, 11\}$$
$$i = 2$$

- Partition the input array:

$$A = \{\ \underbrace{2, 3, 5,}_{\text{left subarray}}\ \underbrace{13, 8, 10, 6, 11}_{\text{right subarray}}\}$$

- make a recursive call to select the 2^{nd} **smallest** element in **left subarray**

Selection in Expected Linear Time: Example 2

- Select the 7^{th} smallest element:

$$A = \{6, 10, 13, 5, 8, 3, 2, 11\}$$
$$i = 7$$

- Partition the input array:

$$A = \{\ \underbrace{2, 3, 5,}_{\text{left subarray}}\ \underbrace{13, 8, 10, 6, 11}_{\text{right subarray}}\}$$

- make a recursive call to select the 4^{th} **smallest** element in **right subarray**

Selection in Expected Linear Time

```
R-SELECT(A,p,r,i)
  if p == r then
    return A[p];
  q = R-PARTITION(A, p, r)
  k = q-p+1;
  if i <= k then
    return R-SELECT(A, p, q, i);
  else
    return R-SELECT(A, q+1, r, i-k);
```

$$A = \{ \underbrace{\mid}_{p} \cdots \leq x (\text{k smallest elements}) \dots \underbrace{\mid}_{q} \cdots \geq x \dots \underbrace{\mid}_{r} \}$$

$$x = pivot$$

Selection in Expected Linear Time

$$A = \{ \overbrace{|}^{p} \underbrace{\dots \leq x \dots}_{L} \overbrace{|}^{q} \underbrace{\dots \geq x \dots}_{R} \overbrace{|}^{r} \}$$

$$x = pivot$$

- All elements in $L \leq$ all elements in R
- L contains:
- $|L| = q–p + 1 =$ k smallest elements of $A[p...r]$
- if $i \leq |L| = k$ then
  - **search** L recursively for its i^{th} smallest element
- else
  - **search** R recursively for its $(i - k)^{th}$ smallest element

Runtime Analysis

- **Worst case:**
  - Imbalanced partitioning at every level and the recursive call always to the larger partition

$$= \{1, \underbrace{2, 3, 4, 5, 6, 7, 8}_{\text{recursive call}}\} \quad i = 8$$

$$= \{2, \underbrace{3, 4, 5, 6, 7, 8}_{\text{recursive call}}\} \qquad i = 7$$

Runtime Analysis

- **Worst case:** Worse than the naïve method (based on sorting)

$$T(n) = T(n-1) + \Theta(n)$$
$$T(n) = \Theta(n^2)$$

- **Best case:** Balanced partitioning at every recursive level

$$T(n) = T(n/2) + \Theta(n)$$
$$T(n) = \Theta(n)$$

- **Avg case:** Expected runtime – need analysis T.B.D.

Reminder: Various Outcomes of H-PARTITION

- $x : pivot$
- $|L|$: size of left region
- $P(rank(x) = i) = 1/n$ for $1 \leq i \leq n$
  - if $rank(x) = 1$ then $|L| = 1$
  - if $rank(x) > 1$ then $|L| = rank(x) - 1$
- $P(|L| = 1) = P(rank(x) = 1) + P(rank(x) = 2)$
  - $P(|L| = 1) = 2/n$
- $P(|L| = i) = P(rank(x) = i + 1)$ for $1 < i < n$
  - $P(|L| = i) = 1/n$ for $1 < i < n$

Average Case Analysis of Randomized Select

- To compute the **upper bound** for the **avg case**, assume that the i^{th} element always falls into the **larger partition**.

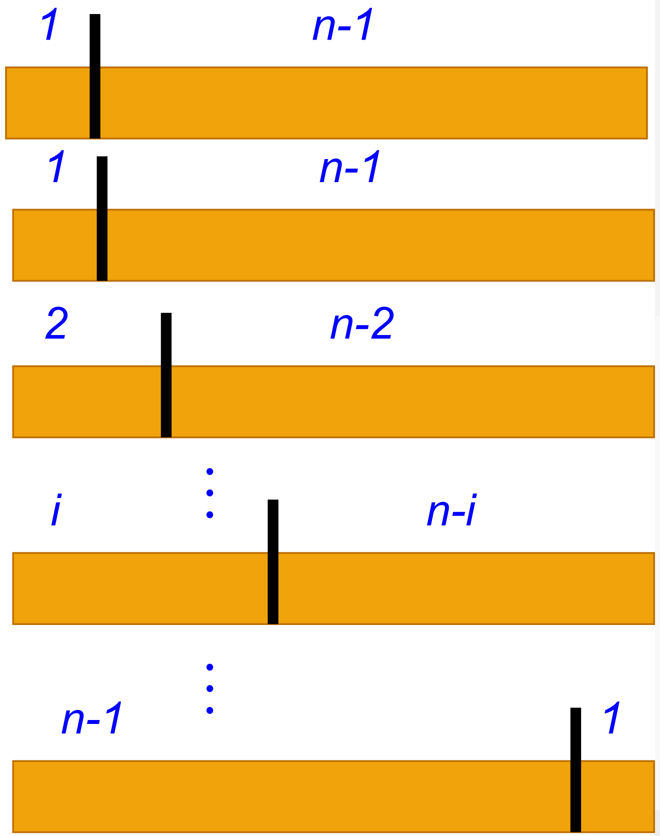
$$A = \{ \overbrace{|}^{p} \underbrace{\dots \leq x \dots}_{LeftPartition} \overbrace{|}^{q} \underbrace{\dots \geq x \dots}_{RightPartition} \overbrace{|}^{r} \}$$

$$x = pivot$$

- We will analyze the case where the recursive call is always made to the larger partition
  - *This will give us an upper bound for the avg case*

Various Outcomes of H-PARTITION

$rank(x)$	$probability$	$T(n)$
1	$\frac{1}{n}$	$\leq T(max(1, n-1)) + \Theta(n)$
2	$\frac{1}{n}$	$\leq T(max(1, n-1)) + \Theta(n)$
3	$\frac{1}{n}$	$\leq T(max(2, n-2)) + \Theta(n)$
$\vdots$	$\vdots$	$\vdots$
$i+1$	$\frac{1}{n}$	$\leq T(max(i, n-i)) + \Theta(n)$
$\vdots$	$\vdots$	$\vdots$
n	$\frac{1}{n}$	$\leq T(max(n-1, 1)) + \Theta(n)$



Average-Case Analysis of Randomized Select

$$\text{Recall:} P(|L| = i) = \begin{cases} 2/n & \text{for } i = 1 \\ 1/n & \text{for } i = 2, 3, \ldots, n-1 \end{cases}$$

Upper bound: Assume i^{th} element always falls into the larger part.

$$T(n) \leq \frac{1}{n} T(max(1, n-1)) + \frac{1}{n} \sum_{q=1}^{n-1} T(max(q, n-q)) + O(n)$$

$$Note : \frac{1}{n} T(max(1, n-1)) = \frac{1}{n} T(n-1) = \frac{1}{n} O(n^2) = O(n)$$

$$\therefore \text{(3 dot mean therefore) } T(n) \leq \frac{1}{n} \sum_{q=1}^{n-1} T(max(q, n-q)) + O(n)$$

Average-Case Analysis of Randomized Select

$$\therefore T(n) \leq \frac{1}{n} \sum_{q=1}^{n-1} T(max(q, n-q)) + O(n)$$

$$max(q, n–q) = \begin{cases} q & \text{if } q \geq \lceil n/2 \rceil \\ n-q & \text{if } q < \lceil n/2 \rceil \end{cases}$$

- n is odd: $T(k)$ appears twice for $k = \lceil n/2 \rceil + 1, \lceil n/2 \rceil + 2, \dots, n–1$
- n is even:$T(\lceil n/2 \rceil)$ appears once $T(k)$ appears twice for $k = \lceil n/2 \rceil + 1, \lceil n/2 \rceil + 2, \dots, n–1$

Average-Case Analysis of Randomized Select

- Hence, in both cases:

$$\sum_{q=1}^{n-1} T(max(q, n-q)) + O(n) \leq 2 \sum_{q=\lceil n/2 \rceil}^{n-1} T(q) + O(n)$$

$$\therefore T(n) \leq \frac{2}{n} \sum_{q=\lceil n/2 \rceil}^{n-1} T(q) + O(n)$$

Average-Case Analysis of Randomized Select

$$T(n) \leq \frac{2}{n} \sum_{q=\lceil n/2 \rceil}^{n-1} T(q) + O(n)$$

- By substitution guess $T(n) = O(n)$
- Inductive hypothesis: $T(k) \leq ck, \forall k < n$

$$T(n) \leq \frac{2}{n} \sum_{q=\lceil n/2 \rceil}^{n-1} ck + O(n)$$

$$= \frac{2c}{n} \left(\sum_{k=1}^{n-1} k - \sum_{k=1}^{\lceil n/2 \rceil - 1} k \right) + O(n)$$

$$= \frac{2c}{n} \left(\frac{1}{2} n(n-1) - \frac{1}{2} \lceil \frac{n}{2} \rceil \left(\frac{n}{2} - 1 \right) \right) + O(n)$$

Average-Case Analysis of Randomized Select

$$
\begin{aligned}
T(n) &\leq \frac{2c}{n}\left(\frac{1}{2}n(n-1) - \frac{1}{2}\left\lceil\frac{n}{2}\right\rceil\left(\frac{n}{2}-1\right)\right) + O(n) \\
&\leq c(n-1) - \frac{c}{4}n + \frac{c}{2} + O(n) \\
&= cn - \frac{c}{4}n - \frac{c}{2} + O(n) \\
&= cn - \left(\left(\frac{c}{4}n + \frac{c}{2}\right) + O(n)\right) \\
&\leq cn
\end{aligned}
$$

- since we can choose c large enough so that $(cn/4 + c/2)$ dominates $O(n)$

Summary of Randomized Order-Statistic Selection

- Works fast: linear expected time
- Excellent algorithm in practise
- But, the worst case is very bad: $\Theta(n^2)$
- **Blum, Floyd, Pratt, Rivest & Tarjan[1973]** algorithms are runs in **linear time** in the **worst case**.
- Generate a **good pivot** recursively

Selection in Worst Case Linear Time

```
//return i-th element in set S with n elements
SELECT(S, n, i)

  if n <= 5 then

    SORT S and return the i-th element

  DIVIDE S into ceil(n/5) groups
  //first ceil(n/5) groups are of size 5, last group is of size n mod 5

  FIND median set M={m , …, m_ceil(n/5)}
  // m_j : median of j-th group

  x = SELECT(M,ceil(n/5),floor((ceil(n/5)+1)/2))

  PARTITION set S around the pivot x into L and R

  if i <= |L| then
    return  SELECT(L, |L|, i)
  else
    return  SELECT(R, n-|L|, i-|L|)
```

Selection in Worst Case Linear Time - Example

- **Input:** Array S and index i
- **Output:** The i^{th} smallest value

25	9	16	8	11	27	39	42	15	632	14	36	20	33	22	31	4	17	3	30	41
2	13	19	7	21	10	34	1	37	23	40	5	29	18	24	12	38	28	26	35	43

Selection in Worst Case Linear Time - Example

Step 1: Divide the input array into groups of size 5

$$\overbrace{\begin{array}{ccccc} 25 & 9 & 16 & 8 & 11 \\ 27 & 39 & 42 & 15 & 6 \\ 32 & 14 & 36 & 20 & 33 \\ 22 & 31 & 4 & 17 & 3 \\ 30 & 41 & 2 & 13 & 19 \\ 7 & 21 & 10 & 34 & 1 \\ 37 & 23 & 40 & 5 & 29 \\ 18 & 24 & 12 & 38 & 28 \\ 26 & 35 & 43 & & \end{array}}^{\text{group size=5}}$$

Selection in Worst Case Linear Time - Example

Step 2: Compute the median of each group ($\Theta(n)$)

		$Medians$		
25	16	11	8	9
39	42	27	6	15
36	33	32	20	14
22	31	17	3	4
41	30	19	13	2
21	34	10	1	7
37	40	29	23	5
38	28	24	12	18
	26	35	43	

- Let M be the set of the medians computed:
  - $M = \{11, 27, 32, 17, 19, 10, 29, 24, 35\}$

Selection in Worst Case Linear Time - Example

Step 3: Compute the median of the median group M

$x \leftarrow SELECT(M, |M|, \lfloor (|M|+1)/2 \rfloor)$ where $|M| = \lceil n/5 \rceil$

- Let M be the set of the medians computed:
  - $M = \{11, 27, 32, 17, 19, 10, 29, \overbrace{24}^{Median}, 35\}$
- $Median = 24$
- The runtime of the recursive call: $T(|M|) = T(\lceil n/5 \rceil)$

Selection in Worst Case Linear Time - Example

Step 4: Partition the input array S around the median-of-medians x

25	9	16	8	11	27	39	42	15	632	14	36	20	33	22	31	4	17	3	30	41
2	13	19	7	21	10	34	1	37	23	40	5	29	18	24	12	38	28	26	35	43

Partition S around $x = 24$

Claim: Partitioning around x is guaranteed to be **well-balanced.**

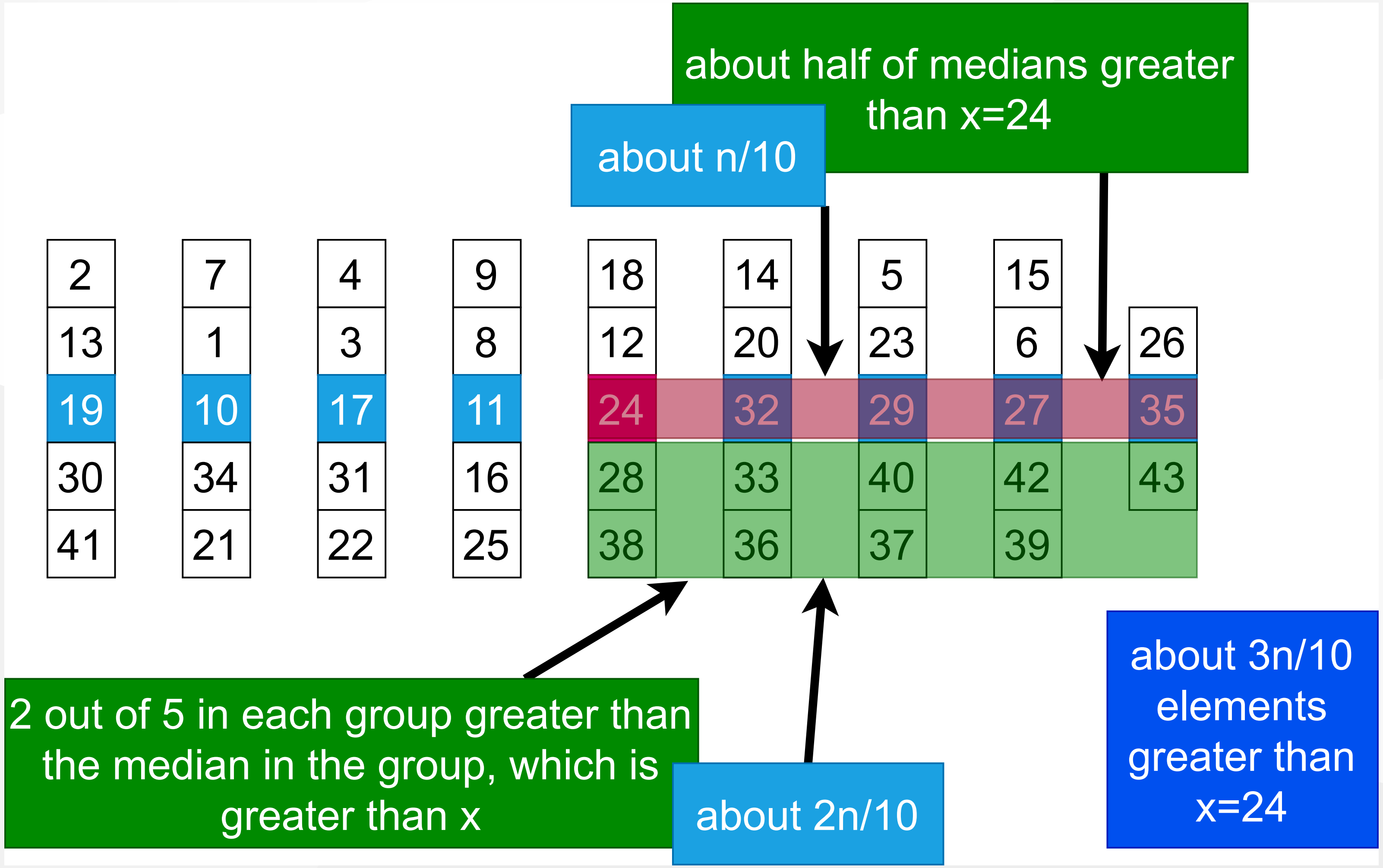
Selection in Worst Case Linear Time - Example

- M : Median, M^* : Median of Medians

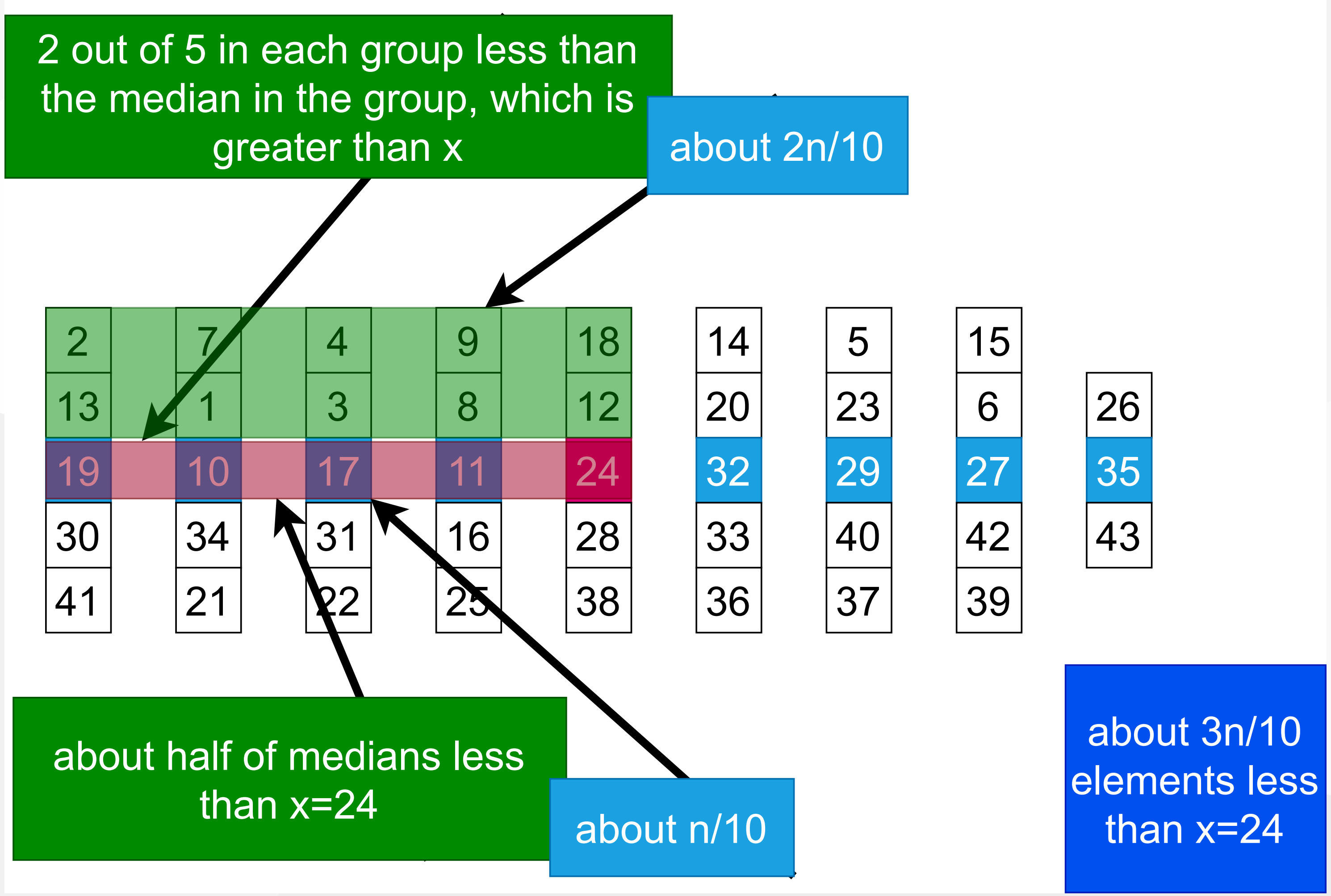
$$
\begin{array}{ccccc}
41 & 30 & \overbrace{19}^{M} & 13 & 2 \\
21 & 34 & 10 & 1 & 7 \\
22 & 31 & 17 & 3 & 4 \\
25 & 16 & 11 & 8 & 9 \\
38 & 28 & \overbrace{24}^{M^*} & 12 & 18 \\
36 & 33 & 32 & 20 & 14 \\
37 & 40 & 29 & 23 & 5 \\
39 & 42 & 27 & 6 & 15 \\
 & 26 & 35 & 43 &
\end{array}
$$

- About half of the medians greater than $x = 24$ (about $n/10$)

Selection in Worst Case Linear Time - Example



Selection in Worst Case Linear Time - Example



Selection in Worst Case Linear Time - Example

$$S = \begin{matrix} \{25 & 9 & 16 & 8 & 11 & 27 & 39 & 42 & 15 & 632 & 14 & 36 & 20 & 33 & 22 & 31 & 4 & 17 & 3 & 30 & 41 \\ 2 & 13 & 19 & 7 & 21 & 10 & 34 & 1 & 37 & 23 & 40 & 5 & 29 & 18 & 24 & 12 & 38 & 28 & 26 & 35 & 43\} \end{matrix}$$

- Partitioning S around $x = 24$ will lead to partitions of sizes $\sim 3n/10$ and $\sim 7n/10$ in the **worst case**.

Step 5: Make a recursive call to one of the partitions

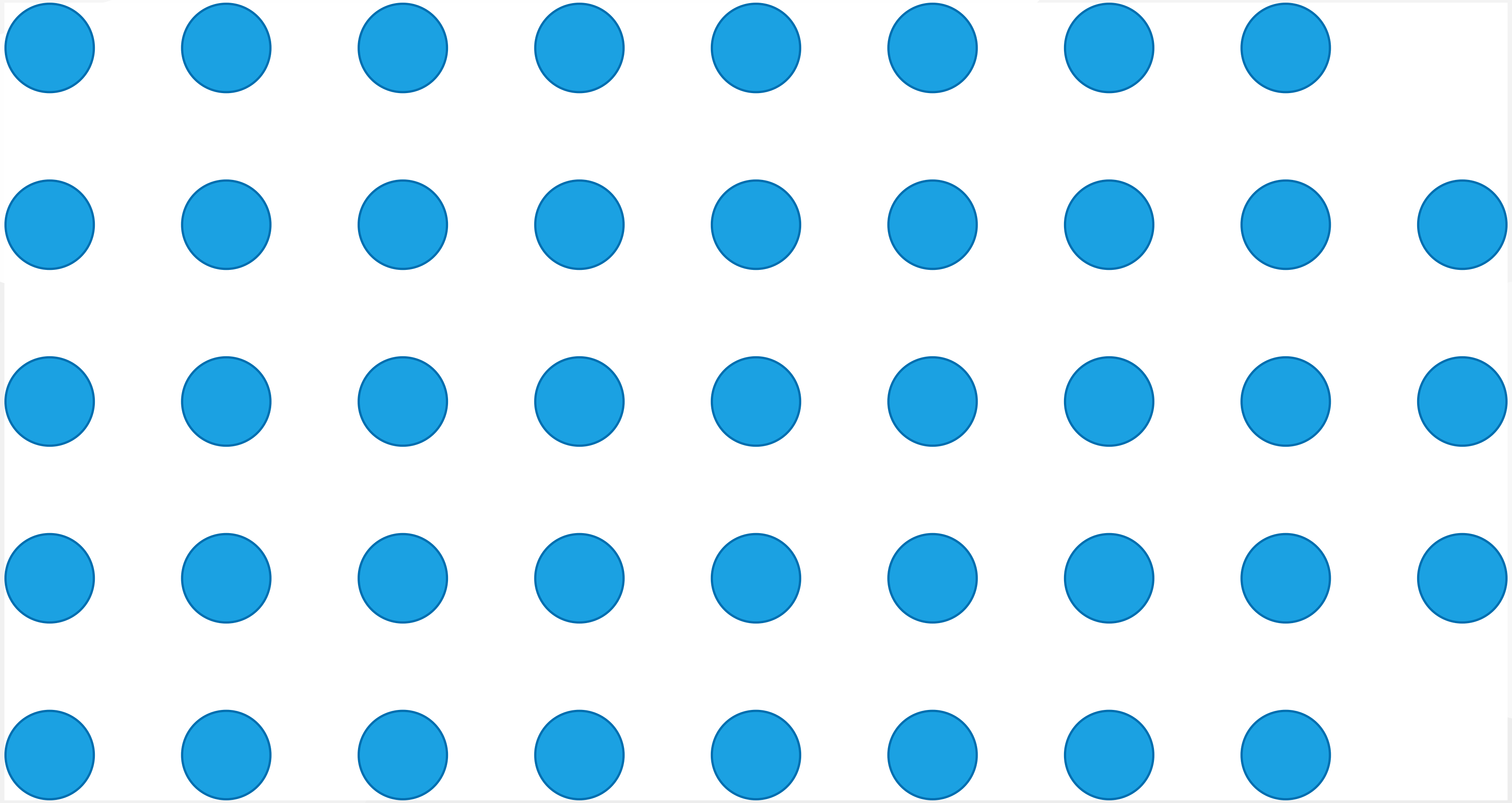
```
if i <= |L| then
  return SELECT(L,|L|,i)
else
  return SELECT(R,n-|L|,i-|L|)
```

Selection in Worst Case Linear Time

```
//return i-th element in set S with n elements
SELECT(S, n, i)

  if n <= 5 then

    SORT S and return the i-th element

  DIVIDE S into ceil(n/5) groups
  //first ceil(n/5) groups are of size 5, last group is of size n mod 5

  FIND median set M={m , …, m_ceil(n/5)}
  // m_j : median of j-th group

  x = SELECT(M,ceil(n/5),floor((ceil(n/5)+1)/2))

  PARTITION set S around the pivot x into L and R

  if i <= |L| then
    return  SELECT(L, |L|, i)
  else
    return  SELECT(R, n-|L|, i-|L|)
```

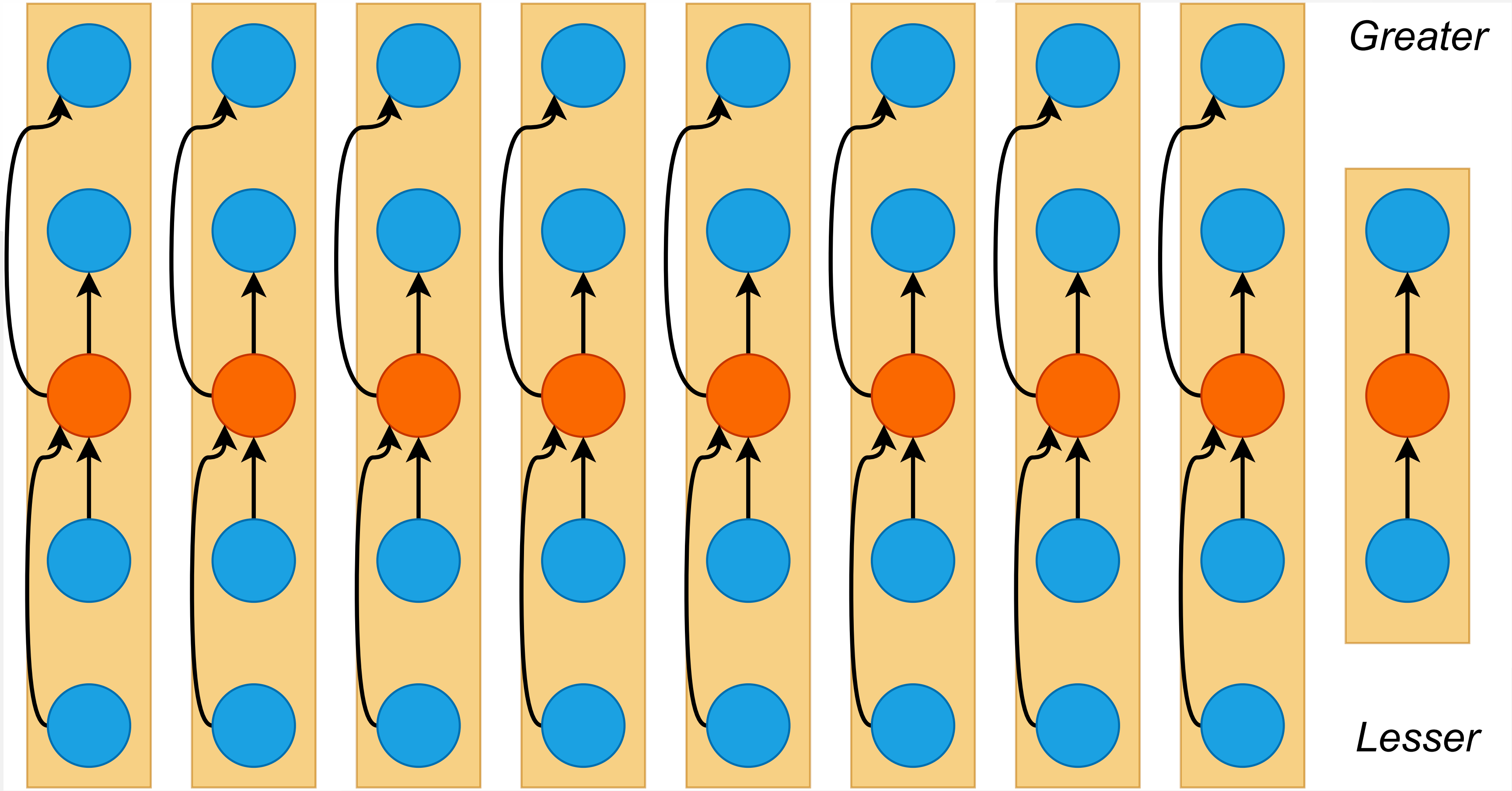
Choosing the Pivot

1. Divide S into groups of size 5



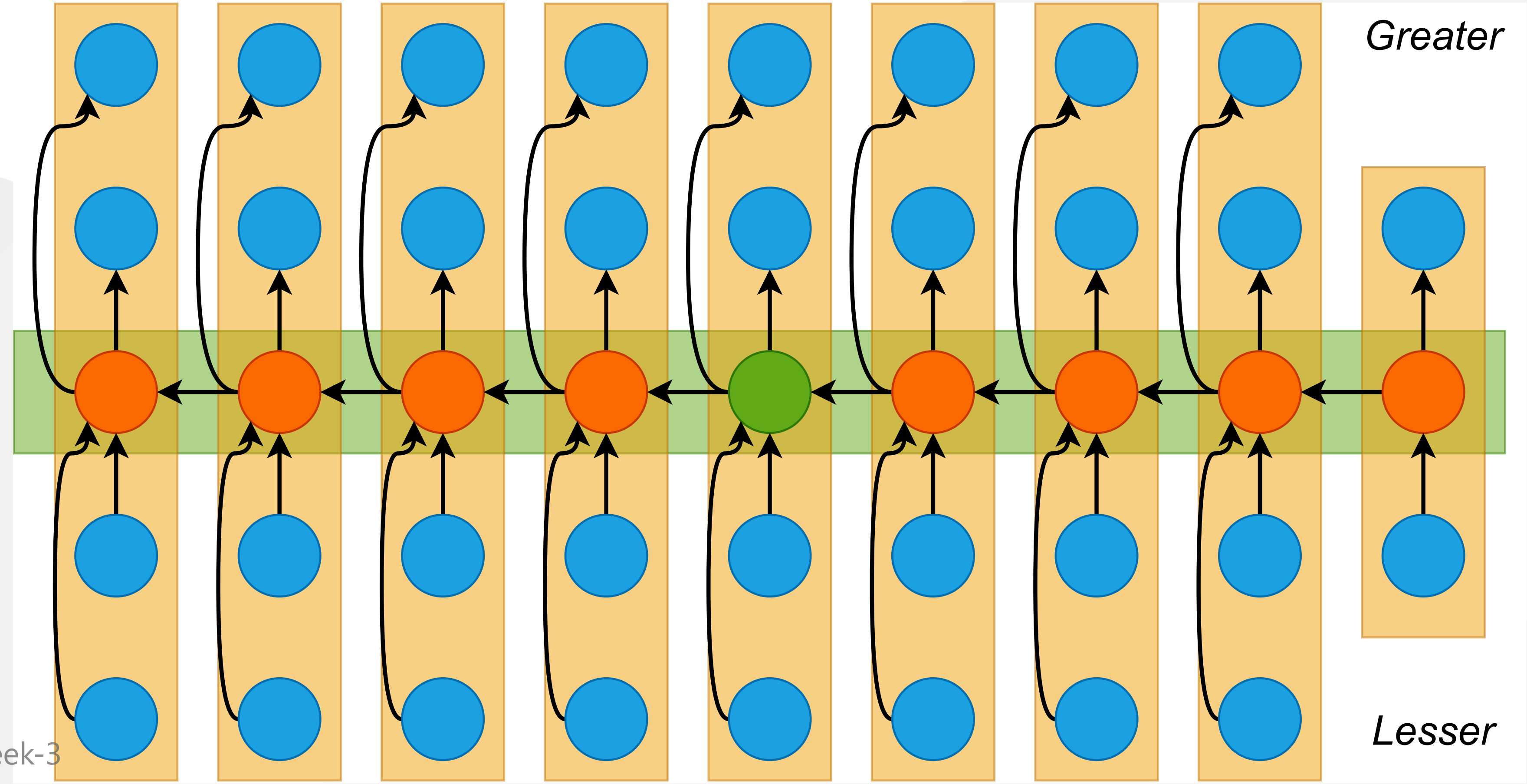
Choosing the Pivot

- Divide S into groups of size 5
- Find the median of each group



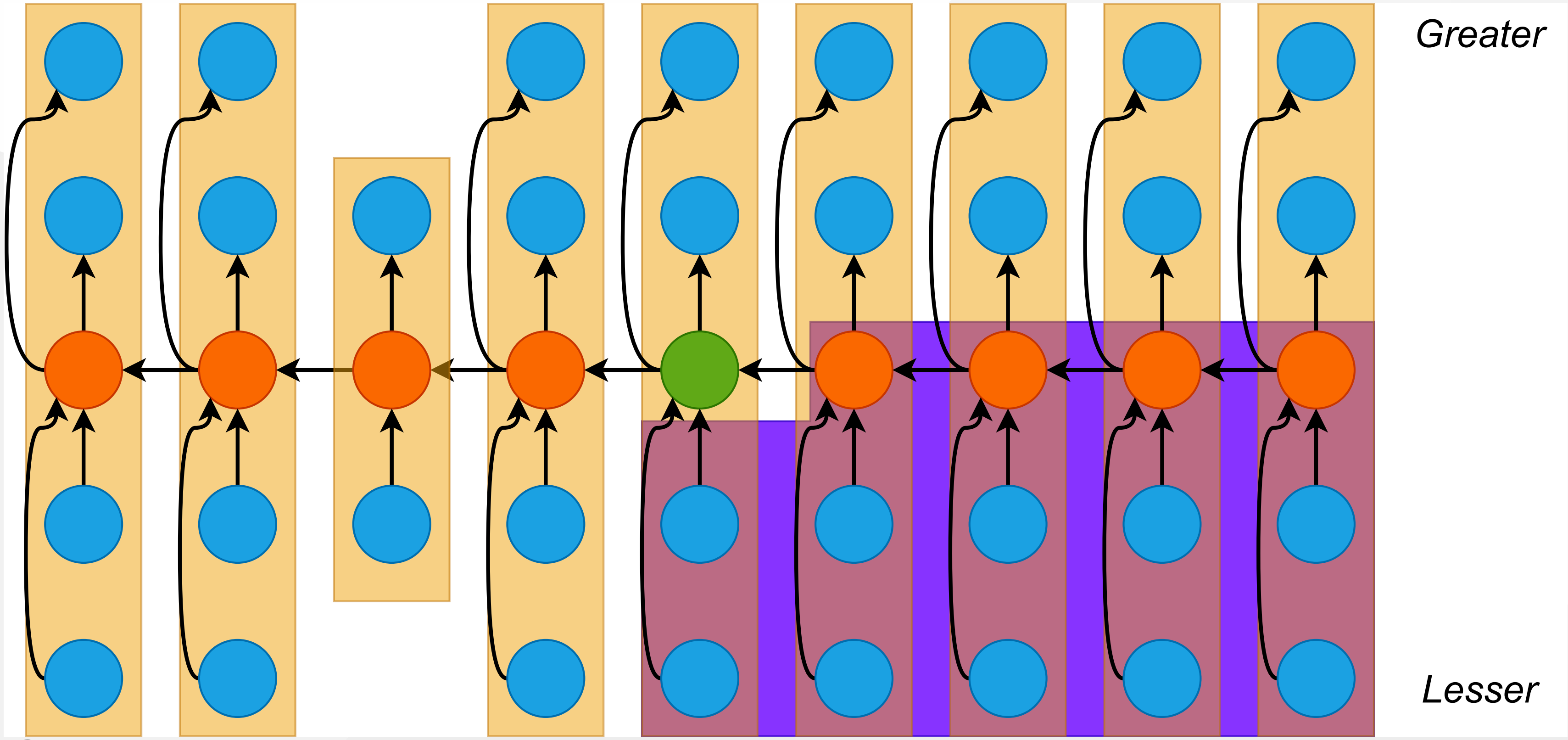
Choosing the Pivot

- Divide S into groups of size 5
- Find the median of each group
- Recursively select the median x of the medians



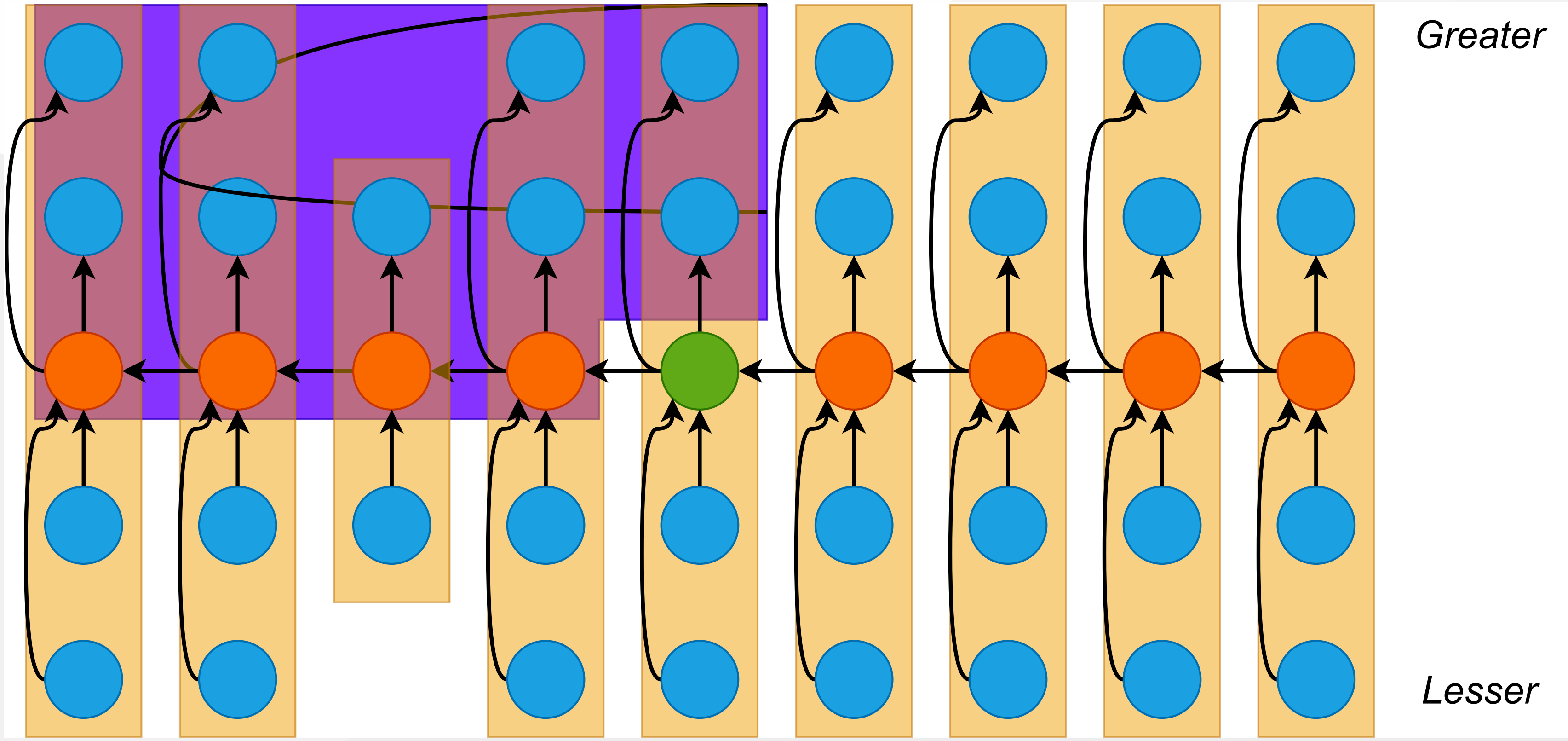
Choosing the Pivot

- At least half of the medians $\geq x$
- Thus $m = \lceil \lceil n/5 \rceil / 2 \rceil$ groups contribute 3 elements to R except possibly the last group and the group that contains x, $|R| \geq 3(m-2) \geq \frac{3n}{10} - 6$



Choosing the Pivot

- Similarly $|L| \geq \frac{3n}{10} - 6$
- Therefore, **SELECT** is recursively called on at most $n - (\frac{3n}{10} - 6) = \frac{7n}{10} + 6$ elements



Selection in Worst Case Linear Time

```
//return i-th element in set S with n elements
SELECT(S, n, i)
  if n <= 5 then
    SORT S and return the i-th element
```

$\Theta(n)$

```
  DIVIDE S into ceil(n/5) groups
  //first ceil(n/5) groups are of size 5, last group is of size n mod 5
```

$\Theta(n)$

```
  FIND median set M={m , …, m_ceil(n/5)}
  // m_j : median of j-th group
```

$T(\lceil n/5 \rceil)$

```
  x = SELECT(M,ceil(n/5),floor((ceil(n/5)+1)/2))
```

$\Theta(n)$

```
  PARTITION set S around the pivot x into L and R
```

$T(\frac{7n}{10}+6)$

```
  if i <= |L| then
    return  SELECT(L, |L|, i)
  else
    return  SELECT(R, n–|L|, i–|L|)
```

Selection in Worst Case Linear Time

- Thus recurrence becomes
  - $T(n) \leq T\left(\lceil \frac{n}{5} \rceil\right) + T\left(\frac{7n}{10} + 6\right) + \Theta(n)$
- Guess $T(n) = O(n)$ and prove by induction
- Inductive step:

$$
\begin{aligned}
T(n) &\leq c\lceil n/5 \rceil + c(7n/10 + 6) + \Theta(n) \\
&\leq cn/5 + c + 7cn/10 + 6c + \Theta(n) \\
&= 9cn/10 + 7c + \Theta(n) \\
&= cn - [c(n/10 - 7) - \Theta(n)] \leq cn \quad (\text{ for large c})
\end{aligned}
$$

- Work at each level of recursion is a constant factor $(9/10)$ smaller

References

- Introduction to Algorithms, Third Edition | The MIT Press
- Bilkent CS473 Course Notes (new)
- Bilkent CS473 Course Notes (old)
- Insertion Sort - GeeksforGeeks
- NIST Dictionary of Algorithms and Data Structures
- NIST - Dictionary of Algorithms and Data Structures
- NIST - big-O notation
- NIST - big-Omega notation