

CE100 Algorithms and Programming II

Week-2 (Solving Recurrences)

Spring Semester, 2021-2022

Download [DOC](#), [SLIDE](#), [PPTX](#)

`<iframe width=700, height=500 frameBorder=0 src='../ce100-week-2-recurrence.md_slide.html'> </iframe>`

Solving Recurrences

Outline

- Solving Recurrences
 - Recursion Tree
 - Master Method
 - Back-Substitution

- Divide-and-Conquer Analysis
 - Merge Sort
 - Binary Search
 - Merge Sort Analysis
 - Complexity

- Recurrence Solution

Solving Recurrences

- Reminder: Runtime ($T(n)$) of *MergeSort* was expressed as a recurrence

$$T(n) = \begin{cases} \Theta(1) & \text{if } n=1 \\ 2T(n/2) + \Theta(n) & \text{otherwise} \end{cases}$$

- Solving recurrences is like solving differential equations, integrals, etc.
 - Need to learn a few tricks

Recurrences

<u>Recurrence</u>: An equation or inequality that describes a function in terms of its value on smaller inputs.

Example :

$$T(n) = \begin{cases} 1 & \text{if } n=1 \\ T(\lceil n/2 \rceil) + 1 & \text{if } n>1 \end{cases}$$

Recurrence Example

$$T(n) = \begin{cases} 1 & \text{if } n=1 \\ T(\lceil n/2 \rceil) + 1 & \text{if } n>1 \end{cases}$$

- Simplification: Assume $n = 2^k$
- Claimed answer : $T(n) = \lg n + 1$
- Substitute claimed answer in the recurrence:

$$\lg n + 1 = \begin{cases} 1 & \text{if } n=1 \\ \lg(\lceil n/2 \rceil) + 2 & \text{if } n>1 \end{cases}$$

- True when $n = 2^k$

Technicalities: Floor / Ceiling

Technically, should be careful about the floor and ceiling functions (as in the book).

e.g. For merge sort, the recurrence should in fact be,

$$T(n) = \begin{cases} \Theta(1) & \text{if } n=1 \\ T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor) + \Theta(n) & \text{if } n>1 \end{cases}$$

But, it's usually ok to:

- ignore floor/ceiling
- solve for the exact power of 2 (or another number)

Technicalities: Boundary Conditions

- Usually assume: $T(n) = \Theta(1)$ for sufficiently small n
 - Changes the exact solution, but usually the asymptotic solution is not affected (e.g. if polynomially bounded)
- For convenience, the boundary conditions generally implicitly stated in a recurrence
 - $T(n) = 2T(n/2) + \Theta(n)$ assuming that
 - $T(n) = \Theta(1)$ for sufficiently small n

Example: When Boundary Conditions Matter

Exponential function: $T(n) = (T(n/2))^2$

Assume

$T(1) = c$ (where c is a positive constant)

$$T(2) = (T(1))^2 = c^2$$

$$T(4) = (T(2))^2 = c^4$$

$$T(n) = \Theta(c^n)$$

e.g.

$$\left. \begin{array}{l} T(1) = 2 \Rightarrow T(n) = \Theta(2^n) \\ T(1) = 3 \Rightarrow T(n) = \Theta(3^n) \end{array} \right\} \text{However } \Theta(2^n) \neq \Theta(3^n)$$

The difference in solution more dramatic when:

$$T(1) = 1 \Rightarrow T(n) = \Theta(1^n) = \Theta(1)$$

Solving Recurrences

We will focus on 3 techniques

- Substitution method
- Recursion tree approach
- Master method

Substitution Method

The most general method:

- Guess
- Prove by induction
- Solve for constants

Substitution Method: Example

Solve $T(n) = 4T(n/2) + n$ (assume $T(1) = \Theta(1)$)

1. Guess $T(n) = O(n^3)$ (need to prove O and Ω separately)
2. Prove by induction that $T(n) \leq cn^3$ for large n (i.e. $n \geq n_0$)
 - Inductive hypothesis: $T(k) \leq ck^3$ for any $k < n$
 - Assuming ind. hyp. holds, prove $T(n) \leq cn^3$

Substitution Method: Example – cont'd

Original recurrence: $T(n) = 4T(n/2) + n$

From inductive hypothesis: $T(n/2) \leq c(n/2)^3$

Substitute this into the original recurrence:

- $T(n) \leq 4c(n/2)^3 + n$
- $= (c/2)n^3 + n$
- $= cn^3 - ((c/2)n^3 - n) \Leftarrow \text{desired - residual}$
- $\leq cn^3$
when $((c/2)n^3 - n) \geq 0$

Substitution Method: Example – cont'd

So far, we have shown:

$$T(n) \leq cn^3 \text{ when } ((c/2)n^3 - n) \geq 0$$

We can choose $c \geq 2$ and $n_0 \geq 1$

But, the proof is not complete yet.

Reminder: Proof by induction:

1. *Prove the base cases* \Leftarrow haven't proved the base cases yet
2. *Inductive hypothesis for smaller sizes*
3. *Prove the general case*

Substitution Method: Example – cont'd

- We need to prove the base cases
 - Base: $T(n) = \Theta(1)$ for small n (e.g. for $n = n_0$)
- We should show that:
 - $\Theta(1) \leq cn^3$ for $n = n_0$, This holds if we pick c big enough
- So, the proof of $T(n) = O(n^3)$ is complete
- But, is this a tight bound?

Example: A tighter upper bound?

- Original recurrence: $T(n) = 4T(n/2) + n$
- Try to prove that $T(n) = O(n^2)$,
 - i.e. $T(n) \leq cn^2$ for all $n \geq n_0$
- Ind. hyp: Assume that $T(k) \leq ck^2$ for $k < n$
- Prove the general case: $T(n) \leq cn^2$

Example (cont'd)

Original recurrence: $T(n) = 4T(n/2) + n$

Ind. hyp: Assume that $T(k) \leq ck^2$ for $k < n$

Prove the general case: $T(n) \leq cn^2$

$$\begin{aligned} T(n) &= 4T(n/2) + n \\ &\leq 4c(n/2)^2 + n \\ &= cn^2 + n \end{aligned}$$

$= O(n^2) \Leftarrow$ Wrong! We must prove exactly

Example (cont'd)

Original recurrence: $T(n) = 4T(n/2) + n$

Ind. hyp: Assume that $T(k) \leq ck^2$ for $k < n$

Prove the general case: $T(n) \leq cn^2$

- So far, we have:

$$T(n) \leq cn^2 + n$$

- No matter which positive c value we choose, this does not show that $T(n) \leq cn^2$
- Proof failed?

Example (cont'd)

- What was the problem?
 - The inductive hypothesis was not strong enough
- **Idea:** Start with a stronger inductive hypothesis
 - Subtract a low-order term
- **Inductive hypothesis:** $T(k) \leq c_1 k^2 - c_2 k$ for $k < n$
- **Prove the general case:** $T(n) \leq c_1 n^2 - c_2 n$

Example (cont'd)

Original recurrence: $T(n) = 4T(n/2) + n$

Ind. hyp: Assume that $T(k) \leq c_1 k^2 - c_2 k$ for $k < n$

Prove the general case: $T(n) \leq c_1 n^2 - c_2 n$

$$\begin{aligned}
 T(n) &= 4T(n/2) + n \\
 &\leq 4(c_1 (n/2)^2 - c_2 (n/2)) + n \\
 &= c_1 n^2 - 2c_2 n + n \\
 &= c_1 n^2 - c_2 n - (c_2 n - n) \\
 &\leq c_1 n^2 - c_2 n \text{ for } n(c_2 - 1) \geq 0 \\
 &\text{choose } c_2 \geq 1
 \end{aligned}$$

Example (cont'd)

We now need to prove

$$T(n) \leq c_1 n^2 - c_2 n$$

for the base cases.

$T(n) = \Theta(1)$ for $1 \leq n \leq n_0$ (implicit assumption)

$\Theta(1) \leq c_1 n^2 - c_2 n$ for n small enough (e.g. $n = n_0$)

- We can choose c_1 large enough to make this hold

We have proved that $T(n) = O(n^2)$

Substitution Method: Example 2

For the recurrence $T(n) = 4T(n/2) + n$,

prove that $T(n) = \Omega(n^2)$

i.e. $T(n) \geq cn^2$ for any $n \geq n_0$

Ind. hyp: $T(k) \geq ck^2$ for any $k < n$

Prove general case: $T(n) \geq cn^2$

$$T(n) = 4T(n/2) + n$$

$$\geq 4c(n/2)^2 + n$$

$$= cn^2 + n$$

$$\geq cn^2 \text{ since } n > 0$$

Example 2 (cont'd)

We now need to prove that

$$T(n) \geq cn^2$$

for the base cases

$$T(n) = \Theta(1) \text{ for } 1 \leq n \leq n_0 \text{ (implicit assumption)}$$

$$\Theta(1) \geq cn^2 \text{ for } n = n_0$$

n_0 is sufficiently small (i.e. constant)

We can choose c small enough for this to hold

We have proved that $T(n) = \Omega(n^2)$

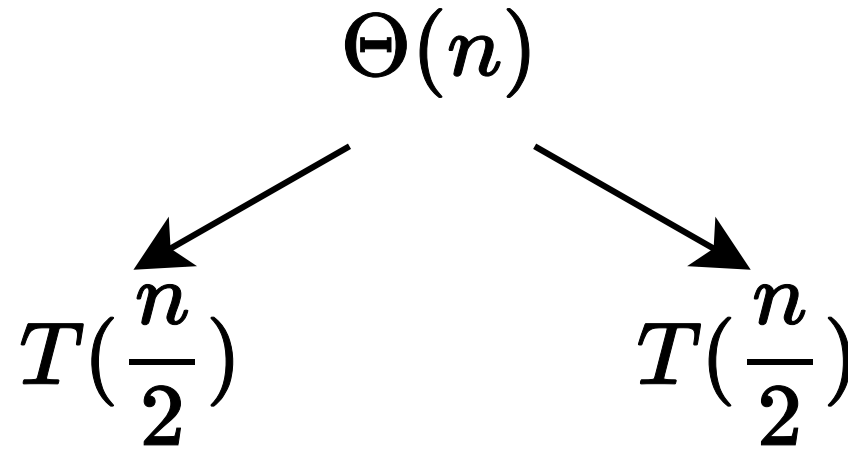
Substitution Method - Summary

- Guess the asymptotic complexity
- Prove your guess using induction
 - Assume inductive hypothesis holds for $k < n$
 - Try to prove the general case for n
 - Note: *MUST* prove the *EXACT* inequality *CANNOT* ignore lower order terms, If the proof fails, strengthen the ind. hyp. and try again
- Prove the base cases (usually straightforward)

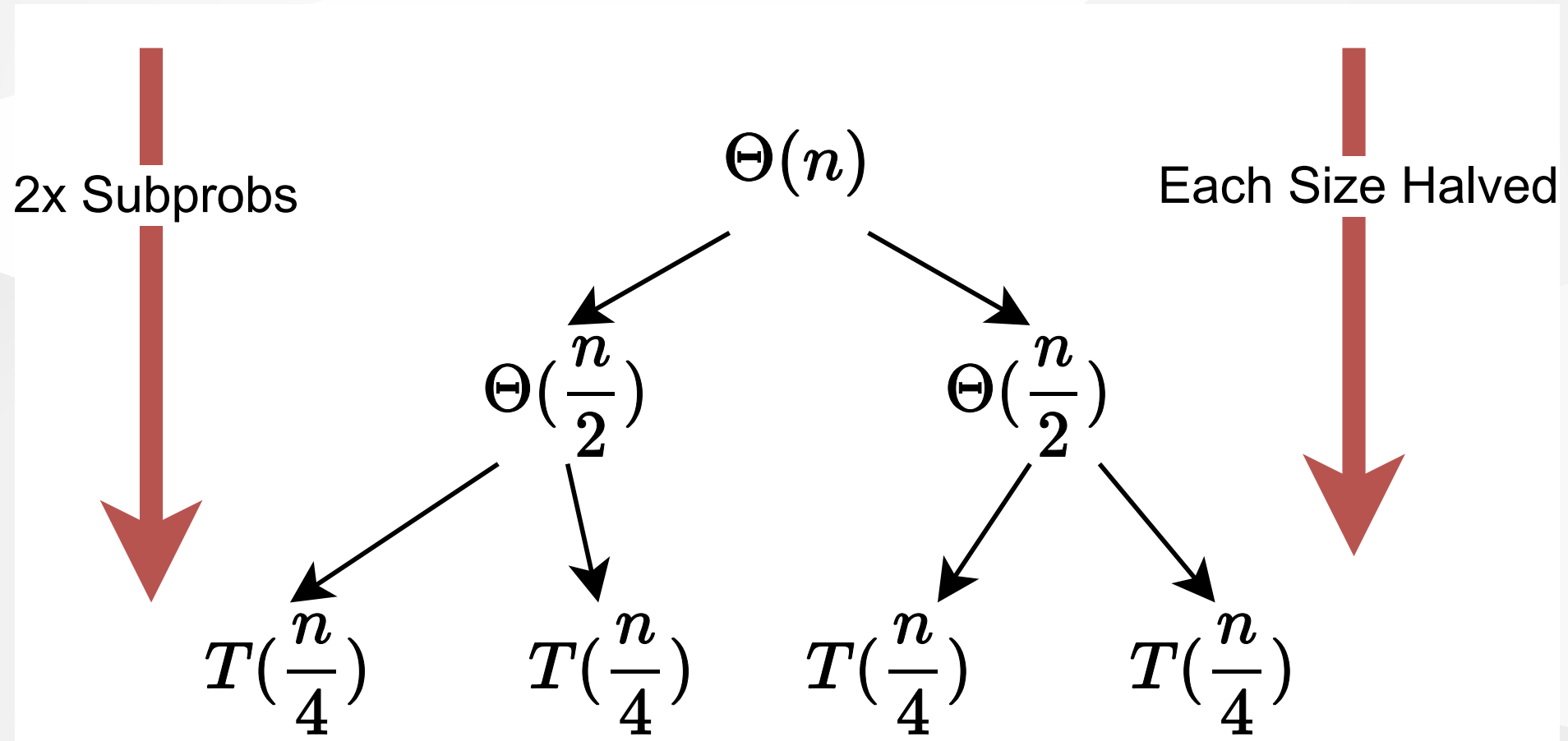
Recursion Tree Method

- A recursion tree models the runtime costs of a recursive execution of an algorithm.
- The recursion tree method is good for generating guesses for the substitution method.
- The recursion-tree method can be unreliable.
 - Not suitable for formal proofs
- The recursion-tree method promotes intuition, however.

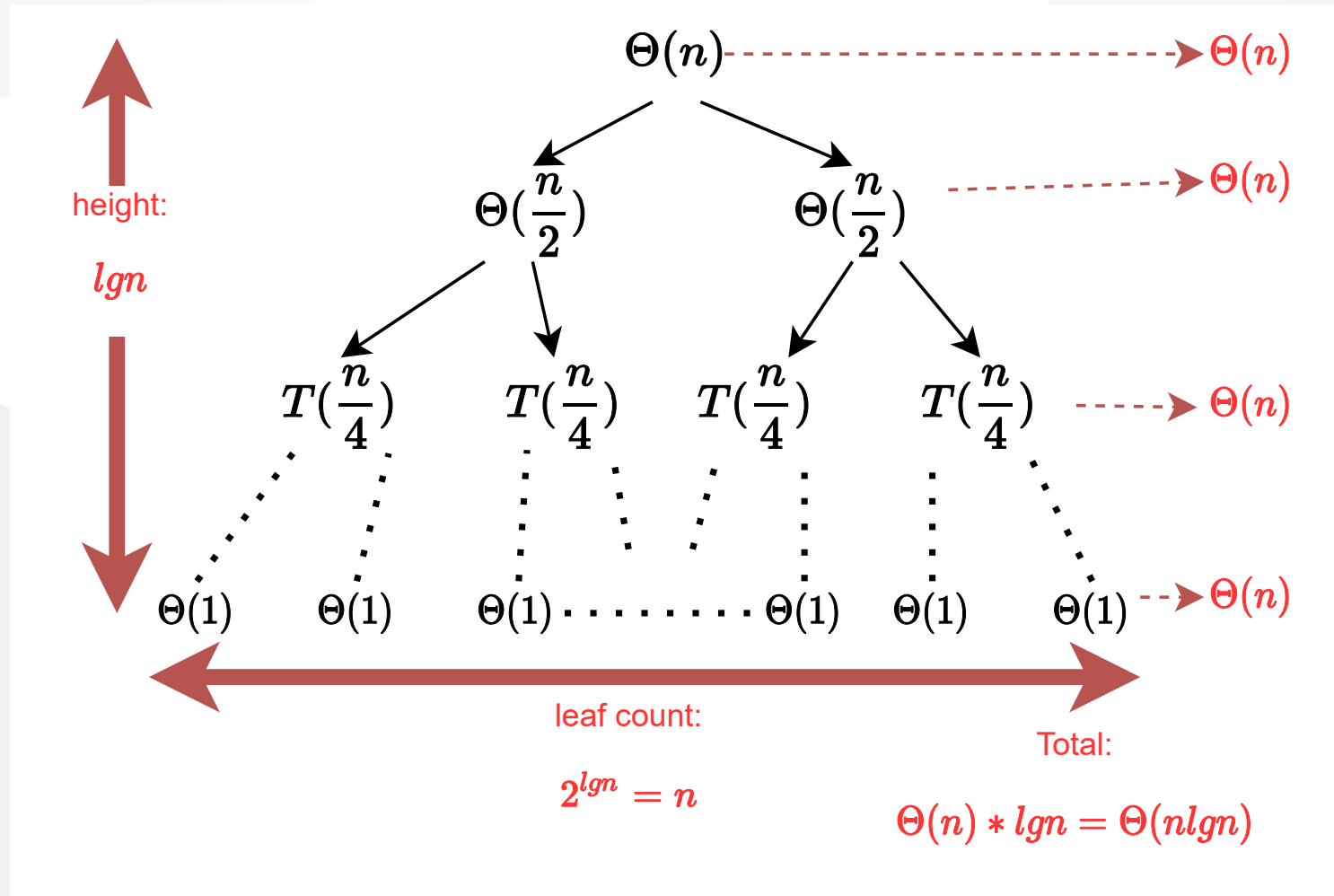
Solve Recurrence : $T(n) = 2T(n/2) + \Theta(n)$



Solve Recurrence : $T(n) = 2T(n/2) + \Theta(n)$

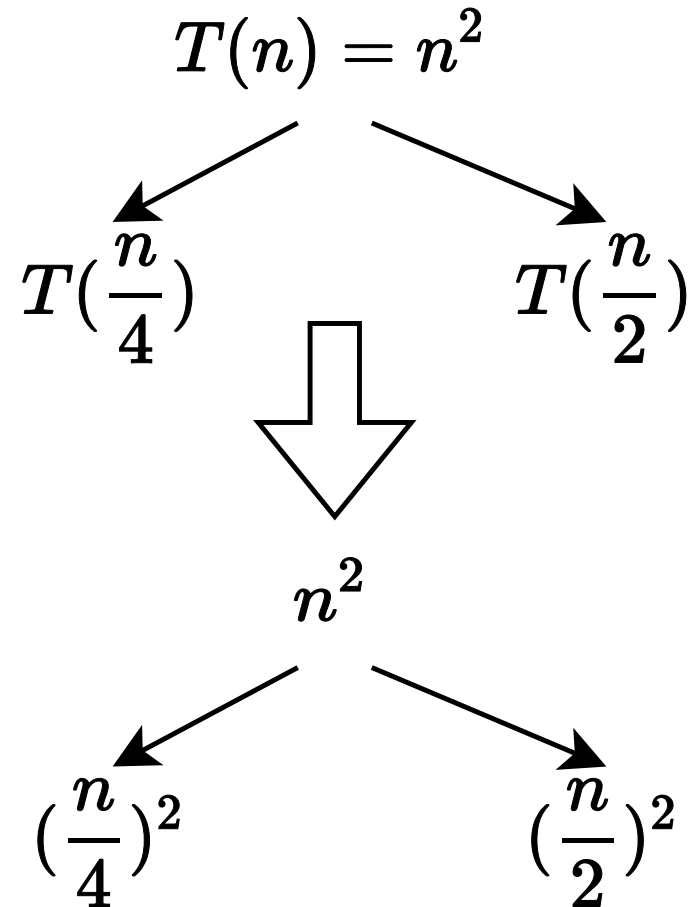


Solve Recurrence : $T(n) = 2T(n/2) + \Theta(n)$



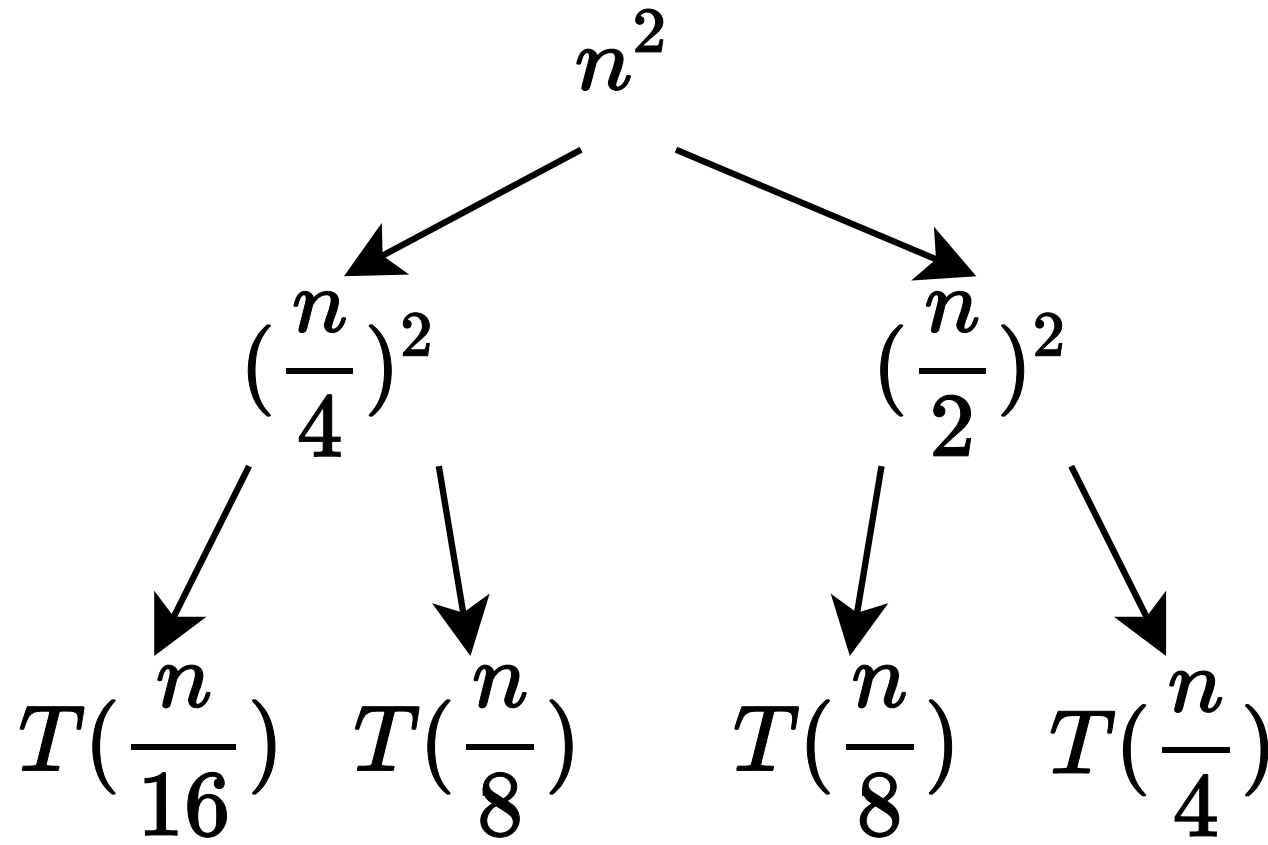
Example of Recursion Tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$



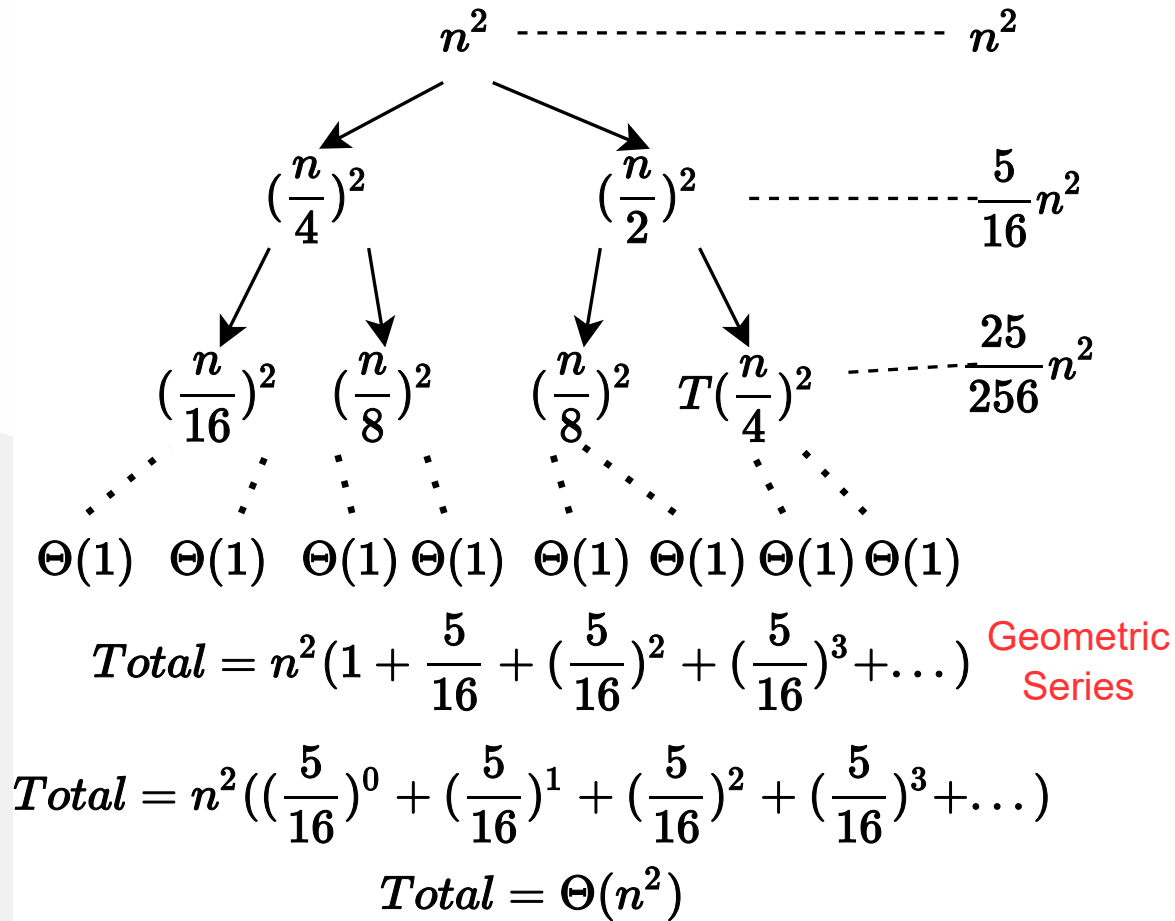
Example of Recursion Tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$



Example of Recursion Tree

Solve $T(n) = T(n/4) + T(n/2) + n^2$



The Master Method

- A powerful black-box method to solve recurrences.
- The master method applies to recurrences of the form
 - $T(n) = aT(n/b) + f(n)$
- where $a \geq 1$, $b > 1$, and f is asymptotically positive.

The Master Method: 3 Cases

- Recurrence: $T(n) = aT(n/b) + f(n)$
- Compare $f(n)$ with $n^{\log_b a}$
- Intuitively:
 - **Case 1:** $f(n)$ grows polynomially slower than $n^{\log_b a}$
 - **Case 2:** $f(n)$ grows at the same rate as $n^{\log_b a}$
 - **Case 3:** $f(n)$ grows polynomially faster than $n^{\log_b a}$

The Master Method: Case 1

- Recurrence: $T(n) = aT(n/b) + f(n)$
- Case 1: $\frac{n^{\log_b^a}}{f(n)} = \Omega(n^\varepsilon)$ for some constant $\varepsilon > 0$
- i.e., $f(n)$ grows polynomially slower than $n^{\log_b^a}$ (by an n^ε factor)
- Solution: $T(n) = \Theta(n^{\log_b^a})$

The Master Method: Case 2 (Simple Version)

- Recurrence: $T(n) = aT(n/b) + f(n)$
- Case 2: $\frac{f(n)}{n^{\log_b a}} = \Theta(1)$
- i.e., $f(n)$ and $n^{\log_b a}$ grow at similar rates
- Solution: $T(n) = \Theta(n^{\log_b a} \lg n)$

The Master Method: Case 3

- Case 3: $\frac{f(n)}{n^{\log_b^a}} = \Omega(n^\varepsilon)$ for some constant $\varepsilon > 0$
- i.e., $f(n)$ grows polynomially faster than $n^{\log_b^a}$ (by an n^ε factor)
- and the following regularity condition holds:
 - $af(n/b) \leq cf(n)$ for some constant $c < 1$
- Solution: $T(n) = \Theta(f(n))$

Example : $T(n) = 4T(n/2) + n$

- $a = 4$
- $b = 2$
- $f(n) = n$
- $n^{\log_b^a} = n^{\log_2^4} = n^{\log_2^{2^2}} = n^{2\log_2^2} = n^2$
- $f(n) = n$ grows polynomially slower than $n^{\log_b^a} = n^2$
 - $\frac{n^{\log_b^a}}{f(n)} = \frac{n^2}{n} = n = \Omega(n^\epsilon)$
- CASE-1:
 - $T(n) = \Theta(n^{\log_b^a}) = \Theta(n^{\log_2^4}) = \Theta(n^2)$

Example : $T(n) = 4T(n/2) + n^2$

- $a = 4$
- $b = 2$
- $f(n) = n^2$
- $n^{\log_b^a} = n^{\log_2^4} = n^{\log_2^{2^2}} = n^{2\log_2^2} = n^2$
- $f(n) = n^2$ grows at similar rate as $n^{\log_b^a} = n^2$
 - $f(n) = \Theta(n^{\log_b^a}) = n^2$
- CASE-2:
 - $T(n) = \Theta(n^{\log_b^a} \lg n) = \Theta(n^{\log_2^4} \lg n) = \Theta(n^2 \lg n)$

Example : $T(n) = 4T(n/2) + n^3$

- $a = 4$
- $b = 2$
- $f(n) = n^3$
- $n^{\log_b^a} = n^{\log_2^4} = n^{\log_2^{2^2}} = n^{2\log_2^2} = n^2$
- $f(n) = n^3$ grows polynomially faster than $n^{\log_b^a} = n^2$
 - $\frac{f(n)}{n^{\log_b^a}} = \frac{n^3}{n^2} = n = \Omega(n^\epsilon)$

Example : $T(n) = 4T(n/2) + n^3$ (con't)

- Seems like CASE 3, but need to check the regularity condition
- Regularity condition $af(n/b) \leq cf(n)$ for some constant $c < 1$
- $4(n/2)^3 \leq cn^3$ for $c = 1/2$
- CASE-3:
 - $T(n) = \Theta(f(n)) \implies T(n) = \Theta(n^3)$

Example : $T(n) = 4T(n/2) + n^2 \lg n$

- $a = 4$
- $b = 2$
- $f(n) = n^2 \lg n$
- $n^{\log_b^a} = n^{\log_2^4} = n^{\log_2^{2^2}} = n^{2\log_2^2} = n^2$
- $f(n) = n^2 \lg n$ grows slower than $n^{\log_b^a} = n^2$
 - but is it polynomially slower?
 - $\frac{n^{\log_b^a} f(n)}{\frac{n^2}{\lg n}} = \lg n \neq \Omega(n^\varepsilon)$ for any $\varepsilon > 0$
 - is not CASE-1
 - Master Method does not apply!

The Master Method : Case 2 (General Version)

- Recurrence : $T(n) = aT(n/b) + f(n)$
- Case 2: $\frac{f(n)}{n^{\log_b a}} = \Theta(\lg^k n)$ for some constant $k \geq 0$
- Solution : $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$

General Method (Akra-Bazzi)

$$T(n) = \sum_{i=1}^k a_i T(n/b_i) + f(n)$$

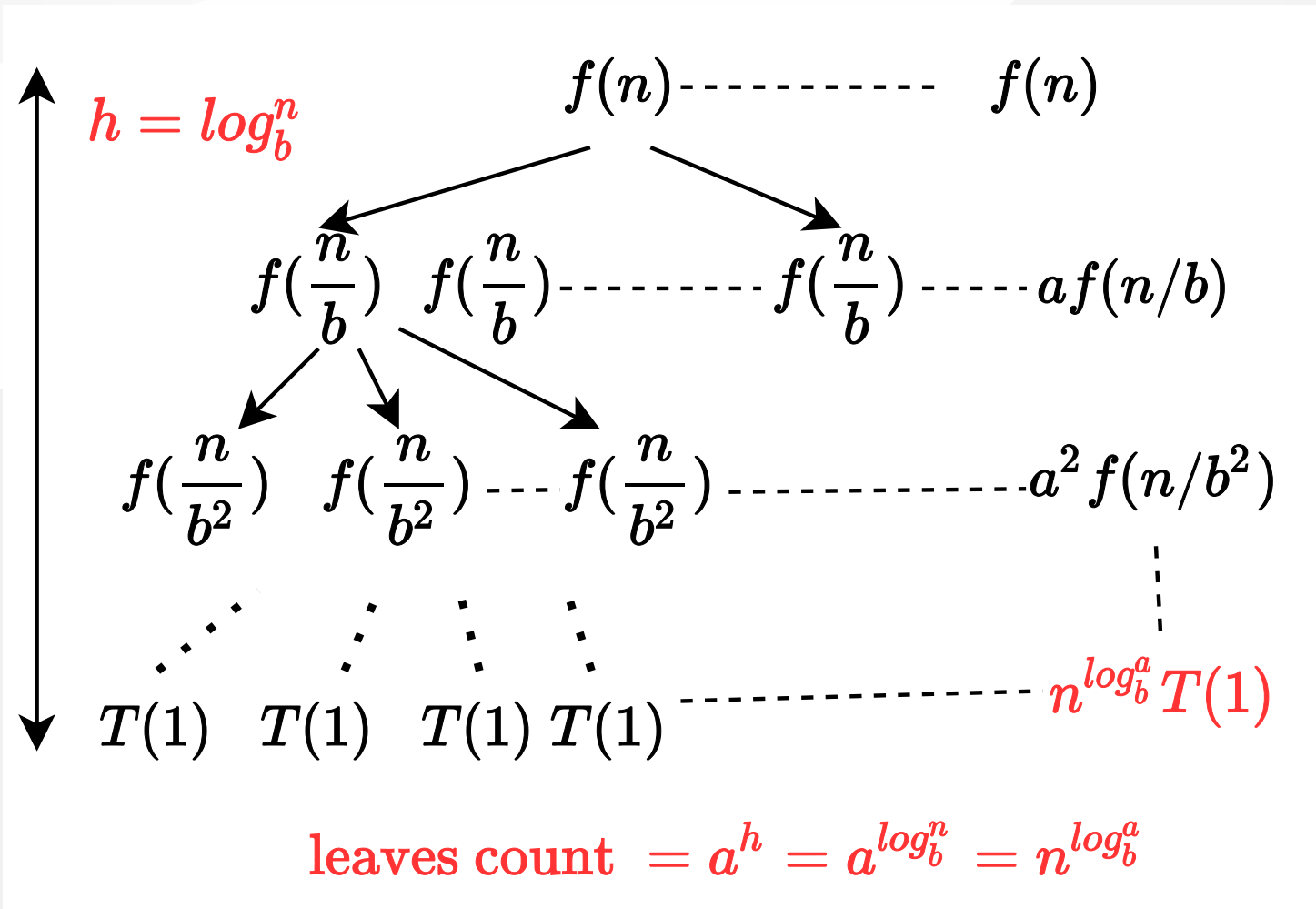
Let p be the unique solution to

$$\sum_{i=1}^k (a_i / b_i^p) = 1$$

Then, the answers are the same as for the master method, but with n^p instead of $n^{\log_b^a}$
(Akra and Bazzi also prove an even more general result.)

Idea of Master Theorem

Recursion Tree:



Idea of Master Theorem

CASE 1 : The weight increases geometrically from the root to the leaves. The leaves hold a constant fraction of the total weight.

$$n^{\log_b^a} T(1) = \Theta(n^{\log_b^a})$$

Idea of Master Theorem

CASE 2 : ($k = 0$) The weight is approximately the same on each of the $\log_b n$ levels.

$$n^{\log_b^a} T(1) = \Theta(n^{\log_b^a} \lg n)$$

Idea of Master Theorem

CASE 3 : The weight decreases geometrically from the root to the leaves. The root holds a constant fraction of the total weight.

$$n^{\log_b^a} T(1) = \Theta(f(n))$$

Proof of Master Theorem: Case 1 and Case 2

- Recall from the recursion tree (note $h = \lg_b n = \text{tree height}$)

$$\text{Leaf Cost} = \Theta(n^{\log_b^a})$$

$$\text{Non-leaf Cost} = g(n) = \sum_{i=0}^{h-1} a^i f(n/b^i)$$

$$T(n) = \text{Leaf Cost} + \text{Non-leaf Cost}$$

$$T(n) = \Theta(n^{\log_b^a}) + \sum_{i=0}^{h-1} a^i f(n/b^i)$$

Proof Case 1

- $\frac{n^{\log_b^a}}{f(n)} = \Omega(n^\varepsilon)$ for some $\varepsilon > 0$
- $\frac{n^{\log_b^a}}{f(n)} = \Omega(n^\varepsilon) \implies O(n^{-\varepsilon}) \implies f(n) = O(n^{\log_b^{a-\varepsilon}})$
- $g(n) = \sum_{i=0}^{h-1} a^i O((n/b^i)^{\log_b^{a-\varepsilon}}) = O(\sum_{i=0}^{h-1} a^i (n/b^i)^{\log_b^{a-\varepsilon}})$
- $O(n^{\log_b^{a-\varepsilon}} \sum_{i=0}^{h-1} a^i b^{i\varepsilon} / b^{i \log_b^{a-\varepsilon}})$

Proof Case 1 (con't)

$$\bullet \sum_{i=0}^{h-1} \frac{a^i b^{i\varepsilon}}{b^{i \log_b a}} = \sum_{i=0}^{h-1} a^i \frac{(b^\varepsilon)^i}{(b^{\log_b a})^i} = \sum_{i=0}^{h-1} a^i \frac{b^{i\varepsilon}}{a^i} = \sum_{i=0}^{h-1} (b^\varepsilon)^i$$

= An increasing geometric series since $b > 1$

$$\frac{b^{h\varepsilon} - 1}{b^\varepsilon - 1} = \frac{(b^h)^\varepsilon - 1}{b^\varepsilon - 1} = \frac{(b^{\log_b n})^\varepsilon - 1}{b^\varepsilon - 1} = \frac{n^\varepsilon - 1}{b^\varepsilon - 1} = O(n^\varepsilon)$$

Proof Case 1 (con't)

- $g(n) = O(n^{\log_b a - \varepsilon} O(n^\varepsilon)) = O(\frac{n^{\log_b a}}{n^\varepsilon} O(n^\varepsilon)) = O(n^{\log_b a})$
- $T(n) = \Theta(n^{\log_b a}) + g(n) = \Theta(n^{\log_b a}) + O(n^{\log_b a}) = \Theta(n^{\log_b a})$

Q.E.D.

(Quod Erat Demonstrandum)

Proof of Case 2 (limited to $k=0$)

- $\frac{f(n)}{n^{\log_b^a}} = \Theta(\lg^0 n) = \Theta(1) \implies f(n) = \Theta(n^{\log_b^a}) \implies f(n/b^i) = \Theta((n/b^i)^{\log_b^a})$
- $g(n) = \sum_{i=0}^{h-1} a^i \Theta((n/b^i)^{\log_b^a})$
- $= \Theta(\sum_{i=0}^{h-1} a^i \frac{n^{\log_b^a}}{b^{i \log_b^a}})$
- $= \Theta(n^{\log_b^a} \sum_{i=0}^{h-1} a^i \frac{1}{(b^{\log_b^a})^i})$
- $= \Theta(n^{\log_b^a} \sum_{i=0}^{h-1} a^i \frac{1}{a^i})$
- $= \Theta(n^{\log_b^a} \sum_{i=0}^{\log_b^{n-1}} 1) = \Theta(n^{\log_b^a} \log_b n) = \Theta(n^{\log_b^a} \lg n)$
- $T(n) = n^{\log_b^a} + \Theta(n^{\log_b^a} \lg n)$
- $= \Theta(n^{\log_b^a} \lg n)$

References

TODO