

# CE100 Algorithms and Programming II

## Week-5 (Dynamic Programming)

Spring Semester, 2021-2022

Download [DOC](#), [SLIDE](#), [PPTX](#)

<iframe width=700, height=500 frameBorder=0 src="../ce100-week-5-dp.md\_slide.html"> </iframe>

# Quicksort Sort

## Outline

- Convex Hull (Divide & Conquer)
- Dynamic Programming
  - Introduction
  - Divide-and-Conquer (DAC) vs Dynamic Programming (DP)

- Fibonacci Numbers
  - Recursive Solution
  - Bottom-Up Solution
- Optimization Problems
- Development of a DP Algorithms

- Matrix-Chain Multiplication
  - Matrix Multiplication and Row Columns Definitions
  - Cost of Multiplication Operations (pxqxr)
  - Counting the Number of Parenthesizations

- The Structure of Optimal Parenthesization
  - Characterize the structure of an optimal solution
  - A Recursive Solution
    - Direct Recursion Inefficiency.
  - Computing the optimal Cost of Matrix-Chain Multiplication
  - Bottom-up Computation

- Algorithm for Computing the Optimal Costs
  - MATRIX-CHAIN-ORDER
- Construction and Optimal Solution
  - MATRIX-CHAIN-MULTIPLY
- Summary

# Dynamic Programming - Introduction

- An algorithm design paradigm like divide-and-conquer
- **Programming:** A tabular method (not writing computer code)
  - Older sense of planning or scheduling, typically by filling in a table
- **Divide-and-Conquer (DAC):** subproblems are independent
- **Dynamic Programming (DP):** subproblems are not independent
- **Overlapping subproblems:** subproblems share sub-subproblems
  - In solving problems with overlapping subproblems
    - A DAC algorithm **does redundant** work
      - Repeatedly solves common subproblems
    - A DP algorithm solves each problem just once
      - **Saves its result in a table**

# Problem 1: Fibonacci Numbers

## Recursive Solution

- Reminder:

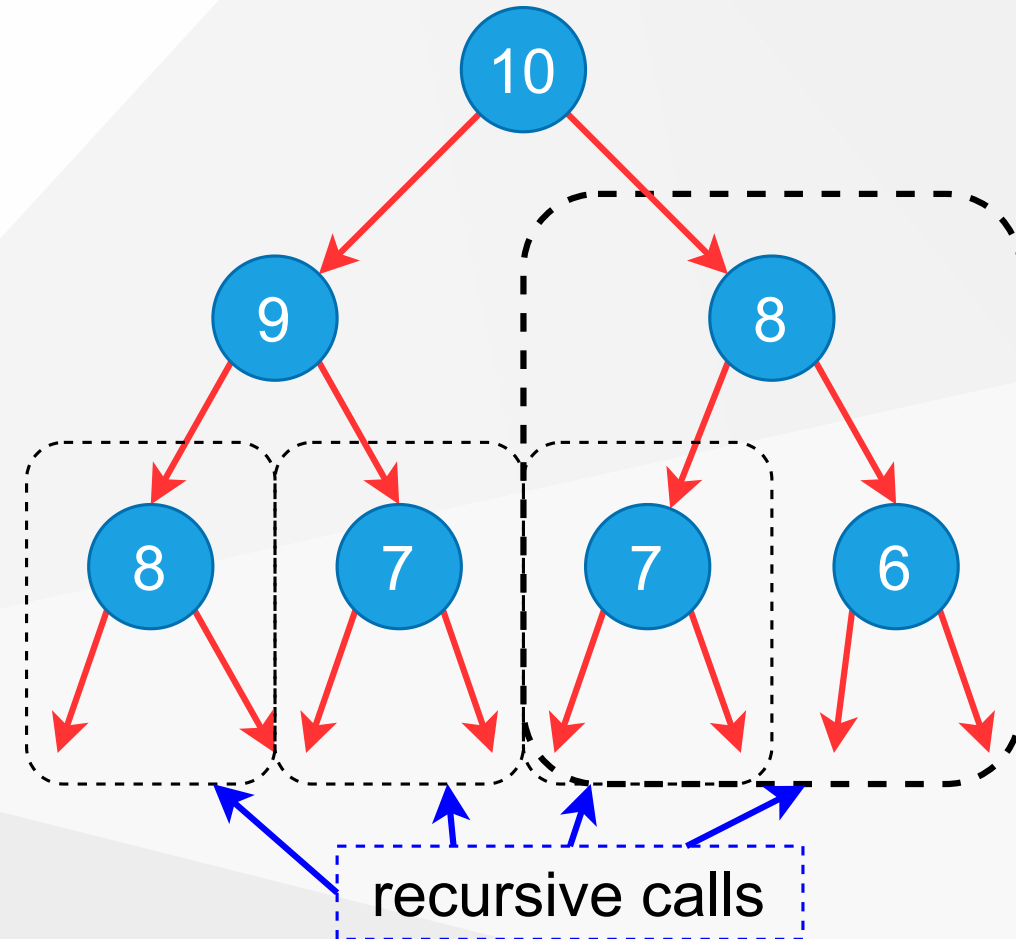
$$F(0) = 0 \text{ and } F(1) = 1$$

$$F(n) = F(n - 1) + F(n - 2)$$

```

REC-FIBO( $n$ ){
  if  $n < 2$ 
    return  $n$ 
  else
    return REC-FIBO( $n - 1$ ) + REC-FIBO( $n - 2$ ) }
  
```

- Overlapping subproblems in different recursive calls. Repeated work!





## Problem 1: Fibonacci Numbers Recursive Solution

- Recurrence:
  - *exponential runtime*

$$T(n) = T(n - 1) + T(n - 2) + 1$$

- Recursive algorithm inefficient because it recomputes the same  $F(i)$  repeatedly in different branches of the recursion tree.

# Problem 1: Fibonacci Numbers

## Bottom-up Computation

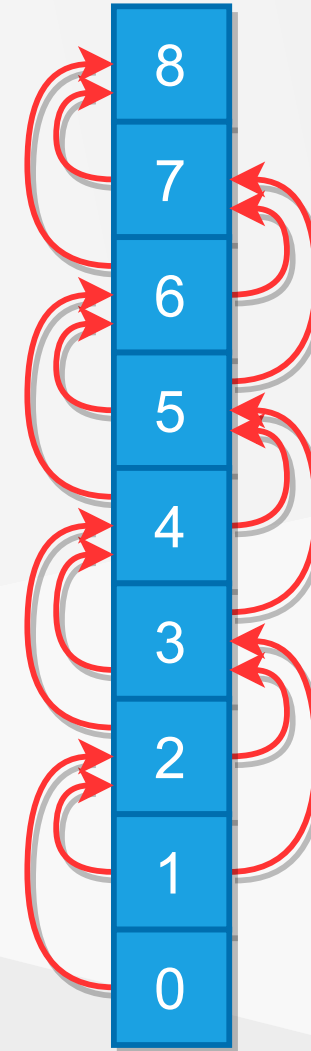
- Reminder:

$$F(0) = 0 \text{ and } F(1) = 1$$

$$F(n) = F(n - 1) + F(n - 2)$$

- Runtime  $\Theta(n)$

```
ITER-FIBO(n)
  F[0] = 0
  F[1] = 1
  for i = 2 to n do
    F[i] = F[i-1] + F[i-2]
  return F[n]
```



# Optimization Problems

- **DP** typically applied to optimization problems
- In an optimization problem
  - There are many possible solutions (feasible solutions)
  - Each solution has a value
  - Want to find an optimal solution to the problem
    - *A solution with the optimal value (min or max value)*
  - Wrong to say **the** optimal solution to the problem
    - *There may be several solutions with the same optimal value*

## Development of a DP Algorithm

**Step-1.** Characterize the structure of an optimal solution

**Step-2.** Recursively define the value of an optimal solution

**Step-3.** Compute the value of an optimal solution in a bottom-up fashion

**Step-4.** Construct an optimal solution from the information computed in **Step 3**

## Problem 2: Matrix Chain Multiplication

- **Input:** a sequence (chain)  $\langle A_1, A_2, \dots, A_n \rangle$  of  $n$  matrices
- **Aim:** compute the product  $A_1 \cdot A_2 \cdot \dots \cdot A_n$
- A product of matrices is fully parenthesized if
  - It is either a single matrix
  - Or, the product of two fully parenthesized matrix products surrounded by a pair of parentheses.

$$\left( \begin{array}{l} A_i(A_{i+1}A_{i+2} \dots A_j) \\ (A_iA_{i+1}A_{i+2} \dots A_{j-1})A_j \\ (A_iA_{i+1}A_{i+2} \dots A_k)(A_{k+1}A_{k+2} \dots A_j) \end{array} \right) \text{ for } i \leq k < j$$

- All parenthesizations yield the same product; matrix product is associative

# Matrix-chain Multiplication: An Example Parenthesization

- Input:  $\langle A_1, A_2, A_3, A_4 \rangle$  (5 distinct ways of full parenthesization)

$$\left( A_1 \left( A_2 (A_3 A_4) \right) \right)$$

$$\left( A_1 \left( (A_2 A_3) A_4 \right) \right)$$

$$\left( (A_1 A_2) (A_3 A_4) \right)$$

$$\left( \left( A_1 (A_2 A_3) A_4 \right) \right)$$

$$\left( \left( (A_1 A_2) A_3 \right) A_4 \right)$$

- The way we parenthesize a chain of matrices can have a dramatic effect on the cost of computing the product

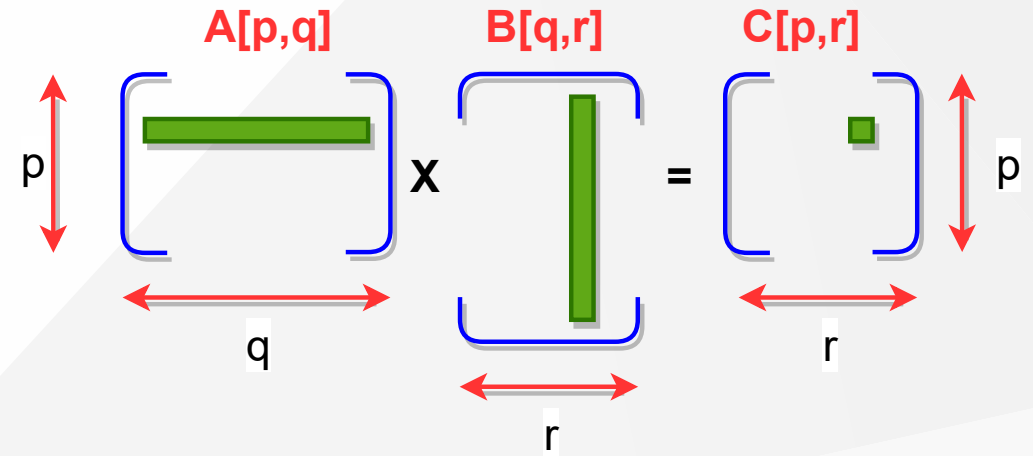
# Matrix-chain Multiplication:

## Reminder

```

MATRIX-MULTIPLY(A, B)
  if cols[A] != rows[B] then
    error("incompatible dimensions")
  for i=1 to rows[A] do
    for j=1 to cols[B] do
      C[i,j]=0
      for k=1 to cols[A] do
        C[i,j]=C[i,j]+A[i,k]·B[k,j]
      return C

```



$\text{rows}(A) = p$      $\text{rows}(B) = q$      $\text{rows}(C) = p$   
 $\text{cols}(A) = q$      $\text{cols}(B) = r$      $\text{cols}(C) = r$

*Note : matrix[row,column]*

A:  $p \times q$

B:  $q \times r$

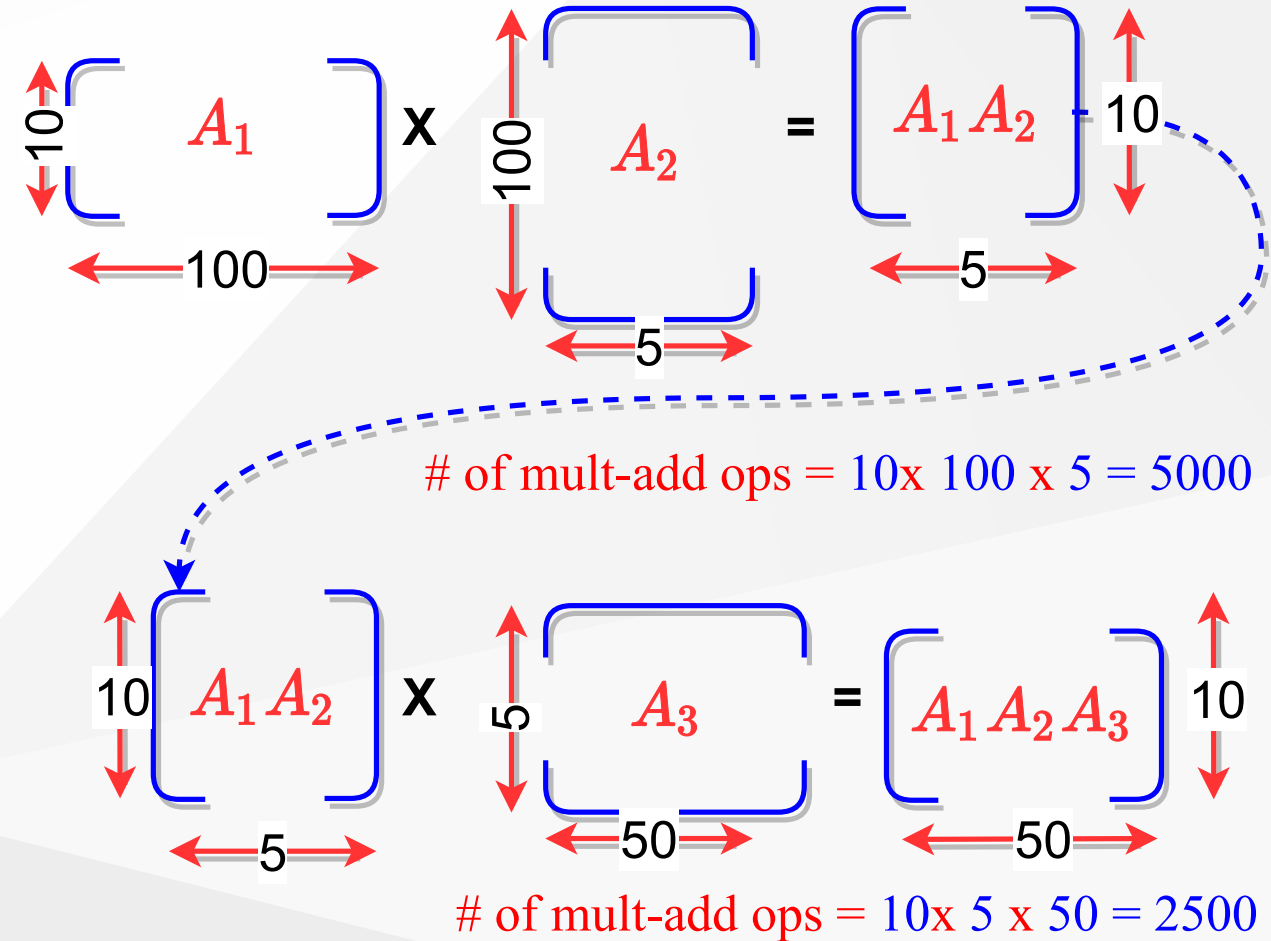
C:  $p \times r$

# of mult-add ops =  $\text{rows}[A] \times \text{cols}[B] \times \text{cols}[A]$

# of mult-add ops =  $p \times q \times r$

## Matrix Chain Multiplication: Example

- $A_1 : 10 \times 100$ ,  $A_2 : 100 \times 5$ ,  
 $A_3 : 5 \times 50$ 
  - Which paranthesization is better?  $(A_1 A_2) A_3$  or  $A_1 (A_2 A_3)$ ?

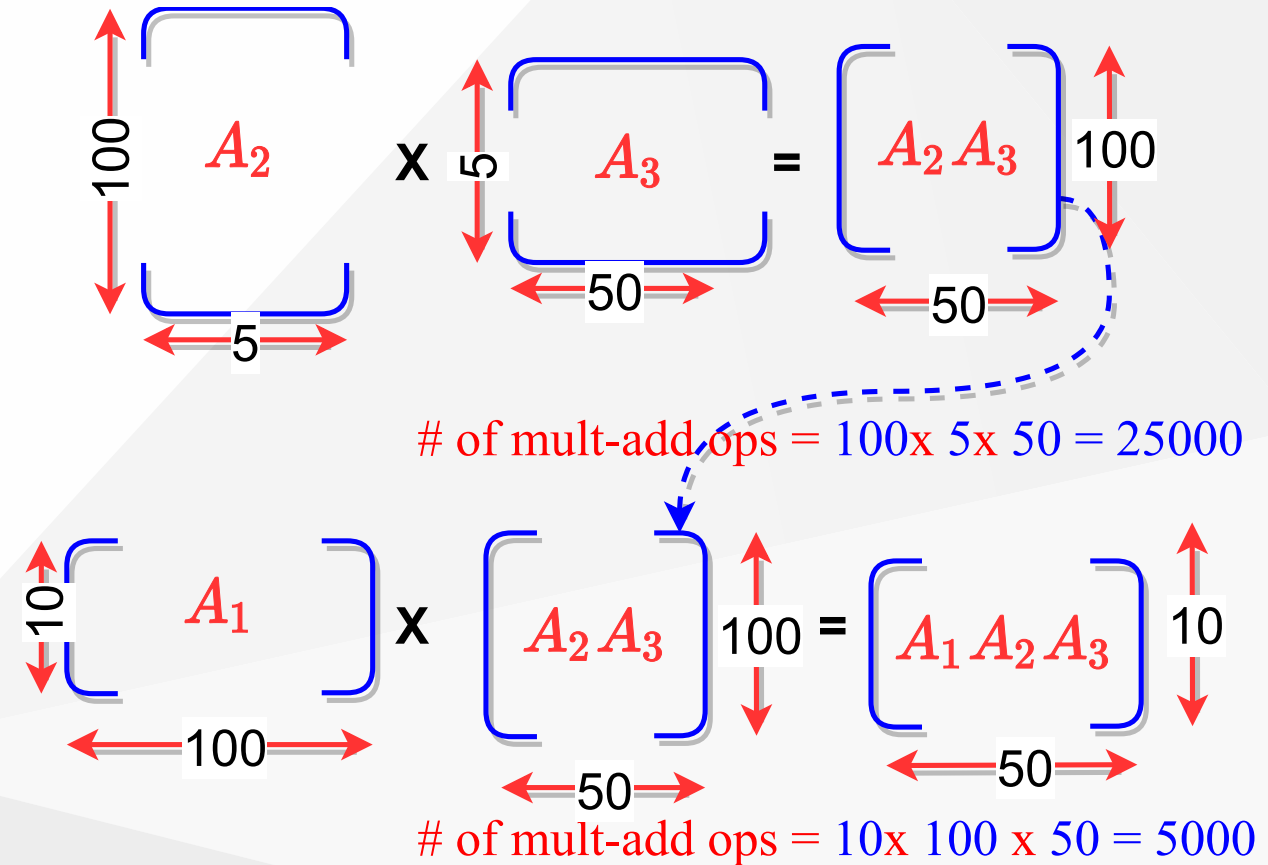


# of mult-add ops =  $5000 + 2500 = 7500$



## Matrix Chain Multiplication: Example

- $A_1 : 10 \times 100$ ,  $A_2 : 100 \times 5$ ,  
 $A_3 : 5 \times 50$ 
  - Which paranthesization is better?  $(A_1 A_2) A_3$  or  $A_1 (A_2 A_3)$ ?



# of mult-add ops =  $25000 + 5000 = 75000$

## Matrix Chain Multiplication: Example

- $A1 : 10 \times 100, A2 : 100 \times 5, A3 : 5 \times 50$ 
  - Which paranthesization is better?  $(A1A2)A3$  or  $A1(A2A3)$ ?

In summary:

- $(A1A2)A3 = \#$  of multiply-add ops: 7500
- $A1(A2A3) = \#$  of multiple-add ops: 75000

First parenthesization yields **10x faster** computation

## Matrix-chain Multiplication Problem

- **Input:** A chain  $\langle A_1, A_2, \dots, A_n \rangle$  of  $n$  matrices,
  - where  $A_i$  is a  $p_{i-1} \times p_i$  matrix
- **Objective:** Fully parenthesize the product
  - $A_1 \cdot A_2 \dots A_n$ 
    - such that the number of **scalar mult-adds** is minimized.

## Counting the Number of Parenthesizations

- **Brute force approach:** exhaustively check all parenthesizations
- $P(n)$ : # of parenthesizations of a sequence of  $n$  matrices
- We can split sequence between  $k^{th}$  and  $(k + 1)^{st}$  matrices for any  $k = 1, 2, \dots, n - 1$ , then parenthesize the two resulting sequences independently, i.e.,

$$(A_1 A_2 A_3 \dots A_k \quad \overbrace{\hspace{1cm}}^{\text{break-point}}) (A_{k+1} A_{k+2} \dots A_n)$$

- We obtain the recurrence

$$P(1) = 1 \text{ and } P(n) = \sum_{k=1}^{n-1} P(k)P(n - k)$$

## Number of Parenthesizations:

- $P(1) = 1$  and  $P(n) = \sum_{k=1}^{n-1} P(k)P(n-k)$
- The recurrence generates the sequence of **Catalan Numbers** Solution is  $P(n) = C(n-1)$  where

$$C(n) = \frac{1}{n+1} \binom{2n}{n} = \Omega(4^n / n^{3/2})$$

- The number of solutions is **exponential** in  $n$
- Therefore, brute force approach is a poor strategy

# The Structure of Optimal Parenthesization

- **Notation:**  $A_{i..j}$ : The matrix that results from evaluation of the product:  
 $A_i A_{i+1} A_{i+2} \dots A_j$
- **Observation:** Consider the last multiplication operation in any parenthesization:  
 $(A_1 A_2 \dots A_k) \cdot (A_{k+1} A_{k+2} \dots A_n)$ 
  - There is a  $k$  value ( $1 \leq k < n$ ) such that:
    - First, the product  $A_1 \dots k$  is computed
    - Then, the product  $A_{k+1..n}$  is computed
    - Finally, the matrices  $A_{1..k}$  and  $A_{k+1..n}$  are multiplied

## Step 1: Characterize the Structure of an Optimal Solution

- An optimal parenthesization of product  $A_1 A_2 \dots A_n$  will be:  
 $(A_1 A_2 \dots A_k) \cdot (A_{k+1} A_{k+2} \dots A_n)$  for some  $k$  value
- The cost of this optimal parenthesization will be:
  - $=$  Cost of computing  $A_{1\dots k}$
  - $+$  Cost of computing  $A_{k+1\dots n}$
  - $+$  Cost of multiplying  $A_{1\dots k} \cdot A_{k+1\dots n}$

## Step 1: Characterize the Structure of an Optimal Solution

- **Key observation:** Given optimal parenthesization
  - $(A_1 A_2 A_3 \dots A_k) \cdot (A_{k+1} A_{k+2} \dots A_n)$
- Parenthesization of the subchain  $A_1 A_2 A_3 \dots A_k$
- Parenthesization of the subchain  $A_{k+1} A_{k+2} \dots A_n$

should both be optimal

- Thus, optimal solution to an instance of the problem contains optimal solutions to subproblem instances
  - **i.e.**, optimal substructure within an optimal solution exists.



## Step 2: A Recursive Solution

- **Step 2:** Define the value of an optimal solution recursively in terms of optimal solutions to the subproblems
- Assume we are trying to determine the min cost of computing  $A_{i..j}$
- $m_{i,j}$ : min # of scalar multiply-add opns needed to compute  $A_{i..j}$ 
  - **Note:** *The optimal cost of the original problem:  $m_{1,n}$*
- How to compute  $m_{i,j}$  recursively?

## Step 2: A Recursive Solution

- Base case:  $m_{i,i} = 0$  (single matrix, no multiplication)
- Let the size of matrix  $A_i$  be  $(p_{i-1} \times p_i)$
- Consider an optimal parenthesization of chain
  - $A_i \dots A_j : (A_i \dots A_k) \cdot (A_{k+1} \dots A_j)$
- The optimal cost:  $m_{i,j} = m_{i,k} + m_{k+1,j} + p_{i-1} \times p_k \times p_j$
- where:
  - $m_{i,k}$ : Optimal cost of computing  $A_{i..k}$
  - $m_{k+1,j}$ : Optimal cost of computing  $A_{k+1..j}$
  - $p_{i-1} \times p_k \times p_j$ : Cost of multiplying  $A_{i..k}$  and  $A_{k+1..j}$

## Step 2: A Recursive Solution

- In an optimal parenthesization:  $k$  must be chosen to minimize  $m_{ij}$
- The recursive formulation for  $m_{ij}$ :

$$m_{ij} = \begin{cases} 0 & \text{if } i = j \\ \underset{i \leq k < j}{MIN} \{m_{ik} + m_{k+1,j} + p_{i-1}p_kp_j\} & \text{if } i < j \end{cases}$$

## Step 2: A Recursive Solution

- The  $m_{ij}$  values give the **costs of optimal solutions** to subproblems
- In order to keep track of how to construct an optimal solution
  - Define  $s_{ij}$  to be the value of  $k$  which yields the optimal split of the subchain  $A_{i..j}$ 
    - That is,  $s_{ij} = k$  such that
      - $m_{ij} = m_{ik} + m_{k+1,j} + p_{i-1}p_kp_j$  holds

## Direct Recursion: Inefficient!

- Recursive Matrix-Chain (RMC) Order

```
RMC(p,i,j)

  if (i == j) then
    return 0

  m[i, j] = INF

  for k=i to j-1 do

    q = RMC(p, i, k) + RMC(p, k+1, j) + p_{i-1} p_k p_j

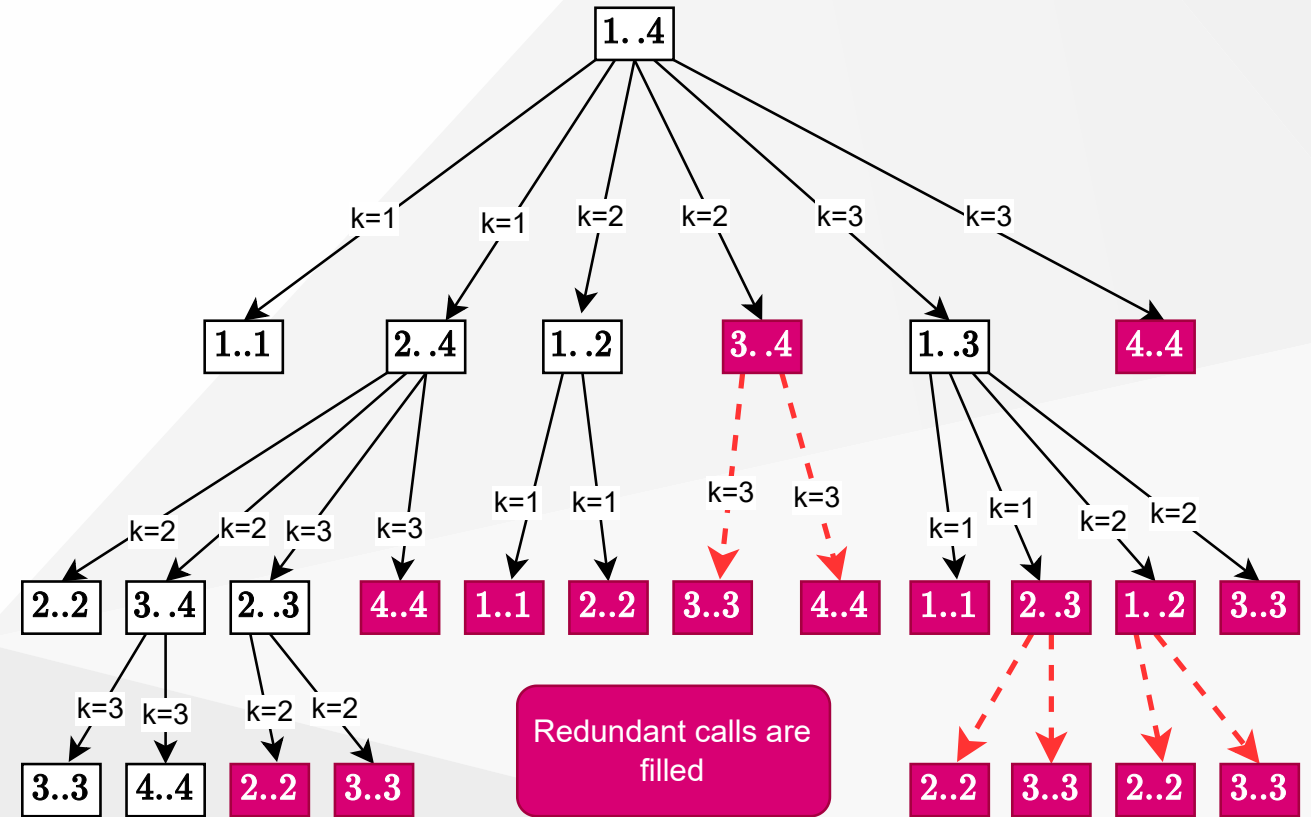
    if q < m[i, j] then
      m[i, j] = q

  endfor

  return m[i, j]
```

## Direct Recursion: Inefficient!

- Recursion tree for  $RMC(p, 1, 4)$
- Nodes are labeled with  $i$  and  $j$  values



# Computing the Optimal Cost (Matrix-Chain Multiplication)

An important observation:

- We have **relatively few subproblems**
  - one problem for each choice of  $i$  and  $j$  satisfying  $1 \leq i \leq j \leq n$
  - total  $n + (n - 1) + \dots + 2 + 1 = \frac{1}{2}n(n + 1) = \Theta(n^2)$  subproblems
- We can write a **recursive** algorithm based on recurrence.
- However, a recursive algorithm may encounter each subproblem many times in different branches of the recursion tree
- This property, **overlapping subproblems**, is the **second important feature** for applicability of **dynamic programming**

## Computing the Optimal Cost (Matrix-Chain Multiplication)

- Compute the value of an optimal solution in a **bottom-up** fashion
  - matrix  $A_i$  has dimensions  $p_{i-1} \times p_i$  for  $i = 1, 2, \dots, n$
  - the input is a sequence  $\langle p_0, p_1, \dots, p_n \rangle$  where  $length[p] = n + 1$
- Procedure uses the following auxiliary tables:
  - $m[1 \dots n, 1 \dots n]$ : for storing the  $m[i, j]$  costs
  - $s[1 \dots n, 1 \dots n]$ : records which index of  $k$  achieved the optimal cost in computing  $m[i, j]$



## Bottom-Up Computation

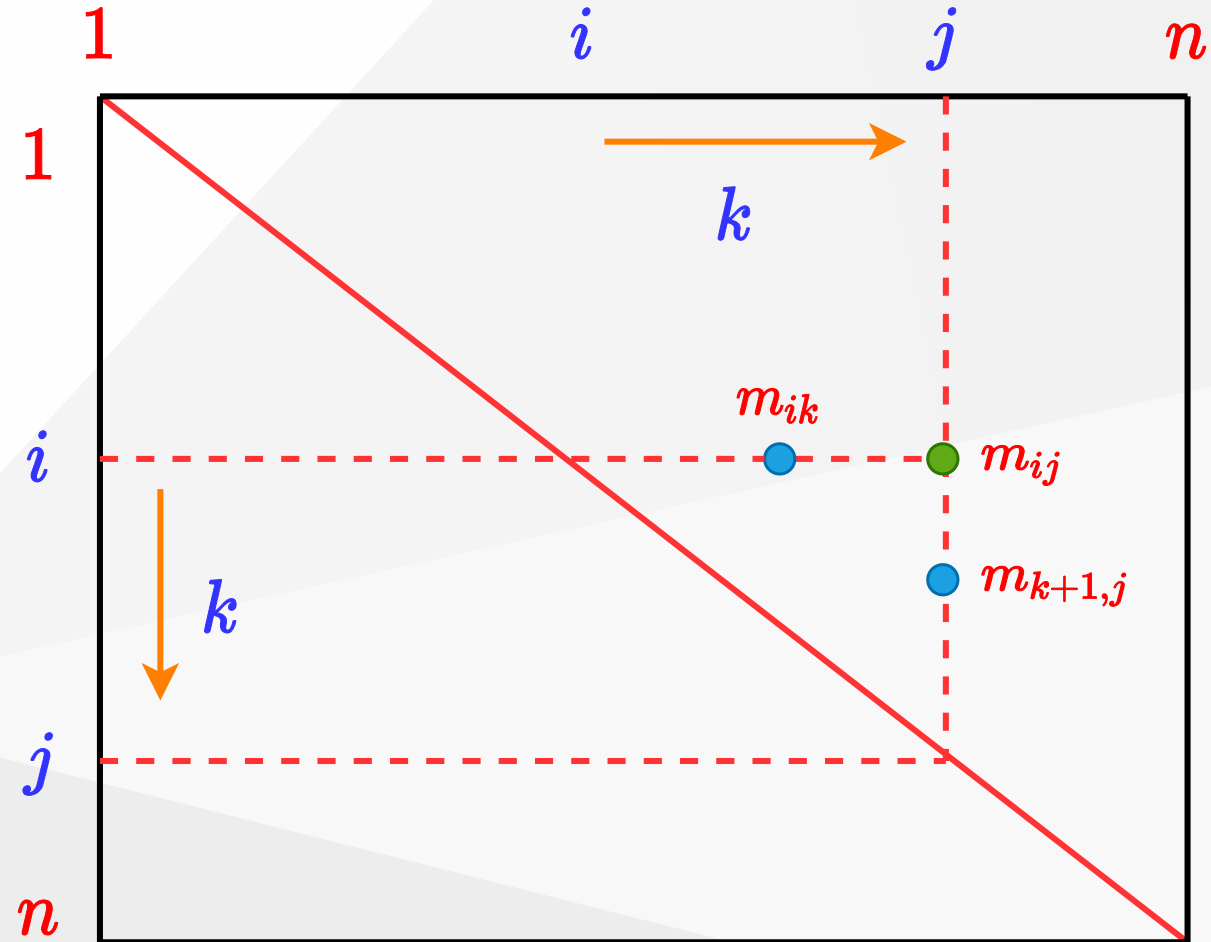
- How to choose the order in which we process  $m_{ij}$  values?
- Before computing  $m_{ij}$ , we have to make sure that the values for  $m_{ik}$  and  $m_{k+1,j}$  have been computed for all  $k$ .

$$m_{ij} = \underset{i \leq k < j}{MIN} \{m_{ik} + m_{k+1,j} + p_{i-1}p_kp_j\}$$

## Bottom-Up Computation

$$m_{ij} = \underset{i \leq k < j}{\text{MIN}} \{m_{ik} + m_{k+1,j} + p_{i-1}p_kp_j\}$$

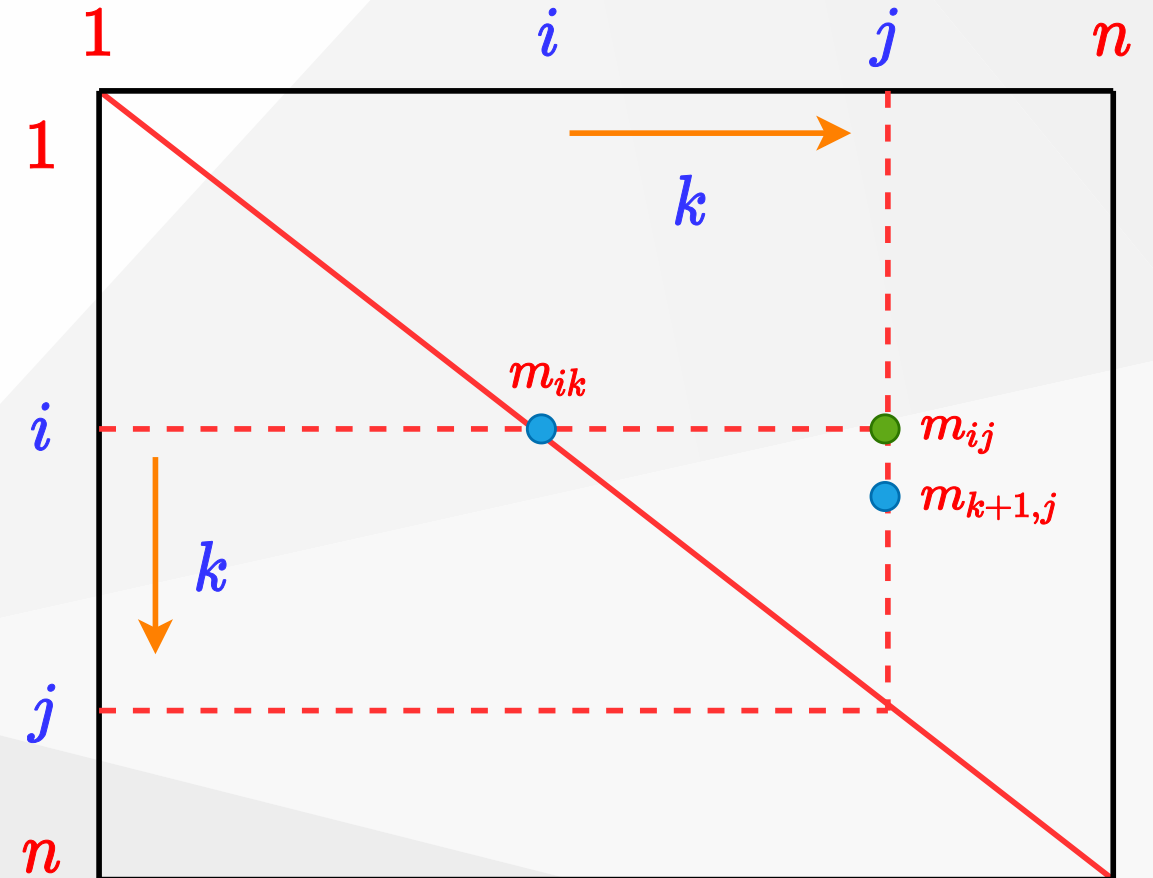
- $m_{ij}$  must be processed after  $m_{ik}$  and  $m_{j,k+1}$
- **Reminder:**  $m_{ij}$  computed only for  $j > i$



## Bottom-Up Computation

$$m_{ij} = \underset{i \leq k < j}{MIN} \{m_{ik} + m_{k+1,j} + p_{i-1}p_kp_j\}$$

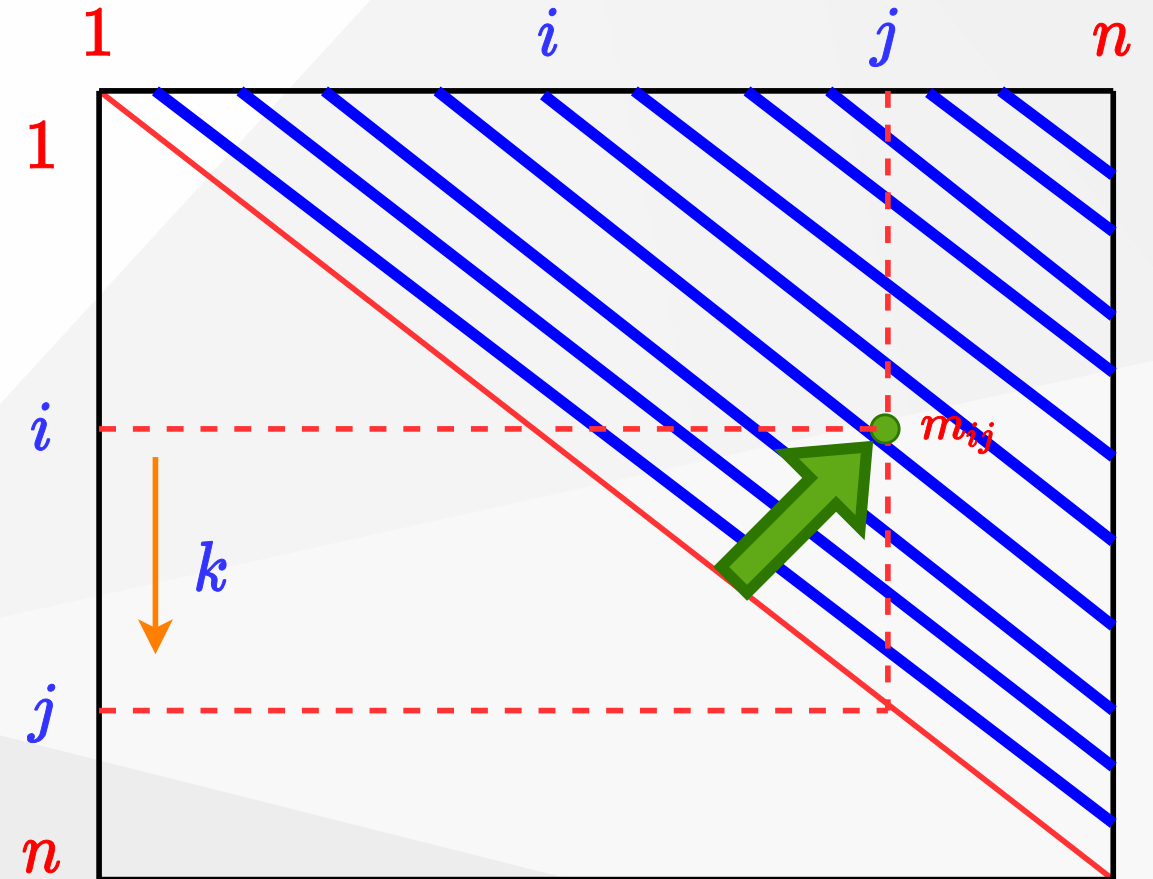
- $m_{ij}$  must be processed after  $m_{ik}$  and  $m_{j,k+1}$
- How to set up the iterations over  $i$  and  $j$  to compute  $m_{ij}$ ?



## Bottom-Up Computation

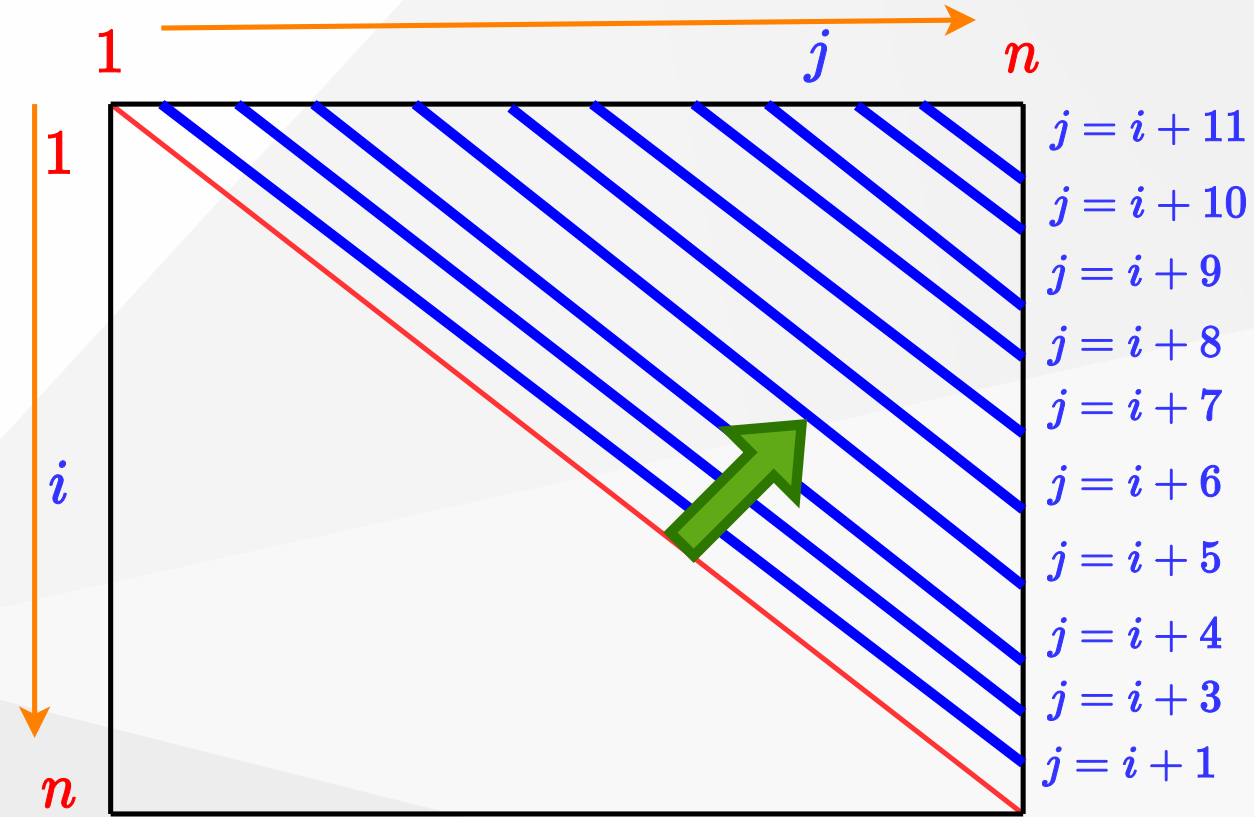
$$m_{ij} = \underset{i \leq k < j}{\text{MIN}} \{m_{ik} + m_{k+1,j} + p_{i-1}p_kp_j\}$$

- If the entries  $m_{ij}$  are computed in the shown order, then  $m_{ik}$  and  $m_{k+1,j}$  values are guaranteed to be computed before  $m_{ij}$ .



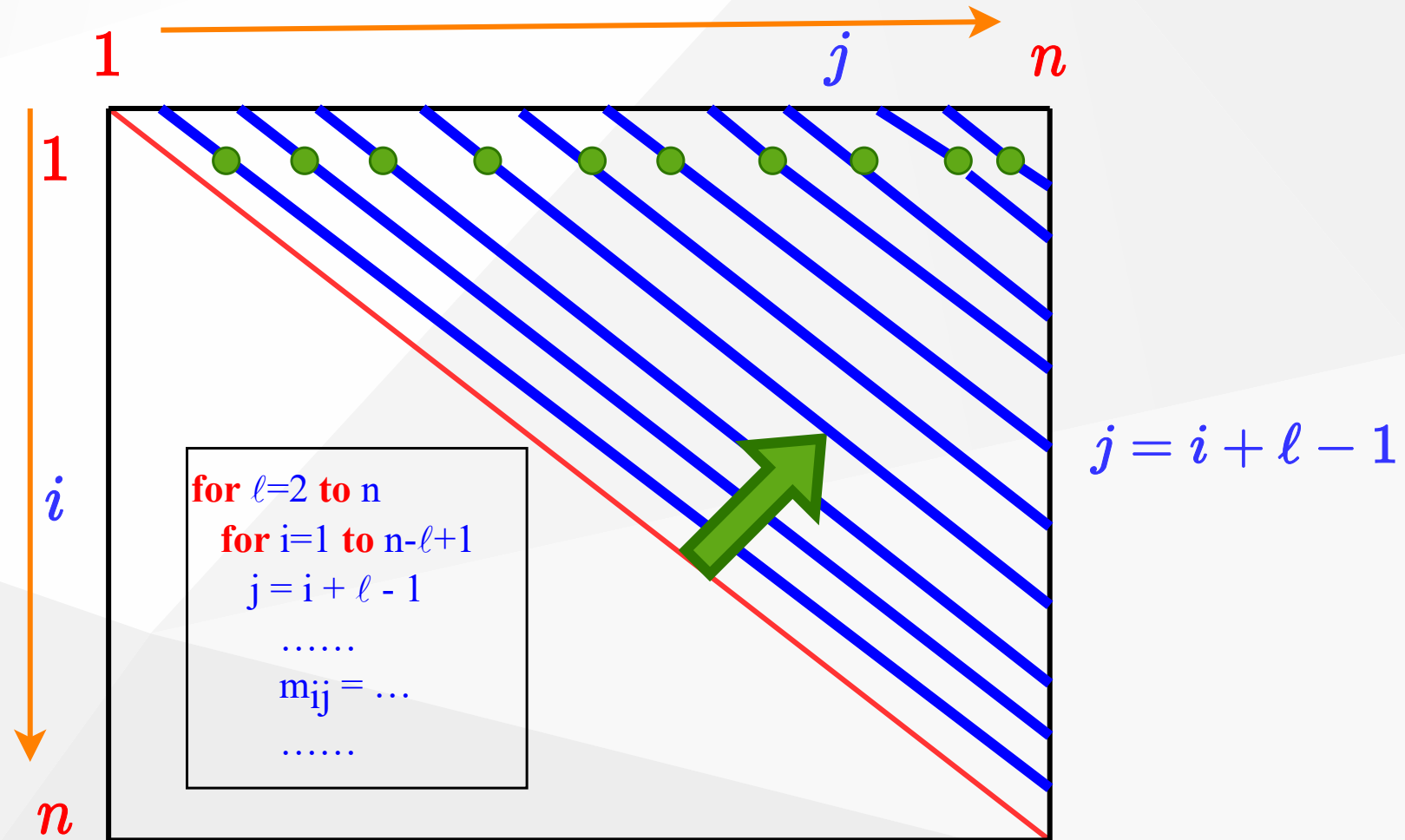
## Bottom-Up Computation

$$m_{ij} = \underset{i \leq k < j}{MIN} \{m_{ik} + m_{k+1,j} + p_{i-1}p_kp_j\}$$



## Bottom-Up Computation

$$m_{ij} = \underset{i \leq k < j}{\text{MIN}} \{m_{ik} + m_{k+1,j} + p_{i-1}p_kp_j\}$$



# Algorithm for Computing the Optimal Costs

- Note:  $l = \ell$  and  $p_{\{i-1\}} p_k p_j = p_{i-1} p_k p_j$

```

MATRIX-CHAIN-ORDER(p)
  n = length[p]-1
  for i=1 to n do
    m[i, i]=0
  endfor
  for l=2 to n do
    for i=1 to n-l+1 do
      j=i+l-1
      m[i, j]=INF
      for k=i to j-1 do
        q=m[i,k]+m[k+1, j]+p_{i-1} p_k p_j
        if q < m[i,j] then
          m[i,j]=q
          s[i,j]=k
        endif
      endfor
    endfor
  endfor
  return m and s

```

## Algorithm for Computing the Optimal Costs

- The algorithm first computes
  - $m[i, i] \leftarrow 0$  for  $i = 1, 2, \dots, n$  min costs for all chains of length 1
- Then, for  $\ell = 2, 3, \dots, n$  computes
  - $m[i, i + \ell - 1]$  for  $i = 1, \dots, n - \ell + 1$  min costs for all chains of length  $\ell$
- For each value of  $\ell = 2, 3, \dots, n$ ,
  - $m[i, i + \ell - 1]$  depends only on table entries  $m[i, k]$  &  $m[k + 1, i + \ell - 1]$  for  $i \leq k < i + \ell - 1$ , which are already computed



# Algorithm for Computing the Optimal Costs

$$\begin{array}{l}
 \underbrace{\{m[1, 2], m[2, 3], \dots, m[n-1, n]\}}_{(n-1) \text{ values}} \left\{ \begin{array}{l} \ell = 2 \\ \text{for } i = 1 \text{ to } n-1 \text{ do} \\ \quad m[i, i+1] = \infty \quad (1) \\ \quad \text{for } k = i \text{ to } i \text{ do} \\ \quad \quad \vdots \end{array} \right. \\
 \\
 \underbrace{\{m[1, 3], m[2, 4], \dots, m[n-2, n]\}}_{(n-2) \text{ values}} \left\{ \begin{array}{l} \ell = 3 \\ \text{for } i = 1 \text{ to } n-2 \text{ do} \\ \quad m[i, i+2] = \infty \quad (1) \\ \quad \text{for } k = i \text{ to } i+1 \text{ do} \\ \quad \quad \vdots \end{array} \right. \\
 \\
 \underbrace{\{m[1, 4], m[2, 5], \dots, m[n-3, n]\}}_{(n-3) \text{ values}} \left\{ \begin{array}{l} \ell = 4 \\ \text{for } i = 1 \text{ to } n-3 \text{ do} \\ \quad m[i, i+3] = \infty \quad (1) \\ \quad \text{for } k = i \text{ to } i+2 \text{ do} \\ \quad \quad \vdots \end{array} \right.
 \end{array}$$

## Table access pattern in computing $m[i, j]$ s for

$$\ell = j - i + 1$$

for  $k \leftarrow i$  to  $j - 1$  do

$$q \leftarrow m[i, k] + m[k + 1, j] + p_{i-1}p_kp_j$$

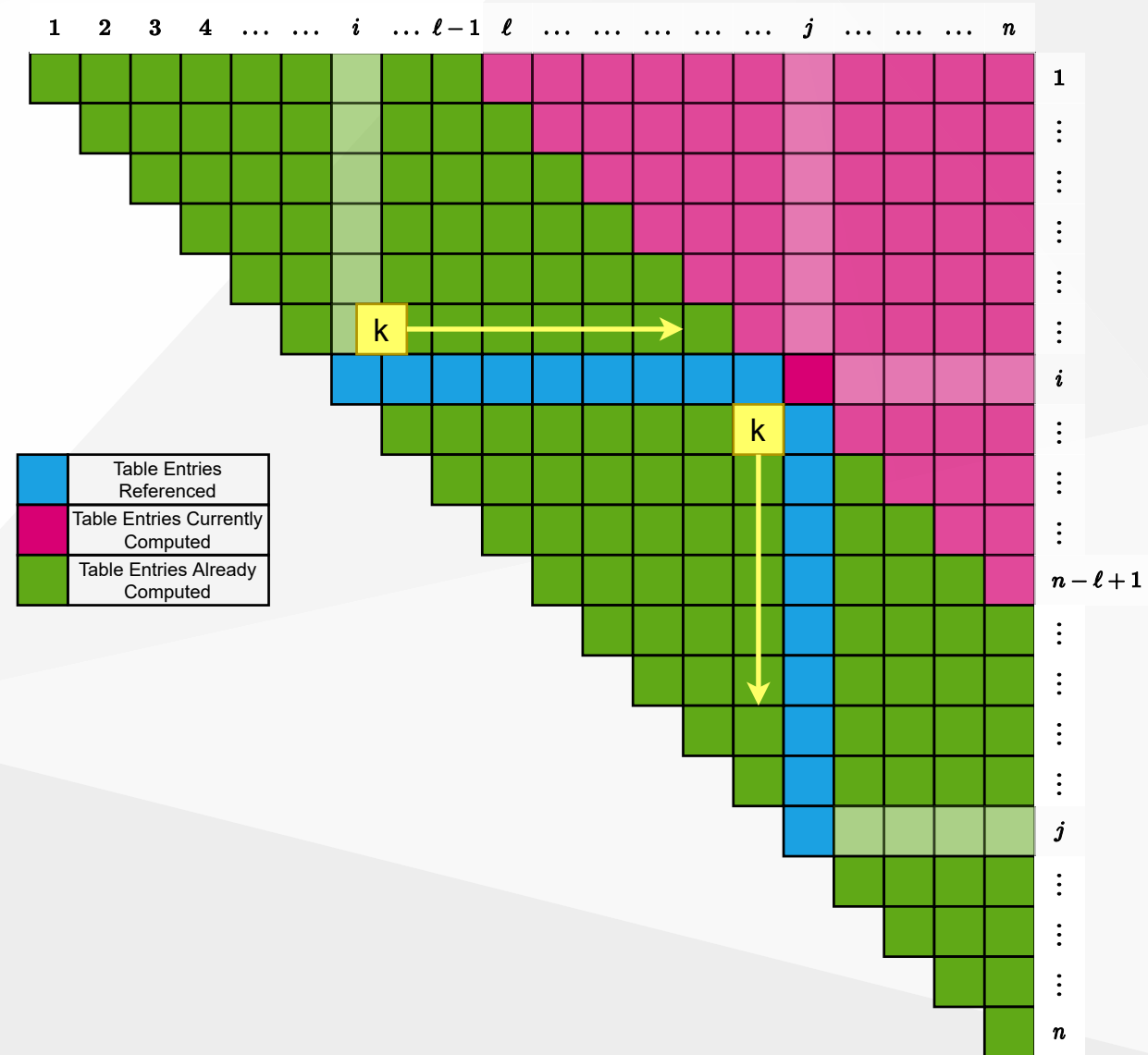
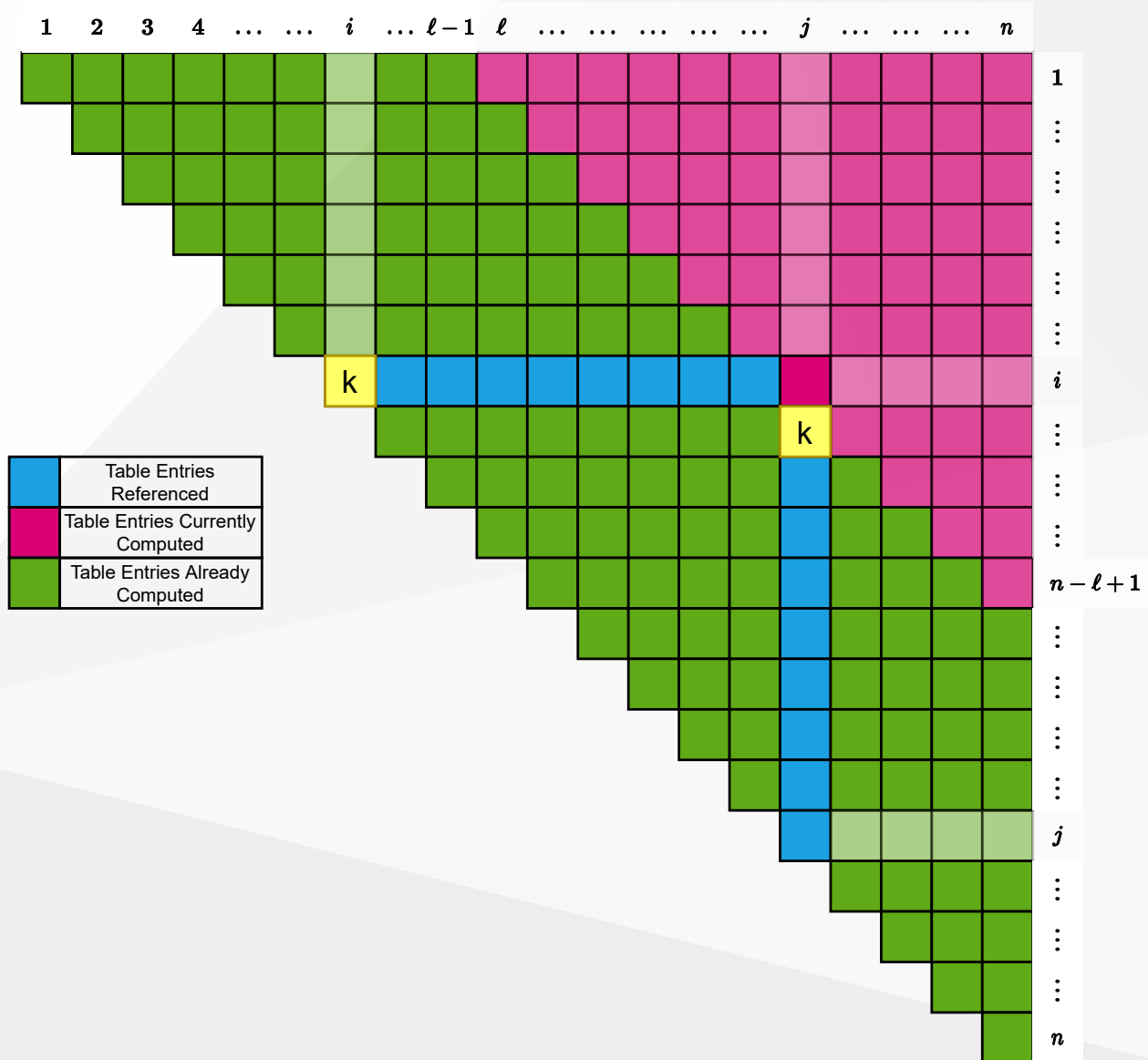


Table access pattern in  
computing  $m[i, j]$ s for  
 $\ell = j - i + 1$

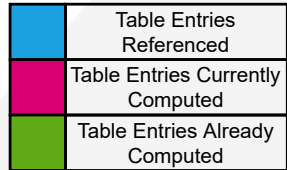
*mult.*

$$\left( (A_i) \vdots (A_{i+1} A_{i+2} \dots A_j) \right)$$



*mult.*

$$\left( \begin{array}{ccc} (A_i A_{i+1}) & \vdots & (A_{i+2} \dots A_j) \end{array} \right)$$

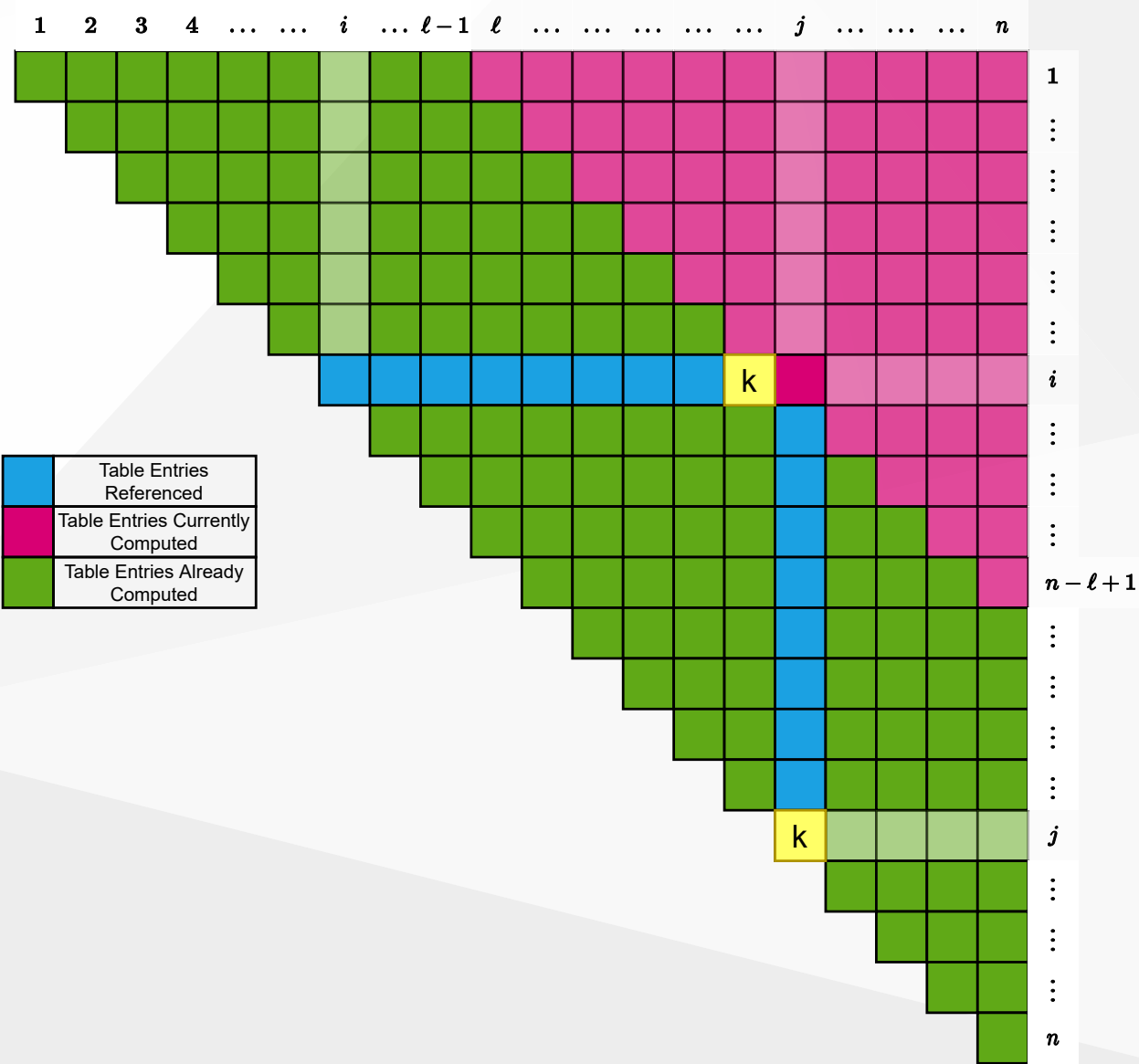


$$\begin{array}{c} \text{mult.} \\ \left( \begin{array}{ccc} (A_i A_{i+1} A_{i+2}) & \vdots & (A_{i+3} \dots A_j) \end{array} \right) \end{array}$$


Table access pattern in computing  $m[i, j]$ s for  $\ell = j - i + 1$

*mult.*

$$\left( (A_i A_{i+1} \dots A_{j-1}) \vdots (A_j) \right)$$



## Table access pattern Example

- Compute  $m_{25}$
- Choose the  $k$  value that leads to min cost

$$m_{ij} = \underset{i \leq k < j}{MIN} \{m_{ik} + m_{k+1,j} + p_{i-1}p_kp_j\}$$

$$\begin{array}{ll}
 A_1 : (30 \times 35) & \\
 A_2 : (35 \times 15) & \underbrace{(k=2)} \\
 A_3 : (15 \times 5) & ((A_2) \vdots (A_3 A_4 A_5)) \\
 A_4 : (5 \times 10) & \\
 A_5 : (10 \times 20) & \text{cost} = m_{22} + m_{35} + p_1 p_2 p_5 \\
 A_6 : (20 \times 25) & = 0 + 2500 + 35 \times 15 \times 20 \\
 & = 13000
 \end{array}$$

	1	2	3	4	5	6	
1	0	15750	7875	9375			1
2		0	2625	4375	???		2
3			0	750	2500		3
4				0	1000	3500	4
5					0	5000	5
6						0	6

## Table access pattern Example

- Compute  $m_{25}$
- Choose the  $k$  value that leads to min cost

$$m_{ij} = \underset{i \leq k < j}{MIN} \{m_{ik} + m_{k+1,j} + p_{i-1}p_kp_j\}$$

$A_1 : (30 \times 35)$

$A_2 : (35 \times 15)$

$A_3 : (15 \times 5)$

$A_4 : (5 \times 10)$

$A_5 : (10 \times 20)$

$A_6 : (20 \times 25)$

$$\overbrace{((A_2 A_3) \vdots (A_4 A_5))}^{(k=3)}$$

$$\begin{aligned} \text{cost} &= m_{23} + m_{45} + p_1 p_3 p_5 \\ &= 2625 + 1000 + 35 \times 5 \times 20 \\ &= 7125 \end{aligned}$$

	1	2	3	4	5	6	
1	0	15750	7875	9375			1
2		0	2625	4375	???		2
3			0	750	2500		3
4				0	1000	3500	4
5					0	5000	5
6						0	6



## Table access pattern Example

- Compute  $m_{25}$
- Choose the  $k$  value that leads to min cost

$$m_{ij} = \underset{i \leq k < j}{MIN} \{m_{ik} + m_{k+1,j} + p_{i-1}p_kp_j\}$$

$A_1 : (30 \times 35)$

$A_2 : (35 \times 15)$

$A_3 : (15 \times 5)$

$A_4 : (5 \times 10)$

$A_5 : (10 \times 20)$

$A_6 : (20 \times 25)$

$(k=4)$   
 $\underbrace{\hspace{1cm}}$   
 $((A_2 A_3 A_4) \vdots (A_5))$

$$\begin{aligned}
 \text{cost} &= m_{24} + m_{55} + p_1 p_4 p_5 \\
 &= 4375 + 0 + 35 \times 10 \times 20 \\
 &= 11375
 \end{aligned}$$

	1	2	3	4	5	6	
	0	15750	7875	9375			1
		0	2625	4375	???		2
			0	750	2500		3
				0	1000	3500	4
					0	5000	5
						0	6

## Table access pattern Example

- Compute  $m_{25}$
- Choose the  $k$  value that leads to min cost

$$m_{ij} = \underset{i \leq k < j}{MIN} \{m_{ik} + m_{k+1,j} + p_{i-1}p_kp_j\}$$

$(k=2)$

$$((A_2) \vdots (A_3 A_4 A_5)) \rightarrow m_{22} + m_{35} + p_1 p_2 p_5 = 13000$$

$A_1 : (30 \times 35)$

$A_2 : (35 \times 15)$

$A_3 : (15 \times 5)$

$A_4 : (5 \times 10)$

$A_5 : (10 \times 20)$

$A_6 : (20 \times 25)$

$$((A_2 A_3) \vdots (A_4 A_5)) \rightarrow m_{23} + m_{45} + p_1 p_3 p_5 = \overbrace{7125}^{\text{selected}} \Leftarrow \min$$

$$((A_2 A_3 A_4) \vdots (A_5)) \rightarrow m_{24} + m_{55} + p_1 p_4 p_5 = 11375$$

$$m_{25} = 7125$$

$$s_{25} = 3$$

	1	2	3	4	5	6	
1	0	15750	7875	9375			1
2		0	2625	4375	7125		2
3			0	750	2500		3
4				0	1000	3500	4
5					0	5000	5
6						0	6

# Constructing an Optimal Solution

- **MATRIX-CHAIN-ORDER** determines the optimal # of scalar **mults/adds**
  - needed to compute a matrix-chain product
  - it does not directly show how to multiply the matrices
- That is,
  - it determines the cost of the optimal solution(s)
  - it does not show how to obtain an optimal solution
- Each entry  $s[i, j]$  records the value of  $k$  such that optimal parenthesization of  $A_i \dots A_j$  splits the product between  $A_k$  &  $A_{k+1}$
- We know that the final matrix multiplication in computing  $A_{1..n}$  optimally is  $A_{1..s[1,n]} \times A_{s[1,n]+1,n}$


## Example: Constructing an Optimal Solution

- **Reminder:**  $s_{ij}$  is the optimal top-level split of  $A_i \dots A_j$
- What is the optimal top-level split for:
  - $A_1 A_2 A_3 A_4 A_5 A_6$
  - $s_{16} = 3$

2	3	4	5	6	
1	1	3	3	3	1
	2	3	3	3	2
		3	3	3	3
			4	5	4
				5	5

## Example: Constructing an Optimal Solution

- Reminder:  $s_{ij}$  is the optimal top-level split of  $A_i \dots A_j$

( $k=4$ )  


- $(A_1 A_2 A_3) \vdots (A_4 A_5 A_6)$ 
  - What is the optimal split for  $A_1 \dots A_3$  ? ( $s_{13} = 1$ )
  - What is the optimal split for  $A_4 \dots A_6$  ? ( $s_{46} = 5$ )

	2	3	4	5	6	
1	1	3	3	3	1	
	2	3	3	3	2	
		3	3	3	3	
			4	5	4	
				5	5	

## Example: Constructing an Optimal Solution

- Reminder:  $s_{ij}$  is the optimal top-level split of  $A_i \dots A_j$

$(k=1)$

$(k=5)$

- $\left( (A_1) \vdots (A_2 A_3) \right) \left( (A_4 A_5) \vdots (A_6) \right)$ 
  - What is the optimal split for  $A_1 \dots A_3$  ? ( $s_{13} = 1$ )
  - What is the optimal split for  $A_4 \dots A_6$  ? ( $s_{46} = 5$ )

	2	3	4	5	6	
1	1	3	3	3	1	
	2	3	3	3	2	
		3	3	3	3	
			4	5	4	
				5	5	

## Example: Constructing an Optimal Solution

- **Reminder:**  $s_{ij}$  is the optimal top-level split of  $A_i \dots A_j$
- $\left( (A_1)(A_2 A_3) \right) \left( (A_4 A_5)(A_6) \right)$ 
  - What is the optimal split for  $A_2 A_3$  ? (  $s_{23} = 2$  )
  - What is the optimal split for  $A_4 A_5$  ? (  $s_{45} = 4$  )

	2	3	4	5	6	
	1	1	3	3	3	1
		2	3	3	3	2
			3	3	3	3
				4	5	4
					5	5

## Example: Constructing an Optimal Solution

- **Reminder:**  $s_{ij}$  is the optimal top-level split of  $A_i \dots A_j$

$$\left( \left( A_1 \right) \overbrace{\left( \left( A_2 \right) \vdots \left( A_3 \right) \right)}^{(k=2)} \right) \left( \overbrace{\left( \left( A_4 \right) \vdots \left( A_5 \right) \right)}^{(k=4)} \left( A_6 \right) \right)$$

- What is the optimal split for  $A_2 A_3$  ? (  $s_{23} = 2$  )
- What is the optimal split for  $A_4 A_5$  ? (  $s_{45} = 4$  )

	2	3	4	5	6	
1	1	3	3	3	1	
	2	3	3	3	2	
		3	3	3	3	
			4	5	4	
				5	5	



## Constructing an **Optimal Solution**

- Earlier optimal matrix multiplications can be computed recursively
- **Given:**
  - the chain of matrices  $A = \langle A_1, A_2, \dots, A_n \rangle$   
the  $s$  table computed by MATRIX-CHAIN-ORDER
  - The following recursive procedure computes the **matrix-chain product**  $A_{i \dots j}$

MATRIX-CHAIN-MULTIPLY( $A, s, i, j$ )

if  $j > i$  then

$X \leftarrow \text{MATRIX-CHAIN-MULTIPLY}(A, s, i, s[i, j])$

$Y \leftarrow \text{MATRIX-CHAIN-MULTIPLY}(A, s, s[i, j] + 1, j)$

return MATRIX-MULTIPLY( $X, Y$ )

else

*return*  $A_i$

- **Invocation:** MATRIX-CHAIN-MULTIPLY( $A, s, 1, n$ )

## Example: Recursive Construction of an Optimal Solution

$MCM(1, 6)$

$X \leftarrow MCM(1, 3) = (A_1 A_2 A_3) \dots \rightarrow MCM(1, 3)$

$Y \leftarrow MCM(4, 6) = (A_4 A_5 A_6)$

*return(?)*

$X \leftarrow MCM(1, 1) = (A_1)$

$Y \leftarrow MCM(2, 3) = (A_2 A_3)$

*return(?)*

*return A<sub>1</sub>*

$s[1 \dots 6, 1 \dots 6]$

	2	3	4	5	6	
	1	1	3	3	3	1
		2	3	3	3	2
			3	3	3	3
				4	5	4
					5	5

## Example: Recursive Construction of an Optimal Solution

$MCM(1, 6)$

$X \leftarrow MCM(1, 3) = (A_1(A_2 A_3)) \dots \rightarrow MCM(1, 3)$

$Y \leftarrow MCM(4, 6) = (A_4 A_5 A_6)$

$return(?)$

$X \leftarrow MCM(1, 1) = (A_1) \dots \rightarrow return A_1$

$Y \leftarrow MCM(2, 3) = (A_2 A_3) \dots \rightarrow MCM(2, 3)$

$Y \leftarrow MCM(2, 3) = (A_2 A_3) \dots \rightarrow MCM(2, 3)$

$X \leftarrow MCM(2, 2) = A_2 \dots \rightarrow return(A_2)$

$Y \leftarrow MCM(3, 3) = A_3 \dots \rightarrow return(A_3)$

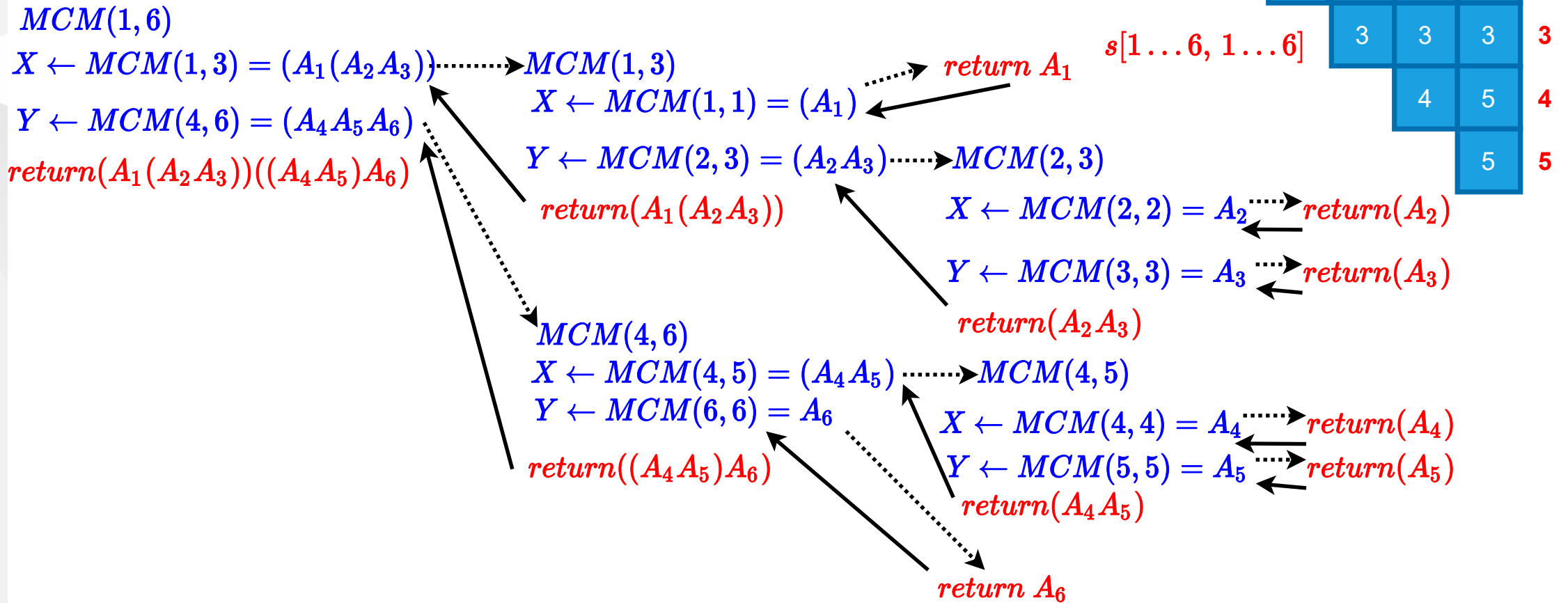
$return(A_2 A_3)$

$return(A_1(A_2 A_3))$

$s[1 \dots 6, 1 \dots 6]$

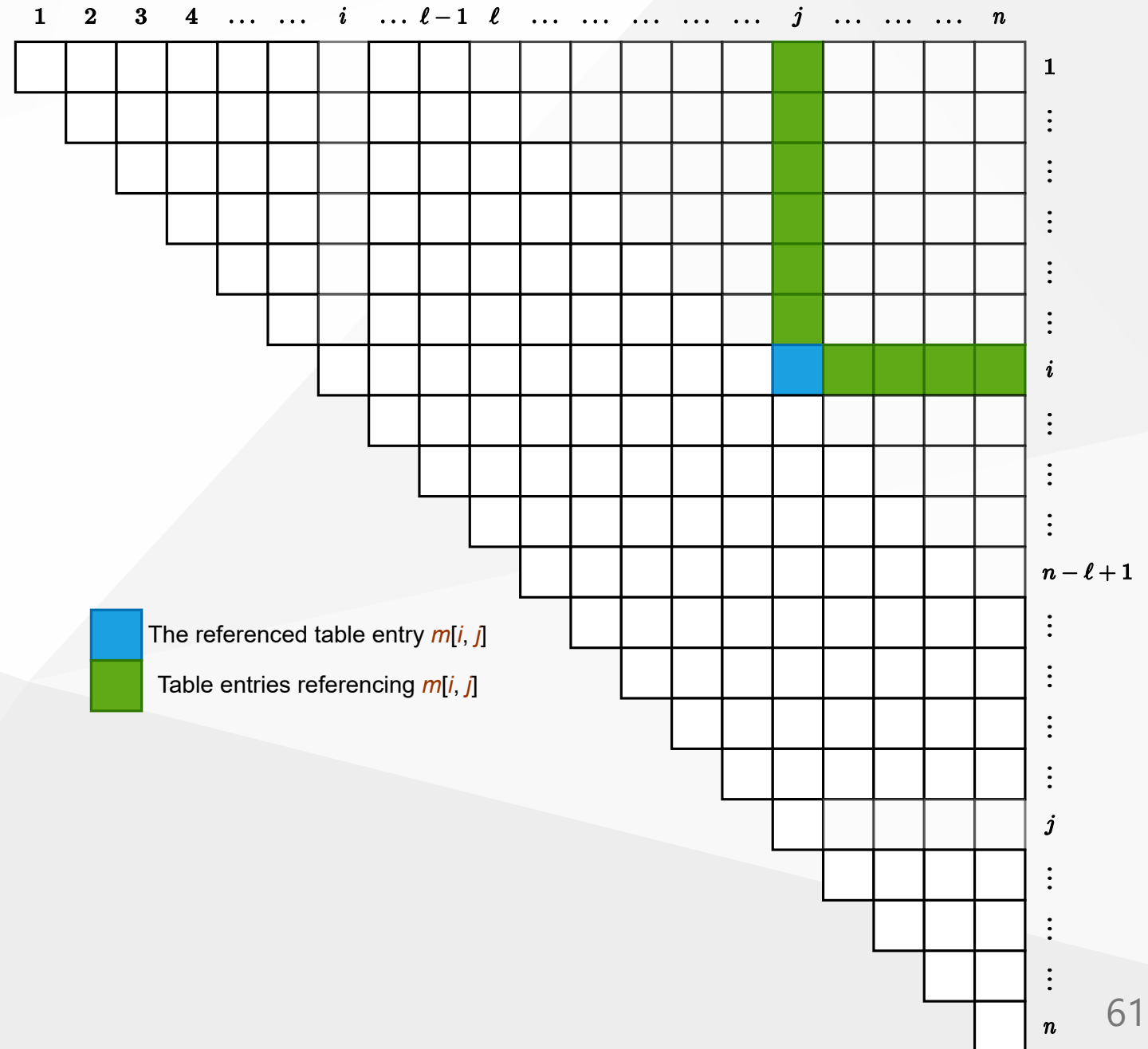
	2	3	4	5	6	
1	1	1	3	3	3	1
2		2	3	3	3	2
3			3	3	3	3
4				4	5	4
5					5	5

# Example: Recursive Construction of an Optimal Solution



# Table reference pattern for $m[i, j]$ ( $1 \leq i \leq j \leq n$ )

- $m[i, j]$  is referenced for the computation of
  - $m[i, r]$  for  $j < r \leq n$  ( $n - j$ ) times
  - $m[r, j]$  for  $1 \leq r < i$  ( $i - 1$ ) times



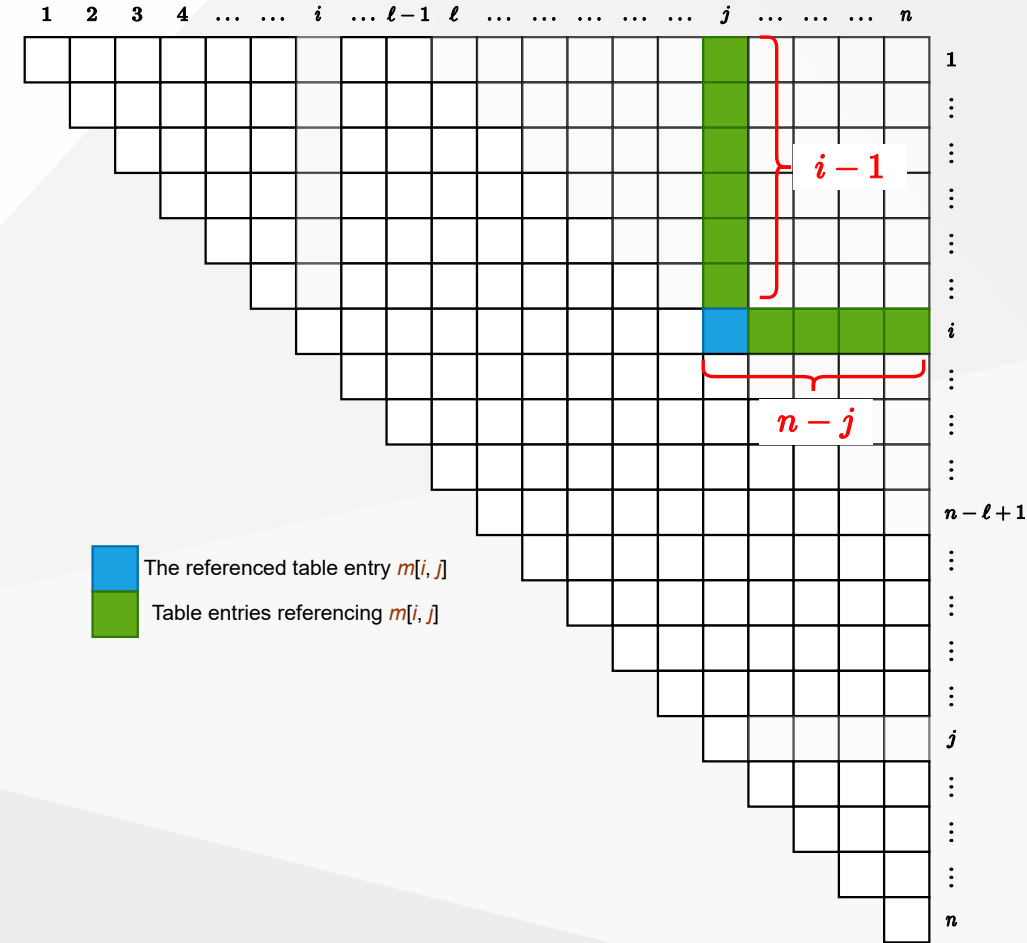
## Table reference pattern for $m[i, j]$ $(1 \leq i \leq j \leq n)$

- $R(i, j) = \#$  of times that  $m[i, j]$  is referenced in computing other entries

$$\begin{aligned} R(i, j) &= (n - j) + (i - 1) \\ &= (n - 1) - (j - i) \end{aligned}$$

- The total  $\#$  of references for the entire

table is:  $\sum_{i=1}^n \sum_{j=i}^n R(i, j) = \frac{n^3 - n}{3}$



## Summary

- Identification of the optimal substructure property
- Recursive formulation to compute the cost of the optimal solution
- Bottom-up computation of the table entries
- Constructing the optimal solution by backtracing the table entries

## References

- [Introduction to Algorithms, Third Edition | The MIT Press](#)
- [Bilkent CS473 Course Notes \(new\)](#)
- [Bilkent CS473 Course Notes \(old\)](#)



*–End – Of – Week – 5 – Course – Module–*