CE100 Algorithms and Programming II

Week-3 (Matrix Multiplication/ Quick Sort)

Spring Semester, 2021-2022

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Matrix Multiplication / Quick Sort

Outline

- Matrix Multiplication
  - Traditional
  - Recursive
  - Strassen

Outline

- Quicksort
  - Hoare Partitioning
  - Lomuto Partitioning
  - Recursive Sorting

Outline

- Quicksort Analysis
  - Randomized Quicksort
  - Randomized Selection
    - Recursive
    - Medians

Matrix Multiplication

- **Input:** $A = [a_{ij}], B = [b_{ij}]$
- **Output:** $C = [c_{ij}] = A \cdot B \implies i, j = 1, 2, 3, \dots, n$

$$\begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \vdots & \ddots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \vdots & \ddots \\ b_{n1} & a_{n2} & \dots & b_{nn} \end{bmatrix}$$

Matrix Multiplication

$$\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ \boxed{c_{21}} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$$

- $c_{ij} = \sum_{1 \le k \le n} a_{ik} . b_{kj}$

Matrix Multiplication: Standard Algorithm

Running Time: $\Theta(n^3)$

```
for i=1 to n do
    for j=1 to n do
        C[i,j] = 0
        for k=1 to n do
            C[i,j] = C[i,j] + A[i,k] + B[k,j]
        endfor
    endfor
endfor
```

Matrix Multiplication: Divide & Conquer

IDEA: Divide the nxn matrix into $2x2$ matrix of $(n/2)x(n/2)$ submatrices.

$$\begin{pmatrix} \boxed{c_{11}} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$c_{11} = a_{11}b_{11} + a_{12}b_{21}$$

$$\begin{pmatrix} c_{11} & c_{12} \\ \boxed{c_{21}} & c_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$c_{21} = a_{21}b_{11} + a_{22}b_{21}$$

$$\begin{pmatrix} c_{11} & \boxed{c_{12}} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$c_{12} = a_{11}b_{12} + a_{12}b_{22}$$

$$\begin{pmatrix} c_{11} & c_{12} \\ \boxed{c_{21}} & c_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$c_{22} = a_{21}b_{12} + a_{22}b_{22}$$

Matrix Multiplication: Divide & Conquer

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\text{8 mults and 4 adds of (n/2)*(n/2) submatrices} = \begin{cases} c_{11} = a_{11}b_{11} + a_{12}b_{21} \\ c_{21} = a_{21}b_{11} + a_{22}b_{21} \\ c_{12} = a_{11}b_{12} + a_{12}b_{22} \\ c_{22} = a_{21}b_{12} + a_{22}b_{22} \end{cases}$$

Matrix Multiplication: Divide & Conquer

```
MATRIX-MULTIPLY(A, B)
    // Assuming that both A and B are nxn matrices
    if n == 1 then
        return A * B
    else
        //partition A, B, and C as shown before
        C[1,1] = MATRIX-MULTIPLY (A[1,1], B[1,1]) +
                 MATRIX-MULTIPLY (A[1,2], B[2,1]);

        C[1,2] = MATRIX-MULTIPLY (A[1,1], B[1,2]) +
                MATRIX-MULTIPLY (A[1,2], B[2,2]);

        C[2,1] = MATRIX-MULTIPLY (A[2,1], B[1,1]) +
        MATRIX-MULTIPLY (A[2,2], B[2,1]);

        C[2,2] = MATRIX-MULTIPLY (A[2,1], B[1,2]) +
        MATRIX-MULTIPLY (A[2,2], B[2,2]);
    endif

    return C
```

Matrix Multiplication: Divide & Conquer Analysis

$$T(n) = 8T(n/2) + \Theta(n^2)$$

- 8 recursive calls $\implies 8T(\cdots)$
- each problem has size $n/2 \implies \cdots T(n/2)$
- Submatrix addition $\implies \Theta(n^2)$

Matrix Multiplication: Solving the Recurrence

- $T(n) = 8T(n/2) + \Theta(n^2)$
  - $a = 8, b = 2$
  - $f(n) = \Theta(n^2)$
  - $n^{log_b^a} = n^3$
- Case 1: $\frac{n^{log_b^a}}{f(n)} = \Omega(n^{\varepsilon}) \Longrightarrow T(n) = \Theta(n^{log_b^a})$

Similar with ordinary (iterative) algorithm.

Matrix Multiplication: Strassen's Idea

Compute $c_{11}, c_{12}, c_{21}, c_{22}$ using 7 recursive multiplications.

In normal case we need 8 as below.

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\text{8 mults and 4 adds of (n/2)*(n/2) submatrices} = \begin{cases} c_{11} = a_{11}b_{11} + a_{12}b_{21} \\ c_{21} = a_{21}b_{11} + a_{22}b_{21} \\ c_{12} = a_{11}b_{12} + a_{12}b_{22} \\ c_{22} = a_{21}b_{12} + a_{22}b_{22} \end{cases}$$

Matrix Multiplication: Strassen’s Idea

- **Reminder:**
  - Each submatrix is of size $(n/2) * (n/2)$
  - Each add/sub operation takes $\Theta(n^2)$ time
- Compute $P1 \dots P7$ using 7 recursive calls to matrix-multiply

$$
\begin{aligned}
P_1 &= a_{11} * (b_{12} - b_{22}) \\
P_2 &= (a_{11} + a_{12}) * b_{22} \\
P_3 &= (a_{21} + a_{22}) * b_{11} \\
P_4 &= a_{22} * (b_{21} - b_{11}) \\
P_5 &= (a_{11} + a_{22}) * (b_{11} + b_{22}) \\
P_6 &= (a_{12} - a_{22}) * (b_{21} + b_{22}) \\
P_7 &= (a_{11} - a_{21}) * (b_{11} + b_{12})
\end{aligned}
$$

Matrix Multiplication: Strassen's Idea

$$
\begin{aligned}
P_1 &= a_{11} * (b_{12} - b_{22}) \\
P_2 &= (a_{11} + a_{12}) * b_{22} \\
P_3 &= (a_{21} + a_{22}) * b_{11} \\
P_4 &= a_{22} * (b_{21} - b_{11}) \\
P_5 &= (a_{11} + a_{22}) * (b_{11} + b_{22}) \\
P_6 &= (a_{12} - a_{22}) * (b_{21} + b_{22}) \\
P_7 &= (a_{11} - a_{21}) * (b_{11} + b_{12})
\end{aligned}
$$

- How to compute c_{ij} using $P1 \ldots P7$?

$$
\begin{aligned}
c_{11} &= P_5 + P_4 - P_2 + P_6 \\
c_{12} &= P_1 + P_2 \\
c_{21} &= P_3 + P_4 \\
c_{22} &= P_5 + P_1 - P_3 - P_7
\end{aligned}
$$

Matrix Multiplication: Strassen’s Idea

- 7 recursive multiply calls
- 18 add/sub operations

Matrix Multiplication: Strassen's Idea

e.g. Show that $c_{12} = P_1 + P_2$

$$
\begin{aligned}
c_{12} &= P_1 + P_2 \\
&= a_{11}(b_{12} - b_{22}) + (a_{11} + a_{12})b_{22} \\
&= a_{11}b_{12} - a_{11}b_{22} + a_{11}b_{22} + a_{12}b_{22} \\
&= a_{11}b_{12} + a_{12}b_{22}
\end{aligned}
$$

Strassen's Algorithm

- **Divide:** Partition A and B into $(n/2) * (n/2)$ submatrices. Form terms to be multiplied using $+$ and $-$.
- **Conquer:** Perform 7 multiplications of $(n/2) * (n/2)$ submatrices recursively.
- **Combine:** Form C using $+$ and $-$ on $(n/2) * (n/2)$submatrices.

Recurrence: $T(n) = 7T(n/2) + \Theta(n^2)$

Strassen's Algorithm: Solving the Recurrence

- $T(n) = 7T(n/2) + \Theta(n^2)$
  - $a = 7, b = 2$
  - $f(n) = \Theta(n^2)$
  - $n^{log_b^a} = n^{lg7}$
- Case 1: $\frac{n^{log_b^a}}{f(n)} = \Omega(n^{\varepsilon}) \implies T(n) = \Theta(n^{log_b^a})$

$T(n) = \Theta(n^{log_2^7})$

$2^3 = 8, 2^2 = 4$ so $\implies log_2^7 \approx 2.81$

or use https://www.omnicalculator.com/math/log

Strassen's Algorithm

- The number 2.81 may not seem much smaller than 3
- But, it is significant because the difference is in the exponent.
- Strassen's algorithm beats the ordinary algorithm on today's machines for $n \geq 30$ or so.
- Best to date: $\Theta(n^{2.376\ldots})$ (of theoretical interest only)

Maximum Subarray Problem

Input: An array of values

Output: The contiguous subarray that has the largest sum of elements

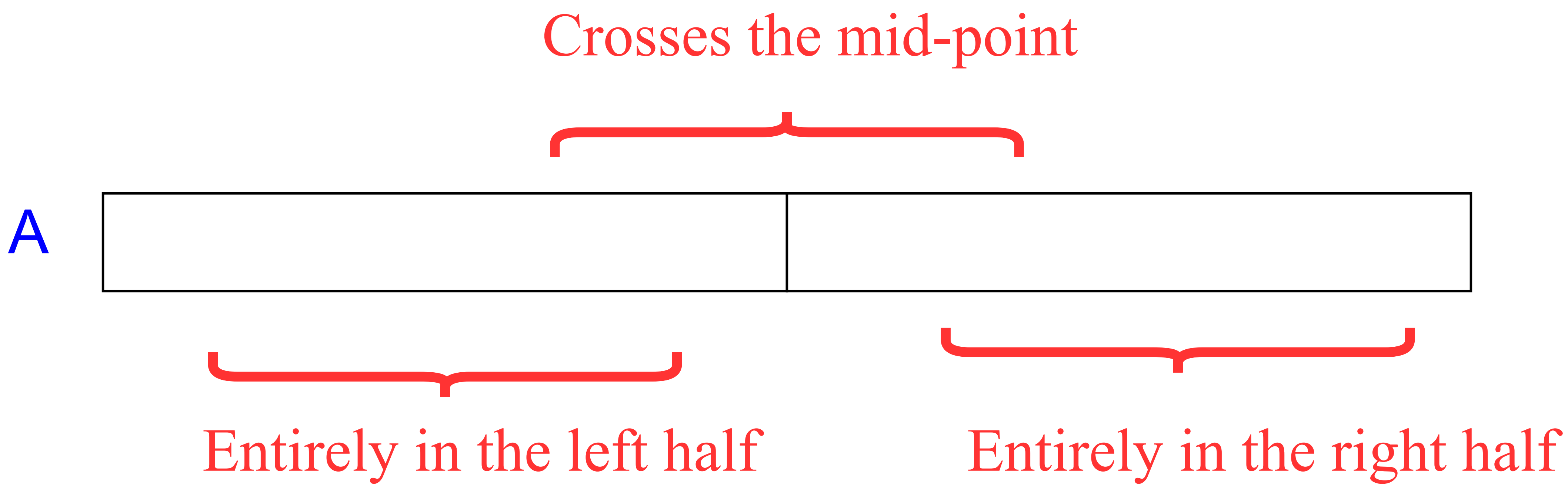
- Input array:

$$[13][-3][-25][20][-3][-16][-23]\ \overbrace{[18][20][-7][12]}^{\text{max. contiguous subarray}}\ [-22][-4][7]$$

Maximum Subarray Problem: Divide & Conquer

- **Basic idea:**
- **Divide** the input array into 2 from the middle
- Pick the **best** solution among the following:
  - The max subarray of the **left half**
  - The max subarray of the **right half**
  - The max subarray **crossing the mid-point**

Maximum Subarray Problem: Divide & Conquer



Maximum Subarray Problem: Divide & Conquer

- **Divide:** Trivial (divide the array from the middle)
- **Conquer:** Recursively compute the max subarrays of the left and right halves
- **Combine:** Compute the max-subarray crossing the $mid - point$
  - (can be done in $\Theta(n)$ time).
  - Return the max among the following:
    - the max subarray of the left-subarray
    - the max subarray of the rightsubarray
    - the max subarray crossing the mid-point

TODO : detailed solution in textbook...

Conclusion : Divide & Conquer

- Divide and conquer is just one of several powerful techniques for algorithm design.
- Divide-and-conquer algorithms can be analyzed using recurrences and the master method (so practice this math).
- Can lead to more efficient algorithms

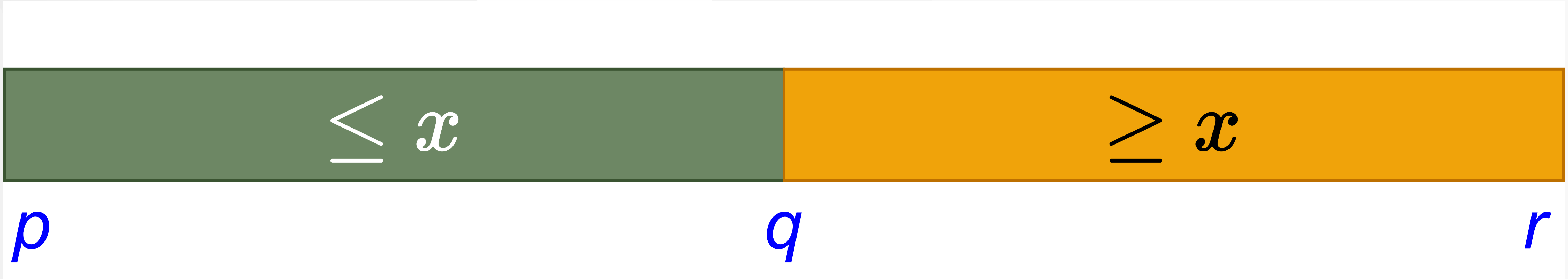
Quicksort

- One of the most-used algorithms in practice
- Proposed by **C.A.R.** *Hoare* in 1962.
- Divide-and-conquer algorithm
- In-place algorithm
  - The additional space needed is O(1)
  - The sorted array is returned in the input array
  - *Reminder: Insertion-sort is also an in-place algorithm, but Merge-Sort is not in-place.*
- Very practical

Quicksort

- **Divide:** Partition the array into 2 subarrays such that elements in the lower part $\leq$ elements in the higher part
- **Conquer:** Recursively sort 2 subarrays
- **Combine:** Trivial (because in-place)

Key: Linear-time $(\Theta(n))$ partitioning algorithm



References

TODO