CE100 Algorithms and Programming II

Week-4 (Heap/Heap Sort)

Spring Semester, 2021-2022

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Heap/Heap Sort

Outline (1)

- Heaps
  - Max / Min Heap
- Heap Data Structure
  - Heapify
    - Iterative
    - Recursive

Outline (2)

- Extract-Max
- Build Heap

Outline (3)

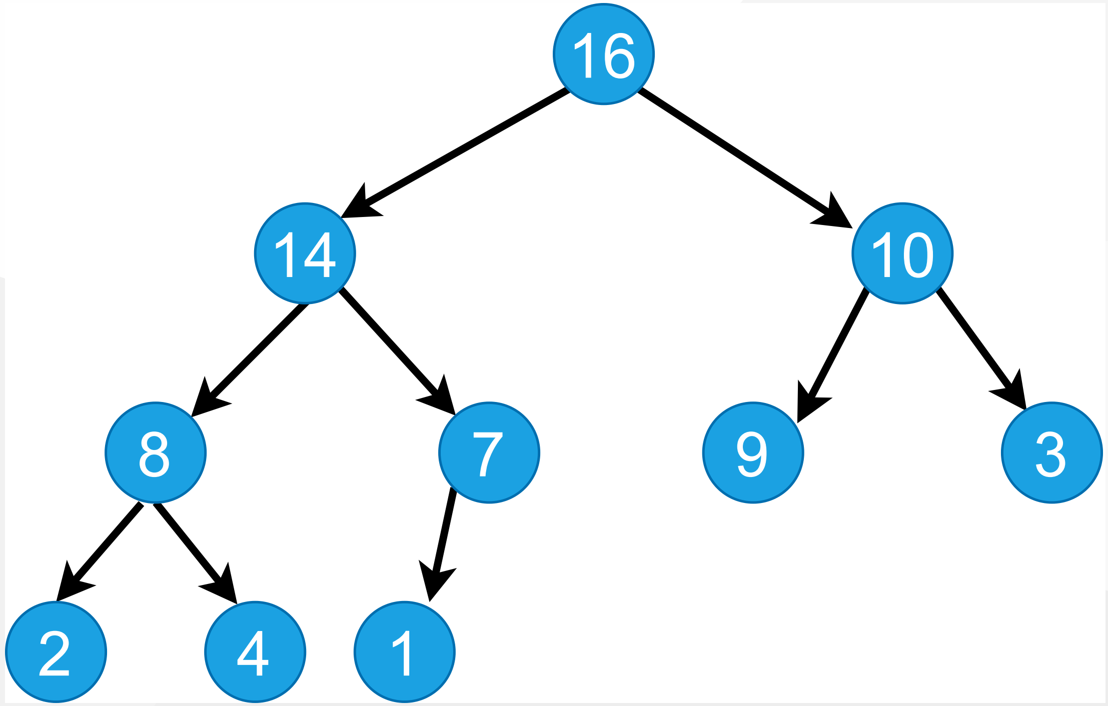
- Heap Sort
- Priority Queues
- Linked Lists
- Radix Sort
- Counting Sort

Heapsort

- Worst-case runtime: $O(nlgn)$
- Sorts in-place
- Uses a special data structure (heap) to manage information during execution of the algorithm
    - Another design paradigm

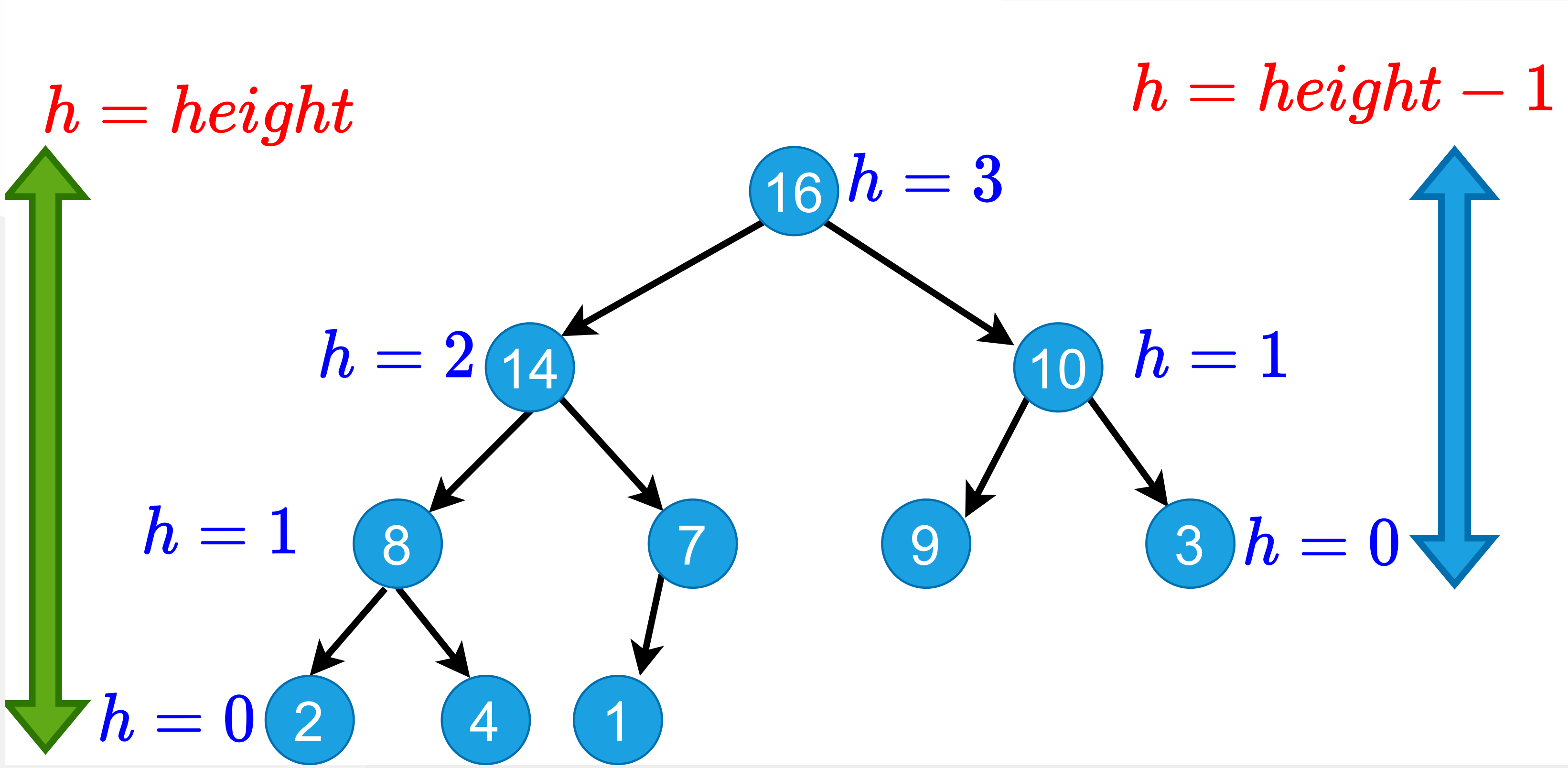
Heap Data Structure (1)

- Nearly complete binary tree
  - Completely filled on all levels except possibly the lowest level



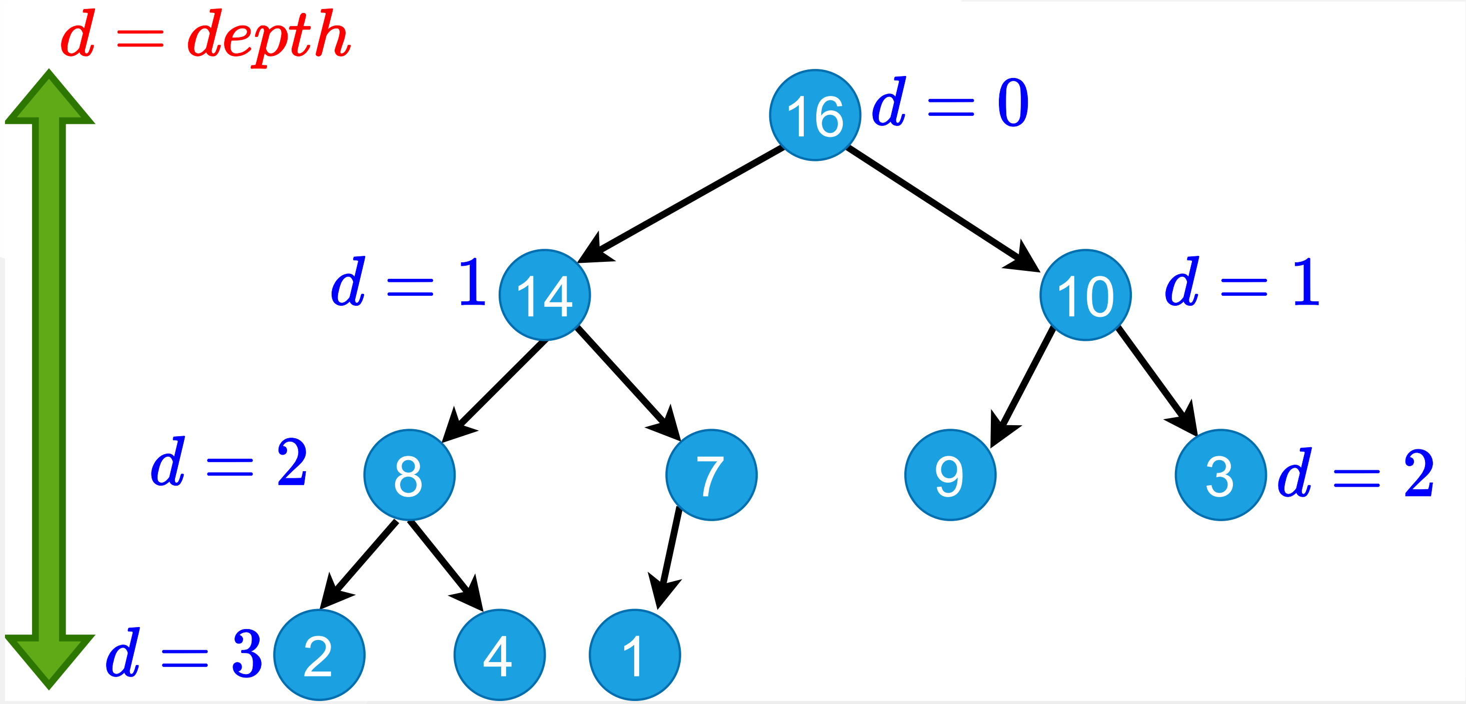
Heap Data Structure (2)

- **Height of node i:** Length of the longest simple downward path from **i** to a **leaf**
- **Height of the tree:** height of the **root**



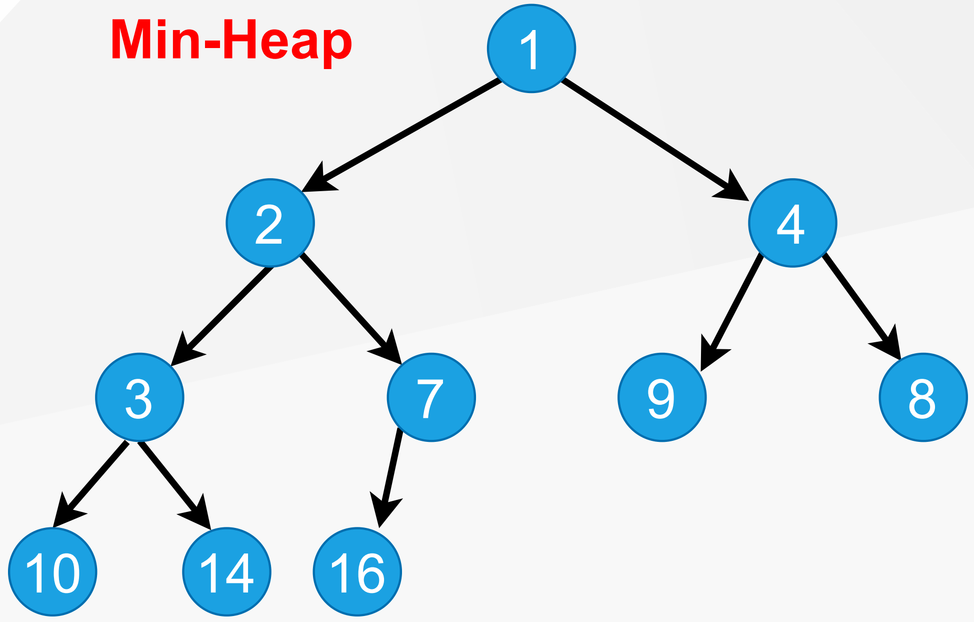
Heap Data Structures (3)

- **Depth of node i:** Length of the simple downward path from the **root** to node i



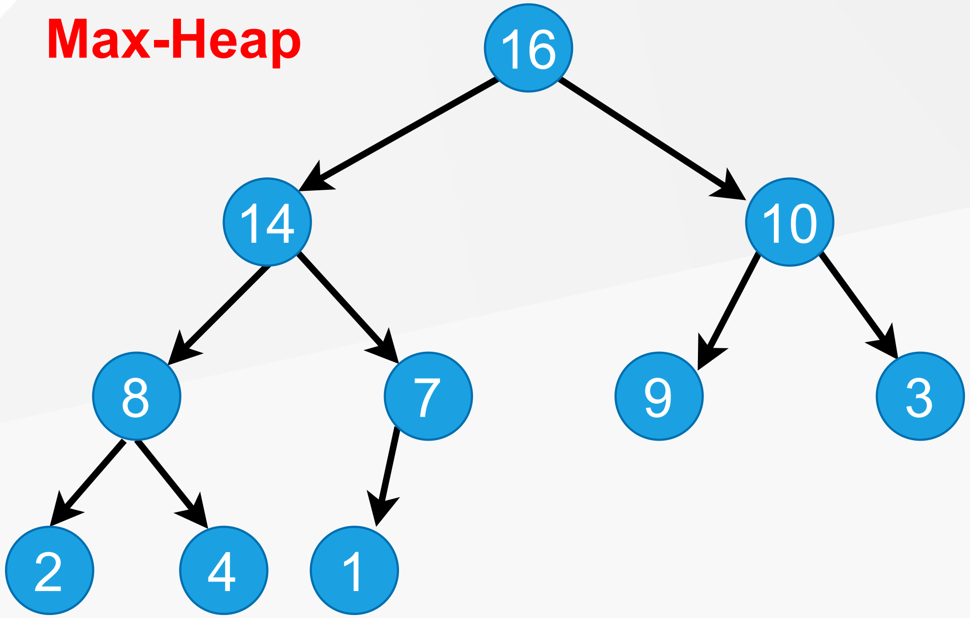
Heap Property: Min-Heap

- The **smallest** element in any subtree is the **root** element in a **min-heap**
- **Min heap:** For every node **i** other than **root**, $A[parent(i)] \leq A[i]$
    - Parent node is always smaller than the child nodes

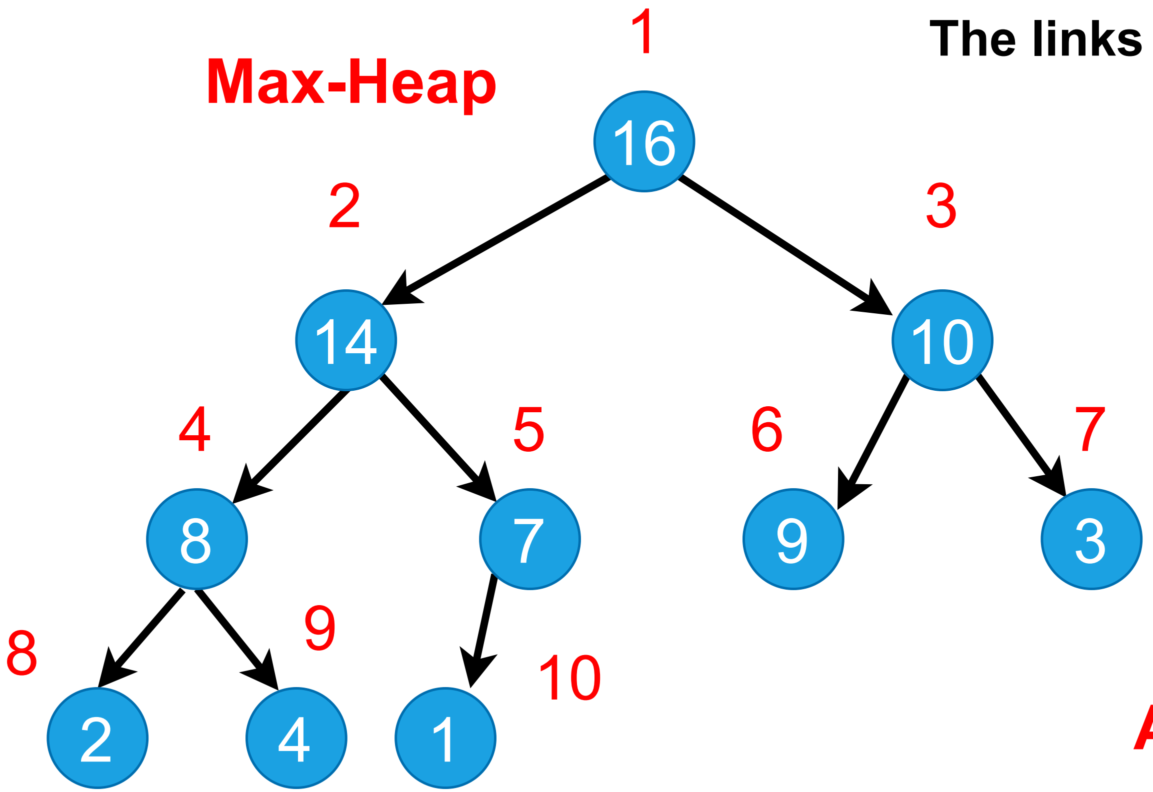


Heap Property: Max-Heap

- The **largest** element in any subtree is the **root** element in a **max-heap**
  - We will focus on max-heaps
- **Max heap:** For every node **i** other than **root**, $A[parent(i)] \geq A[i]$
  - Parent node is always larger than the child nodes



Heap Data Structures (4)



The links in the heap are implicit

$$left(i) = 2i$$

e.g.Left child of node 4 has index 8

$$right(i) = 2i + 1$$

e.g. Right child of node 2 has index 5

$$parent(i) = \lfloor i/2 \rfloor$$

e.g.Parent of node 7 has index 3

Array Storage

Heap can be stored in a linear array

1	2	3	4	5	6	7	8	9	10
16	14	10	8	7	9	3	2	4	1

Heap Data Structures (5)

- Computing left child, right child, and parent indices very fast
  - **left(i) = 2i** $\Longrightarrow$ binary left shift
  - **right(i) = 2i+1** $\Longrightarrow$ binary left shift, then set the lowest bit to 1
  - **parent(i) = floor(i/2)** $\Longrightarrow$ right shift in binary
- $A[1]$ is always the **root** element
- Array A has two attributes:
  - **length(A):** The number of elements in A
  - **n = heap-size(A):** The number elements in $heap$
    - $n \leq length(A)$

Heap Operations : EXTRACT-MAX (1)

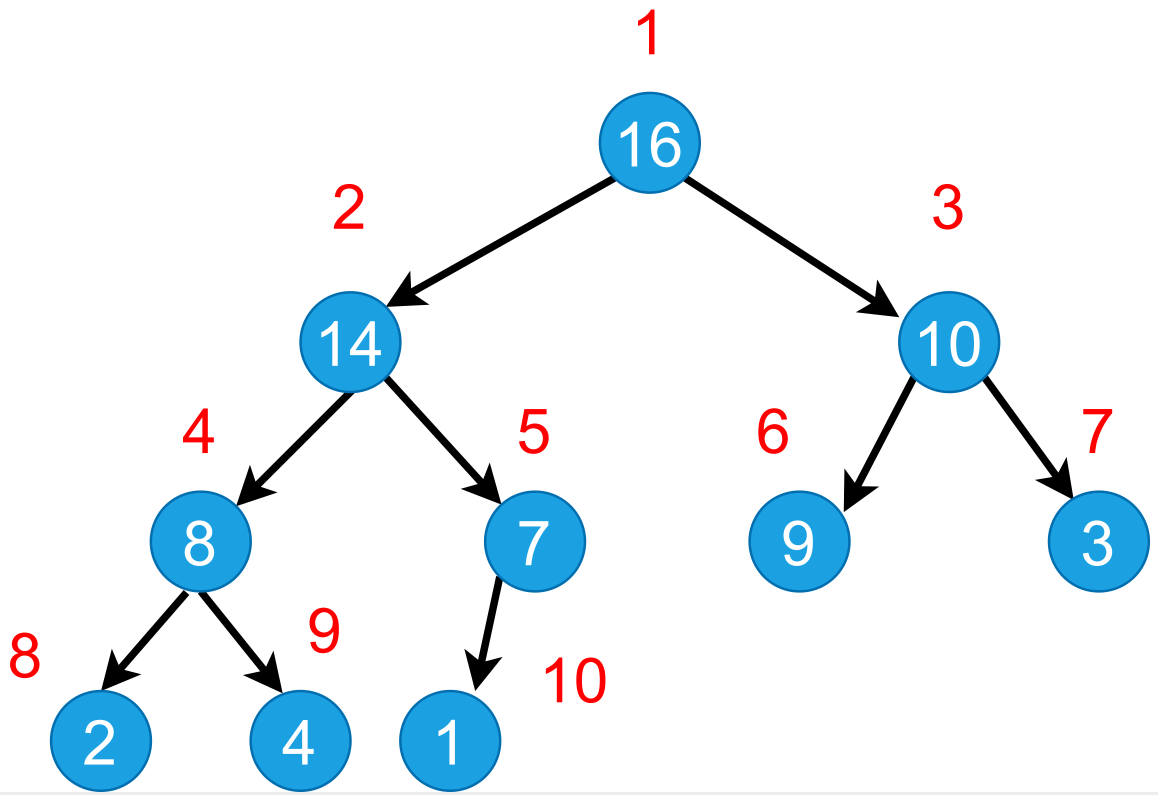
```
EXTRACT-MAX(A, n)
  max = A[1]
  A[1] = A[n]
  n = n - 1
  HEAPIFY(A, 1,n)
  return max
```

Heap Operations : EXTRACT-MAX (2)

- Return the max element,and reorganize the heap to maintain heap property

```
EXTRACT-MAX(A, n)
   max    = A[1]
   A[1]   = A[n]
   n = n - 1
   HEAPIFY(A, 1,n)
   return max
```

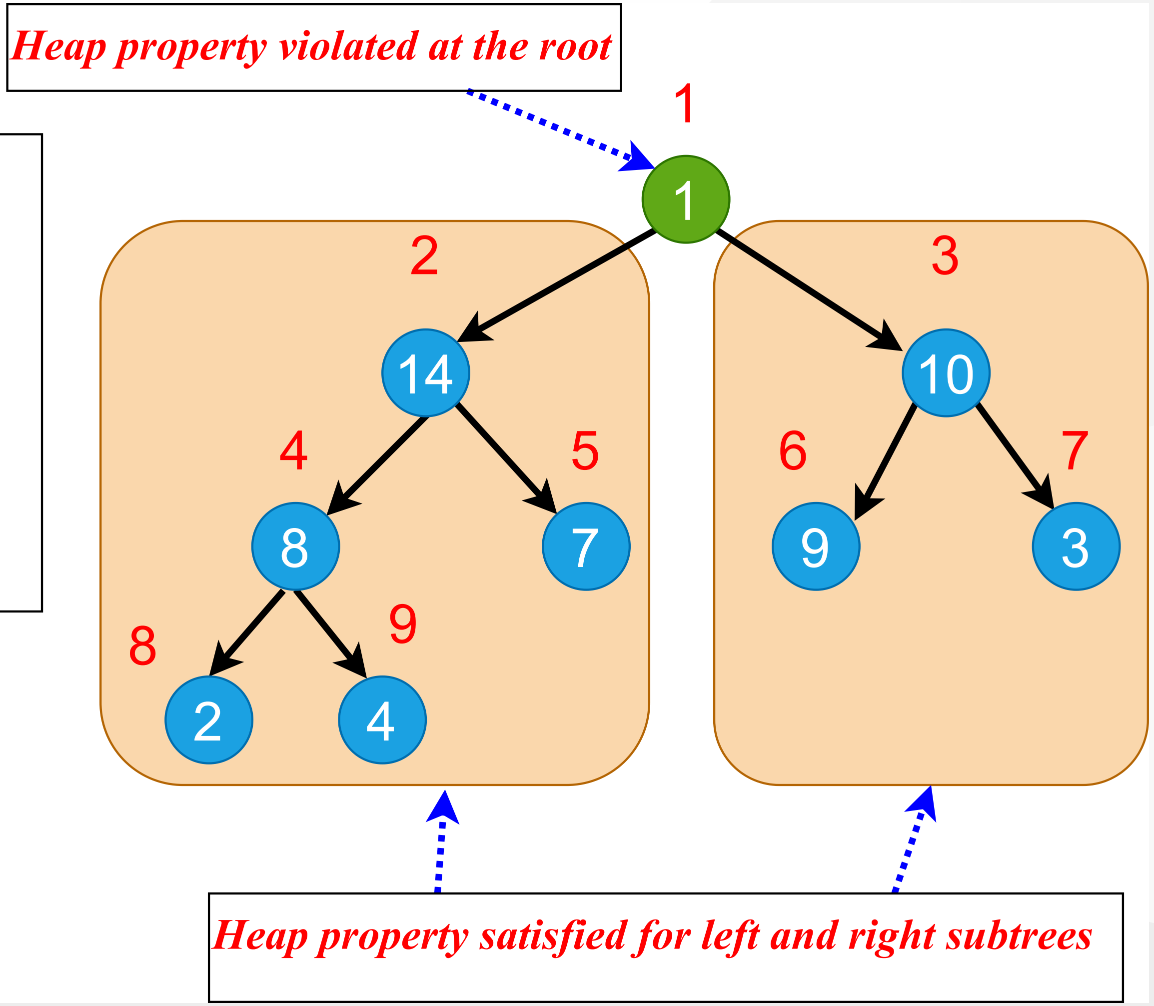
max=?



Heap Operations: HEAPIFY (1)

```
EXTRACT-MAX(A, n)
   max    = A[1]
   A[1]   = A[n]
   n = n - 1
   HEAPIFY(A, 1,n)
   return max
```

max= 16



Heap Operations: HEAPIFY (2)

- Maintaining heap property:
    - Subtrees rooted at $left[i]$ and $right[i]$ are already heaps.
    - But, $A[i]$ may violate the heap property (i.e., may be smaller than its children)
- **Idea:** Float down the value at $A[i]$ in the heap so that subtree rooted at i becomes a heap.

Heap Operations: HEAPIFY (2)

```
HEAPIFY(A, i, n)
  largest = i

  if 2i <= n and A[2i] > A[i] then
    largest = 2i;
  endif

  if 2i+1 <= n and A[2i+1] > A[largest] then
    largest = 2i+1;
  endif

  if largest != i  then
    exchange A[i] with A[largest];
    HEAPIFY(A, largest, n);
  endif
```

Heap Operations: HEAPIFY (3)

```
HEAPIFY(A,i,n)
  largest=i
  if 2i<=n and A[2i]>A[i]
   then largest=2i;
  if 2i+1<=n and A[2i+1]>A[largest]
    then largest=2i+1;
  if largest!=i  then
    exchange A[i] with A[largest];
    HEAPIFY(A,largest,n);
  endif
```

initialize *largest* to be the *node i*

check the *left child* of *node i*

check the *right child* of *node i*

exchange the *largest* of the 3 with *node i*

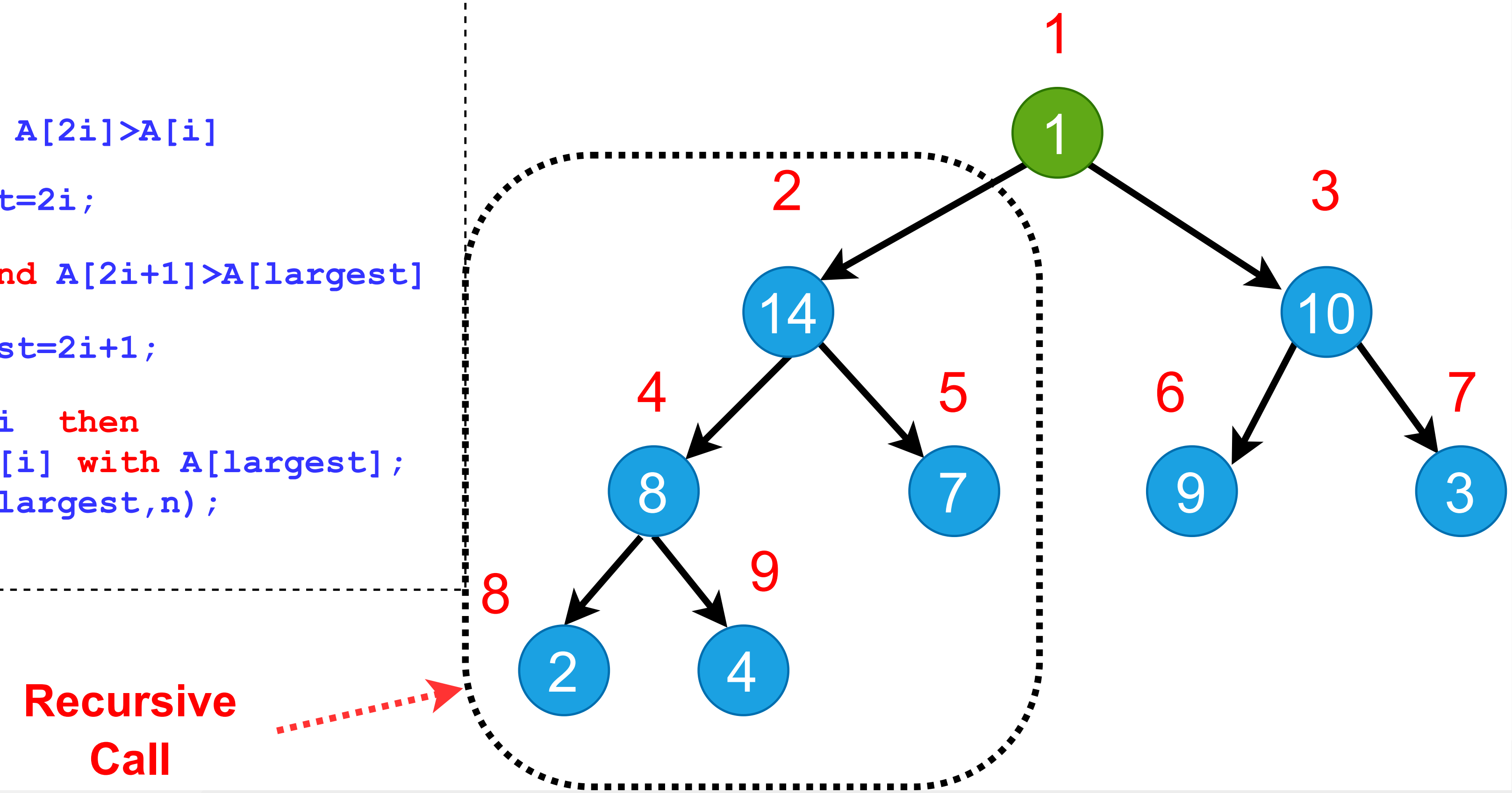
recursive call on the subtree

compute the largest of:
1) node i
2) left child of node i
3) right child of node i

Heap Operations: HEAPIFY (4)

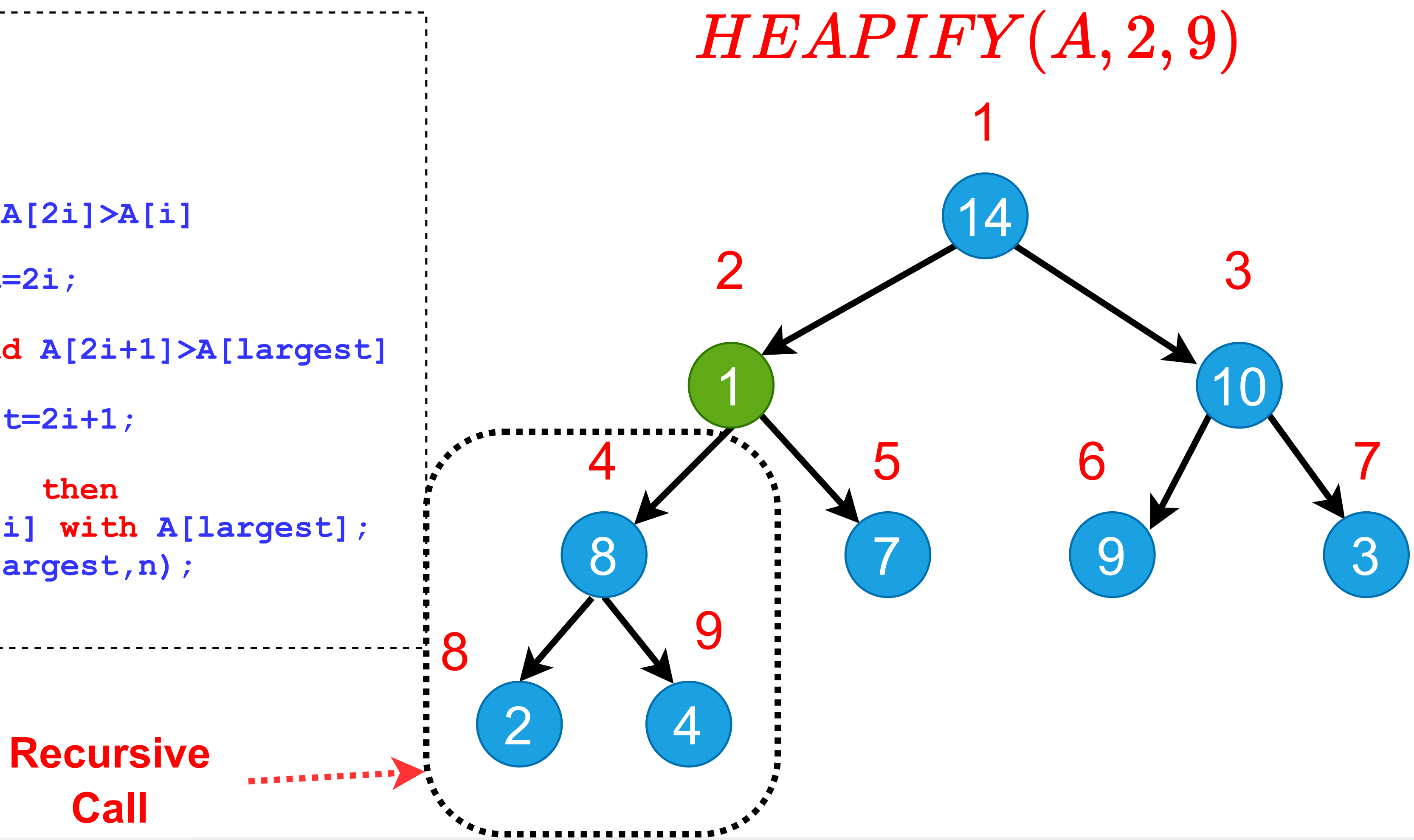
```
HEAPIFY(A,i,n)
  largest=i
  if 2i<=n and A[2i]>A[i]
   then largest=2i;
  if 2i+1<=n and A[2i+1]>A[largest]
    then largest=2i+1;
  if largest!=i  then
    exchange A[i] with A[largest];
    HEAPIFY(A,largest,n);
  endif
```

$HEAPIFY(A,1,9)$



Heap Operations: HEAPIFY (5)

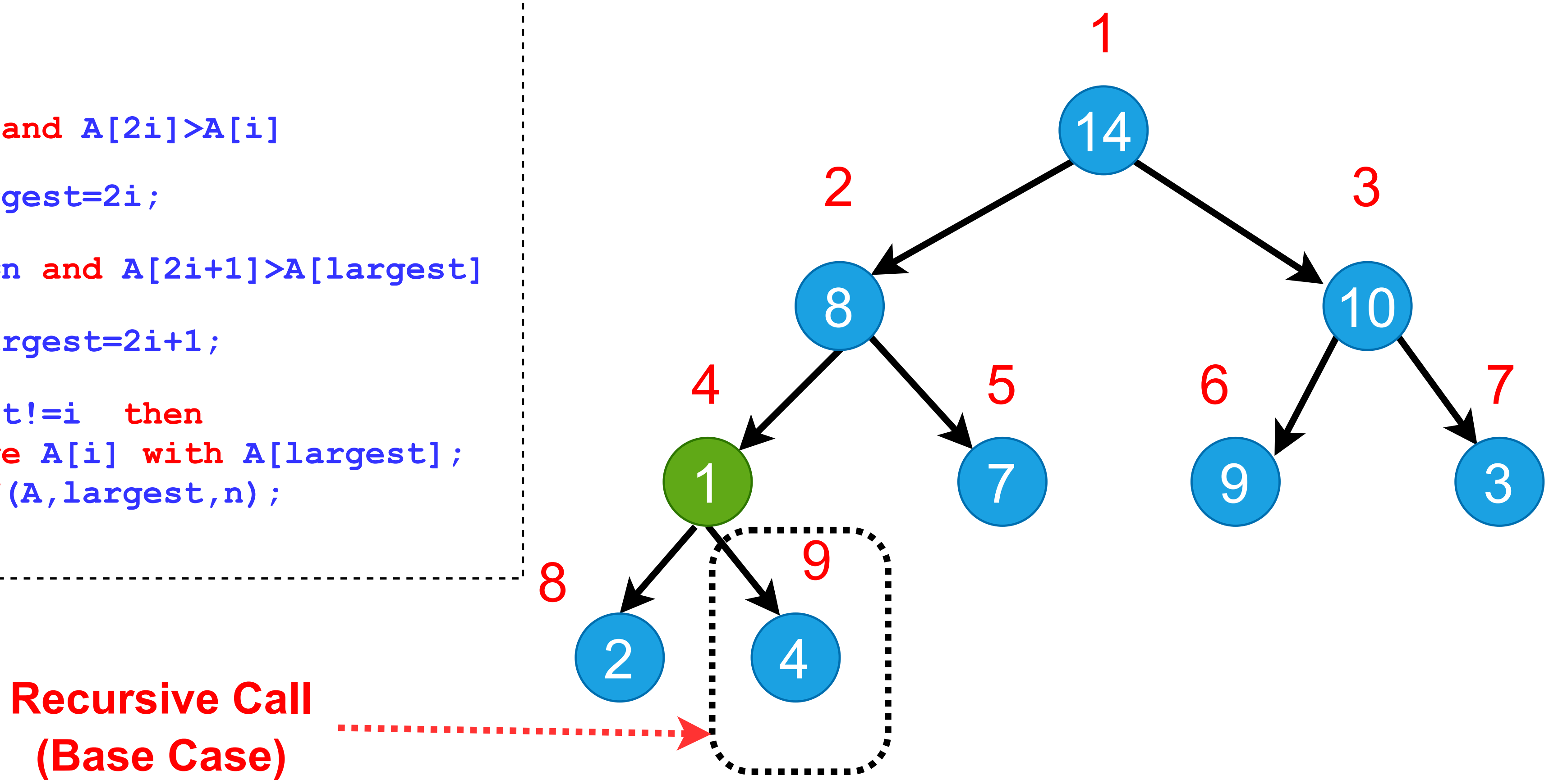
```
HEAPIFY(A,i,n)

  largest=i

  if 2i<=n and A[2i]>A[i]

   then largest=2i;

  if 2i+1<=n and A[2i+1]>A[largest]

    then largest=2i+1;

  if largest!=i  then
    exchange A[i] with A[largest];
    HEAPIFY(A,largest,n);
  endif
```



Heap Operations: HEAPIFY (6)

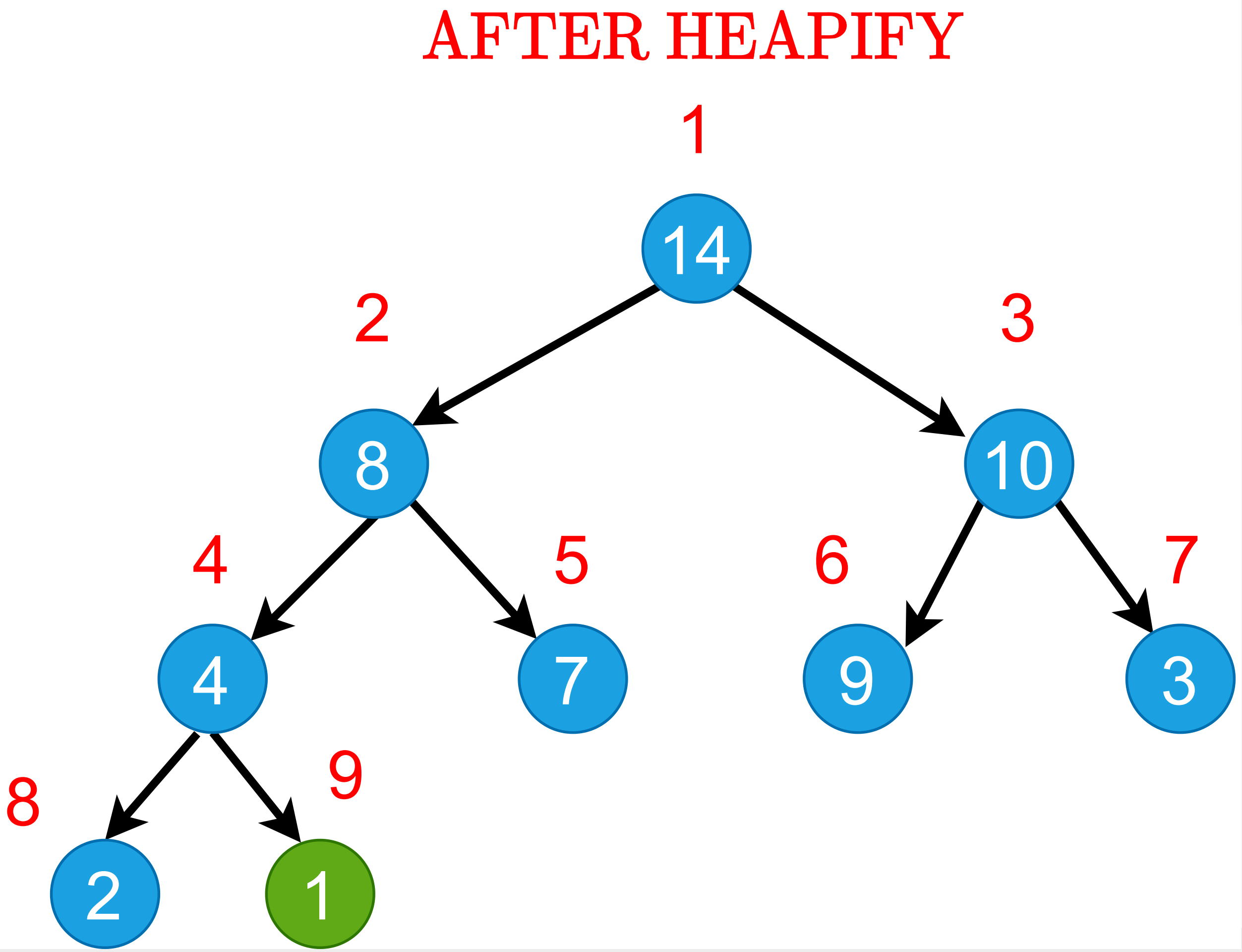
```
HEAPIFY(A,i,n)

  largest=i

  if 2i<=n and A[2i]>A[i]

   then largest=2i;

  if 2i+1<=n and A[2i+1]>A[largest]

    then largest=2i+1;

  if largest!=i  then
    exchange A[i] with A[largest];
    HEAPIFY(A,largest,n);
  endif
```

$HEAPIFY(A, 4, 9)$

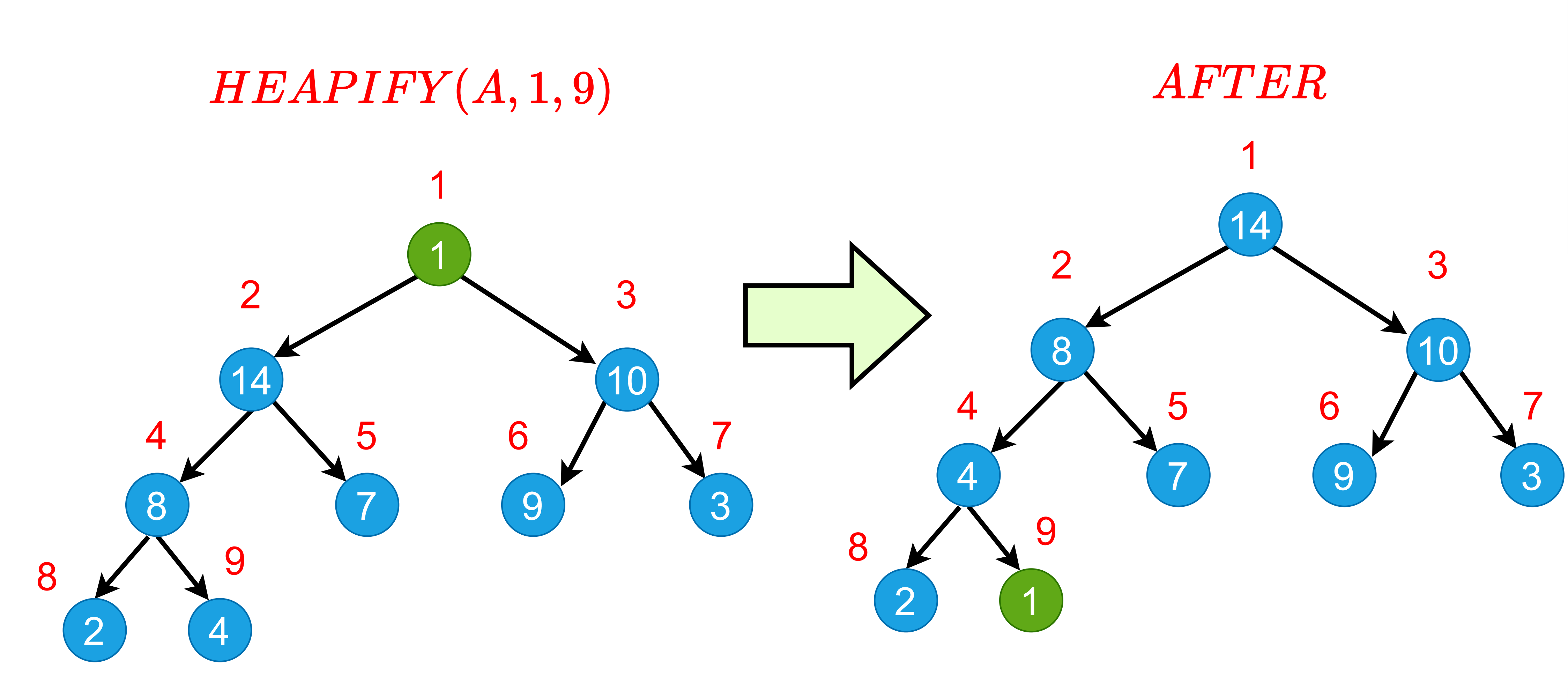


Heap Operations: HEAPIFY (7)

```
HEAPIFY(A,i,n)
  largest=i
  if 2i<=n and A[2i]>A[i]
   then largest=2i;
  if 2i+1<=n and A[2i+1]>A[largest]
    then largest=2i+1;
  if largest!=i  then
    exchange A[i] with A[largest];
    HEAPIFY(A,largest,n);
  endif
```



Heap Operations: HEAPIFY (8)



Intuitive Analysis of HEAPIFY

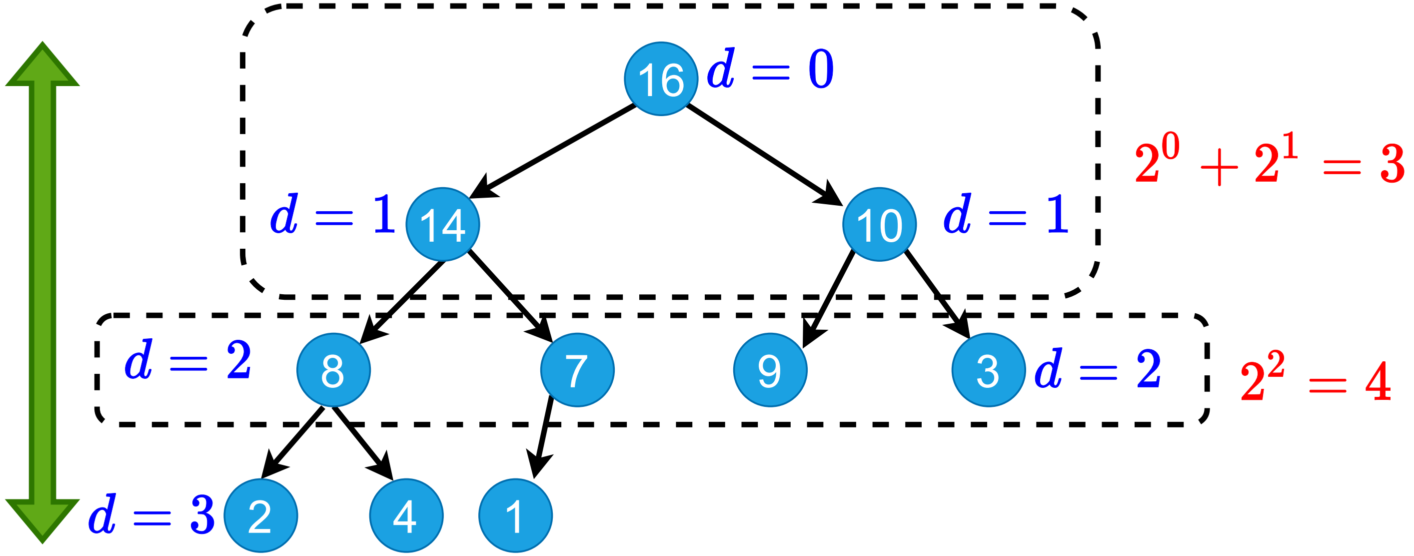
- Consider $HEAPIFY(A, i, n)$
  - let $h(i)$ be the height of node i
  - at most $h(i)$ recursion levels
    - Constant work at each level: $\Theta(1)$
  - Therefore $T(i) = O(h(i))$
- Heap is almost-complete binary tree
  - $h(n) = O(lgn)$
- Thus $T(n) = O(lgn)$

Formal Analysis of HEAPIFY

- **What is the recurrence?**
    - Depends on the size of the **subtree** on which recursive call is made
        - In the next, we try to compute an **upper bound** for this **subtree**.

Reminder: Binary trees

- For a complete binary tree:
  - $\#$ of nodes at depth d: 2^d
  - $\#$ of nodes with depths less than d: $2^d - 1$



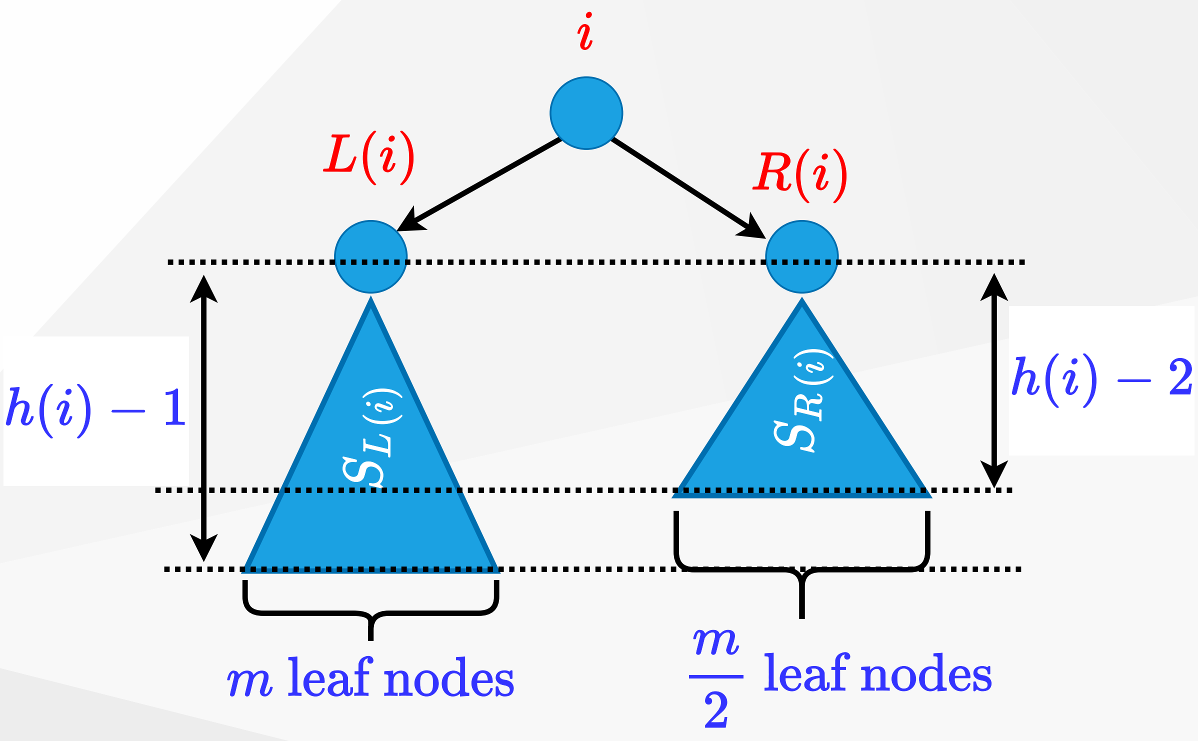
$d = depth$ for example $d = 2$

2^d = node size at d $\quad 2^2 = 4$

$2^d - 1$ = node size less than d $\quad 2^2 - 1 = 3 \Longrightarrow 2^0 + 2^1$

Formal Analysis of HEAPIFY (1)

- Worst case occurs when last row of the subtree S_i rooted at node i is **half full**
- $T(n) \leq T(|S_{L(i)}|) + \Theta(1)$
- $S_{L(i)}$ and $S_{R(i)}$ are complete binary trees of heights $h(i) - 1$ and $h(i) - 2$, respectively



Formal Analysis of HEAPIFY (2)

- Let m be the number of **leaf nodes** in $S_{L(i)}$
  - $|S_{L(i)}| = \overbrace{m}^{ext.} + \overbrace{(m–1)}^{int.} = 2m–1$
  - $|S_{R(i)}| = \overbrace{\frac{m}{2}}^{ext.} + \overbrace{(\frac{m}{2}–1)}^{int.} = m–1$
  - $|S_{L(i)}| + |S_{R(i)}| + 1 = n$

Formal Analysis of HEAPIFY (2)

$$
\begin{aligned}
(2m–1) + (m–1) + 1 &= n \\
m &= (n+1)/3 \\
|S_{L(i)}| &= 2m–1 \\
&= 2(n+1)/3–1 \\
&= (2n/3 + 2/3)–1 \\
&= \frac{2n}{3} - \frac{1}{3} \leq \frac{2n}{3} \\
T(n) &\leq T(2n/3) + \Theta(1) \\
T(n) &= O(lgn)
\end{aligned}
$$

- By CASE-2 of Master Theorem $\Longrightarrow T(n) = \Theta(n^{log_b^a} lgn)$

Formal Analysis of HEAPIFY (2)

- Recurrence: $T(n) = aT(n/b) + f(n)$
- *Case 2:* $\frac{f(n)}{n^{log_b^a}} = \Theta(1)$
- i.e., $f(n)$ and $n^{log_b^a}$ grow at similar rates
- **Solution:** $T(n) = \Theta(n^{log_b^a} lgn)$
  - $T(n) \leq T(2n/3) + \Theta(1)$ (drop constants.)
  - $T(n) \leq \Theta(n^{log_3^1} lgn)$
  - $T(n) \leq \Theta(n^0 lgn)$
  - $T(n) = O(lgn)$

HEAPIFY: Efficiency Issues

- **Recursion vs Iteration:**
  - In the absence of tail recursion, **iterative version** is in general **more efficient** because of the **pop/push** operations **to/from** stack at each **level of recursion**.

Heap Operations: HEAPIFY (1)

Recursive

```
HEAPIFY(A, i, n)
largest = i

if 2i <= n and A[2i] > A[i] then
  largest = 2i

if 2i+1 <= n and A[2i+1] > A[largest] then
  largest = 2i+1

if largest != i  then
  exchange A[i] with A[largest]
  HEAPIFY(A, largest, n)
```

Heap Operations: HEAPIFY (2)

Iterative

```
HEAPIFY(A, i, n)
  j = i
  while(true) do
    largest = j

  if 2j <= n and A[2j] > A[j] then
    largest = 2j

  if 2j+1 <= n and A[2j+1] > A[largest] then
    largest = 2j+1

  if largest != j  then
    exchange A[j] with A[largest]
    j = largest
  else return
```

Heap Operations: HEAPIFY (3)

Recursive

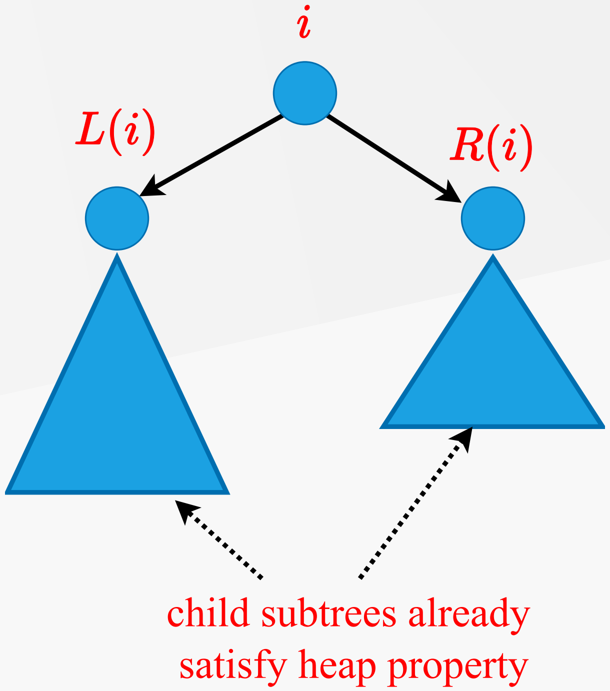
```
HEAPIFY(A,i, n)
 largest ← i
  if 2i ≤ n and A[2i] > A[i] then largest ← 2i
  if 2i +1 ≤ n and A[2i+1] > A[largest] then largest ← 2i +1
  if largest ≠ i  then
     exchange A[i]↔ A[largest]
     HEAPIFY(A, largest, n)
```

Iterative

```
HEAPIFY(A,i, n)
 j ← i
 while (true) do
    largest ← j

if 2j ≤ n and A[2j] > A[j] then largest ← 2j
if 2j +1 ≤ n and A[2j+1] > A[largest] then largest ← 2j +1
if largest ≠ j  then
   exchange A[j]↔ A[largest]
   j ← largest
else return
```

Heap Operations: Building Heap

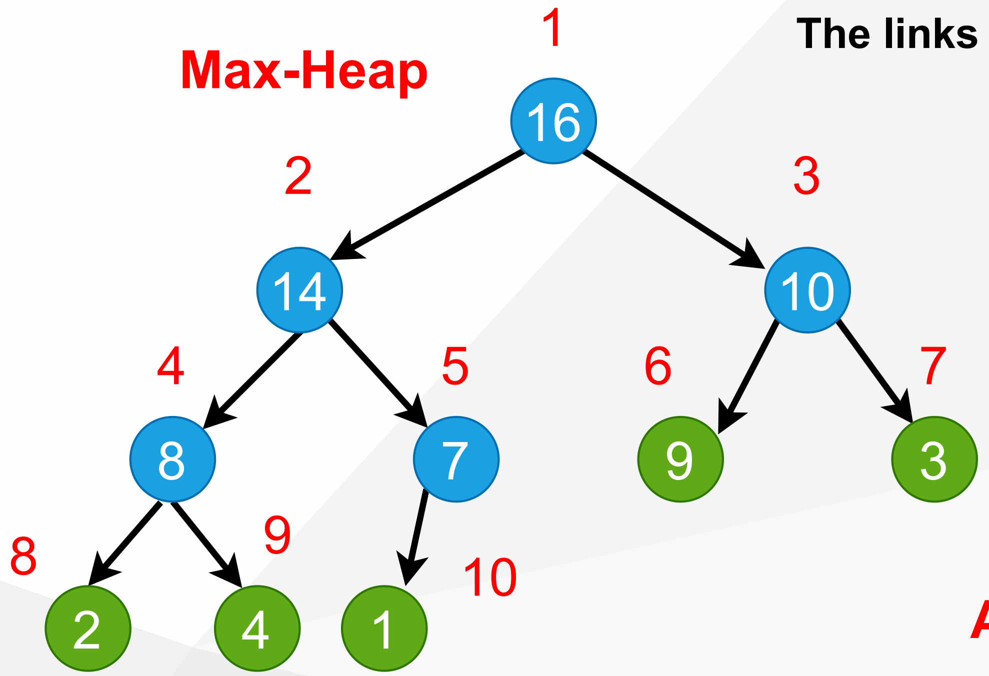
- Given an arbitrary array, how to build a heap from scratch?
- **Basic idea:** Call $HEAPIFY$ on each node bottom up
  - Start from the leaves (which trivially satisfy the heap property)
  - Process nodes in bottom up order.
  - When $HEAPIFY$ is called on node i, the subtrees connected to the $left$ and $right$ subtrees already satisfy the heap property.



Storage of the leaves (Lemma)

- **Lemma:** The last $\left\lceil \frac{n}{2} \right\rceil$ nodes of a heap are all leaves.

Max-Heap



The links in the heap are implicit

$$left(i) = 2i$$

e.g.Left child of node 4 has index 8

$$right(i) = 2i + 1$$

e.g. Right child of node 2 has index 5

$$parent(i) = \lfloor i/2 \rfloor$$

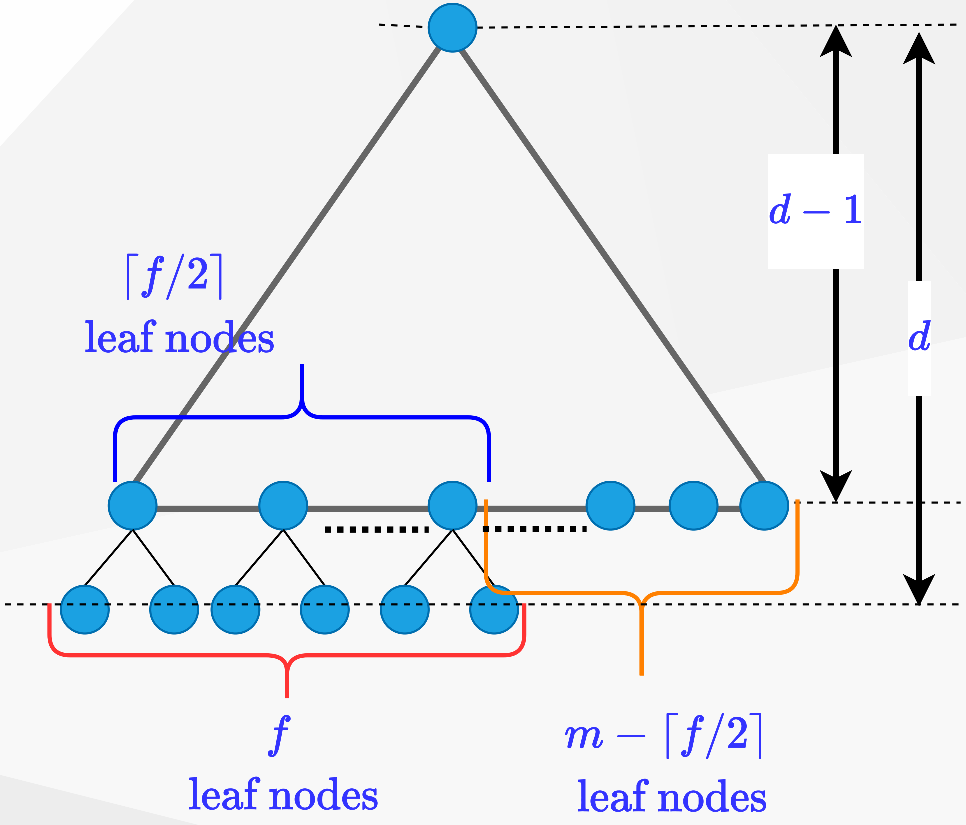
e.g.Parent of node 7 has index 3

Array Storage

1	2	3	4	5	6	7	8	9	10
16	14	10	8	7	9	3	2	4	1

Storage of the leaves (Proof of Lemma) (1)

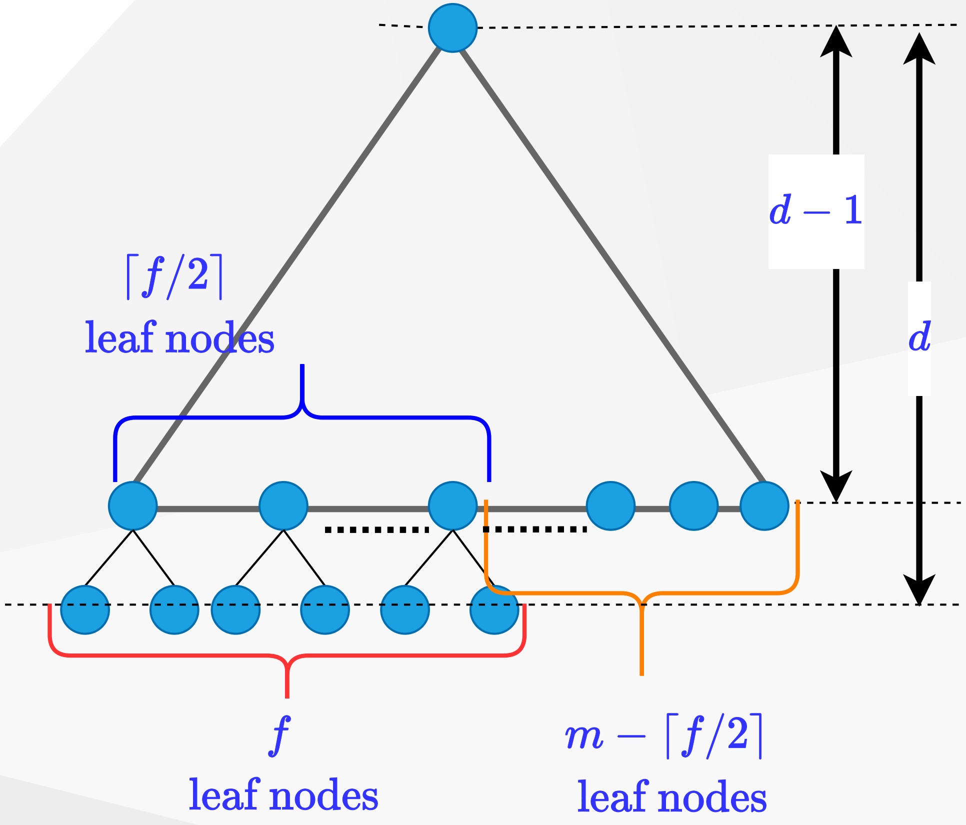
- **Lemma:** last $\lceil n/2 \rceil$ nodes of a heap are all leaves
- Proof :
  - $m = 2^{d-1}$: $\#$ nodes at level $d-1$
  - f: $\#$ nodes at level d (last level)
- $\#$ of nodes with depth $d-1 : m$
- $\#$ of nodes with depth $< d-1 : m-1$
- $\#$ of nodes with depth $d : f$
- **Total** $\#$ of nodes :$n = f + 2m - 1$



Storage of the leaves (Proof of Lemma) (2)

- **Total** $\#$ of nodes : $f = n - 2m + 1$

$$
\begin{aligned}
\#\text{ of leaves: } &= f + m - \lceil f/2 \rceil \\
&= m + \lfloor f/2 \rfloor \\
&= m + \lfloor (n - 2m + 1)/2 \rfloor \\
&= \lfloor (n + 1)/2 \rfloor \\
&= \lceil n/2 \rceil
\end{aligned}
$$



Proof is Completed

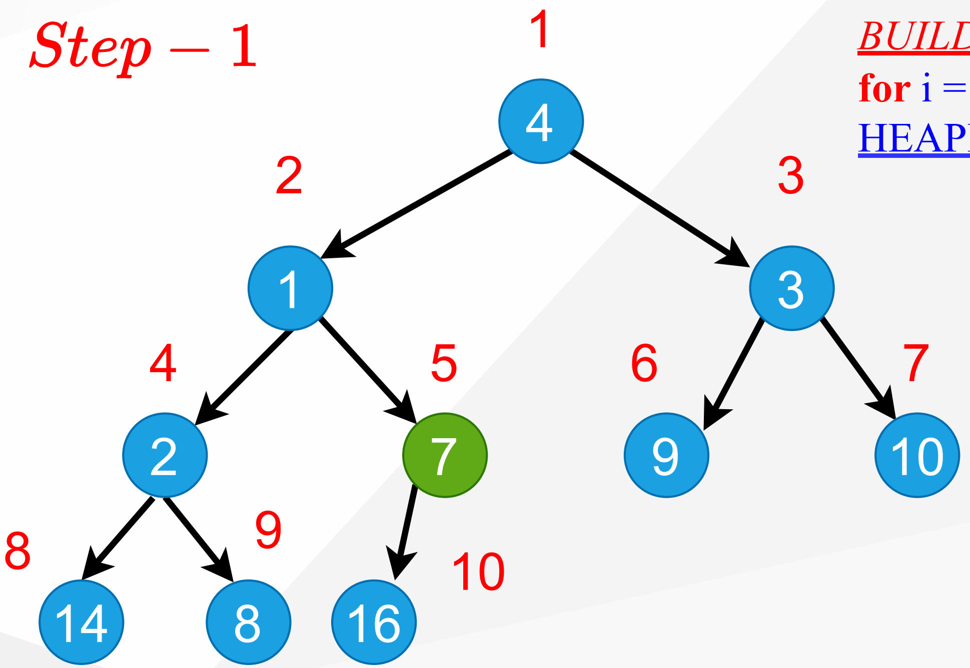
Heap Operations: Building Heap

```
BUILD-HEAP (A, n)
  for i = ceil(n/2) downto 1 do
    HEAPIFY(A, i, n)
```

- **Reminder:** The last $\lceil n/2 \rceil$ nodes of a heap are **all leaves**, which trivially satisfy the heap property

Build-Heap Example (Step-1)

$Step - 1$



BUILD-HEAP(A, n)
for i = ⌊n/2⌋ **downto** 1 **do**
HEAPIFY(A, i, n)

$$i = \lfloor n/2 \rfloor - 0$$

$$i = \lfloor 10/2 \rfloor - 0$$

$$i = 5$$

$$HEAPIFY(A, 5, 10)$$

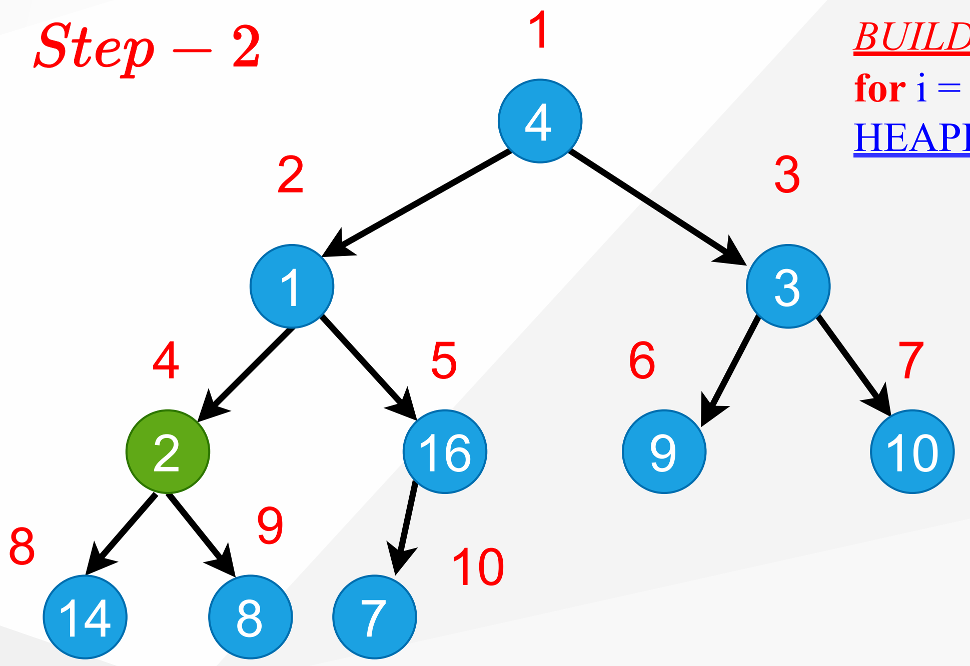
Array Storage

	1	2	3	4	5	6	7	8	9	10
$A =$	4	1	3	2	7	9	10	14	8	16

Exchange

Build-Heap Example (Step-2)

$Step - 2$



BUILD-HEAP(A, n)
for i = ⌊n/2⌋ **downto** 1 **do**
HEAPIFY(A, i, n)

$i = \lfloor n/2 \rfloor - 1$

$i = \lfloor 10/2 \rfloor - 1$

$i = 4$

$HEAPIFY(A, 4, 10)$

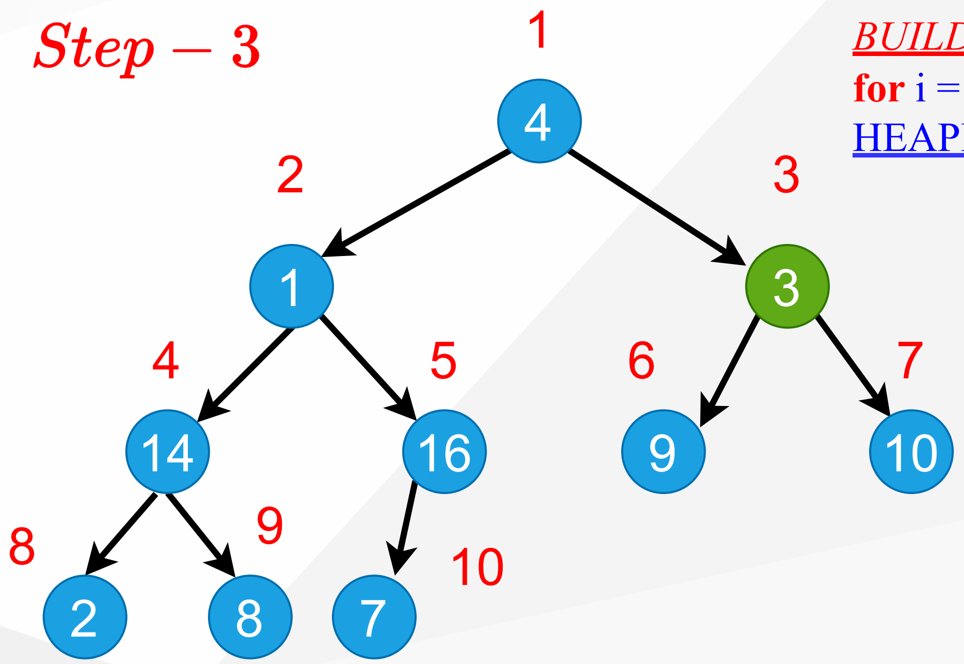
Array Storage

	1	2	3	4	5	6	7	8	9	10
$A =$	4	1	3	2	16	9	10	14	8	7

Exchange

Build-Heap Example (Step-3)

Step − 3



<u>*BUILD-HEAP(A, n)*</u>
for i = ⌊n/2⌋ **downto** 1 **do**
<u>HEAPIFY</u>(A, i, n)

$i = \lfloor n/2 \rfloor - 2$

$i = \lfloor 10/2 \rfloor - 2$

$i = 3$

$HEAPIFY(A, 3, 10)$

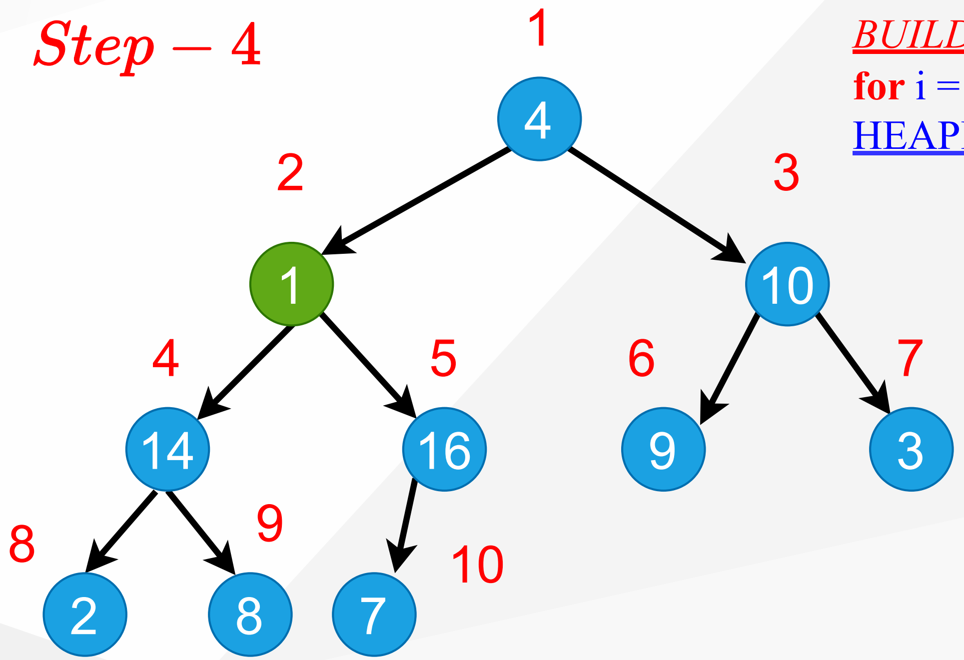
Array Storage

	1	2	3	4	5	6	7	8	9	10
$A =$	4	1	3	14	16	9	10	2	8	7

Exchange

Build-Heap Example (Step-4)

$Step - 4$



BUILD-HEAP(A, n)
for i = ⌊n/2⌋ **downto** 1 **do**
HEAPIFY(A, i, n)

$i = \lfloor n/2 \rfloor - 3$

$i = \lfloor 10/2 \rfloor - 3$

$i = 2$

$HEAPIFY(A, 2, 10)$

Array Storage

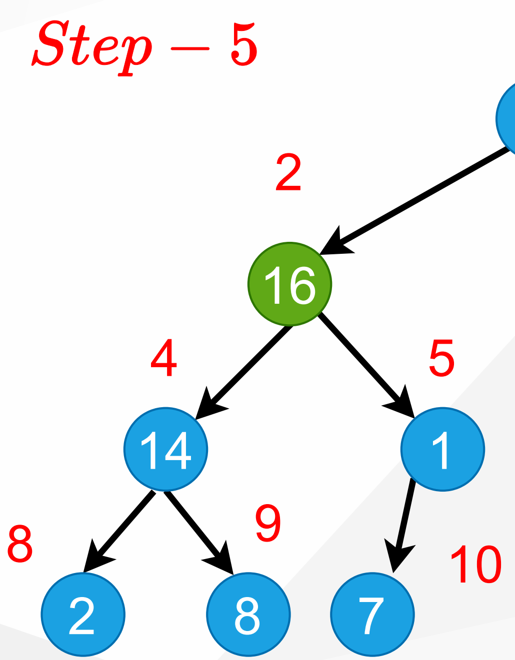
$A =$

1	2	3	4	5	6	7	8	9	10
4	1	10	14	16	9	3	2	8	7

Exchange

Build-Heap Example (Step-5)

$Step - 5$



BUILD-HEAP(A, n)
for i = ⌊n/2⌋ **downto** 1 **do**
HEAPIFY(A, i, n)

$i = \lfloor n/2 \rfloor - 3$

$i = \lfloor 10/2 \rfloor - 3$

$i = 2$

$HEAPIFY(A, 2, 10)$

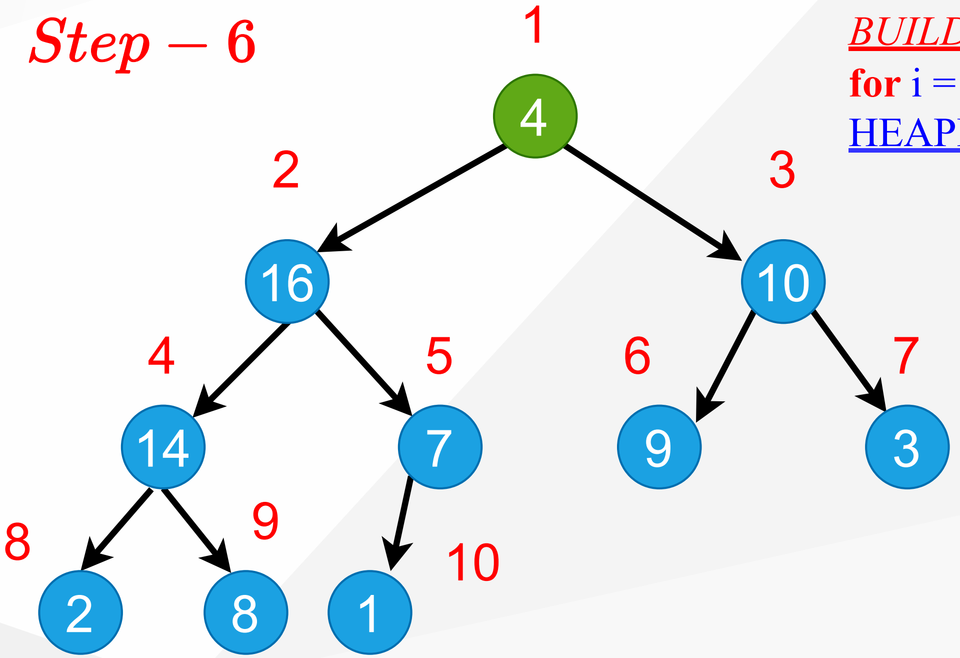
Array Storage

	1	2	3	4	5	6	7	8	9	10
$A =$	4	16	10	14	1	9	3	2	8	7

Exchange

Build-Heap Example (Step-6)

$Step - 6$



<u>*BUILD-HEAP(A, n)*</u>
for i = ⌊n/2⌋ **downto** 1 **do**
<u>HEAPIFY</u>(A, i, n)

$i = \lfloor n/2 \rfloor - 4$

$i = \lfloor 10/2 \rfloor - 4$

$i = 1$

$HEAPIFY(A, 1, 10)$

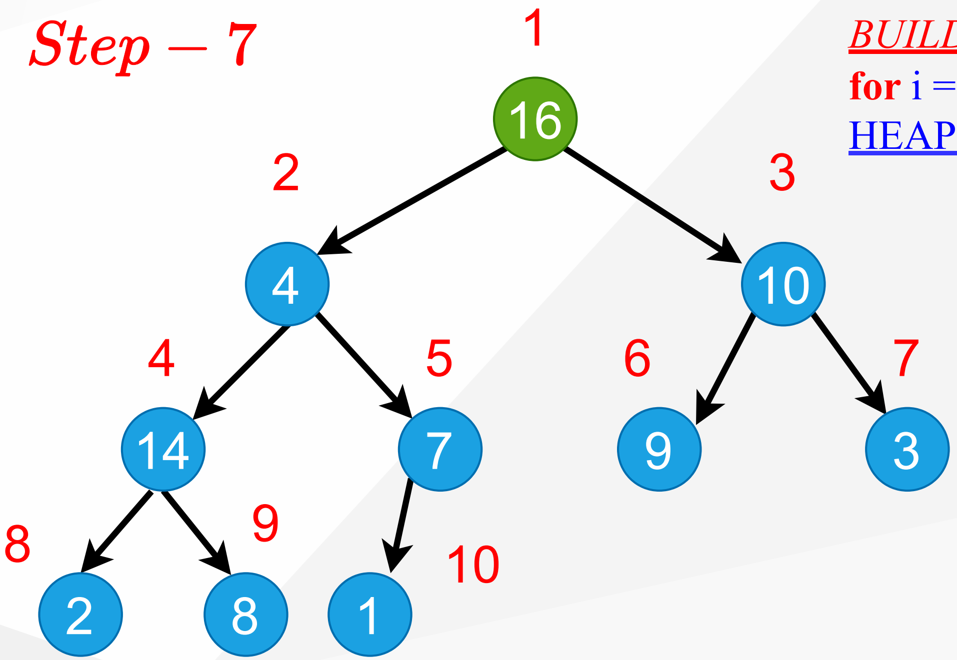
Array Storage

	1	2	3	4	5	6	7	8	9	10
$A =$	4	16	10	14	7	9	3	2	8	1

Exchange

Build-Heap Example (Step-7)

$Step - 7$



BUILD-HEAP(A, n)
for i = ⌊n/2⌋ **downto** 1 **do**
HEAPIFY(A, i, n)

$i = \lfloor n/2 \rfloor - 4$

$i = \lfloor 10/2 \rfloor - 4$

$i = 1$

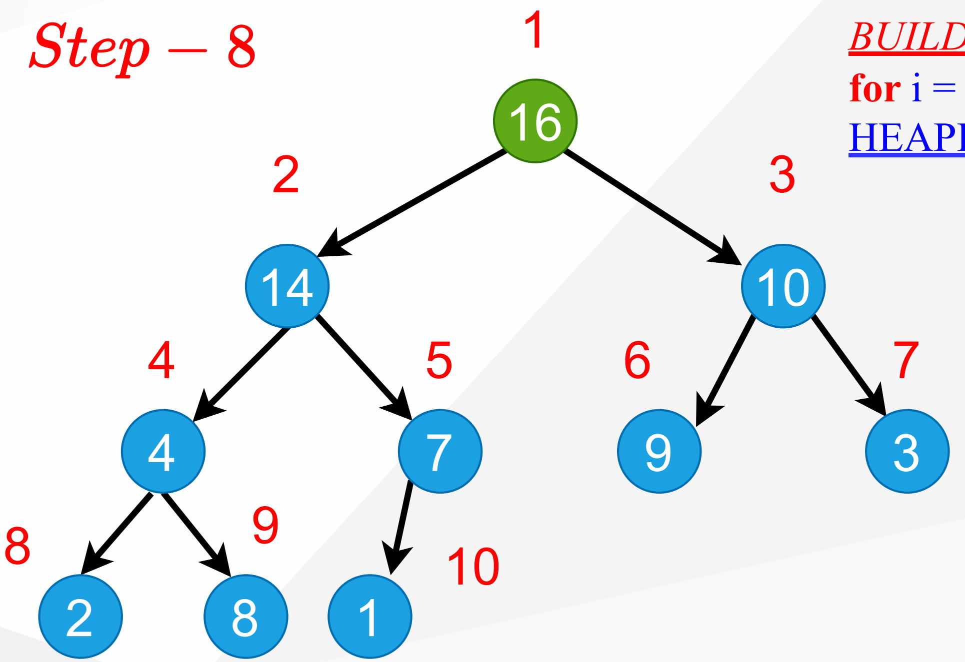
$HEAPIFY(A, 1, 10)$

Array Storage

	1	2	3	4	5	6	7	8	9	10
$A =$	16	4	10	14	7	9	3	2	8	1

Exchange

Build-Heap Example (Step-8)



BUILD-HEAP(A, n)
for i = ⌊n/2⌋ **downto** 1 **do**
HEAPIFY(A, i, n)

$i = \lfloor n/2 \rfloor - 4$

$i = \lfloor 10/2 \rfloor - 4$

$i = 1$

$HEAPIFY(A, 1, 10)$

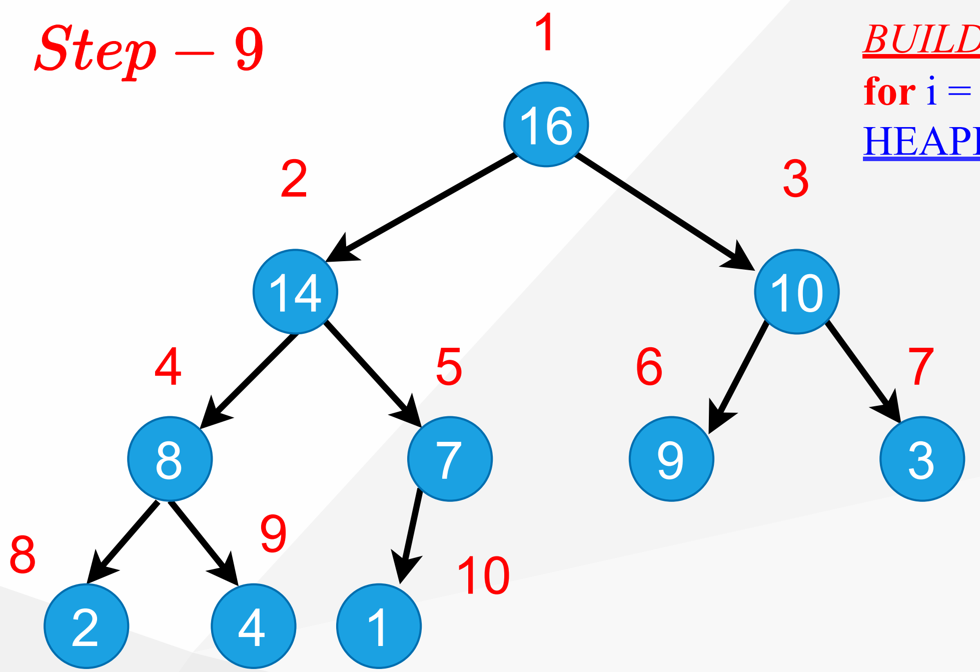
Array Storage

	1	2	3	4	5	6	7	8	9	10
$A =$	16	14	10	4	7	9	3	2	8	1

$Exchange$

Build-Heap Example (Step-9)

Step – 9



<u>*BUILD-HEAP(A, n)*</u>
for i = ⌊n/2⌋ **downto** 1 **do**
<u>HEAPIFY</u>(A, i, n)

After Build-Heap

Array Storage

$A =$

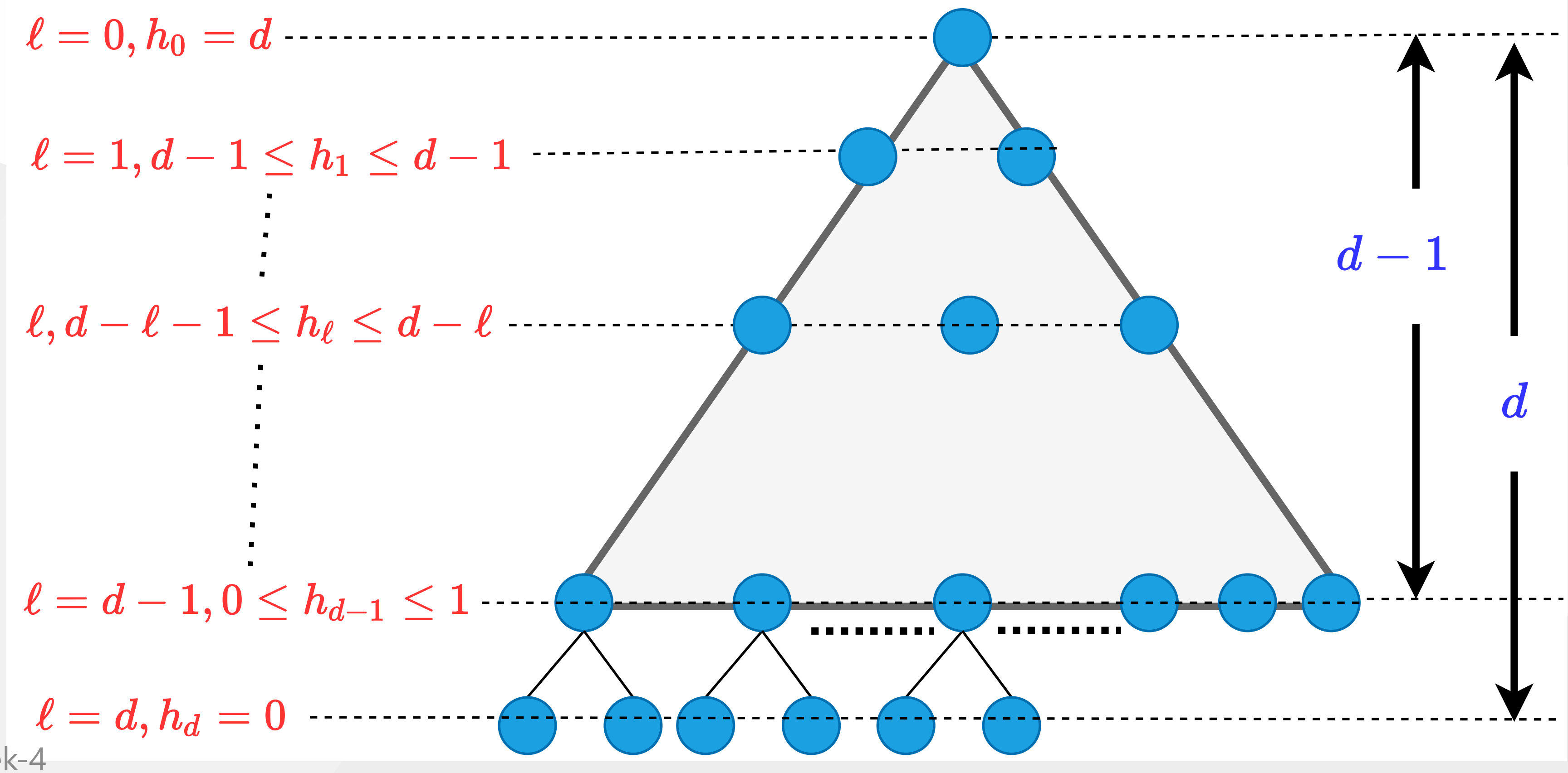
1	2	3	4	5	6	7	8	9	10
16	14	10	8	7	9	3	2	4	1

Build-Heap: Runtime Analysis

- Simple analysis:
  - $O(n)$ calls to $HEAPIFY$, each of which takes $O(lgn)$ time
  - $O(nlgn) \implies$ loose bound
- In general, a good approach:
  - Start by proving an easy bound
  - Then, try to tighten it
- Is there a tighter bound?

Build-Heap: Tighter Running Time Analysis

- If the heap is complete binary tree then $h_\ell = d – \ell$
- Otherwise, nodes at a given level do not all have the same height, But we have $d – \ell – 1 \leq h_\ell \leq d – \ell$



Build-Heap: Tighter Running Time Analysis

- Assume that all nodes at level $\ell = d{-}1$ are processed

$$T(n) = \sum_{\ell=0}^{d-1} n_\ell O(h_\ell) = O(\sum_{\ell=0}^{d-1} n_\ell h_\ell) \begin{cases} n_\ell = 2^\ell = \# \text{ of nodes at level } \ell \\ h_\ell = \text{height of nodes at level } \ell \end{cases}$$

$$\therefore T(n) = O\left(\sum_{\ell=0}^{d-1} 2^\ell (d-\ell)\right)$$

$$\text{Let } h = d - \ell \implies \ell = d - h \text{ change of variables}$$

$$T(n) = O\left(\sum_{h=1}^{d} h 2^{d-h}\right) = O\left(\sum_{h=1}^{d} h \frac{2^d}{2^h}\right) = O\left(2^d \sum_{h=1}^{d} h(1/2)^h\right)$$

$$\text{but } 2^d = \Theta(n) \implies O\left(n \sum_{h=1}^{d} h(1/2)^h\right)$$

Build-Heap: Tighter Running Time Analysis

$$\sum_{h=1}^{d} h(1/2)^h \leq \sum_{h=0}^{d} h(1/2)^h \leq \sum_{h=0}^{\infty} h(1/2)^h$$

- recall infinite decreasing geometric series

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \text{ where } |x| < 1$$

- differentiate both sides

$$\sum_{k=0}^{\infty} k x^{k-1} = \frac{1}{(1-x)^2}$$

Build-Heap: Tighter Running Time Analysis

$$\sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

- then, multiply both sides by x

$$\sum_{k=0}^{\infty} kx^{k} = \frac{x}{(1-x)^2}$$

- in our case: $x = 1/2$ and $k = h$

$$\therefore \sum_{h=0}^{\infty} h(1/2)^h = \frac{1/2}{(1-(1/2))^2} = 2 = O(1)$$

$$\therefore T(n) = O(n\sum_{h=1}^{d} h(1/2)^h) = O(n)$$

Heapsort Algorithm Steps

- **(1)** Build a heap on array $A[1 \ldots n]$ by calling $BUILD-HEAP(A, n)$
- **(2)** The largest element is stored at the root $A[1]$
  - Put it into its correct final position $A[n]$ by $A[1] \longleftrightarrow A[n]$
- **(3)** Discard node n from the heap
- **(4)** Subtrees $(S2\&S3)$ rooted at children of root remain as heaps, but the new root element may violate the heap property.
  - Make $A[1 \ldots n-1]$ a heap by calling $HEAPIFY(A, 1, n-1)$
- **(5)** $n \leftarrow n - 1$
- **(6)** Repeat steps **(2-4)** until $n = 2$

References

- Introduction to Algorithms, Third Edition | The MIT Press
- Bilkent CS473 Course Notes (new)
- Bilkent CS473 Course Notes (old)
- Insertion Sort - GeeksforGeeks
- NIST Dictionary of Algorithms and Data Structures
- NIST - Dictionary of Algorithms and Data Structures

TODO