

CE100 Algorithms and Programming II

Week-4 (Heap/Heap Sort)

Spring Semester, 2021-2022

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Heap/Heap Sort

Outline (1)

- Heaps
 - Max / Min Heap
- Heap Data Structure
 - Heapify
 - Iterative
 - Recursive

Outline (2)

- Extract-Max
- Build Heap

Outline (3)

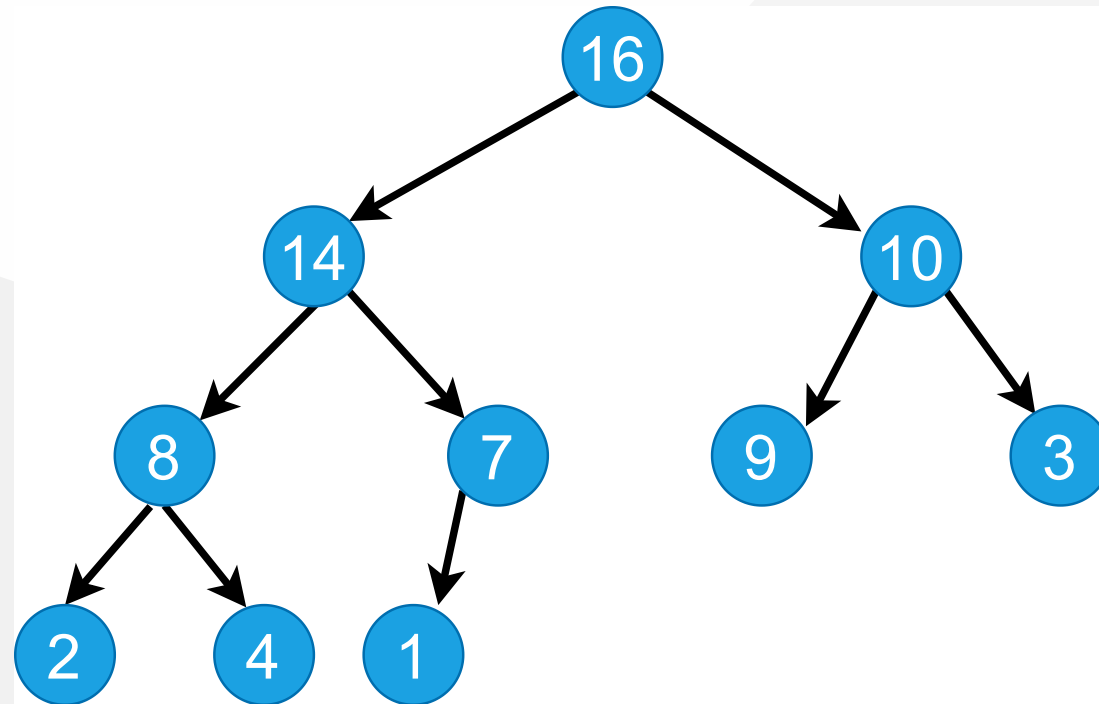
- Heap Sort
- Priority Queues
- Linked Lists
- Radix Sort
- Counting Sort

Heapsort

- Worst-case runtime: $O(n \lg n)$
- Sorts in-place
- Uses a special data structure (heap) to manage information during execution of the algorithm
 - Another design paradigm

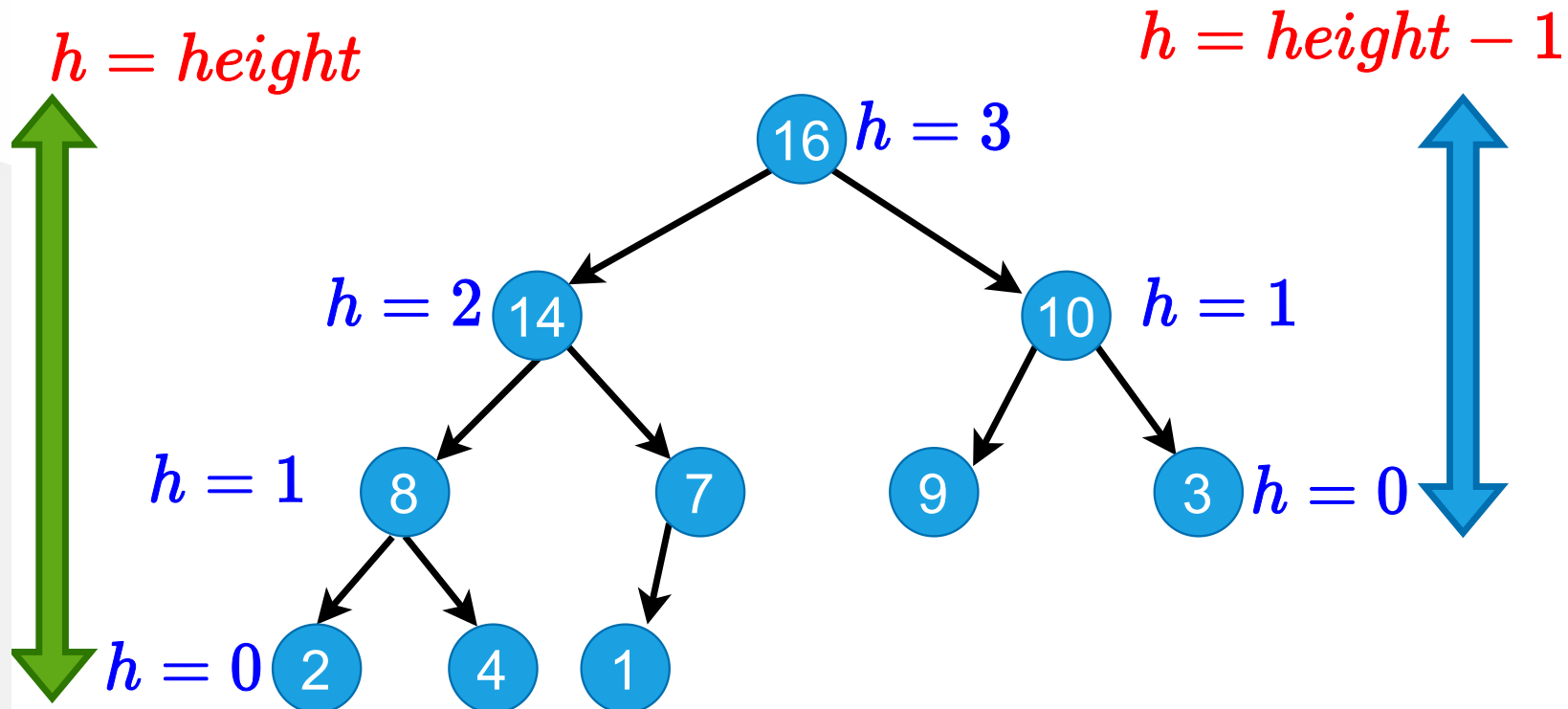
Heap Data Structure (1)

- Nearly complete binary tree
 - Completely filled on all levels except possibly the lowest level



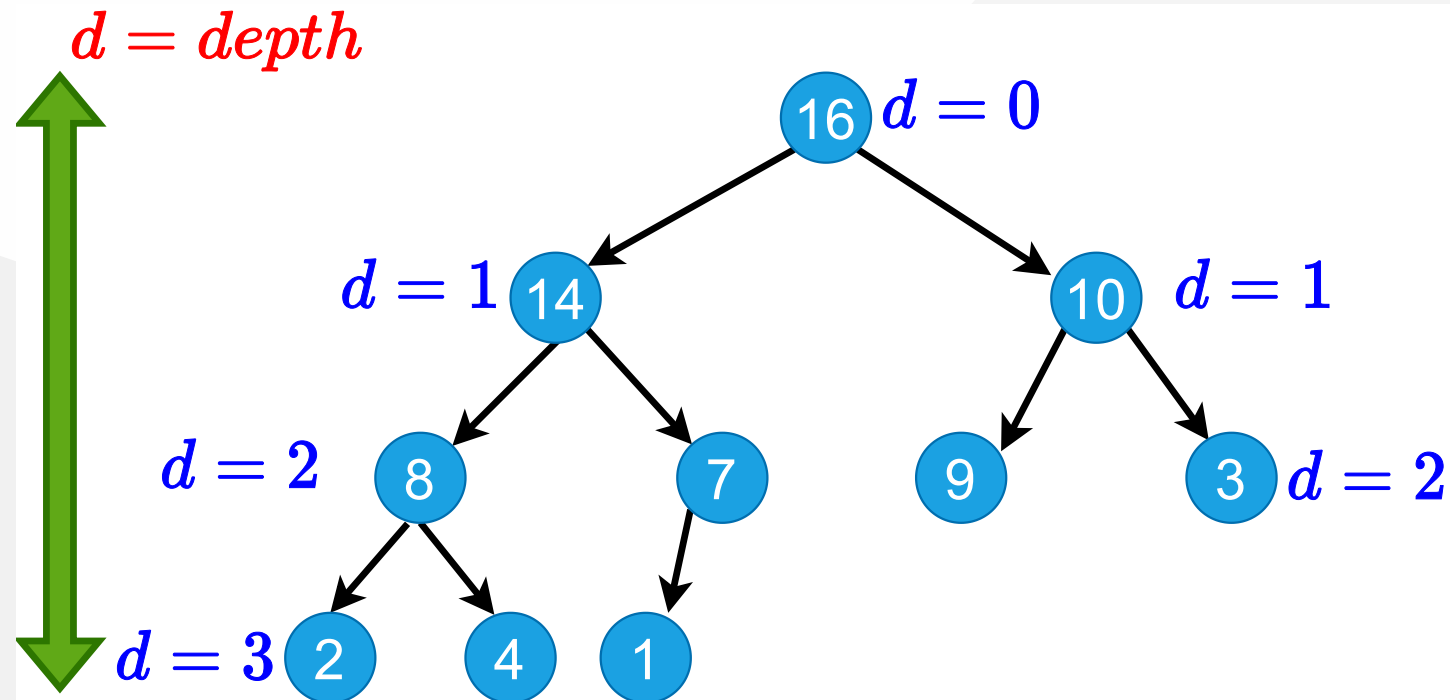
Heap Data Structure (2)

- Height of node i : Length of the longest simple downward path from i to a leaf
- Height of the tree: height of the root



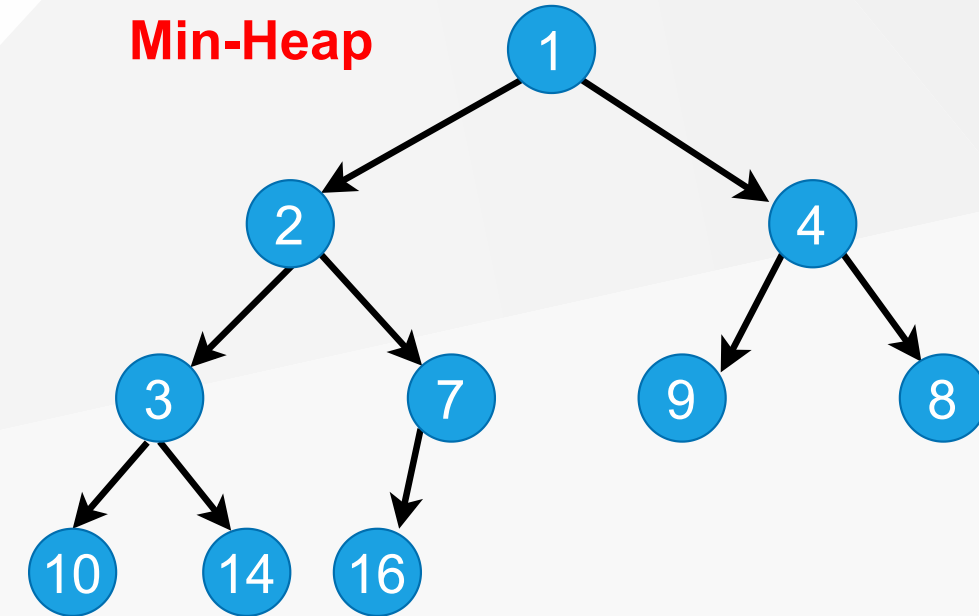
Heap Data Structures (3)

- Depth of node i : Length of the simple downward path from the **root** to node i



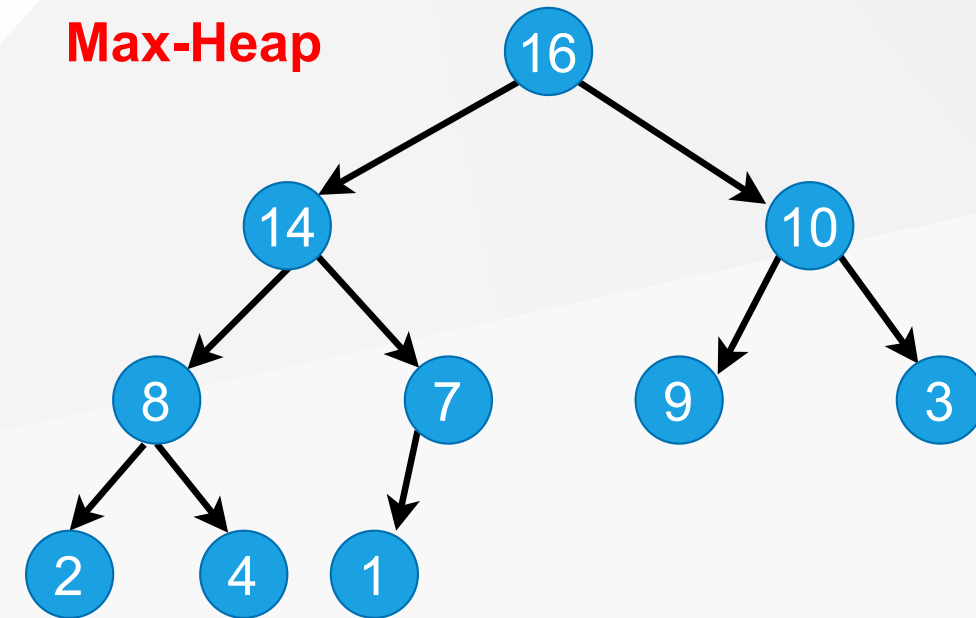
Heap Property: Min-Heap

- The **smallest** element in any subtree is the **root** element in a **min-heap**
- **Min heap:** For every node i other than **root**, $A[\text{parent}(i)] \leq A[i]$
 - Parent node is always smaller than the child nodes

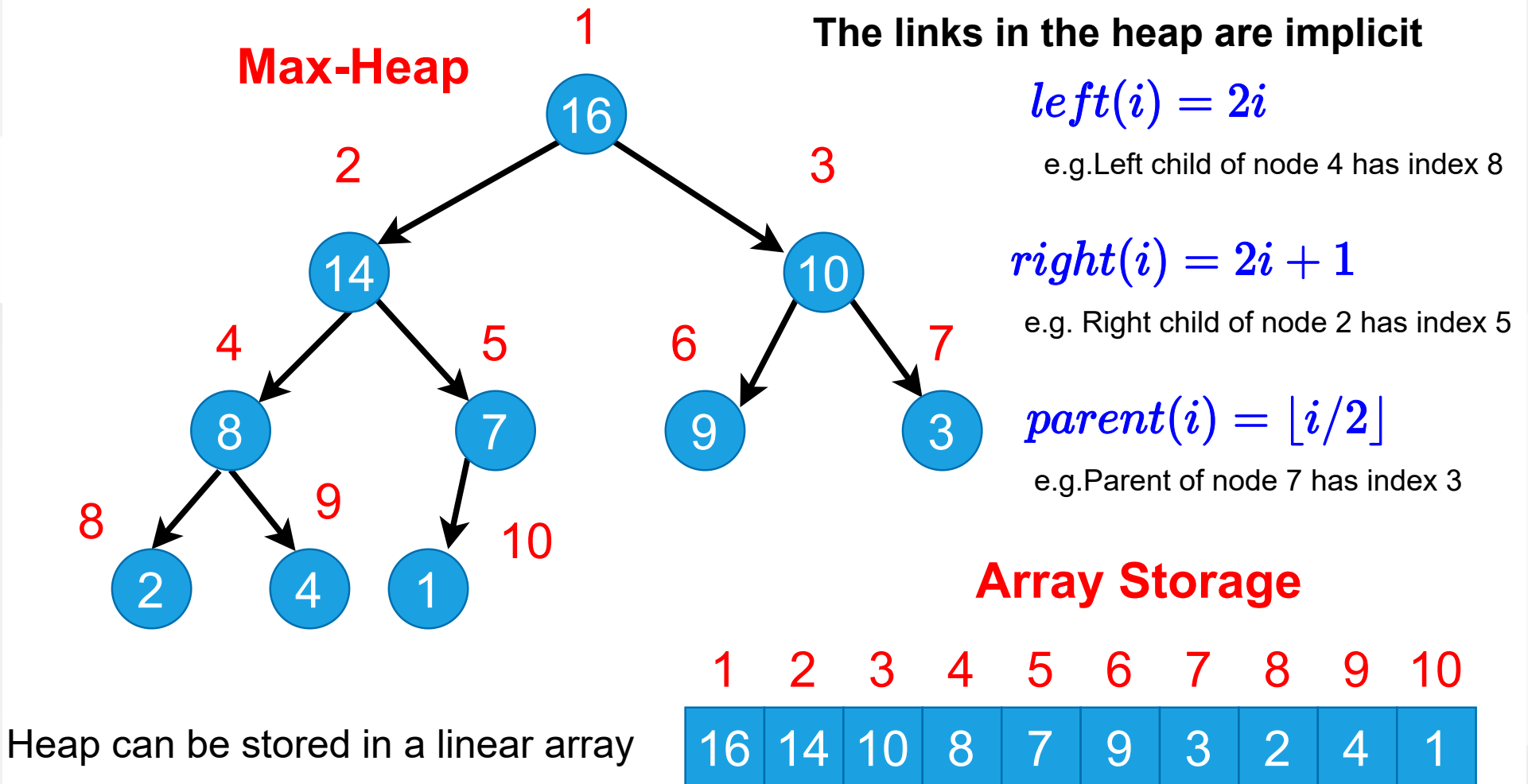


Heap Property: Max-Heap

- The **largest** element in any subtree is the **root** element in a **max-heap**
 - We will focus on max-heaps
- **Max heap:** For every node i other than **root**, $A[\text{parent}(i)] \geq A[i]$
 - Parent node is always larger than the child nodes



Heap Data Structures (4)



Heap Data Structures (5)

- Computing left child, right child, and parent indices very fast
 - $\text{left}(i) = 2i \implies$ binary left shift
 - $\text{right}(i) = 2i+1 \implies$ binary left shift, then set the lowest bit to 1
 - $\text{parent}(i) = \text{floor}(i/2) \implies$ right shift in binary
- $A[1]$ is always the **root** element
- Array A has two attributes:
 - $\text{length}(A)$: The number of elements in A
 - $n = \text{heap-size}(A)$: The number elements in *heap*
 - $n \leq \text{length}(A)$

Heap Operations : EXTRACT-MAX (1)

```
EXTRACT-MAX(A, n)
  max = A[1]
  A[1] = A[n]
  n = n - 1
  HEAPIFY(A, 1, n)
  return max
```

Heap Operations : EXTRACT-MAX (2)

- Return the max element, and reorganize the heap to maintain heap property

EXTRACT-MAX(A, n)

max = A[1]

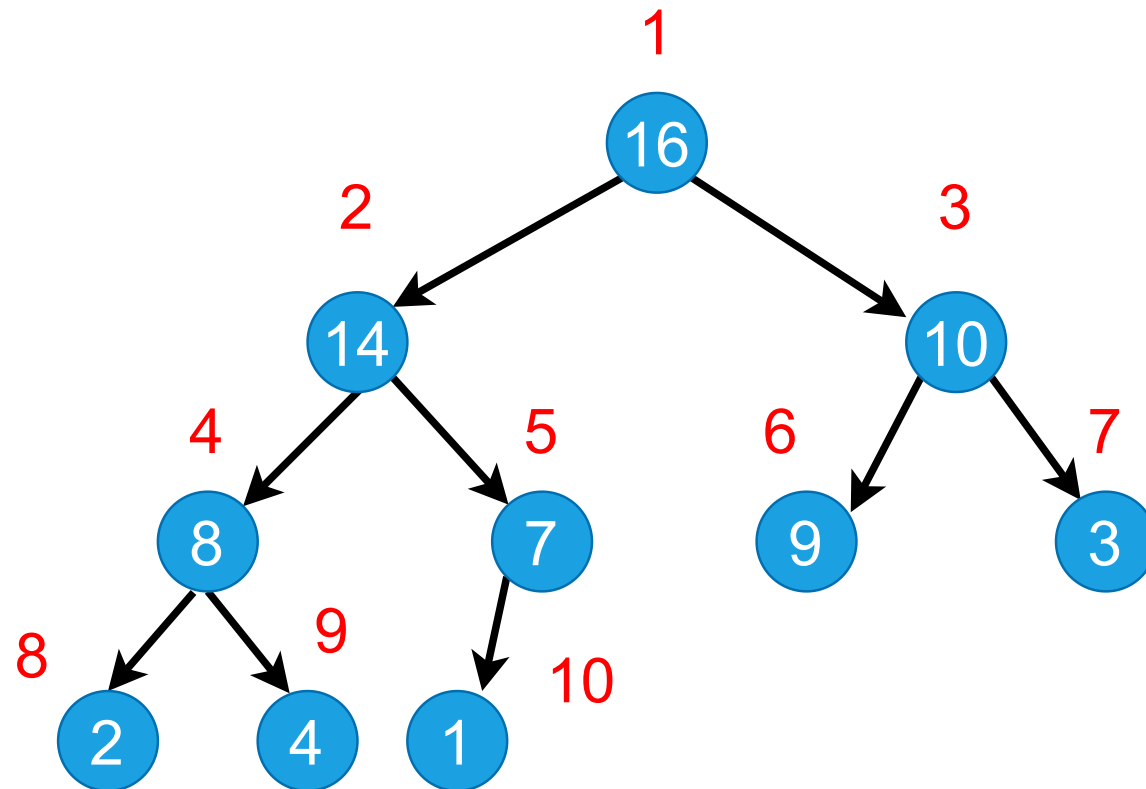
A[1] = A[n]

n = n - 1

HEAPIFY(A, 1, n)

return max

max=?



Heap Operations: HEAPIFY (1)

EXTRACT-MAX(A, n)

max = A[1]

A[1] = A[n]

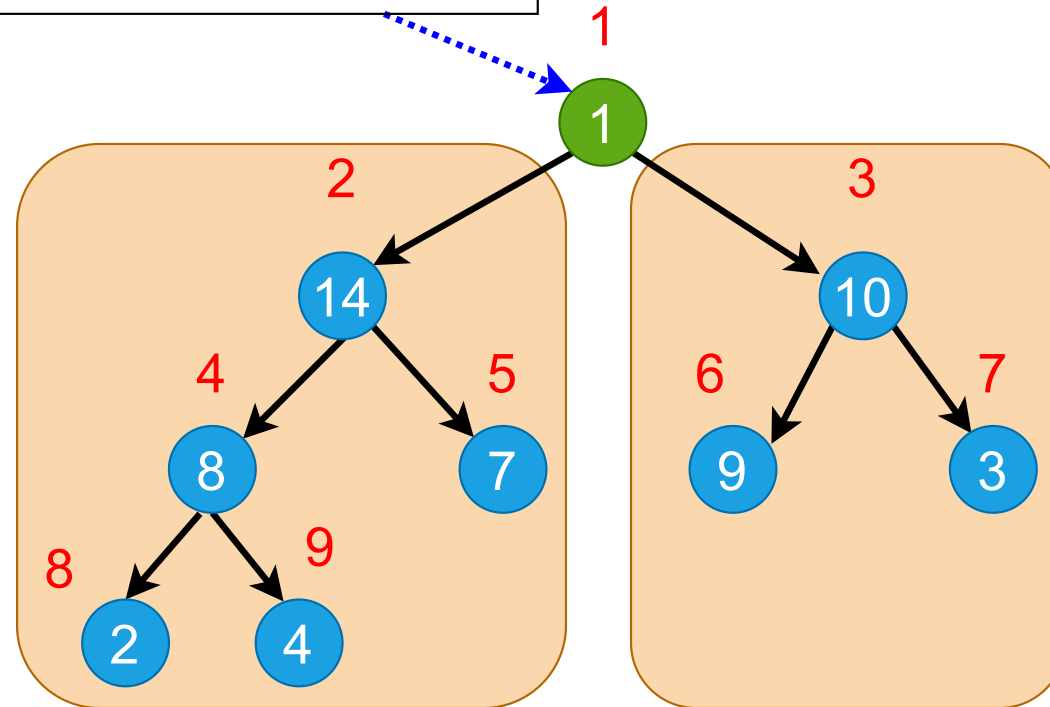
n = n - 1

HEAPIFY(A, 1, n)

return max

max= 16

Heap property violated at the root



Heap property satisfied for left and right subtrees

Heap Operations: HEAPIFY (2)

- Maintaining heap property:
 - Subtrees rooted at $left[i]$ and $right[i]$ are already heaps.
 - But, $A[i]$ may violate the heap property (i.e., may be smaller than its children)
- **Idea:** Float down the value at $A[i]$ in the heap so that subtree rooted at i becomes a heap.

Heap Operations: HEAPIFY (2)

```
HEAPIFY(A, i, n)
    largest = i

    if 2i <= n and A[2i] > A[i] then
        largest = 2i;
    endif

    if 2i+1 <= n and A[2i+1] > A[largest] then
        largest = 2i+1;
    endif

    if largest != i then
        exchange A[i] with A[largest];
        HEAPIFY(A, largest, n);
    endif
```

Heap Operations: HEAPIFY (3)

HEAPIFY(A,i,n)

largest=i

if $2i \leq n$ and $A[2i] > A[i]$

then largest=2i;

if $2i+1 \leq n$ and $A[2i+1] > A[\text{largest}]$

then largest=2i+1;

if largest!=i then

exchange A[i] with A[largest];

HEAPIFY(A,largest,n);

endif

initialize *largest*
to be the *node i*

check the *left*
child of node i

check the *right*
child of node i

exchange the *largest*
of the 3 with *node i*

recursive call on the
subtree

compute the
largest of:

- 1) node i
- 2) left child of node i
- 3) right child of node i

Heap Operations: HEAPIFY (4)

```
HEAPIFY(A,i,n)
```

```
  largest=i
```

```
  if 2i<=n and A[2i]>A[i]
```

```
    then largest=2i;
```

```
  if 2i+1<=n and A[2i+1]>A[largest]
```

```
    then largest=2i+1;
```

```
  if largest!=i then
```

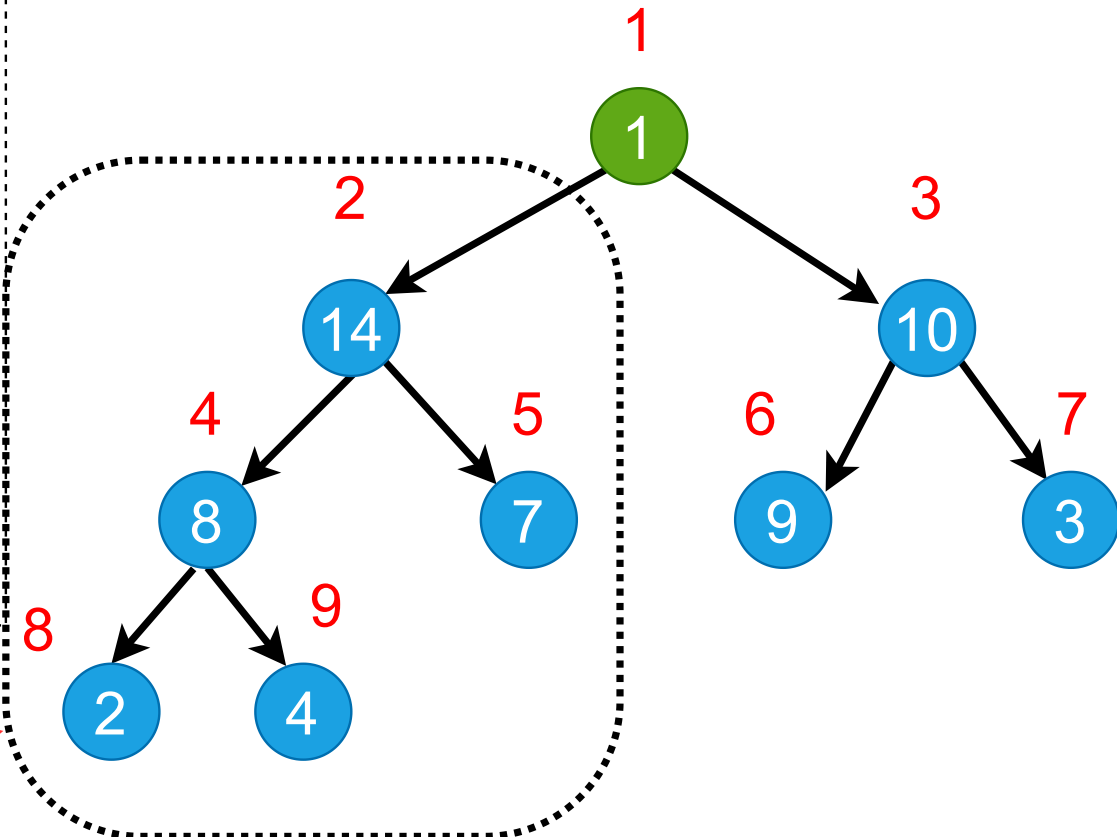
```
    exchange A[i] with A[largest];
```

```
    HEAPIFY(A,largest,n);
```

```
  endif
```

**Recursive
Call**

HEAPIFY(A, 1, 9)



Heap Operations: HEAPIFY (5)

HEAPIFY(A,i,n)

largest=i

if $2i \leq n$ and $A[2i] > A[i]$

then largest=2i;

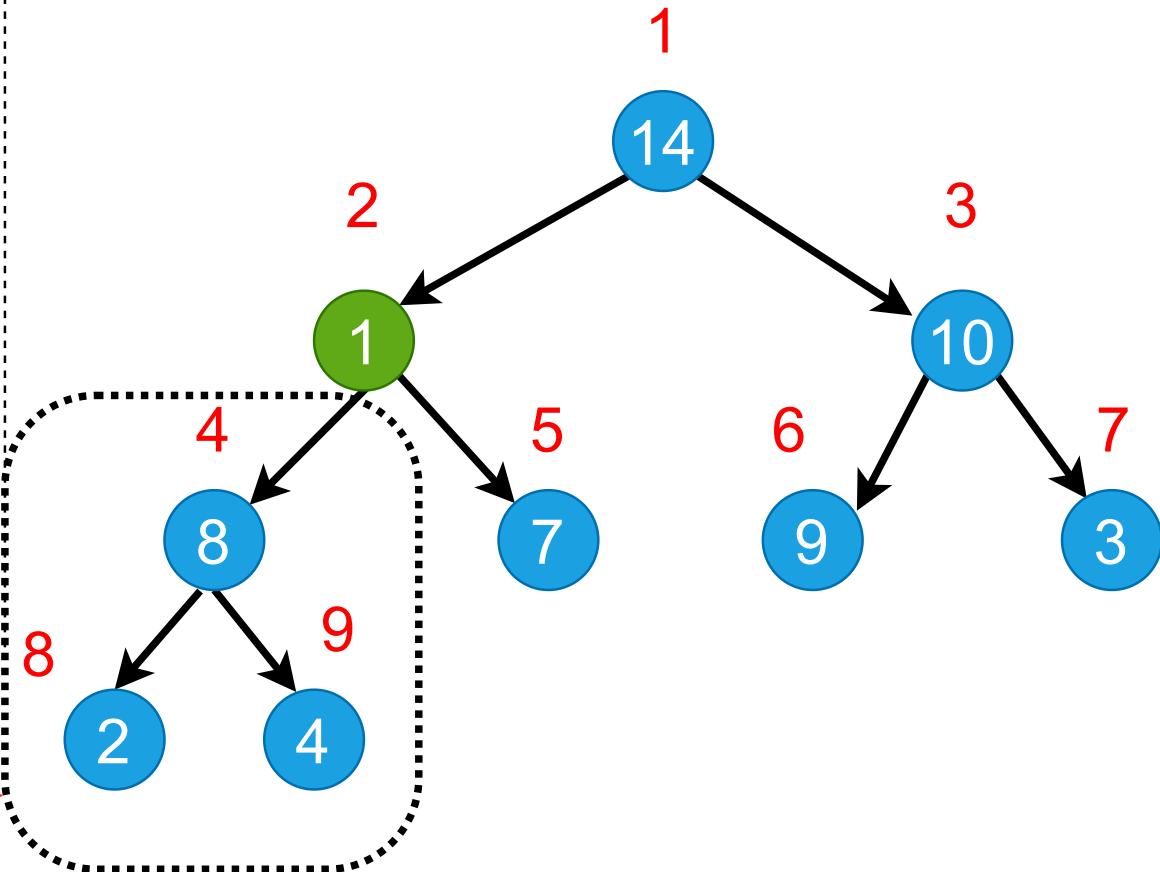
if $2i+1 \leq n$ and $A[2i+1] > A[\text{largest}]$

then largest=2i+1;

if largest!=i then
 exchange A[i] with A[largest];
 HEAPIFY(A,largest,n);
 endif

**Recursive
Call**

HEAPIFY(A, 2, 9)



Heap Operations: HEAPIFY (6)

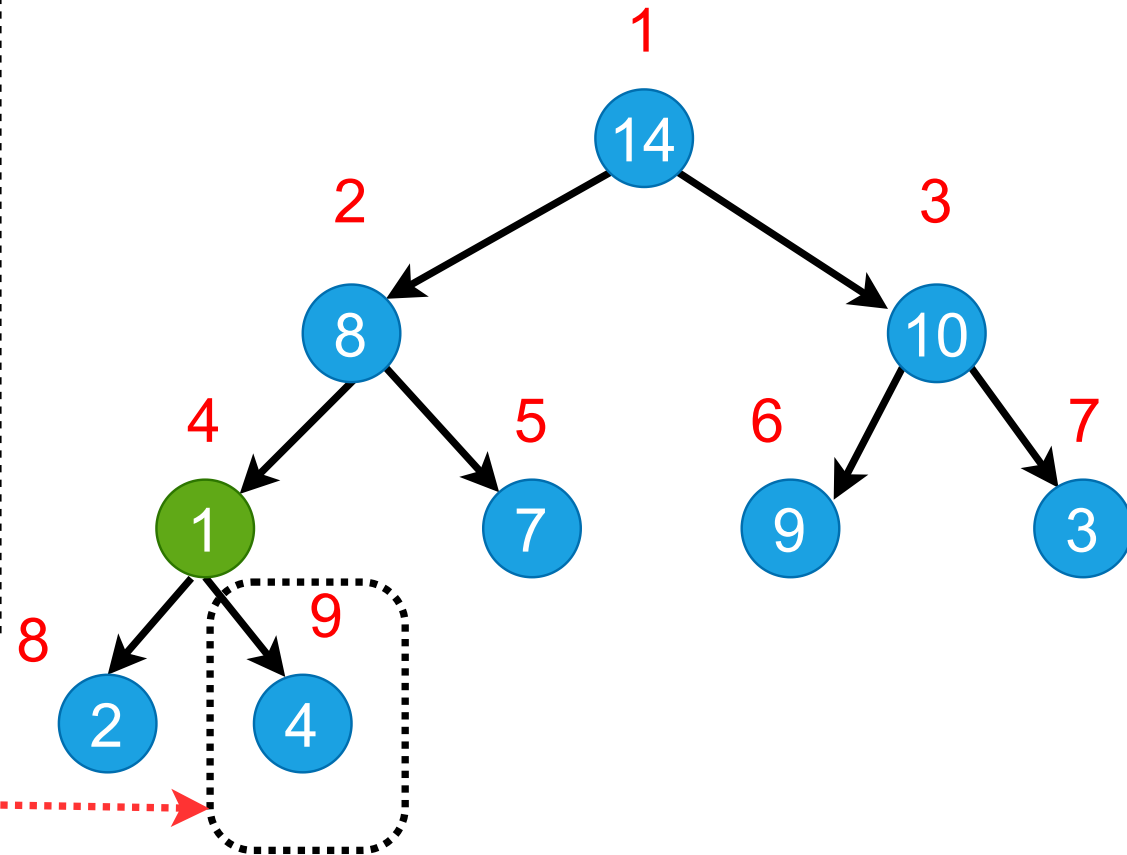
```

HEAPIFY(A,i,n)
    largest=i
    if 2i≤n and A[2i]>A[i]
        then largest=2i;
    if 2i+1≤n and A[2i+1]>A[largest]
        then largest=2i+1;
    if largest≠i then
        exchange A[i] with A[largest];
        HEAPIFY(A,largest,n);
    endif

```

**Recursive Call
(Base Case)**

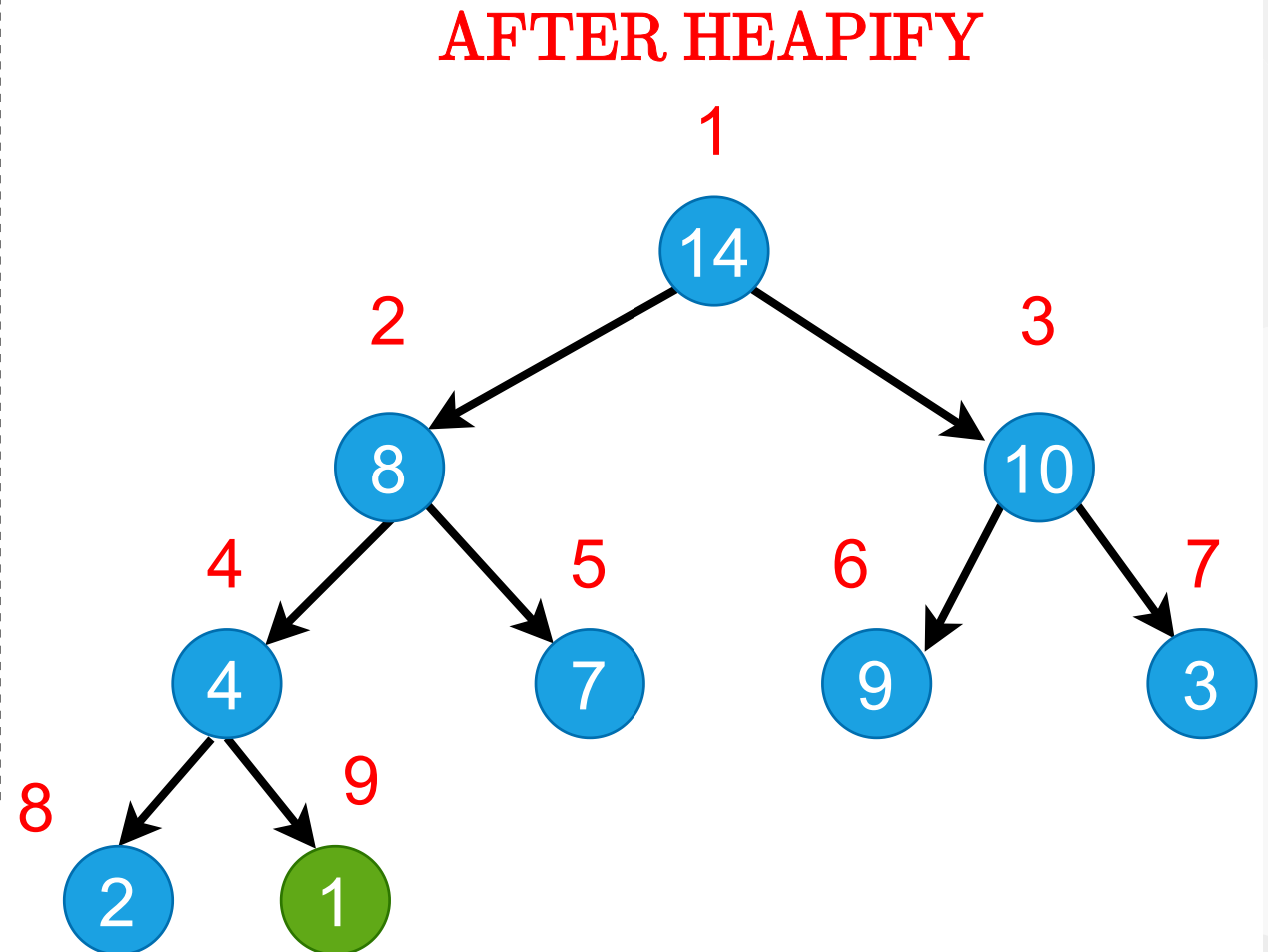
HEAPIFY(A, 4, 9)



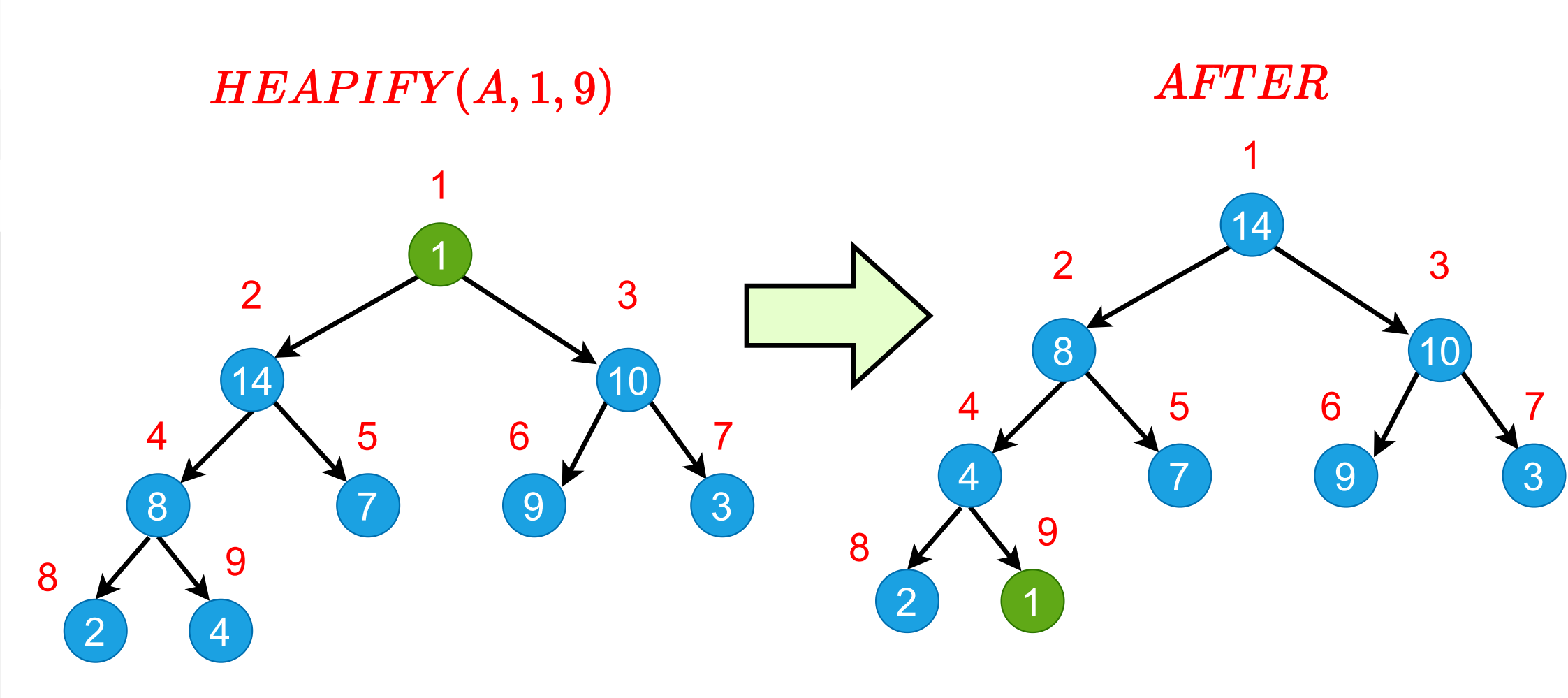
Heap Operations: HEAPIFY (7)

```

HEAPIFY(A,i,n)
    largest=i
    if 2i<=n and A[2i]>A[i]
        then largest=2i;
    if 2i+1<=n and A[2i+1]>A[largest]
        then largest=2i+1;
    if largest!=i then
        exchange A[i] with A[largest];
        HEAPIFY(A,largest,n);
    endif
  
```



Heap Operations: HEAPIFY (8)



Intuitive Analysis of HEAPIFY

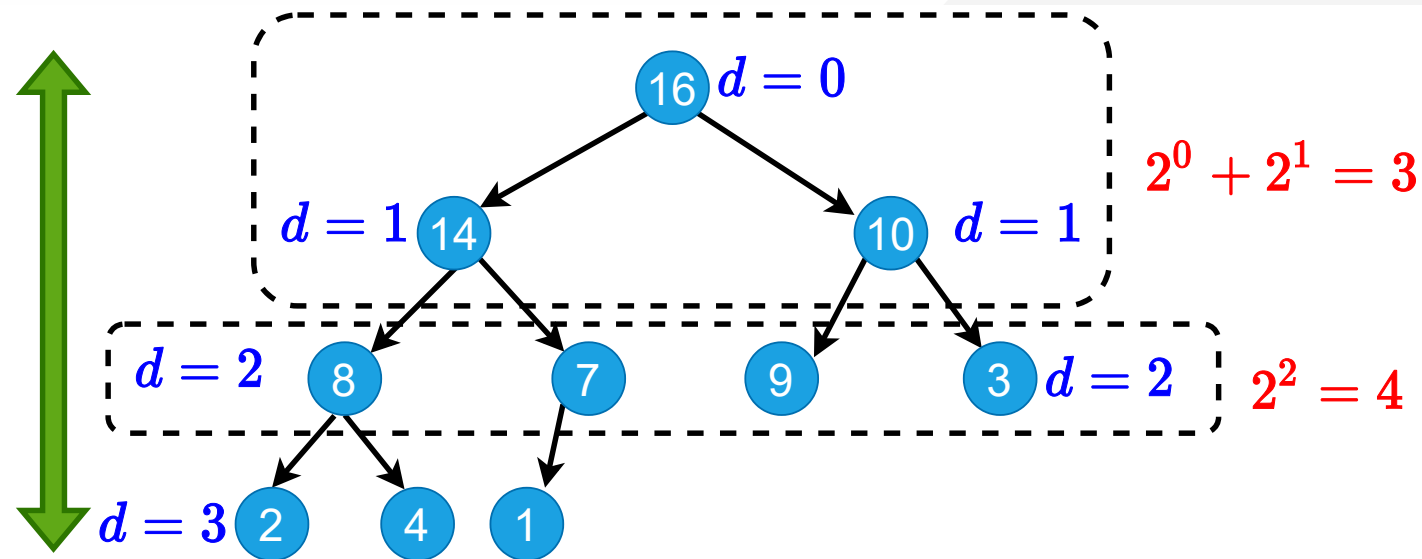
- Consider $HEAPIFY(A, i, n)$
 - let $h(i)$ be the height of node i
 - at most $h(i)$ recursion levels
 - Constant work at each level: $\Theta(1)$
 - Therefore $T(i) = O(h(i))$
- Heap is almost-complete binary tree
 - $h(n) = O(\lg n)$
- Thus $T(n) = O(\lg n)$

Formal Analysis of HEAPIFY

- What is the recurrence?
 - Depends on the size of the **subtree** on which recursive call is made
 - In the next, we try to compute an **upper bound** for this **subtree**.

Reminder: Binary trees

- For a complete binary tree:
 - # of nodes at depth d : 2^d
 - # of nodes with depths less than d : $2^d - 1$



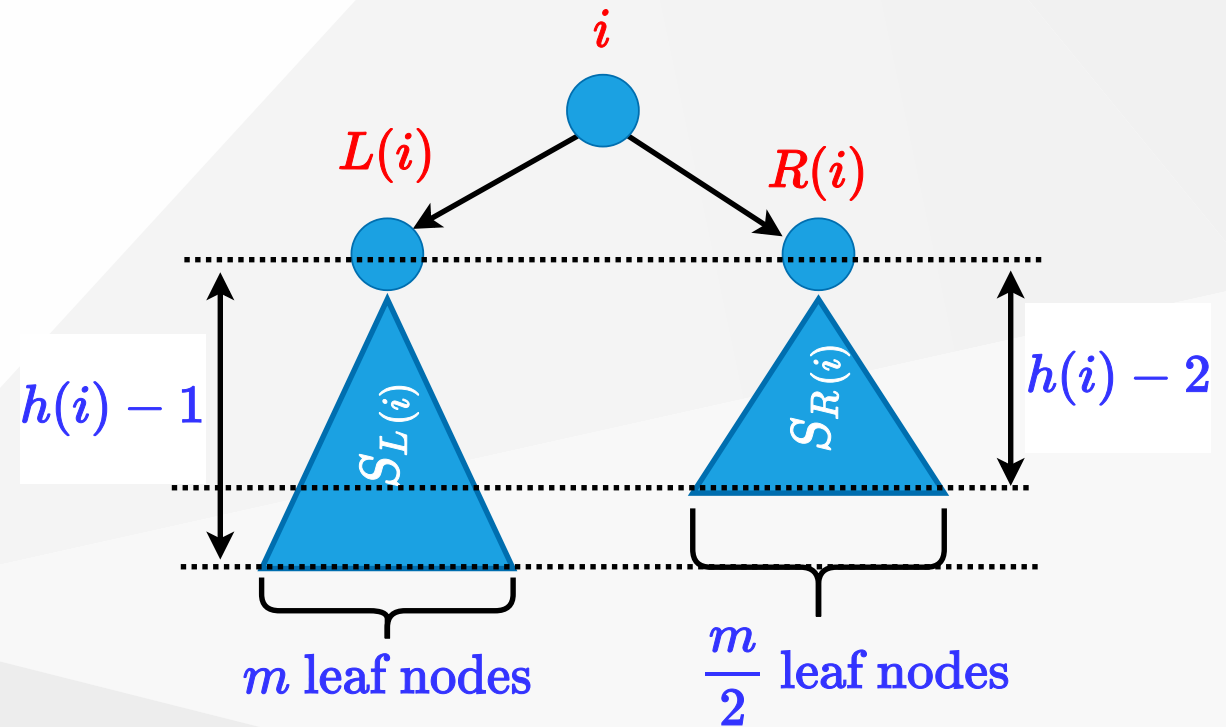
$d = \text{depth}$ for example $d = 2$

$2^d = \text{node size at } d$ $2^2 = 4$

$2^d - 1 = \text{node size less than } d$ $2^2 - 1 = 3 \implies 2^0 + 2^1$

Formal Analysis of HEAPIFY (1)

- Worst case occurs when last row of the subtree S_i rooted at node i is **half full**
- $T(n) \leq T(|S_{L(i)}|) + \Theta(1)$
- $S_{L(i)}$ and $S_{R(i)}$ are complete binary trees of heights $h(i) - 1$ and $h(i) - 2$, respectively



Formal Analysis of HEAPIFY (2)

- Let m be the number of leaf nodes in $S_{L(i)}$

$$\circ |S_{L(i)}| = \overbrace{m}^{ext.} + \overbrace{(m-1)}^{int.} = 2m-1$$

$$\circ |S_{R(i)}| = \frac{\overbrace{m}^{ext.}}{2} + \left(\frac{\overbrace{m}^{int.}}{2} - 1 \right) = m-1$$

$$\circ |S_{L(i)}| + |S_{R(i)}| + 1 = n$$

Formal Analysis of HEAPIFY (2)

$$(2m-1) + (m-1) + 1 = n$$

$$m = (n + 1)/3$$

$$|S_{L(i)}| = 2m-1$$

$$= 2(n + 1)/3 - 1$$

$$= (2n/3 + 2/3) - 1$$

$$= \frac{2n}{3} - \frac{1}{3} \leq \frac{2n}{3}$$

$$T(n) \leq T(2n/3) + \Theta(1)$$

$$T(n) = O(\lg n)$$

- By CASE-2 of Master Theorem $\implies T(n) = \Theta(n^{\log_b^a} \lg n)$

Formal Analysis of HEAPIFY (2)

- Recurrence: $T(n) = aT(n/b) + f(n)$
- Case 2: $\frac{f(n)}{n^{\log_b^a}} = \Theta(1)$
- i.e., $f(n)$ and $n^{\log_b^a}$ grow at similar rates
- Solution: $T(n) = \Theta(n^{\log_b^a} \lg n)$
 - $T(n) \leq T(2n/3) + \Theta(1)$ (drop constants.)
 - $T(n) \leq \Theta(n^{\log_3^1} \lg n)$
 - $T(n) \leq \Theta(n^0 \lg n)$
 - $T(n) = O(\lg n)$

HEAPIFY: Efficiency Issues

- Recursion vs Iteration:
 - In the absence of tail recursion, **iterative version** is in general **more efficient** because of the **pop/push** operations **to/from** stack at each **level of recursion**.

Heap Operations: HEAPIFY (1)

Recursive

```
HEAPIFY(A, i, n)
largest = i

if 2i <= n and A[2i] > A[i] then
    largest = 2i

if 2i+1 <= n and A[2i+1] > A[largest] then
    largest = 2i+1

if largest != i then
    exchange A[i] with A[largest]
    HEAPIFY(A, largest, n)
```


Heap Operations: HEAPIFY (2)

Iterative

```
HEAPIFY(A, i, n)
  j = i
  while(true) do
    largest = j

    if 2j <= n and A[2j] > A[j] then
      largest = 2j

    if 2j+1 <= n and A[2j+1] > A[largest] then
      largest = 2j+1

    if largest != j then
      exchange A[j] with A[largest]
      j = largest
    else return
```

Heap Operations: HEAPIFY (3)

Recursive

HEAPIFY(A, i, n)

largest $\leftarrow i$

if $2i \leq n$ **and** $A[2i] > A[i]$ **then** largest $\leftarrow 2i$

if $2i + 1 \leq n$ **and** $A[2i+1] > A[\text{largest}]$ **then** largest $\leftarrow 2i + 1$

if largest $\neq i$ **then**

exchange $A[i] \leftrightarrow A[\text{largest}]$

HEAPIFY($A, \text{largest}, n$)

Iterative

HEAPIFY(A, i, n)

$j \leftarrow i$

while (true) **do**

largest $\leftarrow j$

if $2j \leq n$ **and** $A[2j] > A[j]$ **then** largest $\leftarrow 2j$

if $2j + 1 \leq n$ **and** $A[2j+1] > A[\text{largest}]$ **then** largest $\leftarrow 2j + 1$

if largest $\neq j$ **then**

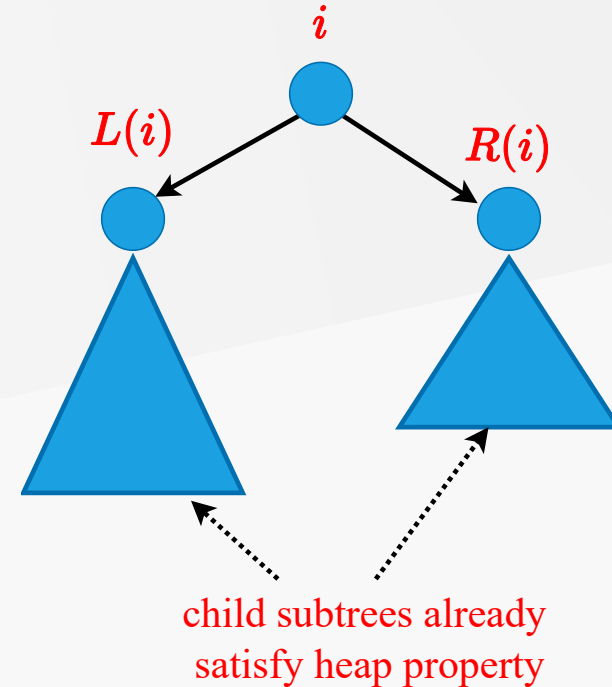
exchange $A[j] \leftrightarrow A[\text{largest}]$

$j \leftarrow \text{largest}$

else return

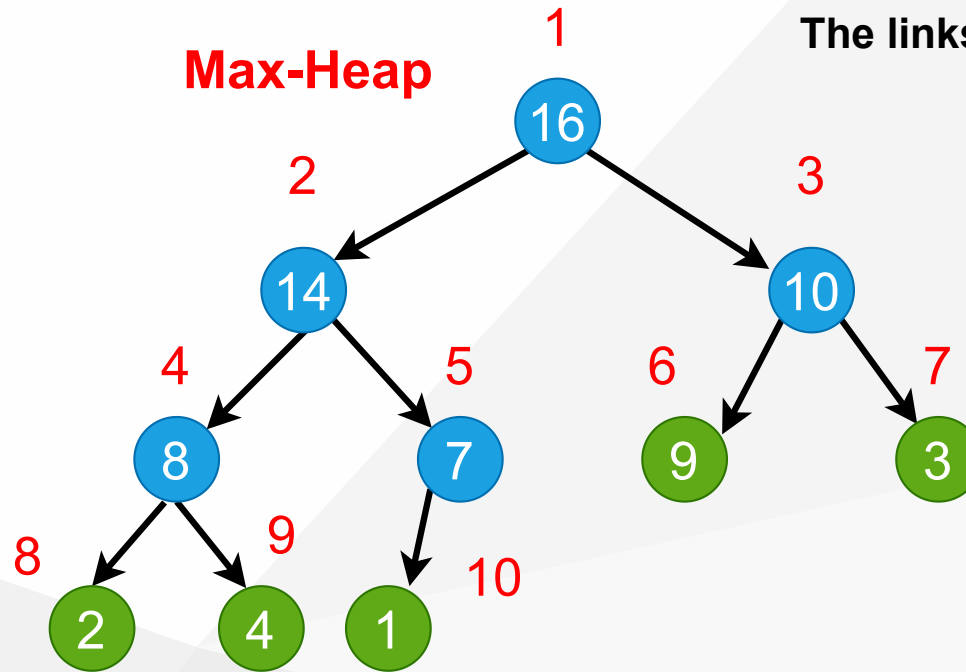
Heap Operations: Building Heap

- Given an arbitrary array, how to build a heap from scratch?
- **Basic idea:** Call *HEAPIFY* on each node bottom up
 - Start from the leaves (which trivially satisfy the heap property)
 - Process nodes in bottom up order.
 - When *HEAPIFY* is called on node i , the subtrees connected to the *left* and *right* subtrees already satisfy the heap property.



Storage of the leaves (Lemma)

- Lemma: The last $\lceil \frac{n}{2} \rceil$ nodes of a heap are all leaves.



The links in the heap are implicit

$$\text{left}(i) = 2i$$

e.g. Left child of node 4 has index 8

$$\text{right}(i) = 2i + 1$$

e.g. Right child of node 2 has index 5

$$\text{parent}(i) = \lfloor i/2 \rfloor$$

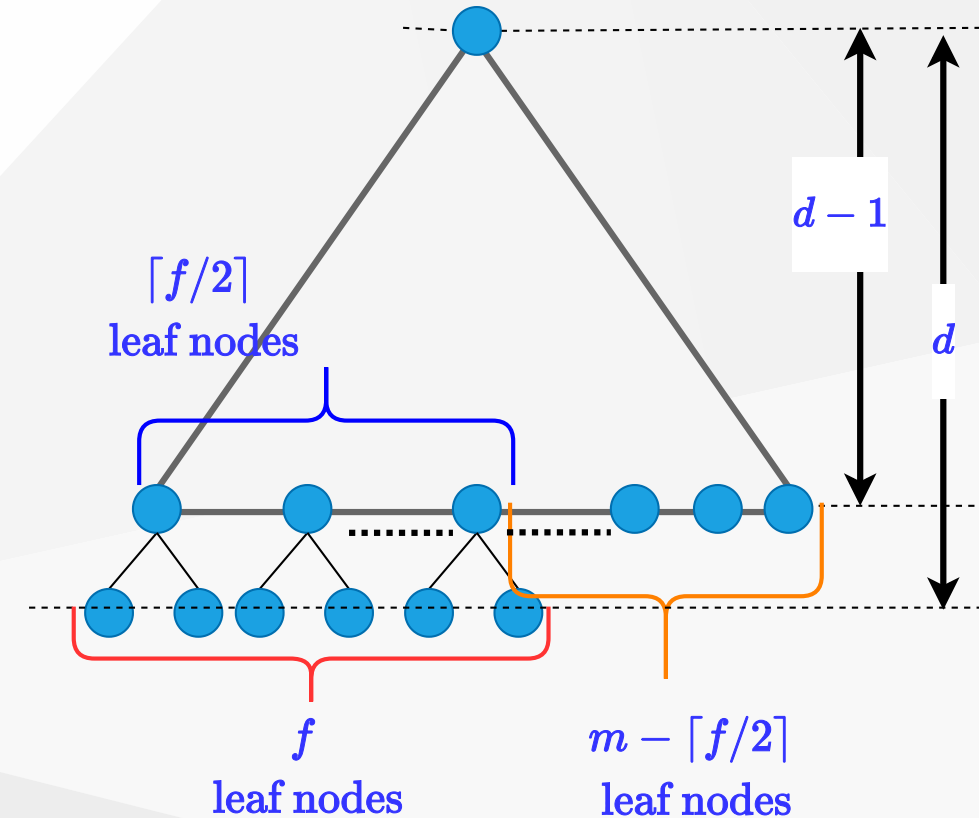
e.g. Parent of node 7 has index 3

Array Storage

1	2	3	4	5	6	7	8	9	10
16	14	10	8	7	9	3	2	4	1

Storage of the leaves (Proof of Lemma) (1)

- **Lemma:** last $\lceil n/2 \rceil$ nodes of a heap are all leaves
- **Proof :**
 - $m = 2^{d-1}$: # nodes at level $d - 1$
 - f : # nodes at level d (last level)
- # of nodes with depth $d - 1 : m$
- # of nodes with depth $< d - 1 : m - 1$
- # of nodes with depth $d : f$
- **Total** # of nodes : $n = f + 2m - 1$

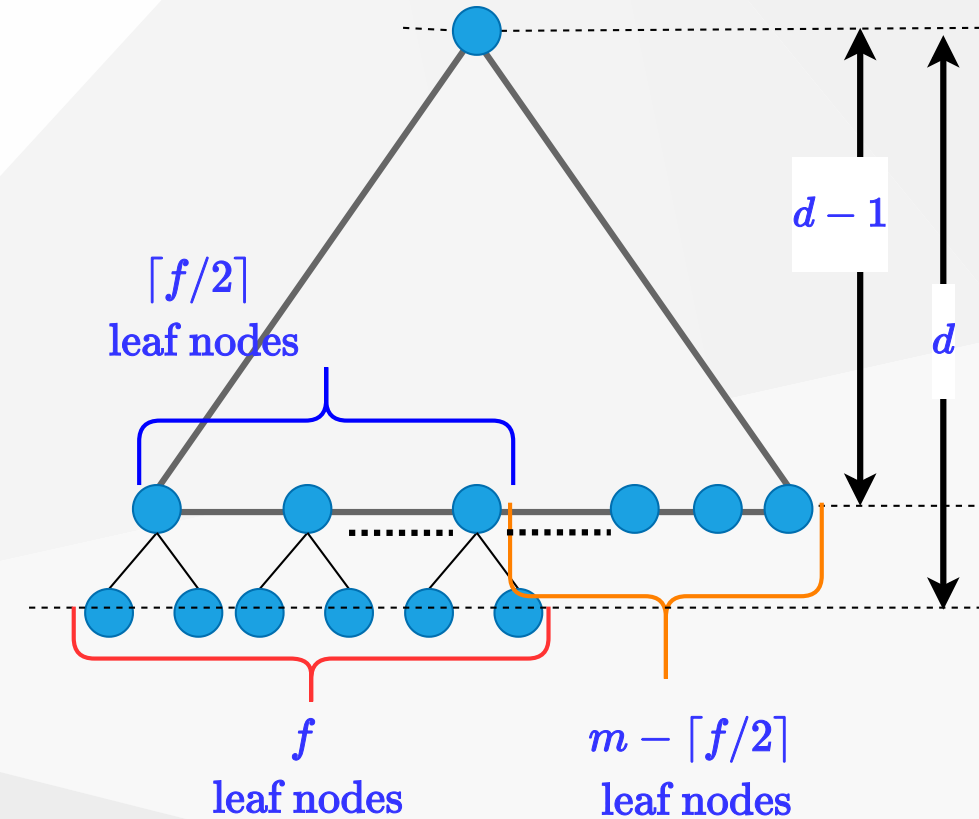


Storage of the leaves (Proof of Lemma) (2)

- Total # of nodes : $f = n - 2m + 1$

$$\begin{aligned}
 \# \text{ of leaves: } &= f + m - \lceil f/2 \rceil \\
 &= m + \lfloor f/2 \rfloor \\
 &= m + \lfloor (n - 2m + 1)/2 \rfloor \\
 &= \lfloor (n + 1)/2 \rfloor \\
 &= \lceil n/2 \rceil
 \end{aligned}$$

Proof is Completed

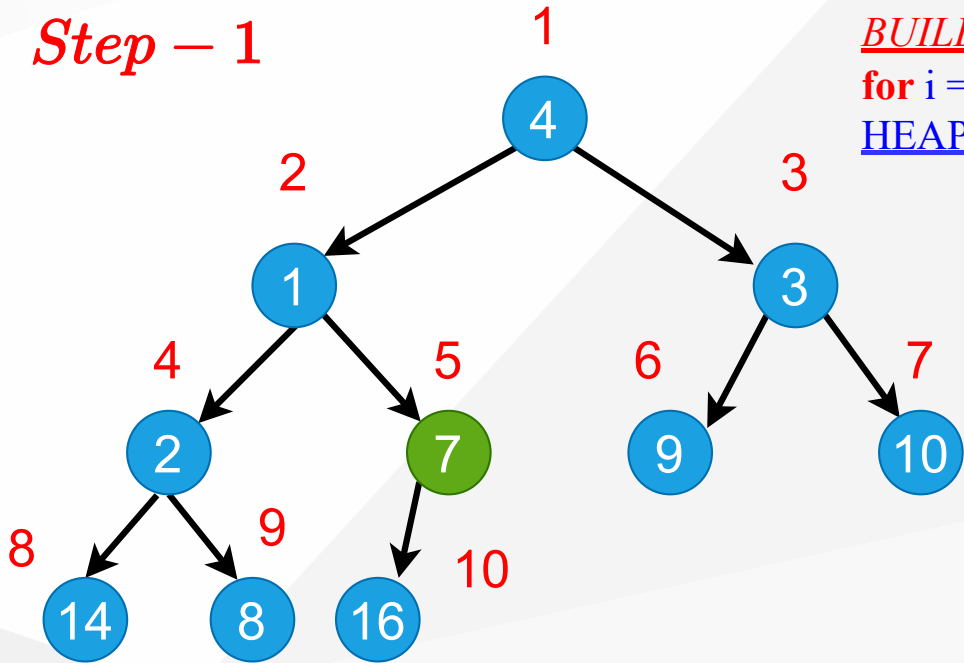


Heap Operations: Building Heap

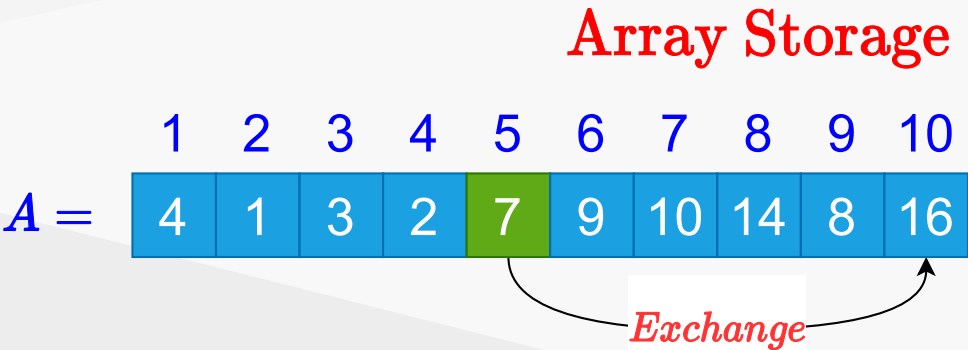
```
BUILD-HEAP (A, n)
  for i = ceil(n/2) downto 1 do
    HEAPIFY(A, i, n)
```

- **Reminder:** The last $\lceil n/2 \rceil$ nodes of a heap are **all leaves**, which trivially satisfy the heap property

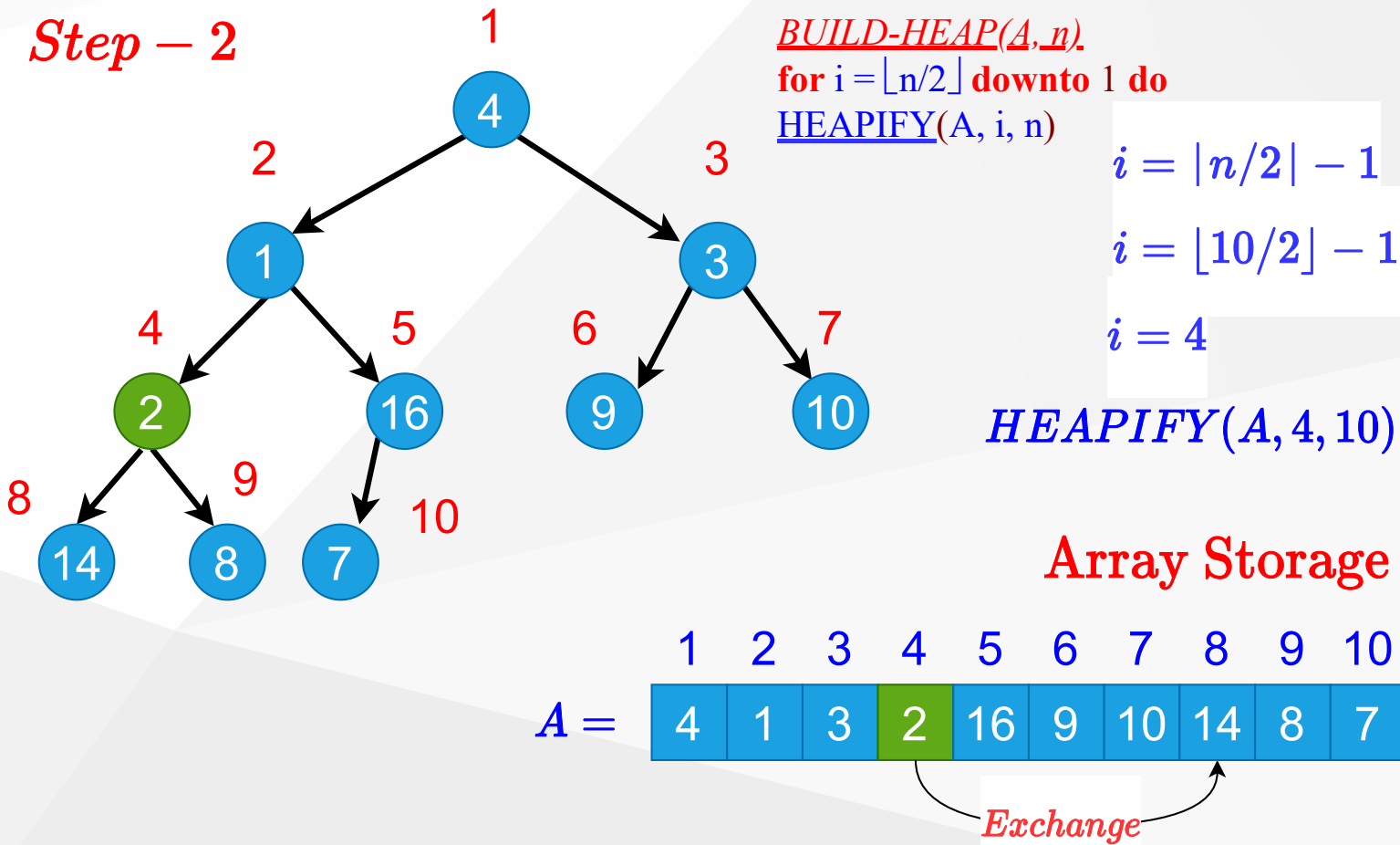
Build-Heap Example (Step-1)



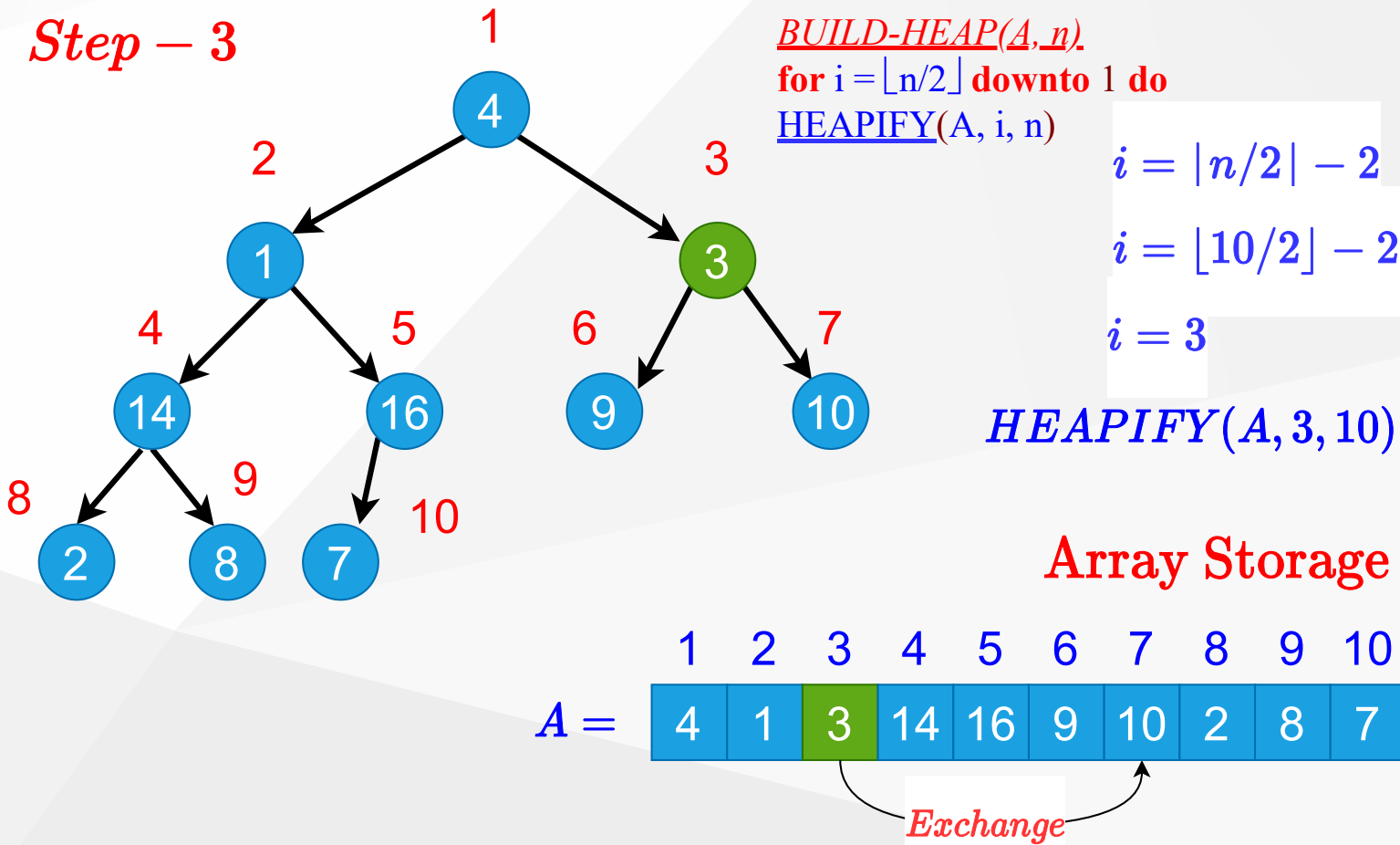
```
BUILD-HEAP(A, n)  
for i =  $\lfloor n/2 \rfloor$  downto 1 do  
  HEAPIFY(A, i, n)  
   $i = \lfloor n/2 \rfloor - 0$   
   $i = \lfloor 10/2 \rfloor - 0$   
   $i = 5$   
HEAPIFY(A, 5, 10)
```



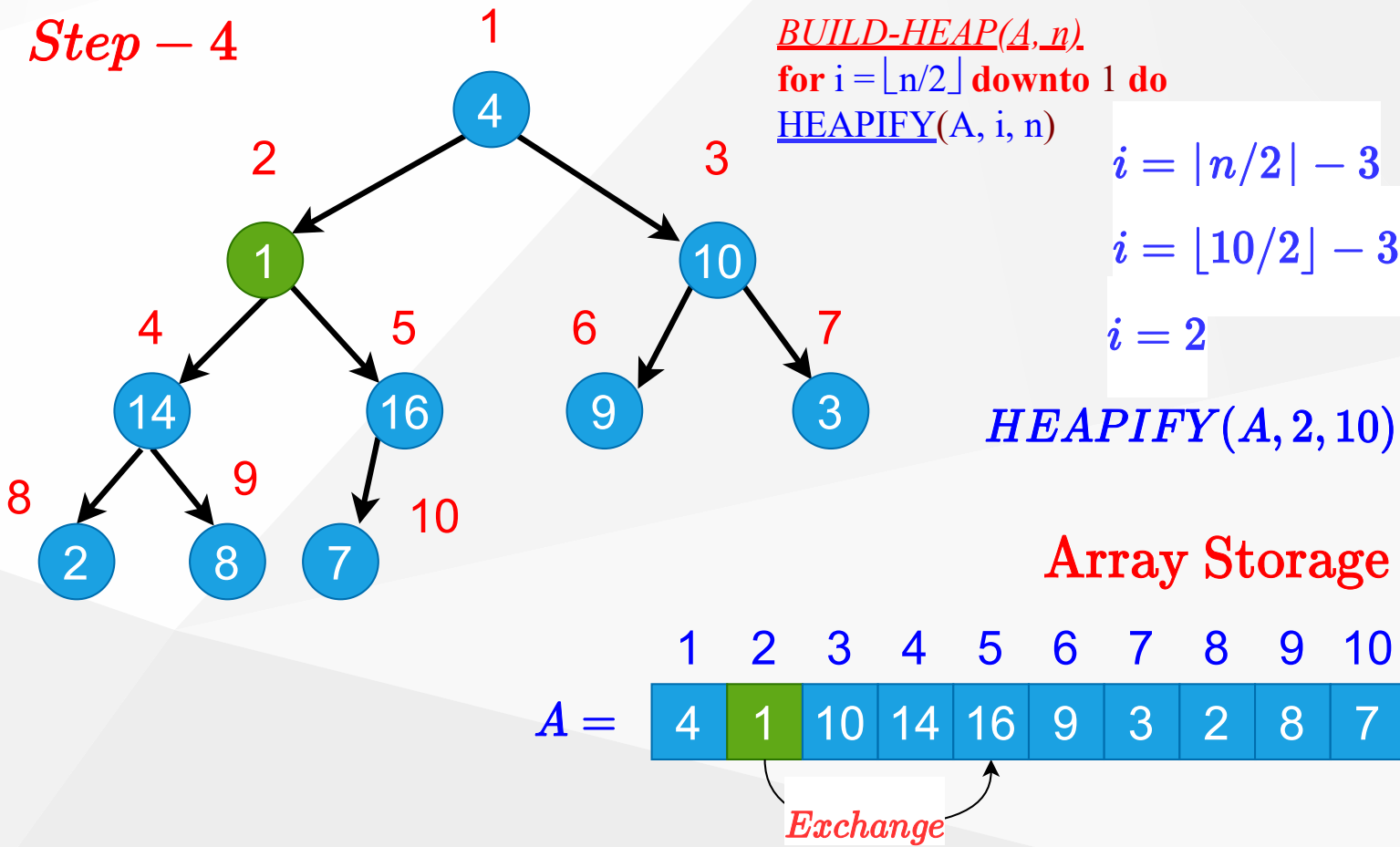
Build-Heap Example (Step-2)



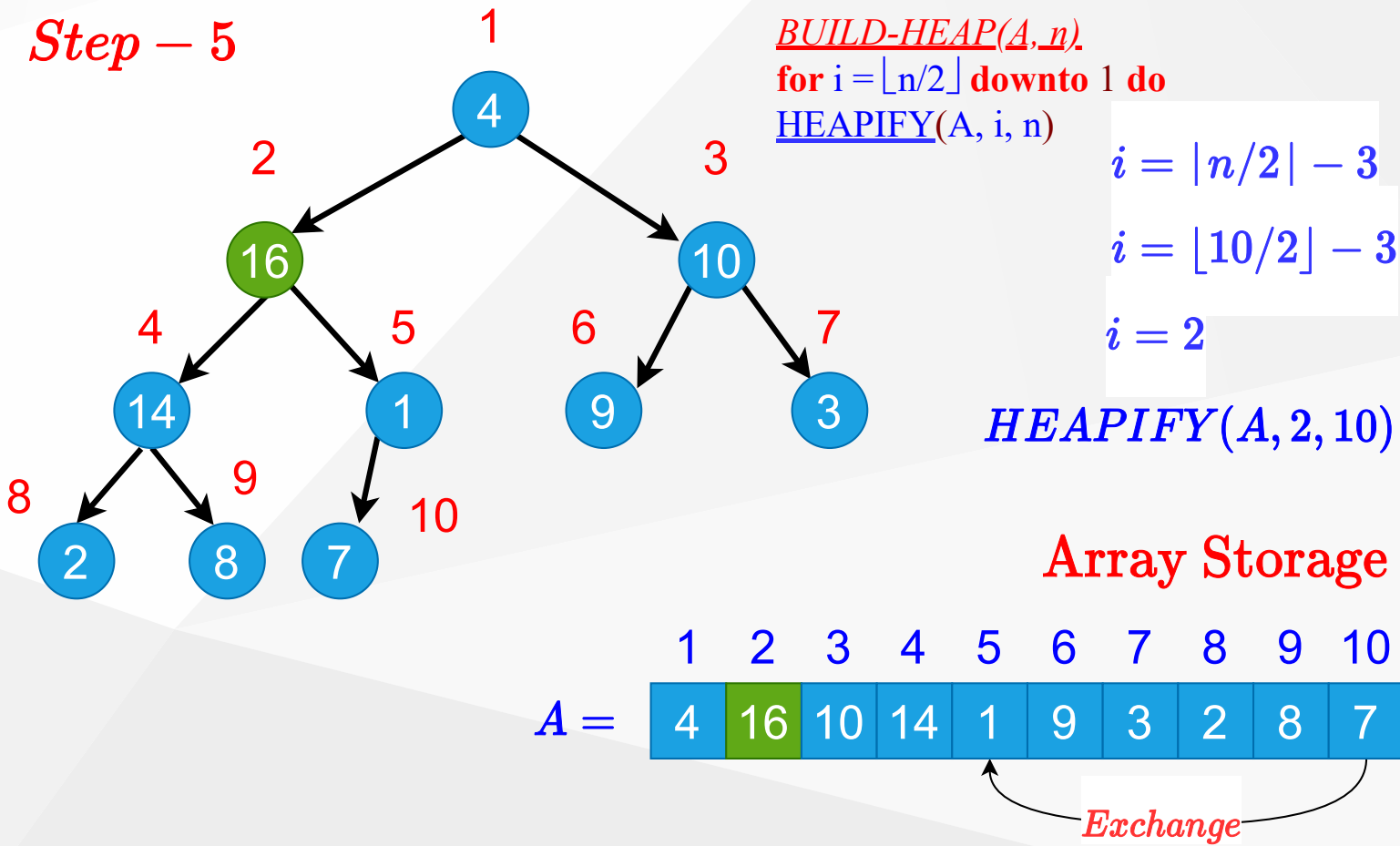
Build-Heap Example (Step-3)



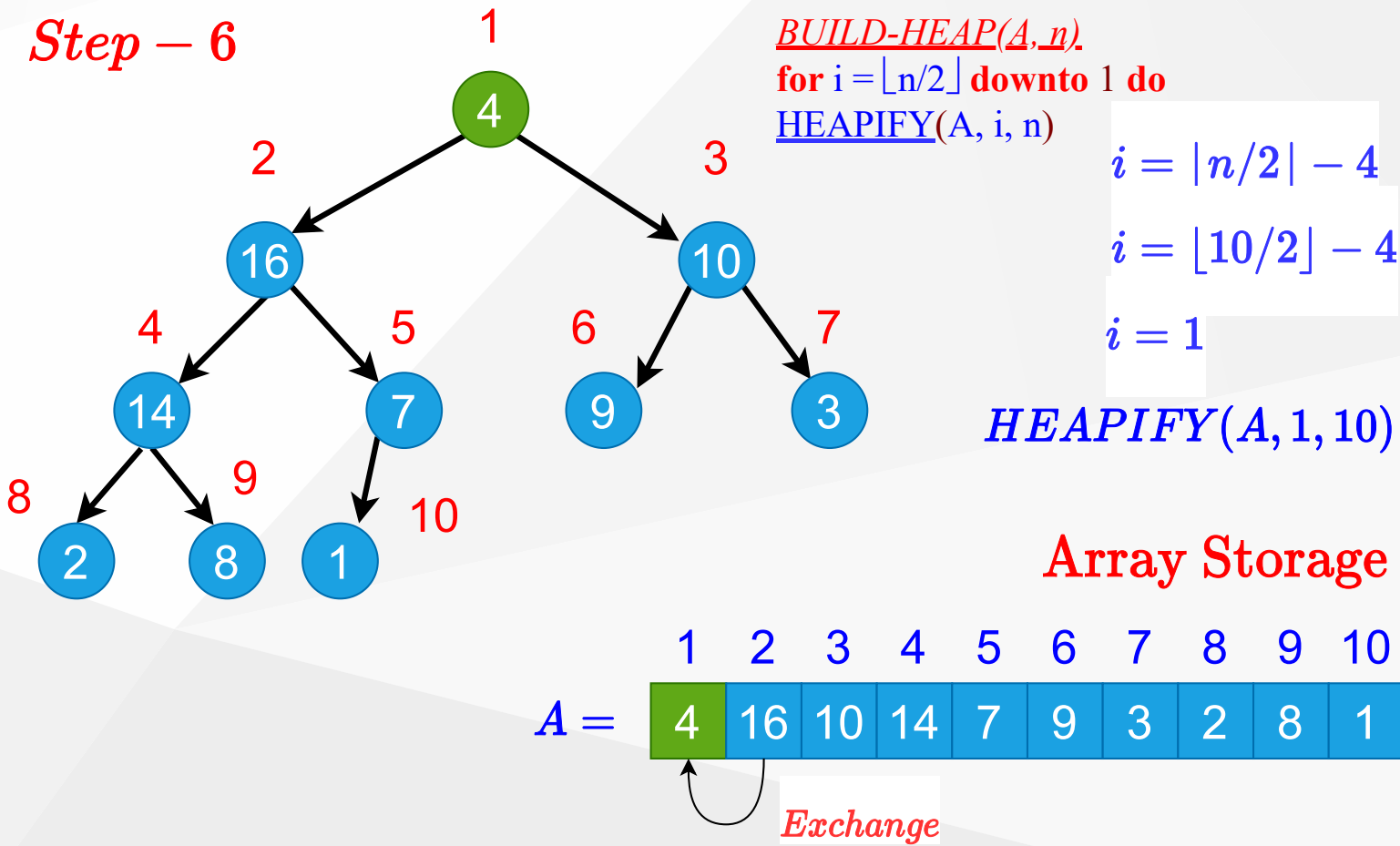
Build-Heap Example (Step-4)



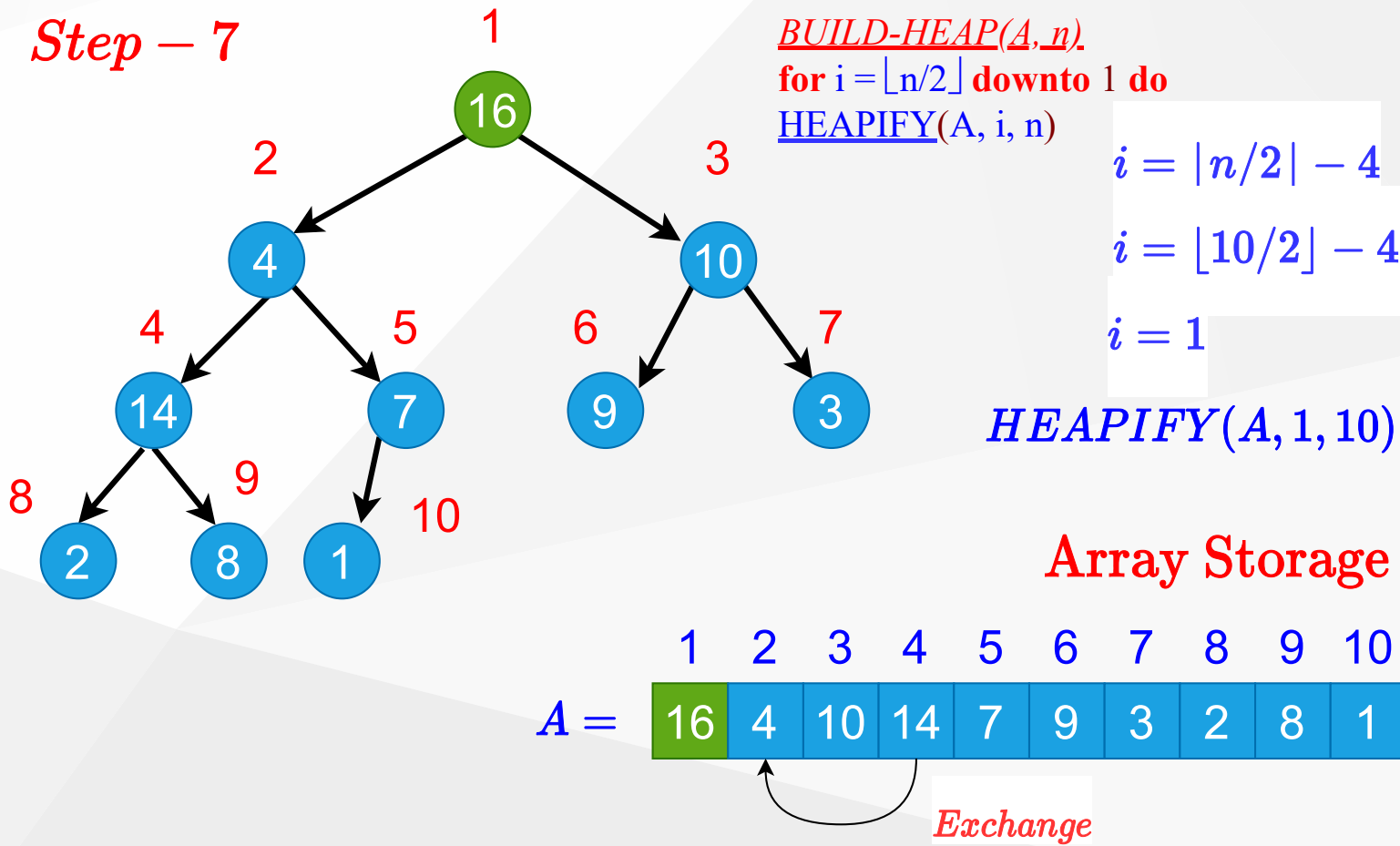
Build-Heap Example (Step-5)



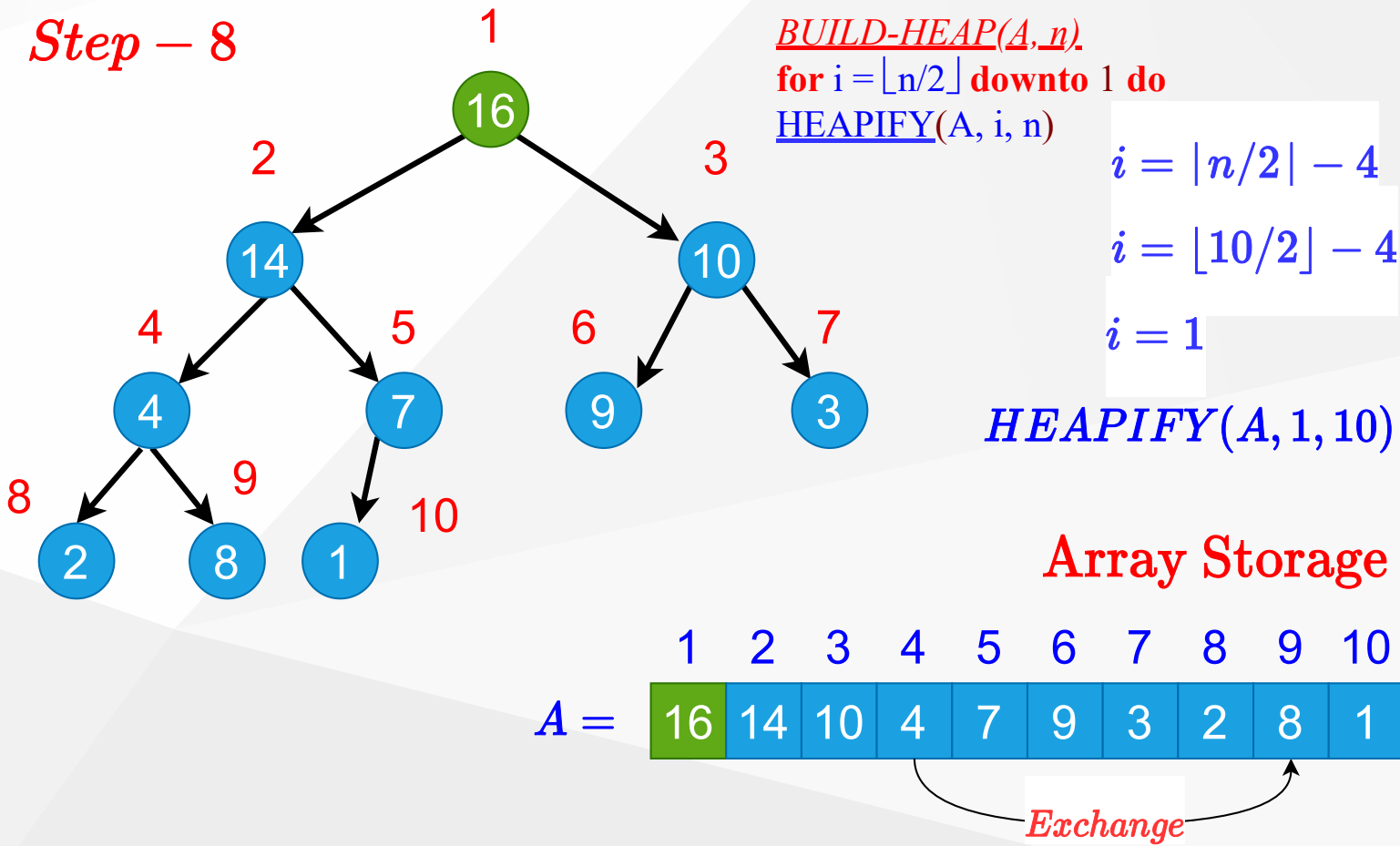
Build-Heap Example (Step-6)



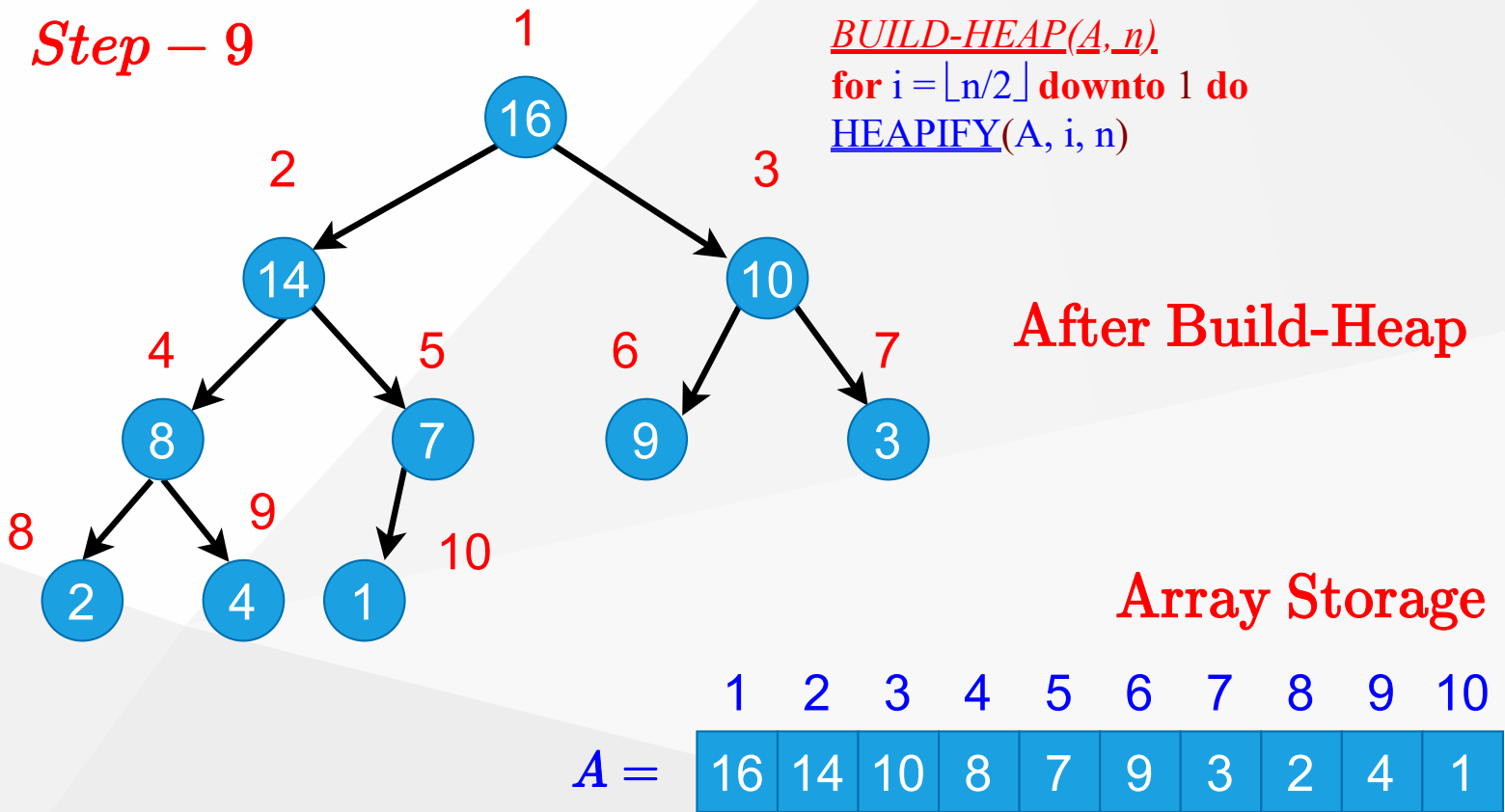
Build-Heap Example (Step-7)



Build-Heap Example (Step-8)



Build-Heap
Example
(Step-9)

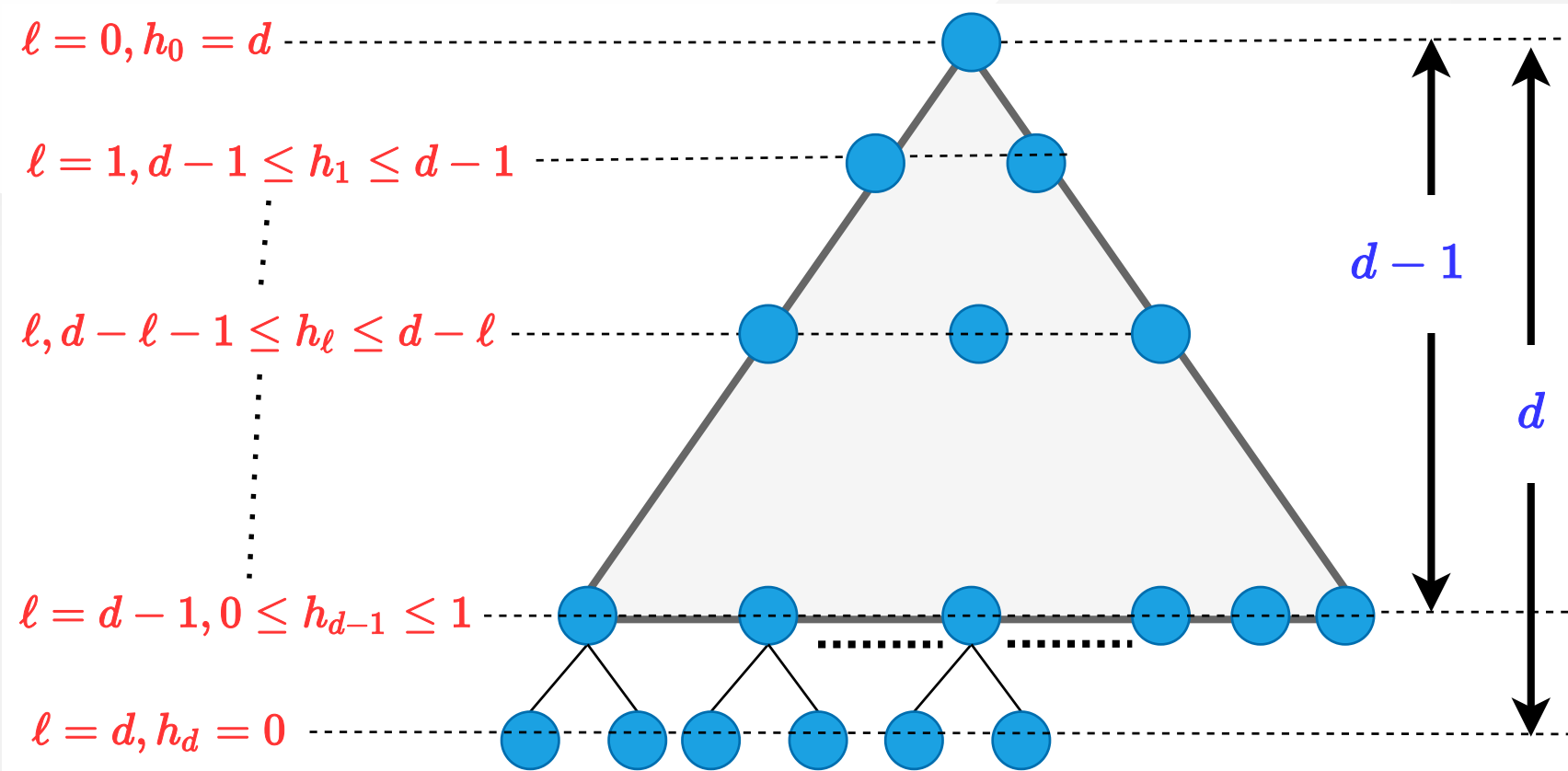


Build-Heap: Runtime Analysis

- Simple analysis:
 - $O(n)$ calls to *HEAPIFY*, each of which takes $O(\lg n)$ time
 - $O(n \lg n) \implies$ loose bound
- In general, a good approach:
 - Start by proving an easy bound
 - Then, try to tighten it
- Is there a tighter bound?

Build-Heap: Tighter Running Time Analysis

- If the heap is complete binary tree then $h_\ell = d - \ell$
- Otherwise, nodes at a given level do not all have the same height, But we have $d - \ell - 1 \leq h_\ell \leq d - \ell$



Build-Heap: **Tighter** Running Time Analysis

- Assume that all nodes at level $\ell = d-1$ are processed

$$T(n) = \sum_{\ell=0}^{d-1} n_{\ell} O(h_{\ell}) = O\left(\sum_{\ell=0}^{d-1} n_{\ell} h_{\ell}\right) \begin{cases} n_{\ell} = 2^{\ell} = \# \text{ of nodes at level } \ell \\ h_{\ell} = \text{height of nodes at level } \ell \end{cases}$$

$$\therefore T(n) = O\left(\sum_{\ell=0}^{d-1} 2^{\ell} (d - \ell)\right)$$

Let $h = d - \ell \implies \ell = d - h$ change of variables

$$T(n) = O\left(\sum_{h=1}^d h 2^{d-h}\right) = O\left(\sum_{h=1}^d h \frac{2^d}{2^h}\right) = O\left(2^d \sum_{h=1}^d h (1/2)^h\right)$$

$$\text{but } 2^d = \Theta(n) \implies O\left(n \sum_{h=1}^d h (1/2)^h\right)$$

Build-Heap: **Tighter** Running Time Analysis

$$\sum_{h=1}^d h(1/2)^h \leq \sum_{h=0}^d h(1/2)^h \leq \sum_{h=0}^{\infty} h(1/2)^h$$

- recall infinite decreasing geometric series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \text{ where } |x| < 1$$

- differentiate both sides

$$\sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

Build-Heap: Tighter Running Time Analysis

$$\sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

- then, multiply both sides by x

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}$$

- in our case: $x = 1/2$ and $k = h$

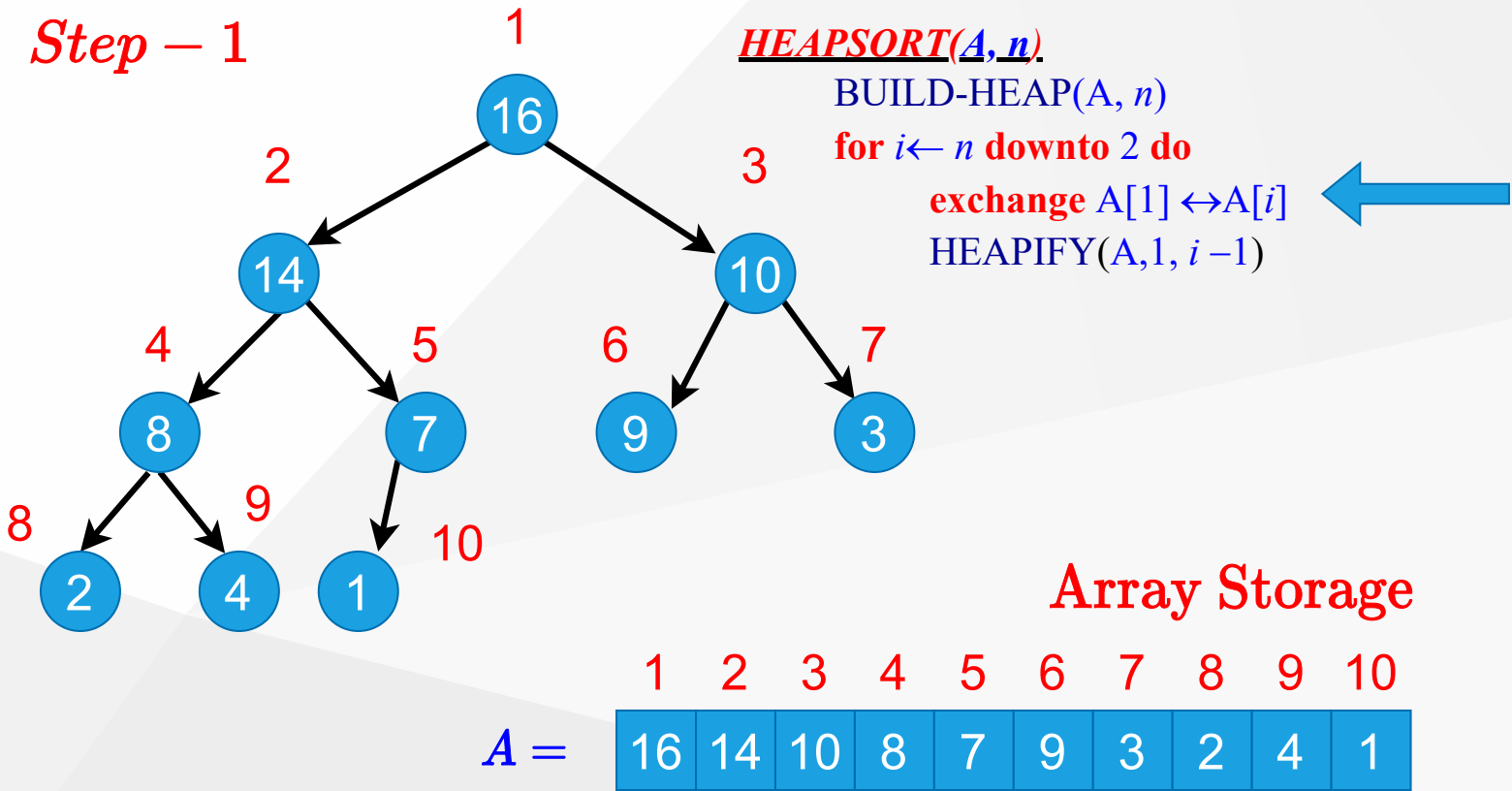
$$\therefore \sum_{h=0}^{\infty} h(1/2)^h = \frac{1/2}{(1 - (1/2))^2} = 2 = O(1)$$

$$\therefore T(n) = O\left(n \sum_{h=1}^d h(1/2)^h\right) = O(n)$$

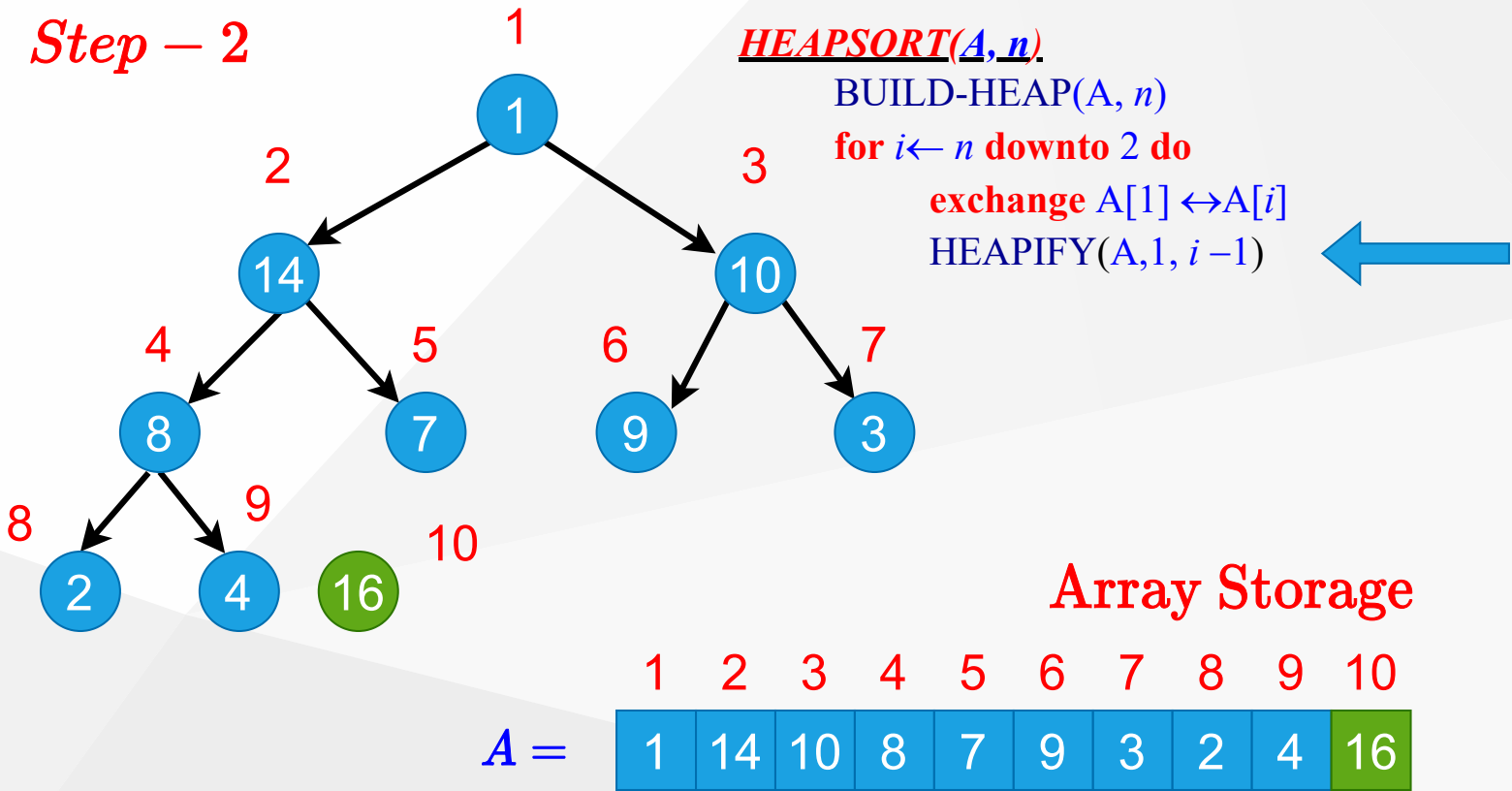
Heapsort Algorithm Steps

- (1) Build a heap on array $A[1 \dots n]$ by calling $BUILD - HEAP(A, n)$
- (2) The largest element is stored at the root $A[1]$
 - Put it into its correct final position $A[n]$ by $A[1] \longleftrightarrow A[n]$
- (3) Discard node n from the heap
- (4) Subtrees ($S2 \& S3$) rooted at children of root remain as heaps, but the new root element may violate the heap property.
 - Make $A[1 \dots n - 1]$ a heap by calling $HEAPIFY(A, 1, n - 1)$
- (5) $n \leftarrow n - 1$
- (6) Repeat steps (2-4) until $n = 2$

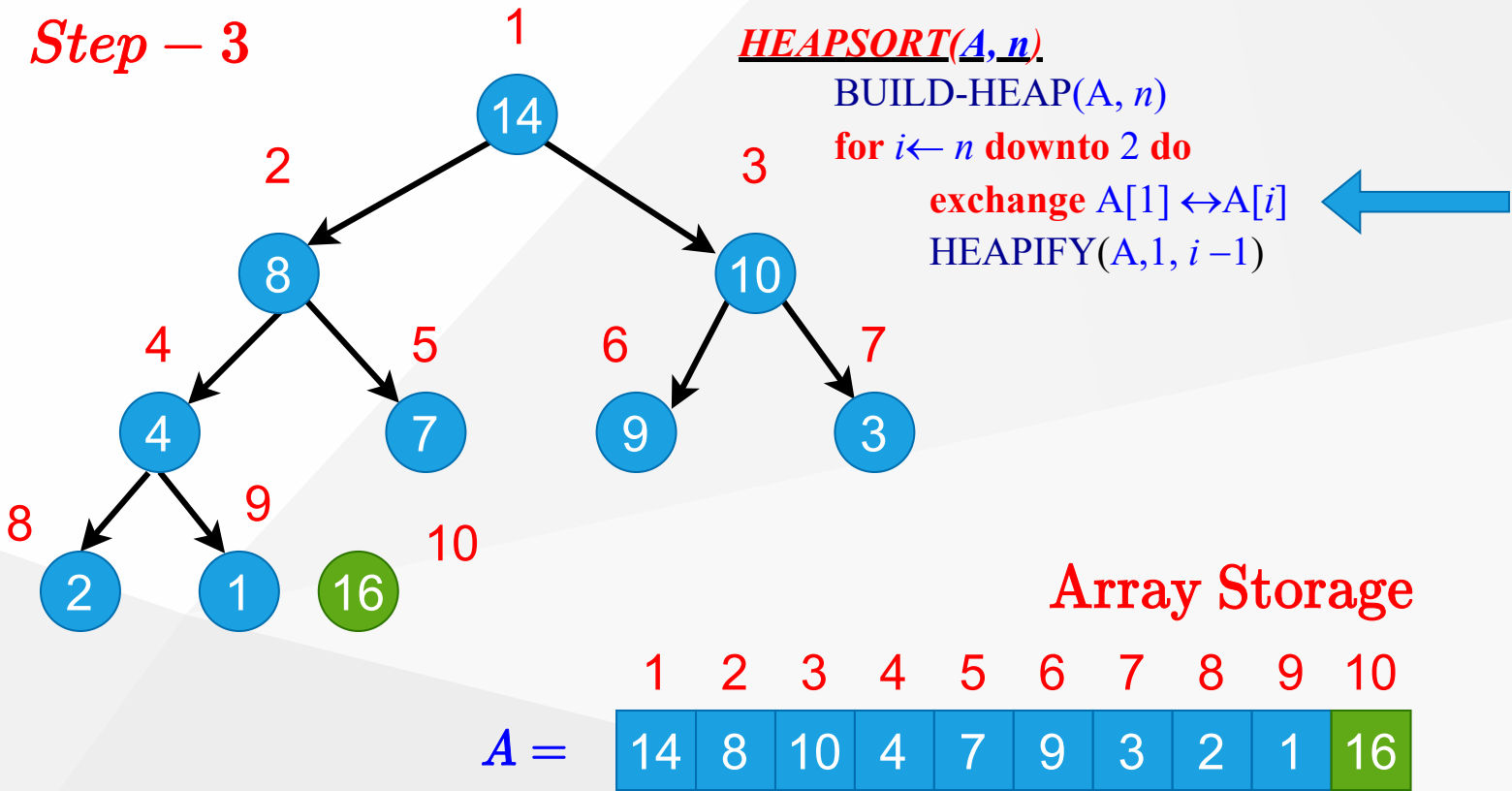
Heapsort
Algorithm
Example
(Step-1)



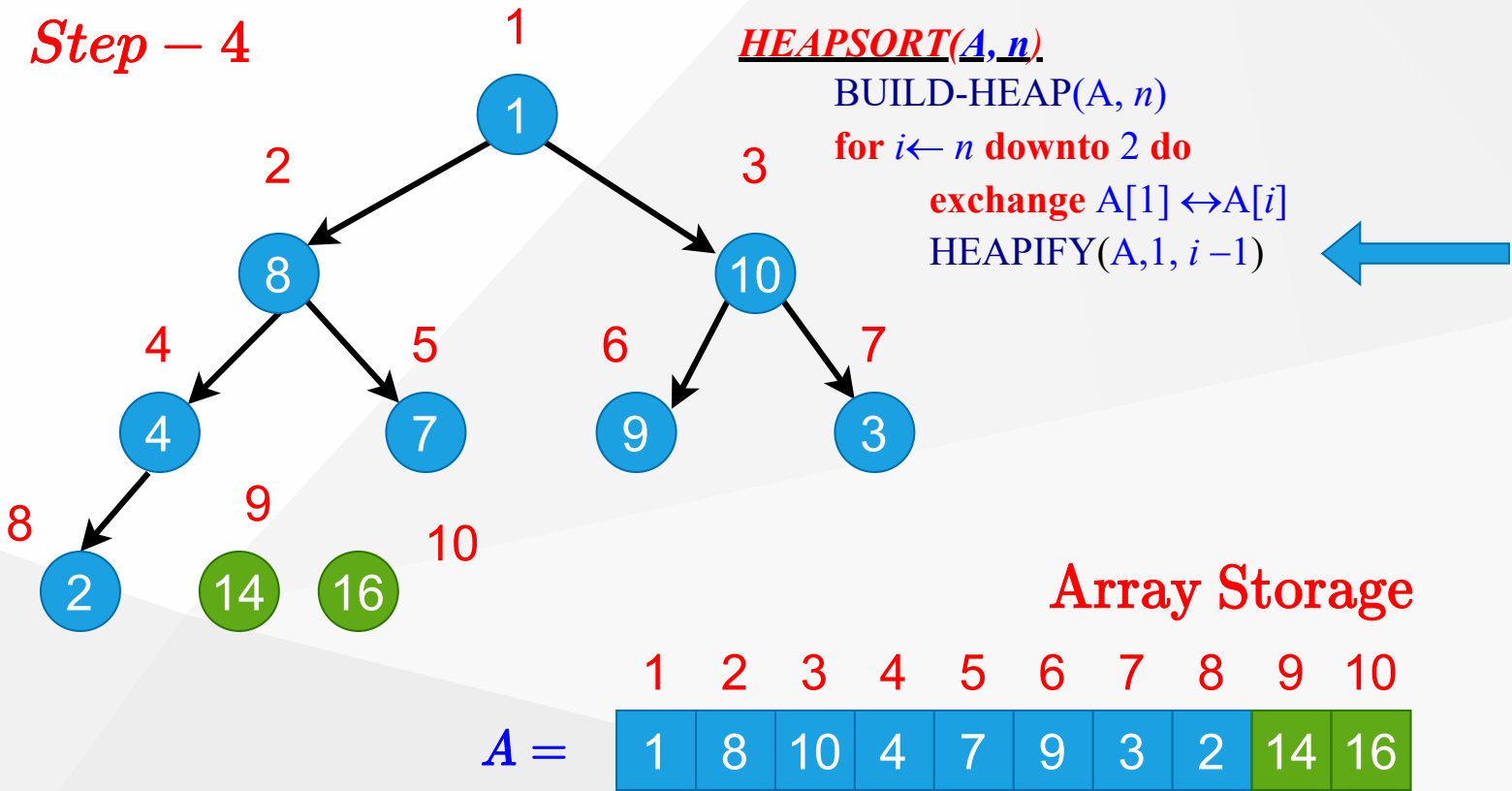
Heapsort Algorithm Example (Step-2)



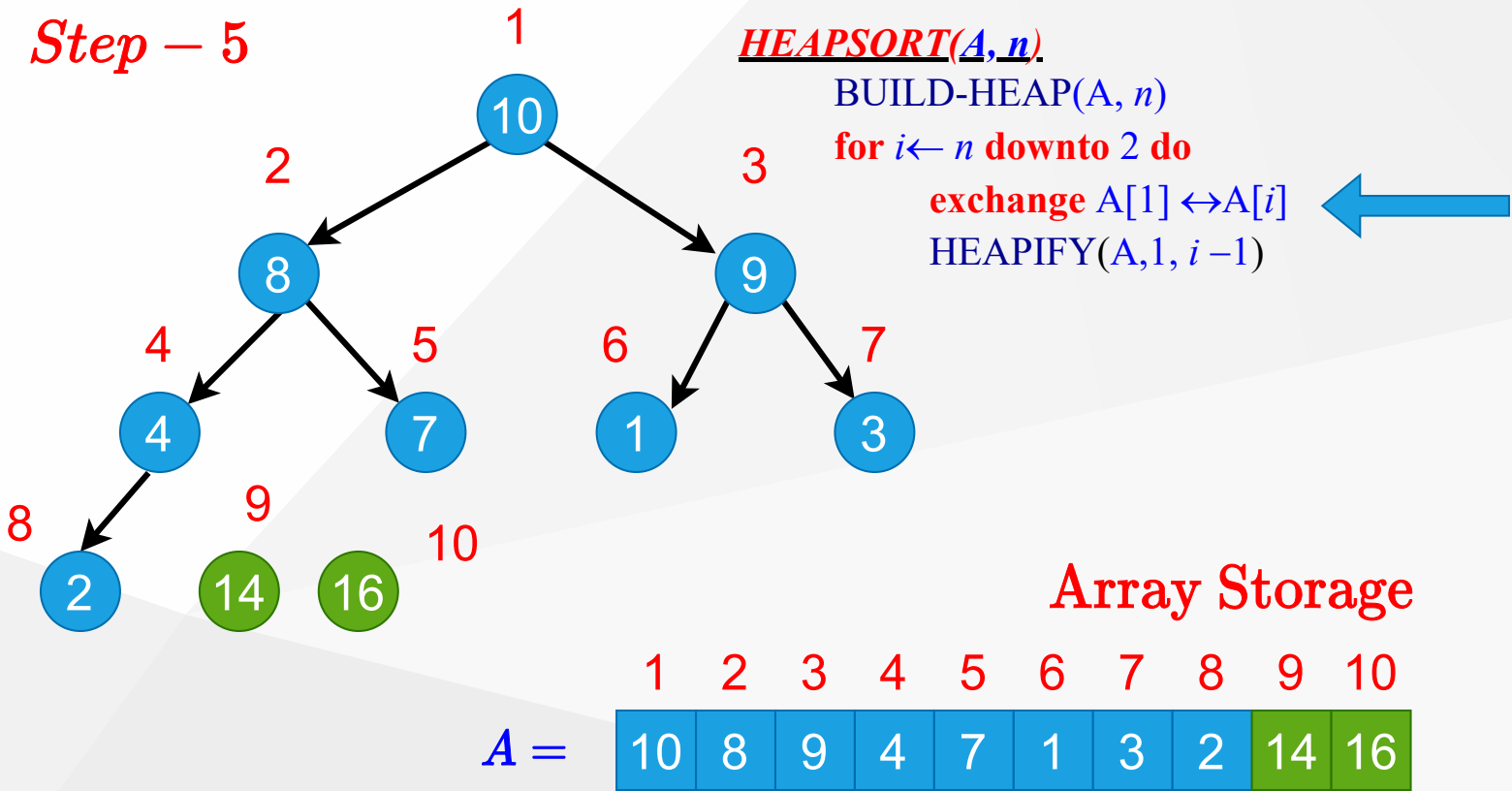
Heapsort Algorithm Example (Step-3)



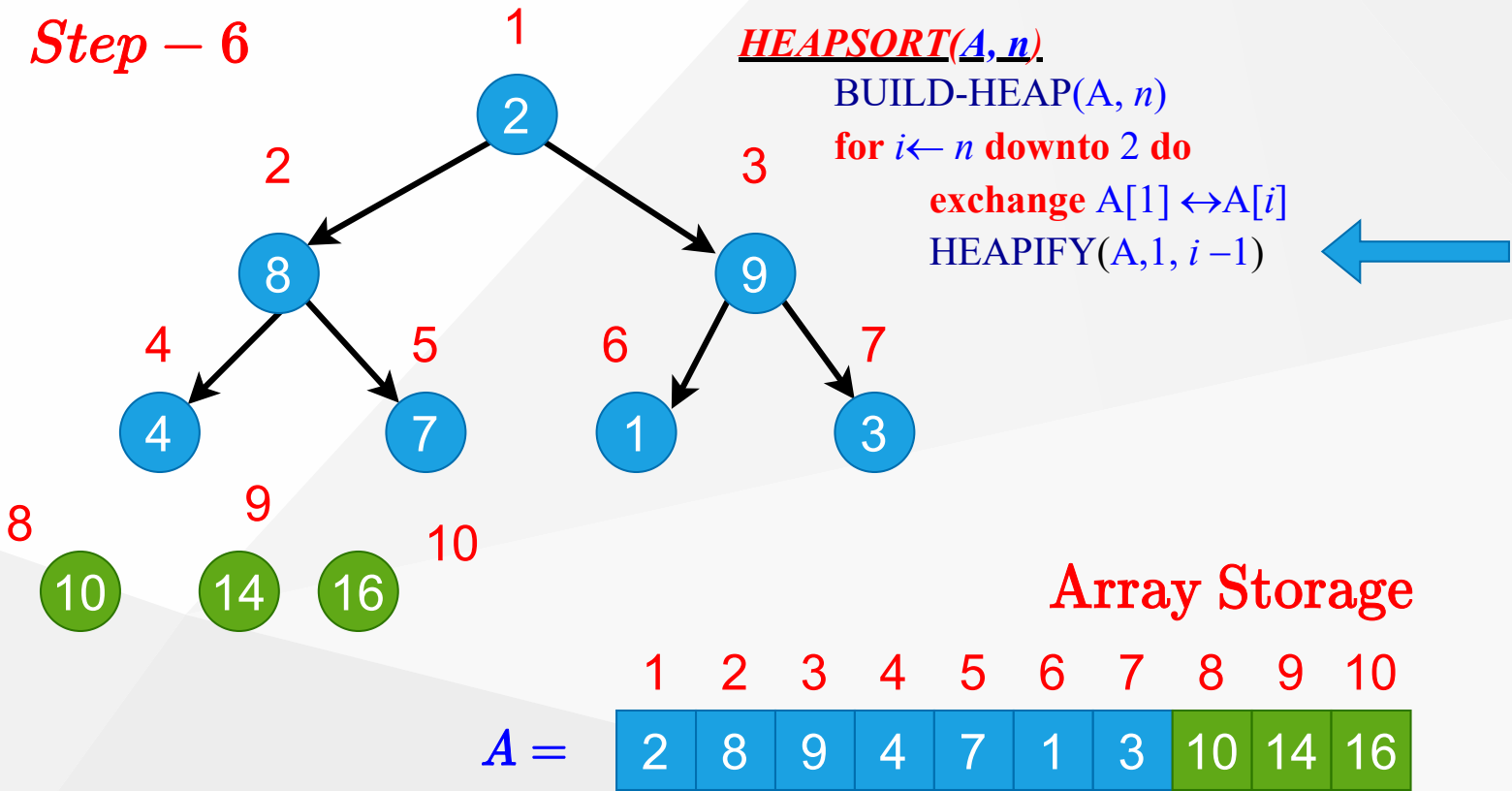
Heapsort Algorithm Example (Step-4)



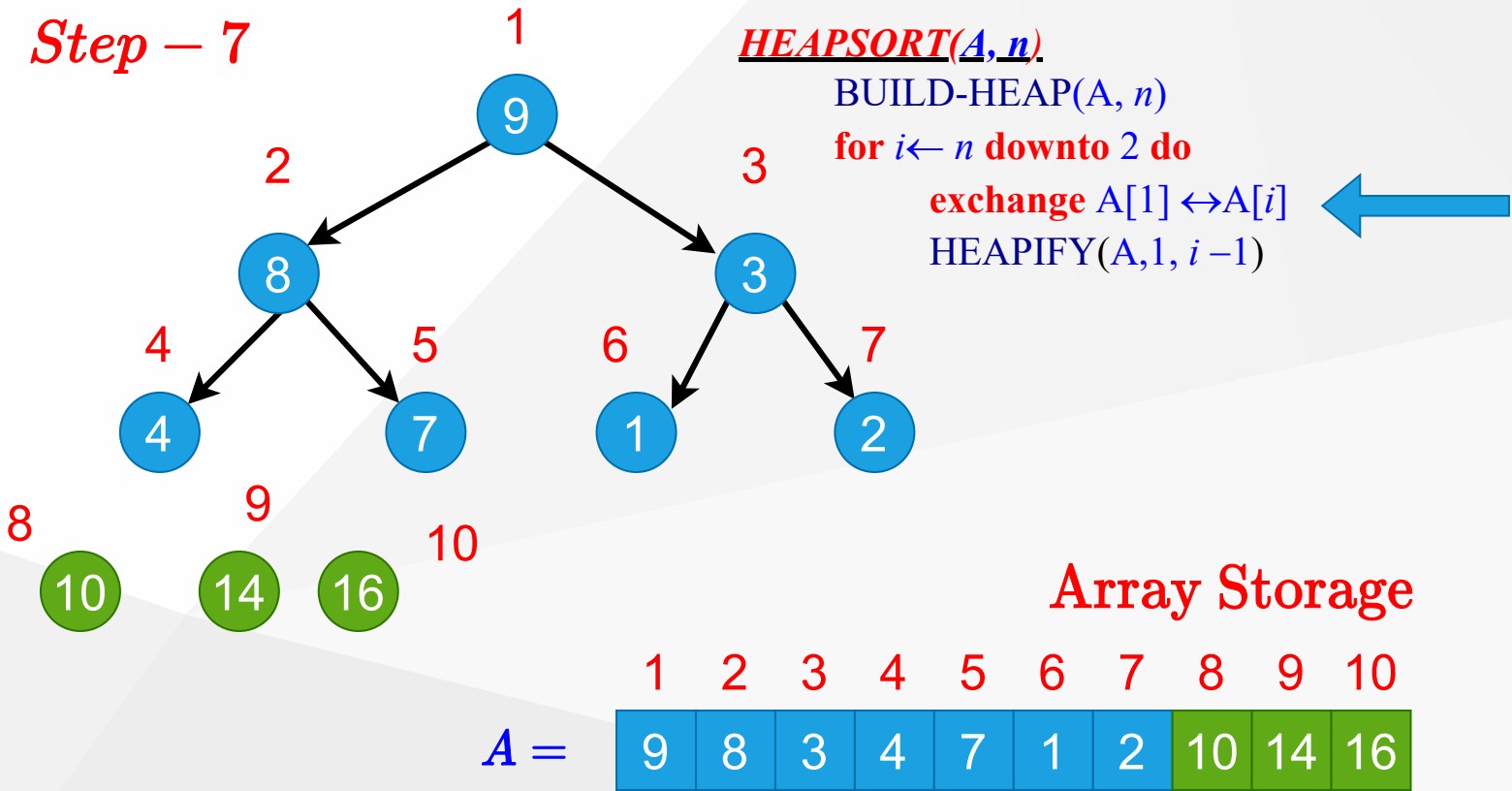
Heapsort Algorithm Example (Step-5)



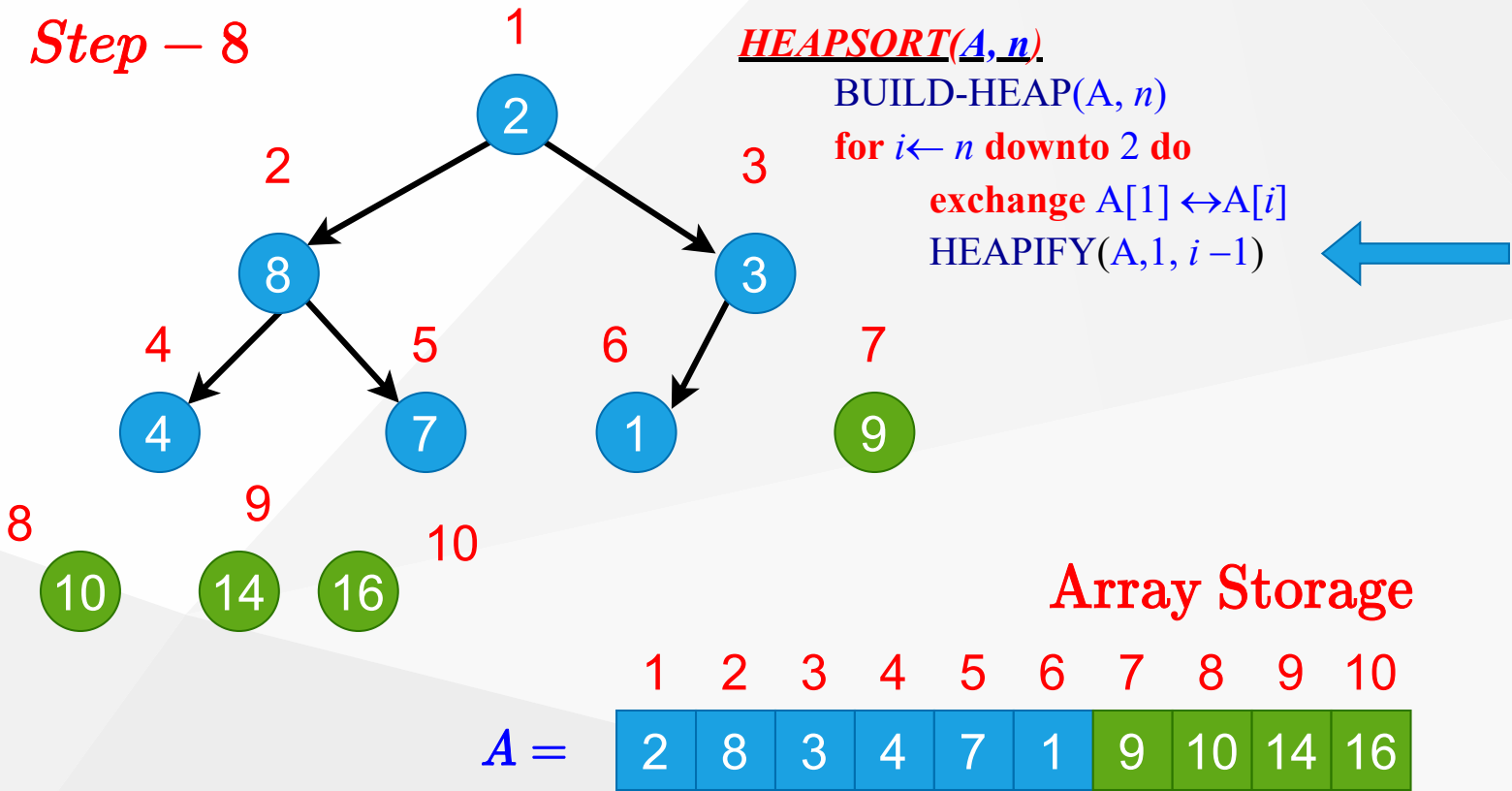
Heapsort
Algorithm
Example
(Step-6)



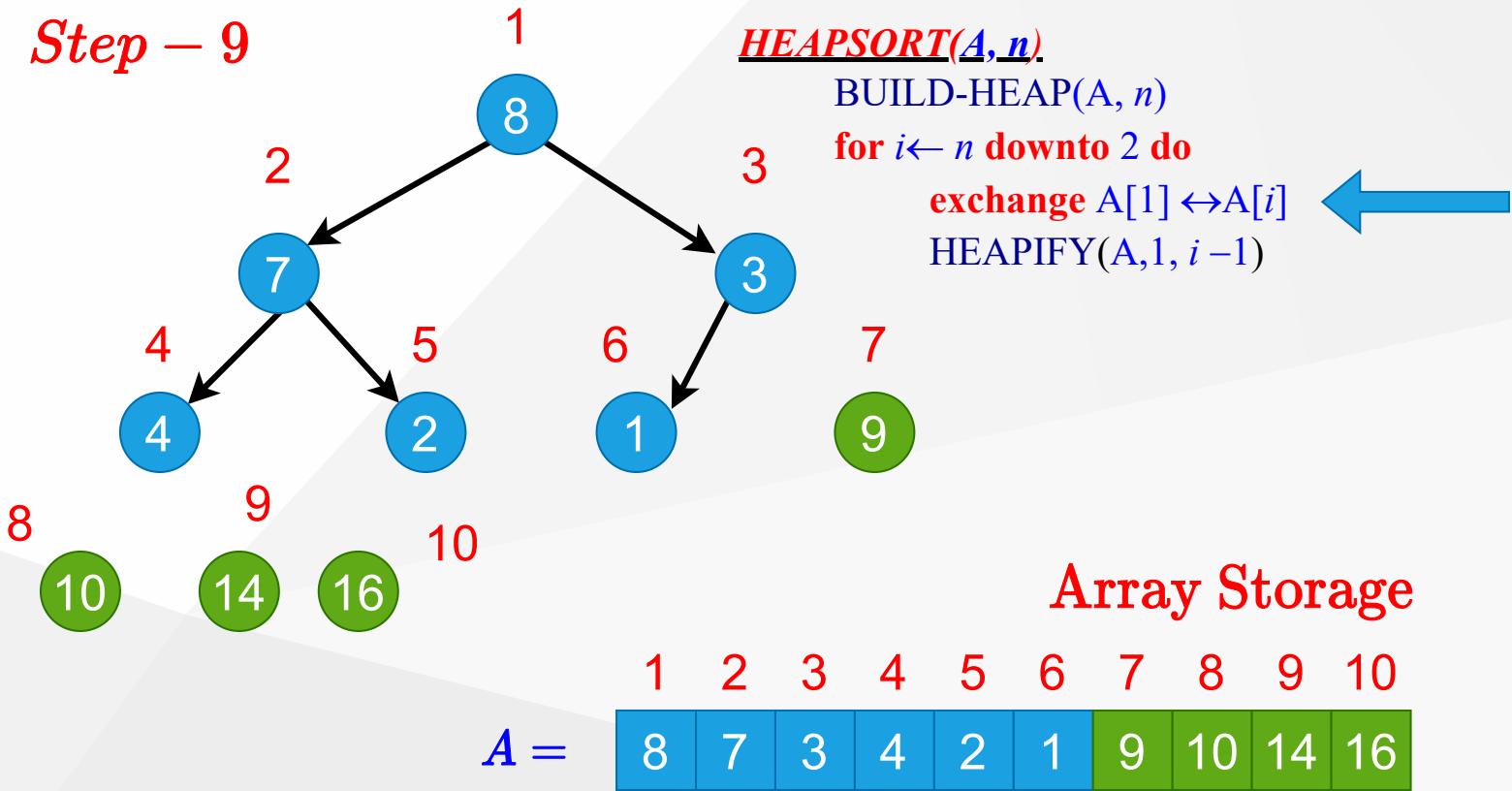
Heapsort Algorithm Example (Step-7)



Heapsort Algorithm Example (Step-8)

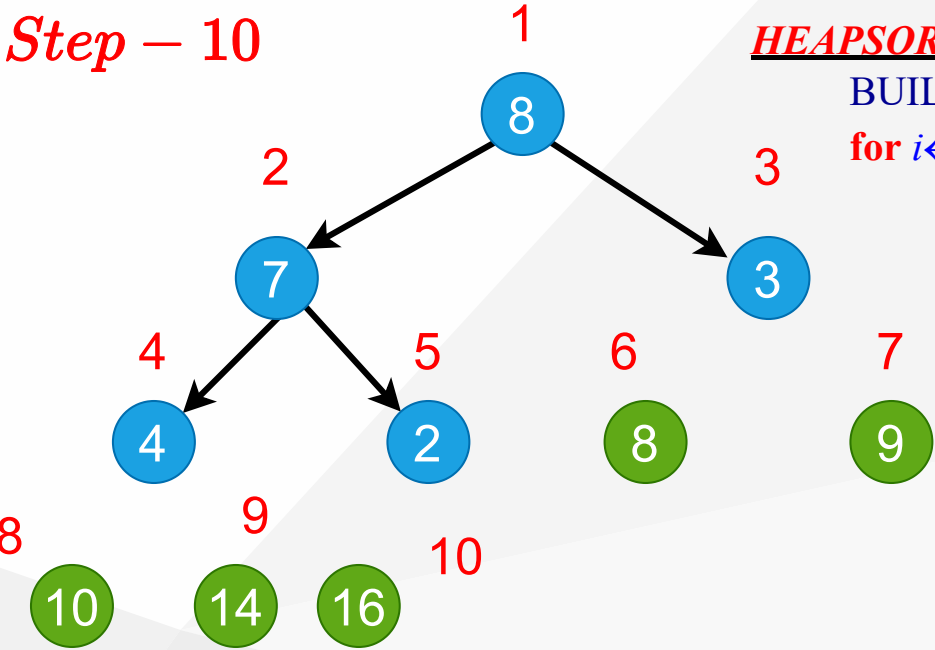


Heapsort
Algorithm
Example
(Step-9)



Heapsort Algorithm Example (Step-10)

Step – 10



```
HEAPSORT(A, n)  
  BUILD-HEAP(A, n)  
  for i ← n downto 2 do  
    exchange A[1] ↔ A[i]  
    HEAPIFY(A, 1, i - 1)
```



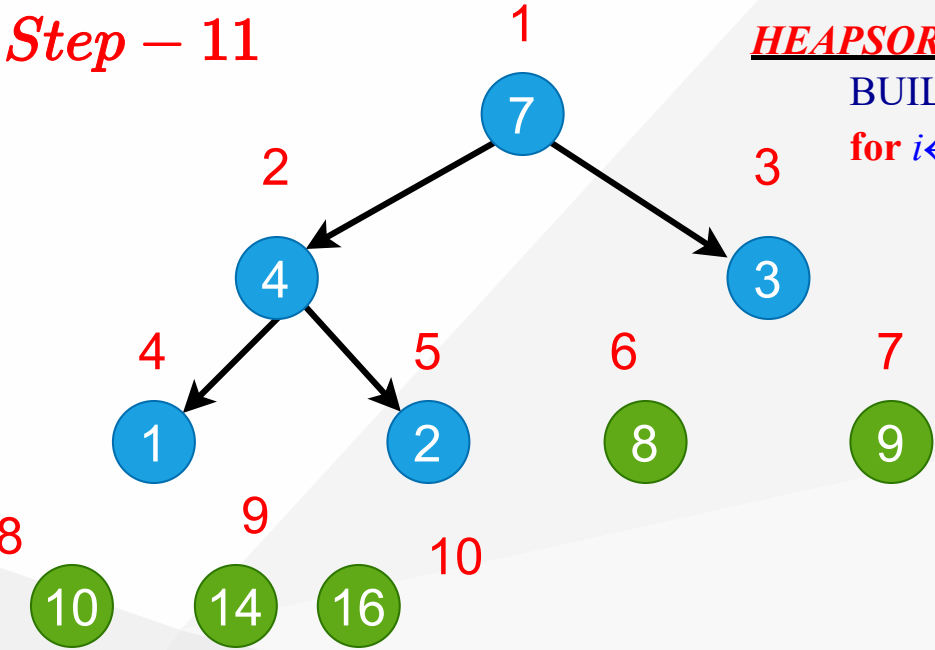
Array Storage

A =

1	2	3	4	5	6	7	8	9	10
1	7	3	4	2	8	9	10	14	16

Heapsort
Algorithm
Example
(Step-11)

Step – 11



```
HEAPSORT(A, n).  
  BUILD-HEAP(A, n)  
  for i ← n downto 2 do  
    exchange A[1] ↔ A[i]  
    HEAPIFY(A, 1, i - 1)
```



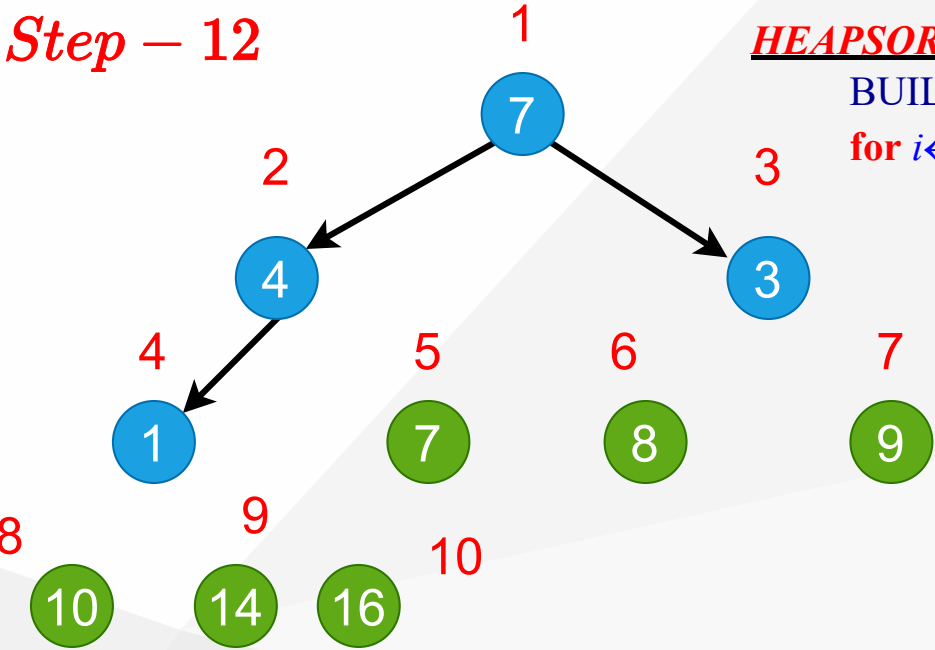
Array Storage

A =

1	2	3	4	5	6	7	8	9	10
7	4	3	1	2	8	9	10	14	16

Heapsort Algorithm Example (Step-12)

Step – 12



```
HEAPSORT(A, n)  
  BUILD-HEAP(A, n)  
  for i ← n downto 2 do  
    exchange A[1] ↔ A[i]  
    HEAPIFY(A, 1, i - 1)
```



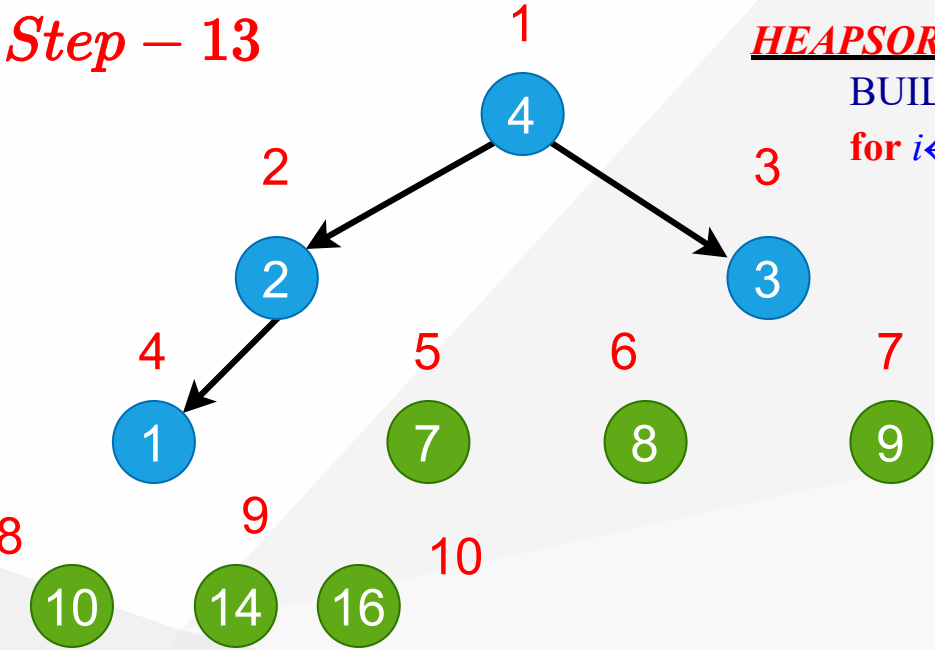
Array Storage

A =

1	2	3	4	5	6	7	8	9	10
2	4	3	1	7	8	9	10	14	16

Heapsort Algorithm Example (Step-13)

Step – 13



```
HEAPSORT(A, n)  
  BUILD-HEAP(A, n)  
  for i ← n downto 2 do  
    exchange A[1] ↔ A[i]  
    HEAPIFY(A, 1, i - 1)
```



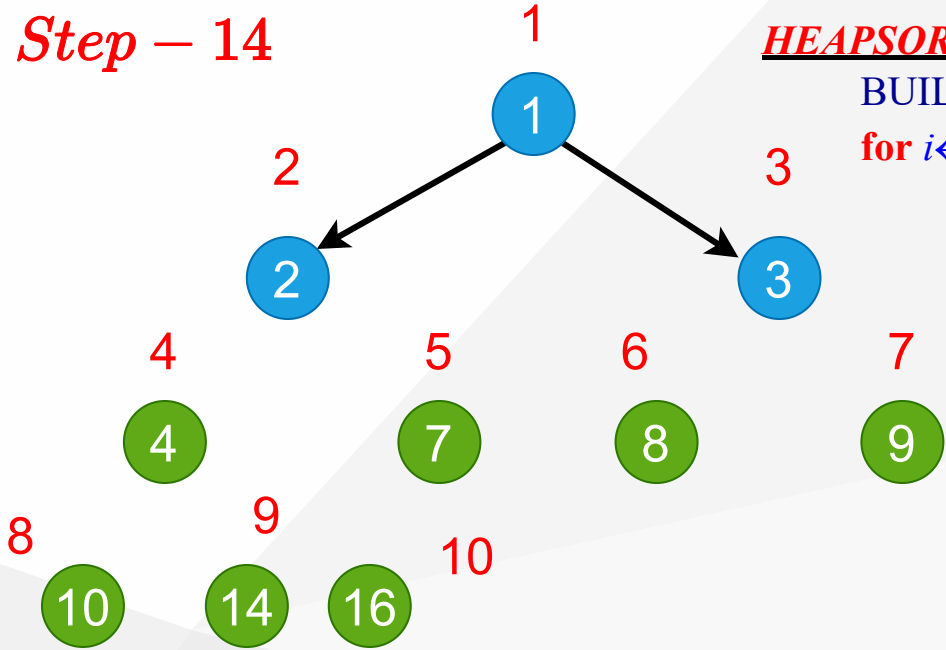
Array Storage

A =

1	2	3	4	5	6	7	8	9	10
4	2	3	1	7	8	9	10	14	16

Heapsort
Algorithm
Example
(Step-14)

Step – 14



```
HEAPSORT(A, n).  
  BUILD-HEAP(A, n)  
  for i ← n downto 2 do  
    exchange A[1] ↔ A[i]  
    HEAPIFY(A, 1, i - 1)
```



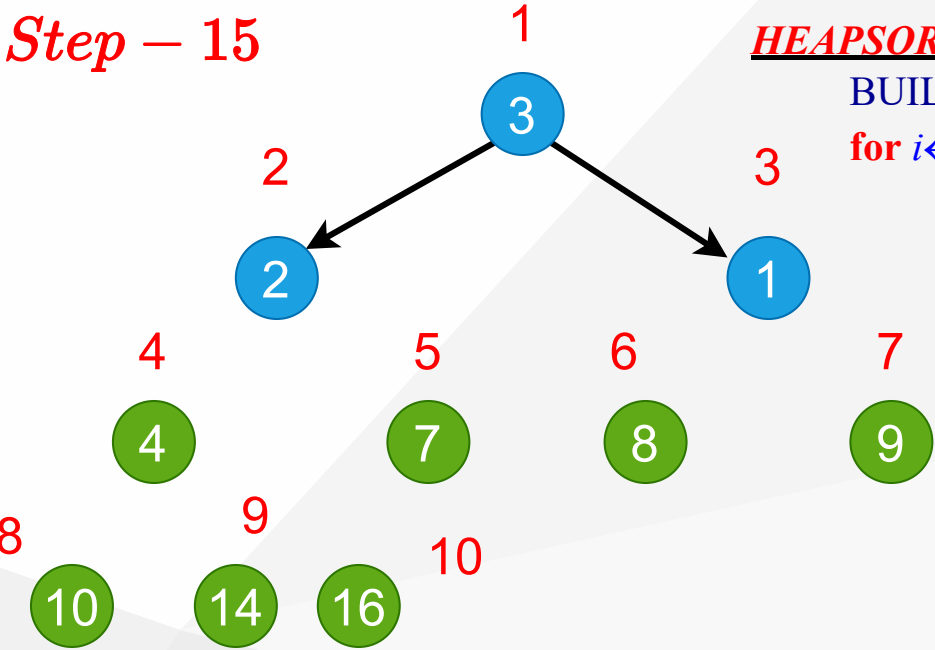
Array Storage

A =

1	2	3	4	5	6	7	8	9	10
1	2	3	4	7	8	9	10	14	16

Heapsort
Algorithm
Example
(Step-15)

Step – 15



```
HEAPSORT(A, n)  
  BUILD-HEAP(A, n)  
  for i ← n downto 2 do  
    exchange A[1] ↔ A[i]  
    HEAPIFY(A, 1, i - 1)
```



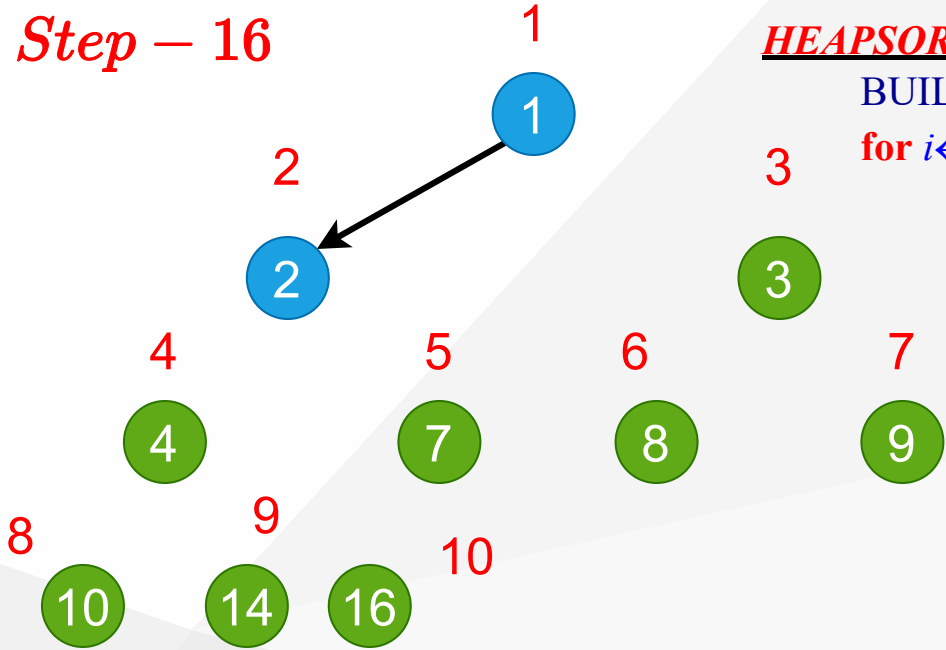
Array Storage

A =

1	2	3	4	5	6	7	8	9	10
3	2	1	4	7	8	9	10	14	16

Heapsort Algorithm Example (Step-16)

Step – 16



```
HEAPSORT(A, n)  
  BUILD-HEAP(A, n)  
  for i ← n downto 2 do  
    exchange A[1] ↔ A[i]  
    HEAPIFY(A, 1, i - 1)
```



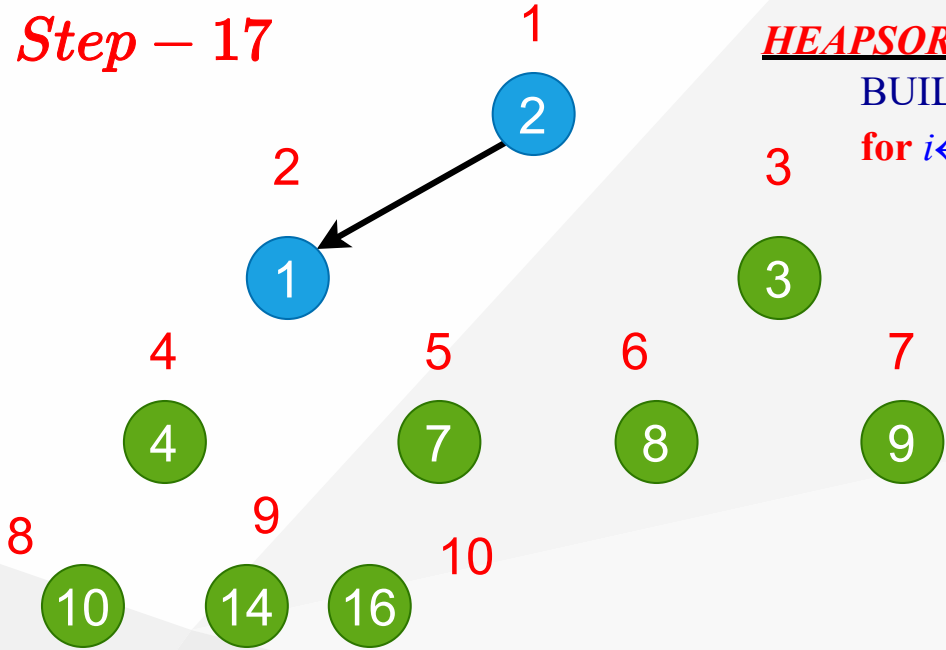
Array Storage

A =

1	2	3	4	5	6	7	8	9	10
1	2	3	4	7	8	9	10	14	16

Heapsort Algorithm Example (Step-17)

Step – 17



```
HEAPSORT(A, n)  
  BUILD-HEAP(A, n)  
  for i ← n downto 2 do  
    exchange A[1] ↔ A[i]  
    HEAPIFY(A, 1, i - 1)
```

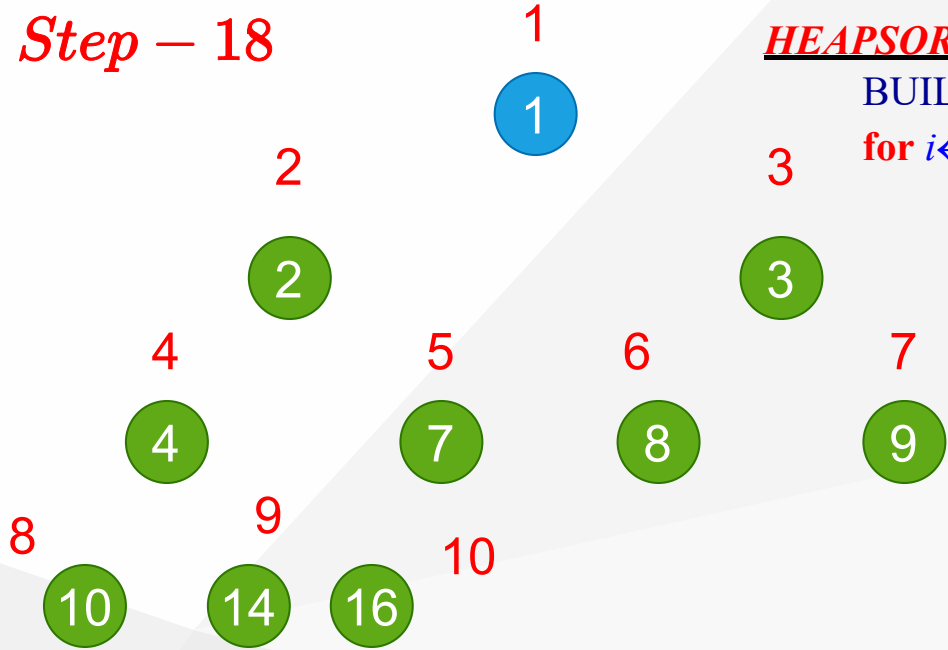
Array Storage

A =

1	2	3	4	5	6	7	8	9	10
2	1	3	4	7	8	9	10	14	16

Heapsort Algorithm Example (Step-18)

Step – 18



```
HEAPSORT(A, n)  
  BUILD-HEAP(A, n)  
  for i ← n downto 2 do  
    exchange A[1] ↔ A[i]  
    HEAPIFY(A, 1, i - 1)
```



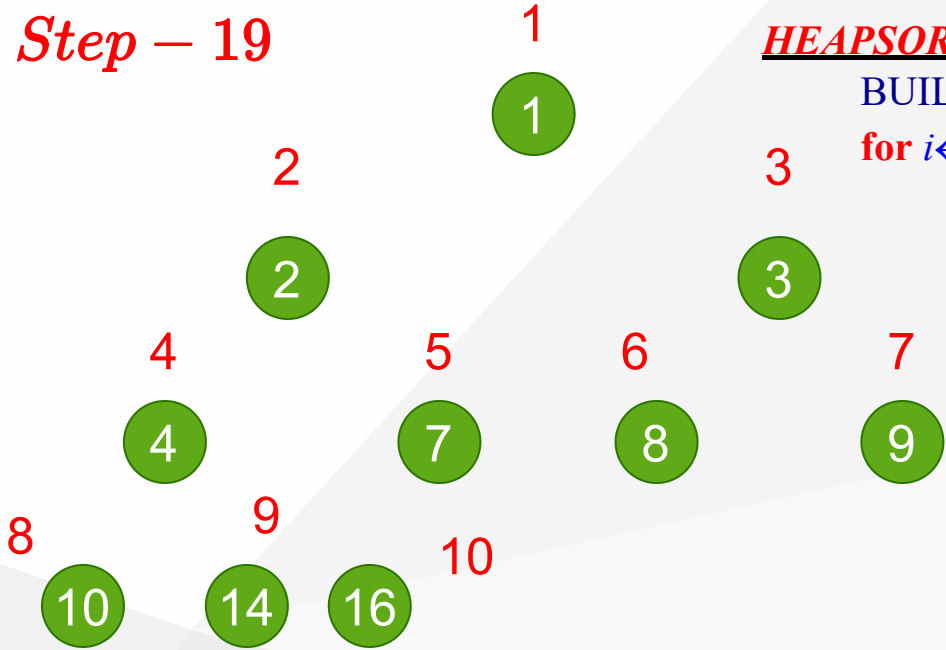
Array Storage

A =

1	2	3	4	5	6	7	8	9	10
1	2	3	4	7	8	9	10	14	16

Heapsort Algorithm Example (Step-19)

Step – 19



```
HEAPSORT(A, n)  
  BUILD-HEAP(A, n)  
  for i ← n downto 2 do  
    exchange A[1] ↔ A[i]  
    HEAPIFY(A, 1, i - 1)
```



Array Storage

A =

1	2	3	4	5	6	7	8	9	10
1	2	3	4	7	8	9	10	14	16

Heapsort Algorithm: Runtime Analysis

HEAPSORT(*A*, *n*)

BUILD-HEAP(*A*, *n*) $\Theta(n)$

for $i \leftarrow n$ **downto** 2 **do**

exchange $A[1] \leftrightarrow A[i]$ $\Theta(1)$

 HEAPIFY(*A*, 1, $i - 1$) $O(\lg(i - 1))$

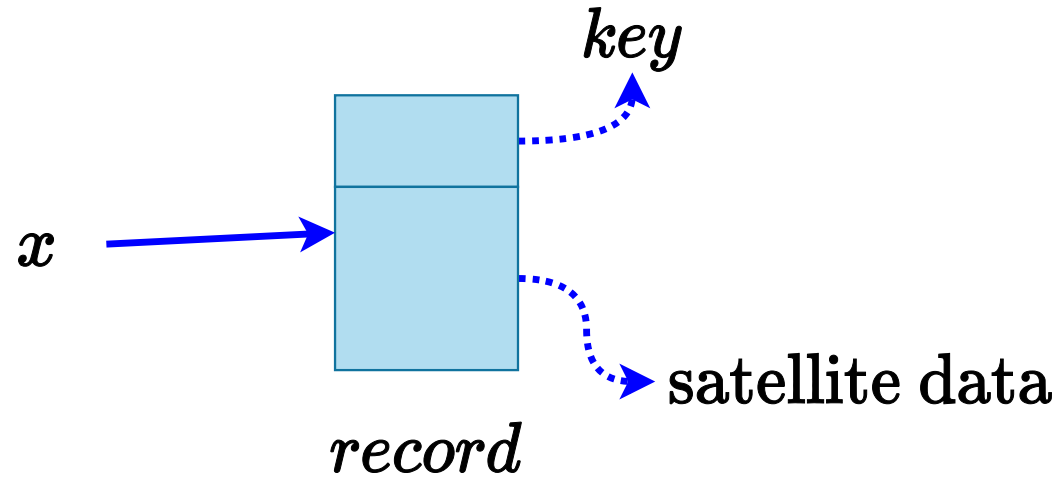
$$\begin{aligned}
 T(n) &= \Theta(n) + \sum_{i=2}^n O(\lg i) \\
 &= \Theta(n) + O\left(\sum_{i=2}^n O(\lg n)\right) \\
 &= O(n \lg n)
 \end{aligned}$$

Heapsort - Notes

- **Heapsort** is a very good algorithm but, a good implementation of **quicksort** always beats heapsort in practice
- However, **heap data structure** has many popular applications, and it can be efficiently used for implementing **priority queues**

Data structures for **Dynamic Sets**

- Consider sets of records having **key** and **satellite data**



Operations on **Dynamic Sets**

- **Queries:** Simply return info;
 - $MAX(S)/MIN(S)$: (Query) return $x \in S$ with the **largest/smallest key**
 - $SEARCH(S, k)$: (Query) return $x \in S$ with $key[x] = k$
 - $SUCCESSOR(S, x)/PREDECESSOR(S, x)$: (Query) return $y \in S$ which is the next **larger/smaller** element after x
- **Modifying operations:** Change the set
 - $INSERT(S, x)$: (Modifying) $S \leftarrow S \cup \{x\}$
 - $DELETE(S, x)$: (Modifying) $S \leftarrow S - \{x\}$
 - $EXTRACT-MAX(S)/EXTRACT-MIN(S)$: (Modifying) return and delete $x \in S$ with the largest/smallest **key**
- Different data structures support/optimize different operations

Priority Queues (PQ)

- Supports
 - *INSERT*
 - *MAX/MIN*
 - *EXTRACT-MAX/EXTRACT-MIN*

Priority Queues (PQ)

- **One application:** Schedule jobs on a shared resource
 - PQ keeps track of jobs and their relative priorities
 - When a job is finished or interrupted, highest priority job is selected from those pending using **EXTRACT-MAX**
 - A new job can be added at any time using *INSERT*

Priority Queues (PQ)

- **Another application:** Event-driven simulation
 - Events to be simulated are the items in the **PQ**
 - Each event is associated with a time of occurrence which serves as a *key*
 - Simulation of an event can cause other events to be simulated in the future
 - Use **EXTRACT-MIN** at each step to choose the next event to simulate
 - As new events are produced insert them into the **PQ** using *INSERT*

Implementation of **Priority Queue**

- **Sorted linked list:** Simplest implementation
 - *INSERT*
 - $O(n)$ time
 - Scan the list to find place and splice in the new item
 - **EXTRACT-MAX**
 - $O(1)$ time
 - Take the first element
 - **Fast** extraction but **slow** insertion.

Implementation of Priority Queue

- Unsorted linked list: Simplest implementation
 - *INSERT*
 - $O(1)$ time
 - Put the new item at front
 - **EXTRACT-MAX**
 - $O(n)$ time
 - Scan the whole list
 - **Fast** insertion but **slow** extraction.
- Sorted linked list is better on the average
 - **Sorted list:** on the average, scans $n/2$ element per insertion
 - **Unsorted list:** always scans n element at each extraction

Heap Implementation of PQ

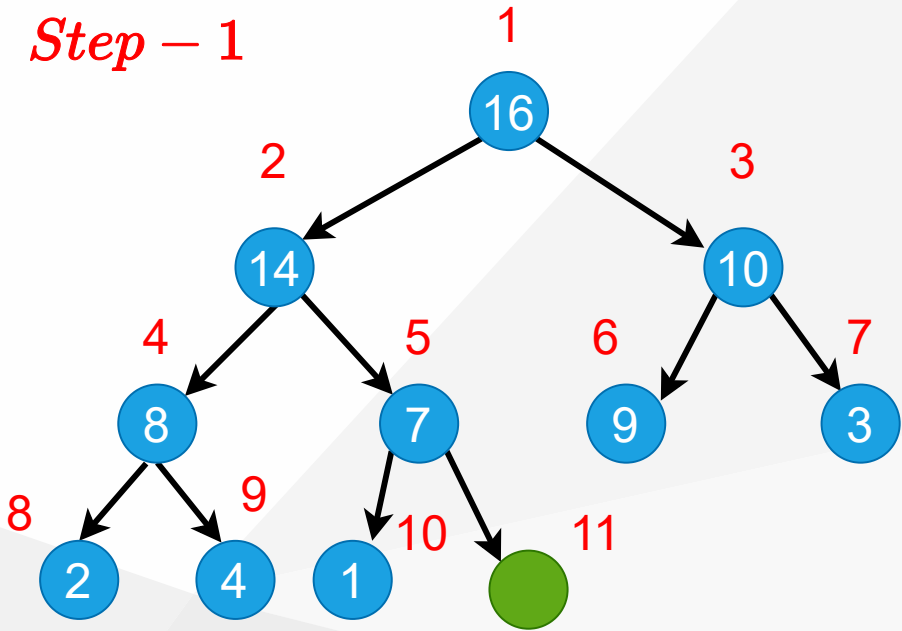
- *INSERT* and *EXTRACT-MAX* are both $O(\lg n)$
 - good compromise between fast insertion but slow extraction and vice versa
- *EXTRACT-MAX*: already discussed *HEAP-EXTRACT-MAX*
- *INSERT*: Insertion is like that of Insertion-Sort.

```
HEAP-INSERT(A, key, n)
  n = n+1
  i=n
  while i>1 and A[floor(i/2)] < key do
    A[i]=A[floor(i/2)]
    i= floor(i/2)
  A[i]=key
```

Heap Implementation of PQ

- Traverses $O(\lg n)$ nodes, as *HEAPIFY* does but makes fewer comparisons and assignments
 - *HEAPIFY*: compares parent with both children
 - *HEAP – INSERT*: with only one

HEAP-INSERT
Example
(Step-1)

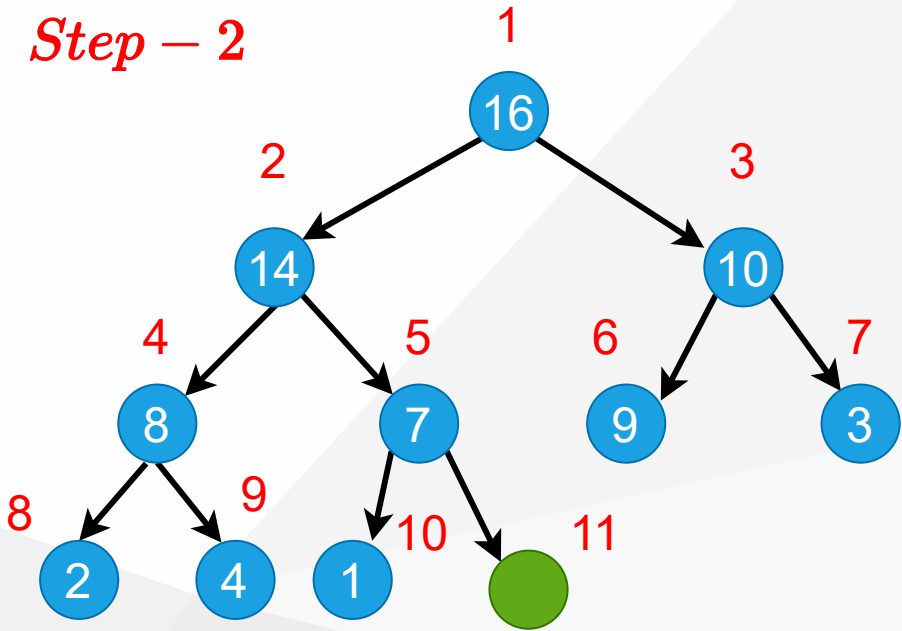


```
HEAP-INSERT(A, key, n).  
n ← n+1  
i ← n  
while i > 1 and A[⌊i/2⌋] < key do  
    A[i] ← A[⌊i/2⌋]  
    i ← ⌊i/2⌋  
A[i] ← key  
HEAP-INSERT(A, 15)  
key=15
```

Array Storage

1	2	3	4	5	6	7	8	9	10	11
16	14	10	8	7	9	3	2	4	1	

HEAP-INSERT
Example
(Step-2)



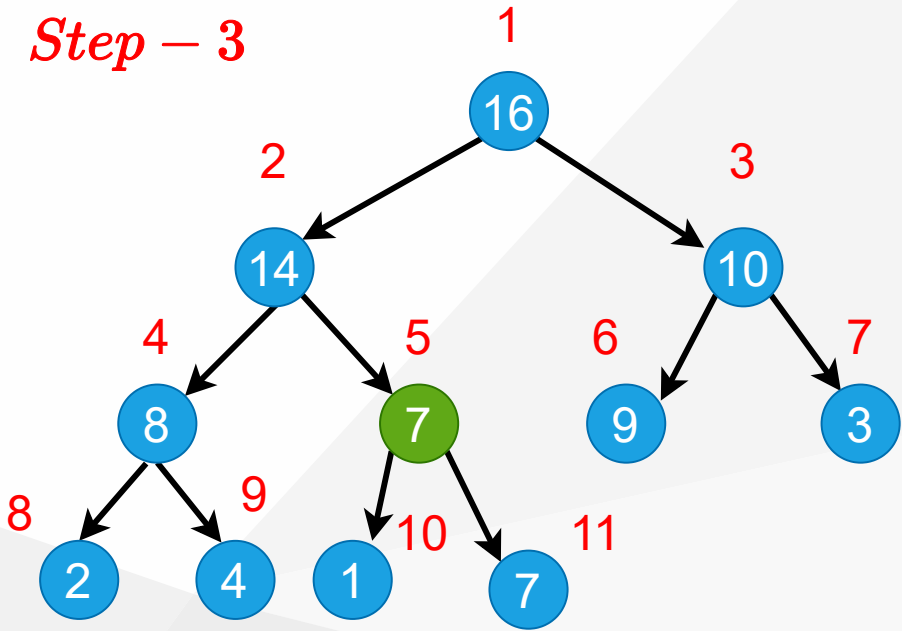
```
HEAP-INSERT(A, key, n).  
n ← n+1  
i ← n  
while i > 1 and A[⌊i/2⌋] < key do  
    A[i] ← A[⌊i/2⌋]  
    i ← ⌊i/2⌋  
A[i] ← key
```

HEAP-INSERT(A, 15)
key=15

Array Storage

	1	2	3	4	5	6	7	8	9	10	11
A =	16	14	10	8	7	9	3	2	4	1	

HEAP-INSERT
Example
(Step-3)

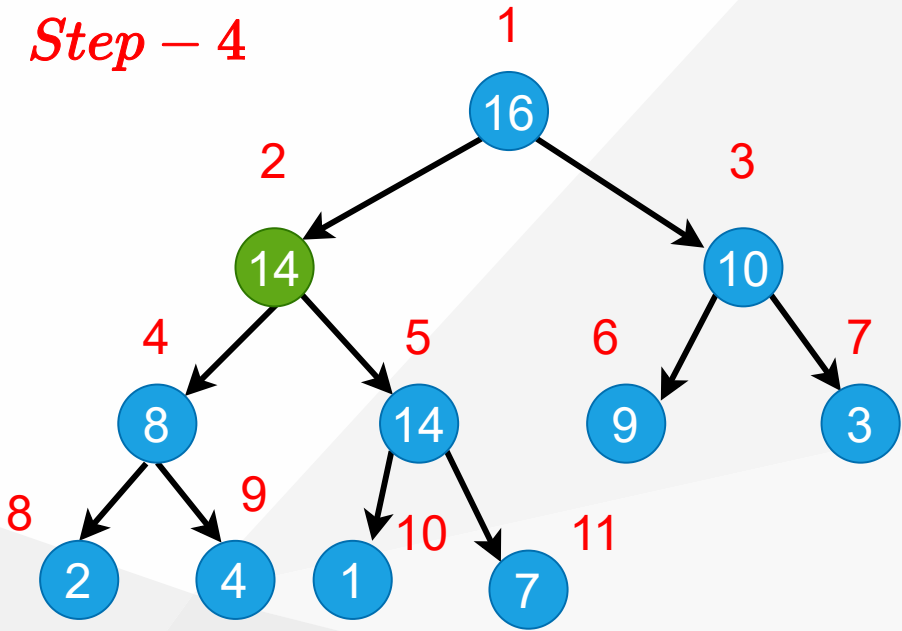


```
HEAP-INSERT(A, key, n).  
  
n ← n+1  
i ← n  
while i > 1 and A[⌊i/2⌋] < key do  
    A[i] ← A[⌊i/2⌋]  
    i ← ⌊i/2⌋  
A[i] ← key  
  
HEAP-INSERT(A, 15)  
key=15
```

Array Storage

	1	2	3	4	5	6	7	8	9	10	11
A =	16	14	10	8	7	9	3	2	4	1	7

HEAP-INSERT
Example
(Step-4)



```
HEAP-INSERT(A, key, n).  
  
n ← n+1  
i ← n  
  
while i > 1 and A[⌊i/2⌋] < key do  
    A[i] ← A[⌊i/2⌋]  
    i ← ⌊i/2⌋  
A[i] ← key
```

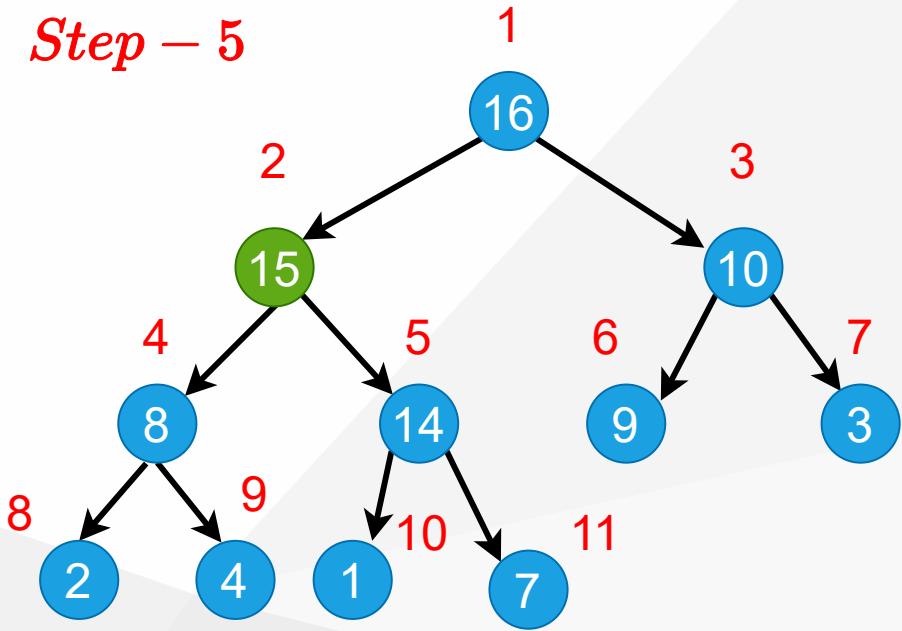
HEAP-INSERT(A, 15)
key=15

Array Storage

	1	2	3	4	5	6	7	8	9	10	11
A =	16	14	10	8	14	9	3	2	4	1	7

HEAP-INSERT
Example
(Step-5)

Step – 5



```
HEAP-INSERT(A, key, n).  
  
n ← n+1  
i ← n  
  
while i > 1 and A[⌊i/2⌋] < key do  
    A[i] ← A[⌊i/2⌋]  
    i ← ⌊i/2⌋  
  
A[i] ← key  
  
HEAP-INSERT(A, 15)  
key=15
```

Array Storage

1	2	3	4	5	6	7	8	9	10	11
16	15	10	8	14	9	3	2	4	1	7

Heap Increase Key

- Key value of i^{th} element of heap is increased from $A[i]$ to key

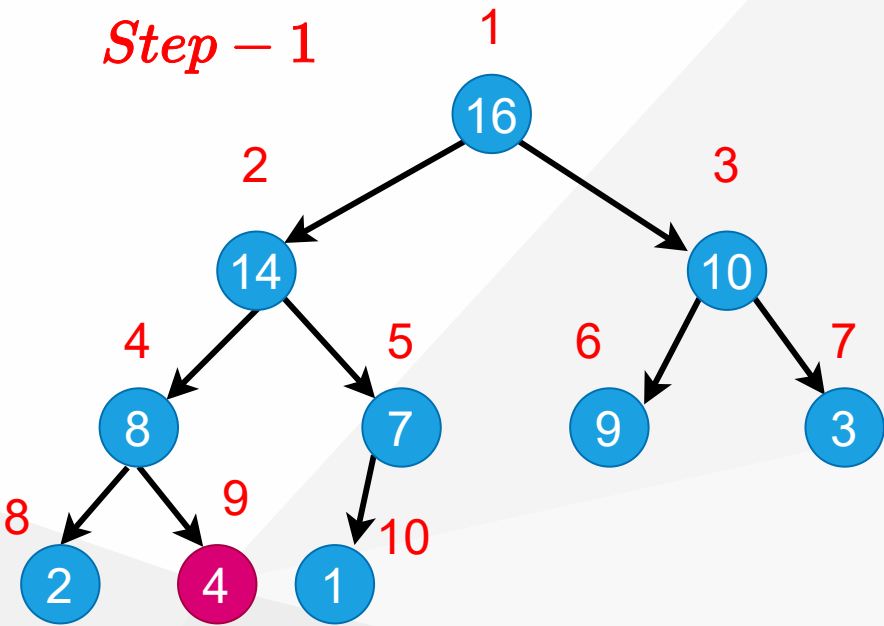
```
HEAP-INCREASE-KEY(A, i, key)
```

```
  if key < A[i] then  
    return error
```

```
  while i > 1 and A[floor(i/2)] < key do  
    A[i] = A[floor(i/2)]  
    i = floor(i/2)
```

```
  A[i] = key
```

HEAP-INCREASE-KEY Example (Step-1)



HEAP-INCREASE-KEY(A, i, key)

```
if key < A[i] then
  return error
while i > 1 and A[⌊i/2⌋] < key do
  A[i] ← A[⌊i/2⌋]
  i ← ⌊i/2⌋
A[i] ← key
```

HEAP-INCREASE-KEY(A, 9, 15)

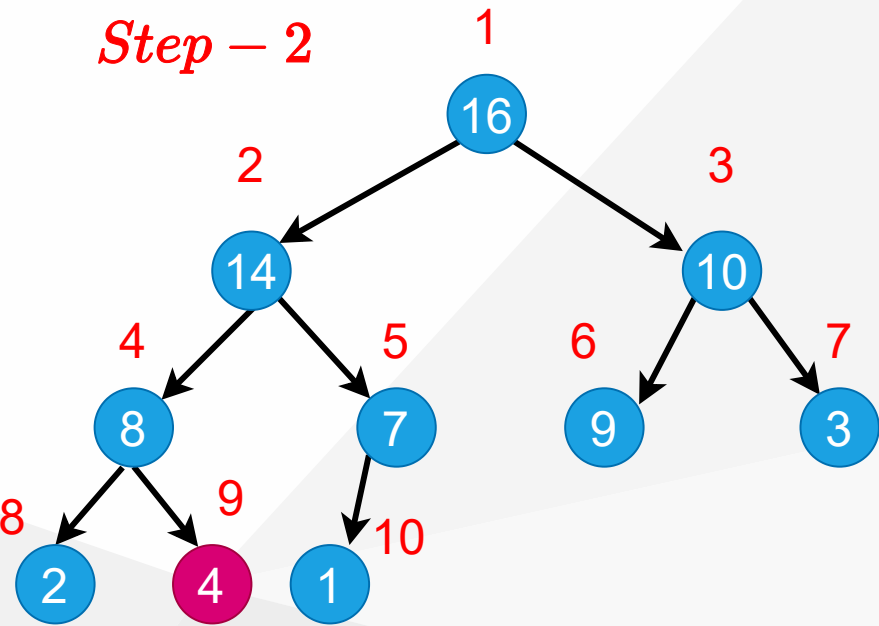
key=15

Array Storage

A =

1	2	3	4	5	6	7	8	9	10
16	14	10	8	7	9	3	2	4	1

HEAP-INCREASE-KEY Example (Step-2)



HEAP-INCREASE-KEY(A, i, key)

```
if key < A[i] then
  return error
while i > 1 and A[⌊i/2⌋] < key do
  A[i] ← A[⌊i/2⌋]
  i ← ⌊i/2⌋
A[i] ← key
```

HEAP-INCREASE-KEY(A, 9, 15)

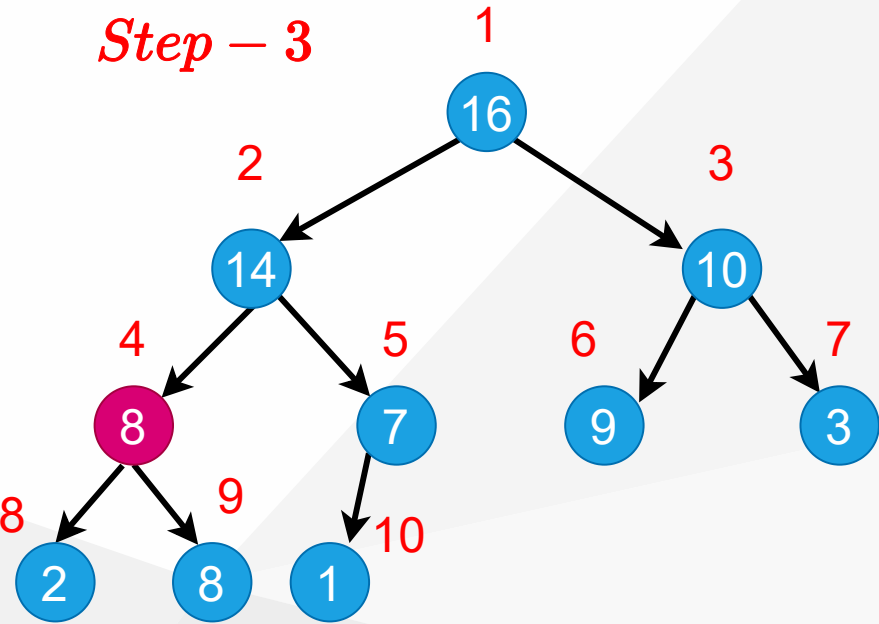
key=15

Array Storage

A =

1	2	3	4	5	6	7	8	9	10
16	14	10	8	7	9	3	2	4	1

HEAP-INCREASE-KEY Example (Step-3)



HEAP-INCREASE-KEY(A, i, key)

```
if key < A[i] then
  return error
while i > 1 and A[⌊i/2⌋] < key do
  A[i] ← A[⌊i/2⌋]
  i ← ⌊i/2⌋
A[i] ← key
```

HEAP-INCREASE-KEY(A, 9, 15)

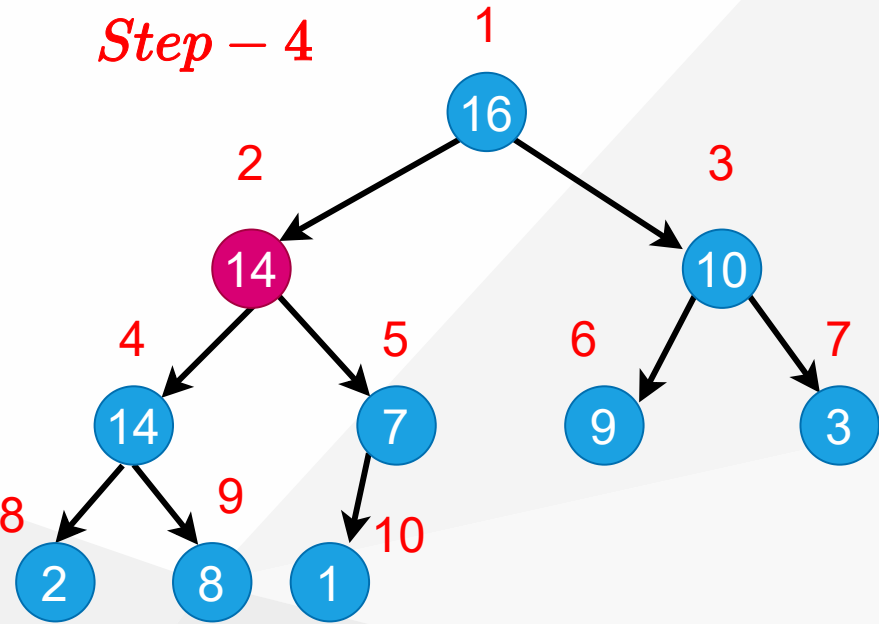
key=15

Array Storage

A =

1	2	3	4	5	6	7	8	9	10
16	14	10	8	7	9	3	2	8	1

HEAP-INCREASE-KEY Example (Step-4)

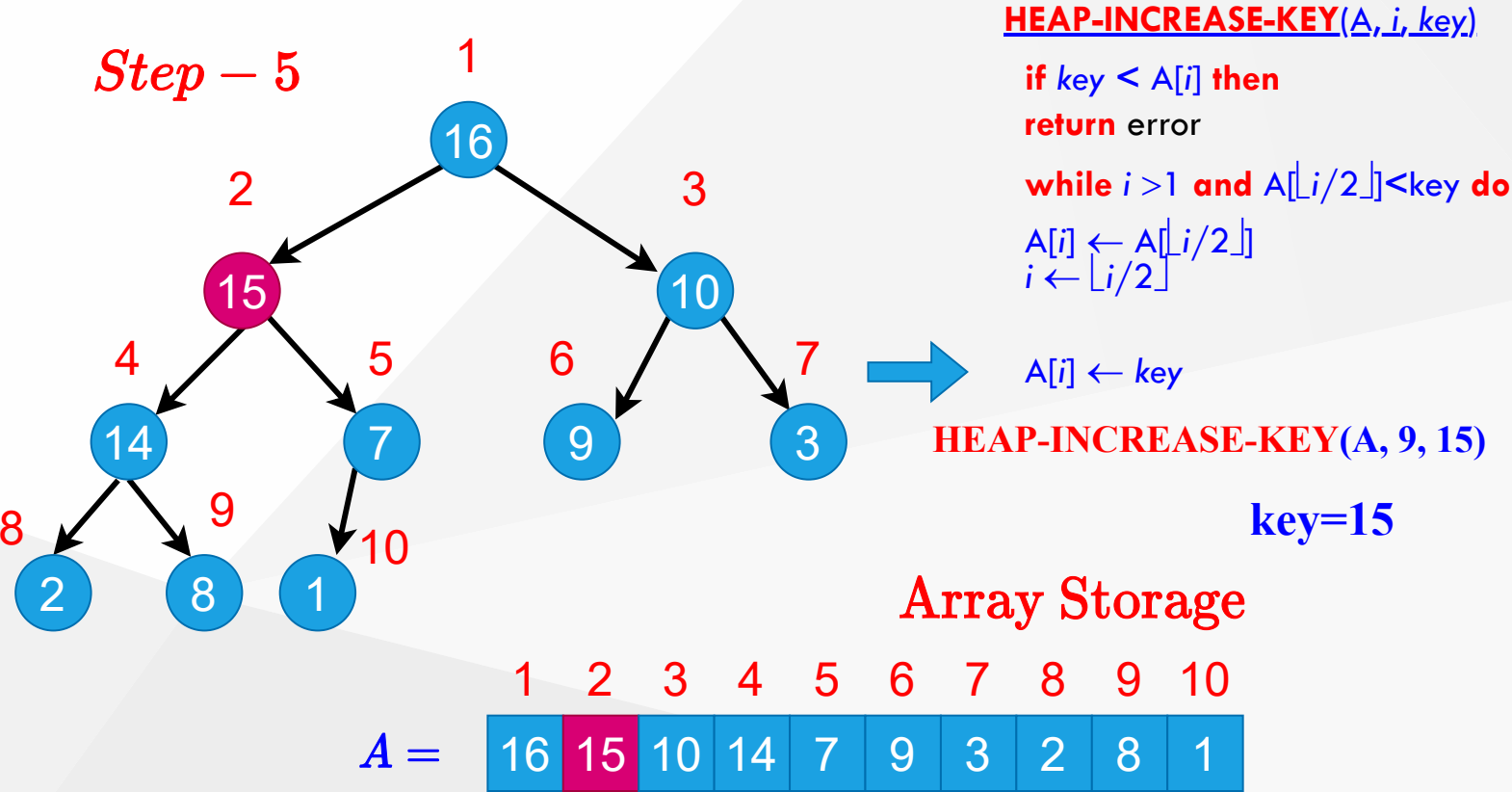


```
HEAP-INCREASE-KEY(A, i, key)
  if key < A[i] then
    return error
  while i > 1 and A[⌊i/2⌋] < key do
    A[i] ← A[⌊i/2⌋]
    i ← ⌊i/2⌋
  A[i] ← key
HEAP-INCREASE-KEY(A, 9, 15)
key=15
```

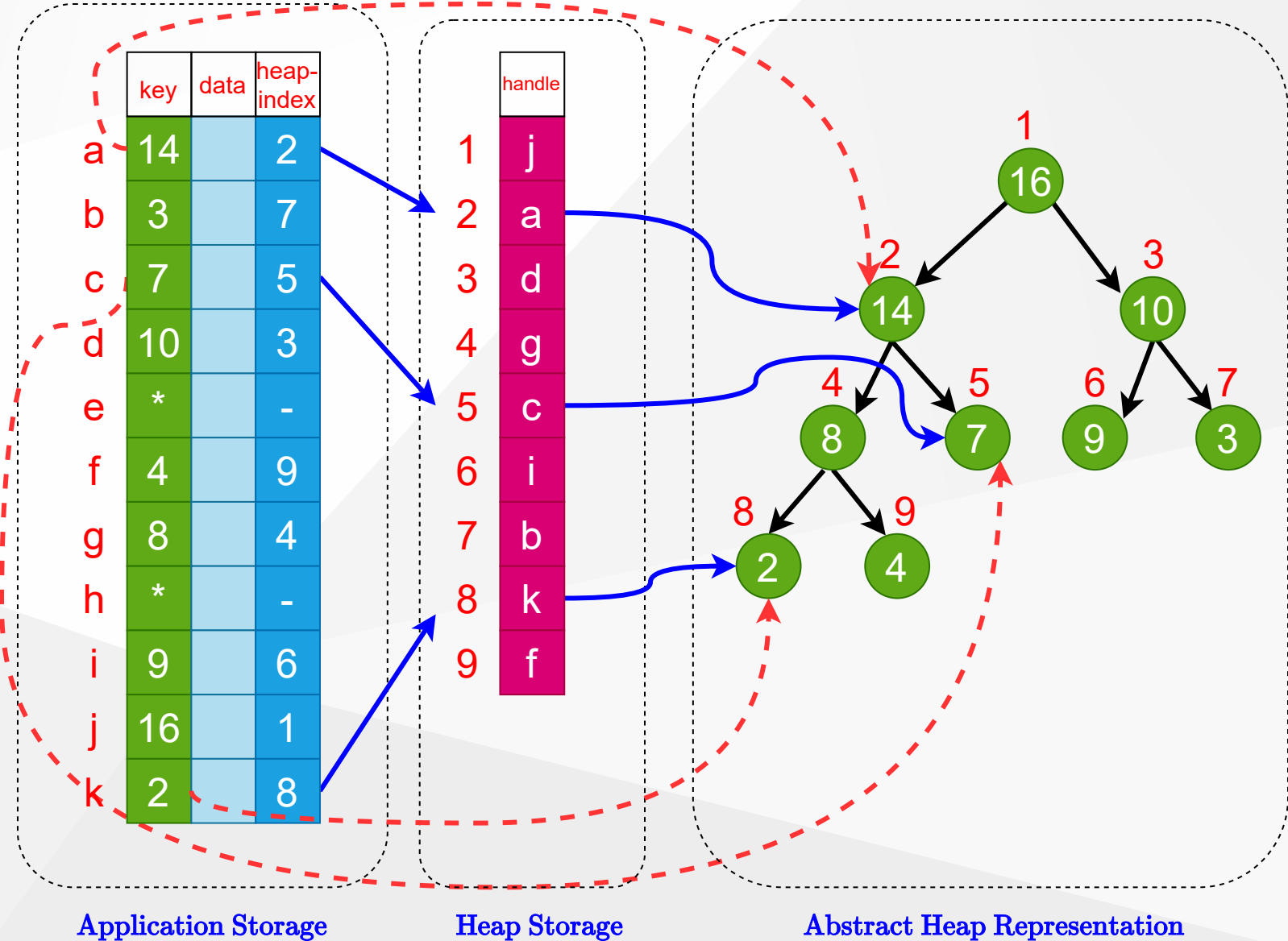
Array Storage

1	2	3	4	5	6	7	8	9	10
16	14	10	14	7	9	3	2	8	1

HEAP-INCREASE-KEY Example (Step-5)



Heap Implementat ion of Priority Queue (PQ)



Summary: Max Heap

- **Heapify(A, i)**
 - Works when both child subtrees of node i are heaps
 - "*Floats down*" node i to satisfy the heap property
 - Runtime: $O(\lg n)$
- **Max(A, n)**
 - Returns the max element of the heap (no modification)
 - Runtime: $O(1)$
- **Extract-Max(A, n)**
 - Returns and removes the max element of the heap
 - Fills the gap in $A[1]$ with $A[n]$, then calls **Heapify(A,1)**
 - Runtime: $O(\lg n)$

Summary: Max Heap

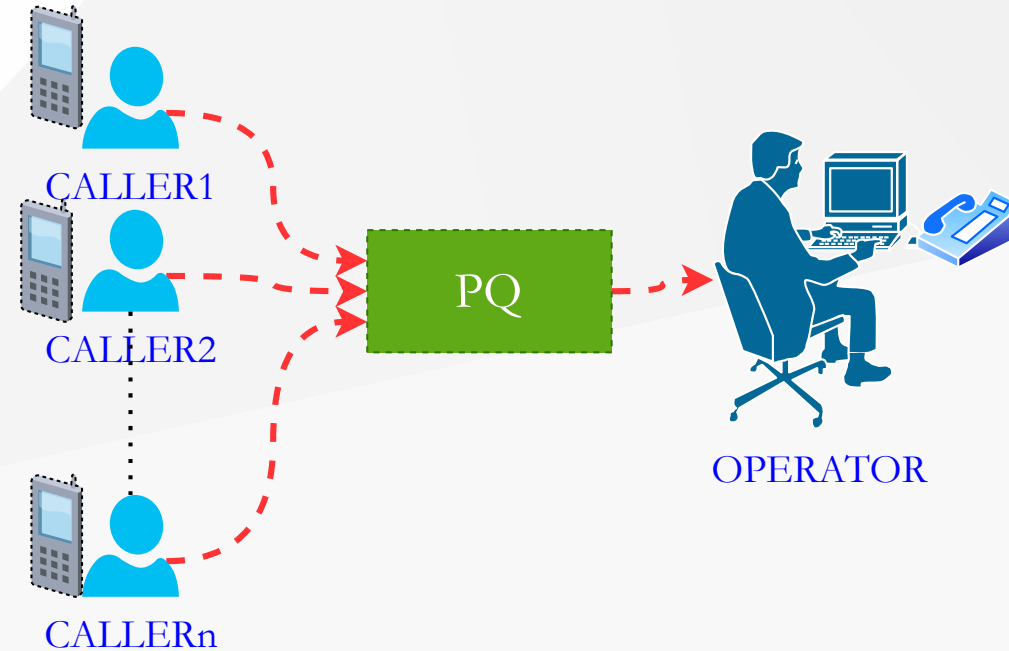
- **Build-Heap(A, n)**
 - Given an arbitrary array, builds a heap from scratch
 - Runtime: $O(n)$
- **Min(A, n)**
 - How to return the min element in a max-heap?
 - Worst case runtime: $O(n)$
 - because ~half of the heap elements are leaf nodes
 - Instead, use a min-heap for efficient min operations
- **Search(A, x)**
 - For an arbitrary x value, the worst-case runtime: $O(n)$
 - Use a sorted array instead for efficient search operations

Summary: Max Heap

- **Increase-Key(A, i, x)**
 - Increase the key of node i (from $A[i]$ to x)
 - "*Float up*" x until heap property is satisfied
 - Runtime: $O(\lg n)$
- **Decrease-Key(A, i, x)**
 - Decrease the key of node i (from $A[i]$ to x)
 - Call **Heapify(A, i)**
 - Runtime: $O(\lg n)$

Phone Operator Problem

- A phone operator answering n phones
- Each phone i has x_i people waiting in line for their calls to be answered.
- Phone operator needs to answer the phone with the largest number of people waiting in line.
- New calls come continuously, and some people hang up after waiting.



Phone Operator Solution

- **Step 1:** Define the following array:
- $A[i]$: the i th element in heap
- $A[i].id$: the index of the corresponding phone
- $A[i].key$: # of people waiting in line for phone with index $A[i].id$

A

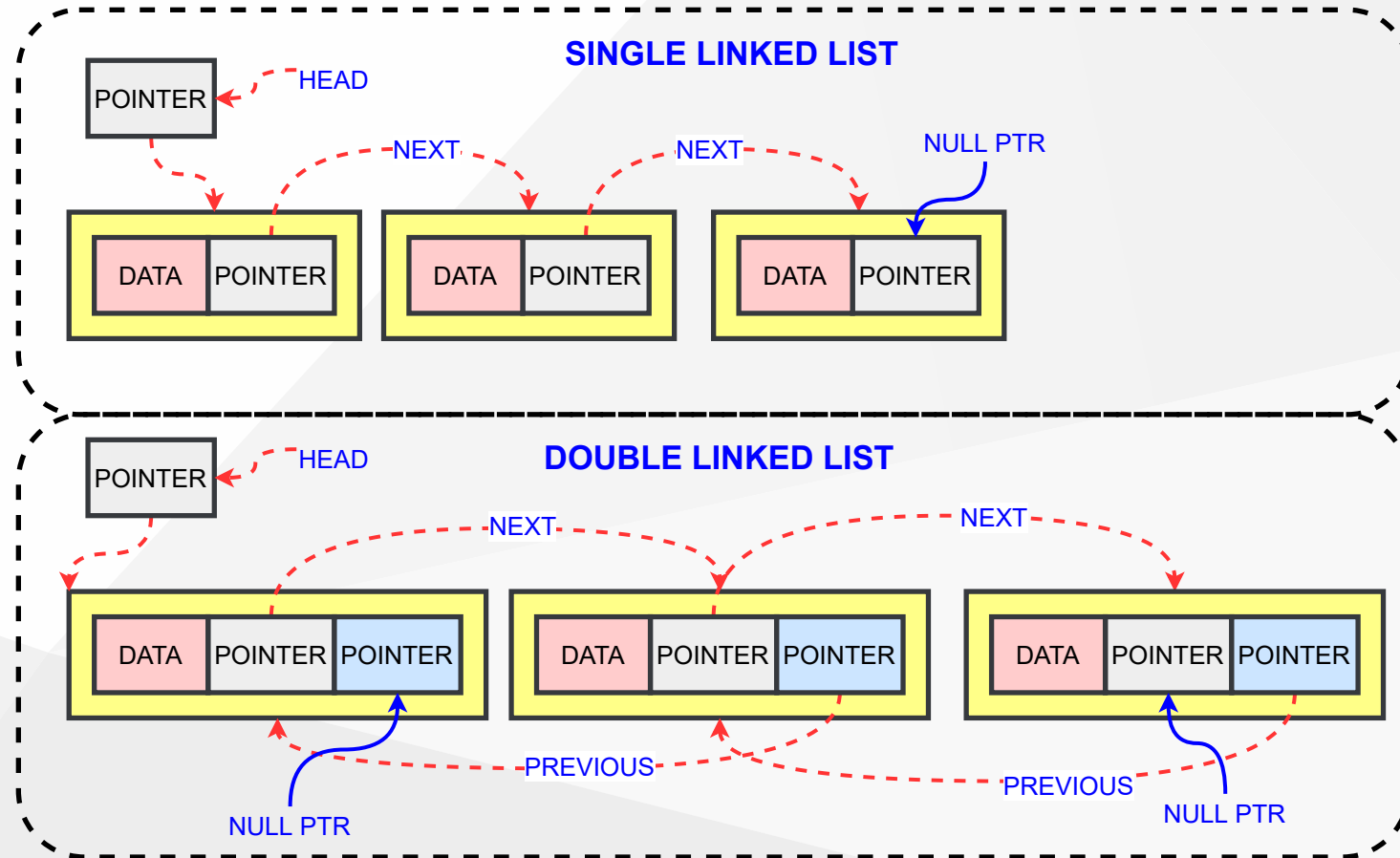
	key	id
1		
n		

Phone Operator Solution

- Step 2: Build-Max-Heap(A, n)
 - Execution:
 - When the operator wants to answer a phone:
 - $id = A[1].id$
 - Decrease-Key($A, 1, A[1].key - 1$)
 - answer phone with index id
 - When a new call comes in to phone i :
 - Increase-Key($A, i, A[i].key + 1$)
 - When a call drops from phone i :
 - Decrease-Key($A, i, A[i].key - 1$)

Linked Lists

- Like arrays, Linked List is a linear data structure.
- Unlike arrays, linked list elements are not stored at a contiguous location; the elements are linked using pointers.



Linked Lists - C Definition

- C

```
// A linked list node
struct Node {
    int data;
    struct Node* next;
};
```


Linked Lists - Cpp Definition

- Cpp

```
class Node {  
public:  
    int data;  
    Node* next;  
};
```

Linked Lists - Java Definition

- Java

```
class LinkedList {  
    Node head; // head of the list  
  
    /* Linked list Node*/  
    class Node {  
        int data;  
        Node next;  
  
        // Constructor to create a new node  
        // Next is by default initialized  
        // as null  
        Node(int d) { data = d; }  
    }  
}
```

Linked Lists - Csharp Definition

- Csharp

```
class LinkedList {  
    // The first node(head) of the linked list  
    // Will be an object of type Node (null by default)  
    Node head;  
  
    class Node {  
        int data;  
        Node next;  
  
        // Constructor to create a new node  
        Node(int d) { data = d; }  
    }  
}
```

Priority Queue using **Linked List** Methods

- Implement Priority Queue using Linked Lists.
 - **push()**: This function is used to insert a new data into the queue.
 - **pop()**: This function removes the element with the highest priority from the queue.
 - **peek()/top()**: This function is used to get the highest priority element in the queue without removing it from the queue.

Priority Queue using **Linked List** Algorithm

```
PUSH(HEAD, DATA, PRIORITY)
  Create NEW.Data = DATA & NEW.Priority = PRIORITY
  If HEAD.priority < NEW.Priority
    NEW -> NEXT = HEAD
    HEAD = NEW
  Else
    Set TEMP to head of the list
  Endif

  WHILE TEMP -> NEXT != NULL and TEMP -> NEXT ->PRIORITY > PRIORITY THEN
    TEMP = TEMP -> NEXT
  ENDWHILE

  NEW -> NEXT = TEMP -> NEXT
  TEMP -> NEXT = NEW
```

Priority Queue using **Linked List** Algorithm

POP(HEAD)

//Set the head of the list to the **next** node **in** the list.

HEAD = HEAD -> NEXT.

Free the node at the head of the list

PEEK(HEAD):

Return HEAD -> DATA

Priority Queue using **Linked List** Notes

- LinkedList is already sorted.
- Time Complexities and Comparison with Binary Heap

	peek()	push()	pop()
Linked List	$O(1)$	$O(n)$	$O(1)$
Binary Heap	$O(1)$	$O(\lg n)$	$O(\lg n)$

Sorting in Linear Time

How Fast Can We Sort?

- The algorithms we have seen so far:
 - Based on comparison of elements
 - We only care about the relative ordering between the elements (not the actual values)
 - The smallest worst-case runtime we have seen so far: $O(n \lg n)$
 - Is $O(n \lg n)$ the best we can do?
- **Comparison sorts:** Only use comparisons to determine the relative order of elements.

Decision Trees for Comparison Sorts

- Represent a sorting algorithm abstractly in terms of a **decision tree**
 - A **binary tree** that represents the **comparisons between** elements in the sorting algorithm
 - Control, data movement, and other aspects are ignored
- One decision tree corresponds to one sorting algorithm and one value of n (*input size*)

Reminder: Insertion Sort Step-By-Step Description (1)

Insertion-Sort(A)

1. **for** $j = 2$ **to** n **do**

2. $\text{key} = A[j];$

3. $i = j - 1;$

4. **while** $i > 0$ **and** $A[i] > \text{key}$ **do**

5. $A[i+1] = A[i];$

6. $i = i - 1;$

endwhile

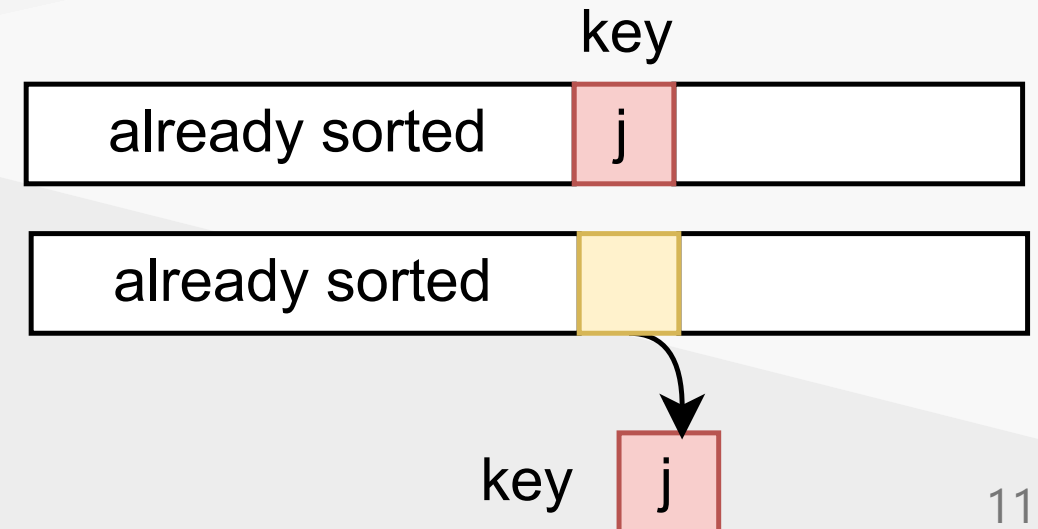
7. $A[i+1] = \text{key};$

endfor

} Iterate over array

Loop invariant:

The subarray $A[1..j - 1]$
is always sorted



Reminder: Insertion Sort Step-By-Step Description (2)

Insertion-Sort(A)

1. **for** $j = 2$ **to** n **do**

2. $\text{key} = A[j]$;

3. $i = j - 1$;

4. **while** $i > 0$ **and** $A[i] > \text{key}$ **do**

5. $A[i+1] = A[i]$;

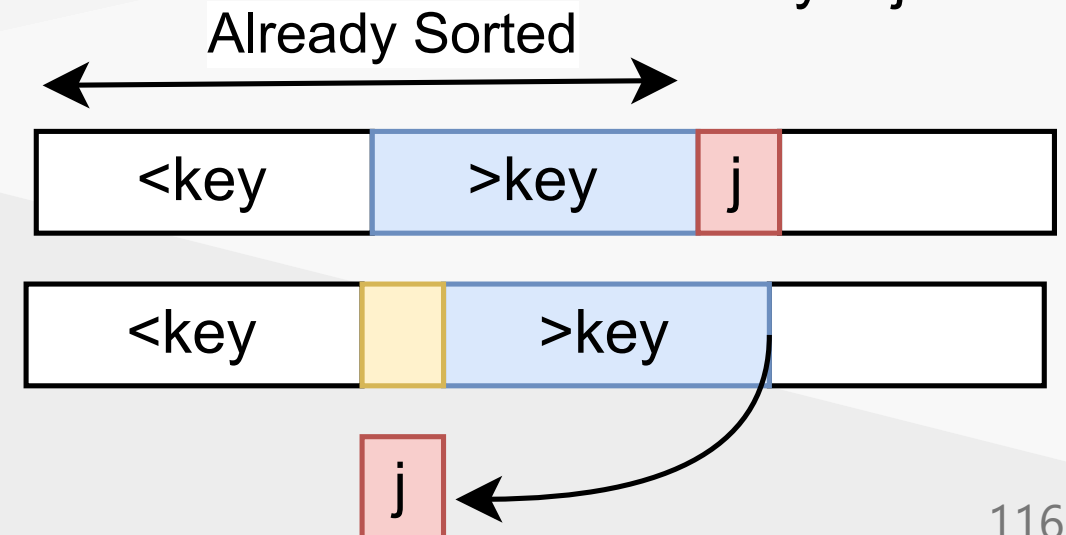
6. $i = i - 1$;

endwhile

7. $A[i+1] = \text{key}$;

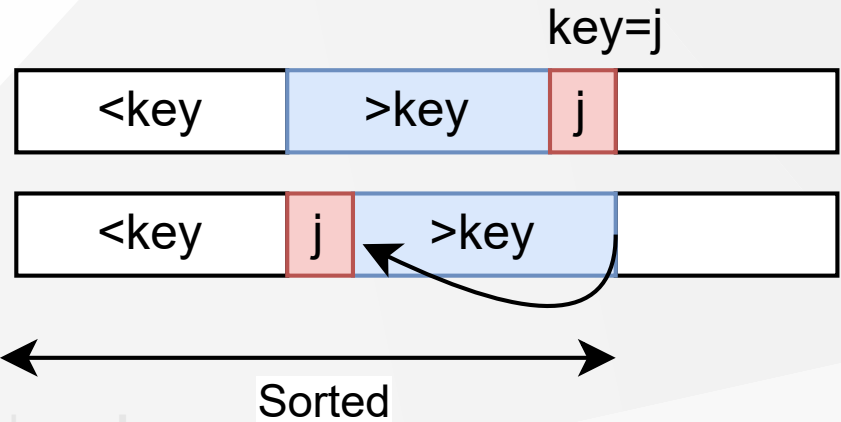
endfor

Shift right the
entries in
 $A[1..j-1]$
that are
bigger than
 $\text{key} = j$



Reminder: Insertion Sort Step-By-Step Description (3)

```
Insertion-Sort(A)
1. for j = 2 to n do
2.   key = A[j];
3.   i = j-1;
4.   while i>0 and A[i]>key do
5.     A[i+1]=A[i];
6.     i = i-1;
7.   endwhile
8.   A[i+1]=key;
9. endfor
```

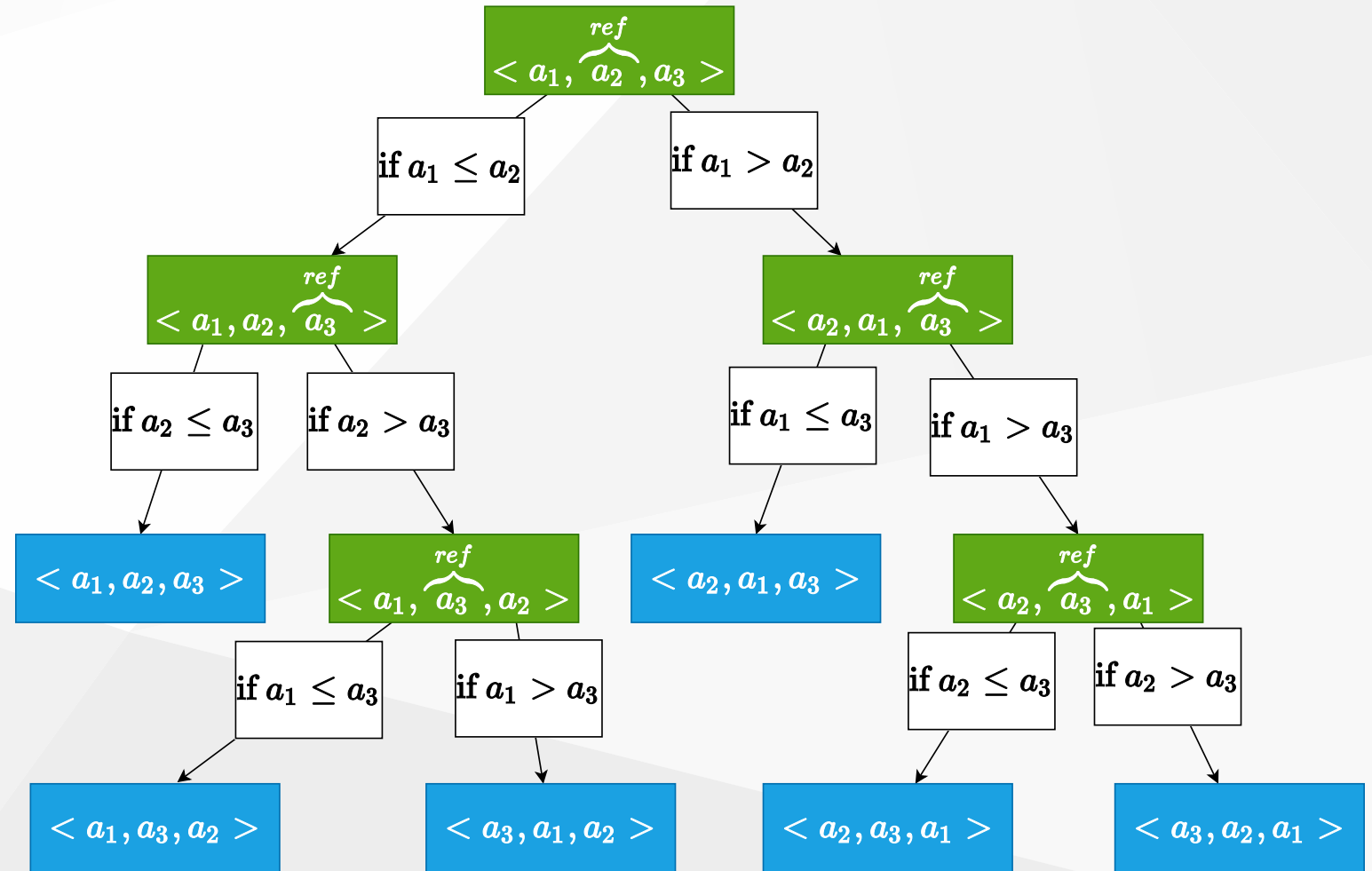


} Insert key to the correct location

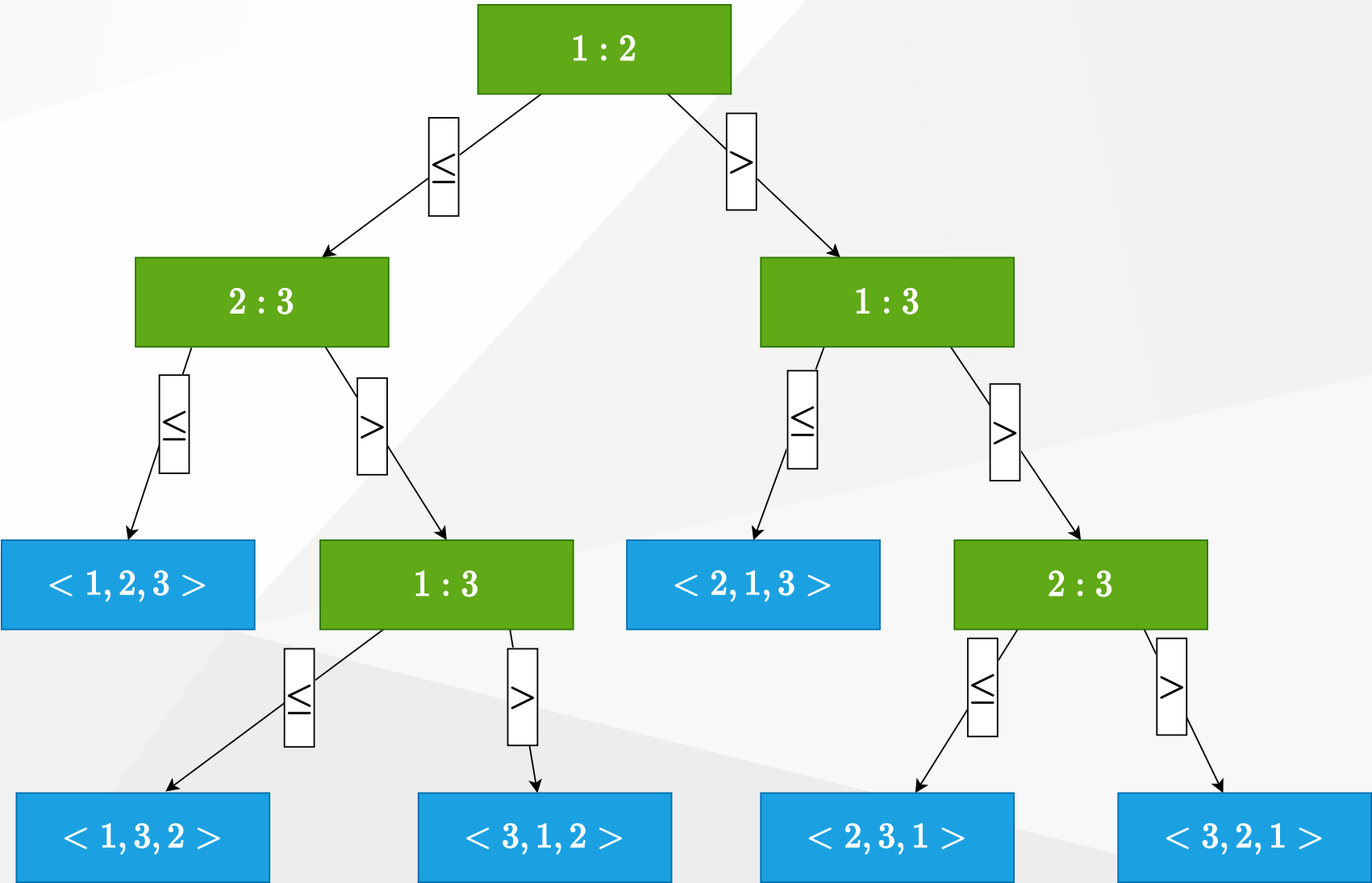
End of iteration $j : A[1..j]$ is sorted

Different Outcomes for Insertion Sort and $n=3$

- Input :
 $\langle a_1, a_2, a_3 \rangle$



Decision Tree
for Insertion
Sort and n=3

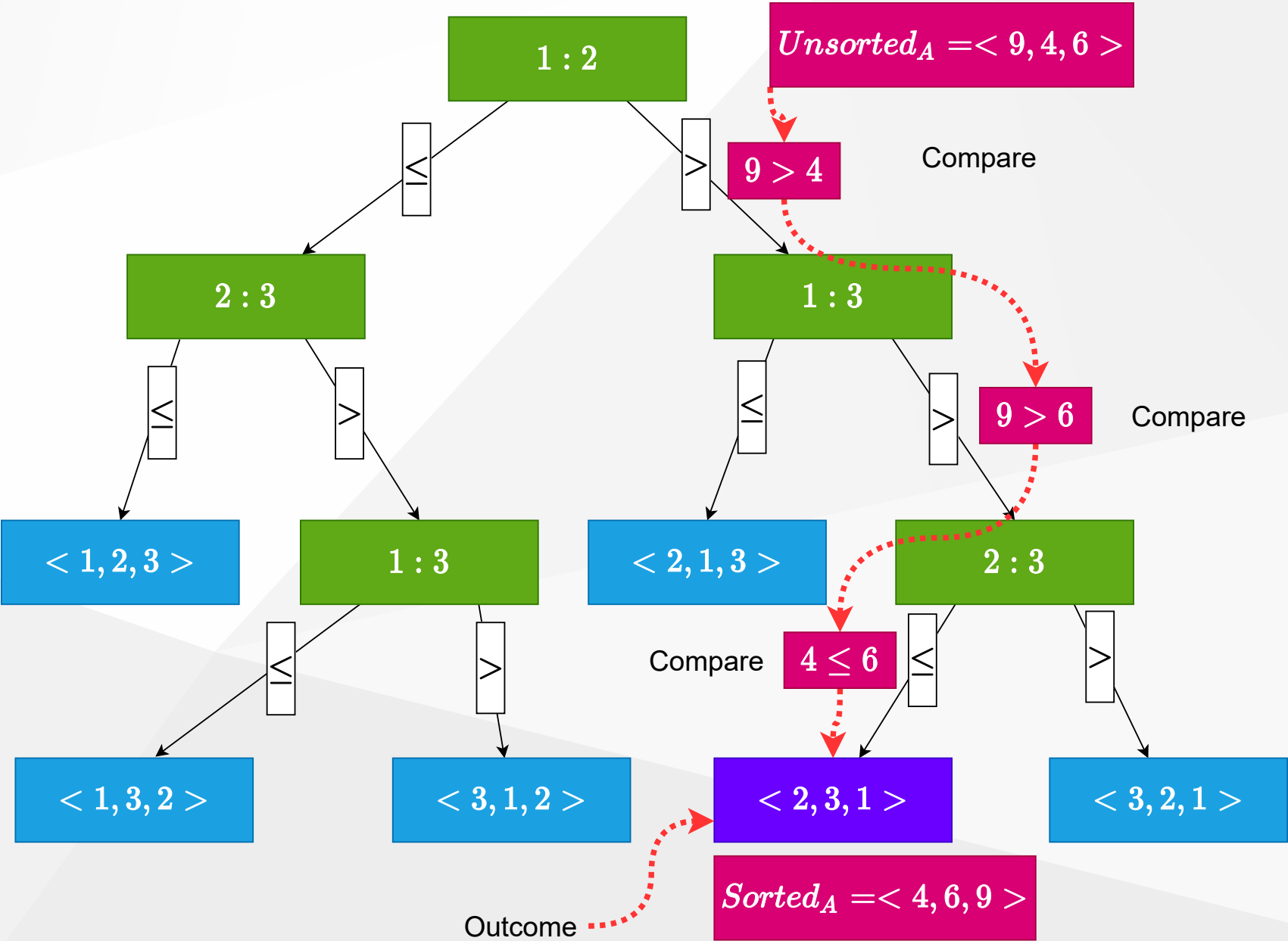


Decision Tree Model for Comparison Sorts

- **Internal node** $(i : j)$: Comparison between elements a_i and a_j
- **Leaf node**: An output of the sorting algorithm
- **Path from root to a leaf**: The execution of the sorting algorithm for a given input
- **All possible executions** are captured by the decision tree
- **All possible outcomes (permutations)** are in the leaf nodes

Decision Tree for Insertion Sort and n=3

- Input:
 $< 9, 4, 6 >$



Decision Tree Model

- A decision tree can model the execution of any comparison sort:
 - One tree for each input size n
 - View the algorithm as **splitting** whenever it compares two elements
 - The tree contains the **comparisons along all possible** instruction traces
- The running time of the algorithm = *the length of the path taken*
- Worst case running time = *height of the tree*

Counting Sort

Lower Bound for Comparison Sorts

- Let n be the number of elements in the input array.
- What is the *min* number of leaves in the decision tree?
 - $n!$ (because there are $n!$ permutations of the input array, and all possible outputs must be captured in the leaves)
- What is the max number of leaves in a binary tree of height h ? $\implies 2^h$
- So, we must have:

$$2^h \geq n!$$

Lower Bound for Decision Tree Sorting

- **Theorem:** Any comparison sort algorithm requires $\Omega(n \lg n)$ comparisons in the worst case.
- **Proof:** We'll prove that any decision tree corresponding to a comparison sort algorithm must have height $\Omega(n \lg n)$

$$2^h \geq n!$$

$$h \geq \lg(n!)$$

$$\geq \lg((n/e)^n) \text{ (Stirling Approximation)}$$

$$= n \lg n - n \lg e$$

$$= \Omega(n \lg n)$$

Lower Bound for Decision Tree Sorting

Corollary: Heapsort and merge sort are asymptotically optimal comparison sorts.

Proof: The $O(n \lg n)$ upper bounds on the runtimes for heapsort and merge sort match the $\Omega(n \lg n)$ **worst-case** lower bound from the previous theorem.

Sorting in Linear Time

- **Counting sort:** No comparisons between elements
 - **Input:** $A[1 \dots n]$, where $A[j] \in \{1, 2, \dots, k\}$
 - **Output:** $B[1 \dots n]$, sorted
 - **Auxiliary storage:** $C[1 \dots k]$

Counting Sort-1

for $i \leftarrow 1$ **to** k **do**

$C[i] \leftarrow 0$

for $j \leftarrow 1$ **to** n **do**

$C[A[j]] \leftarrow C[A[j]] + 1$

// $C[i] = |\{\text{key} = i\}|$

for $i \leftarrow 2$ **to** k **do**

$C[i] \leftarrow C[i] + C[i-1]$

// $C[i] = |\{\text{key} \leq i\}|$

for $j \leftarrow n$ **downto** 1 **do**

$B[C[A[j]]] \leftarrow A[j]$

$C[A[j]] \leftarrow C[A[j]] - 1$

$A =$

4	1	3	4	3
---	---	---	---	---

$B =$

--	--	--	--	--

$C =$

1	2	3	4

Counting Sort-2

- Step 1: Initialize all counts to 0



for $i \leftarrow 1$ **to** k **do**

$C[i] \leftarrow 0$

for $j \leftarrow 1$ **to** n **do**

$C[A[j]] \leftarrow C[A[j]] + 1$

// $C[i] = |\{\text{key} = i\}|$

for $i \leftarrow 2$ **to** k **do**

$C[i] \leftarrow C[i] + C[i-1]$

// $C[i] = |\{\text{key} \leq i\}|$

for $j \leftarrow n$ **downto** 1 **do**

$B[C[A[j]]] \leftarrow A[j]$

$C[A[j]] \leftarrow C[A[j]] - 1$

$A =$

4	1	3	4	3
---	---	---	---	---

$B =$

--	--	--	--	--

$C =$

1	2	3	4
0	0	0	0

Counting Sort-3

- **Step 2:** Count the number of occurrences of each value in the input array



for $i \leftarrow 1$ **to** k **do**

$C[i] \leftarrow 0$

for $j \leftarrow 1$ **to** n **do**

$C[A[j]] \leftarrow C[A[j]] + 1$

// $C[i] = |\{\text{key} = i\}|$

for $i \leftarrow 2$ **to** k **do**

$C[i] \leftarrow C[i] + C[i-1]$

// $C[i] = |\{\text{key} \leq i\}|$

for $j \leftarrow n$ **downto** 1 **do**

$B[C[A[j]]] \leftarrow A[j]$

$C[A[j]] \leftarrow C[A[j]] - 1$

$A =$

4	1	3	4	3
---	---	---	---	---

$B =$

--	--	--	--	--

$C =$

1	2	3	4
0	0	0	0

Counting Sort-4

- **Step 3:** Compute the number of elements less than or equal to each value



for $i \leftarrow 1$ **to** k **do**

$C[i] \leftarrow 0$

for $j \leftarrow 1$ **to** n **do**

$C[A[j]] \leftarrow C[A[j]] + 1$

// $C[i] = |\{\text{key} = i\}|$

for $i \leftarrow 2$ **to** k **do**

$C[i] \leftarrow C[i] + C[i-1]$

// $C[i] = |\{\text{key} \leq i\}|$

for $j \leftarrow n$ **downto** 1 **do**

$B[C[A[j]]] \leftarrow A[j]$

$C[A[j]] \leftarrow C[A[j]] - 1$

$A =$

4	1	3	4	3
---	---	---	---	---

$B =$

--	--	--	--	--

$C =$

i				
1	2	3	4	
1	1	3	5	

Counting Sort-5

- **Step 4:** Populate the output array
 - There are $C[3] = 3$ elements that are ≤ 3



for $i \leftarrow 1$ **to** k **do**

$C[i] \leftarrow 0$

for $j \leftarrow 1$ **to** n **do**

$C[A[j]] \leftarrow C[A[j]] + 1$

// $C[i] = |\{\text{key} = i\}|$

for $i \leftarrow 2$ **to** k **do**

$C[i] \leftarrow C[i] + C[i-1]$

// $C[i] = |\{\text{key} \leq i\}|$

for $j \leftarrow n$ **downto** 1 **do**

$B[C[A[j]]] \leftarrow A[j]$

$C[A[j]] \leftarrow C[A[j]] - 1$

					j
$A =$	4	1	3	4	3
$B =$					
$C =$	1	2	3	4	
	1	1	2	5	

Counting Sort-6

- **Step 4:** Populate the output array
 - There are $C[4] = 5$ elements that are ≤ 4



Step-5: Populate the output array

for $i \leftarrow 1$ **to** k **do**

$C[i] \leftarrow 0$

for $j \leftarrow 1$ **to** n **do**

$C[A[j]] \leftarrow C[A[j]] + 1$

// $C[i] = |\{\text{key} = i\}|$

for $i \leftarrow 2$ **to** k **do**

$C[i] \leftarrow C[i] + C[i-1]$

// $C[i] = |\{\text{key} \leq i\}|$

for $j \leftarrow n$ **downto** 1 **do**

$B[C[A[j]]] \leftarrow A[j]$

$C[A[j]] \leftarrow C[A[j]] - 1$

	j				
$A =$	4	1	3	4	3
	1	2	3	4	5
$B =$			3		
	1	2	3	4	
$C =$	1	1	2	4	

There are $C[4] = 5$ elts that are ≤ 4

Counting Sort-7

- **Step 4:** Populate the output array
 - There are $C[3] = 2$ elements that are ≤ 3



for $i \leftarrow 1$ **to** k **do**

$C[i] \leftarrow 0$

for $j \leftarrow 1$ **to** n **do**

$C[A[j]] \leftarrow C[A[j]] + 1$

// $C[i] = |\{\text{key} = i\}|$

for $i \leftarrow 2$ **to** k **do**

$C[i] \leftarrow C[i] + C[i-1]$

// $C[i] = |\{\text{key} \leq i\}|$

for $j \leftarrow n$ **downto** 1 **do**

$B[C[A[j]]] \leftarrow A[j]$

$C[A[j]] \leftarrow C[A[j]] - 1$

	j				
$A =$	4	1	3	4	3
	1	2	3	4	5
$B =$			3		4
	1	2	3	4	
$C =$	1	1	1	4	

Counting Sort-8

- **Step 4:** Populate the output array
 - There are $C[1] = 1$ elements that are ≤ 1



for $i \leftarrow 1$ **to** k **do**

$C[i] \leftarrow 0$

for $j \leftarrow 1$ **to** n **do**

$C[A[j]] \leftarrow C[A[j]] + 1$

// $C[i] = |\{\text{key} = i\}|$

for $i \leftarrow 2$ **to** k **do**

$C[i] \leftarrow C[i] + C[i-1]$

// $C[i] = |\{\text{key} \leq i\}|$

for $j \leftarrow n$ **downto** 1 **do**

$B[C[A[j]]] \leftarrow A[j]$

$C[A[j]] \leftarrow C[A[j]] - 1$

	j				
$A =$	4	1	3	4	3
	1	2	3	4	5
$B =$		3	3		4
	1	2	3	4	
$C =$	0	1	1	4	

Counting Sort-9

- **Step 4:** Populate the output array
 - There are $C[4] = 4$ elements that are ≤ 4



for $i \leftarrow 1$ **to** k **do**

$C[i] \leftarrow 0$

for $j \leftarrow 1$ **to** n **do**

$C[A[j]] \leftarrow C[A[j]] + 1$

// $C[i] = |\{\text{key} = i\}|$

for $i \leftarrow 2$ **to** k **do**

$C[i] \leftarrow C[i] + C[i-1]$

// $C[i] = |\{\text{key} \leq i\}|$

for $j \leftarrow n$ **downto** 1 **do**

$B[C[A[j]]] \leftarrow A[j]$

$C[A[j]] \leftarrow C[A[j]] - 1$

	j				
$A =$	4	1	3	4	3
	1	2	3	4	5
$B =$	1	3	3		4
	1	2	3	4	
$C =$	0	1	1	3	

Counting Sort: Runtime Analysis

- Total Runtime:
 $\Theta(n + k)$
 - n : size of the input array
 - k : the range of input values

```

for  $i \leftarrow 1$  to  $k$  do
     $C[i] \leftarrow 0$ 
for  $j \leftarrow 1$  to  $n$  do
     $C[A[j]] \leftarrow C[A[j]] + 1$ 
    //  $C[i] = |\{\text{key} = i\}|$ 
for  $i \leftarrow 2$  to  $k$  do
     $C[i] \leftarrow C[i] + C[i-1]$ 
    //  $C[i] = |\{\text{key} \leq i\}|$ 
for  $j \leftarrow n$  downto  $1$  do
     $B[C[A[j]]] \leftarrow A[j]$ 
     $C[A[j]] \leftarrow C[A[j]] - 1$ 
  
```

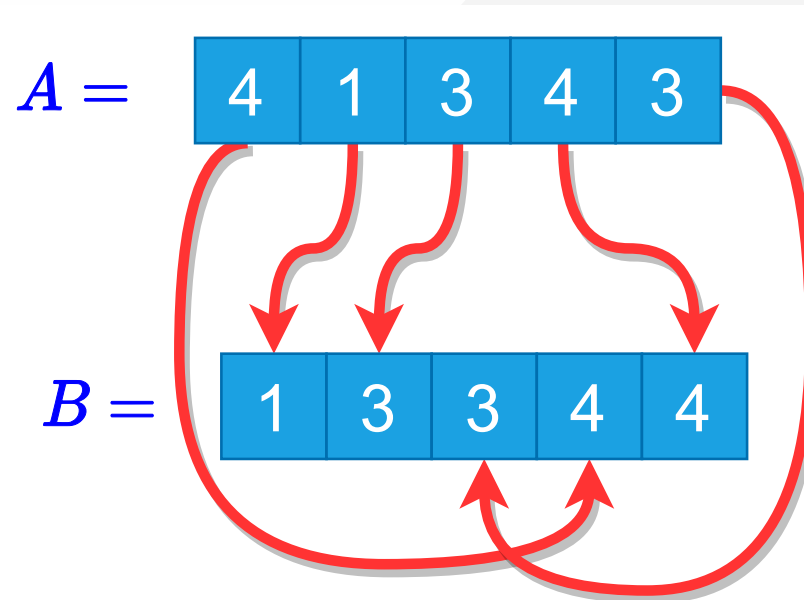
$\Theta(k)$
 $\Theta(n)$
 $\Theta(k)$
 $\Theta(n)$

Counting Sort: Runtime

- Runtime is $\Theta(n + k)$
 - If $k = O(n)$, then counting sort takes $\Theta(n)$
- **Question:** We proved a lower bound of $\Theta(n \lg n)$ before! Where is the fallacy?
- **Answer:**
 - $\Theta(n \lg n)$ lower bound is for comparison-based sorting
 - Counting sort is not a comparison sort
 - In fact, not a single comparison between elements occurs!

Stable Sorting

- Counting sort is a **stable sort**: It preserves the input order among equal elements.
 - i.e. The numbers with the same value appear in the output array in the same order as they do in the input array.
- Note**: Which other sorting algorithms have this property?



Radix Sort

- **Origin:** Herman Hollerith's card-sorting machine for the 1890 US Census.
- **Basic idea:** Digit-by-digit sorting
- Two variations:
 - Sort from **MSD** to **LSD** (bad idea)
 - Sort from **LSD** to **MSD** (good idea)

(LSD/MSD: Least/most significant digit)

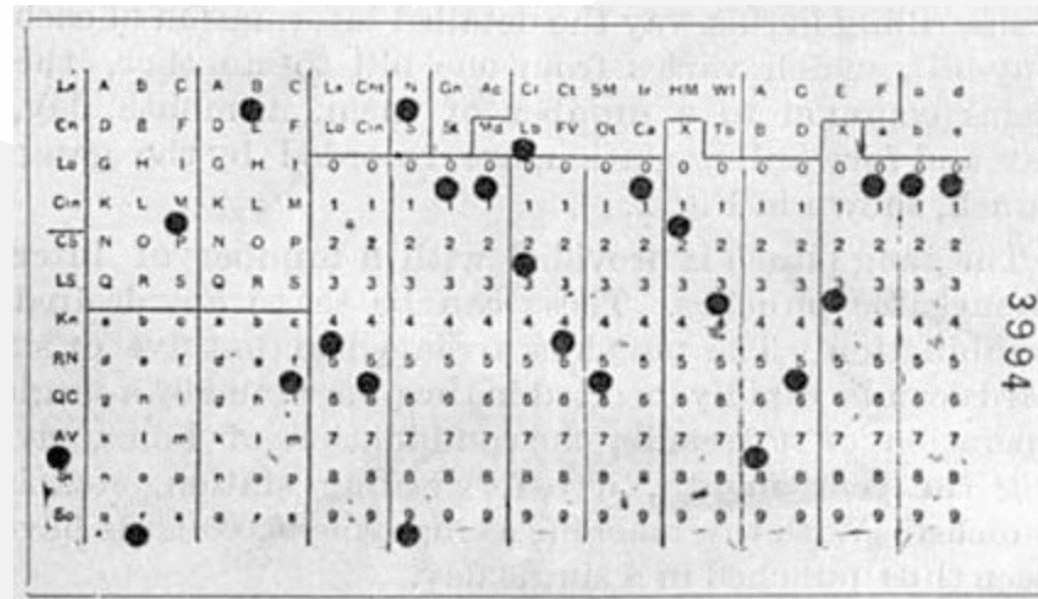
Herman Hollerith (1860-1929)

- The 1880 U.S. Census took **almost 10 years** to process.
- While a lecturer at MIT, Hollerith prototyped **punched-card technology**.
- His machines, including a **card sorter**, allowed the 1890 census total to be reported in **6 weeks**.
- He founded the **Tabulating Machine Company** in 1911, which merged with other companies in 1924 to form **International Business Machines(IBM)**.



Hollerith Punched Card

- **Punched card:** A piece of stiff paper that contains digital information represented by the presence or absence of holes.
 - 12 rows and 24 columns
 - coded for age, state of residency, gender, etc.



Modern IBM card

- One character per column
 - So, that's why text windows have 80 columns!



- for more samples visit https://en.wikipedia.org/wiki/Punched_card

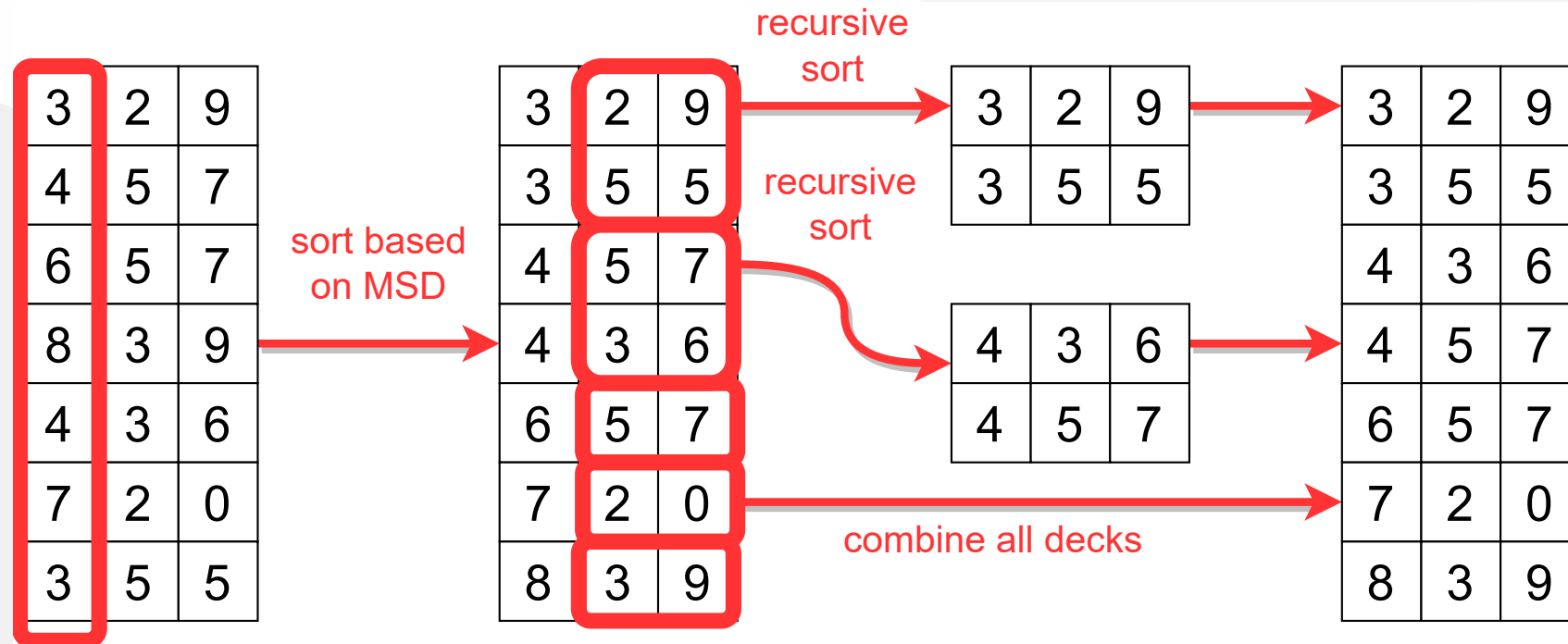
Hollerith Tabulating Machine and Sorter

- Mechanically sorts the cards based on the hole locations.
- Sorting performed for one column at a time
- Human operator needed to load/retrieve/move cards at each stage



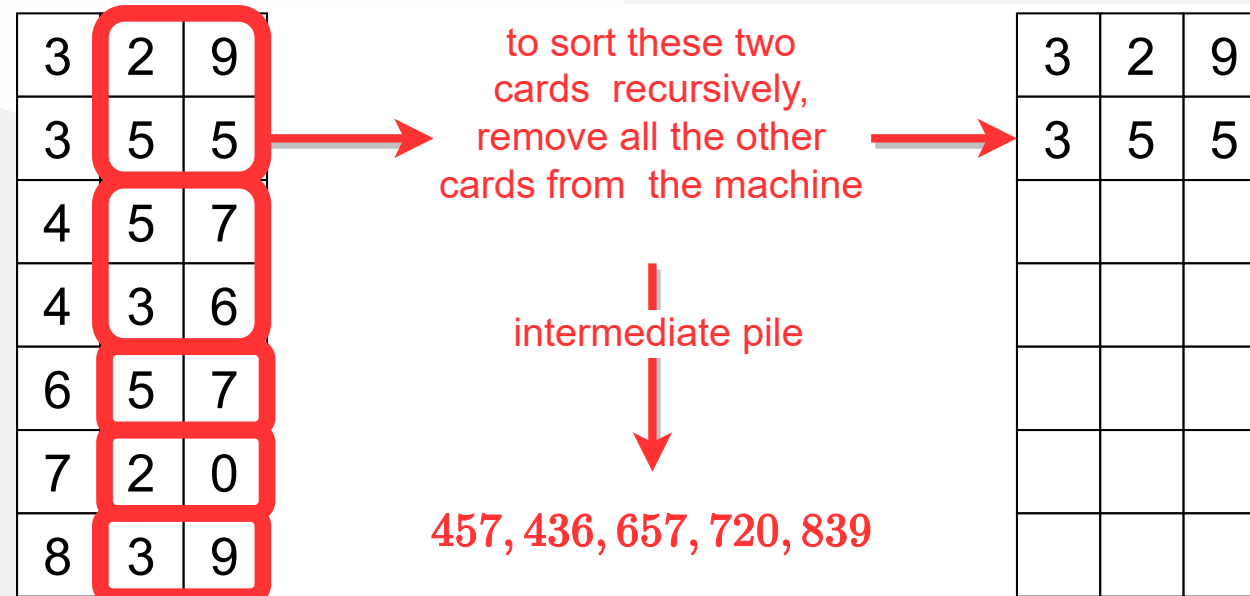
Hollerith's MSD-First Radix Sort

- Sort starting from the most significant digit (MSD)
- Then, sort each of the resulting bins recursively
- At the end, combine the decks in order



Hollerith's MSD-First Radix Sort

- To sort a subset of cards recursively:
 - All the other cards need to be removed from the machine, because the machine can handle only one sorting problem at a time.
 - The human operator needs to keep track of the intermediate card piles



Hollerith's MSD-First Radix Sort

- MSD-first sorting may require:
 - very large number of sorting passes
 - very large number of intermediate card piles to maintain
- $S(d)$:
 - # of passes needed to sort d-digit numbers (worst-case)
- Recurrence:
 - $S(d) = 10S(d - 1) + 1$ with $S(1) = 1$
 - **Reminder:** Recursive call made to each subset with the same most significant digit(MSD)

Hollerith's MSD-First Radix Sort

- Recurrence: $S(d) = 10S(d - 1) + 1$

$$S(d) = 10S(d - 1) + 1$$

$$= 10 \left(10S(d - 2) + 1 \right) + 1$$

$$= 10 \left(10 \left(10S(d - 3) + 1 \right) + 1 \right) + 1$$

$$= 10^i S(d - i) + 10^i - 1 + 10^i - 2 + \dots + 10^1 + 10^0$$

$$= \sum_{i=0}^{d-1} 10^i$$

- Iteration terminates when $i = d - 1$ with $S(d - (d - 1)) = S(1) = 1$

Hollerith's MSD-First Radix Sort

- Recurrence: $S(d) = 10S(d-1) + 1$

$$\begin{aligned} S(d) &= \sum_{i=0}^{d-1} 10^i \\ &= \frac{10^d - 1}{10 - 1} \\ &= \frac{1}{9} (10^d - 1) \\ &\Downarrow \\ S(d) &= \frac{1}{9} (10^d - 1) \end{aligned}$$

Hollerith's MSD-First Radix Sort

- $P(d)$: # of intermediate card piles maintained (worst-case)
- **Reminder:** Each routing pass generates 9 intermediate piles except the sorting passes on least significant digits (LSDs)
 - There are 10^{d-1} sorting calls to LSDs

$$\begin{aligned}P(d) &= 9(S(d) - 10^{d-1}) \\&= 9 \frac{(10^d - 1) - (10^d - 10^{d-1})}{10 - 1} \\&= (10^d - 10^{d-1}) \\&= 10^{d-1} \cdot 9\end{aligned}$$

Hollerith's MSD-First Radix Sort

$$P(d) = 10^{d-1} - 1$$

Alternative solution: Solve the recurrence

$$P(d) = 10P(d-1) + 9$$

$$P(1) = 0$$

Hollerith's MSD-First Radix Sort

- **Example:** To sort 3 digit numbers, in the worst case:
 - $S(d) = (1/9)(10^3 - 1) = 111$ sorting passes needed
 - $P(d) = 10d - 1 - 1 = 99$ intermediate card piles generated
- MSD-first approach has more recursive calls and intermediate storage requirement
 - Expensive for a **tabulating machine** to sort punched cards
 - Overhead of recursive calls in a modern computer

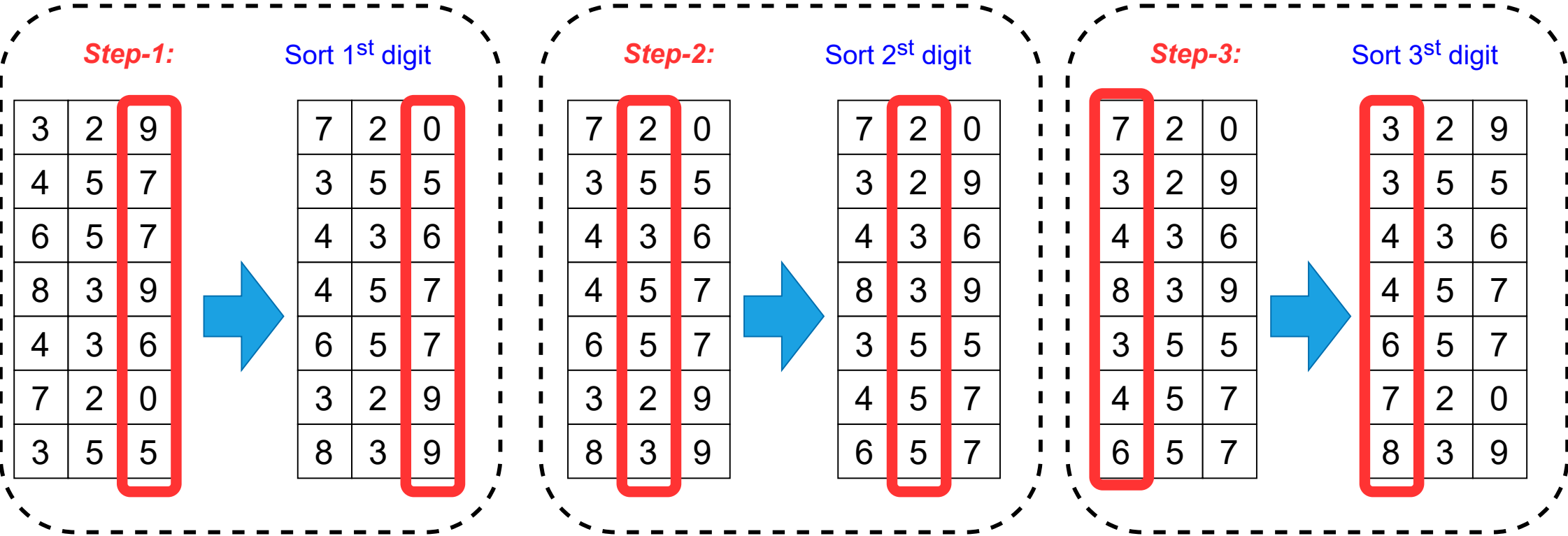
LSD-First Radix Sort

- Least significant digit (**LSD**)-first radix sort seems to be a folk invention originated by machine operators.
- It is the counter-intuitive, but the better algorithm.
- **Basic Algorithm:**

Sort numbers on their LSD first (Stable Sorting Needed)
Combine the cards into a single deck **in** order
Continue this sorting process **for** the other digits
from the LSD to MSD

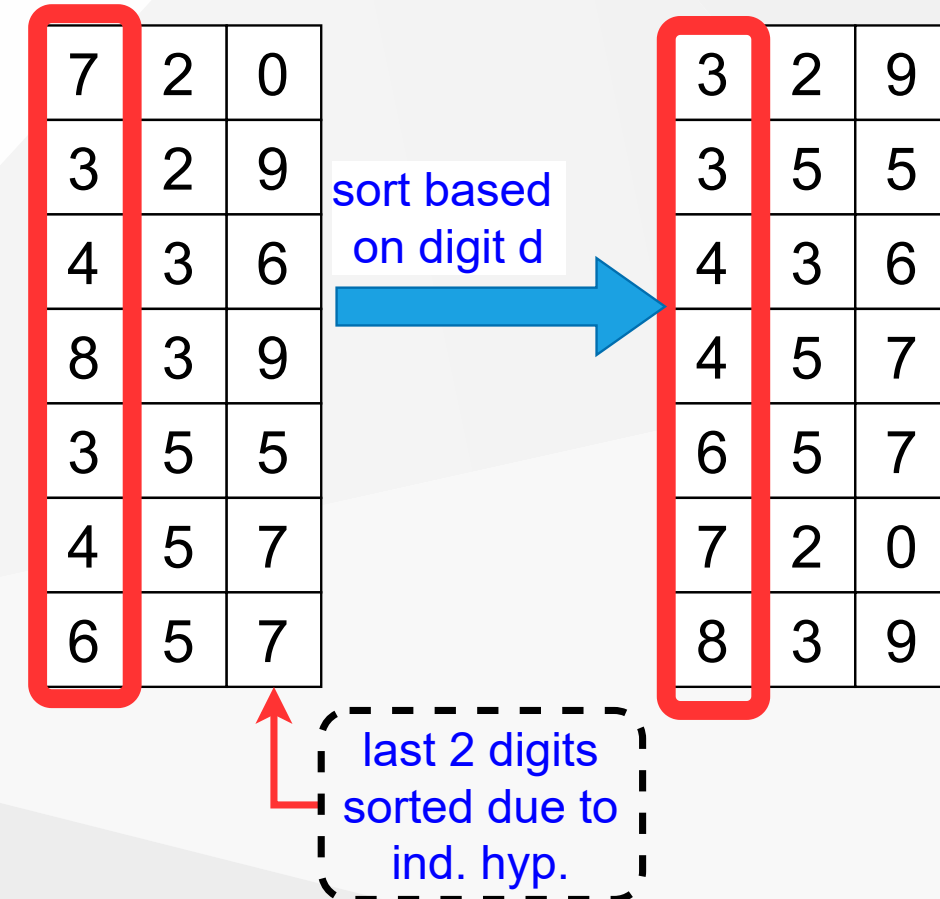
- Requires only d sorting passes
- No intermediate card pile generated

LSD-first Radix Sort Example



Correctness of Radix Sort (LSD-first)

- **Proof by induction:**
 - **Base case:** $d = 1$ is correct (trivial)
 - **Inductive hyp:** Assume the first $d - 1$ digits are sorted correctly
- Prove that all d digits are sorted correctly after sorting digit d
- Two numbers that differ in digit d are correctly sorted (e.g. 355 and 657)
- Two numbers equal in digit d are put in the same order as the input
 - (correct order)



Radix Sort Runtime

- Use counting-sort to sort each digit
- **Reminder:** Counting sort complexity: $\Theta(n + k)$
 - n : size of input array
 - k : the range of the values
- Radix sort runtime: $\Theta(d(n + k))$
 - d : # of digits

How to choose the d and k ?

Radix Sort: Runtime – Example 1

- We have flexibility in choosing d and k
- Assume we are trying to sort 32-bit words
 - We can define each digit to be 4 bits
 - Then, the range for each digit $k = 2^4 = 16$
 - So, counting sort will take $\Theta(n + 16)$
 - The number of digits $d = 32/4 = 8$
 - Radix sort runtime: $\Theta(8(n + 16)) = \Theta(n)$
- $\overbrace{[4bits|4bits|4bits|4bits|4bits|4bits|4bits|4bits]}^{32\text{-bits}}$

Radix Sort: Runtime – Example 2

- We have flexibility in choosing d and k
- Assume we are trying to sort 32-bit words
 - Or, we can define each digit to be 8 bits
 - Then, the range for each digit $k = 2^8 = 256$
 - So, counting sort will take $\Theta(n + 256)$
 - The number of digits $d = 32/8 = 4$
 - Radix sort runtime: $\Theta(4(n + 256)) = \Theta(n)$
- $\overbrace{[8bits|8bits|8bits|8bits]}^{32\text{-bits}}$

Radix Sort: Runtime

- Assume we are trying to sort b -bit words
 - Define each digit to be r bits
 - Then, the range for each digit $k = 2^r$
 - So, counting sort will take $\Theta(n + 2^r)$
 - The number of digits $d = b/r$
 - Radix sort runtime:

$$T(n, b) = \Theta\left(\frac{b}{r}(n + 2^r)\right)$$

- $\overbrace{[r\text{bits} | r\text{bits} | r\text{bits} | r\text{bits}]}^{b/r \text{ bits}}$

Radix Sort: Runtime Analysis

$$T(n, b) = \Theta\left(\frac{b}{r}(n + 2^r)\right)$$

- Minimize $T(n, b)$ by differentiating and setting to 0
- Or, intuitively:
 - We want to balance the terms (b/r) and $(n + 2^r)$
 - **Choose $r \approx \lg n$**
 - If we choose $r \ll \lg n \implies (n + 2^r)$ term **doesn't improve**
 - If we choose $r \gg \lg n \implies (n + 2^r)$ increases **exponentially**

Radix Sort: Runtime Analysis

$$T(n, b) = \Theta\left(\frac{b}{r}(n + 2^r)\right)$$

$$\text{Choose } r = \lg n \implies T(n, b) = \Theta(bn/\lg n)$$

- For numbers in the range from 0 to $n^d - 1$, we have:
 - The number of bits $b = \lg(nd) = d \lg n$
 - Radix sort runs in $\Theta(dn)$

Radix Sort: Conclusions

Choose $r = \lg n \implies T(n, b) = \Theta(bn/\lg n)$

- **Example:** Compare radix sort with merge sort/heapsort
 - 1 million (2^{20}), 32-bit numbers ($n = 2^{20}, b = 32$)
 - Radix sort: $\lfloor 32/20 \rfloor = 2$ passes
 - Merge sort/heap sort: $\lg n = 20$ passes
- **Downsides:**
 - Radix sort has **little locality of reference** (more cache misses)
 - The version that uses counting sort is not in-place
- On modern processors, a well-tuned quicksort implementation typically runs faster.

References

- [Introduction to Algorithms, Third Edition | The MIT Press](#)
- [Bilkent CS473 Course Notes \(new\)](#)
- [Bilkent CS473 Course Notes \(old\)](#)
- [Insertion Sort - GeeksforGeeks](#)
- [Priority Queue Using Linked List - GeeksforGeeks](#)
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- [NIST Dictionary of Algorithms and Data Structures](#)
- [NIST - Dictionary of Algorithms and Data Structures](#)

–End – Of – Week – 4 – Course – Module–