CE100 Algorithms and Programming II

Matrix Multiplication / Quick Sort

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0.2.0.1 Spring Semester, 2021-2022

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0.3 Matrix Multiplication / Quick Sort

0.4 Outline

- Matrix Multiplication
  - Traditional
  - Recursive
  - Strassen

0.5 Outline

- Quicksort
  - Hoare Partitioning
  - Lomuto Partitioning
  - Recursive Sorting

0.6 Outline

- Quicksort Analysis
  - Randomized Quicksort
  - Randomized Selection
    - Recursive
    - Medians

0.7 Matrix Multiplication

- **Input:** $A = [a_{ij}], B = [b_{ij}]$
- **Output:** $C = [c_{ij}] = A \cdot B \Longrightarrow i, j = 1, 2, 3, \dots, n$

$$\begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \vdots & \ddots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \vdots & \ddots \\ b_{n1} & a_{n2} & \dots & b_{nn} \end{bmatrix}$$

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[1] ce100-week-3-matrix.md_doc.pdf
[2] ce100-week-3-matrix.md_slide.pdf
[3] ce100-week-3-matrix.md_slide.pptx

0.8 Matrix Multiplication

\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c__{22} & \cdots & c_{2n} \\...

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- $c_{ij} = \sum_{1 \leq k \leq n} a_{ik}.b_{kj}$

0.9 Matrix Multiplication: Standard Algorithm

Running Time: $\Theta(n^3)$

```
for i=1 to n do
    for j=1 to n do
        C[i,j] = 0
        for k=1 to n do
            C[i,j] = C[i,j] + A[i,k] + B[k,j]
        endfor
    endfor
endfor
```

0.10 Matrix Multiplication: Divide & Conquer

IDEA: Divide the nxn matrix into $2x2$ matrix of $(n/2)x(n/2)$ submatrices.

\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_... \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_...

c_{11}=a_{11}b_{11}+a_{12}b_... c_{21}=a_{21}b_{11}+a_{22}b_...

\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_... \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_...

c_{12}=a_{11}b_{12}+a_{12}b_... c_{22}=a_{21}b_{12}+a_{22}b_...

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0.11 Matrix Multiplication: Divide & Conquer

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$8 \text{ mults and 4 adds of (n/2)*(n/2) submatrices} = \begin{cases} c_{11} = a_{11}b_{11} + a_{12}b_{21} \\ c_{21} = a_{21}b_{11} + a_{22}b_{21} \\ c_{12} = a_{11}b_{12} + a_{12}b_{22} \\ c_{22} = a_{21}b_{12} + a_{22}b_{22} \end{cases}$$

0.12 Matrix Multiplication: Divide & Conquer

```
MATRIX-MULTIPLY(A, B)
    // Assuming that both A and B are nxn matrices
    if n == 1 then
        return A * B
    else
        //partition A, B, and C as shown before
        C[1,1] = MATRIX-MULTIPLY (A[1,1], B[1,1]) +
                 MATRIX-MULTIPLY (A[1,2], B[2,1]);

        C[1,2] = MATRIX-MULTIPLY (A[1,1], B[1,2]) +
                MATRIX-MULTIPLY (A[1,2], B[2,2]);

        C[2,1] = MATRIX-MULTIPLY (A[2,1], B[1,1]) +
        MATRIX-MULTIPLY (A[2,2], B[2,1]);

        C[2,2] = MATRIX-MULTIPLY (A[2,1], B[1,2]) +
        MATRIX-MULTIPLY (A[2,2], B[2,2]);
    endif

    return C
```

0.13 Matrix Multiplication: Divide & Conquer Analysis

$T(n) = 8T(n/2) + \Theta(n^2)$

- 8 recursive calls $\Longrightarrow 8T(\cdots)$
- each problem has size $n/2 \Longrightarrow \cdots T(n/2)$
- Submatrix addition $\Longrightarrow \Theta(n^2)$

0.14 Matrix Multiplication: Solving the Recurrence

- $T(n) = 8T(n/2) + \Theta(n^2)$
  - $a = 8$, $b = 2$
  - $f(n) = \Theta(n^2)$
  - $n^{log_b^a} = n^3$
- Case 1: $\frac{n^{log_b^a}}{f(n)} = \Omega(n^\varepsilon) \Longrightarrow T(n) = \Theta(n^{log_b^a})$

Similar with ordinary (iterative) algorithm.

0.15 Matrix Multiplication: Strassen's Idea

Compute $c_{11}, c_{12}, c_{21}, c_{22}$ using 7 recursive multiplications.

In normal case we need 8 as below.

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\text{8 mults and 4 adds of (n/2)*(n/2) submatrices} = \begin{cases} c_{11} = a_{11}b_{11} + a_{12}b_{21} \\ c_{21} = a_{21}b_{11} + a_{22}b_{21} \\ c_{12} = a_{11}b_{12} + a_{12}b_{22} \\ c_{22} = a_{21}b_{12} + a_{22}b_{22} \end{cases}$$

0.16 Matrix Multiplication: Strassen's Idea

- **Reminder:**
  - Each submatrix is of size $(n/2) * (n/2)$
  - Each add/sub operation takes $\Theta(n^2)$ time
- Compute $P1 \ldots P7$ using 7 recursive calls to matrix-multiply

$P_1 = a_{11} * (b_{12} - b_{22})$
$P_2 = (a_{11} + a_{12}) * b_{22}$
$P_3 = (a_{21} + a_{22}) * b_{11}$
$P_4 = a_{22} * (b_{21} - b_{11})$
$P_5 = (a_{11} + a_{22}) * (b_{11} + b_{22})$
$P_6 = (a_{12} - a_{22}) * (b_{21} + b_{22})$
$P_7 = (a_{11} - a_{21}) * (b_{11} + b_{12})$

0.17 Matrix Multiplication: Strassen's Idea

$P_1 = a_{11} * (b_{12} - b_{22})$
$P_2 = (a_{11} + a_{12}) * b_{22}$
$P_3 = (a_{21} + a_{22}) * b_{11}$
$P_4 = a_{22} * (b_{21} - b_{11})$
$P_5 = (a_{11} + a_{22}) * (b_{11} + b_{22})$
$P_6 = (a_{12} - a_{22}) * (b_{21} + b_{22})$
$P_7 = (a_{11} - a_{21}) * (b_{11} + b_{12})$

- How to compute c_{ij} using $P1 \ldots P7$?

$c_{11} = P_5 + P_4 – P_2 + P_6$
$c_{12} = P_1 + P_2$
$c_{21} = P_3 + P_4$
$c_{22} = P_5 + P_1 – P_3 – P_7$

0.18 Matrix Multiplication: Strassen's Idea

- 7 recursive multiply calls
- 18 add/sub operations

0.19 Matrix Multiplication: Strassen's Idea

e.g. Show that $c_{12} = P_1 + P_2$

$c_{12} = P_1 + P_2$
$= a_{11}(b_{12} – b_{22}) + (a_{11} + a_{12})b_{22}$
$= a_{11}b_{12} - a_{11}b_{22} + a_{11}b_{22} + a_{12}b_{22}$
$= a_{11}b_{12} + a_{12}b_{22}$

0.20 Strassen's Algorithm

- **Divide:** Partition A and B into $(n/2) * (n/2)$ submatrices. Form terms to be multiplied using $+$ and $-$.
- **Conquer:** Perform 7 multiplications of $(n/2) * (n/2)$ submatrices recursively.
- **Combine:** Form C using $+$ and $-$ on $(n/2) * (n/2)$submatrices.

Recurrence: $T(n) = 7T(n/2) + \Theta(n^2)$

0.21 Strassen's Algorithm: Solving the Recurrence

- $T(n) = 7T(n/2) + \Theta(n^2)$
  - $a = 7$, $b = 2$
  - $f(n) = \Theta(n^2)$
  - $n^{log_b^a} = n^{lg7}$
- Case 1: $\frac{n^{log_b^a}}{f(n)} = \Omega(n^{\varepsilon}) \Longrightarrow T(n) = \Theta(n^{log_b^a})$

$T(n) = \Theta(n^{log_2^7})$

$2^3 = 8, 2^2 = 4$ so $\Longrightarrow log_2^7 \approx 2.81$

or use https://www.omnicalculator.com/math/log

0.22 Strassen's Algorithm

- The number 2.81 may not seem much smaller than 3
- But, it is significant because the difference is in the exponent.
- Strassen's algorithm beats the ordinary algorithm on today's machines for $n \geq 30$ or so.
- Best to date: $\Theta(n^{2.376\ldots})$ (of theoretical interest only)

0.23 Maximum Subarray Problem

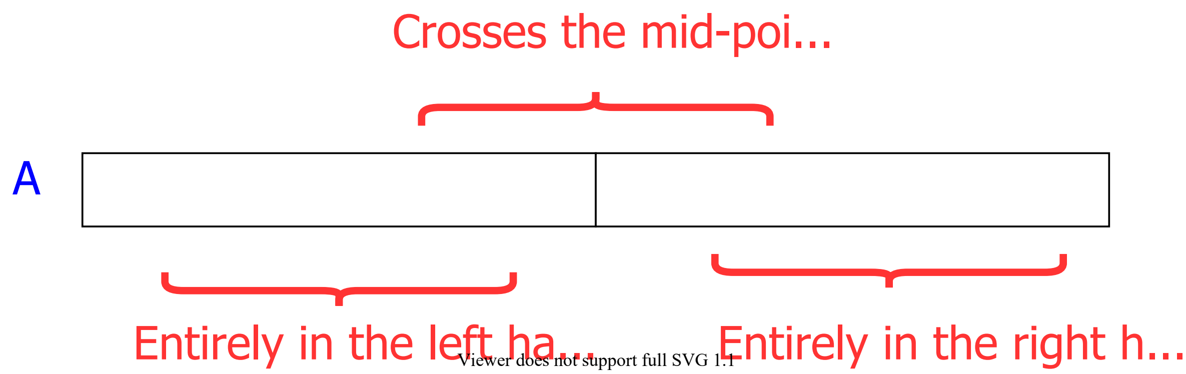
Input: An array of values **Output:** The contiguous subarray that has the largest sum of elements

- Input array: $[13][-3][-25][20][-3][-16][-23] \quad \overbrace{[18][20][-7][12]}^{\text{max. contiguous subarray}} \quad [-22][-4][7]$

0.24 Maximum Subarray Problem: Divide & Conquer

- **Basic idea:**
- **Divide** the input array into 2 from the middle
- Pick the **best** solution among the following:
  - The max subarray of the **left half**
  - The max subarray of the **right half**
  - The max subarray **crossing the mid-point**

0.25 Maximum Subarray Problem: Divide & Conquer



0.26 Maximum Subarray Problem: Divide & Conquer

- **Divide:** Trivial (divide the array from the middle)
- **Conquer:** Recursively compute the max subarrays of the left and right halves
- **Combine:** Compute the max-subarray crossing the $mid - point$
  - (can be done in $\Theta(n)$ time).
  - Return the max among the following:
    * the max subarray of the left-subarray
    * the max subarray of the rightsubarray
    * the max subarray crossing the mid-point

TODO : detailed solution in textbook…

0.27 Conclusion : Divide & Conquer

- Divide and conquer is just one of several powerful techniques for algorithm design.
- Divide-and-conquer algorithms can be analyzed using recurrences and the master method (so practice this math).
- Can lead to more efficient algorithms

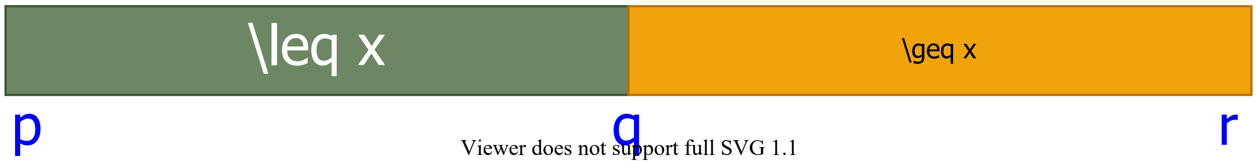
0.28 Quicksort

- One of the most-used algorithms in practice
- Proposed by **C.A.R.** *Hoare* in 1962.
- Divide-and-conquer algorithm
- In-place algorithm
  - The additional space needed is O(1)
  - The sorted array is returned in the input array
  - *Reminder: Insertion-sort is also an in-place algorithm, but Merge-Sort is not in-place.*
- Very practical

0.29 Quicksort

- **Divide:** Partition the array into 2 subarrays such that elements in the lower part $\leq$ elements in the higher part
- **Conquer:** Recursively sort 2 subarrays
- **Combine:** Trivial (because in-place)

Key: Linear-time ($\Theta(n)$) partitioning algorithm



0.30 References

TODO