

# CE100 Algorithms and Programming II

## Week-2 (Solving Recurrences)

Spring Semester, 2021-2022

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# Solving Recurrences

## Outline

- Solving Recurrences
  - Recursion Tree
  - Master Method
  - Back-Substitution

- Divide-and-Conquer Analysis
  - Merge Sort
  - Binary Search
  - Merge Sort Analysis
  - Complexity

- Recurrence Solution

## Solving Recurrences

- Reminder: Runtime ( $T(n)$ ) of *MergeSort* was expressed as a recurrence

$$T(n) = \begin{cases} \Theta(1) & \text{if } n=1 \\ 2T(n/2) + \Theta(n) & \text{otherwise} \end{cases}$$

- Solving recurrences is like solving differential equations, integrals, etc.
  - Need to learn a few tricks

## Recurrences

<u>Recurrence</u>: An equation or inequality that describes a function in terms of its value on smaller inputs.

Example :

$$T(n) = \begin{cases} 1 & \text{if } n=1 \\ T(\lceil n/2 \rceil) + 1 & \text{if } n>1 \end{cases}$$

## Recurrence Example

$$T(n) = \begin{cases} 1 & \text{if } n=1 \\ T(\lceil n/2 \rceil) + 1 & \text{if } n>1 \end{cases}$$

- Simplification: Assume  $n = 2^k$
- Claimed answer :  $T(n) = \lg n + 1$
- Substitute claimed answer in the recurrence:

$$\lg n + 1 = \begin{cases} 1 & \text{if } n=1 \\ \lg(\lceil n/2 \rceil) + 2 & \text{if } n>1 \end{cases}$$

- True when  $n = 2^k$

## Technicalities: Floor / Ceiling

Technically, should be careful about the floor and ceiling functions (as in the book).

e.g. For merge sort, the recurrence should in fact be,

$$T(n) = \begin{cases} \Theta(1) & \text{if } n=1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + \Theta(n) & \text{if } n>1 \end{cases}$$

But, it's usually ok to:

- ignore floor/ceiling
- solve for the exact power of 2 (or another number)



## Technicalities: Boundary Conditions

- Usually assume:  $T(n) = \Theta(1)$  for sufficiently small  $n$ 
  - Changes the exact solution, but usually the asymptotic solution is not affected (e.g. if polynomially bounded)
- For convenience, the boundary conditions generally implicitly stated in a recurrence
  - $T(n) = 2T(n/2) + \Theta(n)$  assuming that
  - $T(n) = \Theta(1)$  for sufficiently small  $n$

## Example: When Boundary Conditions Matter

Exponential function:  $T(n) = (T(n/2))^2$

Assume

$T(1) = c$  (where  $c$  is a positive constant)

$$T(2) = (T(1))^2 = c^2$$

$$T(4) = (T(2))^2 = c^4$$

$$T(n) = \Theta(c^n)$$

e.g.

$$\left. \begin{array}{l} T(1) = 2 \Rightarrow T(n) = \Theta(2^n) \\ T(1) = 3 \Rightarrow T(n) = \Theta(3^n) \end{array} \right\} \text{However } \Theta(2^n) \neq \Theta(3^n)$$

The difference in solution more dramatic when:

$$T(1) = 1 \Rightarrow T(n) = \Theta(1^n) = \Theta(1)$$

# Solving Recurrences

We will focus on 3 techniques

- Substitution method
- Recursion tree approach
- Master method

# Substitution Method

The most general method:

- Guess
- Prove by induction
- Solve for constants

## Substitution Method: Example

Solve  $T(n) = 4T(n/2) + n$  (assume  $T(1) = \Theta(1)$ )

1. Guess  $T(n) = O(n^3)$  (need to prove  $O$  and  $\Omega$  separately)
2. Prove by induction that  $T(n) \leq cn^3$  for large  $n$  (i.e.  $n \geq n_0$ )
  - Inductive hypothesis:  $T(k) \leq ck^3$  for any  $k < n$
  - Assuming ind. hyp. holds, prove  $T(n) \leq cn^3$

## Substitution Method: Example – cont'd

Original recurrence:  $T(n) = 4T(n/2) + n$

From inductive hypothesis:  $T(n/2) \leq c(n/2)^3$

Substitute this into the original recurrence:

- $T(n) \leq 4c(n/2)^3 + n$
- $= (c/2)n^3 + n$
- $= cn^3 - ((c/2)n^3 - n) \Leftarrow \text{desired - residual}$
- $\leq cn^3$   
when  $((c/2)n^3 - n) \geq 0$

## Substitution Method: Example – cont'd

So far, we have shown:

$$T(n) \leq cn^3 \text{ when } ((c/2)n^3 - n) \geq 0$$

We can choose  $c \geq 2$  and  $n_0 \geq 1$

But, the proof is not complete yet.

**Reminder: Proof by induction:**

1. *Prove the base cases*  $\Leftarrow$  haven't proved the base cases yet
2. *Inductive hypothesis for smaller sizes*
3. *Prove the general case*

## Substitution Method: Example – cont'd

- We need to prove the base cases
  - Base:  $T(n) = \Theta(1)$  for small  $n$  (e.g. for  $n = n_0$ )
- We should show that:
  - $\Theta(1) \leq cn^3$  for  $n = n_0$  , This holds if we pick  $c$  big enough
- So, the proof of  $T(n) = O(n^3)$  is complete
- But, is this a tight bound?



## Example: A tighter upper bound?

- Original recurrence:  $T(n) = 4T(n/2) + n$
- Try to prove that  $T(n) = O(n^2)$ ,
  - i.e.  $T(n) \leq cn^2$  for all  $n \geq n_0$
- Ind. hyp: Assume that  $T(k) \leq ck^2$  for  $k < n$
- Prove the general case:  $T(n) \leq cn^2$

## Example (cont'd)

Original recurrence:  $T(n) = 4T(n/2) + n$

Ind. hyp: Assume that  $T(k) \leq ck^2$  for  $k < n$

Prove the general case:  $T(n) \leq cn^2$

$$\begin{aligned} T(n) &= 4T(n/2) + n \\ &\leq 4c(n/2)^2 + n \\ &= cn^2 + n \end{aligned}$$

$= O(n^2) \Leftarrow$  Wrong! We must prove exactly

## Example (cont'd)

Original recurrence:  $T(n) = 4T(n/2) + n$

Ind. hyp: Assume that  $T(k) \leq ck^2$  for  $k < n$

Prove the general case:  $T(n) \leq cn^2$

- So far, we have:

$$T(n) \leq cn^2 + n$$

- No matter which positive  $c$  value we choose, this does not show that  $T(n) \leq cn^2$
- Proof failed?

## Example (cont'd)

- What was the problem?
  - The inductive hypothesis was not strong enough
- **Idea:** Start with a stronger inductive hypothesis
  - Subtract a low-order term
- **Inductive hypothesis:**  $T(k) \leq c_1 k^2 - c_2 k$  for  $k < n$
- **Prove the general case:**  $T(n) \leq c_1 n^2 - c_2 n$

## Example (cont'd)

Original recurrence:  $T(n) = 4T(n/2) + n$

Ind. hyp: Assume that  $T(k) \leq c_1 k^2 - c_2 k$  for  $k < n$

Prove the general case:  $T(n) \leq c_1 n^2 - c_2 n$

$$\begin{aligned}
 T(n) &= 4T(n/2) + n \\
 &\leq 4(c_1(n/2)^2 - c_2(n/2)) + n \\
 &= c_1 n^2 - 2c_2 n + n \\
 &= c_1 n^2 - c_2 n - (c_2 n - n) \\
 &\leq c_1 n^2 - c_2 n \text{ for } n(c_2 - 1) \geq 0 \\
 &\text{choose } c_2 \geq 1
 \end{aligned}$$

## Example (cont'd)

We now need to prove

$$T(n) \leq c_1 n^2 - c_2 n$$

for the base cases.

$T(n) = \Theta(1)$  for  $1 \leq n \leq n_0$  (implicit assumption)

$\Theta(1) \leq c_1 n^2 - c_2 n$  for  $n$  small enough (e.g.  $n = n_0$ )

- We can choose  $c_1$  large enough to make this hold

We have proved that  $T(n) = O(n^2)$

## Substitution Method: Example 2

For the recurrence  $T(n) = 4T(n/2) + n$ ,

prove that  $T(n) = \Omega(n^2)$

i.e.  $T(n) \geq cn^2$  for any  $n \geq n_0$

Ind. hyp:  $T(k) \geq ck^2$  for any  $k < n$

Prove general case:  $T(n) \geq cn^2$

$$T(n) = 4T(n/2) + n$$

$$\geq 4c(n/2)^2 + n$$

$$= cn^2 + n$$

$$\geq cn^2 \text{ since } n > 0$$

## Example 2 (cont'd)

We now need to prove that

$$T(n) \geq cn^2$$

for the base cases

$$T(n) = \Theta(1) \text{ for } 1 \leq n \leq n_0 \text{ (implicit assumption)}$$

$$\Theta(1) \geq cn^2 \text{ for } n = n_0$$

$n_0$  is sufficiently small (i.e. constant)

We can choose  $c$  small enough for this to hold

We have proved that  $T(n) = \Omega(n^2)$



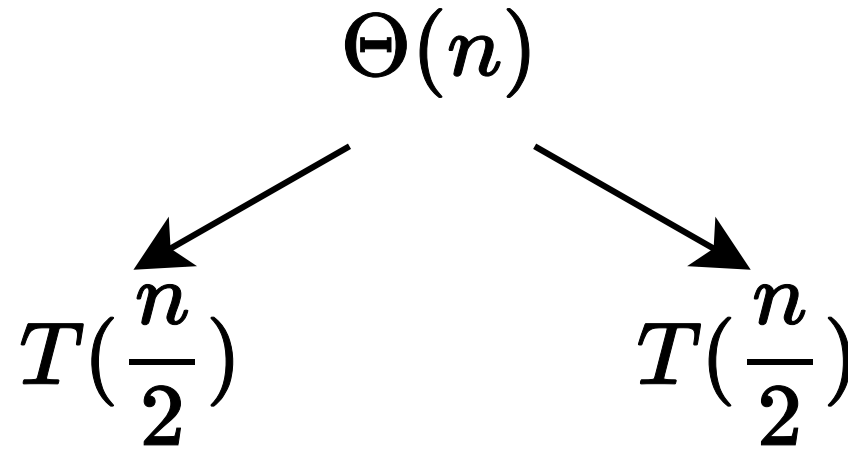
## Substitution Method - Summary

- Guess the asymptotic complexity
- Prove your guess using induction
  - Assume inductive hypothesis holds for  $k < n$
  - Try to prove the general case for  $n$ 
    - Note: *MUST* prove the *EXACT* inequality *CANNOT* ignore lower order terms, If the proof fails, strengthen the ind. hyp. and try again
- Prove the base cases (usually straightforward)

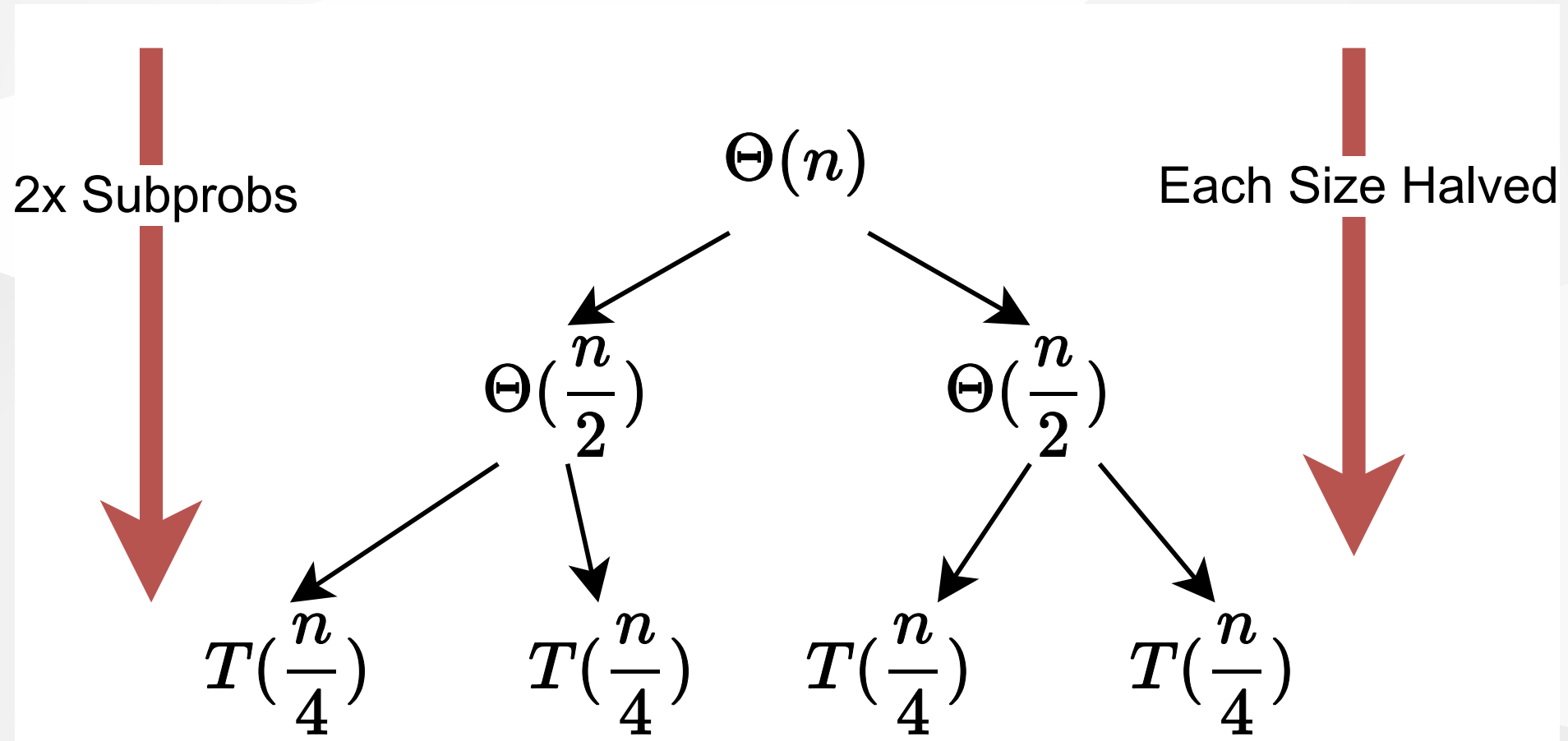
## Recursion Tree Method

- A recursion tree models the runtime costs of a recursive execution of an algorithm.
- The recursion tree method is good for generating guesses for the substitution method.
- The recursion-tree method can be unreliable.
  - Not suitable for formal proofs
- The recursion-tree method promotes intuition, however.

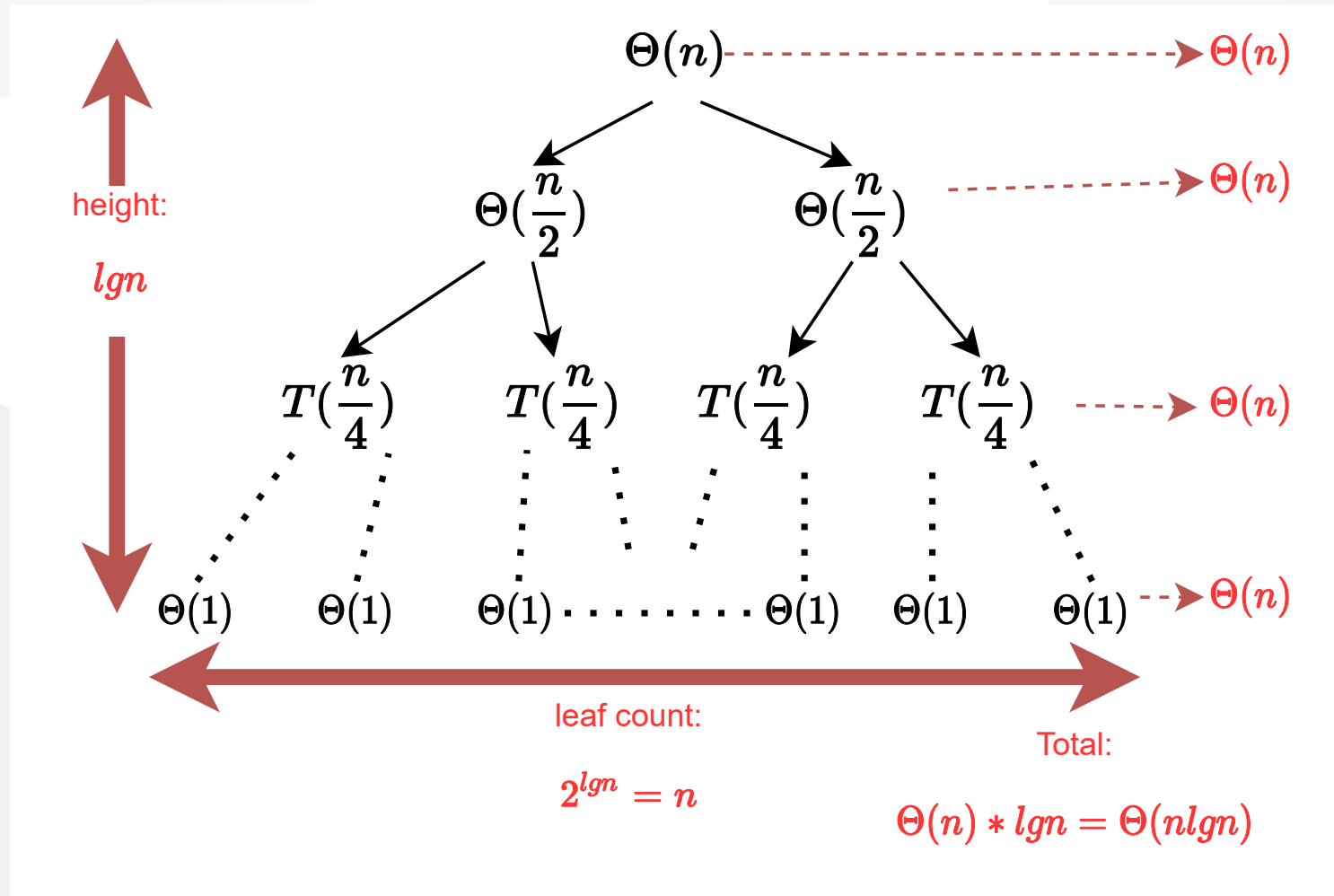
**Solve Recurrence :  $T(n) = 2T(n/2) + \Theta(n)$**



Solve Recurrence :  $T(n) = 2T(n/2) + \Theta(n)$

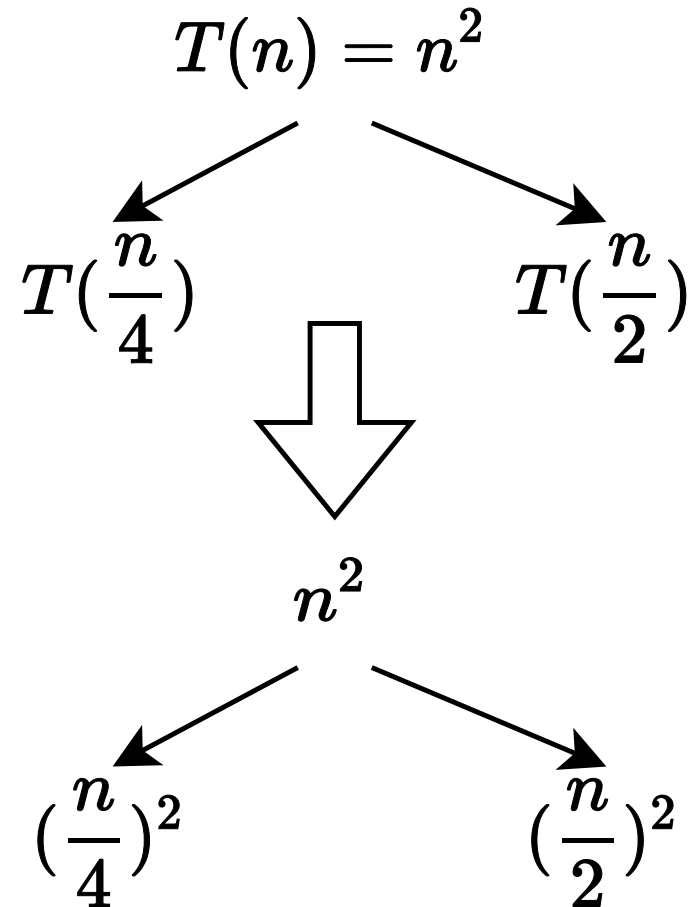


# Solve Recurrence : $T(n) = 2T(n/2) + \Theta(n)$



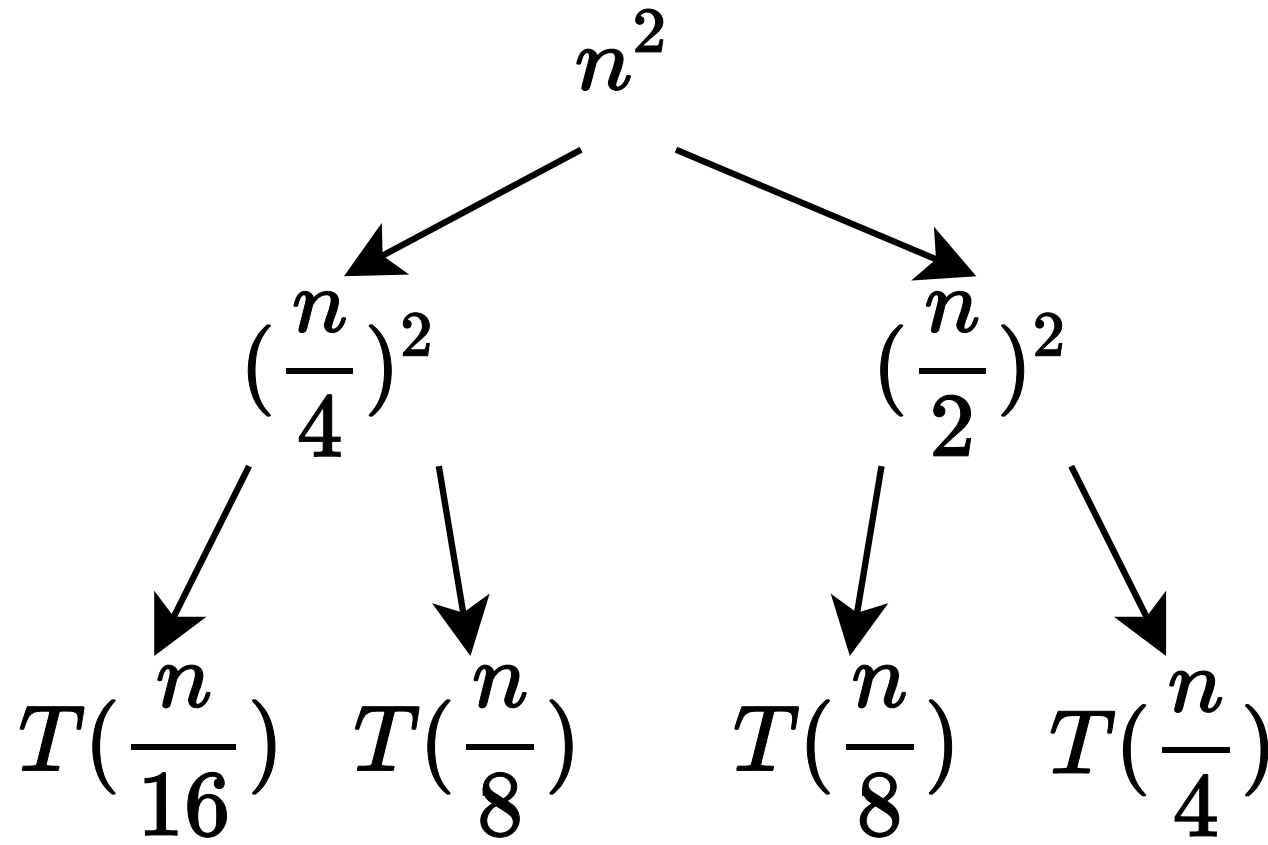
# Example of Recursion Tree

Solve  $T(n) = T(n/4) + T(n/2) + n^2$



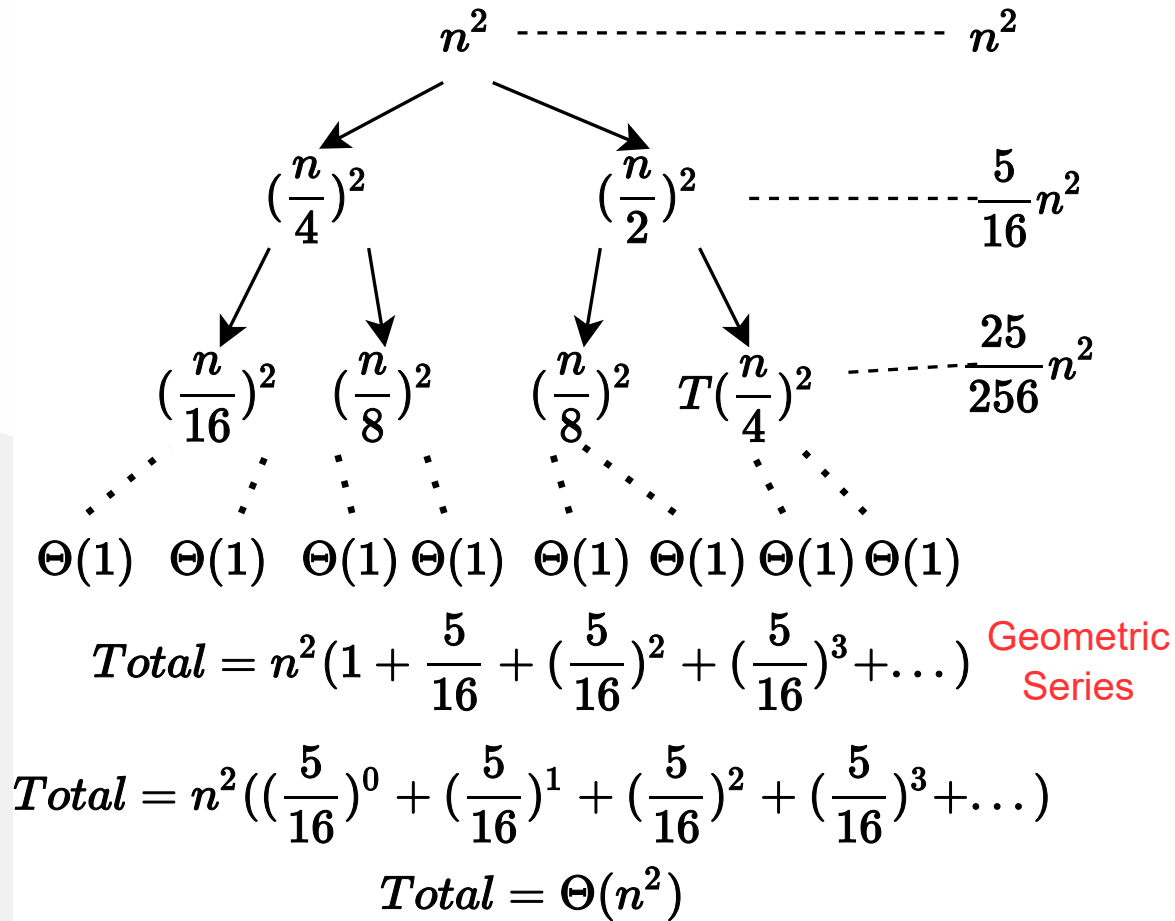
## Example of Recursion Tree

Solve  $T(n) = T(n/4) + T(n/2) + n^2$



# Example of Recursion Tree

Solve  $T(n) = T(n/4) + T(n/2) + n^2$





## The Master Method

- A powerful black-box method to solve recurrences.
- The master method applies to recurrences of the form
  - $T(n) = aT(n/b) + f(n)$
- where  $a \geq 1$ ,  $b > 1$ , and  $f$  is asymptotically positive.

## The Master Method: 3 Cases

- Recurrence:  $T(n) = aT(n/b) + f(n)$
- Compare  $f(n)$  with  $n^{\log_b a}$
- Intuitively:
  - **Case 1:**  $f(n)$  grows polynomially slower than  $n^{\log_b a}$
  - **Case 2:**  $f(n)$  grows at the same rate as  $n^{\log_b a}$
  - **Case 3:**  $f(n)$  grows polynomially faster than  $n^{\log_b a}$

## The Master Method: Case 1

- Recurrence:  $T(n) = aT(n/b) + f(n)$
- Case 1:  $\frac{n^{\log_b^a}}{f(n)} = \Omega(n^\varepsilon)$  for some constant  $\varepsilon > 0$
- i.e.,  $f(n)$  grows polynomially slower than  $n^{\log_b^a}$  (by an  $n^\varepsilon$  factor)
- Solution:  $T(n) = \Theta(n^{\log_b^a})$

## The Master Method: Case 2 (Simple Version)

- Recurrence:  $T(n) = aT(n/b) + f(n)$
- Case 2:  $\frac{f(n)}{n^{\log_b a}} = \Theta(1)$
- i.e.,  $f(n)$  and  $n^{\log_b a}$  grow at similar rates
- Solution:  $T(n) = \Theta(n^{\log_b a} \lg n)$

## The Master Method: Case 3

- Case 3:  $\frac{f(n)}{n^{\log_b^a}} = \Omega(n^\varepsilon)$  for some constant  $\varepsilon > 0$
- i.e.,  $f(n)$  grows polynomially faster than  $n^{\log_b^a}$  (by an  $n^\varepsilon$  factor)
- and the following regularity condition holds:
  - $af(n/b) \leq cf(n)$  for some constant  $c < 1$
- Solution:  $T(n) = \Theta(f(n))$

**Example :**  $T(n) = 4T(n/2) + n$

- $a = 4$
- $b = 2$
- $f(n) = n$
- $n^{\log_b^a} = n^{\log_2^4} = n^{\log_2^{2^2}} = n^{2\log_2^2} = n^2$
- $f(n) = n$  grows polynomially slower than  $n^{\log_b^a} = n^2$ 
  - $\frac{n^{\log_b^a}}{f(n)} = \frac{n^2}{n} = n = \Omega(n^\epsilon)$
- CASE-1:
  - $T(n) = \Theta(n^{\log_b^a}) = \Theta(n^{\log_2^4}) = \Theta(n^2)$

**Example :**  $T(n) = 4T(n/2) + n^2$

- $a = 4$
- $b = 2$
- $f(n) = n^2$
- $n^{\log_b^a} = n^{\log_2^4} = n^{\log_2^{2^2}} = n^{2\log_2^2} = n^2$
- $f(n) = n^2$  grows at similar rate as  $n^{\log_b^a} = n^2$ 
  - $f(n) = \Theta(n^{\log_b^a}) = n^2$
- CASE-2:
  - $T(n) = \Theta(n^{\log_b^a} \lg n) = \Theta(n^{\log_2^4} \lg n) = \Theta(n^2 \lg n)$

**Example :**  $T(n) = 4T(n/2) + n^3$

- $a = 4$
- $b = 2$
- $f(n) = n^3$
- $n^{\log_b^a} = n^{\log_2^4} = n^{\log_2^{2^2}} = n^{2\log_2^2} = n^2$
- $f(n) = n^3$  grows polynomially faster than  $n^{\log_b^a} = n^2$ 
  - $\frac{f(n)}{n^{\log_b^a}} = \frac{n^3}{n^2} = n = \Omega(n^\epsilon)$



## Example : $T(n) = 4T(n/2) + n^3$ (con't)

- Seems like CASE 3, but need to check the regularity condition
- Regularity condition  $af(n/b) \leq cf(n)$  for some constant  $c < 1$
- $4(n/2)^3 \leq cn^3$  for  $c = 1/2$
- CASE-3:
  - $T(n) = \Theta(f(n)) \implies T(n) = \Theta(n^3)$

**Example :**  $T(n) = 4T(n/2) + n^2 \lg n$

- $a = 4$
- $b = 2$
- $f(n) = n^2 \lg n$
- $n^{\log_b^a} = n^{\log_2^4} = n^{\log_2^{2^2}} = n^{2\log_2^2} = n^2$
- $f(n) = n^2 \lg n$  grows slower than  $n^{\log_b^a} = n^2$ 
  - but is it polynomially slower?
  - $\frac{n^{\log_b^a} f(n)}{\frac{n^2}{\lg n}} = \lg n \neq \Omega(n^\varepsilon)$  for any  $\varepsilon > 0$ 
    - is not CASE-1
    - Master Method does not apply!

## The Master Method : Case 2 (General Version)

- Recurrence :  $T(n) = aT(n/b) + f(n)$
- Case 2:  $\frac{f(n)}{n^{\log_b a}} = \Theta(\lg^k n)$  for some constant  $k \geq 0$
- Solution :  $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$

## General Method (Akra-Bazzi)

$$T(n) = \sum_{i=1}^k a_i T(n/b_i) + f(n)$$

Let  $p$  be the unique solution to

$$\sum_{i=1}^k (a_i / b_i^p) = 1$$

Then, the answers are the same as for the master method, but with  $n^p$  instead of  $n^{\log_b^a}$   
(Akra and Bazzi also prove an even more general result.)



## Idea of Master Theorem

CASE 1 : The weight increases geometrically from the root to the leaves. The leaves hold a constant fraction of the total weight.

$$n^{\log_b^a} T(1) = \Theta(n^{\log_b^a})$$

## Idea of Master Theorem

CASE 2 : ( $k = 0$ ) The weight is approximately the same on each of the  $\log_b n$  levels.

$$n^{\log_b^a} T(1) = \Theta(n^{\log_b^a} \lg n)$$

## Idea of Master Theorem

CASE 3 : The weight decreases geometrically from the root to the leaves. The root holds a constant fraction of the total weight.

$$n^{\log_b^a} T(1) = \Theta(f(n))$$



## Proof of Master Theorem: Case 1 and Case 2

- Recall from the recursion tree (note  $h = \lg_b n = \text{tree height}$ )

$$\text{Leaf Cost} = \Theta(n^{\log_b^a})$$

$$\text{Non-leaf Cost} = g(n) = \sum_{i=0}^{h-1} a^i f(n/b^i)$$

$$T(n) = \text{Leaf Cost} + \text{Non-leaf Cost}$$

$$T(n) = \Theta(n^{\log_b^a}) + \sum_{i=0}^{h-1} a^i f(n/b^i)$$

## Proof Case 1

- $\frac{n^{\log_b^a}}{f(n)} = \Omega(n^\varepsilon)$  for some  $\varepsilon > 0$
- $\frac{n^{\log_b^a}}{f(n)} = \Omega(n^\varepsilon) \implies O(n^{-\varepsilon}) \implies f(n) = O(n^{\log_b^{a-\varepsilon}})$
- $g(n) = \sum_{i=0}^{h-1} a^i O((n/b^i)^{\log_b^{a-\varepsilon}}) = O(\sum_{i=0}^{h-1} a^i (n/b^i)^{\log_b^{a-\varepsilon}})$
- $O(n^{\log_b^{a-\varepsilon}} \sum_{i=0}^{h-1} a^i b^{i\varepsilon} / b^{i \log_b^{a-\varepsilon}})$

## Proof Case 1 (con't)

$$\bullet \sum_{i=0}^{h-1} \frac{a^i b^{i\varepsilon}}{b^{i \log_b a}} = \sum_{i=0}^{h-1} a^i \frac{(b^\varepsilon)^i}{(b^{\log_b a})^i} = \sum_{i=0}^{h-1} a^i \frac{b^{i\varepsilon}}{a^i} = \sum_{i=0}^{h-1} (b^\varepsilon)^i$$

= An increasing geometric series since  $b > 1$

$$\frac{b^{h\varepsilon} - 1}{b^\varepsilon - 1} = \frac{(b^h)^\varepsilon - 1}{b^\varepsilon - 1} = \frac{(b^{\log_b n})^\varepsilon - 1}{b^\varepsilon - 1} = \frac{n^\varepsilon - 1}{b^\varepsilon - 1} = O(n^\varepsilon)$$

## Proof Case 1 (con't)

- $g(n) = O(n^{\log_b a - \varepsilon} O(n^\varepsilon)) = O(\frac{n^{\log_b a}}{n^\varepsilon} O(n^\varepsilon)) = O(n^{\log_b a})$
- $T(n) = \Theta(n^{\log_b a}) + g(n) = \Theta(n^{\log_b a}) + O(n^{\log_b a}) = \Theta(n^{\log_b a})$

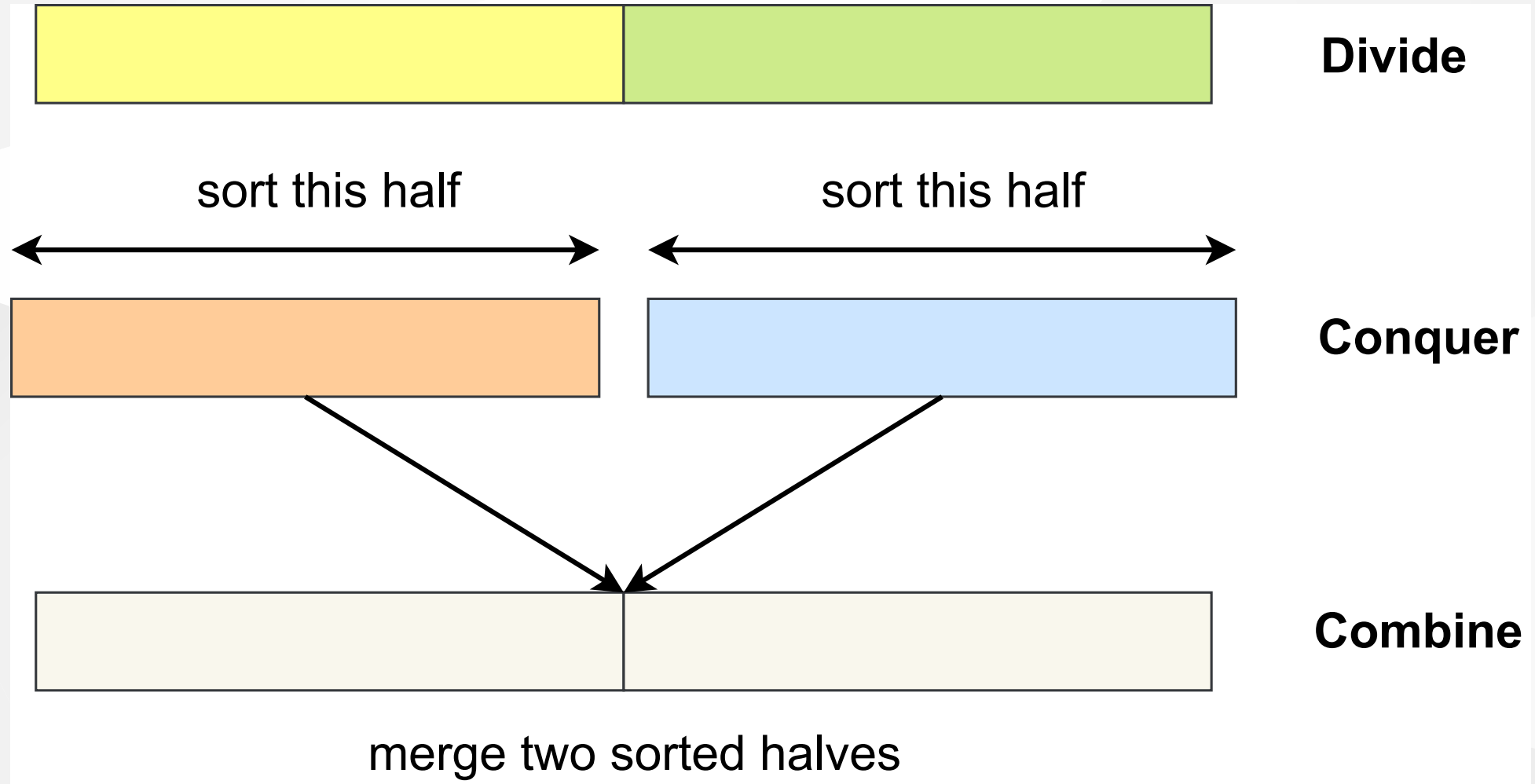
**Q.E.D.**

(Quod Erat Demonstrandum)

## Proof of Case 2 (limited to $k=0$ )

- $\frac{f(n)}{n^{\log_b^a}} = \Theta(\lg^0 n) = \Theta(1) \implies f(n) = \Theta(n^{\log_b^a}) \implies f(n/b^i) = \Theta((n/b^i)^{\log_b^a})$
- $g(n) = \sum_{i=0}^{h-1} a^i \Theta((n/b^i)^{\log_b^a})$
- $= \Theta(\sum_{i=0}^{h-1} a^i \frac{n^{\log_b^a}}{b^{i \log_b^a}})$
- $= \Theta(n^{\log_b^a} \sum_{i=0}^{h-1} a^i \frac{1}{(b^{\log_b^a})^i})$
- $= \Theta(n^{\log_b^a} \sum_{i=0}^{h-1} a^i \frac{1}{a^i})$
- $= \Theta(n^{\log_b^a} \sum_{i=0}^{\log_b^{n-1}} 1) = \Theta(n^{\log_b^a} \log_b n) = \Theta(n^{\log_b^a} \lg n)$
- $T(n) = n^{\log_b^a} + \Theta(n^{\log_b^a} \lg n)$
- $= \Theta(n^{\log_b^a} \lg n)$

# The Divide-and-Conquer Design Paradigm



# The Divide-and-Conquer Design Paradigm

1. **Divide** the problem (instance) into subproblems.
2. **Conquer** the subproblems by solving them recursively.
3. **Combine** subproblem solutions.

## Example : Merge Sort

1. **Divide:** Trivial.
  2. **Conquer:** Recursively sort 2 subarrays.
  3. **Combine:** Linear- time merge.
- $T(n) = 2T(n/2) + \Theta(n)$ 
    - Subproblems  $\implies 2$
    - Subproblemsize  $\implies n/2$
    - Work dividing and combining  $\implies \Theta(n)$



## Master Theorem: Reminder

- $T(n) = aT(n/b) + f(n)$ 
  - Case 1:  $\frac{n^{\log_b^a}}{f(n)} = \Omega(n^\epsilon) \implies T(n) = \Theta(n^{\log_b^a})$
  - Case 2:  $\frac{f(n)}{n^{\log_b^a}} = \Theta(\lg^k n) \implies T(n) = \Theta(n^{\log_b^a} \lg^{k+1} n)$
  - Case 3:  $\frac{n^{\log_b^a}}{f(n)} = \Omega(n^\epsilon) \implies T(n) = \Theta(f(n))$  and  $af(n/b) \leq cf(n)$  for  $c < 1$

## Merge Sort: Solving the Recurrence

$$T(n) = 2T(n/2) + \Theta(n)$$

$$a = 2, b = 2, f(n) = \Theta(n), n^{\log_b^a} = n$$

$$\text{Case-2: } \frac{f(n)}{n^{\log_b^a}} = \Theta(\lg^k n) \implies T(n) = \Theta(n^{\log_b^a} \lg^{k+1} n) \text{ holds for } k = 0$$

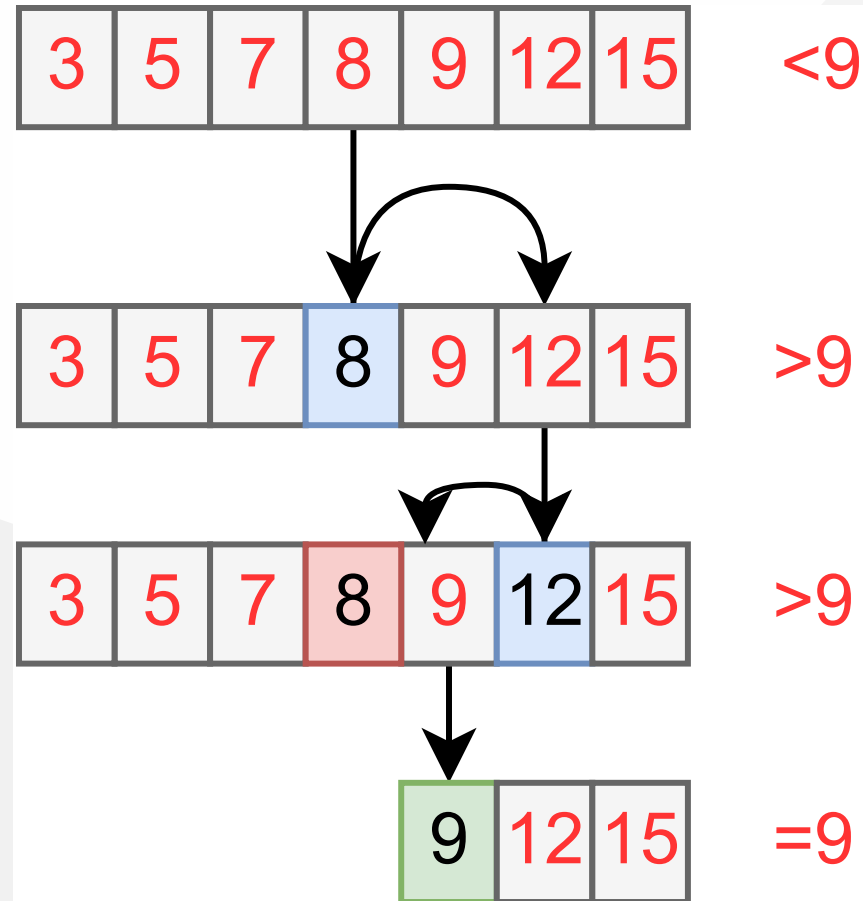
$$T(n) = \Theta(n \lg n)$$

# Binary Search

Find an element in a sorted array:

1. Divide: Check middle element.
2. Conquer: Recursively search 1 subarray.
3. Combine: Trivial.

## Example: Find 9



## Recurrence for Binary Search

$$T(n) = 1T(n/2) + \Theta(1)$$

- Subproblems  $\implies 1$
- Subproblemsize  $\implies n/2$
- Work dividing and combining  $\implies \Theta(1)$

# Binary Search: Solving the Recurrence

**References**

**TODO**