

CE100 Algorithms and Programming II

Week-3 (Matrix Multiplication/ Quick Sort)

Spring Semester, 2021-2022

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<iframe width=700, height=500 frameBorder=0 src="../ce100-week-3-matrix.md_slide.html"> </iframe>

Matrix Multiplication / Quick Sort

Outline (1)

- Matrix Multiplication
 - Traditional
 - Recursive
 - Strassen

Outline (2)

- Quicksort
 - Hoare Partitioning
 - Lomuto Partitioning
 - Recursive Sorting

Outline (3)

- Quicksort Analysis
 - Randomized Quicksort
 - Randomized Selection
 - Recursive
 - Medians

Matrix Multiplication (1)

- Input: $A = [a_{ij}]$, $B = [b_{ij}]$
- Output: $C = [c_{ij}] = A \cdot B \implies i, j = 1, 2, 3, \dots, n$

$$\begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \vdots & \ddots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \vdots & \ddots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{bmatrix}$$

Matrix Multiplication (2)

$$\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ \boxed{c_{21}} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \boxed{a_{21} \quad a_{22} \quad \cdots \quad a_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ \boxed{b_{21}} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$$

- $c_{ij} = \sum_{1 \leq k \leq n} a_{ik} \cdot b_{kj}$

Matrix Multiplication: Standard Algorithm

Running Time: $\Theta(n^3)$

```
for i=1 to n do
  for j=1 to n do
    C[i,j] = 0
    for k=1 to n do
      C[i,j] = C[i,j] + A[i,k] + B[k,j]
    endfor
  endfor
endfor
```

Matrix Multiplication: Divide & Conquer (1)

IDEA: Divide the $n \times n$ matrix into 2×2 matrix of $(n/2) \times (n/2)$ submatrices.

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \quad \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$c_{11} = a_{11}b_{11} + a_{12}b_{21}$$

$$c_{21} = a_{21}b_{11} + a_{22}b_{21}$$

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \quad \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$c_{12} = a_{11}b_{12} + a_{12}b_{22}$$

$$c_{22} = a_{21}b_{12} + a_{22}b_{22}$$

Matrix Multiplication: Divide & Conquer (2)

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$8 \text{ mults and } 4 \text{ adds of } (n/2) \times (n/2) \text{ submatrices} = \begin{cases} c_{11} = a_{11}b_{11} + a_{12}b_{21} \\ c_{21} = a_{21}b_{11} + a_{22}b_{21} \\ c_{12} = a_{11}b_{12} + a_{12}b_{22} \\ c_{22} = a_{21}b_{12} + a_{22}b_{22} \end{cases}$$

Matrix Multiplication: Divide & Conquer (3)

```
MATRIX-MULTIPLY(A, B)
// Assuming that both A and B are nxn matrices
if n == 1 then
    return A * B
else
    //partition A, B, and C as shown before
    C[1,1] = MATRIX-MULTIPLY (A[1,1], B[1,1]) +
             MATRIX-MULTIPLY (A[1,2], B[2,1]);

    C[1,2] = MATRIX-MULTIPLY (A[1,1], B[1,2]) +
             MATRIX-MULTIPLY (A[1,2], B[2,2]);

    C[2,1] = MATRIX-MULTIPLY (A[2,1], B[1,1]) +
             MATRIX-MULTIPLY (A[2,2], B[2,1]);

    C[2,2] = MATRIX-MULTIPLY (A[2,1], B[1,2]) +
             MATRIX-MULTIPLY (A[2,2], B[2,2]);
endif

return C
```

Matrix Multiplication: Divide & Conquer Analysis

$$T(n) = 8T(n/2) + \Theta(n^2)$$

- 8 recursive calls $\implies 8T(\dots)$
- each problem has size $n/2 \implies \dots T(n/2)$
- Submatrix addition $\implies \Theta(n^2)$

Matrix Multiplication: Solving the Recurrence

- $T(n) = 8T(n/2) + \Theta(n^2)$
 - $a = 8, b = 2$
 - $f(n) = \Theta(n^2)$
 - $n^{\log_b^a} = n^3$
- Case 1: $\frac{n^{\log_b^a}}{f(n)} = \Omega(n^\epsilon) \implies T(n) = \Theta(n^{\log_b^a})$

Similar with ordinary (iterative) algorithm.

Matrix Multiplication: Strassen's Idea (1)

Compute $c_{11}, c_{12}, c_{21}, c_{22}$ using 7 recursive multiplications.

In normal case we need 8 as below.

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$8 \text{ mults and } 4 \text{ adds of } (n/2) \times (n/2) \text{ submatrices} = \begin{cases} c_{11} = a_{11}b_{11} + a_{12}b_{21} \\ c_{21} = a_{21}b_{11} + a_{22}b_{21} \\ c_{12} = a_{11}b_{12} + a_{12}b_{22} \\ c_{22} = a_{21}b_{12} + a_{22}b_{22} \end{cases}$$

Matrix Multiplication: Strassen's Idea (2)

- **Reminder:**
 - Each submatrix is of size $(n/2) * (n/2)$
 - Each add/sub operation takes $\Theta(n^2)$ time
- Compute $P_1 \dots P_7$ using 7 recursive calls to matrix-multiply

$$P_1 = a_{11} * (b_{12} - b_{22})$$

$$P_2 = (a_{11} + a_{12}) * b_{22}$$

$$P_3 = (a_{21} + a_{22}) * b_{11}$$

$$P_4 = a_{22} * (b_{21} - b_{11})$$

$$P_5 = (a_{11} + a_{22}) * (b_{11} + b_{22})$$

$$P_6 = (a_{12} - a_{22}) * (b_{21} + b_{22})$$

$$P_7 = (a_{11} - a_{21}) * (b_{11} + b_{12})$$

Matrix Multiplication: Strassen's Idea (3)

$$P_1 = a_{11} * (b_{12} - b_{22})$$

$$P_2 = (a_{11} + a_{12}) * b_{22}$$

$$P_3 = (a_{21} + a_{22}) * b_{11}$$

$$P_4 = a_{22} * (b_{21} - b_{11})$$

$$P_5 = (a_{11} + a_{22}) * (b_{11} + b_{22})$$

$$P_6 = (a_{12} - a_{22}) * (b_{21} + b_{22})$$

$$P_7 = (a_{11} - a_{21}) * (b_{11} + b_{12})$$

- How to compute c_{ij} using $P_1 \dots P_7$?

$$c_{11} = P_5 + P_4 - P_2 + P_6$$

$$c_{12} = P_1 + P_2$$

$$c_{21} = P_3 + P_4$$

$$c_{22} = P_5 + P_1 - P_3 - P_7$$

Matrix Multiplication: Strassen's Idea (4)

- 7 recursive multiply calls
- 18 add/sub operations

Matrix Multiplication: Strassen's Idea (5)

e.g. Show that $c_{12} = P_1 + P_2$:

$$\begin{aligned}c_{12} &= P_1 + P_2 \\&= a_{11}(b_{12}-b_{22}) + (a_{11} + a_{12})b_{22} \\&= a_{11}b_{12} - a_{11}b_{22} + a_{11}b_{22} + a_{12}b_{22} \\&= a_{11}b_{12} + a_{12}b_{22}\end{aligned}$$

Strassen's Algorithm

- **Divide:** Partition A and B into $(n/2) * (n/2)$ submatrices. Form terms to be multiplied using $+$ and $-$.
- **Conquer:** Perform 7 multiplications of $(n/2) * (n/2)$ submatrices recursively.
- **Combine:** Form C using $+$ and $-$ on $(n/2) * (n/2)$ submatrices.

Recurrence: $T(n) = 7T(n/2) + \Theta(n^2)$

Strassen's Algorithm: Solving the Recurrence (1)

- $T(n) = 7T(n/2) + \Theta(n^2)$
 - $a = 7, b = 2$
 - $f(n) = \Theta(n^2)$
 - $n^{\log_b^a} = n^{\lg 7}$
- Case 1: $\frac{n^{\log_b^a}}{f(n)} = \Omega(n^\epsilon) \implies T(n) = \Theta(n^{\log_b^a})$

$$T(n) = \Theta(n^{\log_2^7})$$

$$2^3 = 8, 2^2 = 4 \text{ so } \implies \log_2^7 \approx 2.81$$

or use <https://www.omnicalculator.com/math/log>

Strassen's Algorithm: Solving the Recurrence (2)

- The number 2.81 may not seem much smaller than 3
- But, it is significant because the difference is in the exponent.
- Strassen's algorithm beats the ordinary algorithm on today's machines for $n \geq 30$ or so.
- Best to date: $\Theta(n^{2.376\dots})$ (of theoretical interest only)

Maximum Subarray Problem

Input: An array of values

Output: The contiguous subarray that has the largest sum of elements

- Input array:

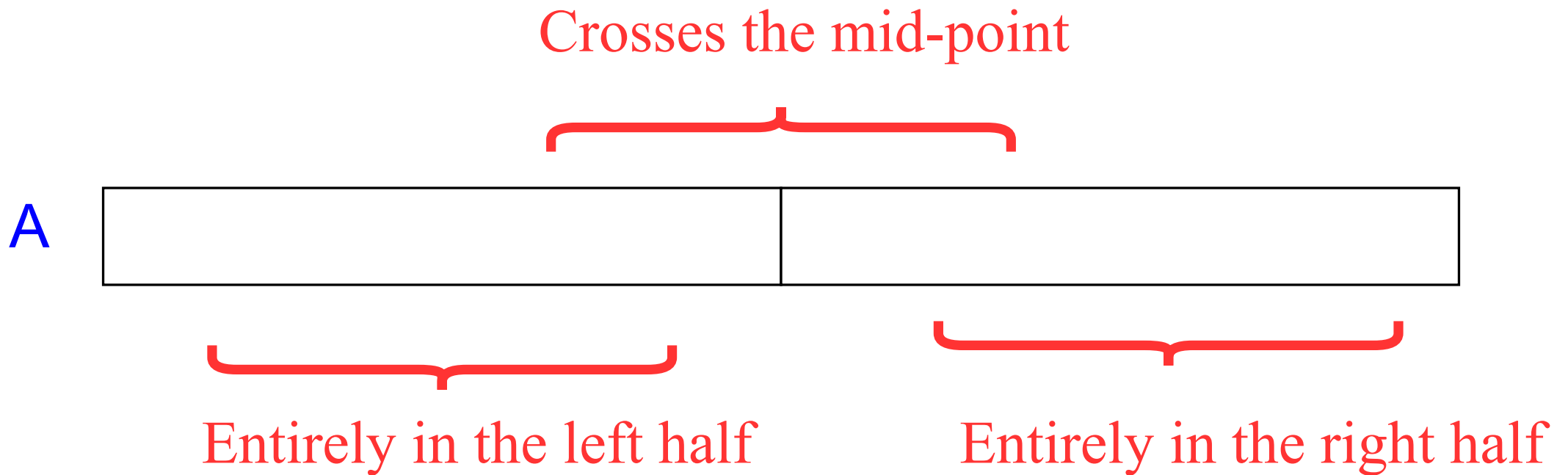
max. contiguous subarray

$[13] [-3] [-25] [20] [-3] [-16] [-23] \quad \overbrace{[18] [20] [-7] [12]} \quad [-22] [-4] [7]$

Maximum Subarray Problem: Divide & Conquer (1)

- **Basic idea:**
- **Divide** the input array into 2 from the middle
- Pick the **best** solution among the following:
 - The max subarray of the **left half**
 - The max subarray of the **right half**
 - The max subarray **crossing the mid-point**

Maximum Subarray Problem: Divide & Conquer (2)



Maximum Subarray Problem: Divide & Conquer (3)

- **Divide:** Trivial (divide the array from the middle)
- **Conquer:** Recursively compute the max subarrays of the left and right halves
- **Combine:** Compute the max-subarray crossing the *mid – point*
 - (can be done in $\Theta(n)$ time).
 - Return the max among the following:
 - the max subarray of the left-subarray
 - the max subarray of the rightsubarray
 - the max subarray crossing the mid-point

TODO : detailed solution in textbook...

Conclusion : Divide & Conquer

- Divide and conquer is just one of several powerful techniques for algorithm design.
- Divide-and-conquer algorithms can be analyzed using recurrences and the master method (so practice this math).
- Can lead to more efficient algorithms

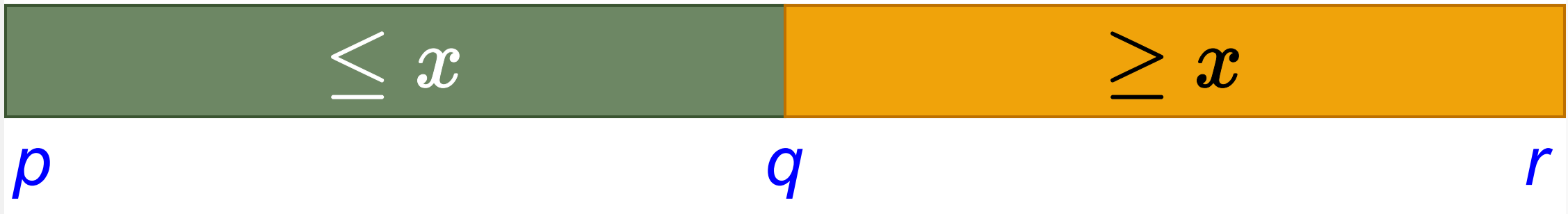
Quicksort (1)

- One of the most-used algorithms in practice
- Proposed by **C.A.R. Hoare** in 1962.
- Divide-and-conquer algorithm
- In-place algorithm
 - The additional space needed is $O(1)$
 - The sorted array is returned in the input array
 - *Reminder: Insertion-sort is also an in-place algorithm, but Merge-Sort is not in-place.*
- Very practical

Quicksort (2)

- **Divide:** Partition the array into 2 subarrays such that elements in the lower part \leq elements in the higher part
- **Conquer:** Recursively sort 2 subarrays
- **Combine:** Trivial (because in-place)

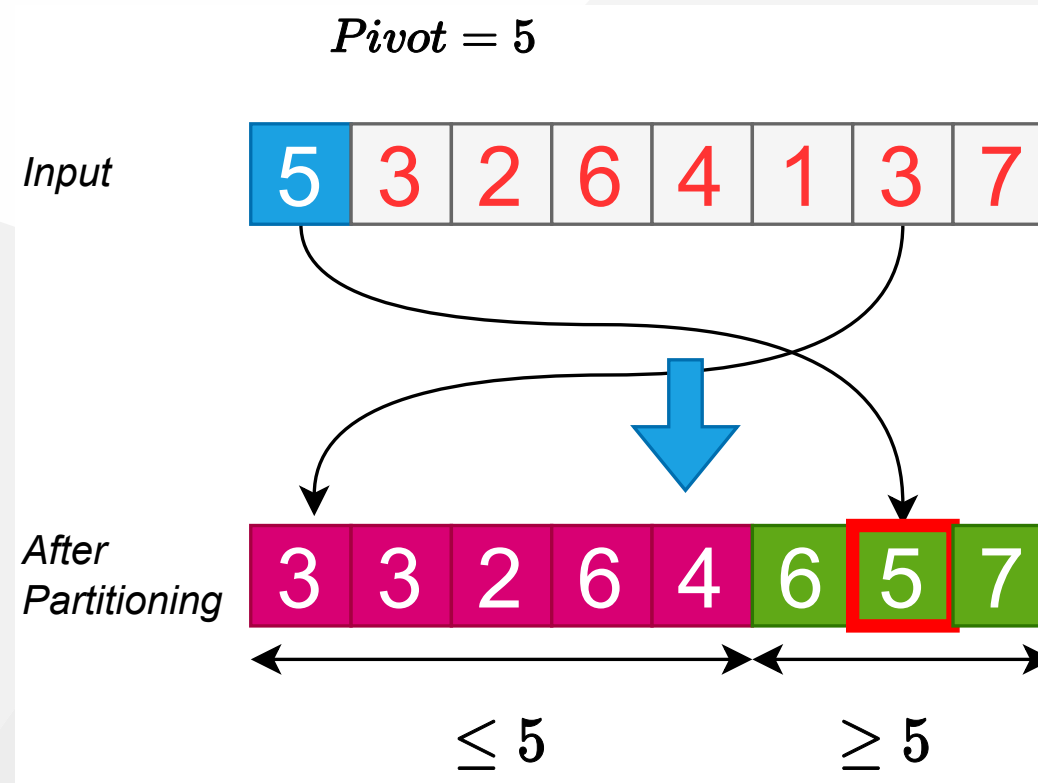
Key: Linear-time ($\Theta(n)$) partitioning algorithm



Divide: Partition the array around a pivot element

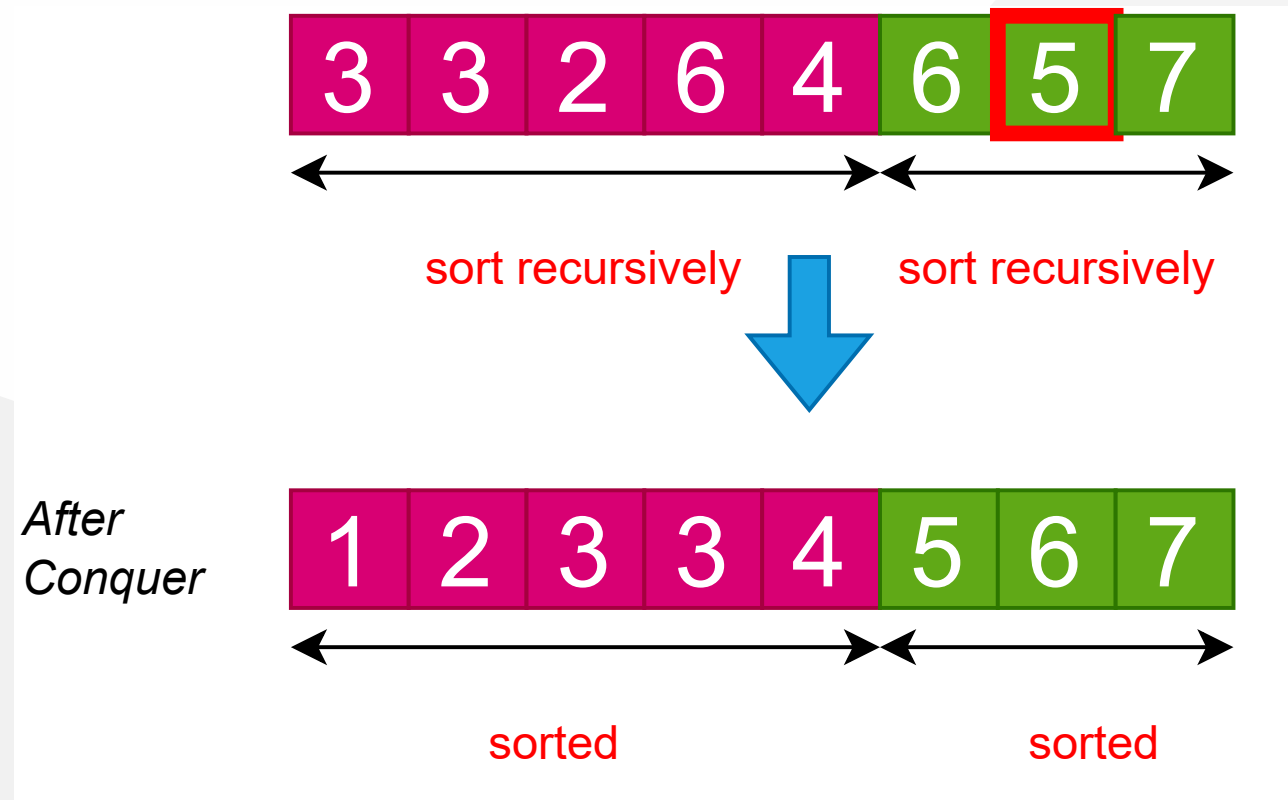
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- Choose a pivot element x
- Rearrange the array such that:
 - Left subarray: All elements $\leq x$
 - Right subarray: All elements $\geq x$



Conquer: Recursively Sort the Subarrays

Note: Everything in the left subarray \leq everything in the right subarray



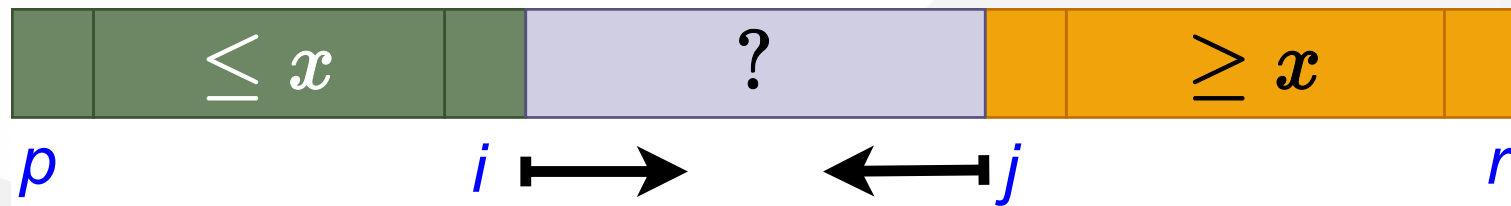
Note: Combine is trivial after conquer. Array already sorted.

Two partitioning algorithms

- **Hoare's algorithm:**

Partitions around the first element of subarray

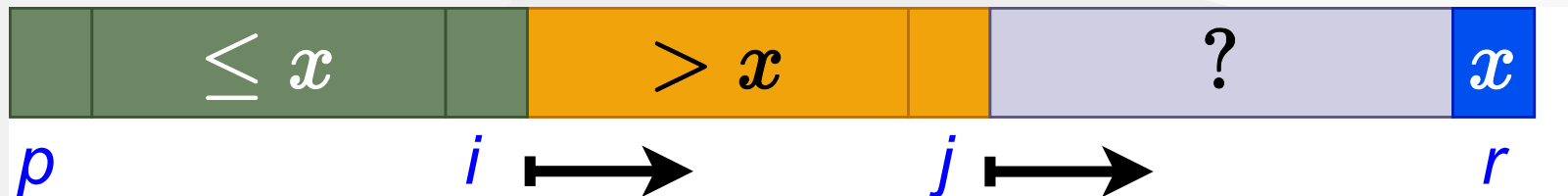
- ($pivot = x = A[p]$)



- **Lomuto's algorithm:**

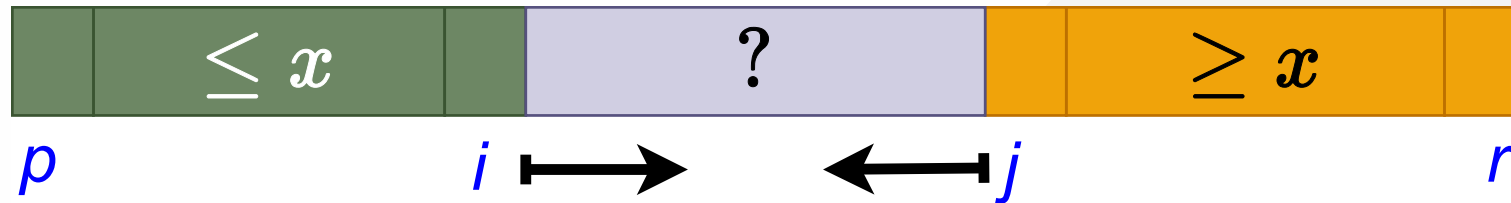
Partitions around the last element of subarray

- ($pivot = x = A[r]$)



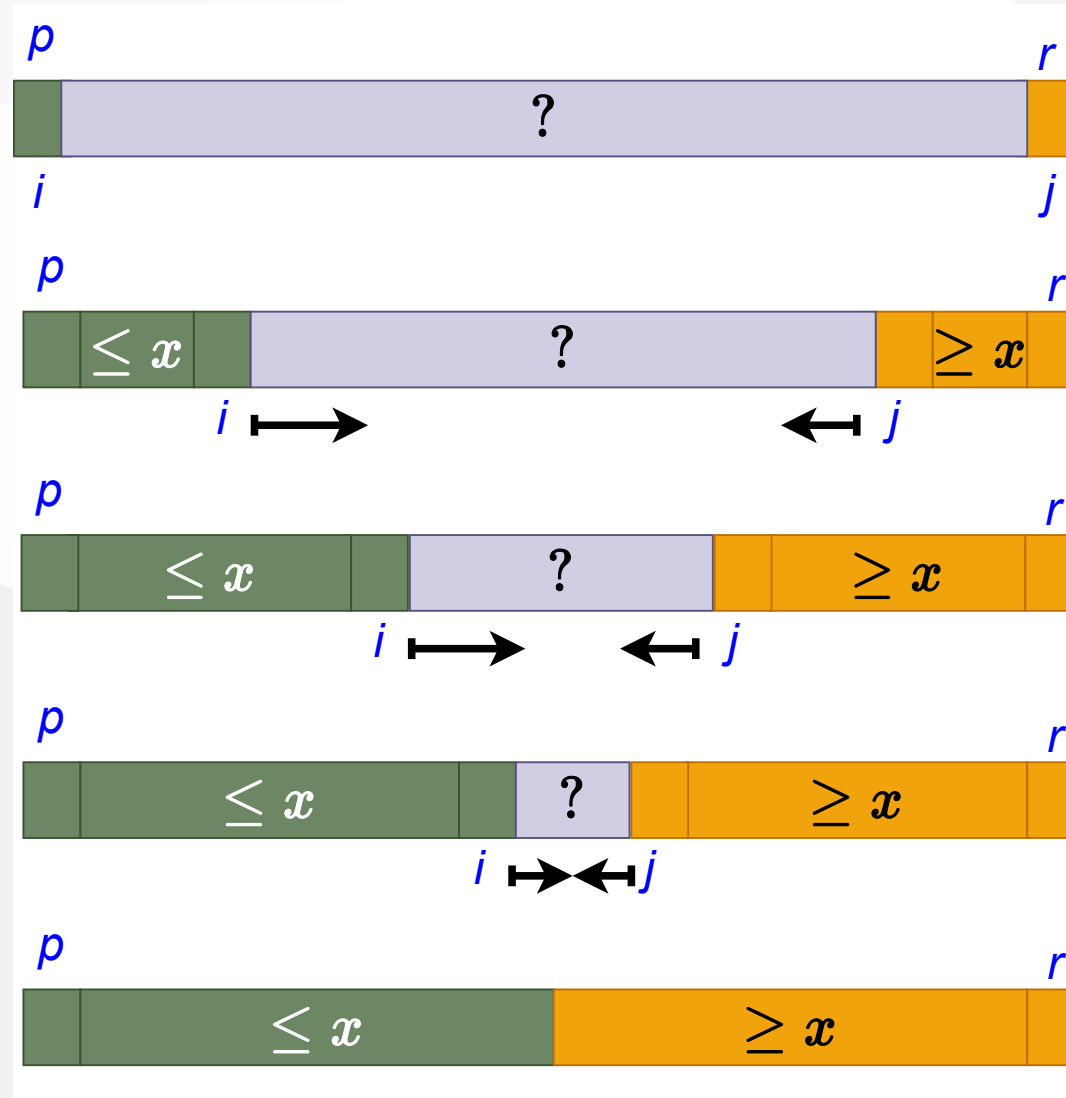
Hoare's Partitioning Algorithm (1)

- Choose a pivot element: $pivot = x = A[p]$



- Grow two regions:
 - from left to right: $A[p \dots i]$
 - from right to left: $A[j \dots r]$
 - such that:
 - every element in $A[p \dots i] \leq pivot$
 - every element in $A[j \dots r] \geq pivot$

Hoare's Partitioning Algorithm (2)



Hoare's Partitioning Algorithm (3)

- Elements are exchanged when
 - $A[i]$ is **too large** to belong to the **left** region
 - $A[j]$ is **too small** to belong to the **right** region
 - assuming that the inequality is strict
- The two regions $A[p \dots i]$ and $A[j \dots r]$ grow until $A[i] \geq pivot \geq A[j]$

```
H-PARTITION(A, p, r)
    pivot = A[p]
    i = p - 1
    j = r - 1
    while true do
        repeat j = j - 1 until A[j] <= pivot
        repeat i = i - 1 until A[i] <= pivot
        if i < j then
            exchange A[i] with A[j]
        else
            return j
```

Hoare's Partitioning Algorithm Example (Step-1)

Pivot = 5

Input

5	3	2	6	4	1	3	7
---	---	---	---	---	---	---	---

STEP – 1

H-PARTITION (A, p, r).

pivot \leftarrow A[p]

i \leftarrow p – 1

j \leftarrow r + 1

while true do

repeat *j* \leftarrow *j* – 1 **until** A[*j*] \leq *pivot*

repeat *i* \leftarrow *i* + 1 **until** A[*i*] \geq *pivot*

if *i* < *j* **then**

 exchange A[*i*] \leftrightarrow A[*j*]

else

return *j*

Hoare's Partitioning Algorithm Example (Step-2)

H-PARTITION (A, p, r).

$pivot \leftarrow A[p]$

$i \leftarrow p - 1$

$j \leftarrow r + 1$

while true do

repeat $j \leftarrow j - 1$ **until** $A[j] \leq pivot$

repeat $i \leftarrow i + 1$ **until** $A[i] \geq pivot$

if $i < j$ **then**

 exchange $A[i] \leftrightarrow A[j]$

else

return j

$Pivot = 5$

Input

5	3	2	6	4	1	3	7
---	---	---	---	---	---	---	---

i

j

$STEP - 2$

Hoare's Partitioning Algorithm Example (Step-3)

H-PARTITION (A, p, r).

$pivot \leftarrow A[p]$

$i \leftarrow p - 1$

$j \leftarrow r + 1$

while true do

repeat $j \leftarrow j - 1$ **until** $A[j] \leq pivot$

repeat $i \leftarrow i + 1$ **until** $A[i] \geq pivot$

if $i < j$ **then**

 exchange $A[i] \leftrightarrow A[j]$

else

return j

Pivot = 5

Input

5	3	2	6	4	1	3	7
---	---	---	---	---	---	---	---

i

j

STEP - 3

Hoare's Partitioning Algorithm Example (Step-4)

H-PARTITION (A, p, r).

$pivot \leftarrow A[p]$

$i \leftarrow p - 1$

$j \leftarrow r + 1$

while true do

repeat $j \leftarrow j - 1$ **until** $A[j] \leq pivot$

repeat $i \leftarrow i + 1$ **until** $A[i] \geq pivot$

if $i < j$ **then**

 exchange $A[i] \leftrightarrow A[j]$

else

return j

Pivot = 5

Input

5	3	2	6	4	1	3	7
---	---	---	---	---	---	---	---

i

j

STEP - 4

Hoare's Partitioning Algorithm Example (Step-5)

H-PARTITION (A, p, r).

$pivot \leftarrow A[p]$

$i \leftarrow p - 1$

$j \leftarrow r + 1$

while true do

repeat $j \leftarrow j - 1$ **until** $A[j] \leq pivot$

repeat $i \leftarrow i + 1$ **until** $A[i] \geq pivot$

if $i < j$ **then**

 exchange $A[i] \leftrightarrow A[j]$

else

return j

Pivot = 5

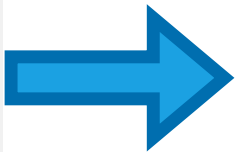
Input

5	3	2	6	4	1	3	7
---	---	---	---	---	---	---	---

i

j

STEP – 5



Hoare's Partitioning Algorithm Example (Step-6)

H-PARTITION (A, p, r)

$pivot \leftarrow A[p]$

$i \leftarrow p - 1$

$j \leftarrow r + 1$

while true do

repeat $j \leftarrow j - 1$ **until** $A[j] \leq pivot$

repeat $i \leftarrow i + 1$ **until** $A[i] \geq pivot$

if $i < j$ **then**

 exchange $A[i] \leftrightarrow A[j]$

else

return j

$Pivot = 5$

Input

3	3	2	6	4	1	5	7
						i	j

STEP - 6

Hoare's Partitioning Algorithm Example (Step-7)

H-PARTITION (A, p, r).

$pivot \leftarrow A[p]$

$i \leftarrow p - 1$

$j \leftarrow r + 1$

while true do

repeat $j \leftarrow j - 1$ **until** $A[j] \leq pivot$

repeat $i \leftarrow i + 1$ **until** $A[i] \geq pivot$

if $i < j$ **then**

 exchange $A[i] \leftrightarrow A[j]$

else

return j

$Pivot = 5$

Input

3	3	2	6	4	1	5	7
---	---	---	---	---	---	---	---

i

j

$STEP - 7$

Hoare's Partitioning Algorithm Example (Step-8)

H-PARTITION (A, p, r).

$pivot \leftarrow A[p]$

$i \leftarrow p - 1$

$j \leftarrow r + 1$

while true do

repeat $j \leftarrow j - 1$ **until** $A[j] \leq pivot$

repeat $i \leftarrow i + 1$ **until** $A[i] \geq pivot$

if $i < j$ **then**

 exchange $A[i] \leftrightarrow A[j]$

else

return j

$Pivot = 5$

Input

3	3	2	6	4	1	5	7
---	---	---	---	---	---	---	---

i

j

$STEP - 8$

Hoare's Partitioning Algorithm Example (Step-9)

H-PARTITION (A, p, r).

$pivot \leftarrow A[p]$

$i \leftarrow p - 1$

$j \leftarrow r + 1$

while true do

repeat $j \leftarrow j - 1$ **until** $A[j] \leq pivot$

repeat $i \leftarrow i + 1$ **until** $A[i] \geq pivot$

if $i < j$ **then**

exchange $A[i] \leftrightarrow A[j]$

else

return j

Pivot = 5

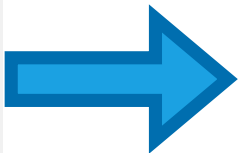
Input

3	3	2	6	4	1	5	7
---	---	---	---	---	---	---	---

i

j

STEP – 9



Hoare's Partitioning Algorithm Example (Step-10)

H-PARTITION (A, p, r).

$pivot \leftarrow A[p]$

$i \leftarrow p - 1$

$j \leftarrow r + 1$

while true do

repeat $j \leftarrow j - 1$ **until** $A[j] \leq pivot$

repeat $i \leftarrow i + 1$ **until** $A[i] \geq pivot$

if $i < j$ **then**

 exchange $A[i] \leftrightarrow A[j]$

else

return j

Pivot = 5

Input

3	3	2	1	4	6	5	7
---	---	---	---	---	---	---	---

i

j

STEP – 10

Hoare's Partitioning Algorithm Example (Step-11)

H-PARTITION (A, p, r).

$pivot \leftarrow A[p]$

$i \leftarrow p - 1$

$j \leftarrow r + 1$

while true do

repeat $j \leftarrow j - 1$ **until** $A[j] \leq pivot$

repeat $i \leftarrow i + 1$ **until** $A[i] \geq pivot$

if $i < j$ **then**

 exchange $A[i] \leftrightarrow A[j]$

else

return j

Pivot = 5

Input

3	3	2	1	4	6	5	7
---	---	---	---	---	---	---	---

i j

STEP – 11

Hoare's Partitioning Algorithm Example (Step-12)

H-PARTITION (A, p, r).

$pivot \leftarrow A[p]$

$i \leftarrow p - 1$

$j \leftarrow r + 1$

while true do

repeat $j \leftarrow j - 1$ **until** $A[j] \leq pivot$

repeat $i \leftarrow i + 1$ **until** $A[i] \geq pivot$

if $i < j$ **then**

 exchange $A[i] \leftrightarrow A[j]$

else

return j

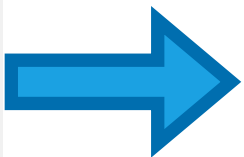
$Pivot = 5$

Input

3	3	2	1	4	6	5	7
---	---	---	---	---	---	---	---

j i

$STEP - 12$



Hoare's Partitioning Algorithm - Notes

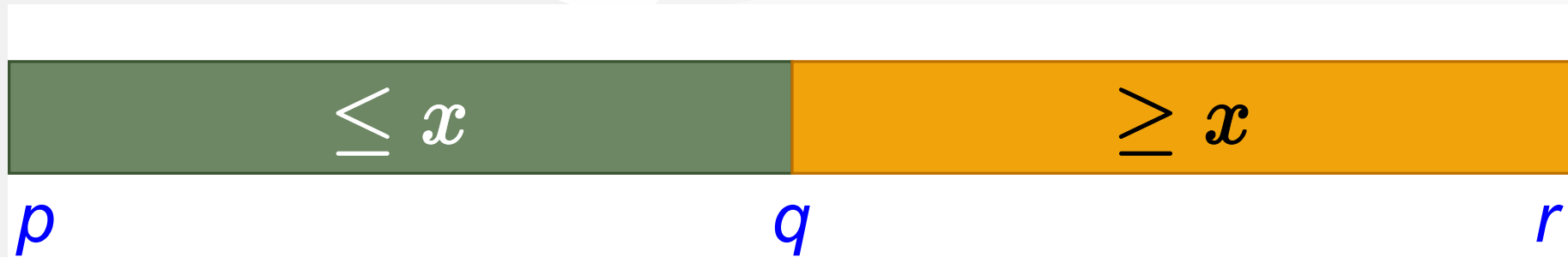
- Elements are exchanged when
 - $A[i]$ is **too large** to belong to the **left** region
 - $A[j]$ is **too small** to belong to the **right** region
 - assuming that the inequality is strict
- The two regions $A[p \dots i]$ and $A[j \dots r]$ grow until $A[i] \geq pivot \geq A[j]$
- The asymptotic runtime of Hoare's partitioning algorithm $\Theta(n)$

```
H-PARTITION(A, p, r)
    pivot = A[p]
    i = p - 1
    j = r - 1
    while true do
        repeat j = j - 1 until A[j] <= pivot
        repeat i = i - 1 until A[i] <= pivot
        if i < j then exchange A[i] with A[j]
        else return j
```

Quicksort with Hoare's Partitioning Algorithm

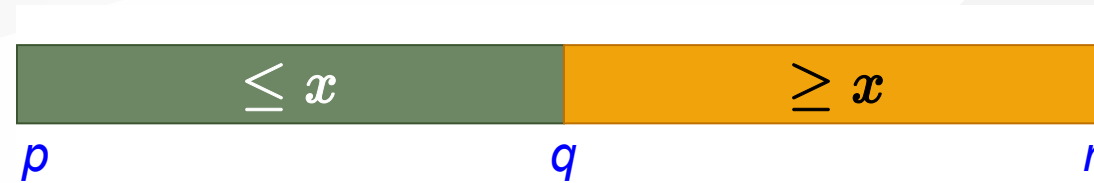
```
QUICKSORT (A, p, r)
  if p < r then
    q = H-PARTITION(A, p, r)
    QUICKSORT(A, p, q)
    QUICKSORT(A, q + 1, r)
  endif
```

Initial invocation: QUICKSORT(A, 1, n)



Hoare's Partitioning Algorithm: Pivot Selection

- if we select pivot to be $A[r]$ instead of $A[p]$ in H-PARTITION



Pivot = 7

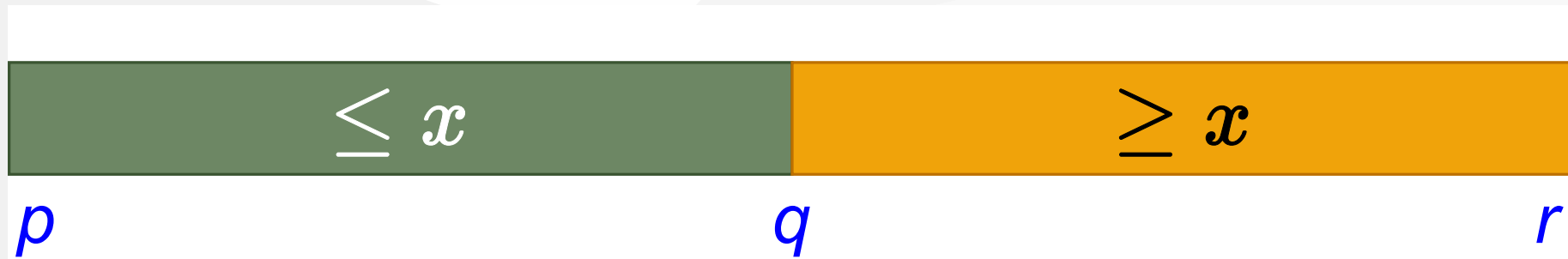


- Consider the example where $A[r]$ is the largest element in the array:
 - End of H-PARTITION: $i = j = r$
 - In QUICKSORT: $q = r$
 - So, recursive call to:
 - QUICKSORT(A , p , $q=r$)
 - infinite loop

Correctness of Hoare's Algorithm (1)

We need to prove 3 claims to show correctness:

- Indices i and j never reference A outside the interval $A[p \dots r]$
- Split is always non-trivial; i.e., $j \neq r$ at termination
- Every element in $A[p \dots j] \leq$ every element in $A[j + 1 \dots r]$ at termination



Correctness of Hoare's Algorithm (2)

- Notations:
 - k : # of times the while-loop iterates until termination
 - i_m : the value of index i at the end of iteration m
 - j_m : the value of index j at the end of iteration m
 - x : the value of the pivot element
- **Note:** We always have $i_1 = p$ and $p \leq j_1 \leq r$
because $x = A[p]$

Correctness of Hoare's Algorithm (3)

Lemma 1: Either $i_k = j_k$ or $i_k = j_k + 1$ at termination

Proof of Lemma 1:

- The algorithm terminates when $i \geq j$ (the else condition).
- So, it is sufficient to prove that $i_k - j_k \leq 1$
- There are 2 cases to consider:
 - Case 1: $k = 1$, i.e. the algorithm terminates in a single iteration
 - Case 2: $k > 1$, i.e. the alg. does not terminate in a single iter.

By contradiction, assume there is a run with $i_k - j_k > 1$

Correctness of Hoare's Algorithm (4)

Original correctness claims:

- Indices i and j never reference A outside the interval $A[p \dots r]$
- Split is always non-trivial; i.e., $j \neq r$ at termination

Proof:

- For $k = 1$:
 - Trivial because $i_1 = j_1 = p$ (see Case 1 in proof of Lemma 2)
- For $k > 1$:
 - $i_k > p$ and $j_k < r$ (due to the repeat-until loops moving indices)
 - $i_k \leq r$ and $j_k \geq p$ (due to Lemma 1 and the statement above)

The proof of claims (a) and (b) complete

Correctness of Hoare's Algorithm (5)

Lemma 2: At the end of iteration m , where $m < k$ (i.e. m is not the last iteration), we must have:

$$A[p \dots i_m] \leq x \text{ and } A[j_m \dots r] \geq x$$

Proof of Lemma 2:

- **Base case:** $m = 1$ and $k > 1$ (i.e. the alg. does not terminate in the first iter.)

Ind. Hyp.: At the end of iteration $m - 1$, where $m < k$ (i.e. m is not the last iteration), we must have:

$$A[p \dots i_m - 1] \leq x \text{ and } A[j_m - 1 \dots r] \geq x$$

General case: The lemma holds for m , where $m < k$

Proof of base case complete!

Correctness of Hoare's Algorithm (6)

Original correctness claim:

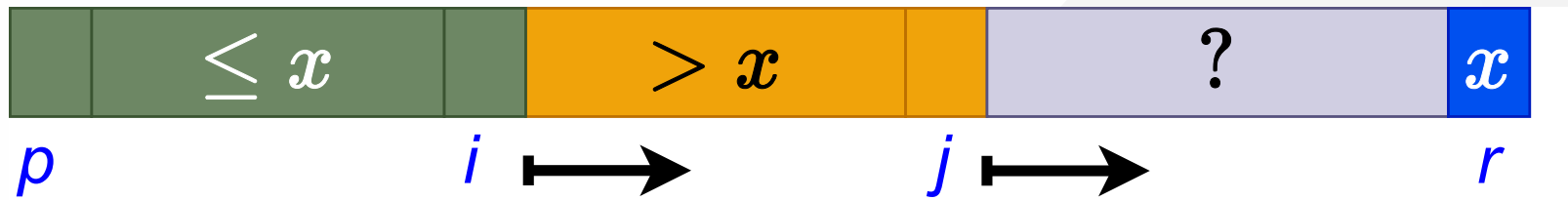
- (c) Every element in $A[\dots j] \leq$ every element in $A[j + \dots r]$ at termination

Proof of claim (c)

- There are 3 cases to consider:
 - **Case 1:** $k = 1$, i.e. the algorithm terminates in a single iteration
 - **Case 2:** $k > 1$ and $i_k = j_k$
 - **Case 3:** $k > 1$ and $i_k = j_k + 1$

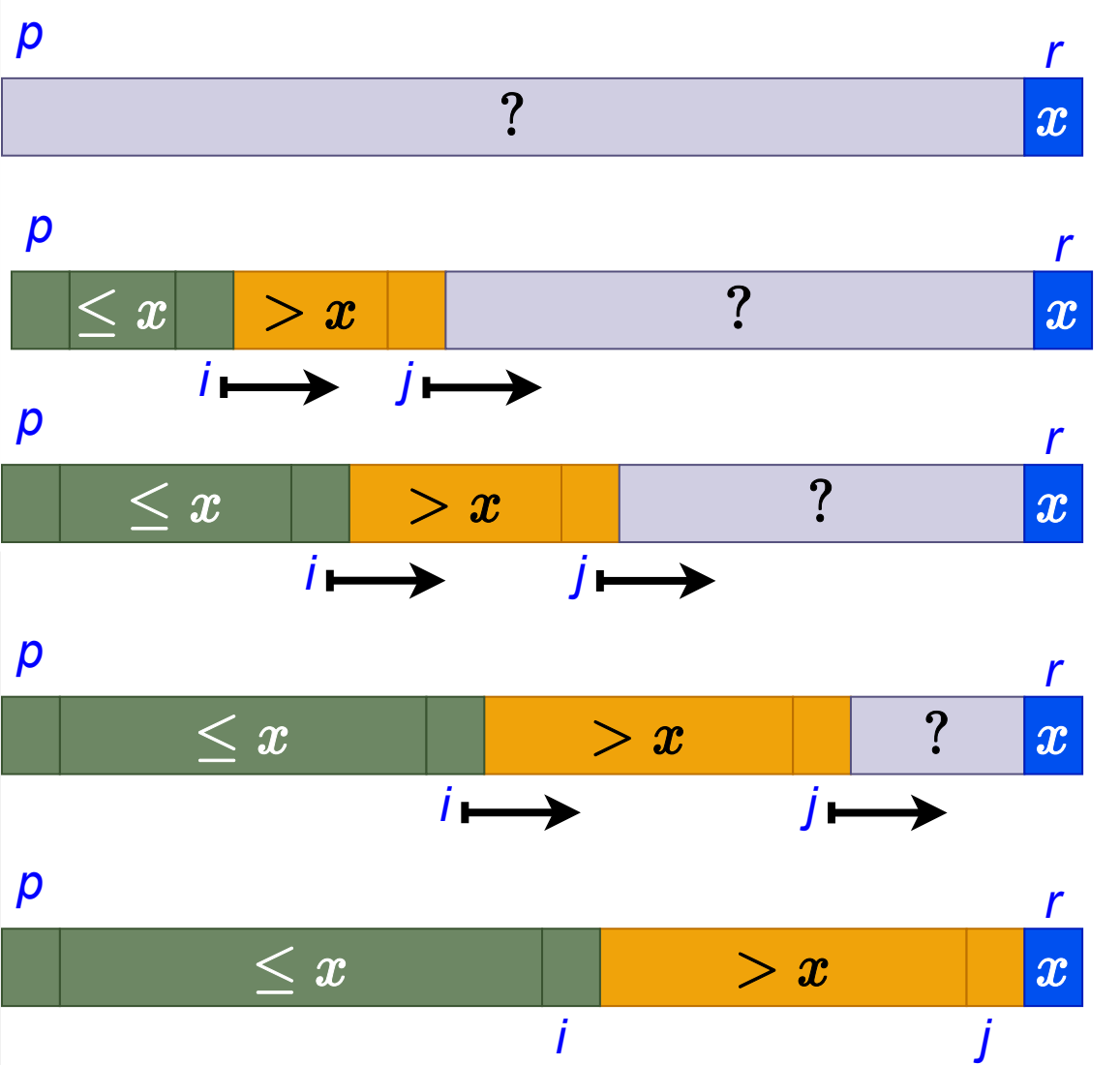
Lomuto's Partitioning Algorithm (1)

- Choose a pivot element: $pivot = x = A[r]$

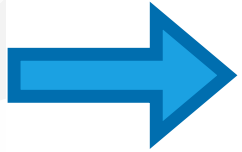


- Grow two regions:
 - from left to right: $A[p \dots i]$
 - from left to right: $A[i + 1 \dots j]$
 - such that:
 - every element in $A[p \dots i] \leq pivot$
 - every element in $A[i + 1 \dots j] > pivot$

Lomuto's Partitioning Algorithm (2)



Lomuto's Partitioning Algorithm Ex. (Step-1)



L-PARTITION (A, p, r).

$pivot \leftarrow A[r]$

$i \leftarrow p - 1$

for $j \leftarrow p$ **to** $r - 1$ **do**

if $A[j] \leq pivot$ **then**

$i \leftarrow i + 1$

exchange $A[i] \leftrightarrow A[j]$

exchange $A[i + 1] \leftrightarrow A[r]$

return $i + 1$

Input

p	$Pivot = 4$						r
7	8	2	6	5	1	3	4

STEP - 1

Lomuto's Partitioning Algorithm Ex. (Step-2)

L-PARTITION (A, p, r).

$pivot \leftarrow A[r]$

$i \leftarrow p - 1$

for $j \leftarrow p$ **to** $r - 1$ **do**

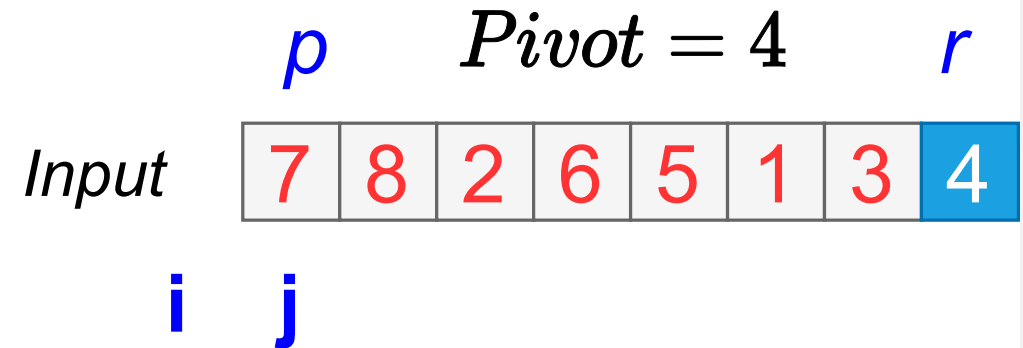
if $A[j] \leq pivot$ **then**

$i \leftarrow i + 1$

exchange $A[i] \leftrightarrow A[j]$

exchange $A[i + 1] \leftrightarrow A[r]$

return $i + 1$



STEP - 2

Lomuto's Partitioning Algorithm Ex. (Step-3)

L-PARTITION (A, p, r).

$pivot \leftarrow A[r]$

$i \leftarrow p - 1$

for $j \leftarrow p$ **to** $r - 1$ **do**

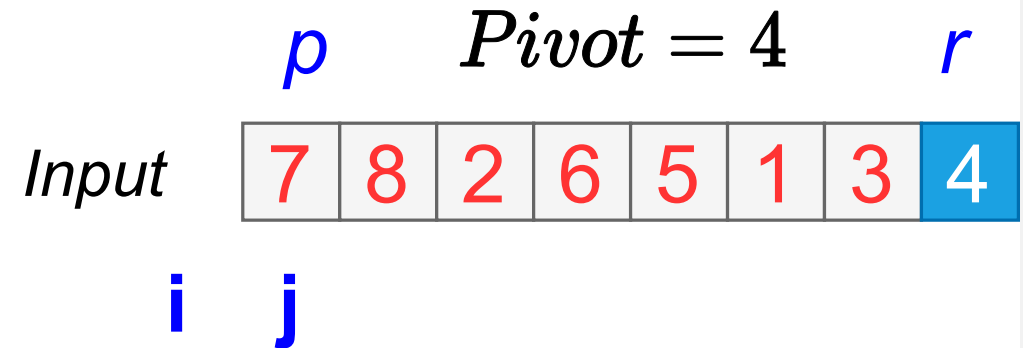
if $A[j] \leq pivot$ **then**

$i \leftarrow i + 1$

exchange $A[i] \leftrightarrow A[j]$

exchange $A[i + 1] \leftrightarrow A[r]$

return $i + 1$



STEP - 3

Lomuto's Partitioning Algorithm Ex. (Step-4)

L-PARTITION (A, p, r).

$pivot \leftarrow A[r]$

$i \leftarrow p - 1$

for $j \leftarrow p$ **to** $r - 1$ **do**

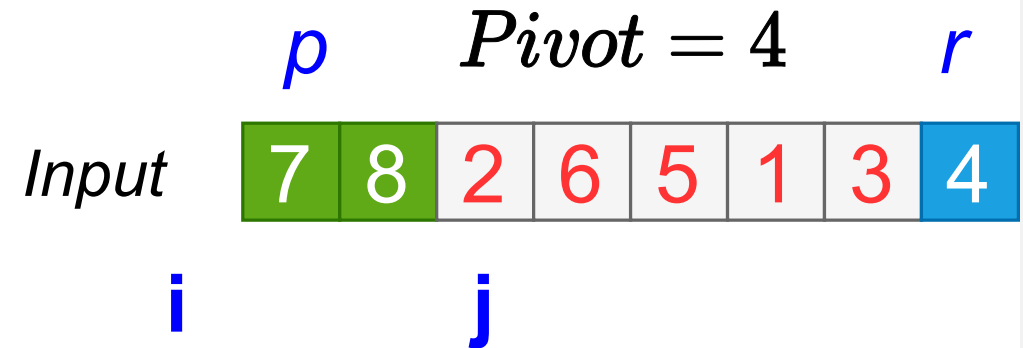
if $A[j] \leq pivot$ **then**

$i \leftarrow i + 1$

exchange $A[i] \leftrightarrow A[j]$

exchange $A[i + 1] \leftrightarrow A[r]$

return $i + 1$



STEP - 4

Lomuto's Partitioning Algorithm Ex. (Step-5)

L-PARTITION (A, p, r).

$pivot \leftarrow A[r]$

$i \leftarrow p - 1$

for $j \leftarrow p$ **to** $r - 1$ **do**

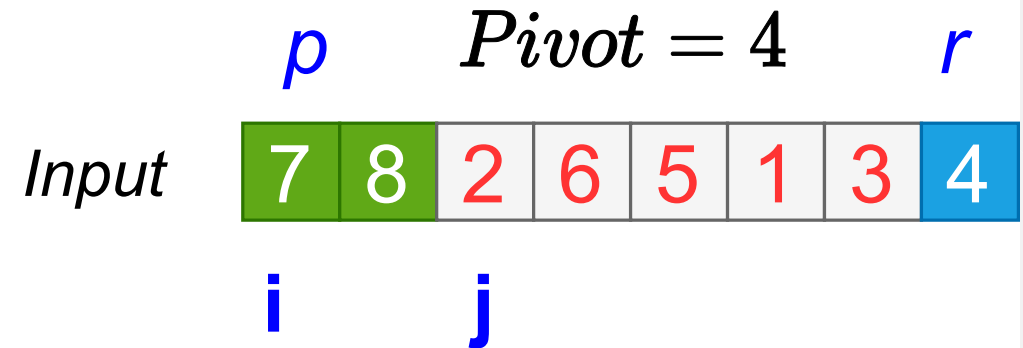
if $A[j] \leq pivot$ **then**

$i \leftarrow i + 1$

exchange $A[i] \leftrightarrow A[j]$

exchange $A[i + 1] \leftrightarrow A[r]$

return $i + 1$



STEP - 5

Lomuto's Partitioning Algorithm Ex. (Step-6)

L-PARTITION (A, p, r).

$pivot \leftarrow A[r]$

$i \leftarrow p - 1$

for $j \leftarrow p$ **to** $r - 1$ **do**

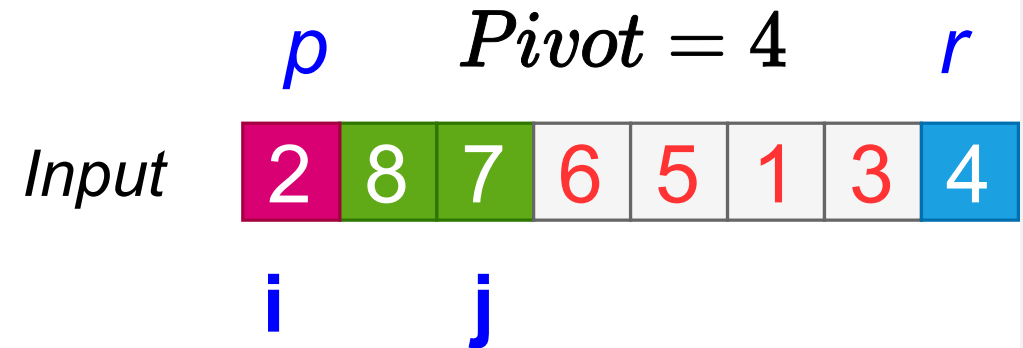
if $A[j] \leq pivot$ **then**

$i \leftarrow i + 1$

exchange $A[i] \leftrightarrow A[j]$

exchange $A[i + 1] \leftrightarrow A[r]$

return $i + 1$



STEP - 6

Lomuto's Partitioning Algorithm Ex. (Step-9)

L-PARTITION (A, p, r).

$pivot \leftarrow A[r]$

$i \leftarrow p - 1$

for $j \leftarrow p$ **to** $r - 1$ **do**

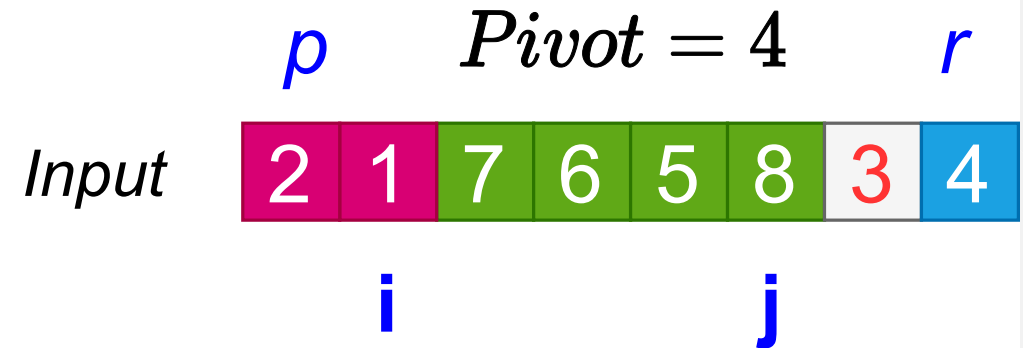
if $A[j] \leq pivot$ **then**

$i \leftarrow i + 1$

exchange $A[i] \leftrightarrow A[j]$

exchange $A[i + 1] \leftrightarrow A[r]$

return $i + 1$



STEP - 9

Lomuto's Partitioning Algorithm Ex. (Step-15)

L-PARTITION (A, p, r).

$pivot \leftarrow A[r]$

$i \leftarrow p - 1$

for $j \leftarrow p$ **to** $r - 1$ **do**

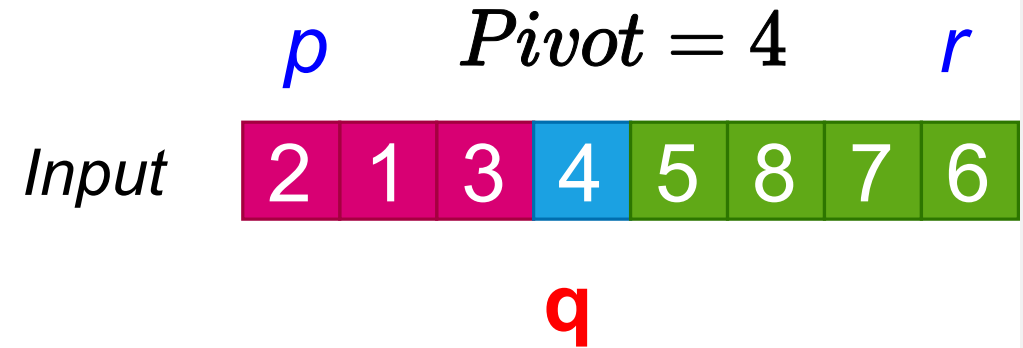
if $A[j] \leq pivot$ **then**

$i \leftarrow i + 1$

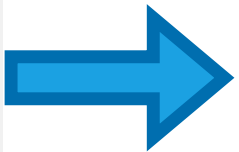
exchange $A[i] \leftrightarrow A[j]$

exchange $A[i + 1] \leftrightarrow A[r]$

return $i + 1$



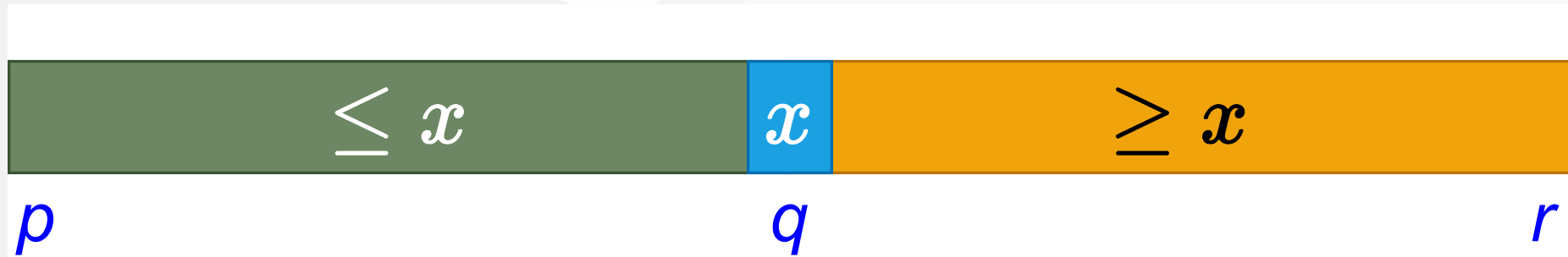
STEP – 15



Quicksort with Lomuto's Partitioning Algorithm

```
QUICKSORT (A, p, r)
  if p < r then
    q = L-PARTITION(A, p, r)
    QUICKSORT(A, p, q - 1)
    QUICKSORT(A, q + 1, r)
  endif
```

Initial invocation: QUICKSORT(A, 1, n)



Comparison of Hoare's & Lomuto's Algorithms (1)

- Notation: $n = r - p + 1$
 - $pivot = A[p]$ (Hoare)
 - $pivot = A[r]$ (Lomuto)
- # of element exchanges: $e(n)$
 - Hoare: $0 \leq e(n) \leq \lfloor \frac{n}{2} \rfloor$
 - Best: $k = 1$ with $i_1 = j_1 = p$ (i.e., $A[p + 1 \dots r] > pivot$)
 - Worst: $A[p + 1 \dots p + \lfloor \frac{n}{2} \rfloor - 1] \geq pivot \geq A[p + \lceil \frac{n}{2} \rceil \dots r]$
 - Lomuto : $1 \leq e(n) \leq n$
 - Best: $A[p \dots r - 1] > pivot$
 - Worst: $A[p \dots r - 1] \leq pivot$

Comparison of Hoare's & Lomuto's Algorithms (2)

- # of element comparisons: $c_e(n)$
 - Hoare: $n + 1 \leq c_e(n) \leq n + 2$
 - Best: $i_k = j_k$
 - Worst: $i_k = j_k + 1$
 - Lomuto: $c_e(n) = n - 1$
- # of index comparisons: $c_i(n)$
 - Hoare: $1 \leq c_i(n) \leq \lfloor \frac{n}{2} \rfloor + 1 \mid (c_i(n) = e(n) + 1)$
 - Lomuto: $c_i(n) = n - 1$

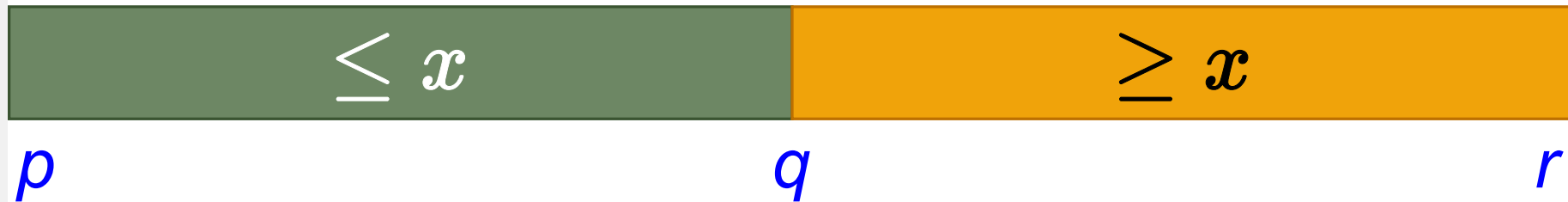
Comparison of Hoare's & Lomuto's Algorithms (3)

- # of index increment/decrement operations: $a(n)$
 - **Hoare:** $n + 1 \leq a(n) \leq n + 2 \mid (a(n) = c_e(n))$
 - **Lomuto:** $n \leq a(n) \leq 2n - 1 \mid (a(n) = e(n) + (n - 1))$
- Hoare's algorithm is in general faster
- Hoare behaves better when pivot is repeated in $A[p \dots r]$
 - **Hoare:** Evenly distributes them between left & right regions
 - **Lomuto:** Puts all of them to the left region

Analysis of Quicksort (1)

```
QUICKSORT (A, p, r)
  if p < r then
    q = H-PARTITION(A, p, r)
    QUICKSORT(A, p, q)
    QUICKSORT(A, q + 1, r)
  endif
```

Initial invocation: QUICKSORT(A, 1, n)



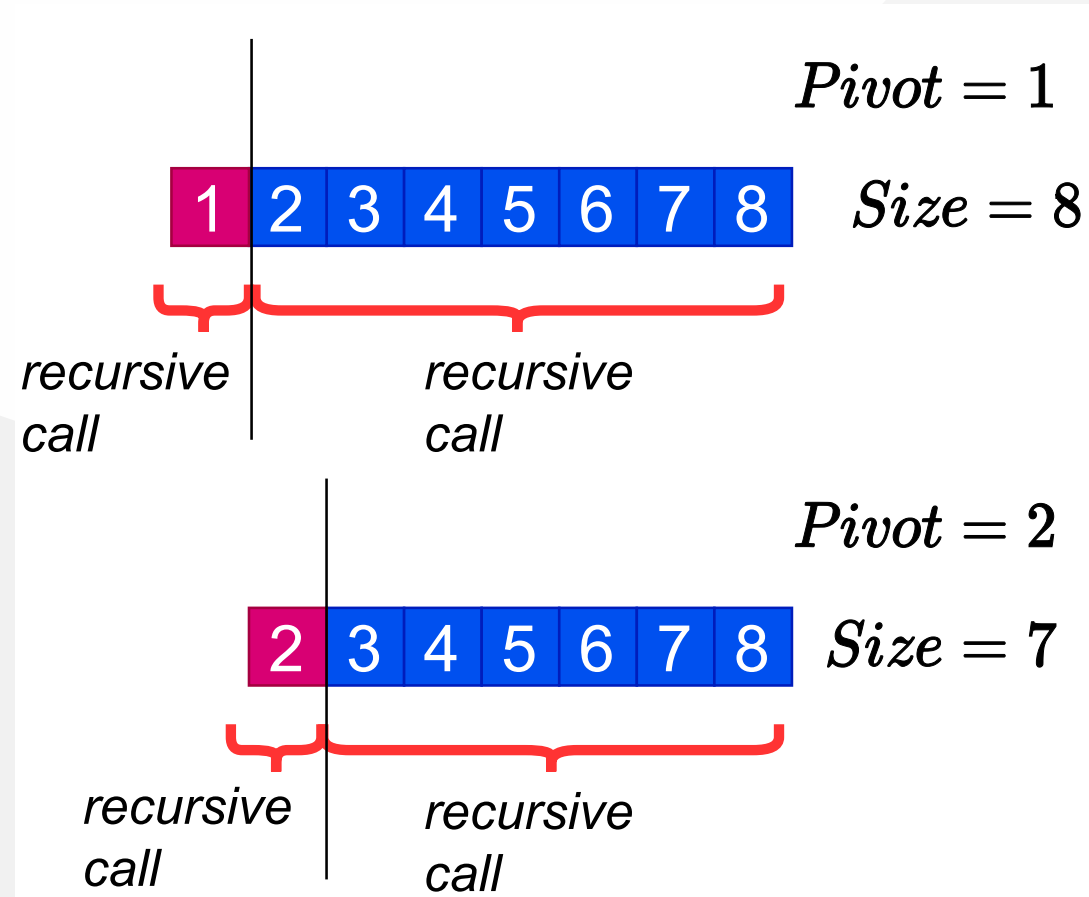
Assume **all** elements are **distinct** in the following analysis

Analysis of Quicksort (2)

- **H-PARTITION** always chooses $A[p]$ (the first element) as the pivot.
- The runtime of **QUICKSORT** on an already-sorted array is $\Theta(n^2)$

Example: An Already Sorted Array

Partitioning always leads to 2 parts of size 1 and $n - 1$



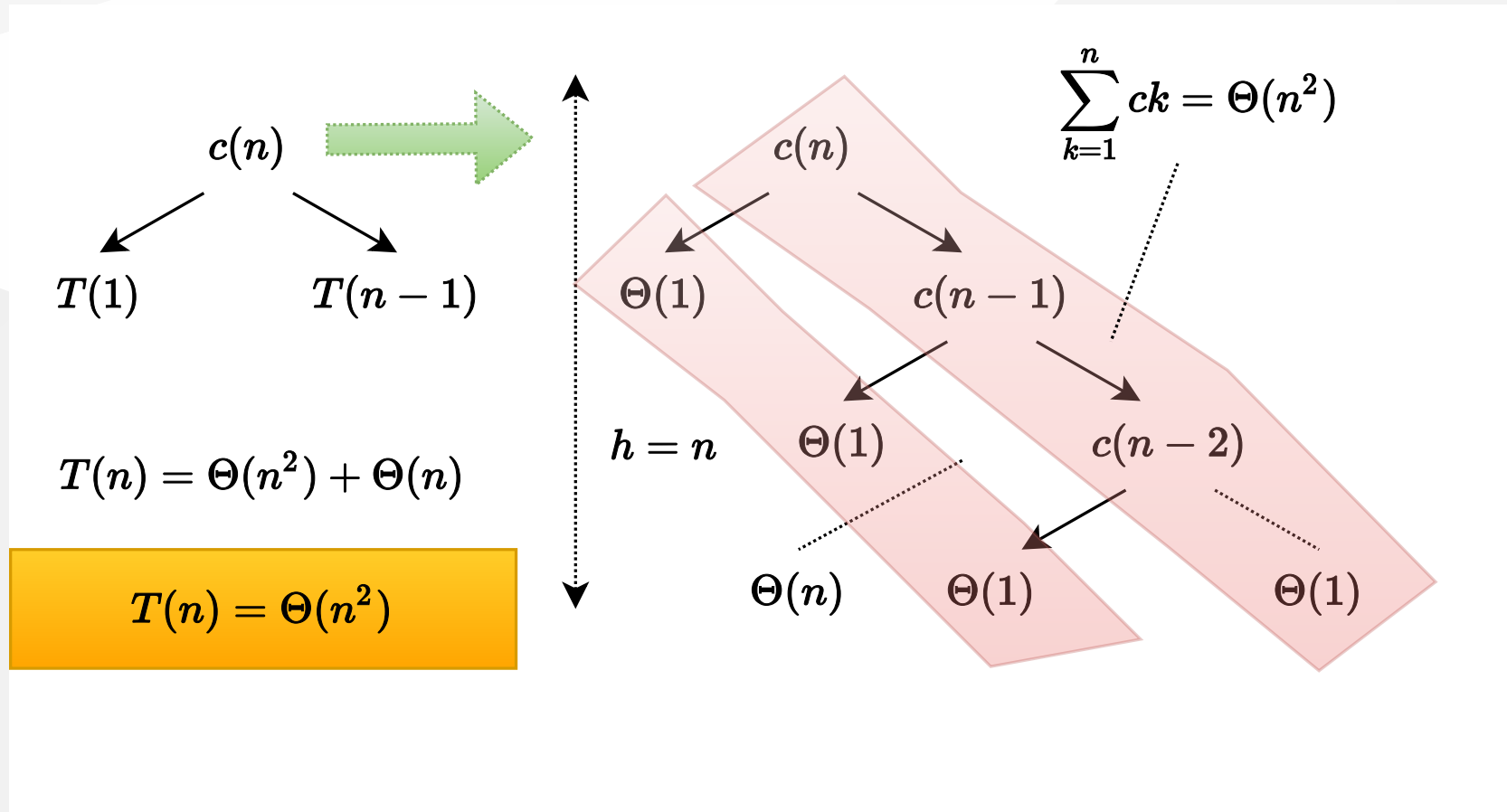
Worst Case Analysis of Quicksort

- **Worst case** is when the **PARTITION** algorithm always returns **imbalanced partitions** (of size 1 and $n - 1$) in every recursive call.
 - This happens when the pivot is selected to be either the min or **max** element.
 - This happens for **H-PARTITION** when the input array is already sorted or reverse sorted

$$\begin{aligned}T(n) &= T(1) + T(n - 1) + \Theta(n) \\&= T(n - 1) + \Theta(n) \\&= \Theta(n^2)\end{aligned}$$

Worst Case Recursion Tree

$$T(n) = T(1) + T(n-1) + cn$$



Best Case Analysis (for intuition only)

- If we're extremely lucky, **H-PARTITION** splits the array evenly at every recursive call

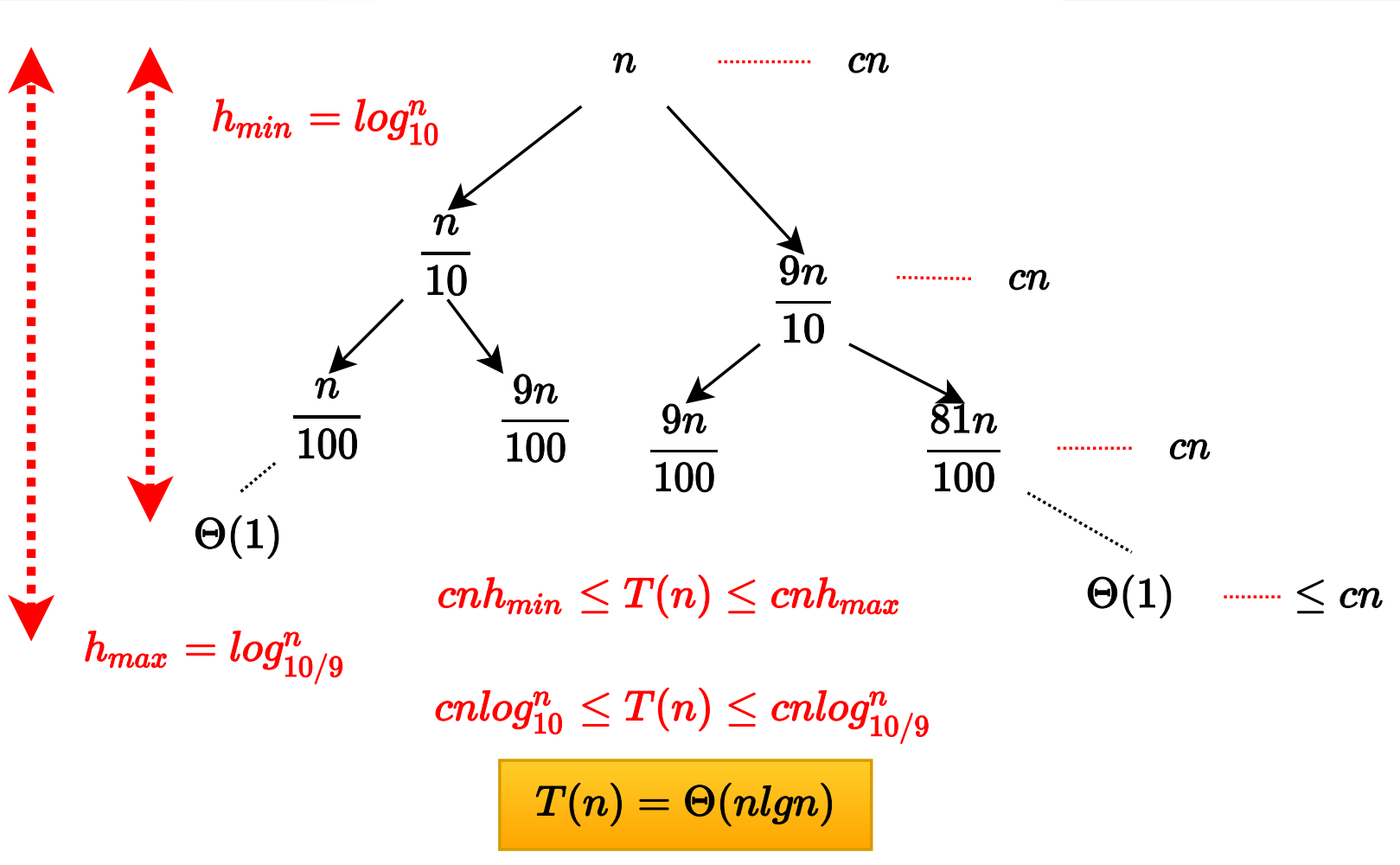
$$\begin{aligned}T(n) &= 2T(n/2) + \Theta(n) \\ &= \Theta(n \lg n)\end{aligned}$$

(same as merge sort)

- Instead of splitting 0.5 : 0.5, if we split 0.1 : 0.9 then we need solve following equation.

$$\begin{aligned}T(n) &= T(n/10) + T(9n/10) + \Theta(n) \\ &= \Theta(n \lg n)\end{aligned}$$

“Almost-Best” Case Analysis



Balanced Partitioning (1)

- We have seen that if **H-PARTITION** always splits the array with $0.1 - to - 0.9$ ratio, the runtime will be $\Theta(n \lg n)$.
- Same is true with a split ratio of $0.01 - to - 0.99$, etc.
- Possible to show that if the split has always constant ($\Theta(1)$) proportionality, then the runtime will be $\Theta(n \lg n)$.
- In other words, for a **constant** $\alpha \mid (0 < \alpha \leq 0.5)$:
 - $\alpha - to - (1 - \alpha)$ proportional split yields $\Theta(n \lg n)$ total runtime

Balanced Partitioning (2)

- In the rest of the analysis, assume that all input permutations are equally likely.
 - This is only to gain some intuition
 - We cannot make this assumption for average case analysis
 - We will revisit this assumption later
- Also, assume that all input elements are distinct.

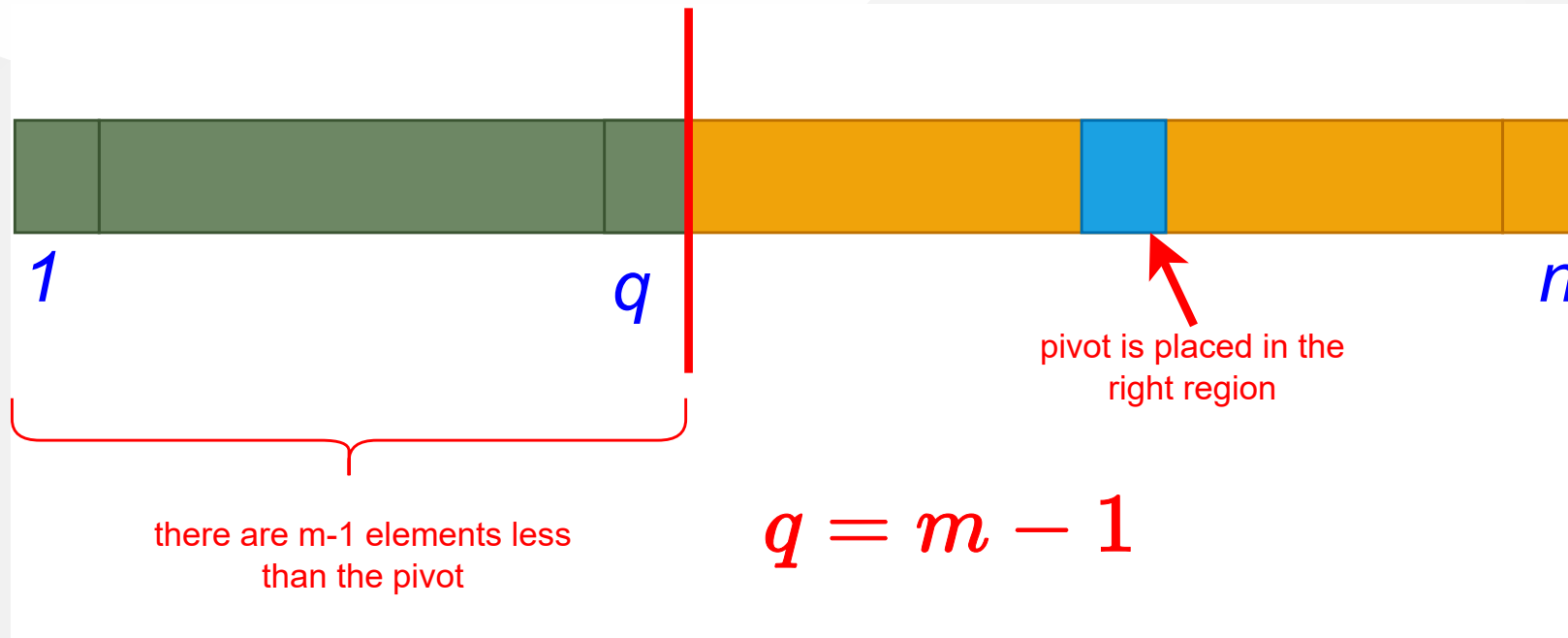
Balanced Partitioning (3)

- **Question:** What is the probability that H-PARTITION returns a split that is more balanced than $0.1 - t_0 - 0.9$?

Balanced Partitioning (4)

Reminder: *H-PARTITION* will place the pivot in the right partition unless the pivot is the smallest element in the arrays.

Question: If the pivot selected is the m th smallest value ($1 < m \leq n$) in the input array, what is the size of the left region after partitioning?

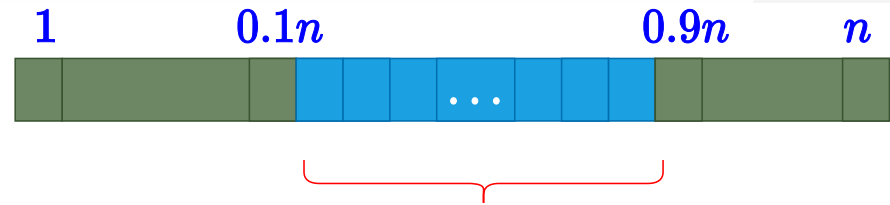


Balanced Partitioning (5)

- **Question:** What is the probability that the **pivot** selected is the m^{th} smallest value in the array of size n ?
 - $1/n$ (*since all input permutations are equally likely*)
- **Question:** What is the probability that the left partition returned by **H-PARTITION** has size m , where $1 < m < n$?
 - $1/n$ (*due to the answers to the previous 2 questions*)

Balanced Partitioning (6)

- **Question:** What is the probability that H-PARTITION returns a split that is more balanced than $0.1 - to - 0.9$?



The partition boundary will be in this region for a more balanced split than

$0.1 - to - 0.9$

$$Probability = \sum_{q=0.1n+1}^{0.9n-1} \frac{1}{n} = \frac{1}{n} (0.9n - 1 - 0.1n - 1 + 1)$$

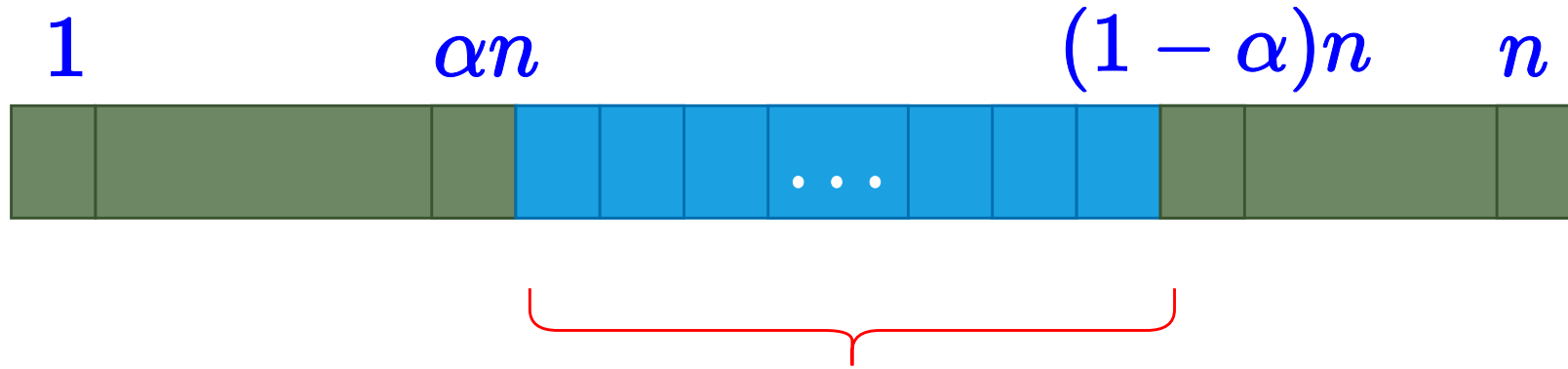
$$= 0.8 - \frac{1}{n}$$

$$\approx 0.8 \text{ for large } n$$

Balanced Partitioning (7)

- The probability that **H-PARTITION** yields a split that is more balanced than $0.1 - to - 0.9$ is 80% on a random array.
- Let $P_{\alpha>}$ be the probability that **H-PARTITION** yields a split more balanced than $\alpha - to - (1 - \alpha)$, where $0 < \alpha \leq 0.5$
- Repeat the analysis to generalize the previous result

balanced than $\alpha - to - (1 - \alpha)$?



The partition boundary will be in this region for a more balanced split than

$$\alpha n - to - (1 - \alpha)n$$

$$\begin{aligned}
 \text{Probability} &= \sum_{q=\alpha n+1}^{(1-\alpha)n-1} \frac{1}{n} \\
 &= \frac{1}{n} ((1 - \alpha)n - 1 - \alpha n - 1 + 1)
 \end{aligned}$$

Balanced Partitioning (9)

- We found $P_{\alpha>} = 1 - 2\alpha$
 - Ex: $P_{0.1>} = 0.8$ and $P_{0.01>} = 0.98$
- Hence, **H-PARTITION** produces a split
 - **more balanced than a**
 - $0.1 - to - 0.9$ split 80% of the time
 - $0.01 - to - 0.99$ split 98% of the time
 - **less balanced than a**
 - $0.1 - to - 0.9$ split 20% of the time
 - $0.01 - to - 0.99$ split 2% of the time

Intuition for the Average Case (1)

- **Assumption:** All permutations are equally likely
 - Only for intuition; we'll revisit this assumption later
- **Unlikely:** Splits always the same way at every level
- **Expectation:**
 - Some splits will be reasonably balanced
 - Some splits will be fairly unbalanced
- **Average case:** A mix of good and bad splits
 - **Good** and **bad** splits distributed randomly thru the tree

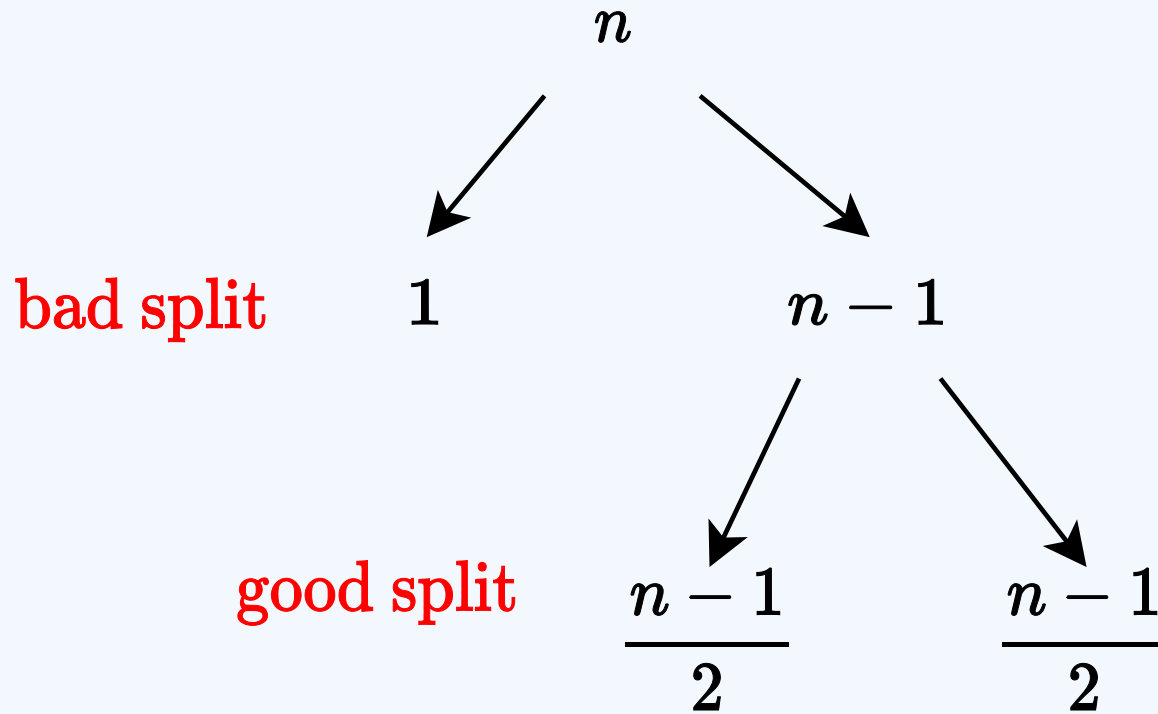
Intuition for the Average Case (2)

- **Assume for intuition:** Good and bad splits occur in the alternate levels of the tree
 - **Good split:** Best case split
 - **Bad split:** Worst case split

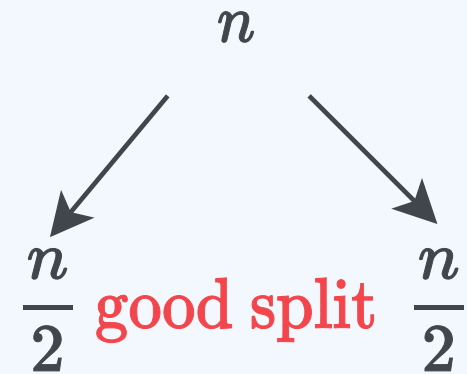
Intuition for the Average Case (3)

Compare 2-successive levels of avg case vs. 1 level of best case

AVERAGE CASE



BEST CASE



Intuition for the Average Case (4)

- In terms of the remaining subproblems, **two levels of avg case** is slightly better than the **single level of the best case**
- The avg case has **extra divide cost of $\Theta(n)$** at alternate levels
- The extra divide cost $\Theta(n)$ of bad splits absorbed into the $\Theta(n)$ of good splits.
- Running time is still $\Theta(n \lg n)$
 - But, slightly larger hidden constants, because the height of the recursion tree is about twice of that of best case.

Intuition for the Average Case (5)

- Another way of looking at it:
 - Suppose we alternate lucky, unlucky, lucky, unlucky, ...
 - We can write the recurrence as:
 - $L(n) = 2U(n/2) + \Theta(n)$ lucky split (best)
 - $U(n) = L(n - 1) + \Theta(n)$ unlucky split (worst)
 - Solving:

$$\begin{aligned} L(n) &= 2(L(n/2 - 1) + \Theta(n/2)) + \Theta(n) \\ &= 2L(n/2 - 1) + \Theta(n) \\ &= \Theta(n \lg n) \end{aligned}$$

- How can we make sure we are usually lucky for all inputs?

Summary: Quicksort Runtime Analysis (1)

- **Worst case:** Unbalanced split at every recursive call

$$T(n) = T(1) + T(n - 1) + \Theta(n)$$

$$T(n) = \Theta(n^2)$$

- **Best case:** Balanced split at every recursive call (*extremely lucky*)

$$T(n) = 2T(n/2) + \Theta(n)$$

$$T(n) = \Theta(n \lg n)$$

Summary: Quicksort Runtime Analysis (2)

- **Almost-best case:** Almost-balanced split at every recursive call

$$T(n) = T(n/10) + T(9n/10) + \Theta(n)$$

$$\text{or } T(n) = T(n/100) + T(99n/100) + \Theta(n)$$

$$\text{or } T(n) = T(\alpha n) + T((1 - \alpha)n) + \Theta(n)$$

for any constant α , $0 < \alpha \leq 0.5$

Summary: Quicksort Runtime Analysis (3)

- For a random input array, the probability of having a split
 - more balanced than 0.1 – to – 0.9 : 80%
 - more balanced than 0.01 – to – 0.99 : 98%
 - more balanced than α – to – $(1 - \alpha)$: $1 - 2\alpha$
for any constant α , $0 < \alpha \leq 0.5$

Summary: Quicksort Runtime Analysis (4)

- **Avg case intuition:** Different splits expected at different levels
 - some balanced (good), some unbalanced (bad)
- **Avg case intuition:** Assume the good and bad splits alternate
 - i.e. good split -> bad split -> good split -> ...
 - $T(n) = \Theta(n \lg n)$
 - (informal analysis for intuition)

Randomized Quicksort

- In the avg-case analysis, we assumed that **all permutations** of the input array are **equally likely**.
 - But, this assumption **does not always hold**
 - e.g. What if **all** the input arrays are **reverse sorted**?
 - **Always worst-case behavior**
- Ideally, the avg-case runtime should be **independent of the input permutation**.
- **Randomness should be within the algorithm**, not based on the distribution of the inputs.
 - i.e. The avg case should hold for all possible inputs

Randomized Algorithms (1)

- Alternative to assuming a uniform distribution:
 - **Impose a uniform distribution**
 - e.g. Choose a random pivot rather than the first element
- Typically useful when:
 - there are many ways that an algorithm can proceed
 - but, it's **difficult** to determine a way that is **always guaranteed to be good**.
 - If there are **many good alternatives**; simply choose one randomly.

Randomized Algorithms (1)

- Ideally:
 - Runtime should be **independent of the specific inputs**
 - No specific input should cause worst-case behavior
 - Worst-case should be determined only by output of a random number generator.

Randomized Quicksort (1)

- Using Hoare's partitioning algorithm:

```
R-QUICKSORT(A, p, r)
  if p < r then
    q = R-PARTITION(A, p, r)
    R-QUICKSORT(A, p, q)
    R-QUICKSORT(A, q+1, r)
```

```
R-PARTITION(A, p, r)
  s = RANDOM(p, r)
  exchange A[p] with A[s]
  return H-PARTITION(A, p, r)
```

- Alternatively, permuting the whole array would also work
 - but, would be more difficult to analyze

Randomized Quicksort (2)

- Using Lomuto's partitioning algorithm:

```
R-QUICKSORT(A, p, r)
  if p < r then
    q = R-PARTITION(A, p, r)
    R-QUICKSORT(A, p, q-1)
    R-QUICKSORT(A, q+1, r)
```

```
R-PARTITION(A, p, r)
  s = RANDOM(p, r)
  exchange A[r] with A[s]
  return L-PARTITION(A, p, r)
```

- Alternatively, permuting the whole array would also work
 - but, would be more difficult to analyze

Notations for Formal Analysis

- Assume all elements in $A[p \dots r]$ are distinct
 - Let $n = r - p + 1$
- Let $rank(x) = |A[i] : p \leq i \leq r \text{ and } A[i] \leq x|$
- i.e. $rank(x)$ is the number of array elements with value less than or equal to x
 - $A = \{5, 9, 7, 6, 8, 1, 4\}$
 - $p = 5, r = 4$
 - $rank(5) = 3$
 - i.e. it is the 3rd smallest element in the array

Formal Analysis for Average Case

- The following analysis will be for **Quicksort** using **Hoare's** partitioning algorithm.
- **Reminder:** The **pivot** is selected **randomly** and exchanged with $A[p]$ before calling **H-PARTITION**
- Let x be the **random pivot** chosen.
- What is the probability that $rank(x) = i$ for $i = 1, 2, \dots, n$?
 - $P(rank(x) = i) = 1/n$

Various Outcomes of H-PARTITION (1)

- Assume that $rank(x) = 1$
 - i.e. the **random pivot** chosen is the **smallest** element
 - What will be the **size of the left partition** ($|L|$)?
 - **Reminder:** Only the elements less than or equal to x will be in the left partition.

$$A = \{ \overset{p=x=pivot}{\underbrace{2}}, \underbrace{9, 7, 6, 8, 5}_{\Rightarrow |L|=1}, \overset{r}{\underbrace{4}} \}$$

$$p = 2, r = 4$$

$$pivot = x = 2$$

Various Outcomes of H-PARTITION (2)

- Assume that $rank(x) > 1$
 - i.e. the random pivot chosen is not the smallest element
 - What will be the size of the left partition ($|L|$)?
 - **Reminder:** Only the elements less than or equal to x will be in the left partition.
 - **Reminder:** The pivot will stay in the right region after H-PARTITION if $rank(x) > 1$

$$A = \{ \overbrace{2}^p, 4, \underbrace{7, 6, 8, \overbrace{5}^{pivot}, \overbrace{9}^r} \}$$

$$\implies |L| = rank(x) - 1$$


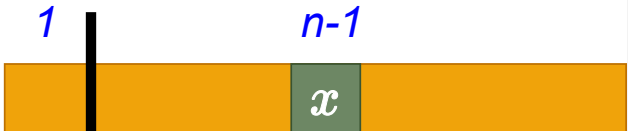
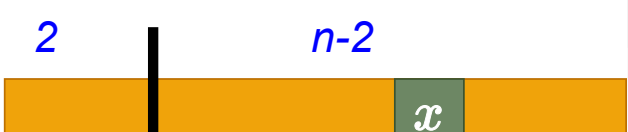


$$p = 2, r = 4$$

$$pivot = x = 5$$

Various Outcomes of H-PARTITION - Summary (1)

- $x : pivot$
- $|L|$: size of left region
- $P(rank(x) = i) = 1/n$ for $1 \leq i \leq n$
 - if $rank(x) = 1$ then $|L| = 1$
 - if $rank(x) > 1$ then $|L| = rank(x) - 1$
- $P(|L| = 1) = P(rank(x) = 1) + P(rank(x) = 2)$
 - $P(|L| = 1) = 2/n$
- $P(|L| = i) = P(rank(x) = i + 1)$ for $1 < i < n$
 - $P(|L| = i) = 1/n$ for $1 < i < n$

Various Outcomes of H-PARTITION - Summary (2)

<i>rank(x)</i>	<i>probability</i>	<i>T(n)</i>	
1	$\frac{1}{n}$	$T(1) + T(n - 1) + \Theta(n)$	
2	$\frac{1}{n}$	$T(1) + T(n - 1) + \Theta(n)$	
3	$\frac{1}{n}$	$T(2) + T(n - 2) + \Theta(n)$	
\vdots	\vdots	\vdots	
$i + 1$	$\frac{1}{n}$	$T(i) + T(n - i) + \Theta(n)$	
\vdots	\vdots	\vdots	
n	$\frac{1}{n}$	$T(n - 1) + T(1) + \Theta(n)$	

Average - Case Analysis: Recurrence (1)

$x = pivot$

$$\begin{aligned}
 T(n) &= \frac{1}{n}(T(1) + t(n-1)) & rank : 1 \\
 &+ \frac{1}{n}(T(1) + t(n-1)) & rank : 2 \\
 &+ \frac{1}{n}(T(2) + t(n-2)) & rank : 3 \\
 &\vdots & \vdots \\
 &+ \frac{1}{n}(T(i) + t(n-i)) & rank : i + 1 \\
 &\vdots & \vdots \\
 &+ \frac{1}{n}(T(n-1) + t(1)) & rank : n \\
 &+ \Theta(n)
 \end{aligned}$$

Average - Case Analysis: Recurrence (2)

$$T(n) = \frac{1}{n} \sum_{q=1}^{n-1} (T(q) + T(n-q)) + \frac{1}{n} (T(1) + T(n-1)) + \Theta(n)$$

Note: $\frac{1}{n} (T(1) + T(n-1)) = \frac{1}{n} (\Theta(1) + O(n^2)) = O(n)$

$$T(n) = \frac{1}{n} \sum_{q=1}^{n-1} (T(q) + T(n-q)) + \Theta(n)$$

for $k = 1, 2, \dots, n-1$ each term $T(k)$ appears twice once for $q = k$ and once for $q = n - k$

$$T(n) = \frac{2}{n} \sum_{k=1}^{n-1} T(k) + \Theta(n)$$

Average - Case Analysis -Solving Recurrence: Substitution

- Guess: $T(n) = O(n \lg n)$
- $T(k) \leq a k \lg k$ for $k < n$, for some constant $a > 0$

$$\begin{aligned} T(n) &= \frac{2}{n} \sum_{k=1}^{n-1} T(k) + \Theta(n) \\ &\leq \frac{2}{n} \sum_{k=1}^{n-1} a k \lg k + \Theta(n) \\ &\leq \frac{2a}{n} \sum_{k=1}^{n-1} k \lg k + \Theta(n) \end{aligned}$$

- Need a tight bound for $\sum k \lg k$

Tight bound for $\sum klgk$ (1)

- Bounding the terms
 - $\sum_{k=1}^{n-1} klgk \leq \sum_{k=1}^{n-1} nlg n = n(n-1)lg n \leq n^2lg n$
 - This bound is **not strong** enough because
 - $T(n) \leq \frac{2a}{n} n^2lg n + \Theta(n)$
 - $= 2anlg n + \Theta(n) \implies$ couldn't prove $T(n) \leq anlg n$

Tight bound for $\sum klgk$ (2)

- **Splitting summations:** ignore ceilings for simplicity
- $\sum_{k=1}^{n-1} klgk \leq \sum_{k=1}^{n/2-1} klgk + \sum_{k=n/2}^{n-1} klgk$
 - **First summation:** $lgk < lg(n/2) = lgn - 1$
 - **Second summation:** $lgk < lgn$

Splitting: $\sum_{k=1}^{n-1} k \lg k \leq \sum_{k=1}^{n/2-1} k \lg k + \sum_{k=n/2}^{n-1} k \lg k \quad (3)$

- $\sum_{k=1}^{n-1} k \lg k \leq (\lg(n-1)) \sum_{k=1}^{n/2-1} k + \lg n \sum_{k=n/2}^{n-1} k$
 - $= \lg n \sum_{k=1}^{n-1} k - \sum_{k=1}^{n/2-1} k$
 - $= \frac{1}{2}n(n-1)\lg n - \frac{1}{2} \frac{n}{2} \left(\frac{n}{2} - 1 \right)$
 - $= \frac{1}{2}n^2 \lg n - \frac{1}{8}n^2 - \frac{1}{2}n(\lg n - 1/2)$
- $\sum_{k=1}^{n-1} k \lg k \leq \frac{1}{2}n^2 \lg n - \frac{1}{8}n^2$ for $\lg n \geq 1/2 \implies n \geq \sqrt{2}$

Substituting: - $\sum_{k=1}^{n-1} k \lg k \leq \frac{1}{2}n^2 \lg n - \frac{1}{8}n^2 \quad (4)$

$$\begin{aligned} T(n) &\leq \frac{2a}{n} \sum_{k=1}^{n-1} k \lg k + \Theta(n) \\ &\leq \frac{2a}{n} \left(\frac{1}{2}n^2 \lg n - \frac{1}{8}n^2 \right) + \Theta(n) \\ &= a n \lg n - \left(\frac{a}{4}n - \Theta(n) \right) \end{aligned}$$

- We can choose a large enough so that $\frac{a}{4}n \geq \Theta(n)$

$$T(n) \leq a n \lg n$$

$$T(n) = O(n \lg n)$$

Q.E.D.

Medians and Order Statistics

- **ith order statistic:** i^{th} smallest element of a set of n elements
- **minimum:** *first* order statistic
- **maximum:** n^{th} order statistic
- **median:** “halfway point” of the set

$$i = \left\lfloor \frac{(n + 1)}{2} \right\rfloor$$

or

$$i = \left\lceil \frac{(n + 1)}{2} \right\rceil$$

Selection Problem

- **Selection problem:** Select the i^{th} smallest of n elements
- **Naïve algorithm:** Sort the input array A ; then return $A[i]$
 - $T(n) = \theta(n \lg n)$
 - *using e.g. merge sort (but not quicksort)*
- Can we do any better?

Selection in Expected Linear Time

- Randomized algorithm using divide and conquer
- Similar to randomized quicksort
 - **Like quicksort:** Partitions input array recursively
 - **Unlike quicksort:** Makes a single recursive call
 - **Reminder:** *Quicksort makes two recursive calls*
- Expected runtime: $\Theta(n)$
 - **Reminder:** *Expected runtime of quicksort: $\Theta(n \lg n)$*

Selection in Expected Linear Time: Example 1

- Select the 2^{nd} smallest element:

$$A = \{6, 10, 13, 5, 8, 3, 2, 11\}$$
$$i = 2$$

- Partition the input array:

$$A = \{ \underbrace{2, 3, 5}_{\text{left subarray}}, \underbrace{13, 8, 10, 6, 11}_{\text{right subarray}} \}$$

- make a recursive call to select the 2^{nd} smallest element in **left subarray**

Selection in Expected Linear Time: Example 2

- Select the 7th smallest element:

$$A = \{6, 10, 13, 5, 8, 3, 2, 11\}$$
$$i = 7$$

- Partition the input array:

$$A = \{ \underbrace{2, 3, 5}_{\text{left subarray}}, \underbrace{13, 8, 10, 6, 11}_{\text{right subarray}} \}$$

- make a recursive call to select the 4th smallest element in right subarray

Selection in Expected Linear Time (1)

```

R-SELECT(A,p,r,i)
  if p == r then
    return A[p];
  q = R-PARTITION(A, p, r)
  k = q-p+1;
  if i <= k then
    return R-SELECT(A, p, q, i);
  else
    return R-SELECT(A, q+1, r, i-k);

```

$$A = \{ \underbrace{\quad}_p \cdots \leq x(\text{k smallest elements}) \cdots \underbrace{\quad}_q \cdots \geq x \cdots \underbrace{\quad}_r \}$$

$x = \text{pivot}$

Selection in Expected Linear Time (2)

$$A = \{ \overbrace{\quad}^p \mid \underbrace{\cdots \leq x \cdots}_L \mid \underbrace{\cdots \geq x \cdots}_R \mid \overbrace{\quad}^r \}$$

$x = \text{pivot}$

- All elements in $L \leq$ all elements in R
- L contains:
- $|L| = q - p + 1 = k$ smallest elements of $A[p \dots r]$
- if $i \leq |L| = k$ then
 - **search** L recursively for its i^{th} smallest element
- else
 - **search** R recursively for its $(i - k)^{\text{th}}$ smallest element

Runtime Analysis (1)

- **Worst case:**
 - Imbalanced partitioning at every level and the recursive call always to the larger partition

$$= \{1, \underbrace{2, 3, 4, 5, 6, 7, 8}_{\text{recursive call}}\} \quad i = 8$$

$$= \{2, \underbrace{3, 4, 5, 6, 7, 8}_{\text{recursive call}}\} \quad i = 7$$

Runtime Analysis (2)

- **Worst case:** Worse than the naïve method (based on sorting)

$$T(n) = T(n - 1) + \Theta(n)$$

$$T(n) = \Theta(n^2)$$

- **Best case:** Balanced partitioning at every recursive level

$$T(n) = T(n/2) + \Theta(n)$$

$$T(n) = \Theta(n)$$

- **Avg case:** Expected runtime – need analysis T.B.D.

Reminder: Various Outcomes of H-PARTITION

- $x : \text{pivot}$
- $|L|$: size of left region
- $P(\text{rank}(x) = i) = 1/n$ for $1 \leq i \leq n$
 - if $\text{rank}(x) = 1$ then $|L| = 1$
 - if $\text{rank}(x) > 1$ then $|L| = \text{rank}(x) - 1$
- $P(|L| = 1) = P(\text{rank}(x) = 1) + P(\text{rank}(x) = 2)$
 - $P(|L| = 1) = 2/n$
- $P(|L| = i) = P(\text{rank}(x) = i + 1)$ for $1 < i < n$
 - $P(|L| = i) = 1/n$ for $1 < i < n$

Average Case Analysis of Randomized Select


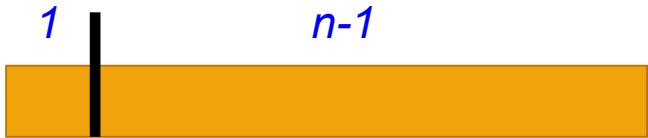
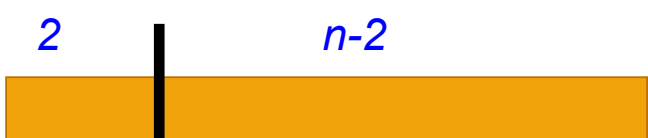


- To compute the **upper bound** for the **avg case**, assume that the i^{th} element always falls into the **larger partition**.

$$A = \left\{ \overbrace{\quad}^p \mid \underbrace{\dots \leq x \dots}_{LeftPartition} \mid \underbrace{\dots \geq x \dots}_{RightPartition} \mid \overbrace{\quad}^r \right\}$$

$$x = pivot$$

- We will analyze the case where the recursive call is always made to the larger partition
 - This will give us an upper bound for the avg case*

Various Outcomes of H-PARTITION

<i>rank(x)</i>	<i>probability</i>	<i>T(n)</i>	
1	$\frac{1}{n}$	$\leq T(\max(1, n - 1)) + \Theta(n)$	
2	$\frac{1}{n}$	$\leq T(\max(1, n - 1)) + \Theta(n)$	
3	$\frac{1}{n}$	$\leq T(\max(2, n - 2)) + \Theta(n)$	
\vdots	\vdots	\vdots	
$i + 1$	$\frac{1}{n}$	$\leq T(\max(i, n - i)) + \Theta(n)$	
\vdots	\vdots	\vdots	
n	$\frac{1}{n}$	$\leq T(\max(n - 1, 1)) + \Theta(n)$	

Average-Case Analysis of Randomized Select (1)

$$\text{Recall: } P(|L| = i) = \begin{cases} 2/n & \text{for } i = 1 \\ 1/n & \text{for } i = 2, 3, \dots, n-1 \end{cases}$$

Upper bound: Assume i^{th} element always falls into the larger part.

$$T(n) \leq \frac{1}{n}T(\max(1, n-1)) + \frac{1}{n} \sum_{q=1}^{n-1} T(\max(q, n-q)) + O(n)$$

$$\text{Note : } \frac{1}{n}T(\max(1, n-1)) = \frac{1}{n}T(n-1) = \frac{1}{n}O(n^2) = O(n)$$

$$\therefore \text{ (3 dot mean therefore) } T(n) \leq \frac{1}{n} \sum_{q=1}^{n-1} T(\max(q, n-q)) + O(n)$$

Average-Case Analysis of Randomized Select (2)

$$\therefore T(n) \leq \frac{1}{n} \sum_{q=1}^{n-1} T(\max(q, n-q)) + O(n)$$

$$\max(q, n-q) = \begin{cases} q & \text{if } q \geq \lceil n/2 \rceil \\ n-q & \text{if } q < \lceil n/2 \rceil \end{cases}$$

- n is odd: $T(k)$ appears twice for $k = \lceil n/2 \rceil + 1, \lceil n/2 \rceil + 2, \dots, n-1$
- n is even: $T(\lceil n/2 \rceil)$ appears once $T(k)$ appears twice for $k = \lceil n/2 \rceil + 1, \lceil n/2 \rceil + 2, \dots, n-1$

Average-Case Analysis of Randomized Select (3)

- Hence, in both cases:

$$\sum_{q=1}^{n-1} T(\max(q, n - q)) + O(n) \leq 2 \sum_{q=\lceil n/2 \rceil}^{n-1} T(q) + O(n)$$

$$\therefore T(n) \leq \frac{2}{n} \sum_{q=\lceil n/2 \rceil}^{n-1} T(q) + O(n)$$

Average-Case Analysis of Randomized Select (4)

$$T(n) \leq \frac{2}{n} \sum_{q=\lceil n/2 \rceil}^{n-1} T(q) + O(n)$$

- By substitution guess $T(n) = O(n)$
- Inductive hypothesis: $T(k) \leq ck, \forall k < n$

$$T(n) \leq \frac{2}{n} \sum_{q=\lceil n/2 \rceil}^{n-1} ck + O(n)$$

$$= \frac{2c}{n} \left(\sum_{k=1}^{n-1} k - \sum_{k=1}^{\lceil n/2 \rceil - 1} k \right) + O(n)$$

$$= \frac{2c}{n} \left(\frac{1}{2}n(n-1) - \frac{1}{2}\lceil \frac{n}{2} \rceil \left(\frac{n}{2} - 1 \right) \right) + O(n)$$

Average-Case Analysis of Randomized Select (5)

$$\begin{aligned}
 T(n) &\leq \frac{2c}{n} \left(\frac{1}{2}n(n-1) - \frac{1}{2} \lceil \frac{n}{2} \rceil \left(\frac{n}{2} - 1 \right) \right) + O(n) \\
 &\leq c(n-1) - \frac{c}{4}n + \frac{c}{2} + O(n) \\
 &= cn - \frac{c}{4}n - \frac{c}{2} + O(n) \\
 &= cn - \left(\left(\frac{c}{4}n + \frac{c}{2} \right) + O(n) \right) \\
 &\leq cn
 \end{aligned}$$

- since we can choose c large enough so that $(cn/4 + c/2)$ dominates $O(n)$

Summary of Randomized Order-Statistic Selection

- Works fast: linear expected time
- Excellent algorithm in practise
- But, the worst case is very bad: $\Theta(n^2)$
- **Blum, Floyd, Pratt, Rivest & Tarjan[1973]** algorithms are runs in **linear time** in the **worst case**.
- Generate a **good pivot** recursively

Selection in Worst Case Linear Time

```
//return i-th element in set S with n elements
SELECT(S, n, i)

    if n <= 5 then

        SORT S and return the i-th element

    DIVIDE S into ceil(n/5) groups
    //first ceil(n/5) groups are of size 5, last group is of size n mod 5

    FIND median set M={m , ..., m_ceil(n/5)}
    // m_j : median of j-th group

    x = SELECT(M,ceil(n/5),floor((ceil(n/5)+1)/2))

    PARTITION set S around the pivot x into L and R

    if i <= |L| then
        return SELECT(L, |L|, i)
    else
        return SELECT(R, n-|L|, i-|L|)
```

Selection in Worst Case Linear Time - Example (1)

- **Input:** Array S and index i
- **Output:** The i^{th} smallest value

25	9	16	8	11	27	39	42	15	63	2	14	36	20	33	22	31	4	17	3	30	41
2	13	19	7	21	10	34	1	37	23	40	5	29	18	24	12	38	28	26	35	43	

Selection in Worst Case Linear Time - Example (2)

Step 1: Divide the input array into groups of size 5

group size=5

25	9	16	8	11
27	39	42	15	6
32	14	36	20	33
22	31	4	17	3
30	41	2	13	19
7	21	10	34	1
37	23	40	5	29
18	24	12	38	28
26	35	43		

Selection in Worst Case Linear Time - Example (3)

Step 2: Compute the median of each group ($\Theta(n)$)

<i>Medians</i>				
25	16	11	8	9
39	42	27	6	15
36	33	32	20	14
22	31	17	3	4
41	30	19	13	2
21	34	10	1	7
37	40	29	23	5
38	28	24	12	18
	26	35	43	

- Let M be the set of the medians computed:
 - $M = \{11, 27, 32, 17, 19, 10, 29, 24, 35\}$

Selection in Worst Case Linear Time - Example (4)

Step 3: Compute the median of the median group M

$x \leftarrow SELECT(M, |M|, \lfloor (|M| + 1)/2 \rfloor)$ where $|M| = \lceil n/5 \rceil$

- Let M be the set of the medians computed:

$$\circ M = \{11, 27, 32, 17, 19, 10, 29, \overset{\text{Median}}{\underbrace{24}}, 35\}$$

- $Median = 24$
- The runtime of the recursive call: $T(|M|) = T(\lceil n/5 \rceil)$

Selection in Worst Case Linear Time - Example (5)

Step 4: Partition the input array S around the median-of-medians x

25	9	16	8	11	27	39	42	15	63	2	14	36	20	33	22	31	4	17	3	30	41
2	13	19	7	21	10	34	1	37	23	40	5	29	18	24	12	38	28	26	35	43	

Partition S around $x = 24$

Claim: Partitioning around x is guaranteed to be **well-balanced**.

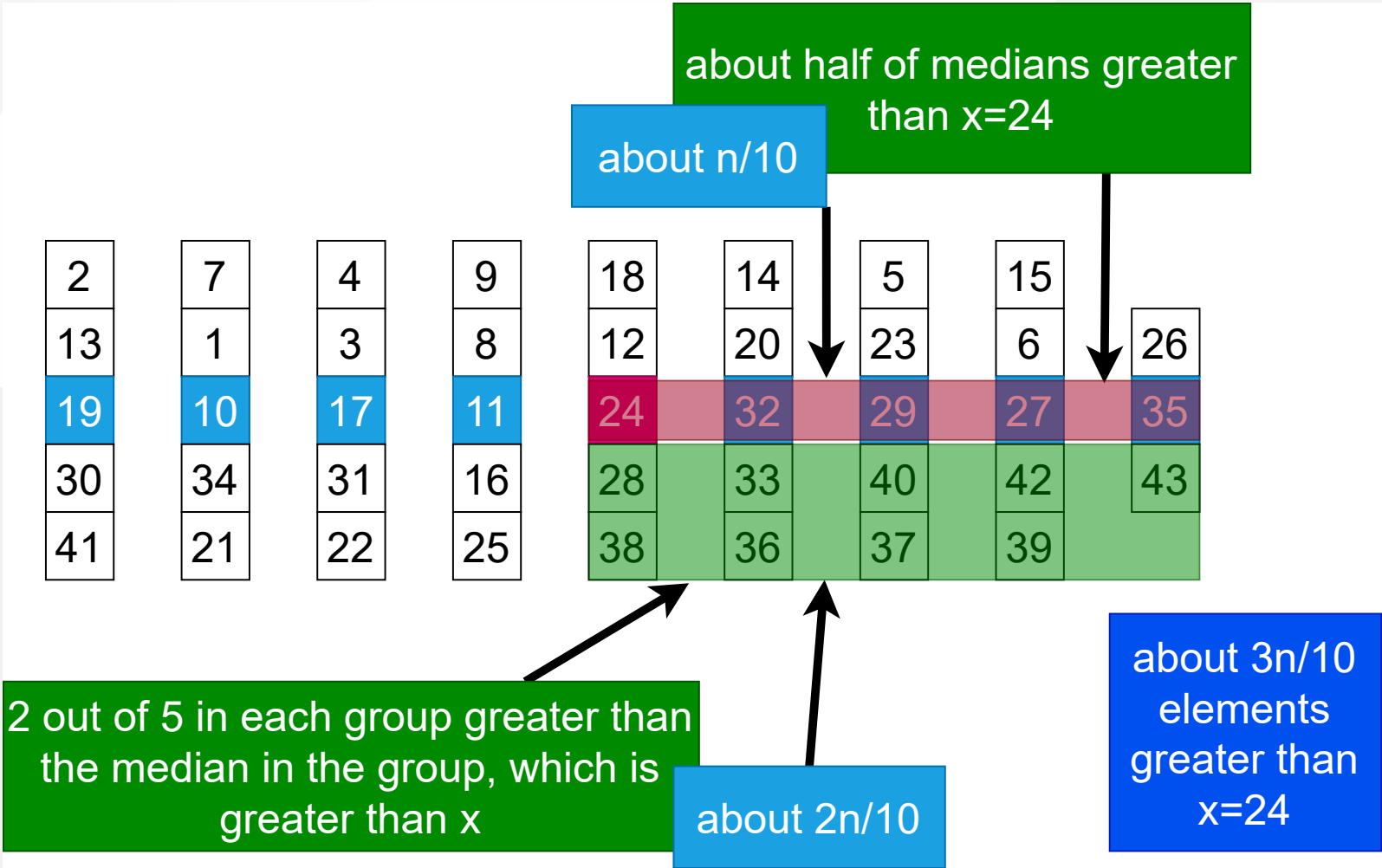
Selection in Worst Case Linear Time - Example (6)

- M : Median, M^* : Median of Medians

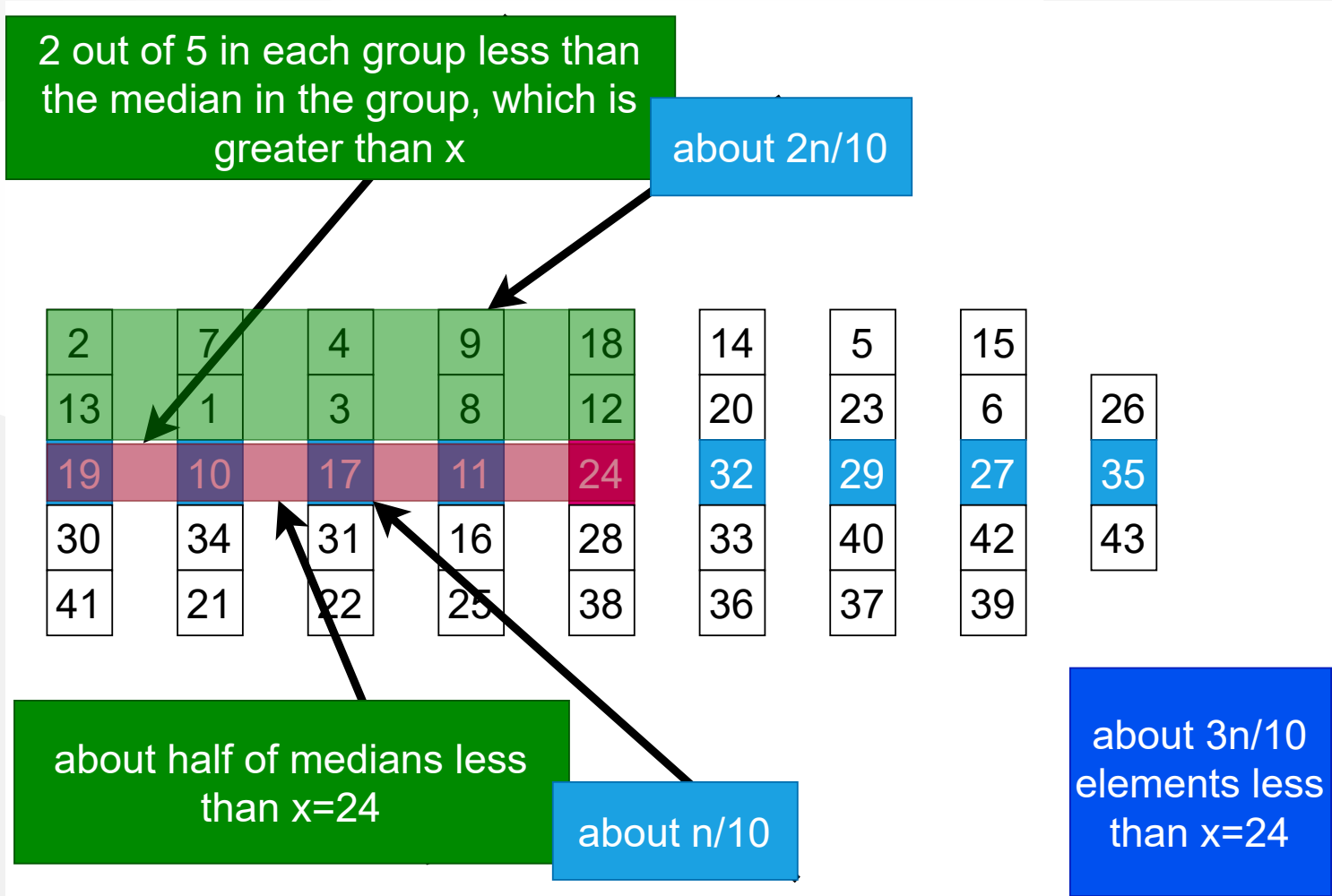
		\overbrace{M}		
41	30	19	13	2
21	34	10	1	7
22	31	17	3	4
25	16	11	8	9
		$\overbrace{M^*}$		
38	28	24	12	18
36	33	32	20	14
37	40	29	23	5
39	42	27	6	15
	26	35	43	

- About half of the medians greater than $x = 24$ (about $n/10$)

Selection in Worst Case Linear Time - Example (7)



Selection in Worst Case Linear Time - Example (8)



Selection in Worst Case Linear Time - Example (9)

$S = \begin{Bmatrix} 25 & 9 & 16 & 8 & 11 & 27 & 39 & 42 & 15 & 63 & 14 & 36 & 20 & 33 & 22 & 31 & 4 & 17 & 3 & 30 & 41 \\ 2 & 13 & 19 & 7 & 21 & 10 & 34 & 1 & 37 & 23 & 40 & 5 & 29 & 18 & 24 & 12 & 38 & 28 & 26 & 35 & 43 \end{Bmatrix}$

- Partitioning S around $x = 24$ will lead to partitions of sizes $\sim 3n/10$ and $\sim 7n/10$ in the **worst case**.

Step 5: Make a recursive call to one of the partitions

```

if  $i \leq |L|$  then
    return SELECT( $L, |L|, i$ )
else
    return SELECT( $R, n - |L|, i - |L|$ )
  
```

Selection in Worst Case Linear Time

```
//return i-th element in set S with n elements
SELECT(S, n, i)

    if n <= 5 then

        SORT S and return the i-th element

    DIVIDE S into ceil(n/5) groups
    //first ceil(n/5) groups are of size 5, last group is of size n mod 5

    FIND median set M={m , ..., m_ceil(n/5)}
    // m_j : median of j-th group

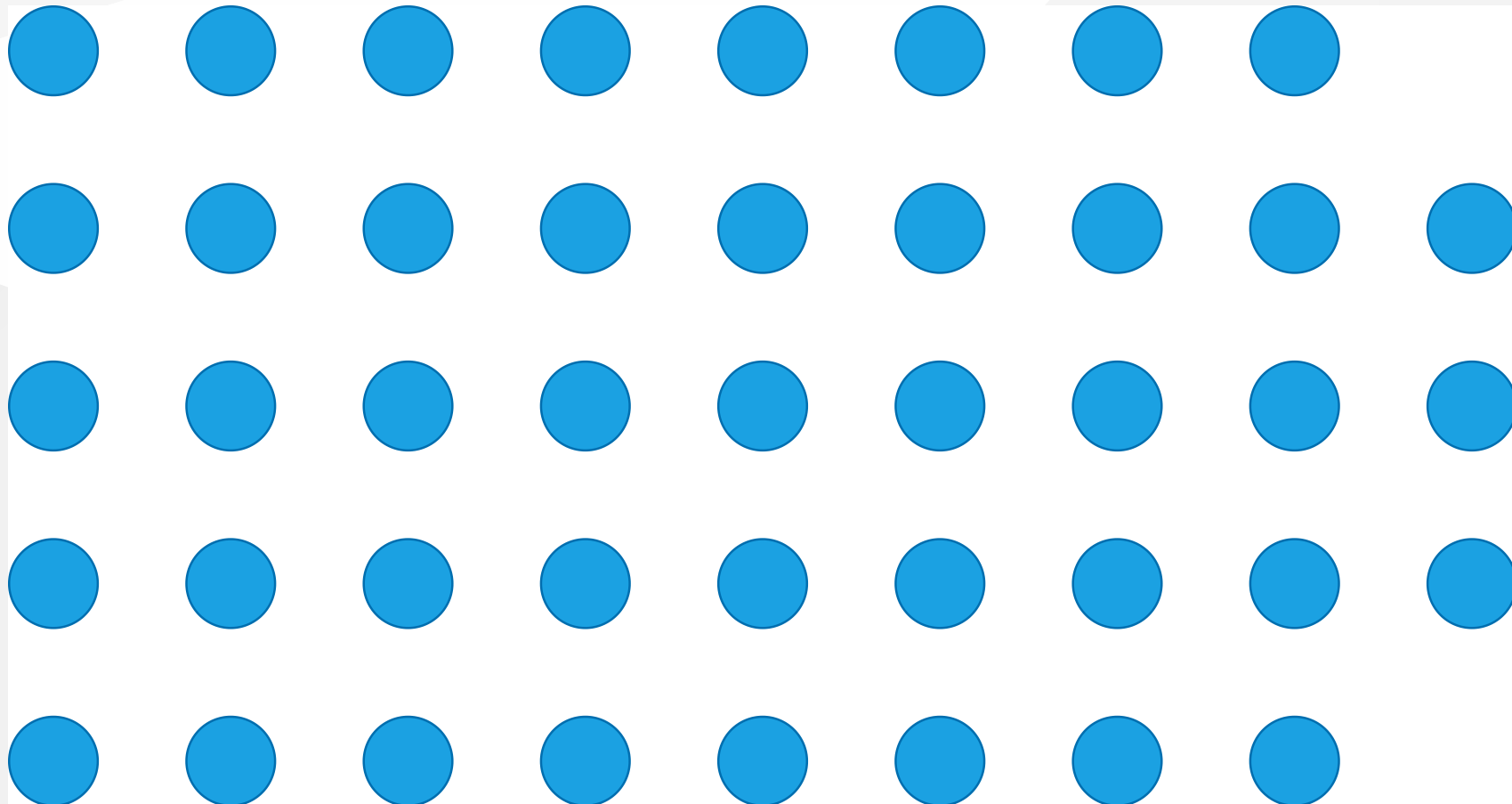
    x = SELECT(M,ceil(n/5),floor((ceil(n/5)+1)/2))

    PARTITION set S around the pivot x into L and R

    if i <= |L| then
        return SELECT(L, |L|, i)
    else
        return SELECT(R, n-|L|, i-|L|)
```

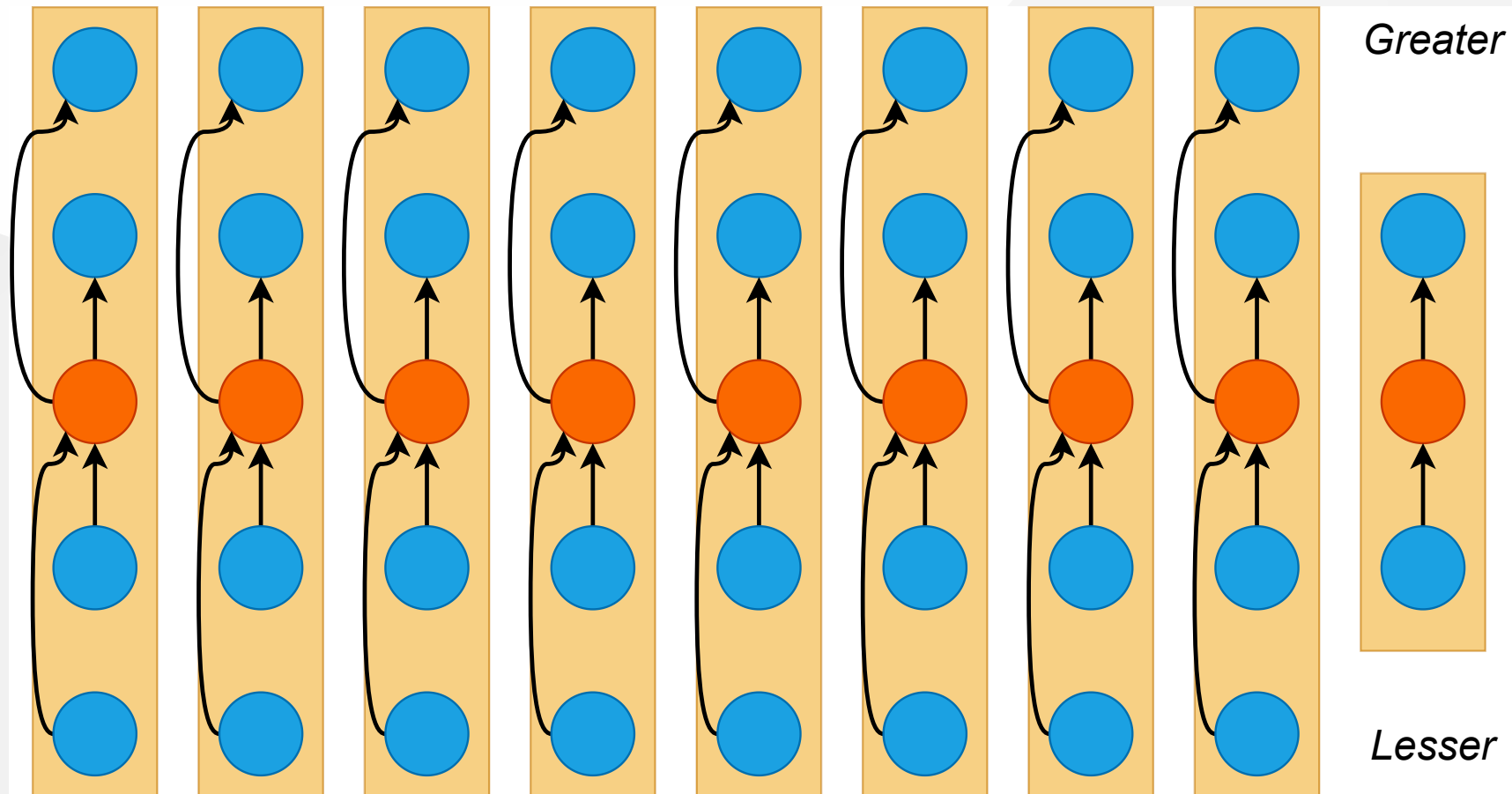
Choosing the Pivot (1)

1. Divide S into groups of size 5



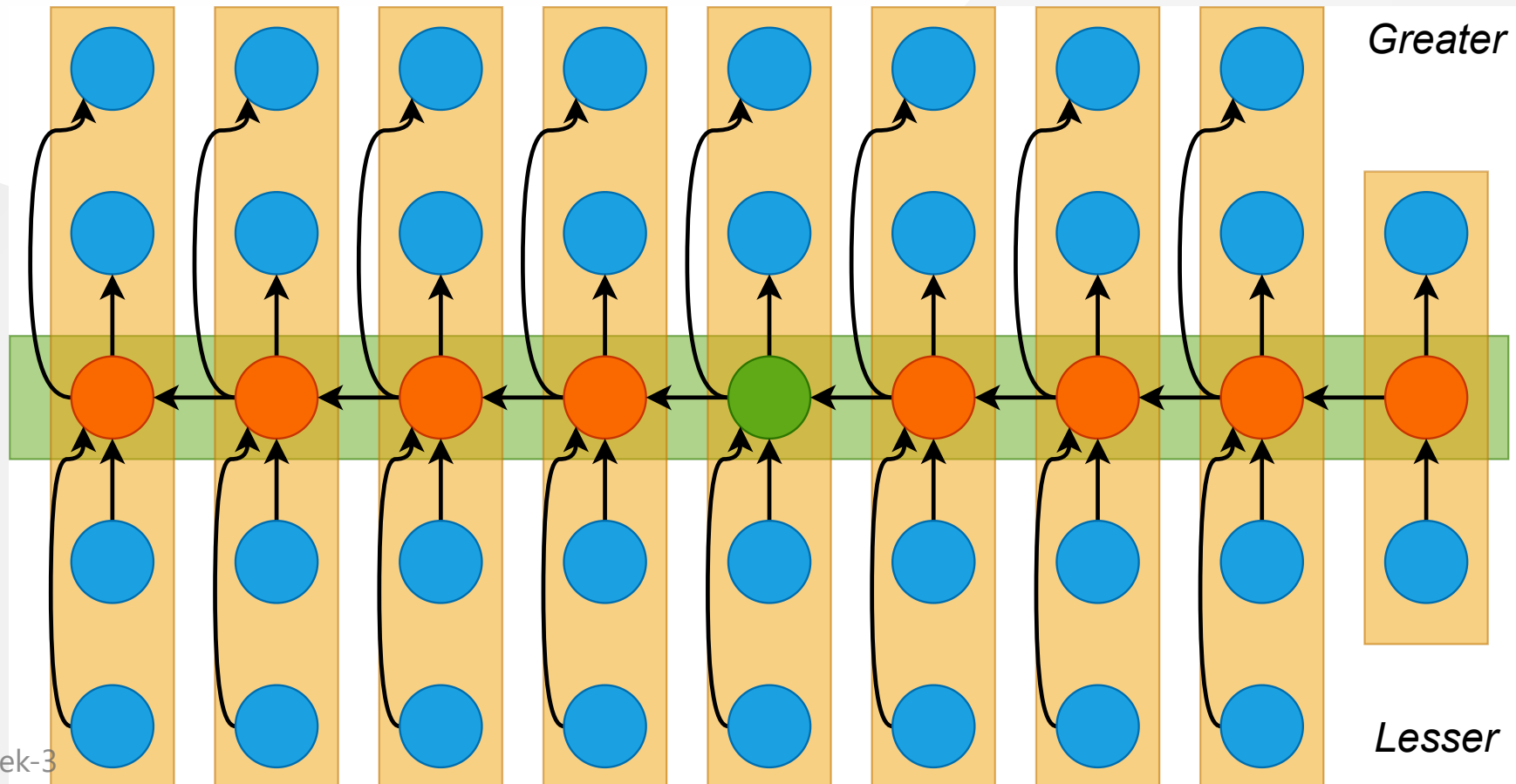
Choosing the Pivot (2)

- Divide S into groups of size 5
- Find the median of each group



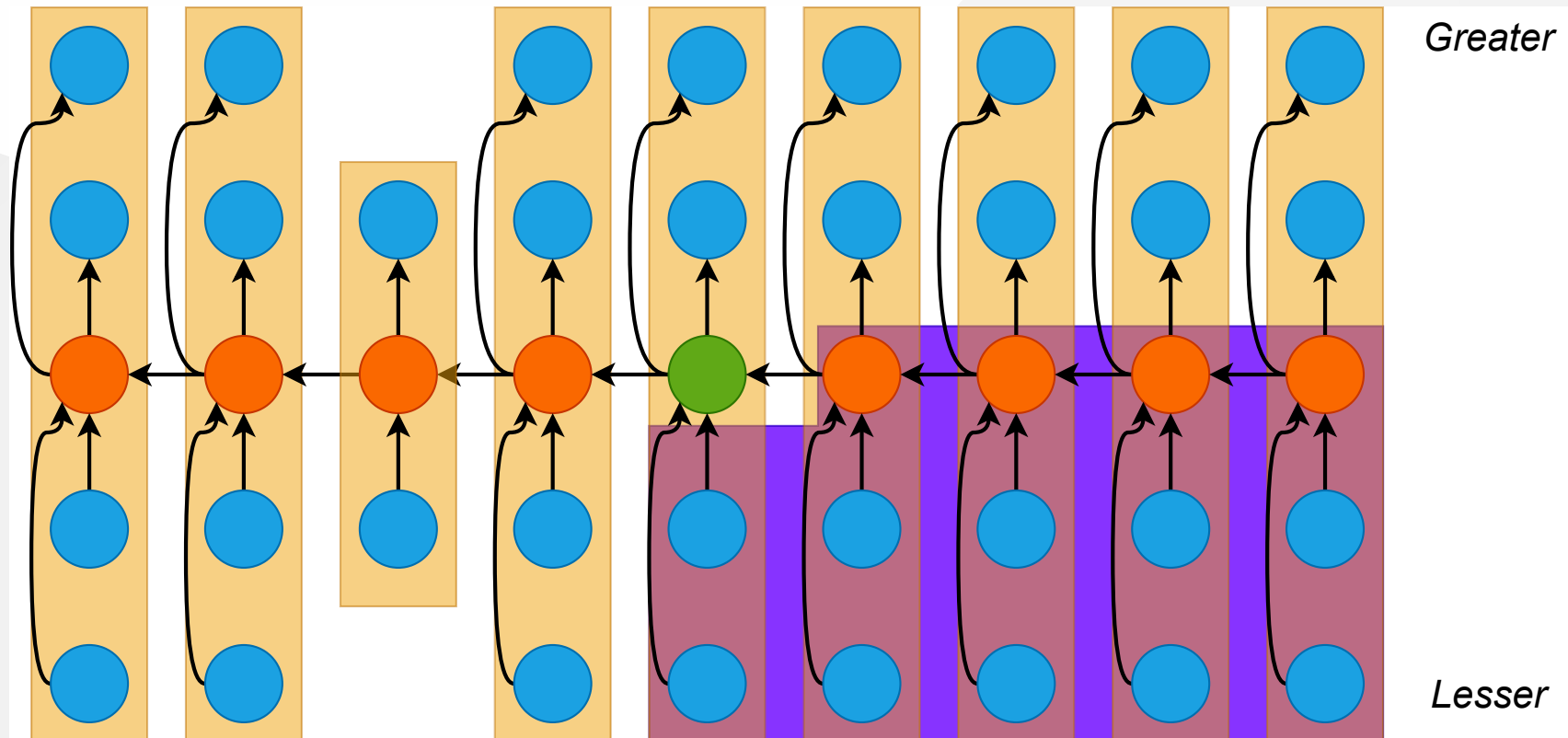
Choosing the Pivot (3)

- Divide S into groups of size 5
- Find the median of each group
- Recursively select the median x of the medians



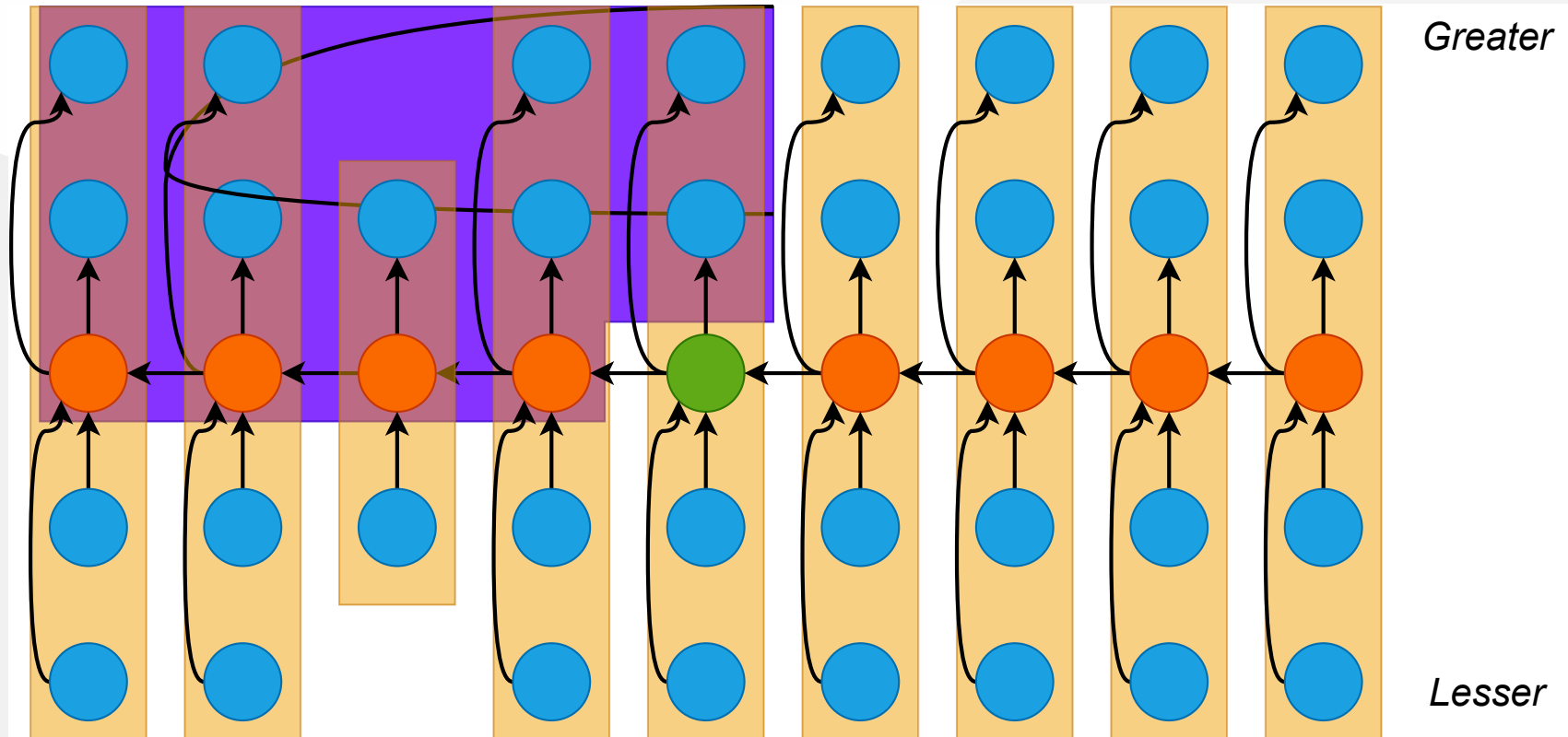
Choosing the Pivot (4)

- At least half of the medians $\geq x$
- Thus $m = \lceil \lceil n/5 \rceil / 2 \rceil$ groups contribute 3 elements to R except possibly the last group and the group that contains x , $|R| \geq 3(m-2) \geq \frac{3n}{10} - 6$



Choosing the Pivot (5)

- Similarly $|L| \geq \frac{3n}{10} - 6$
- Therefore, **SELECT** is recursively called on at most $n - (\frac{3n}{10} - 6) = \frac{7n}{10} + 6$ elements



Selection in Worst Case Linear Time (1)

```

//return i-th element in set S with n elements
SELECT(S, n, i)
  if n <= 5 then
    SORT S and return the i-th element
   $\Theta(n)$  { DIVIDE S into  $\text{ceil}(n/5)$  groups
   $\Theta(n)$  { //first  $\text{ceil}(n/5)$  groups are of size 5, last group is of size  $n \bmod 5$ 
   $T(\lceil n/5 \rceil)$  { FIND median set  $M = \{m_1, \dots, m_{\text{ceil}(n/5)}\}$ 
   $\Theta(n)$  { //  $m_j$  : median of j-th group
   $T(\frac{7n}{10} + 6)$  {  $x = \text{SELECT}(M, \text{ceil}(n/5), \text{floor}((\text{ceil}(n/5)+1)/2))$ 
   $\Theta(n)$  { PARTITION set S around the pivot x into L and R
   $T(\frac{7n}{10} + 6)$  { if  $i \leq |L|$  then
   $\Theta(n)$  {   return SELECT(L, |L|, i)
   $T(\frac{7n}{10} + 6)$  { else
   $\Theta(n)$  {   return SELECT(R, n-|L|, i-|L|)
  
```

Selection in Worst Case Linear Time (2)

- Thus recurrence becomes
 - $T(n) \leq T(\lceil \frac{n}{5} \rceil) + T(\frac{7n}{10} + 6) + \Theta(n)$
- Guess $T(n) = O(n)$ and prove by induction
- Inductive step:

$$\begin{aligned}
 T(n) &\leq c\lceil n/5 \rceil + c(7n/10 + 6) + \Theta(n) \\
 &\leq cn/5 + c + 7cn/10 + 6c + \Theta(n) \\
 &= 9cn/10 + 7c + \Theta(n) \\
 &= cn - [c(n/10 - 7) - \Theta(n)] \leq cn \quad (\text{for large } c)
 \end{aligned}$$

- Work at each level of recursion is a constant factor (9/10) smaller

References

- [Introduction to Algorithms, Third Edition | The MIT Press](#)
- [Bilkent CS473 Course Notes \(new\)](#)
- [Bilkent CS473 Course Notes \(old\)](#)
- [Insertion Sort - GeeksforGeeks](#)
- [NIST Dictionary of Algorithms and Data Structures](#)
- [NIST - Dictionary of Algorithms and Data Structures](#)
- [NIST - big-O notation](#)
- [NIST - big-Omega notation](#)