CE100 Algorithms and Programming II

Week-10 (Graphs)

Spring Semester, 2021-2022

Download DOC-PDF, DOC-DOCX, SLIDE, PPTX

Graphs

Outline

- Introduction to Graphs
- Graphs and Representation
- BFS (Breath-First Search)
- DFS (Depth-First Search)
    - in-order
    - post-order
    - pre-order

- Topological Order
- SCC (Strongly Connected Components)
- MST
  - Prim
  - Kruskal

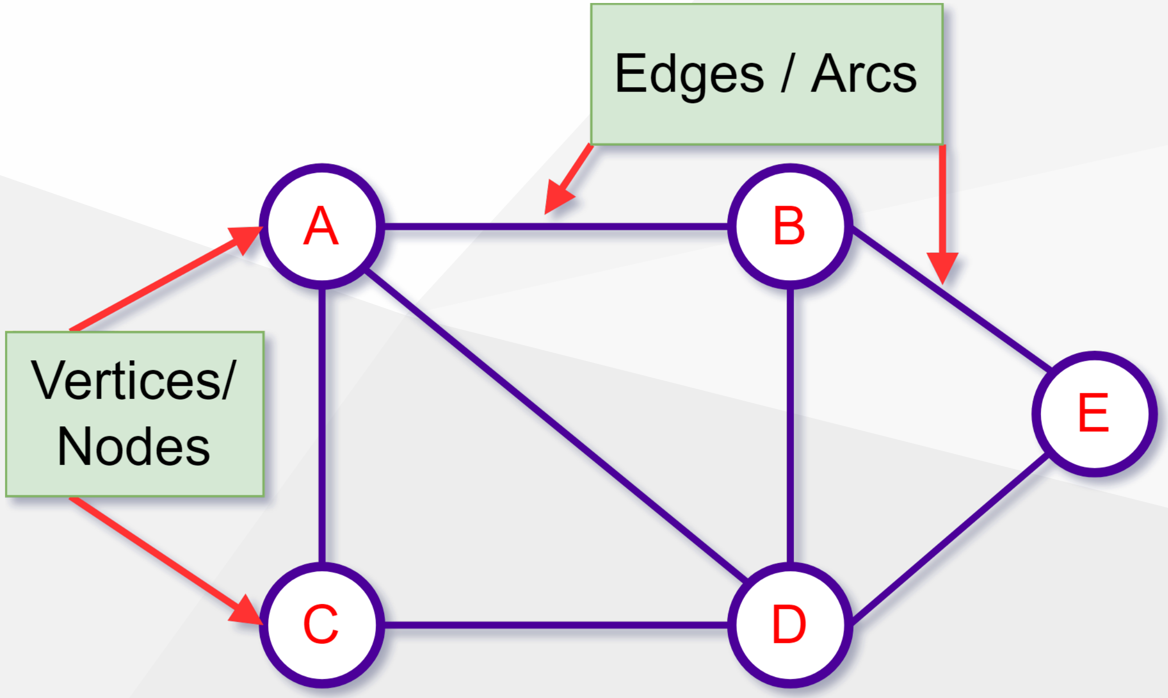
Introduction to Graphs

Introduction to Graphs

- The graph is a non-linear data structure.
- It contains a set of points known as
  - nodes (or vertices) and
  - a set of links known as edges (or Arcs).

Introduction to Graphs

- Here edges are used to connect the vertices. A graph is defined as follows.
- Generally, a graph G is represented as $G = (V, E)$, where
  - V is set of vertices and
  - E is set of edges.



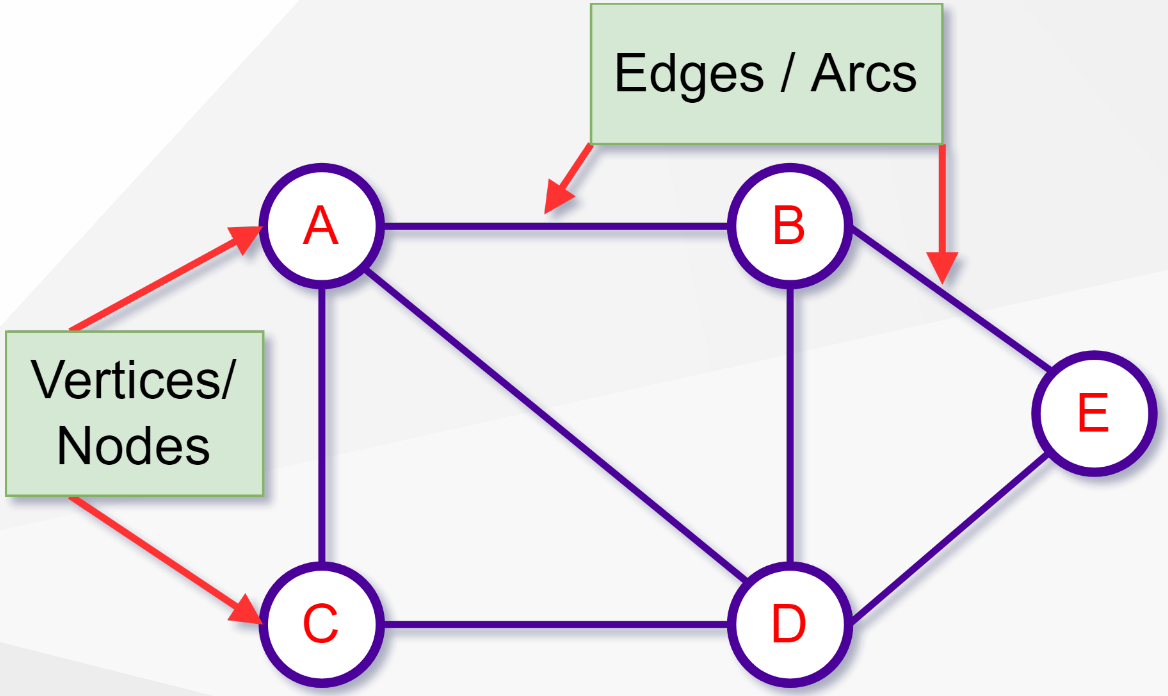
Introduction to Graphs - Example

- The following is a graph with `5 vertices` (V) and `6 edges` (E).
- This graph G can be defined as

$$G = (V, E)$$

$$V = \{A, B, C, D, E\}$$

$$E = \{(A, B), (A, C), (A, D), (B, D), (C, D), (B, E), (E, D)\}$$

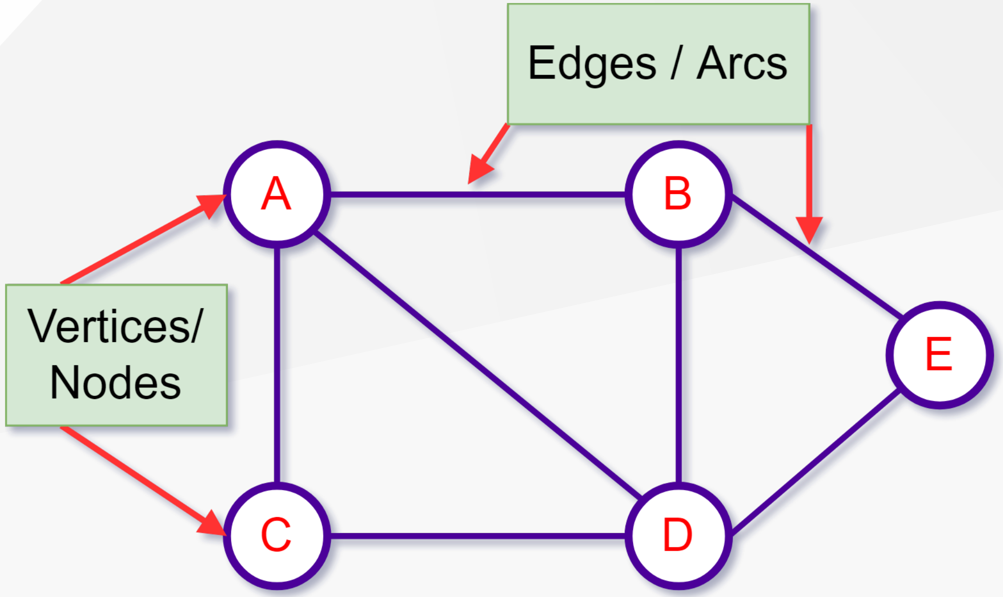


Graph Terminology

Graph Terminology

Vertex

Individual data element of a graph is called as Vertex. Vertex is also known as node. In above example graph, A, B, C, D, E are known as vertices.



Graph Terminology

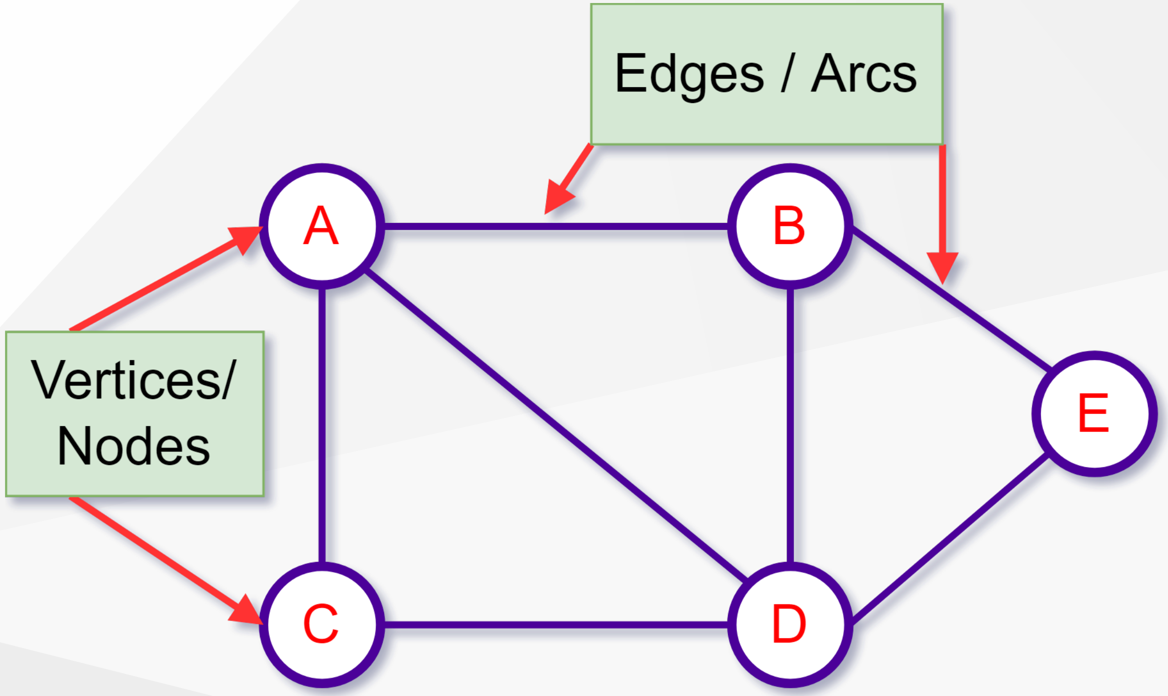
Edge

- An edge is a connecting link between two vertices.
- Edge is also known as Arc.
- An edge is represented as

$$(\text{startingVertex}, \text{endingVertex})$$

- For example, in above graph the link between vertices A and B is represented as

$$(A, B)$$

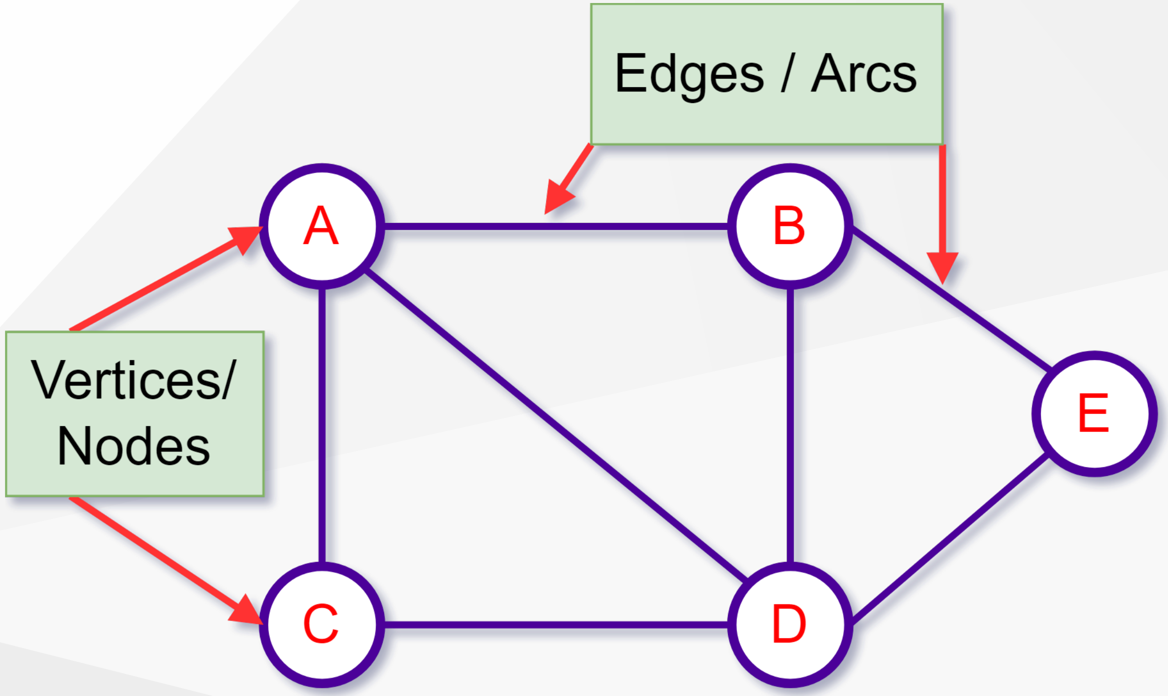


Graph Terminology

Edge

- In example graph, there are 7 edges

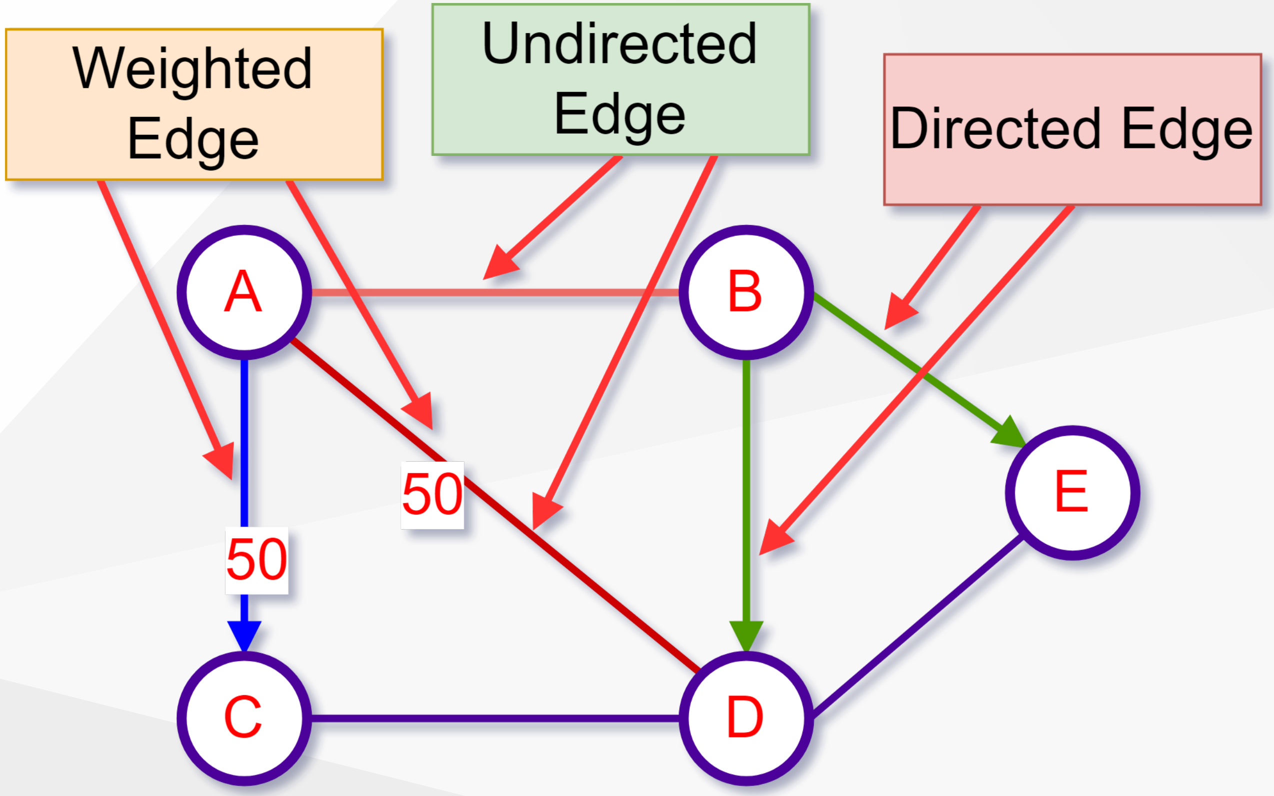
$$(A, B), (A, C), (A, D),$$
$$(B, D), (B, E), (C, D), (D, E)$$



Graph Terminology

Edge

- Edges are three types.
  - Undirected Edge
  - Directed Edge
  - Weighted Edge

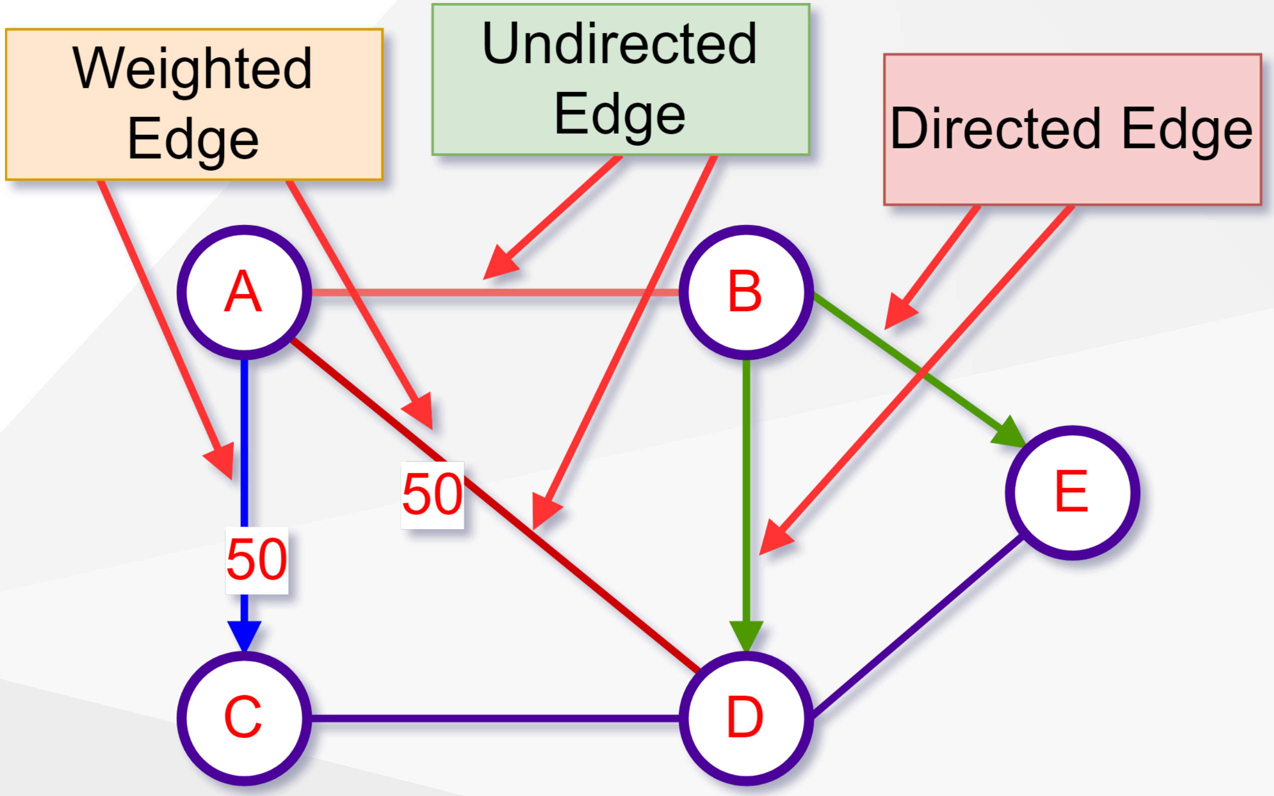


Graph Terminology

Edge

Undirected Edge

- An undirected egde is a bidirectional edge. If there is undirected edge between vertices A and B then edge (A, B) is equal to edge (B, A)

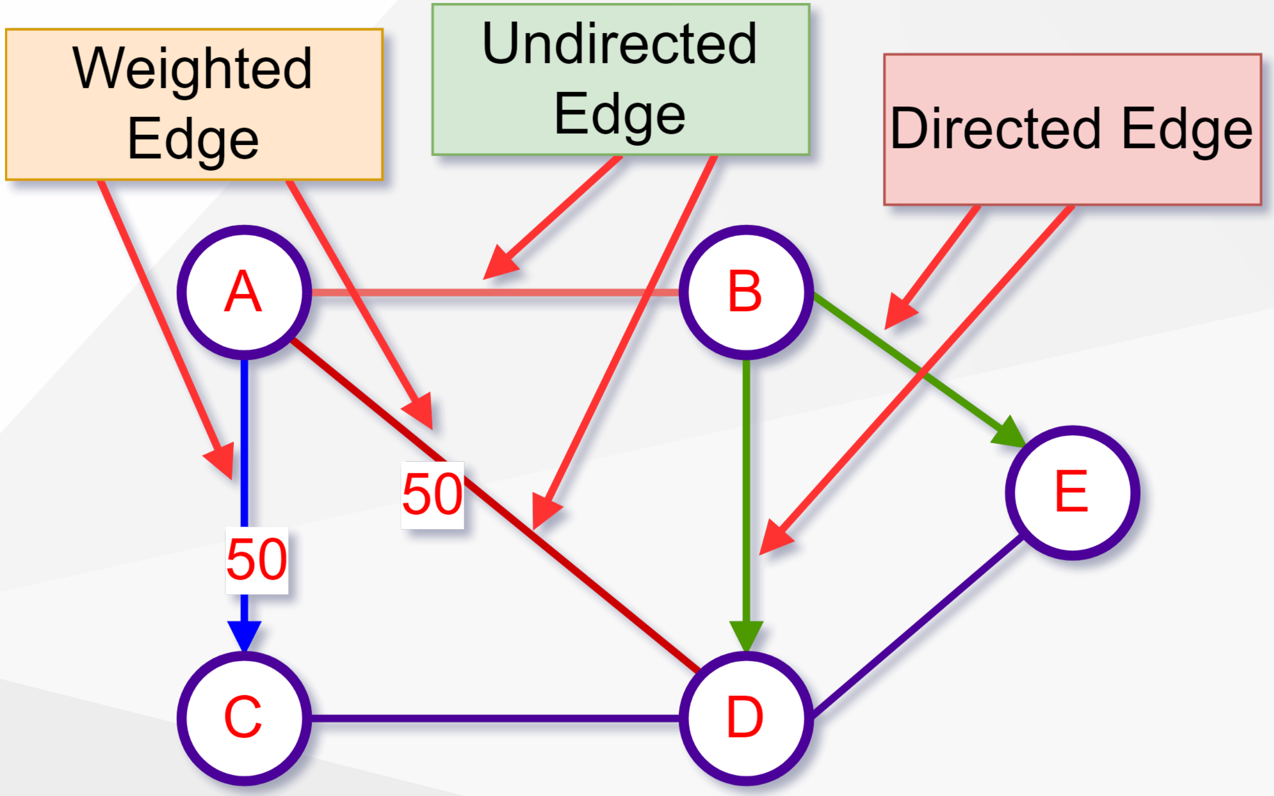


Graph Terminology

Edge

Directed Edge

- A directed egde is a unidirectional edge. If there is directed edge between vertices A and B then edge (A, B) is not equal to edge (B, A).

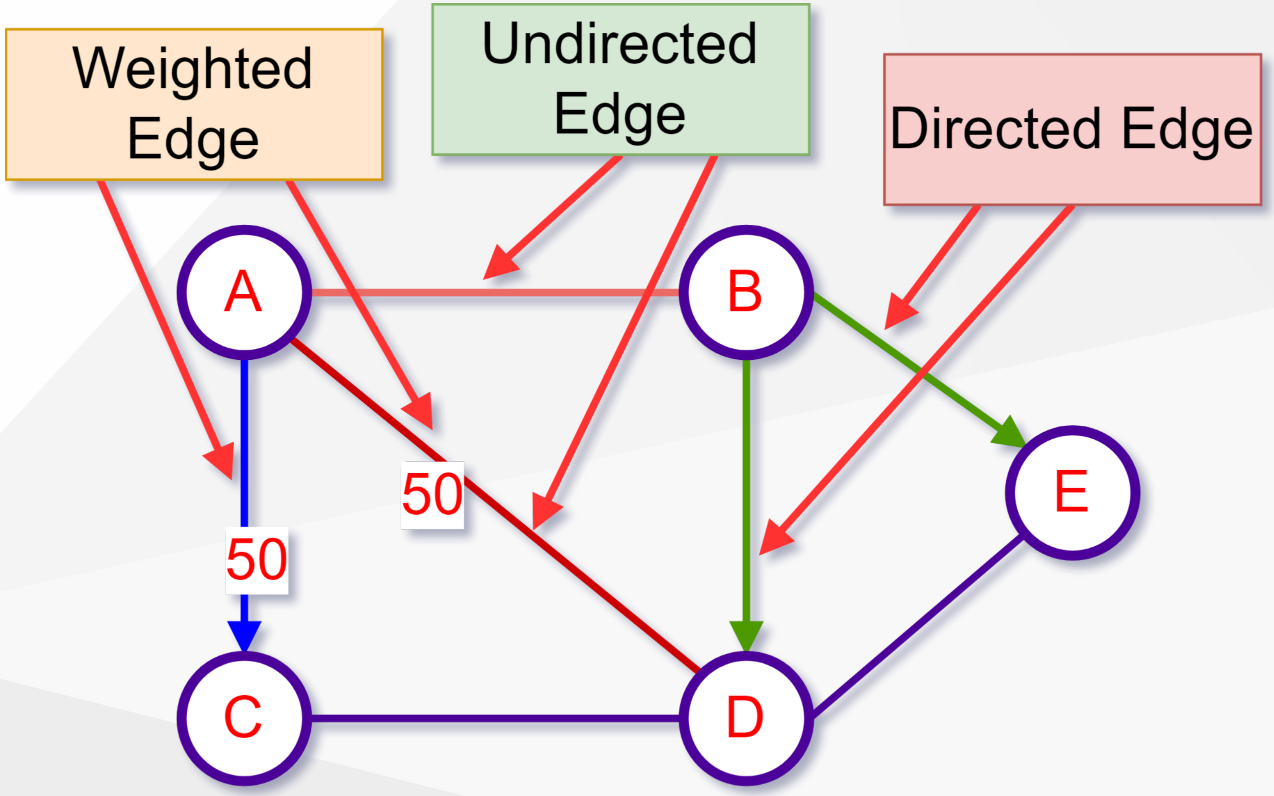


Graph Terminology

Edge

Weighted Edge

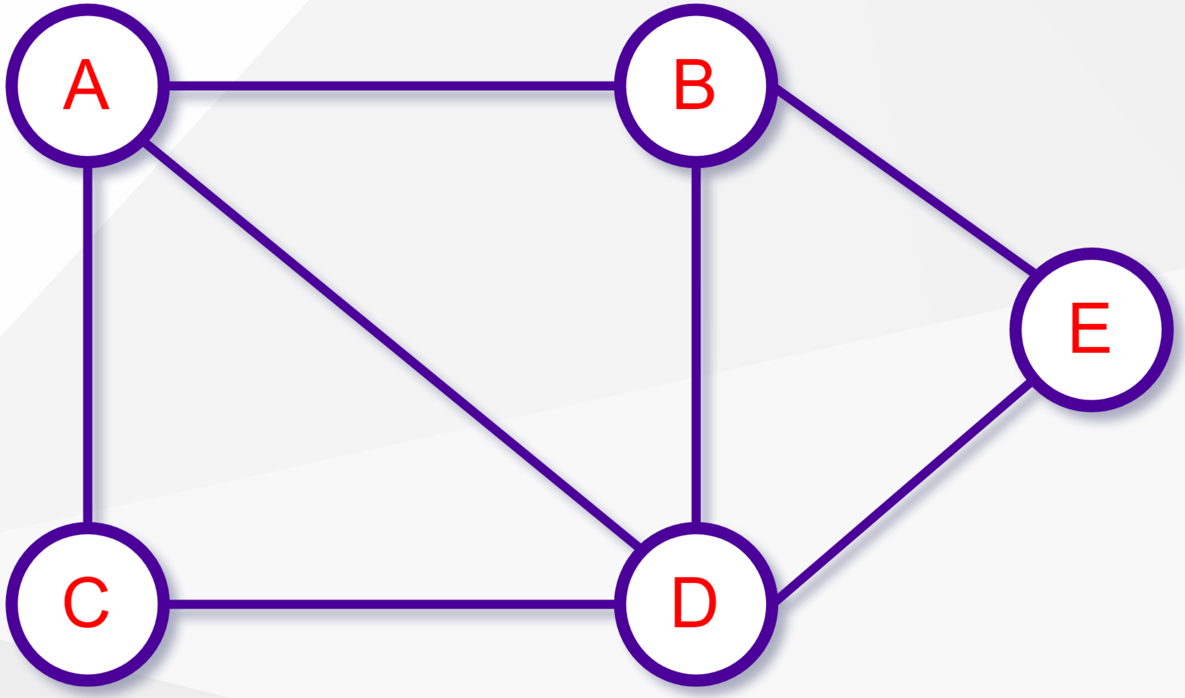
- A weighted egde is a edge with value (cost) on it.



Graph Terminology

Undirected Graph

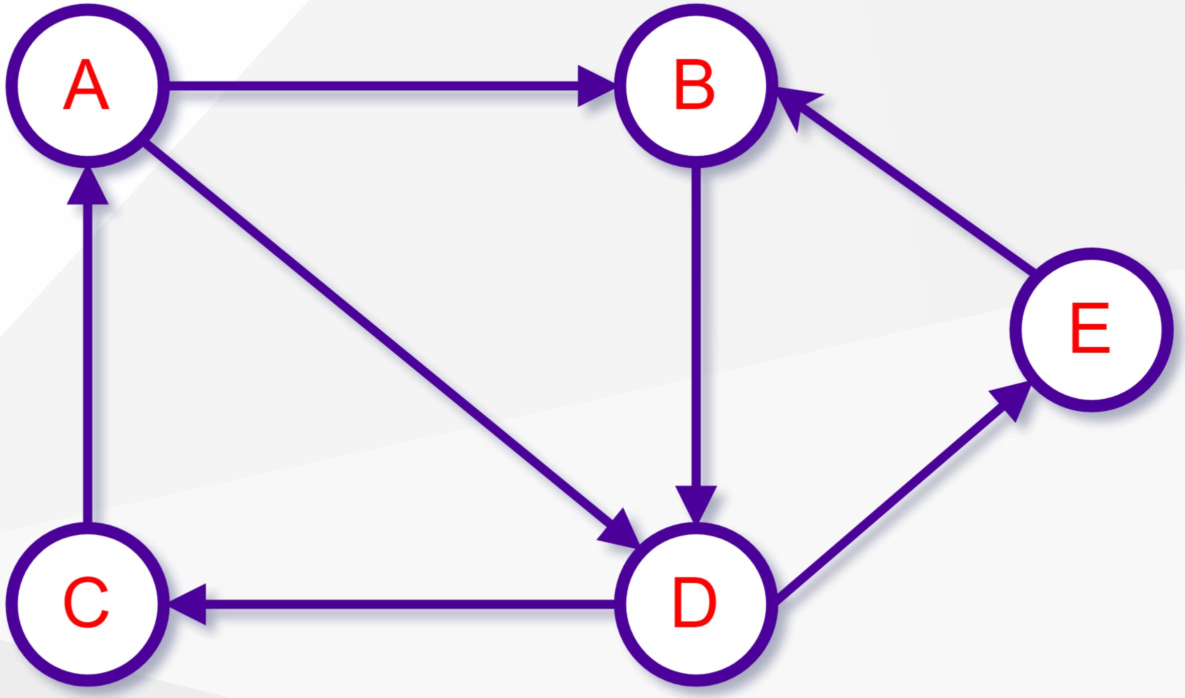
- A graph with only undirected edges is said to be undirected graph.



Graph Terminology

Directed Graph

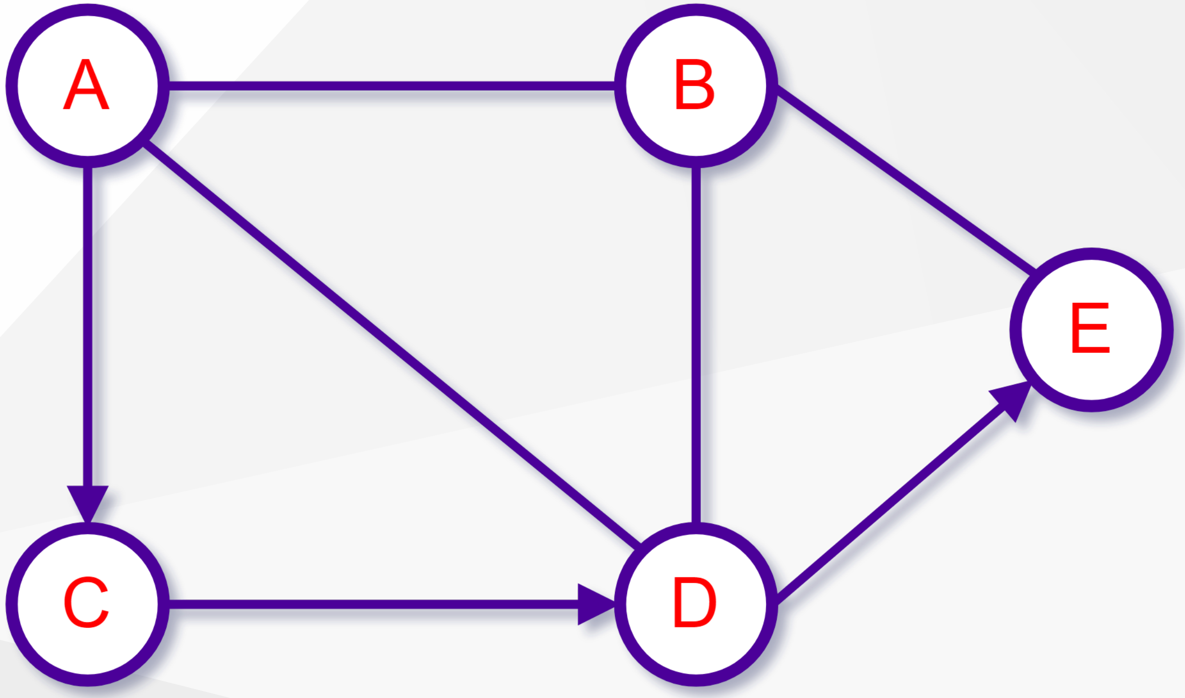
- A graph with only directed edges is said to be directed graph.



Graph Terminology

Mixed Graph

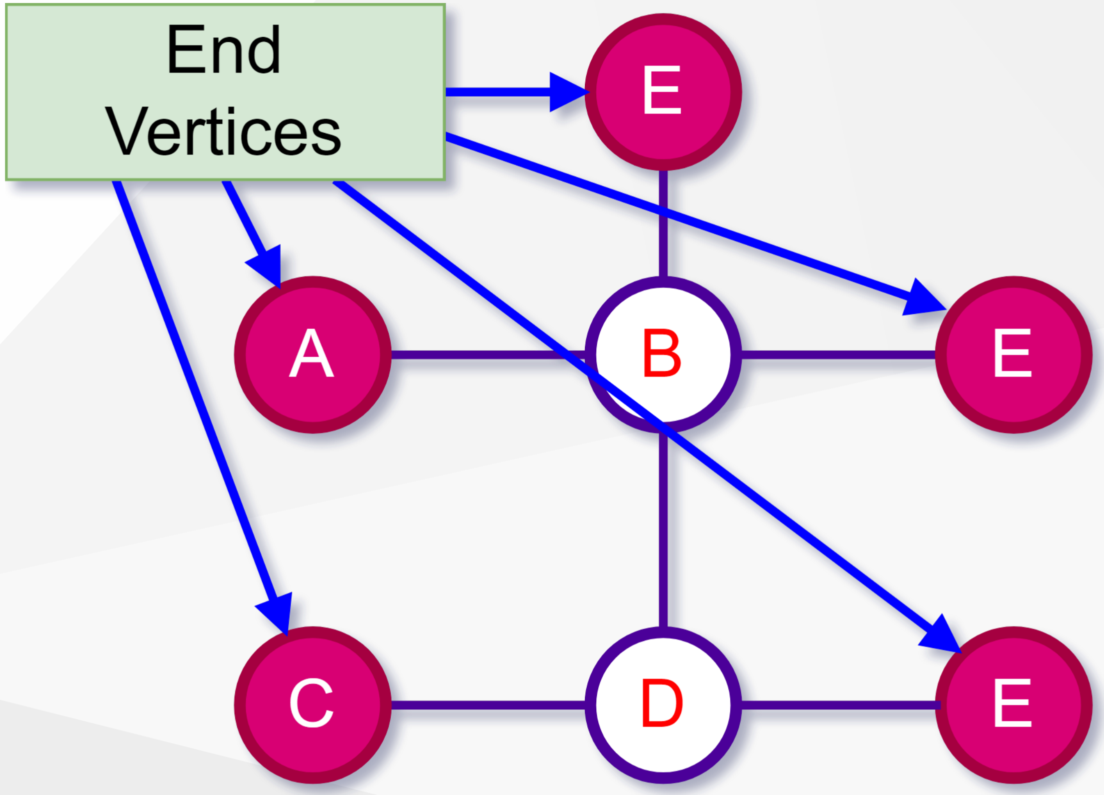
- A graph with both undirected and directed edges is said to be mixed graph.



Graph Terminology

End vertices or Endpoints

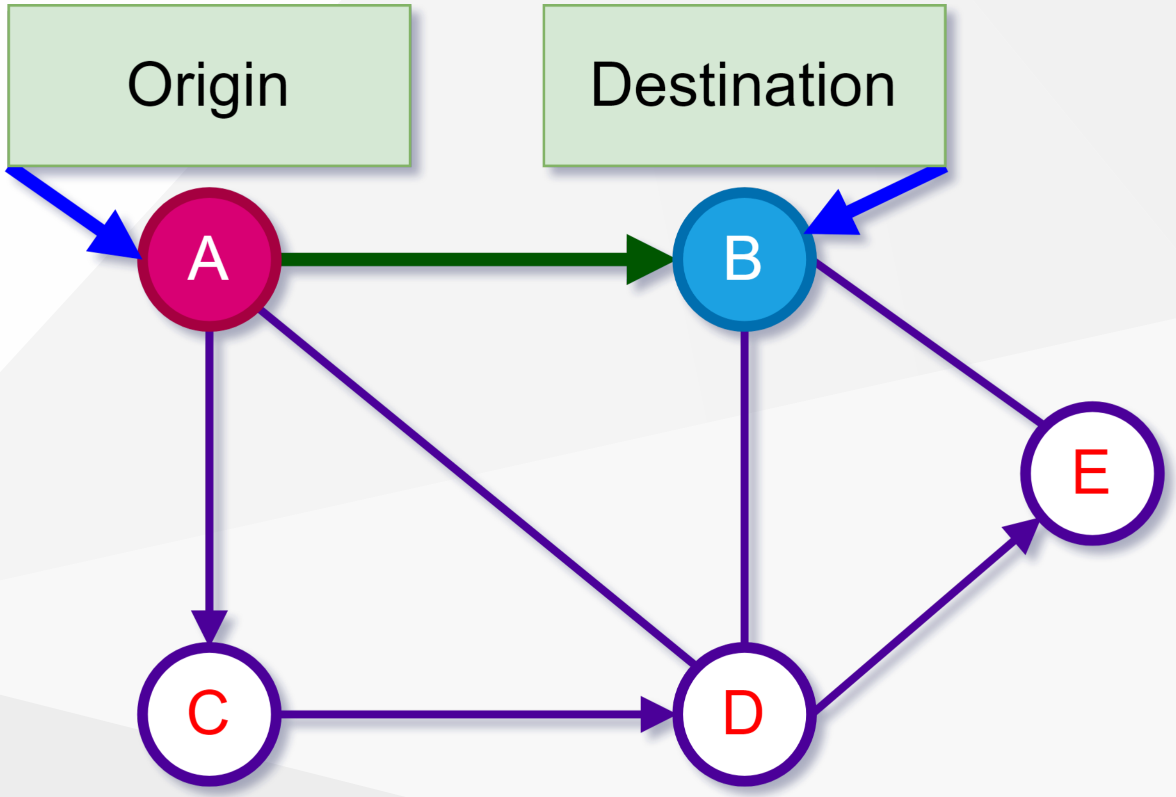
- The two vertices joined by edge are called end vertices (or endpoints) of that edge.
- In graph theory, a vertex with degree 1 is called an end vertex (plural end vertices)



Graph Terminology

Origin

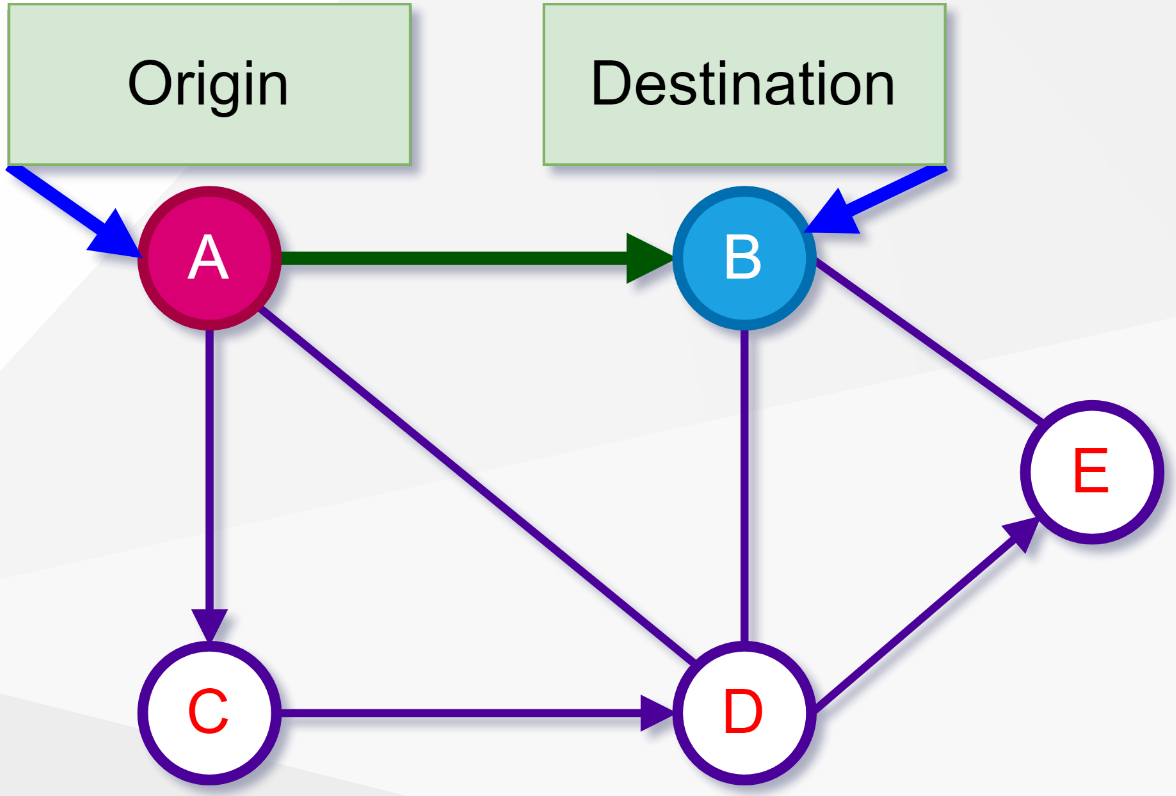
- If a edge is directed, its first endpoint is said to be the origin of it.



Graph Terminology

Destination

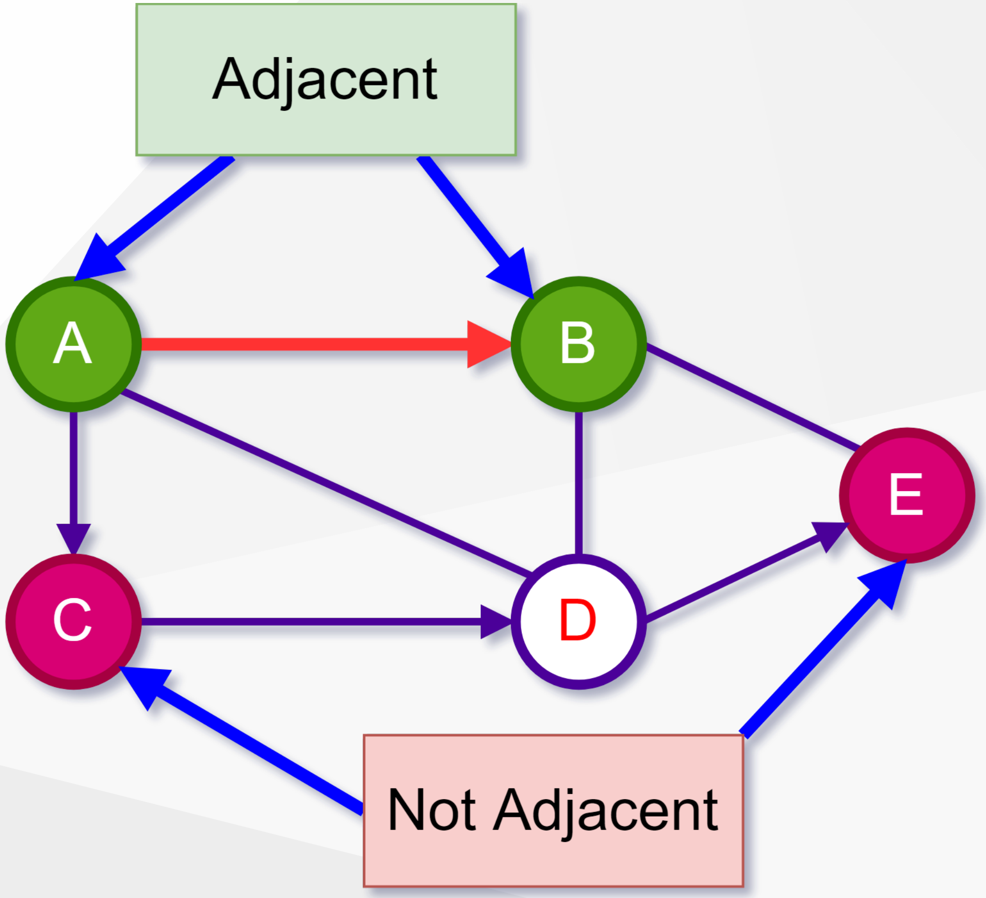
- If a edge is directed, its first endpoint is said to be the origin of it and the other endpoint is said to be the destination of that edge.



Graph Terminology

Adjacent

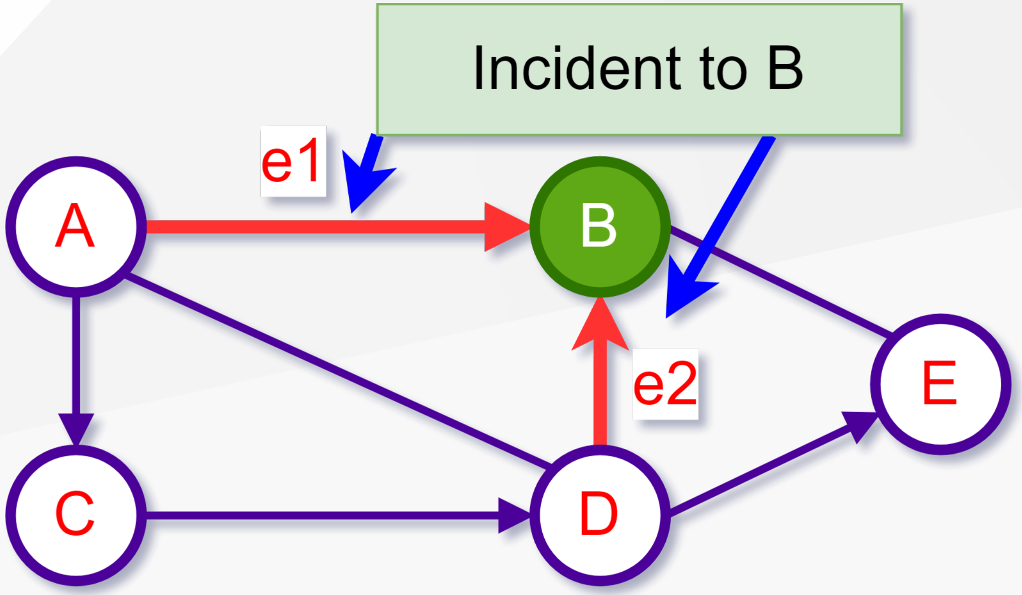
- If there is an edge between vertices A and B then both A and B are said to be adjacent. In other words, vertices A and B are said to be adjacent if there is an edge between them.



Graph Terminology

Incident

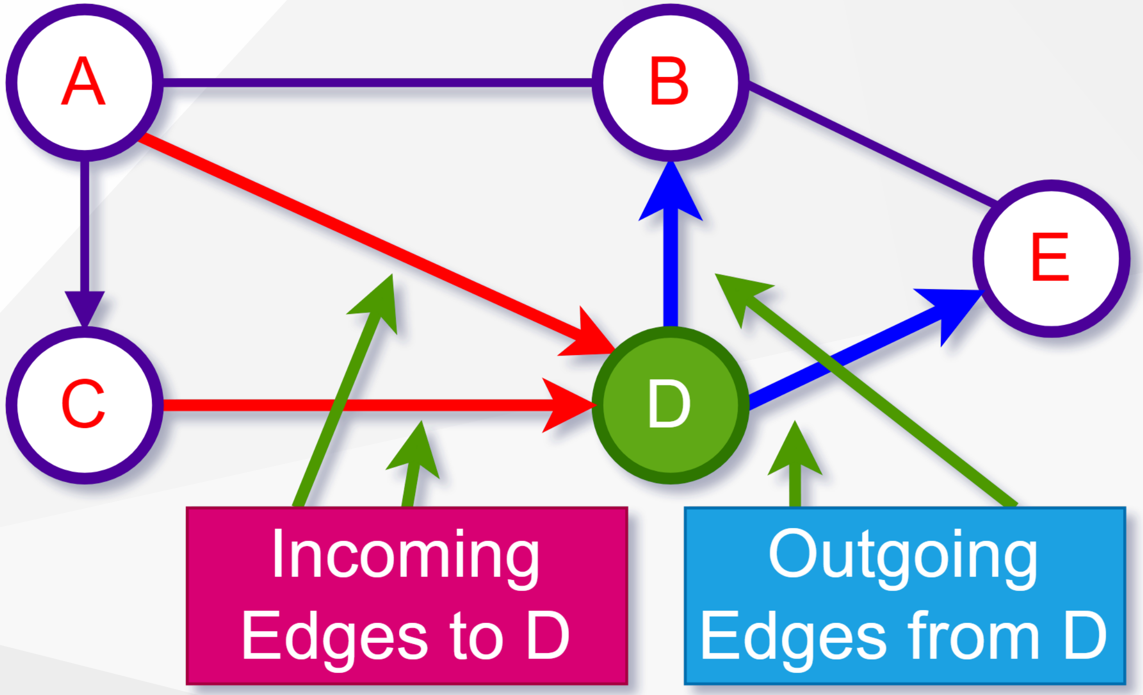
- **Edge/Arc** is said to be **incident** on a **Vertex/Node** if the **Vertex/Node** is one of the endpoints of that **Edge/Arc**.
- An incidence is a pair $(B, e1)$ where B is a vertex and $e1$ is an edge incident to B
- Two distinct incidences $(B, e1)$ and $(v, e2)$ are adjacent if and only if $B = v$, $e1 = e2$ or $BB' = e1$ or $e2$.



Graph Terminology

Outgoing Edge

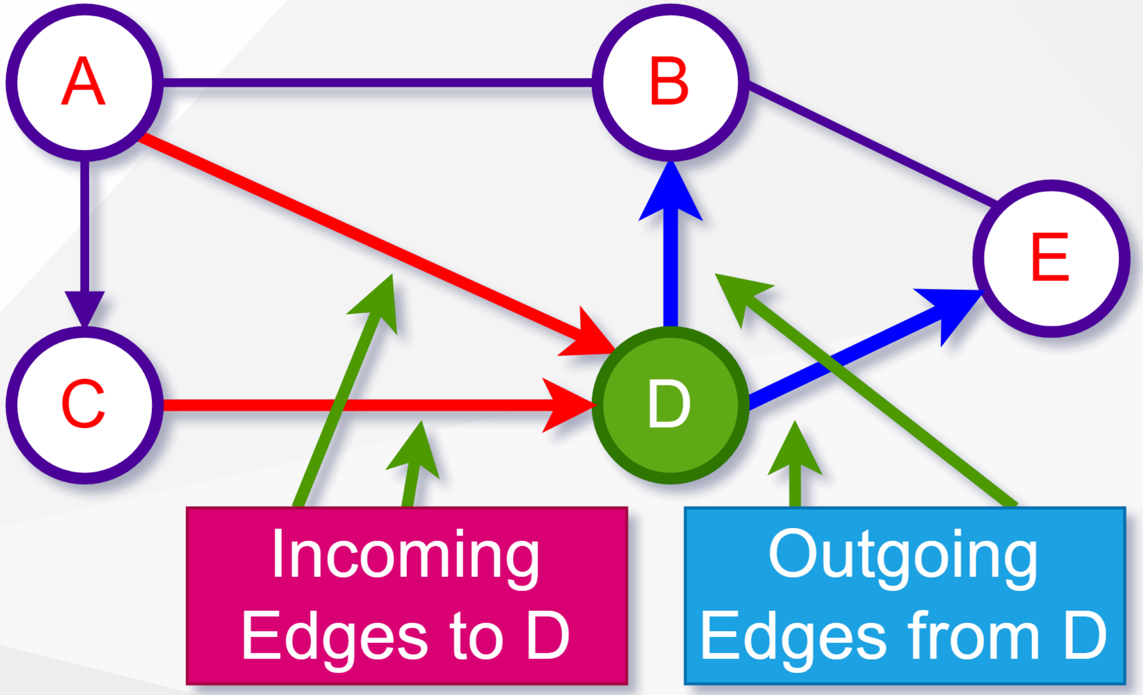
- A directed edge is said to be outgoing edge on its origin vertex.



Graph Terminology

Incoming Edge

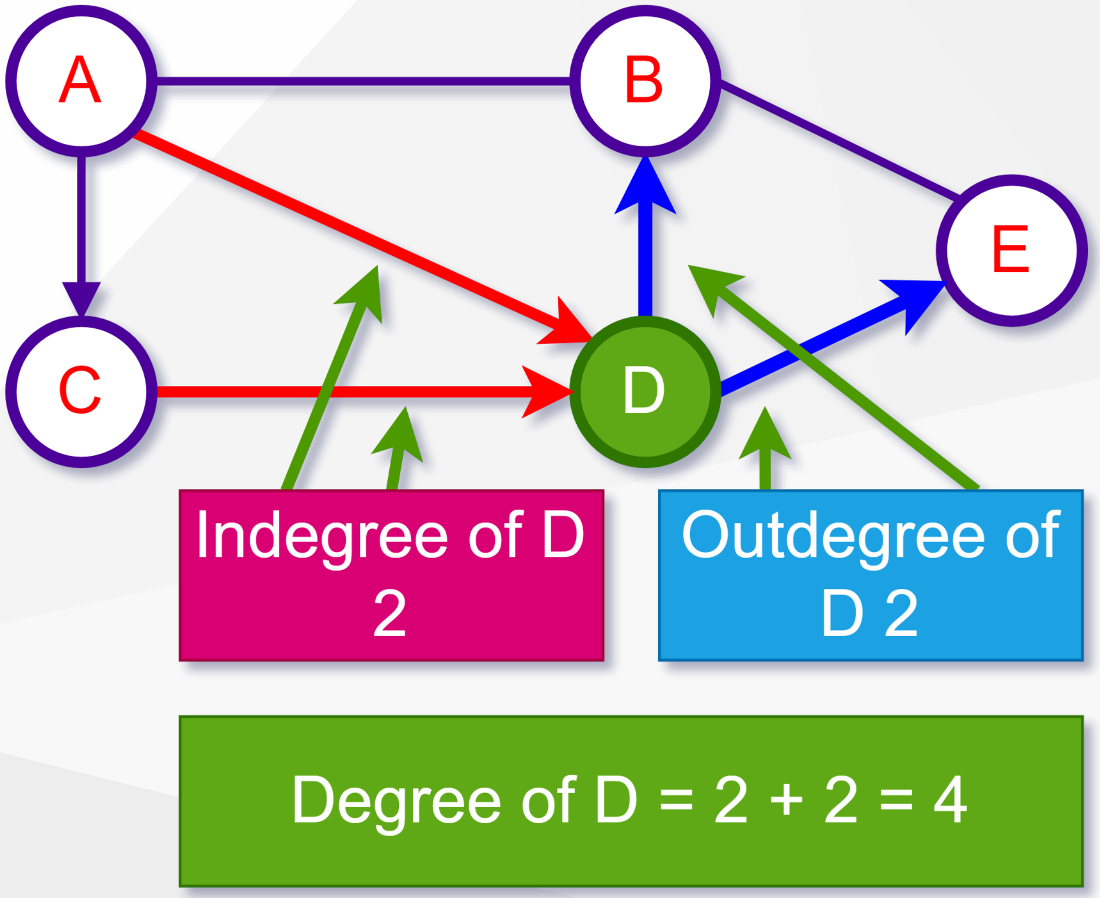
- A directed edge is said to be incoming edge on its destination vertex.



Graph Terminology

Degree

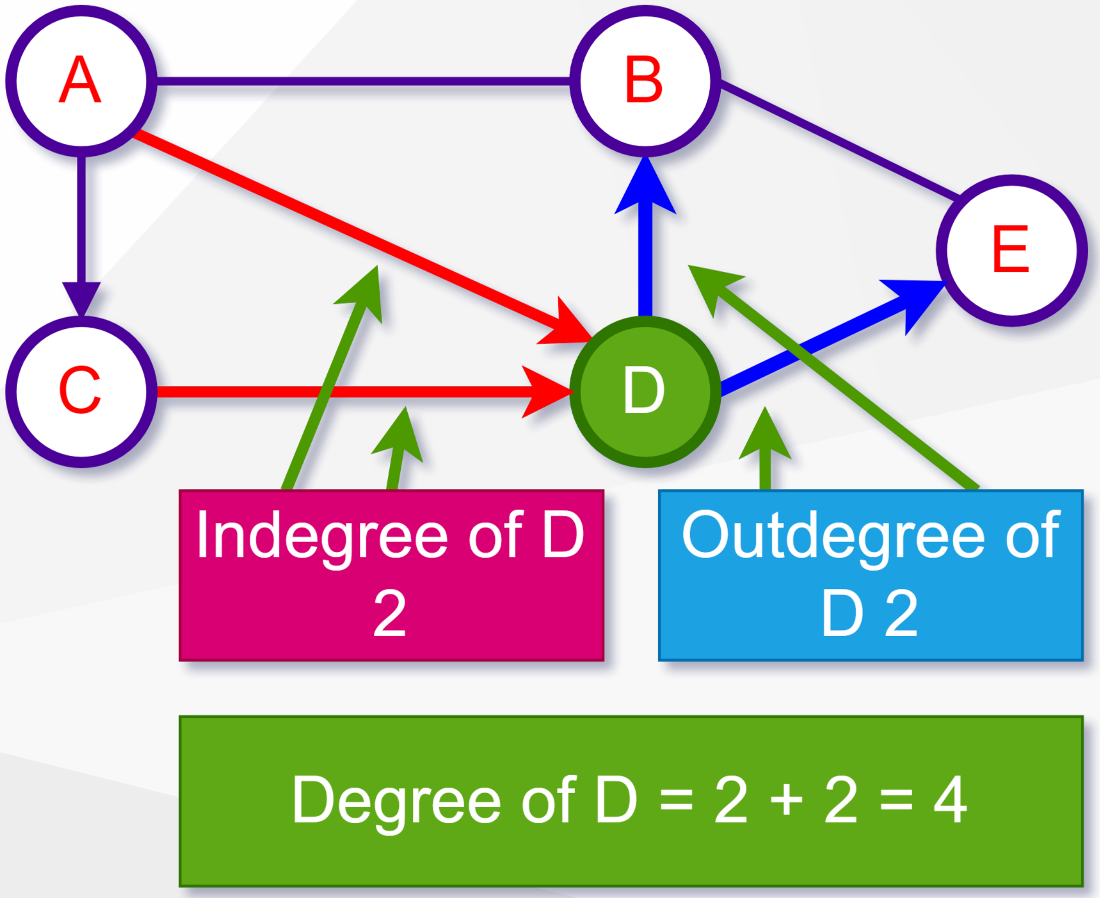
- Total number of edges connected to a vertex is said to be degree of that vertex.



Graph Terminology

Indegree

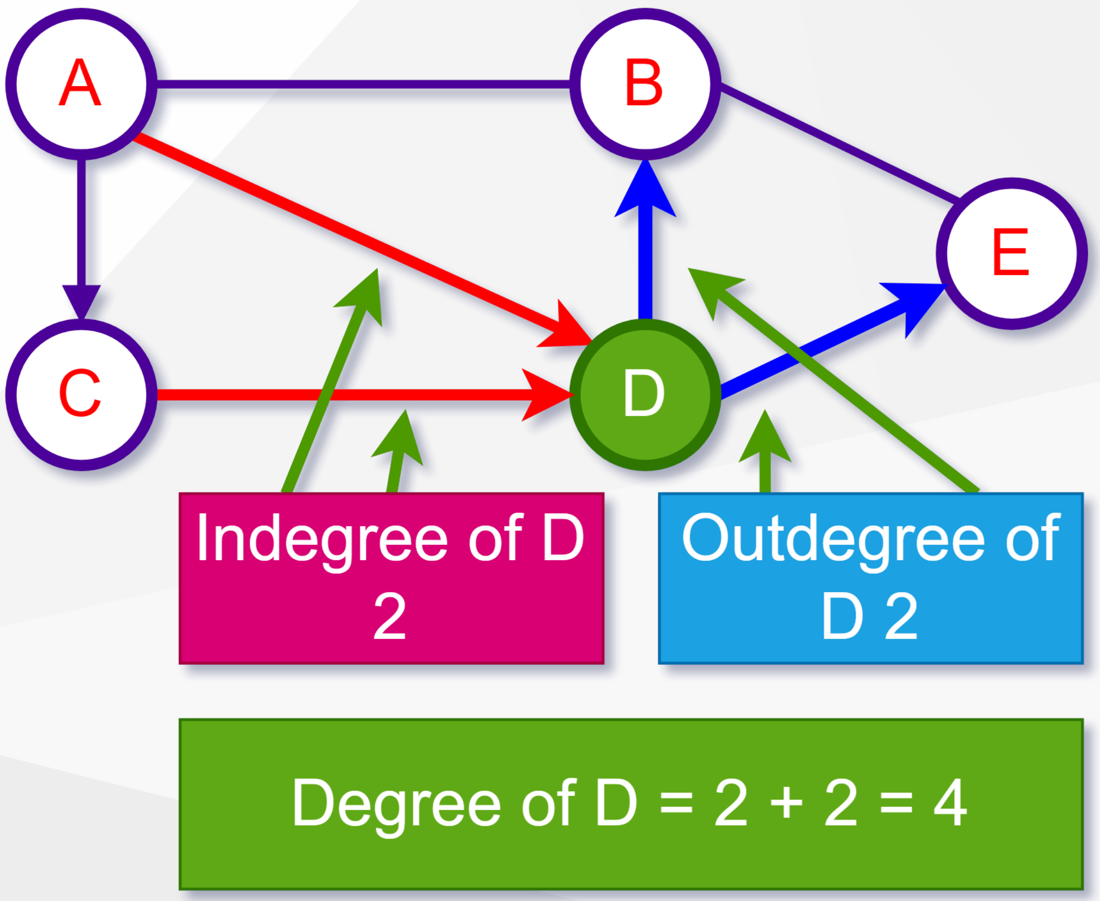
- Total number of incoming edges connected to a vertex is said to be indegree of that vertex.



Graph Terminology

Outdegree

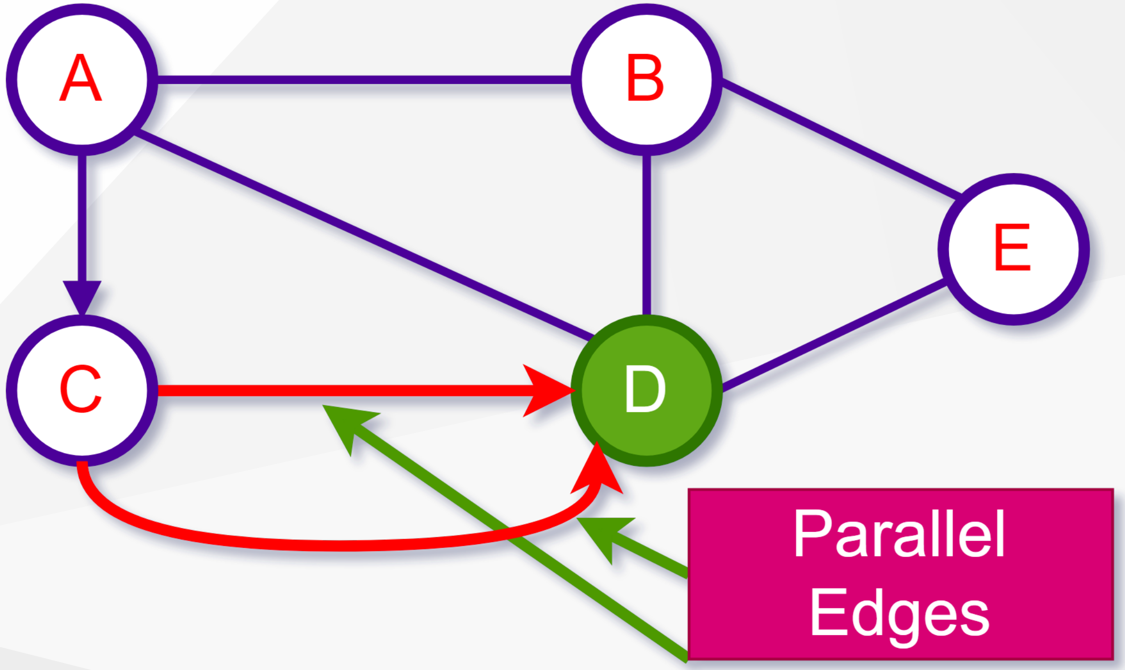
- Total number of outgoing edges connected to a vertex is said to be outdegree of that vertex.



Graph Terminology

Parallel edges or Multiple edges

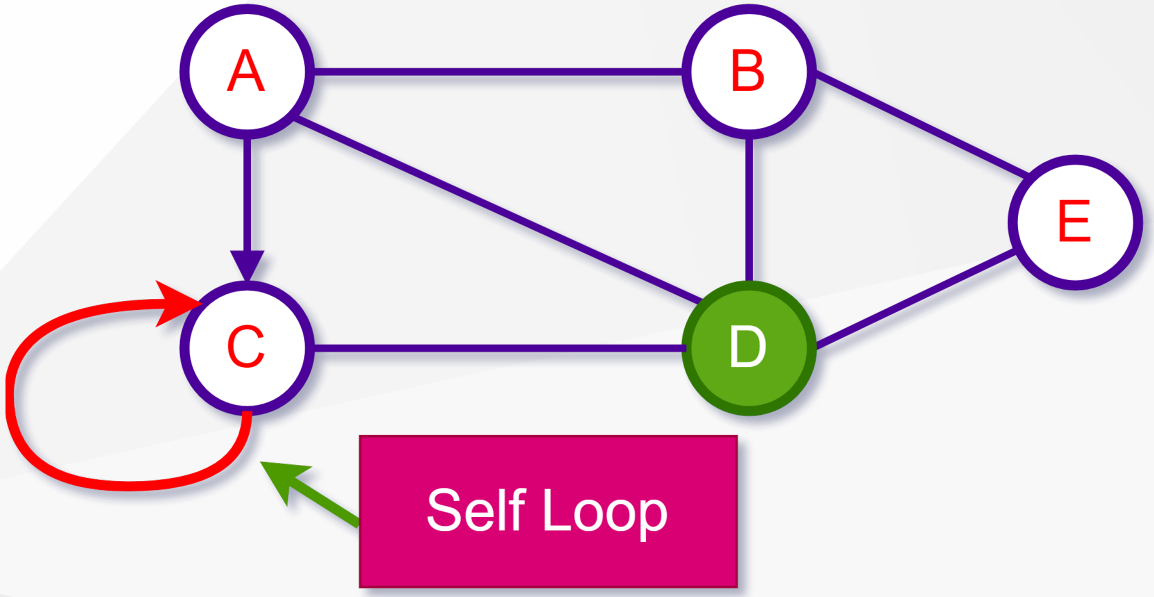
- If there are two undirected edges with same end vertices and two directed edges with same origin and destination, such edges are called parallel edges or multiple edges.



Graph Terminology

Self-loop

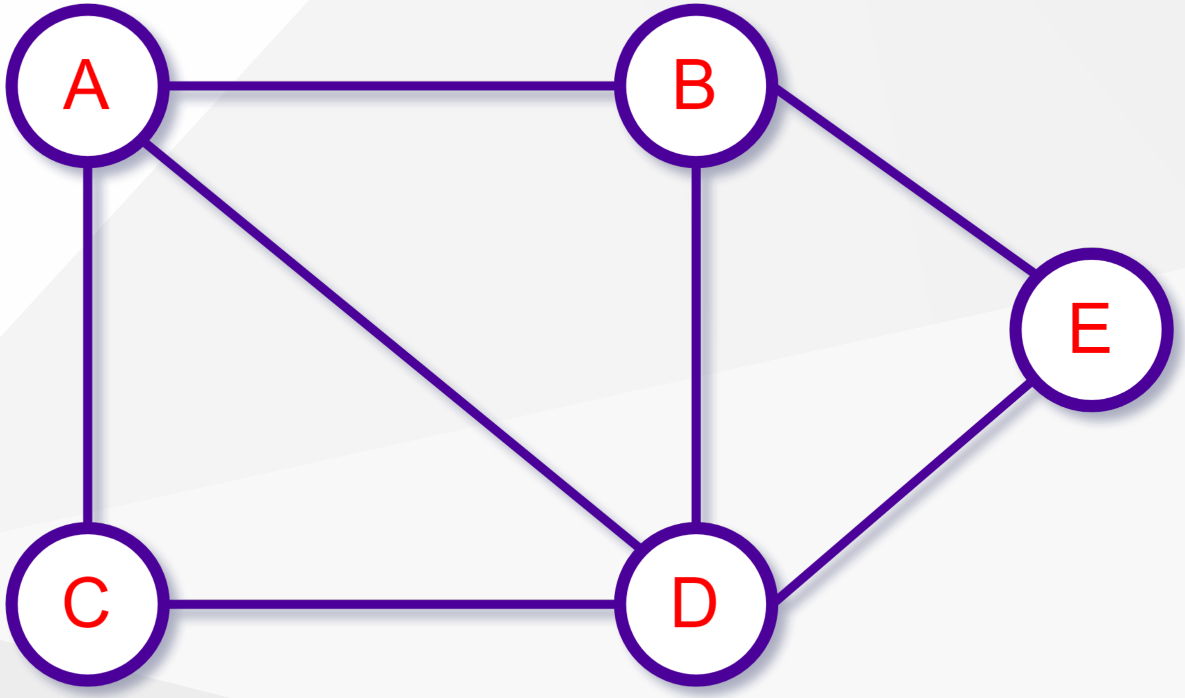
- Edge (undirected or directed) is a self-loop if its two endpoints coincide with each other.



Graph Terminology

Simple Graph

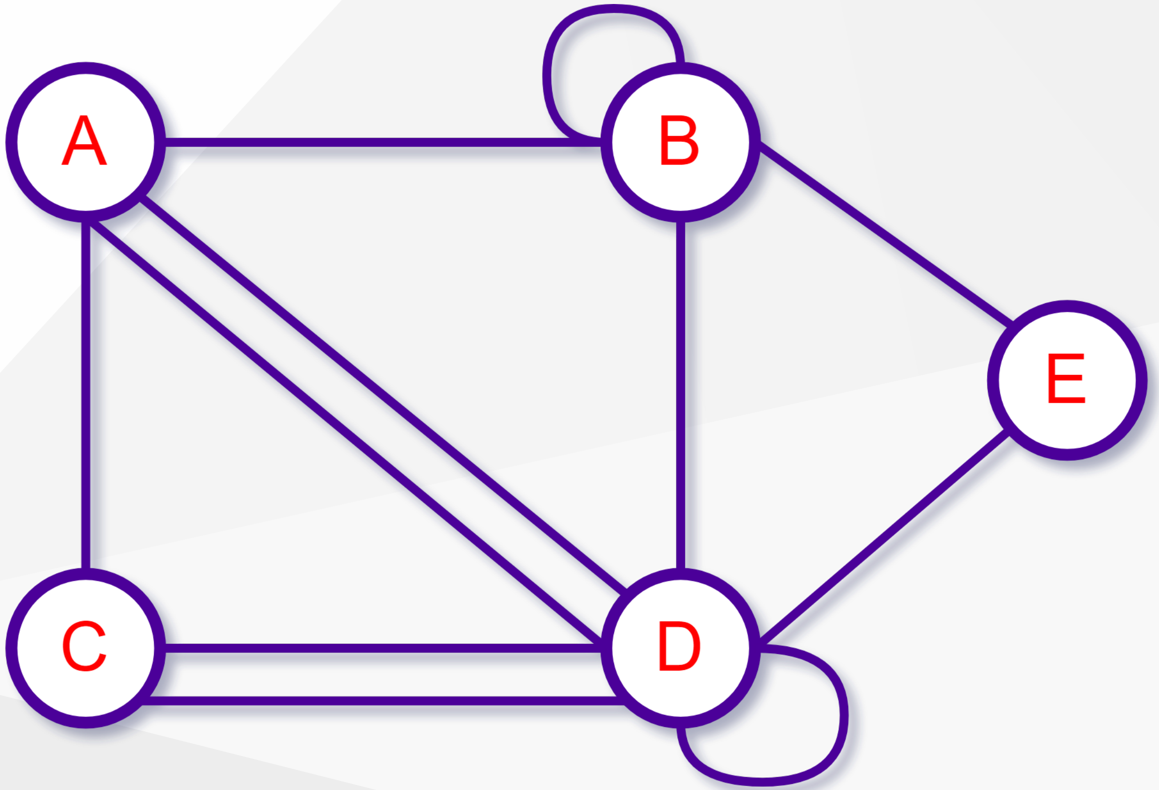
- A graph is said to be simple if there are no parallel and self-loop edges.



Graph Terminology

Complex Graph

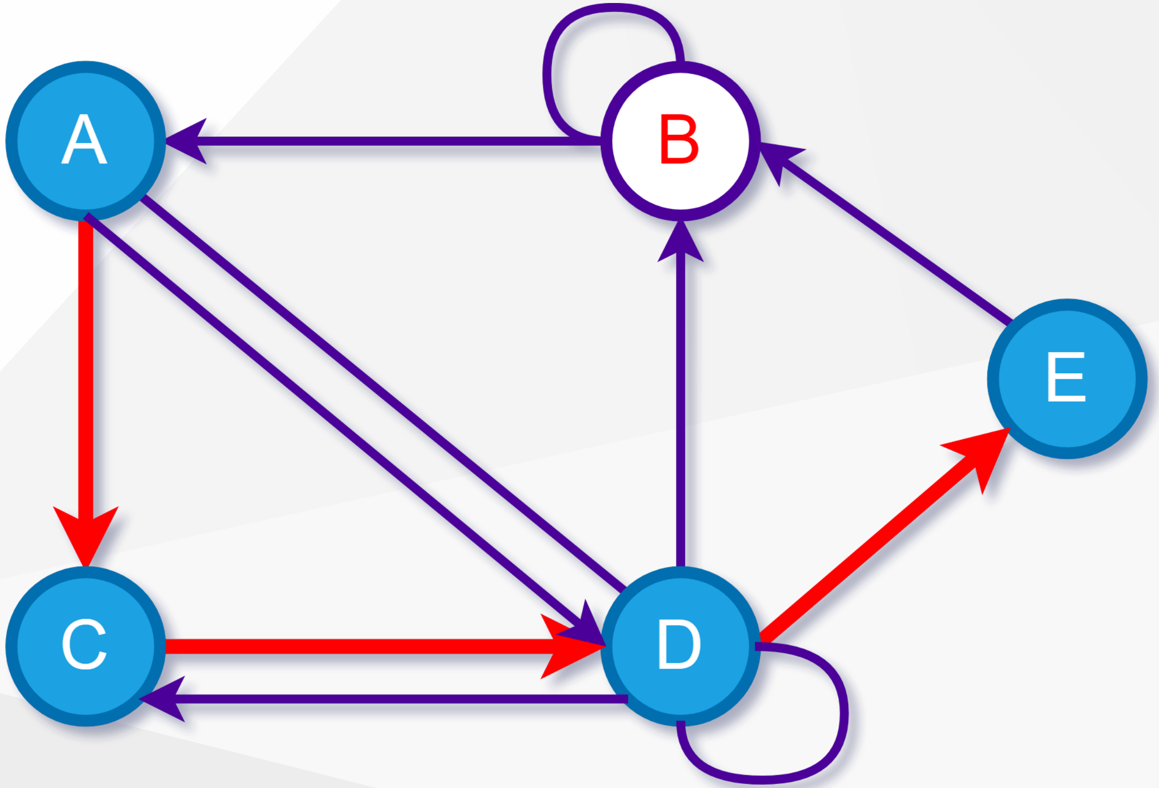
- A graph is said to be complex if there are parallel or self-loop edges.



Graph Terminology

Path

- A path is a sequence of alternate vertices and edges that starts at a vertex and ends at other vertex such that each edge is incident to its predecessor and successor vertex.



Graph Representations

Graph Representations

- Graph data structure is represented using following representations
  - **Adjacency Matrix**
  - **Incidence Matrix**
  - **Adjacency List**

Graph Representations

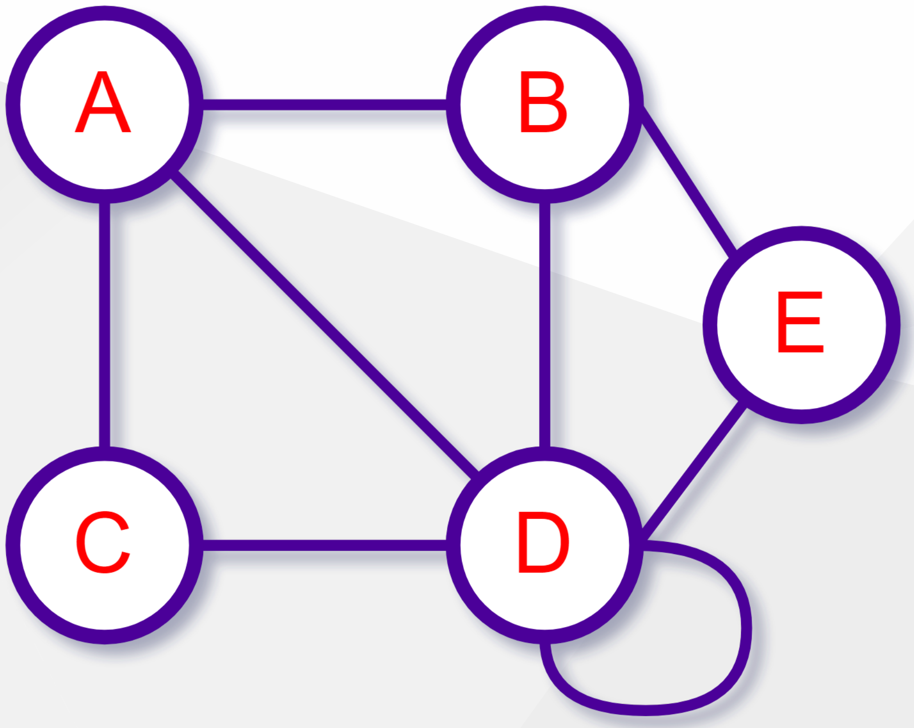
Adjacency Matrix

- In this representation, the graph is represented using a matrix of size total number of vertices by a total number of vertices.
- That means a graph with 4 vertices is represented using a matrix of size 4X4.
- In this matrix, both rows and columns represent vertices.
  - This matrix is filled with either 1 or 0.
  - Here,
    - 1 represents that there is an edge from row vertex to column vertex and
    - 0 represents that there is no edge from row vertex to column vertex.

Graph Representations

Adjacency Matrix

- Undirected Graph

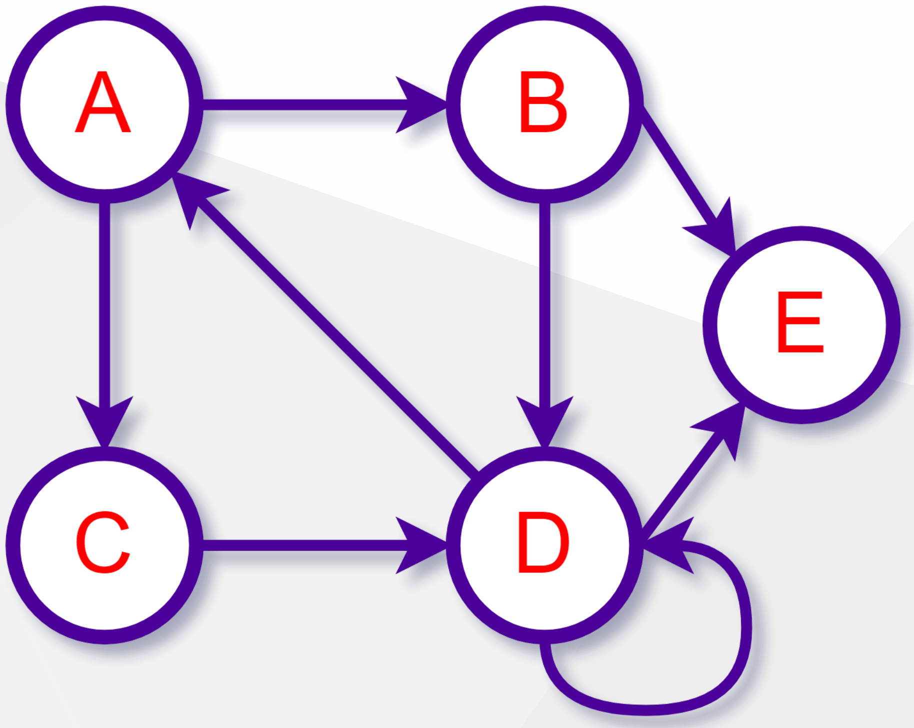


	A	B	C	D	E
A	0	1	1	1	0
B	1	0	0	1	1
C	1	0	0	1	0
D	1	1	1	1	1
E	0	1	0	1	0

Graph Representations

Adjacency Matrix

- Directed Graph



	A	B	C	D	E
A	0	1	1	0	0
B	0	0	0	1	1
C	0	0	0	1	0
D	1	0	0	1	1
E	0	0	0	0	0

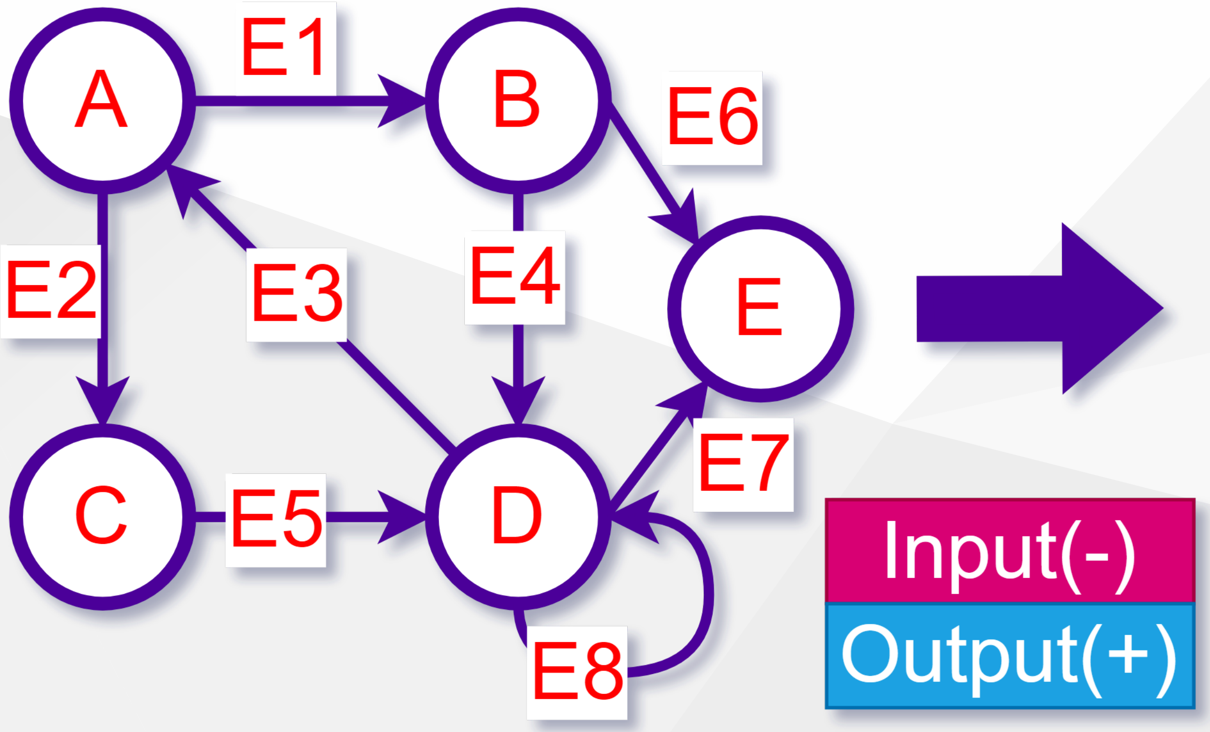
Graph Representations

Incidence Matrix

- In this representation, the graph is represented using a matrix of size total number of vertices by a total number of edges.
- That means graph with 4 vertices and 6 edges is represented using a matrix of size 4X6.
- In this matrix, rows represent vertices and columns represents edges.
- This matrix is filled with 0 or 1 or -1.
  - Here,
    - 0 represents that the row edge is not connected to column vertex,
    - 1 represents that the row edge is connected as the outgoing edge to column vertex and
    - -1 represents that the row edge is connected as the incoming edge to column vertex.

Graph Representations

Incidence Matrix



	E1	E2	E3	E4	E5	E6	E7	E8
A	1	1	-1	0	0	0	0	0
B	-1	0	0	1	0	1	0	0
C	0	-1	0	0	1	0	0	0
D	0	0	1	-1	-1	0	1	1
E	0	0	0	0	0	-1	-1	0

Graph Representations

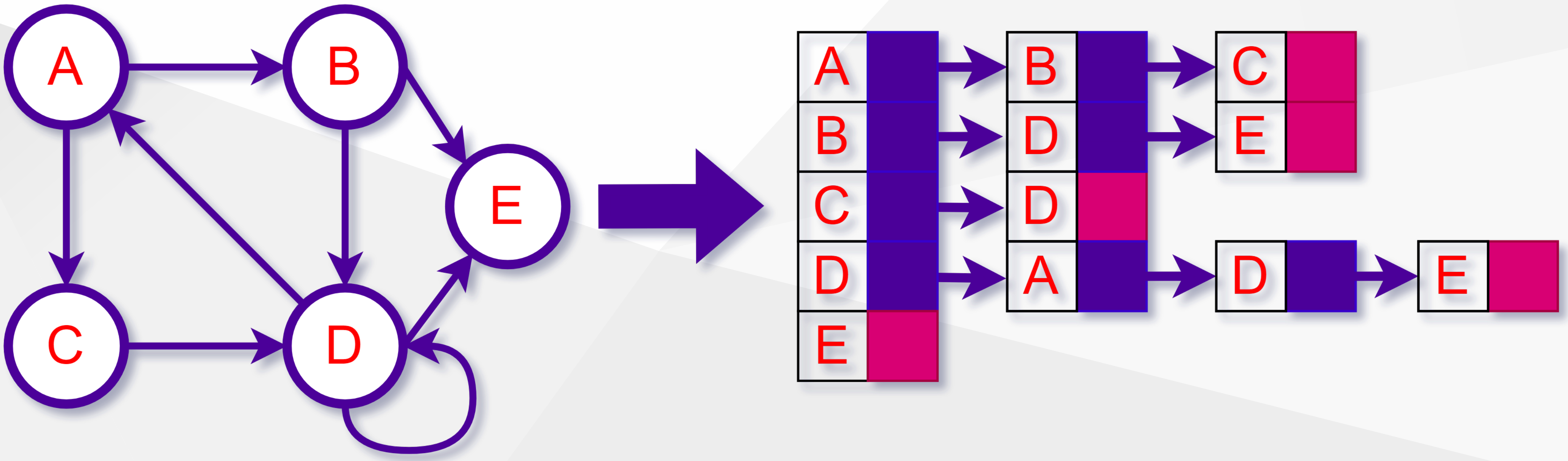
Adjacency List

- In this representation, every vertex of a graph contains list of its adjacent vertices.

Graph Representations

Adjacency List

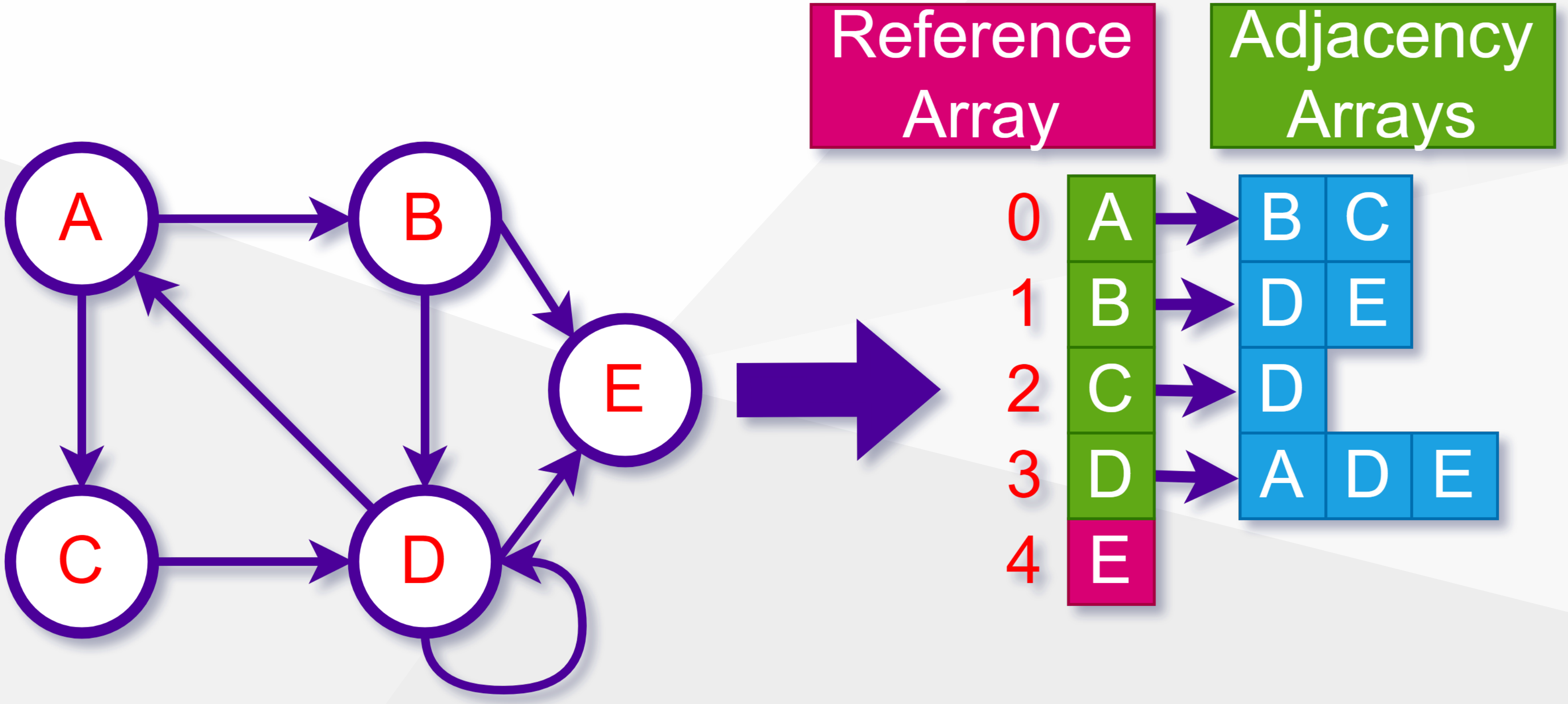
- Linked List Implementation



Graph Representations

Adjacency List

- Reference Array Implementation



Introduction to Graphs - Review

$G = (V, E)$

- Adjency List Complexity $O(degree\ of\ u)$ $(u, v) \in E$
- Sparse Matrix $\rightarrow |E| < |V^2|$
- Dense Matrix $\rightarrow |E|\ close\ to\ |V^2|$
- Space Complexity $\Theta(|V| + |E|)$

Introduction to Graphs - Review

- Many definitions for directed and undirected graphs are the same although certain terms have slightly different meanings
- If $(u, v) \in E$ in a **directed graph** $G = (V, E)$, we say that (u, v) is **incident from** or **leaves** vertex u and is **incident to** or **enters** vertex v
- If $(u, v) \in E$ in an **undirected graph** $G = (V, E)$, we say that (u, v) is **incident on** vertices u and v
- If (u, v) is an edge in a graph $G = (V, E)$, we say that vertex v is **adjacent to** vertex u
- When the graph is **undirected**,the **adjacency relation** is symmetric
- When the graph is **directed**
  - the **adjacency relation** is **not necessarily symmetric**
  - if v is adjacent to u, we sometimes write $u \rightarrow v$

Introduction to Graphs - Review

- The **degree** of a vertex in an **undirected graph** is the number of edges **incident on** it
- In a directed graph,
  - **out-degree of a vertex:** number of edges **leaving** it
  - **in-degree of a vertex:** number of edges **entering** it
  - **degree of a vertex:** its **in-degree** + its **out-degree**
- A **path** of **length** k from a vertex u to a vertex u' in a graph $G = (V, E)$ is a **sequence** $\langle v_0, v_1, v_2, \dots, v_k \rangle$ of vertices such
  - that $v_0 = u$,$v_k = u'$ and $(v_{i-1}, v_i) \in E$, for $i = 1, 2, \dots, k$
- The **length** of a **path** is the **number of edges** in the path

Introduction to Graphs - Review

- If there is a path p from u to u', we say that u' is **reachable** from u via $p : u \xrightarrow{p} u'$
- A **path is simple** if all vertices in the path are **distinct**
- A **subpath** of path $p = \langle v_0, v_1, v_2, \ldots, v_k \rangle$ is a **contiguous subsequence** of its vertices
- That is, for any $0 \leq i \leq j \leq k$, the subsequence of vertices $\langle v_i, v_{i+1}, \ldots, v_j \rangle$ is a **subpath** of p
- In a **directed graph**, a path $\langle v_0, v_1, \ldots, v_k \rangle$ forms a **cycle** if $v_0 = v_k$ and the path contains at least one edge
- The **cycle** is **simple** if, in addition, $v_0, v_1, \ldots, v_k$ are **distinct**
- A **self-loop** is a **cycle** of **length 1**

Introduction to Graphs - Review

Elementary Graph Algorithms

Elementary Graph Algorithms

- Graph Traversal
  - [DONE] Breadth-first search (BFS)
  - [DONE] Depth-first search (DFS)

Elementary Graph Algorithms

- Topological sort
  - [DONE] DFS version
  - [DONE] BFS version (Kahn's algorithm)

Elementary Graph Algorithms

- Bipartite Graph Check
  - [TBD] BFS version
  - [TBD] DFS version

Elementary Graph Algorithms

- Cut Vertex & Bridge
    - [TBD] DFS version

Elementary Graph Algorithms

- Strongly connected components (SCC)
  - [DONE] Kosaraju's algorithm
  - [DONE] Tarjan's algorithm

Elementary Graph Algorithms

- [TBD] 2-SAT Checker

Elementary Graph Algorithms

- Single-source shortest path (SSSP)
  - [TBD] Dijkstra's algorithm
  - [TBD] Bellman-Ford algorithm
  - [TBD] Johnson's algorithm
  - [TBD] DFS
  - [TBD] BFS
  - [TBD] DP

Elementary Graph Algorithms

- Minimum spanning tree
    - [TBD] Kruskal's algorithm
    - [TBD] Prim's algorithm

Elementary Graph Algorithms

- Convex Hull
    - [TBD] Graham Scan
    - [TBD] Jarvis march

Online Visual Animations

Online Visual Animations

- Graph Structures
- https://visualgo.net/en/graphds?slide=1
- Single-Source Shortest Paths (SSSP)
  - https://visualgo.net/en/sssp?slide=1
- Minimum Spanning Tree (MST)
  - https://visualgo.net/en/mst?slide=1
- Convex Hull
  - https://visualgo.net/en/convexhull?slide=1

Online Visual Animations

- Data Structure Visualizations (University of Sout Florida-USF)
  - https://www.cs.usfca.edu/~galles/visualization/Algorithms.html

Online Visual Animations

- Common Graph Algorithms
  - https://algorithm-visualizer.org/

Graph Tools

Graph Tools

- Graphviz Tools
    - https://graphviz.org/download/
- Graphviz (short for Graph Visualization Software) is a package of open-source tools initiated by AT&T Labs Research for drawing graphs specified in DOT language scripts having the file name extension "gv". It also provides libraries for software applications to use the tools. Graphviz is free software licensed under the Eclipse Public License.

Graph Tools

- Graphviz Tools
  - https://graphviz.org/download/
  - https://graphviz.org/doc/info/command.html
  - https://graphviz.org/docs/outputs/svg/
  - http://magjac.com/graphviz-visual-editor/
  - https://graphs.grevian.org/graph
- Graphviz Tutorials
  - https://graphs.grevian.org/example#example-1
  - https://graphs.grevian.org/reference
  -

Graphviz Gallery

Family Tree

- https://graphviz.org/Gallery/directed/kennedyanc.html

UML

- https://graphviz.org/Gallery/directed/UML_Class_diagram.html

Data Structure

- https://graphviz.org/Gallery/gradient/datastruct.html
- https://graphviz.org/Gallery/directed/datastruct.html

Graphviz Gallery

Neural Network (Keras)

- https://graphviz.org/Gallery/directed/neural-network.html

Linux Kernel Diagram

- https://graphviz.org/Gallery/directed/Linux_kernel_diagram.html

Graphviz Tools and Binaries

- Graphviz consists of a graph description language named the DOT language[4] and a set of tools that can generate and/or process DOT files:

Graphviz Layout Engines

dot

a command-line tool to produce layered drawings of directed graphs in a variety of output formats, such as (PostScript, PDF, SVG, annotated text and so on).

Visit: https://graphviz.org/docs/layouts/dot/

Graphviz Layout Engines

neato

useful for undirected graphs. "spring model" layout, minimizes global energy. Useful for graphs up to about 1000 nodes

Visit : https://graphviz.org/docs/layouts/neato/

Graphviz Layout Engines

fdp

useful for undirected graphs. "spring model" which minimizes forces instead of energy

Visit : https://graphviz.org/docs/layouts/fdp/

Graphviz Layout Engines

sfdp

multiscale version of fdp for the layout of large undirected graphs

Visit : https://graphviz.org/docs/layouts/sfdp/

Graphviz Layout Engines

twopi

for radial graph layouts. Nodes are placed on concentric circles depending their distance from a given root node

Visit : https://graphviz.org/docs/layouts/twopi/

Graphviz Layout Engines

circo

circular layout. Suitable for certain diagrams of multiple cyclic structures, such as certain telecommunications networks

Visit : https://graphviz.org/docs/layouts/circo/

Graphviz Layout Engines

osage

osage draws clustered graphs. Suitable for certain diagrams of multiple cyclic structures, such as certain telecommunications networks

Visit : https://graphviz.org/docs/layouts/osage/

Graphviz Layout Engines

patchwork

patchwork draws clustered graphs using a squarified treemap layout.

Visit : https://graphviz.org/docs/layouts/patchwork/

Graphviz Layout Engines

dotty (DEPRECATED)

a graphical user interface to visualize and edit graphs.

Graphviz Tools

lefty (DEPRECATED)

a programmable (in a language inspired by EZ[5]) widget that displays DOT graphs and allows the user to perform actions on them with the mouse. Therefore, Lefty can be used as the view in a model–view–controller GUI application that uses graphs.

Graphviz Tools

gml2gv - gv2gml

convert to/from GML, another graph file format.

Graphviz Tools

graphml2g

convert a GraphML file to the DOT format.

Graphviz Tools

gxl2gv - gv2gxl

convert to/from GXL, another graph file format.

Graphviz Tools

for more information visit

- https://graphviz.org/documentation/#tool-manual-pages

Graphviz API

- Visit
  - https://graphviz.org/documentation/#sample-programs-using-graphviz

Graph Tools

- Plantuml Tools (https://plantuml.com/download)
  - PlantUML is an open-source tool allowing users to create diagrams from a plain text language. Besides various UML diagrams, PlantUML has support for various other software development related formats (such as Archimate, Block diagram, BPMN, C4, Computer network diagram, ERD, Gantt chart, Mind map, and WBD), as well as visualisation of JSON and YAML files.

Graph Tools

- Plantuml Tutorials
  - Visit OOP Plantuml Course Notes

Graph Tools

- Plantuml Graphs and References
  - https://plantuml.com/use-case-diagram
  - https://plantuml.com/deployment-diagram
  - https://plantuml.com/component-diagram
  - https://plantuml.com/mindmap-diagram
  - https://plantuml.com/object-diagram
  - https://plantuml.com/state-diagram
  - https://plantuml.com/wbs-diagram
  - https://plantuml.com/json
  - https://plantuml.com/yaml

Graph Tools

- Plantuml API
  - https://plantuml.com/api

Graph Tools

- Microsoft Graph Layout
- MSAGL is a .NET tool for graph layout and viewing.
- It was developed in Microsoft by Lev Nachmanson, Sergey Pupyrev, Tim Dwyer and Ted Hart.
- MSAGL is available as open source.
  - Demo Project
    - https://github.com/ucoruh/microsoft-graph-layout-cs-demo
  - Library
    - https://github.com/microsoft/automatic-graph-layout
  - Website
    - https://www.microsoft.com/en-us/research/project/microsoft-automatic-graph-layout/

Elementary Graph Algorithms

Graph Traversal

Breadth-first search (BFS)

- Breadth-first search (BFS) is a graph traversal algorithm that starts at a vertex and explores as far as possible along each branch before backtracking.

Graph Traversal

Breadth-first search (BFS)

- In this algorithm, we use a queue to store the vertices that are yet to be visited.
- Complexity of following part is $O(V)$

```
G -> Graph
s -> Source
BFS(G,s)
  // Mark all the vertices as not visited
  for each vertex u in G.V - {s}
      u.color = white;
      u.distance = infinity;
      u.parent = NIL;
...
```

Graph Traversal

Breadth-first search (BFS)

- We enqueue the first vertex and mark it as visited.
- Complexity of following part is $O(1)$

```
...
  s.color = gray;
  s.distance = 0;
  s.parent = NIL;
  // Create a queue for BFS
  Q = empty
  ENQUEUE(Q, s)
...
```

Graph Traversal

Breadth-first search (BFS)

- We dequeue a vertex u and mark it as visited.
- We enqueue all the adjacent vertices of u.
- Complexity of following part is $O(E)$

```
...
  WHILE Q is not empty
      u = DEQUEUE(Q)
      for each vertex v in G.Adj[u]
          if v.color == white
              v.color = gray;
              v.distance = u.distance + 1;
              v.parent = u;
              ENQUEUE(Q, v)
      u.color = black;
```

Graph Traversal

Breadth-first search (BFS)

- Complexity of BFS is $O(V+E) = O(V) + O(E) + O(1)$

Graph Traversal

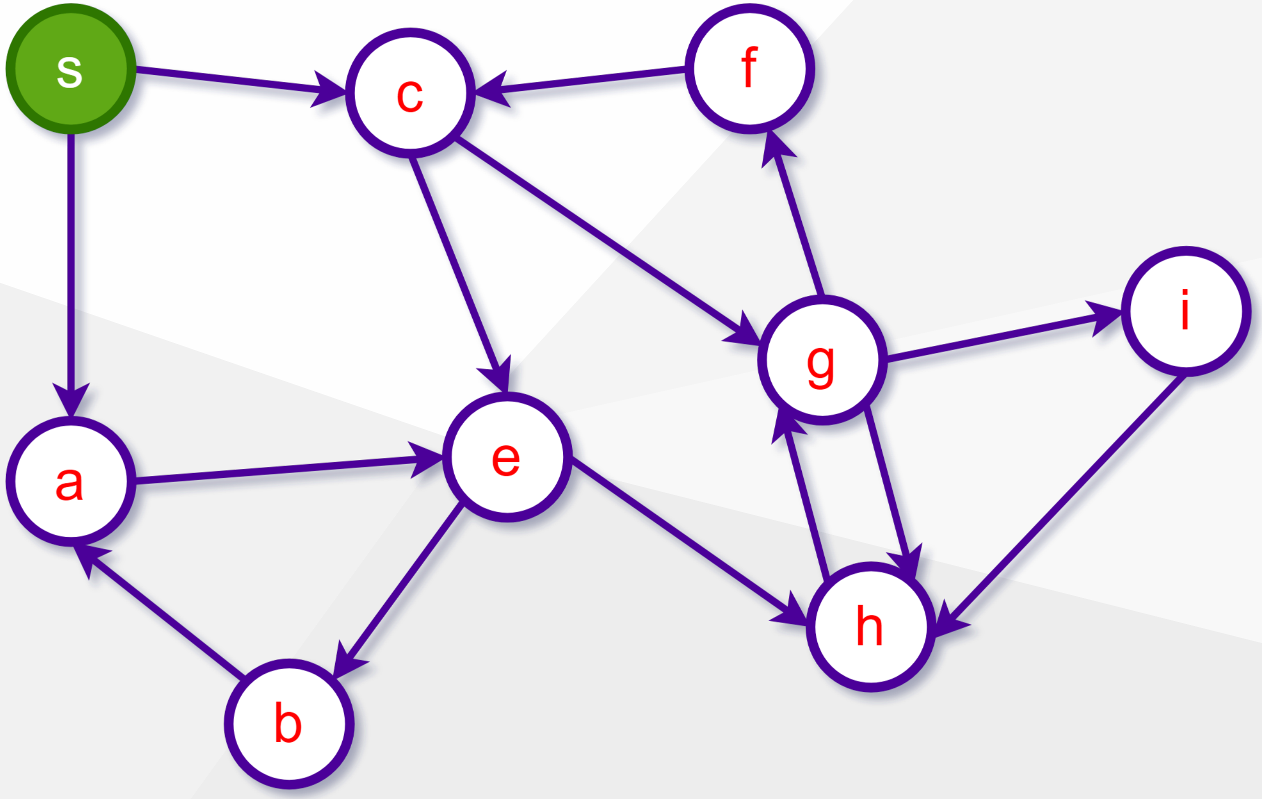
Breadth-first search (BFS) Complete Algorithm

```
G -> Graph
s -> Source
BFS(G,s)
    // Mark all the vertices as not visited
    for each vertex u in G.V - {s}
        u.color = white;
        u.distance = infinity;
        u.parent = NIL;
    s.color = gray;
    s.distance = 0;
    s.parent = NIL;
    // Create a queue for BFS
    Q = empty
    ENQUEUE(Q, s)
    WHILE Q is not empty
        u = DEQUEUE(Q)
        for each vertex v in G.Adj[u]
            if v.color == white
                v.color = gray;
                v.distance = u.distance + 1;
                v.parent = u;
                ENQUEUE(Q, v)
        u.color = black;
```

Graph Traversal

Breadth-first search (BFS) Example-1

- s is the source vertex.

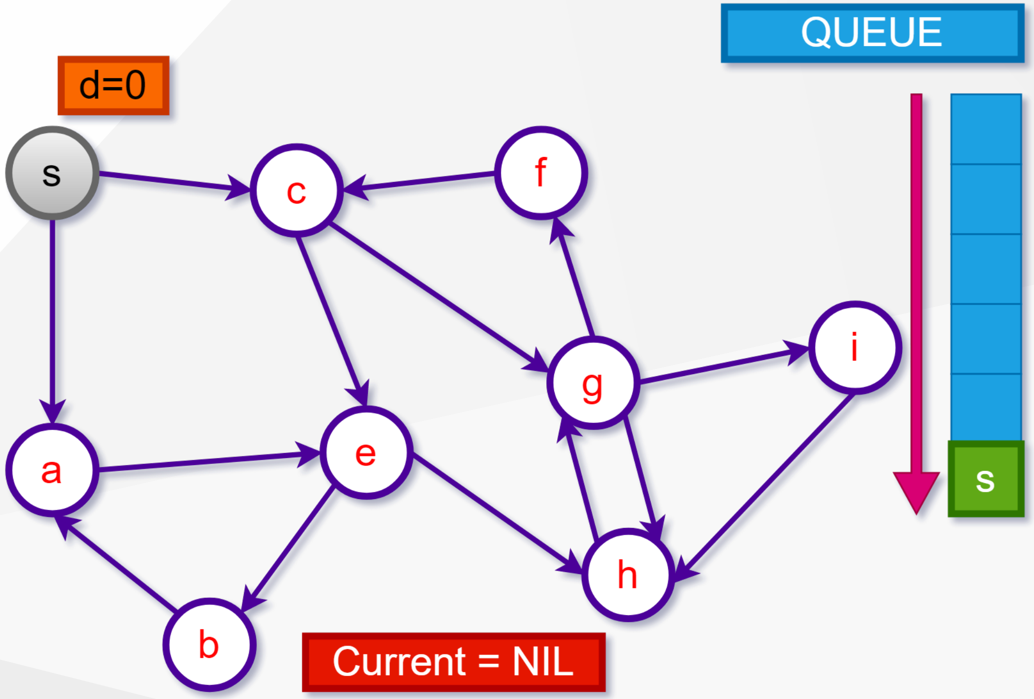


Graph Traversal

Breadth-first search (BFS) Example-1

- STEP-1

```
//init the graph
s.parent = NIL;
s.color = gray;
s.distance = 0;
Q = empty;
ENQUEUE(Q, s)
and
u = DEQUEUE(Q) in the while loop
```

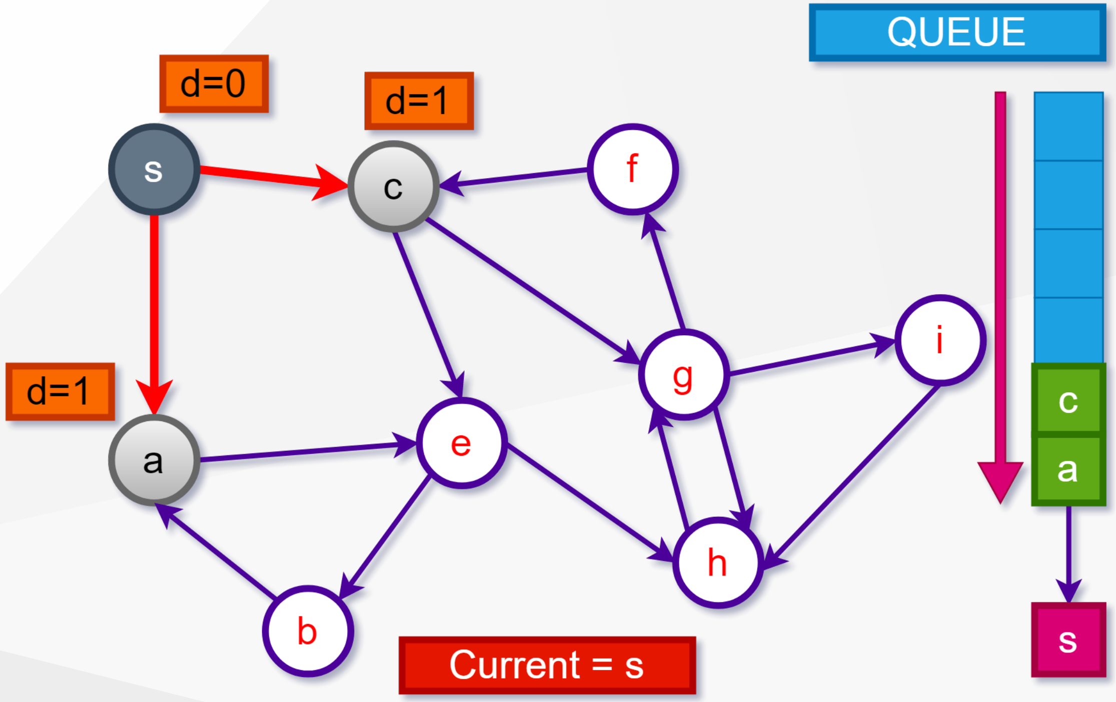


Graph Traversal

Breadth-first search (BFS) Example-1

- STEP-2

```
----------------
Q = {c,a}
s = b
----------------
c.parent = s
c.distance = 1
c.color = gray
----------------
a.parent = s
a.distance = 1
a.color = gray
----------------
```

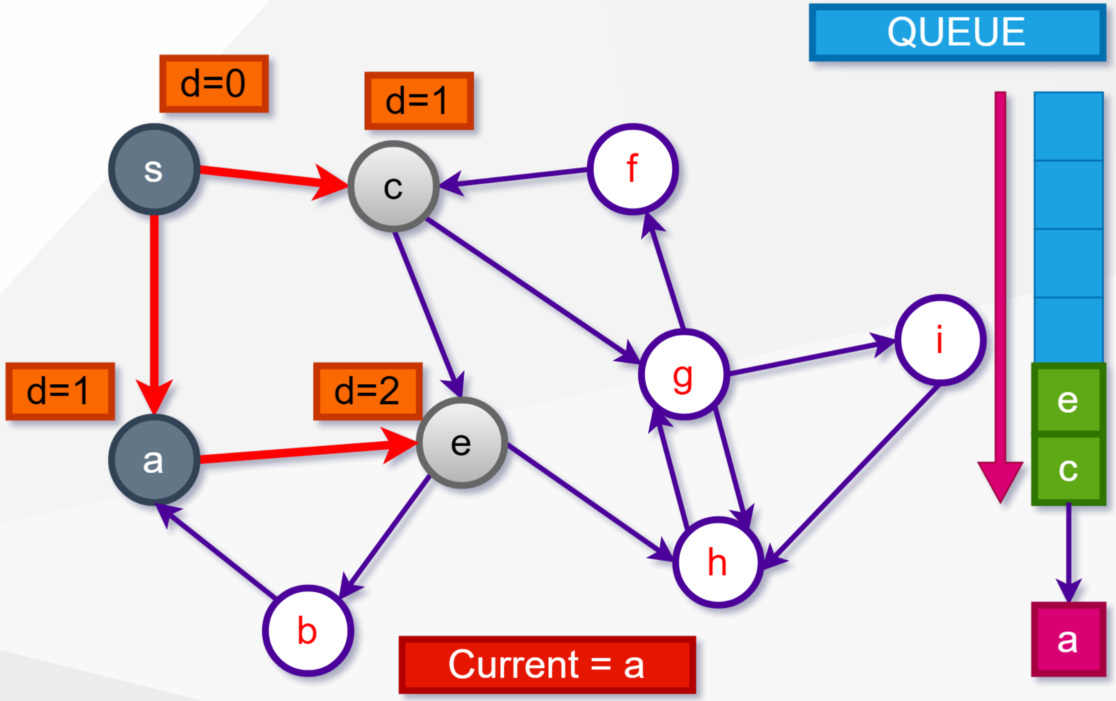


Graph Traversal

Breadth-first search (BFS) Example-1

- STEP-3

```
----------------
Q = {e,c}
a = b
----------------
e.parent = a
e.distance = 2
e.color = gray
----------------
```

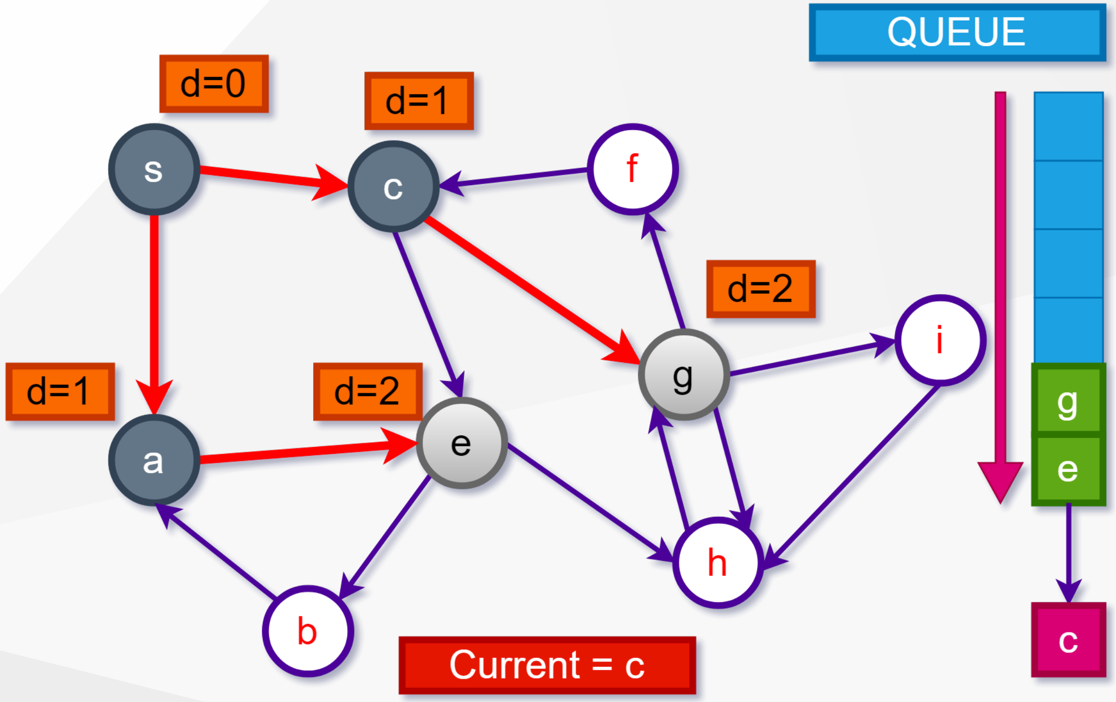


Graph Traversal

Breadth-first search (BFS) Example-1

- STEP-4

```
-----------------
Q = {g,e}
c = b
-----------------
g.parent = c
g.distance = 2
g.color = gray
-----------------
```

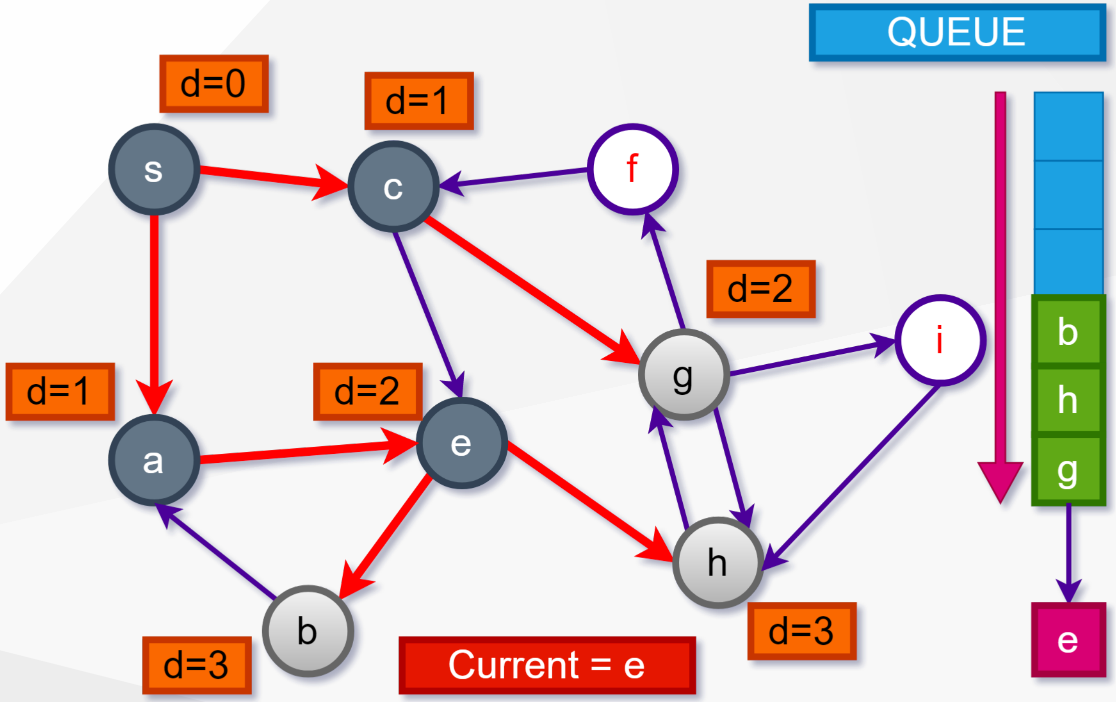


Graph Traversal

Breadth-first search (BFS) Example-1

- STEP-5

```
----------------
Q = {b,h,g}
e = b
----------------
h.parent = e
h.distance = 3
h.color = gray
----------------
b.parent = e
b.distance = 3
b.color = gray
----------------
```

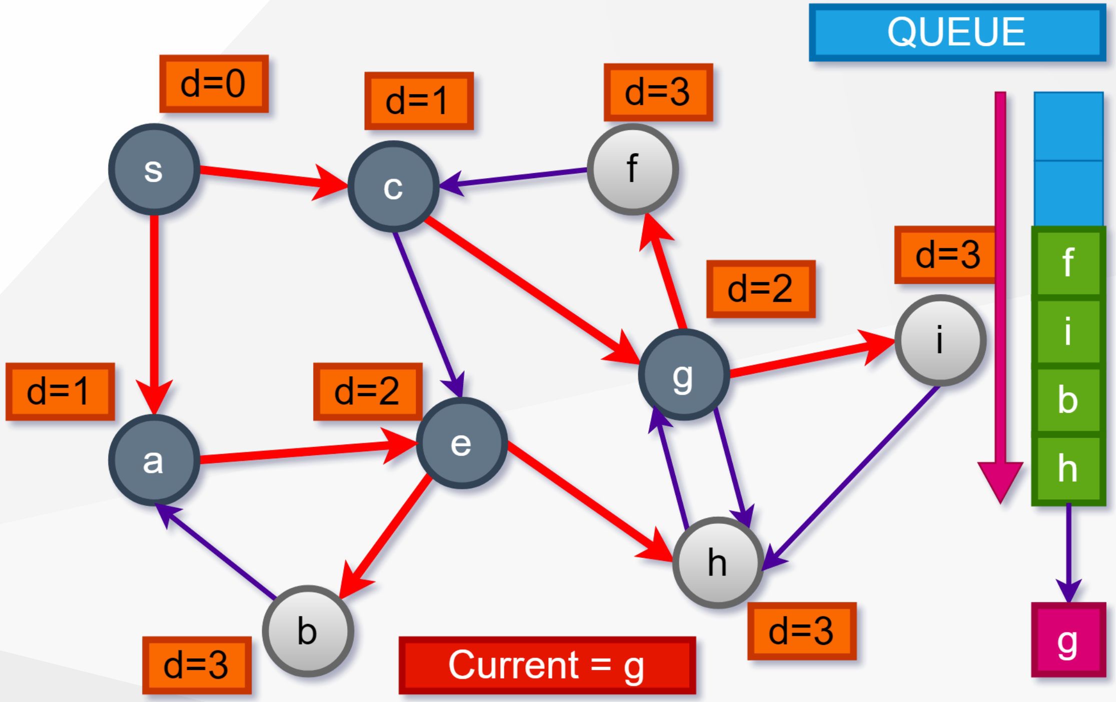


Graph Traversal

Breadth-first search (BFS) Example-1

- STEP-6

```
----------------
Q = {f,i,b,h}
g = b
----------------
i.parent = g
i.distance = 3
i.color = gray
----------------
f.parent = e
f.distance = 3
f.color = gray
----------------
```

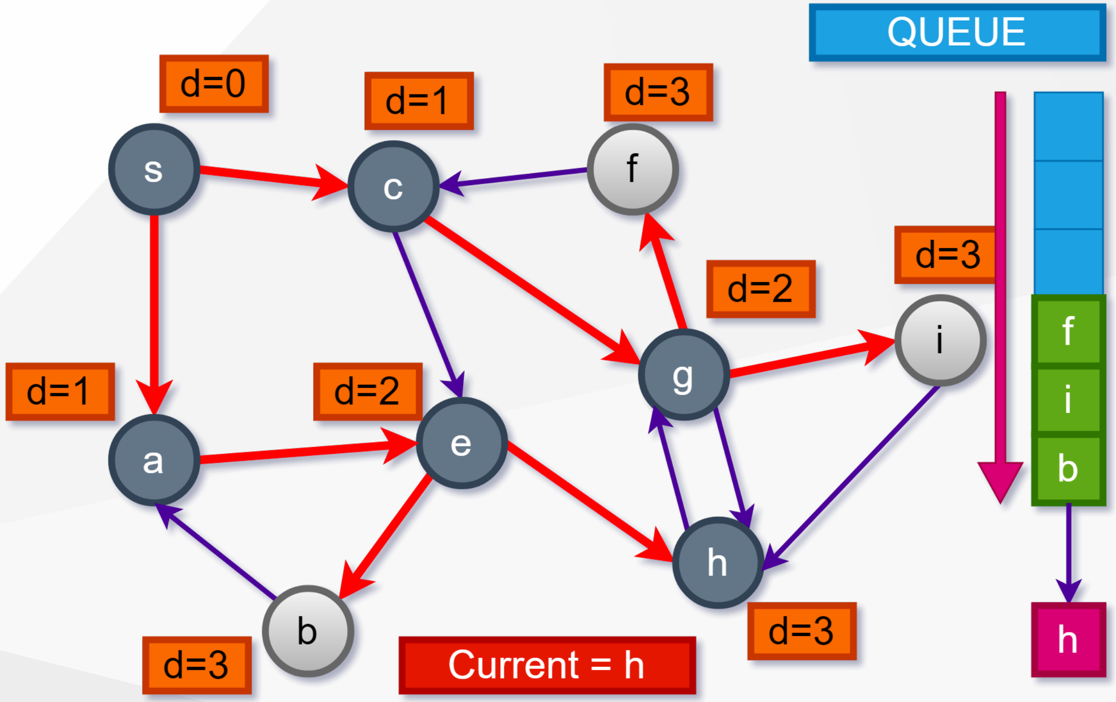


Graph Traversal

Breadth-first search (BFS) Example-1

- STEP-7

```
----------------
Q = {f,i,b}
h = b
----------------
```

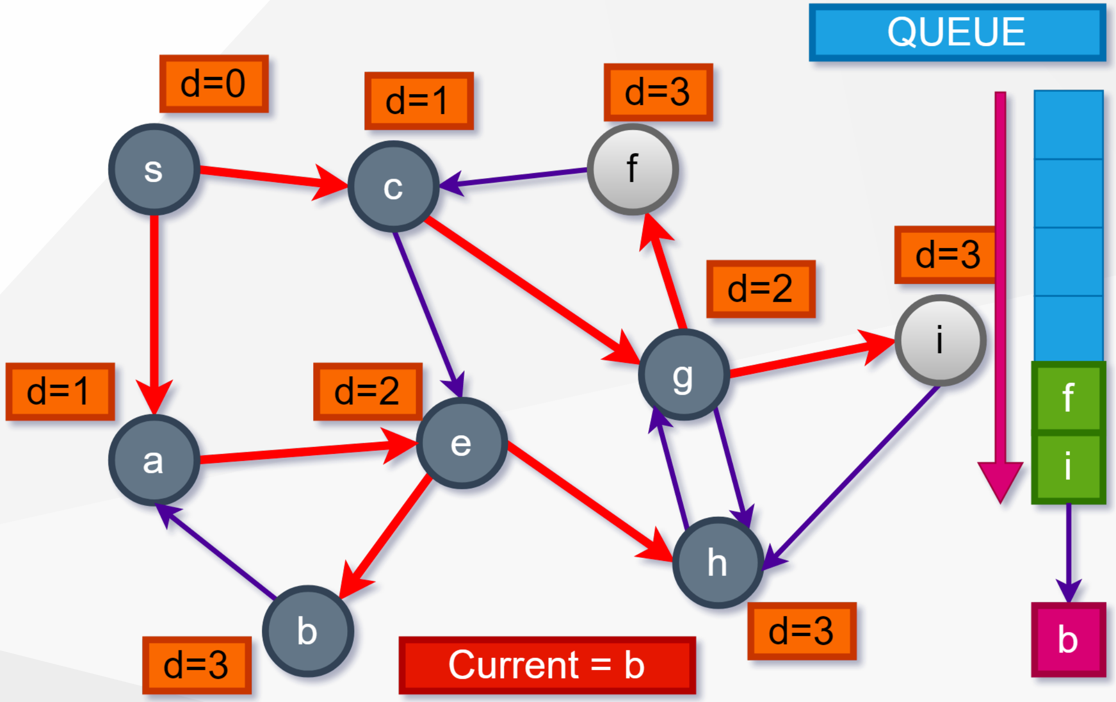


Graph Traversal

Breadth-first search (BFS) Example-1

- STEP-8

```
----------------
Q = {f,i}
b = b
----------------
```

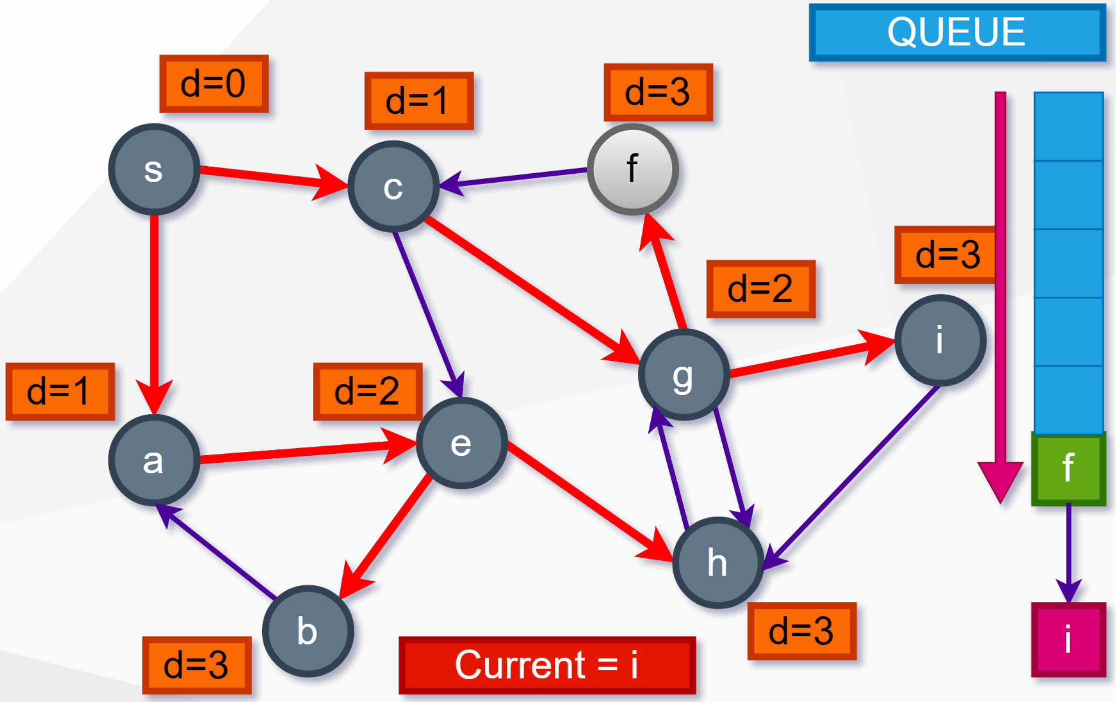


Graph Traversal

Breadth-first search (BFS) Example-1

- STEP-9

```
----------------
Q = {f}
i = b
----------------
```

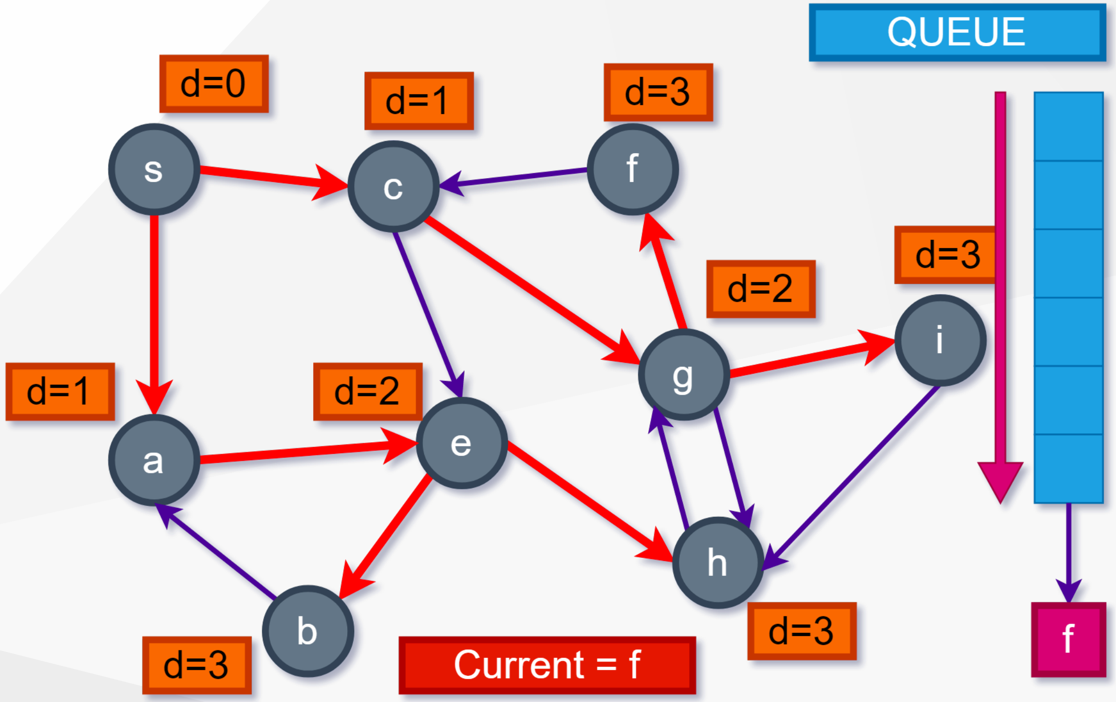


Graph Traversal

Breadth-first search (BFS) Example-1

- STEP-10

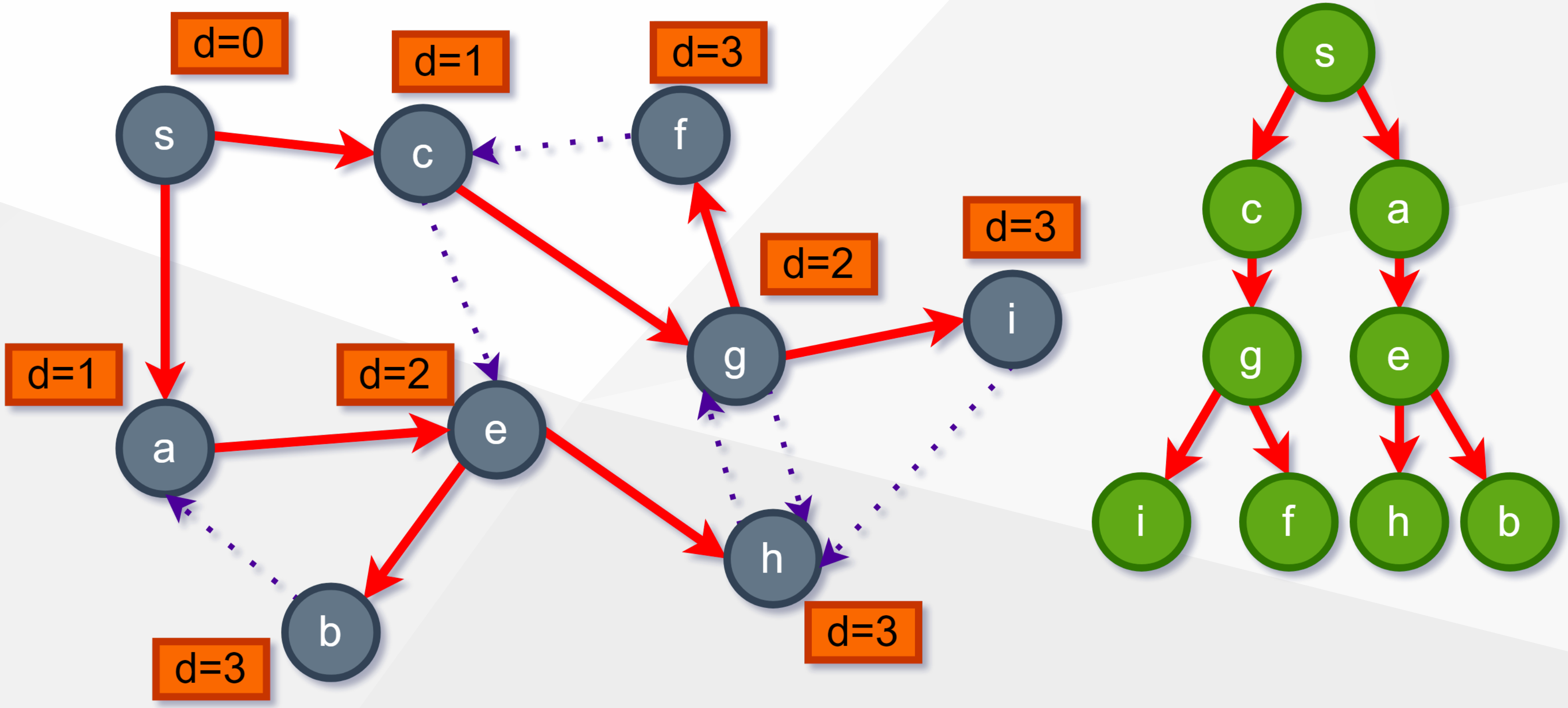
```
----------------
Q = {}
f = b
----------------
```



Graph Traversal

Breadth-first search (BFS) Example-1

- BFS is done and the graph is traversed.



Graph Traversal

Breadth-first search (BFS) Algorithm Summary

- **Step 1** - Define a Queue of size total number of vertices in the graph.
- **Step 2** - Select any vertex as starting point for traversal. Visit that vertex and insert it into the Queue.
- **Step 3** - Visit all the non-visited adjacent vertices of the vertex which is at front of the Queue and insert them into the Queue.
- **Step 4** - When there is no new vertex to be visited from the vertex which is at front of the Queue then delete that vertex.
- **Step 5** - Repeat steps 3 and 4 until queue becomes empty.
- **Step 6** - When queue becomes empty, then produce final spanning tree by removing unused edges from the graph

$TODO - Add\ more\ examples$

Graph Traversal

Depth-first search (DFS)

- DFS is a traversal algorithm that visits each vertex in a graph in a depth-first manner.

Graph Traversal

Depth-first search (DFS)

- Complexity of the following part is $\Theta(V+V)=O(V)$ (two sequential loops)

```
DFS(G)
 for each vertex u in G.V
    u.color = white
    u.parent = nil
 time = 0
  for each vertex u in G.V
    if u.color == white
      DFS-VISIT(G,u)
```

Graph Traversal

Depth-first search (DFS)

```
DFS-VISIT(G,u)
  time = time + 1
  u.discovery = time
  u.color = gray
  for each vertex v in G.Adj[u]
    if v.color == white
      v.parent = u
      DFS-VISIT(G,v)
  u.color = black
  time = time + 1
  u.finish = time
```

SCC - Kosaraju's algorithm

```
for each unvisited vertex u, DFS(u)

  try all free neighbor v of u, DFS(v)

  finish DFS(u), add u to the front of list

transpose the graph

DFS in order of the list, DFS(u)

  try all free neighbor v of u, DFS(v)

each time we complete a DFS, we get an SCC
```

SCC - Tarjan's algorithm

```
for each unvisited vertex u

  DFS(u), s.push(u), num[u] = low[u] = DFSCount

    for each neighbor v of u

      if v is unvisited, DFS(v)

      low[u] = min(low[u], low[v])

    if low[u] == num[u] // root of an SCC

      pop from stack s until we get u
```

Graph Traversal

Depth-first search (DFS)

- DFS complexity is $\Theta(V + E)$
- Note for all $v \rightarrow v.discovery < v.finish$
  - $1 \leq u.discovery < u.finish \leq 2|V|$

Graph Traversal

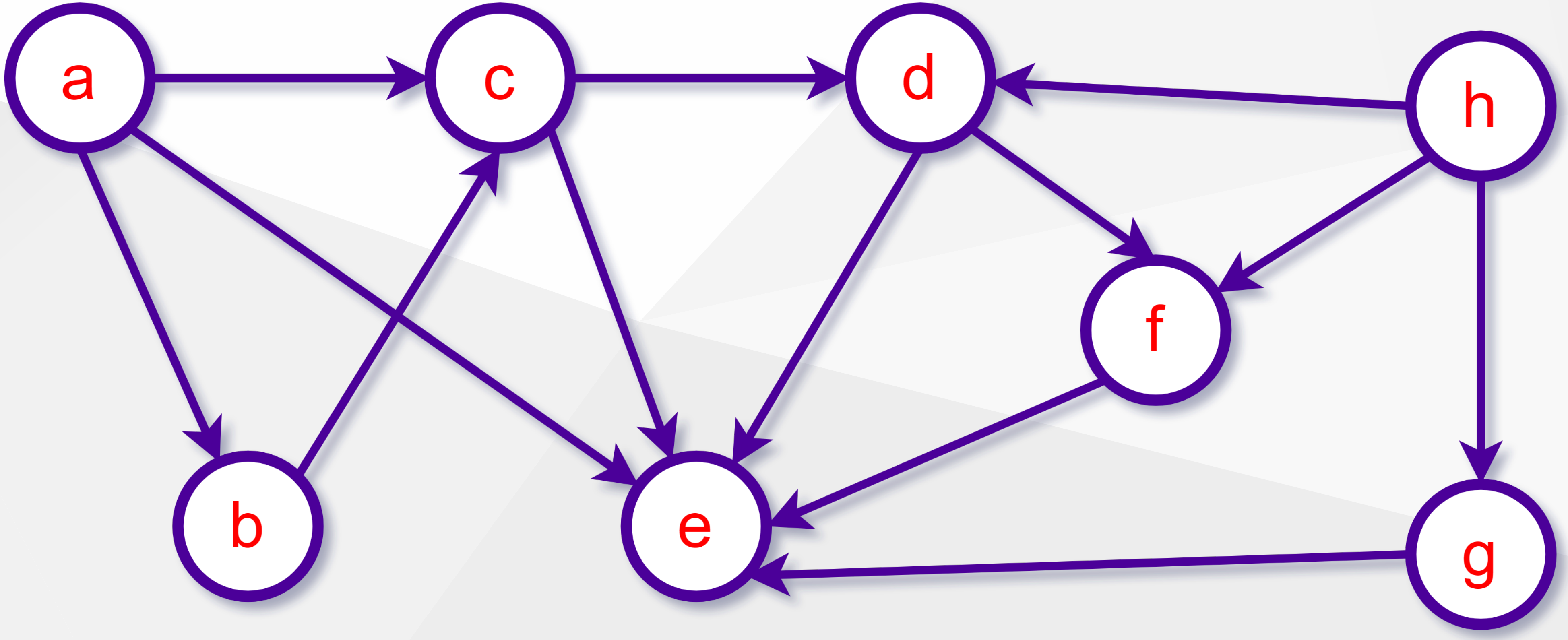
Depth-first search (DFS)

- Edge Types in DFS
  - Tree Edges
  - Back Edges
  - Forward Edges
  - Cross Edges
- Colors in DFS
  - White -> Tree Edges
  - Gray -> Back Edges
  - Black -> Forward Edges

Graph Traversal

Depth-first search (DFS) Example-1

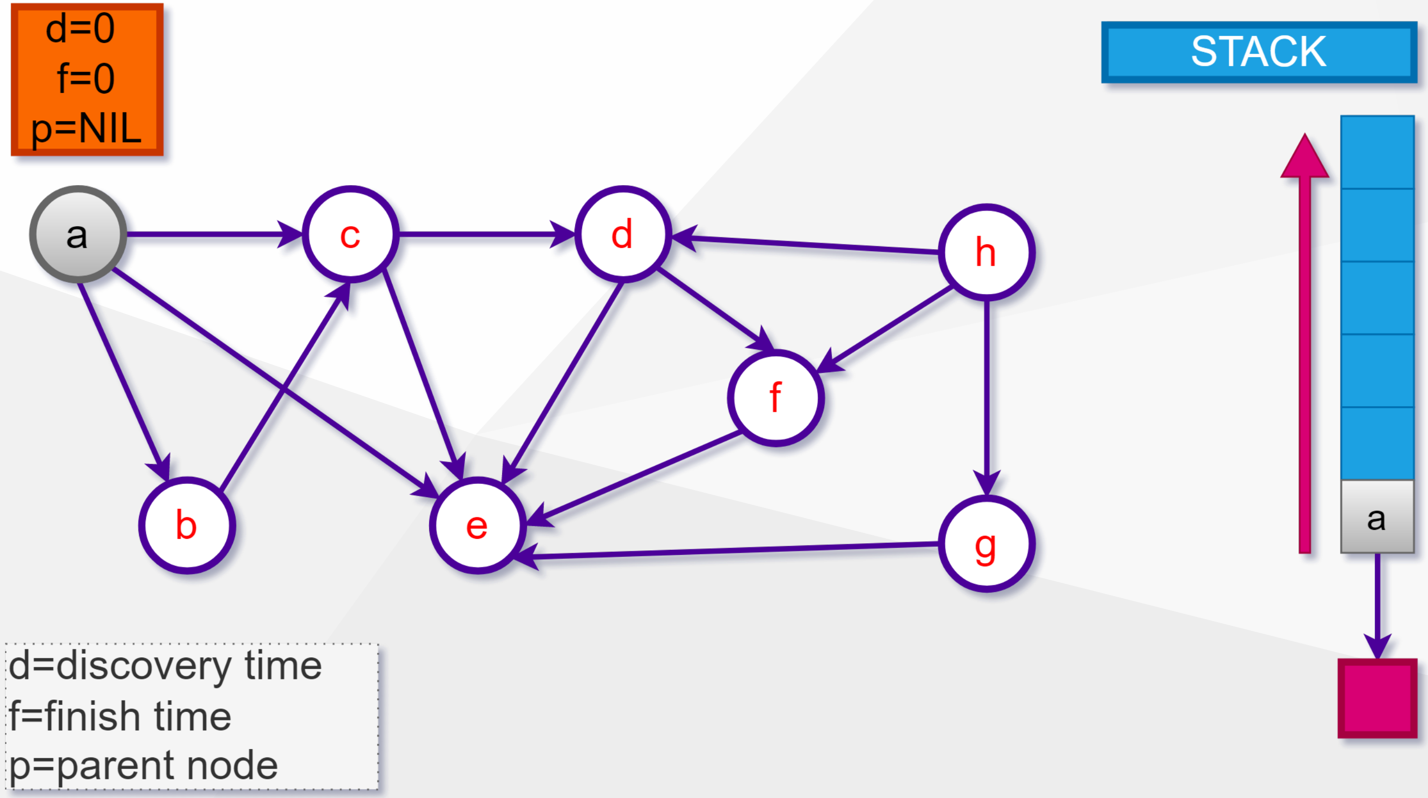
Kosaraju's algorithm



Graph Traversal

Depth-first search (DFS) Example-1

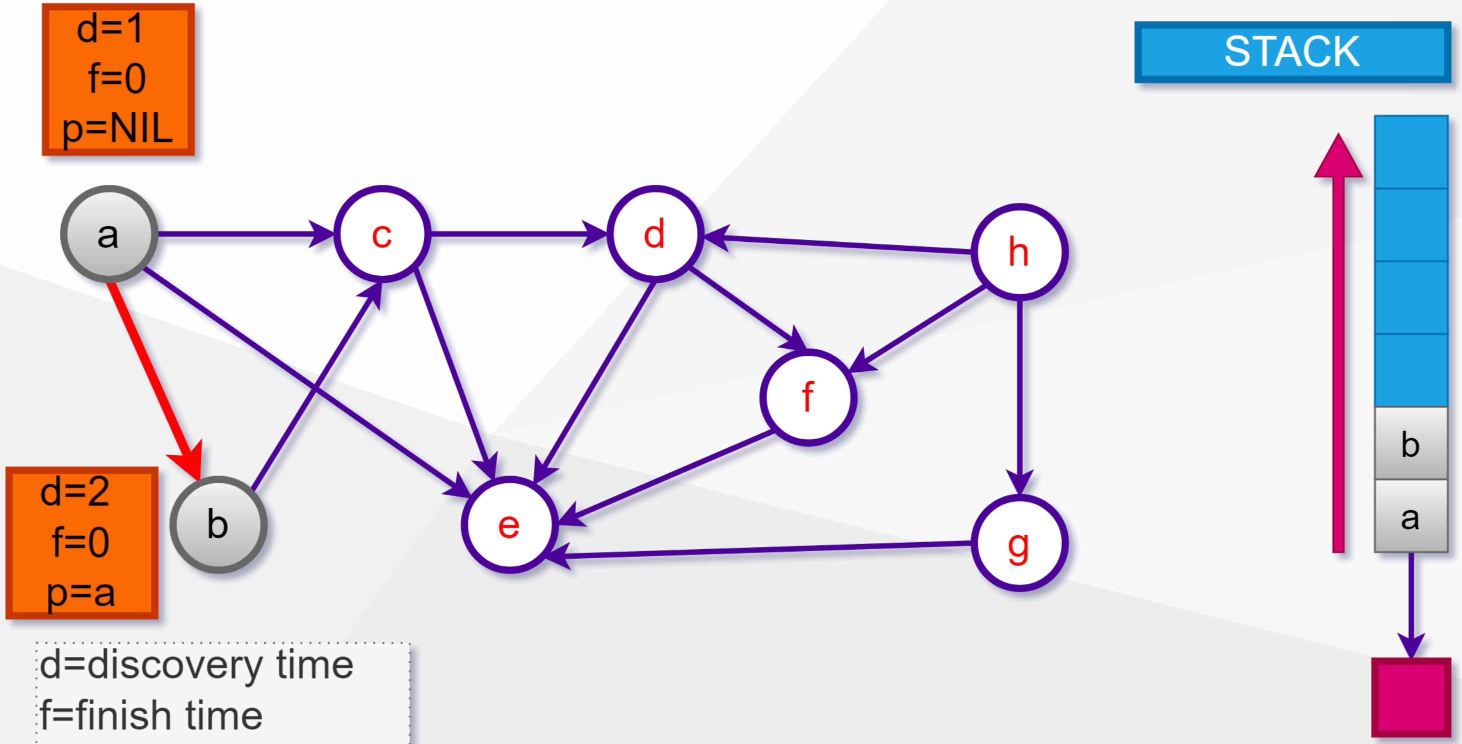
- STEP-1



Graph Traversal

Depth-first search (DFS) Example-1

- STEP-2

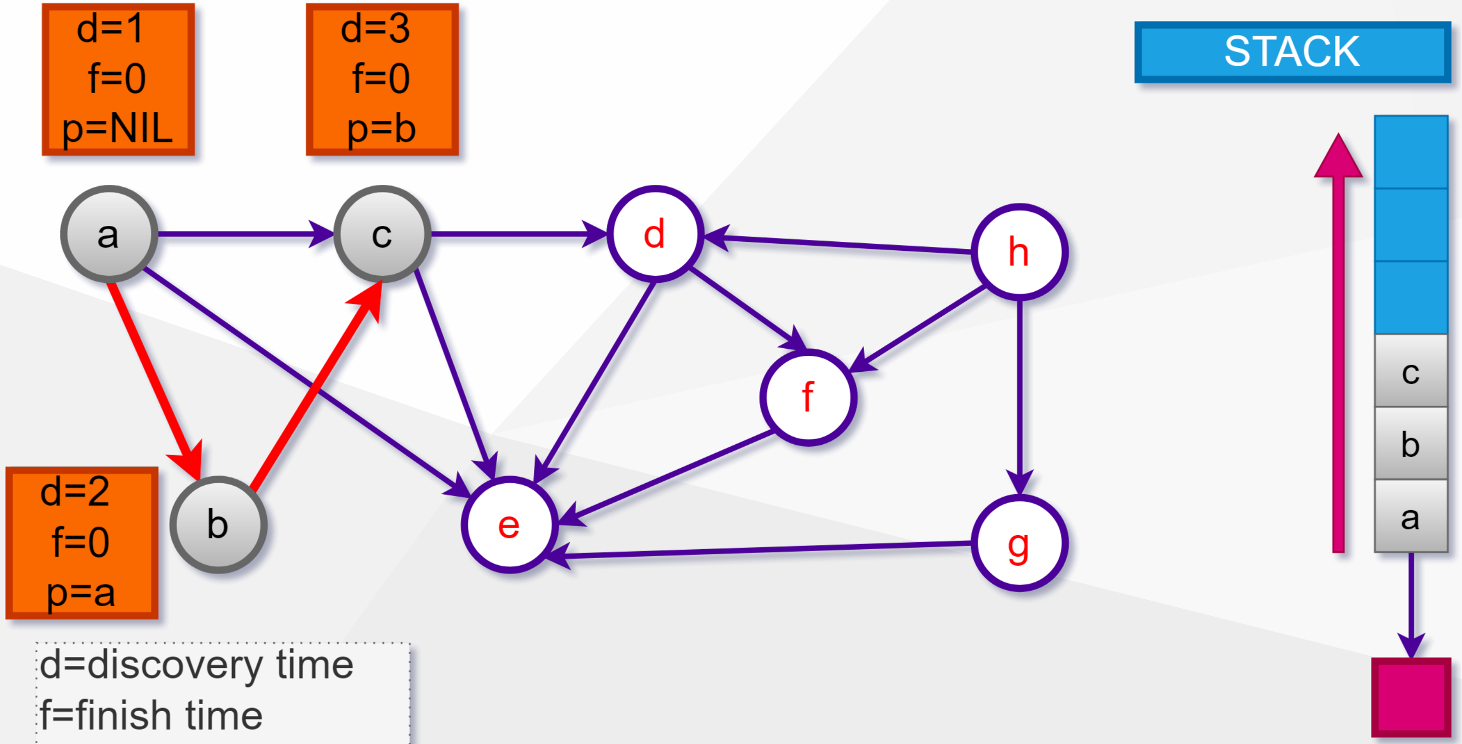


d=discovery time
f=finish time
p=parent node

Graph Traversal

Depth-first search (DFS) Example-1

- STEP-3

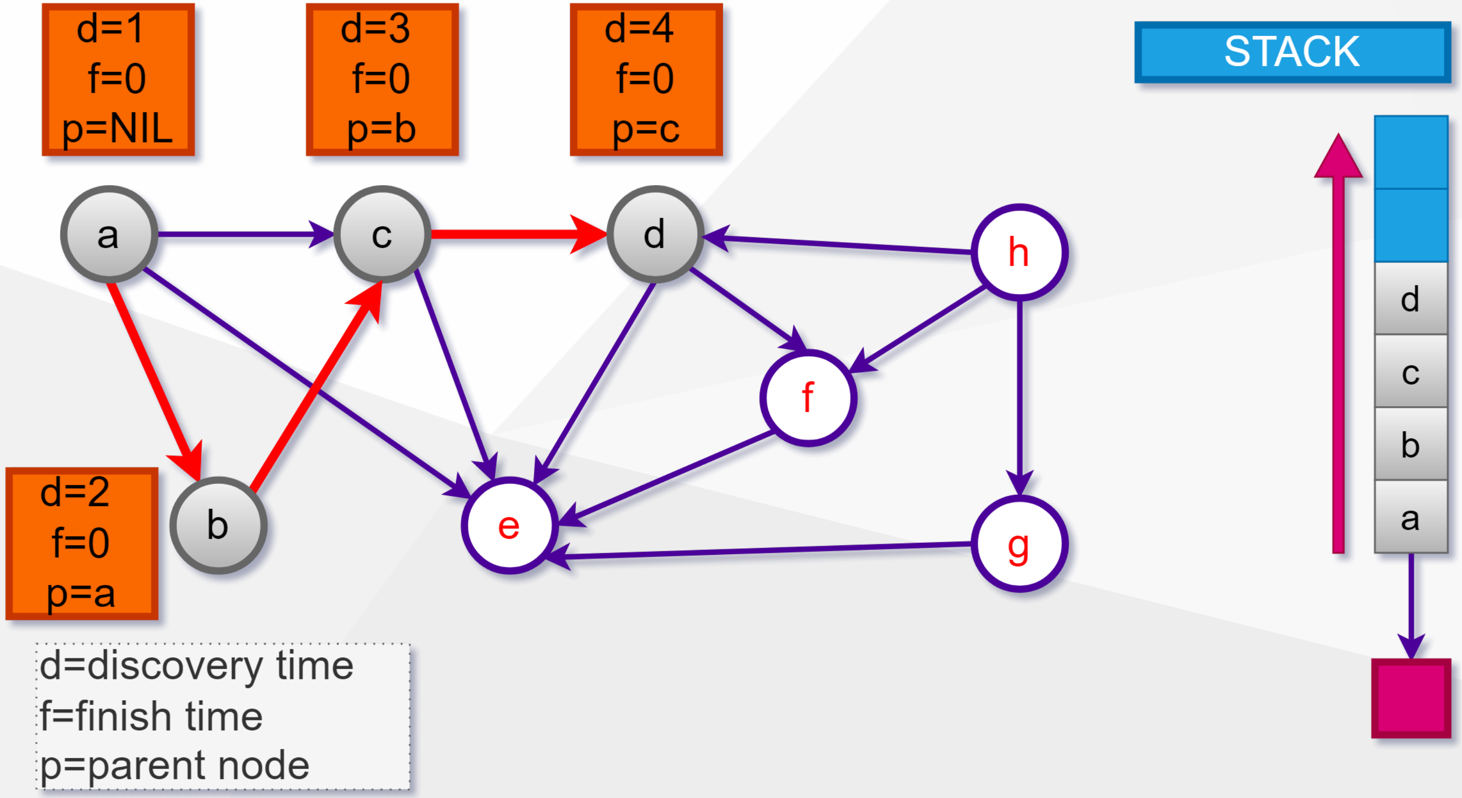


d=discovery time
f=finish time
p=parent node

Graph Traversal

Depth-first search (DFS) Example-1

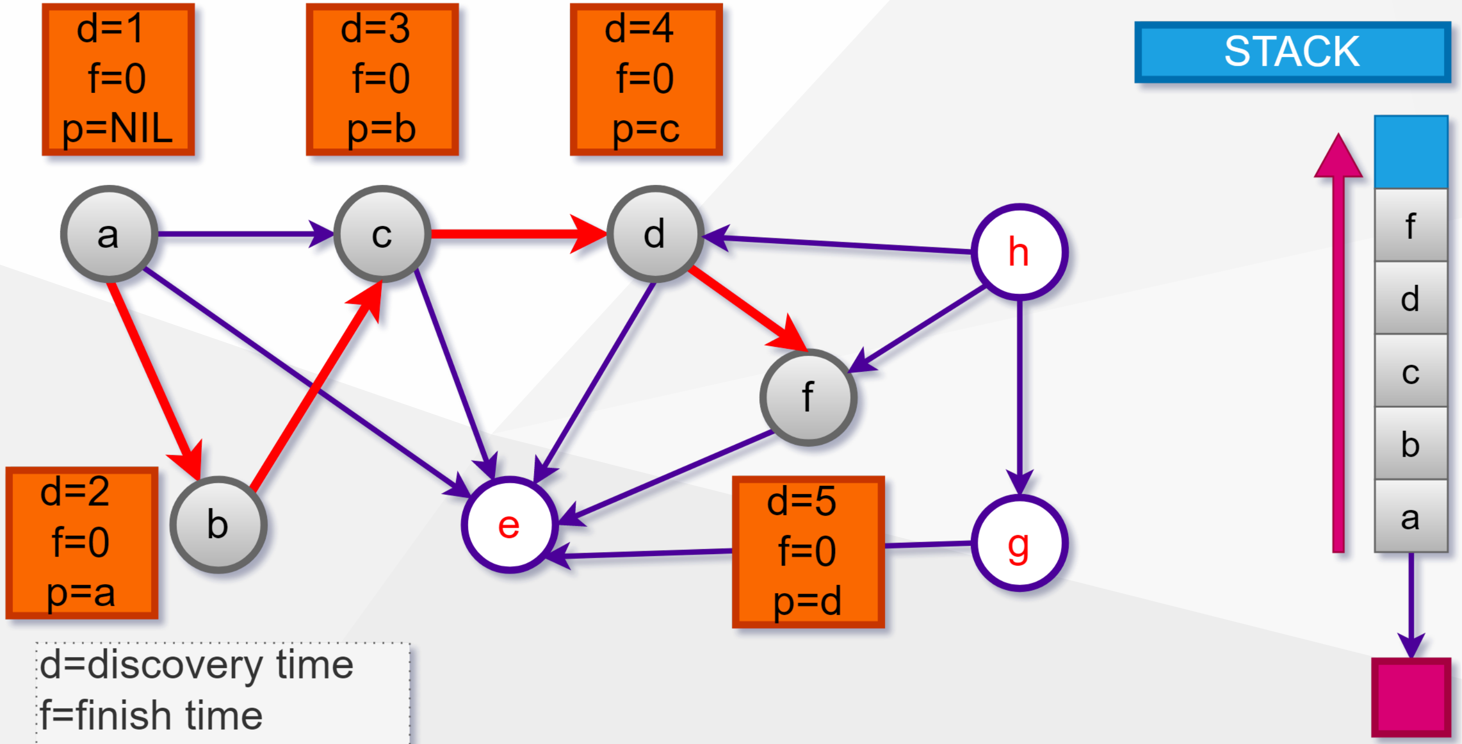
- STEP-4



Graph Traversal

Depth-first search (DFS) Example-1

- STEP-5

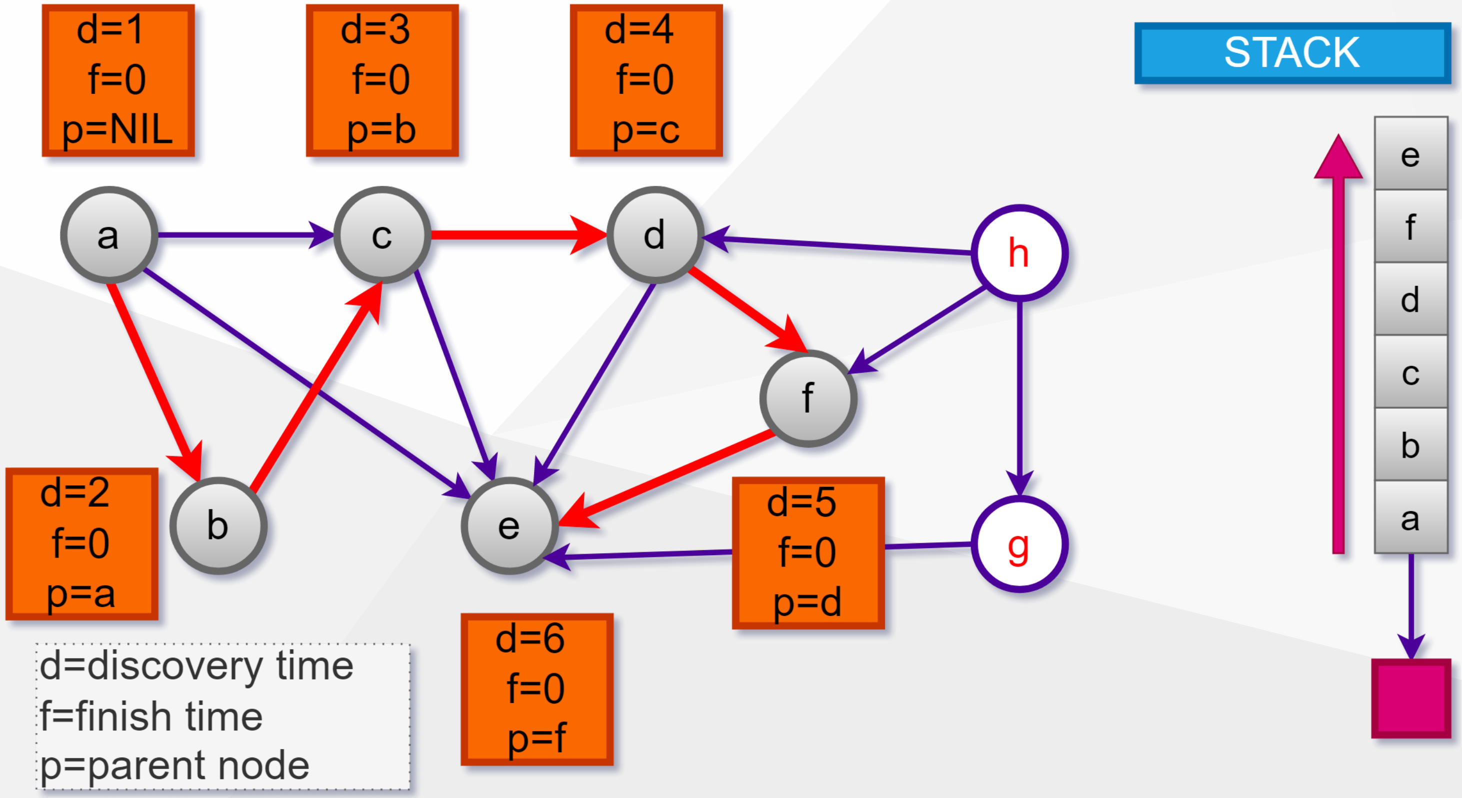


d=discovery time
f=finish time
p=parent node

Graph Traversal

Depth-first search (DFS) Example-1

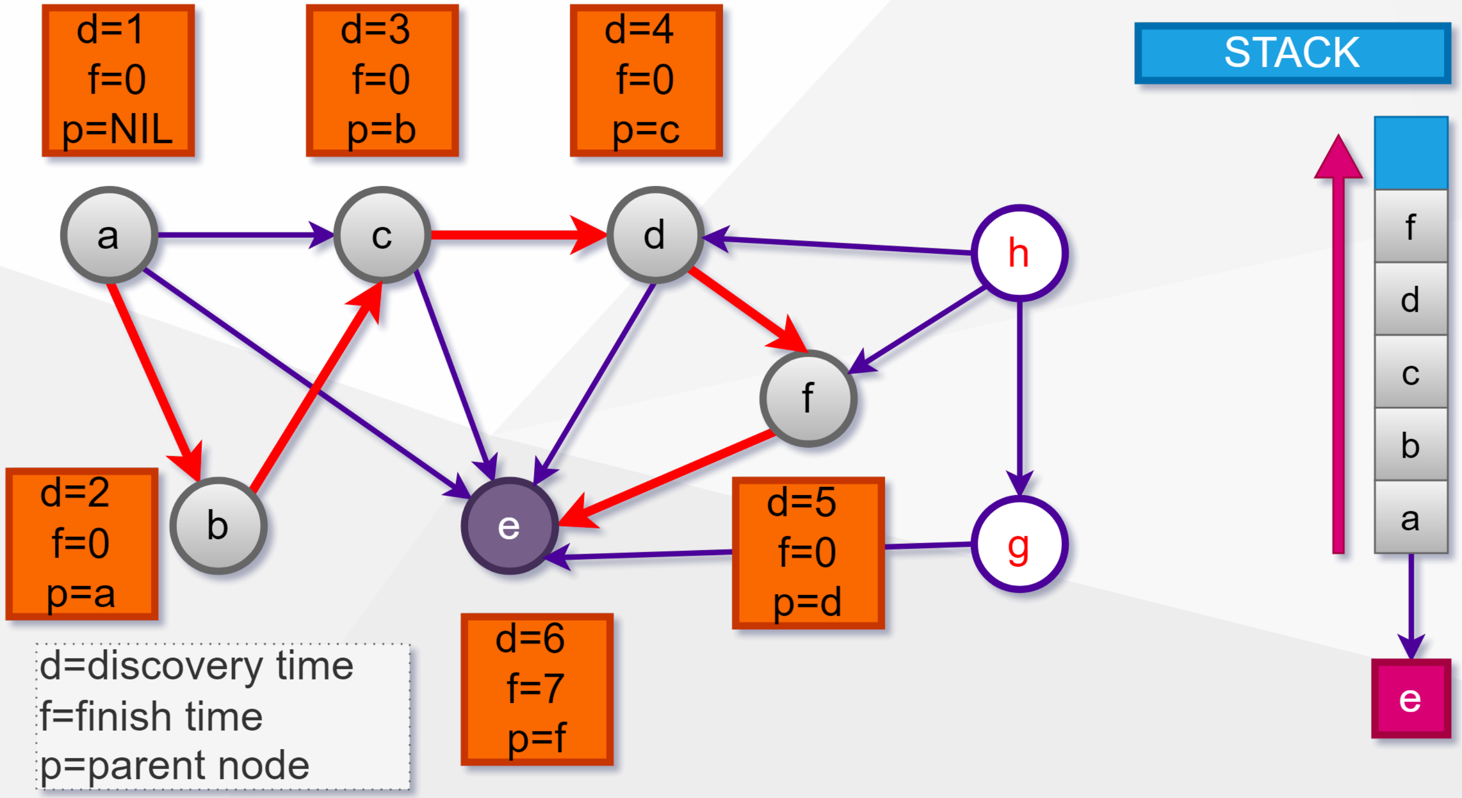
- STEP-6



Graph Traversal

Depth-first search (DFS) Example-1

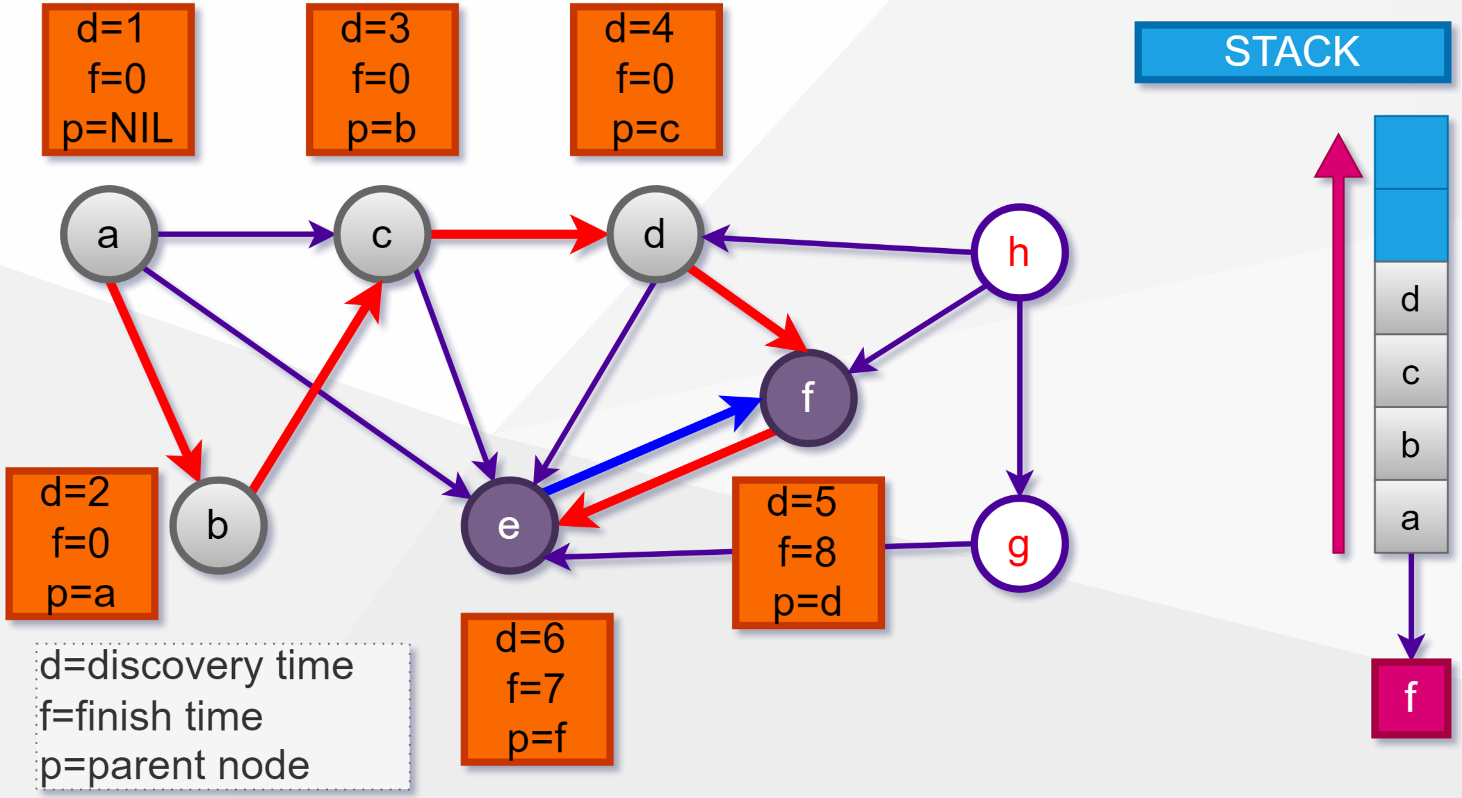
- STEP-7



Graph Traversal

Depth-first search (DFS) Example-1

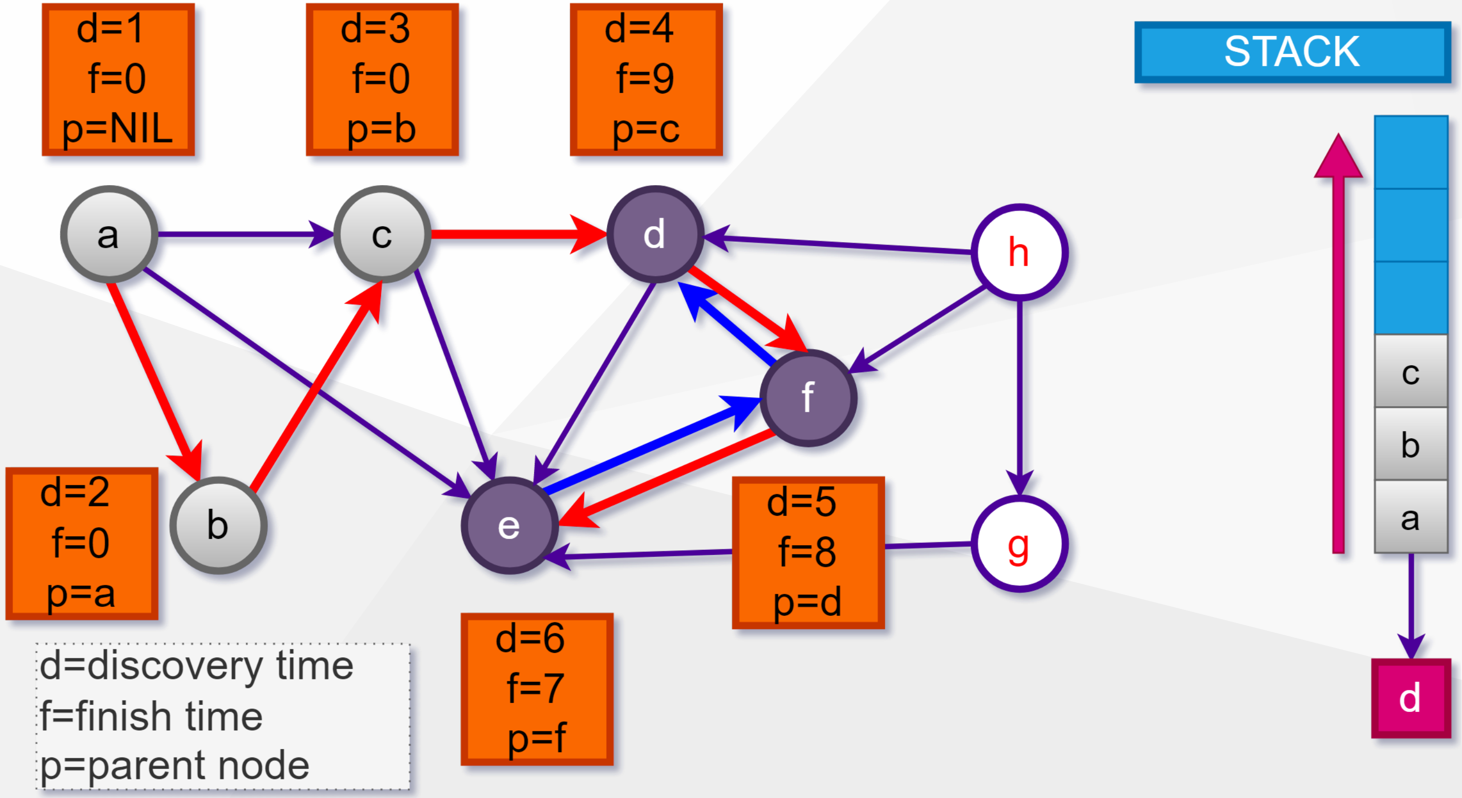
- STEP-8



Graph Traversal

Depth-first search (DFS) Example-1

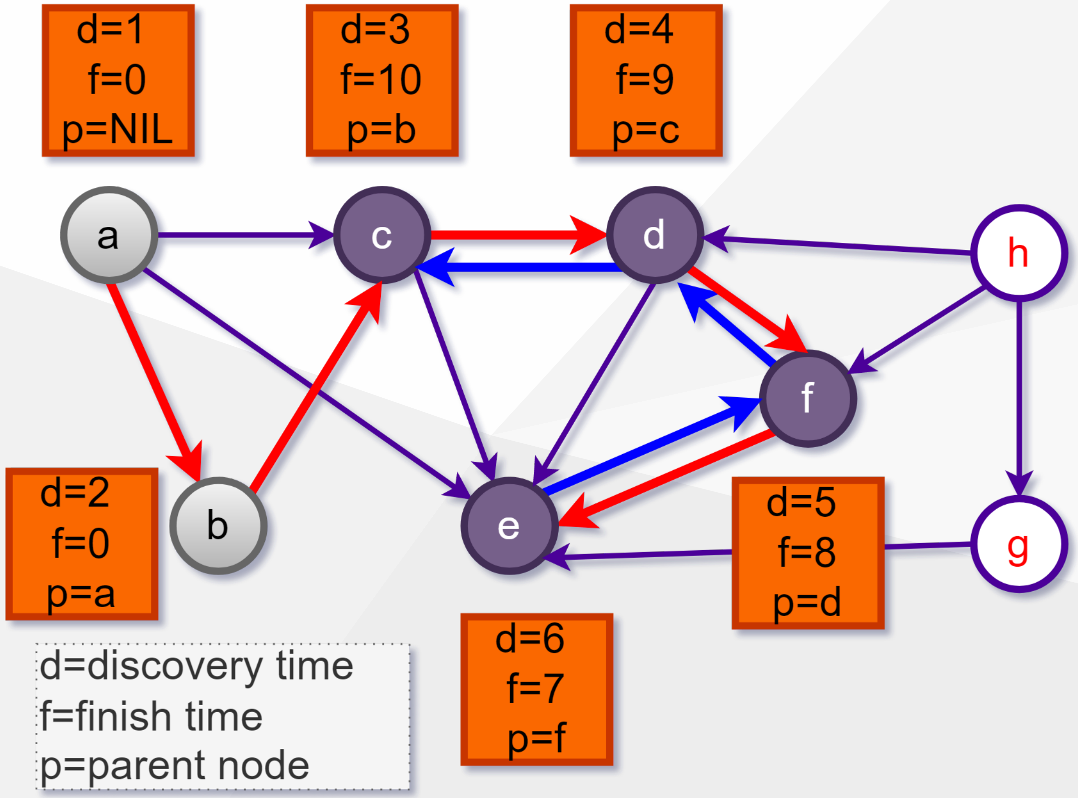
- STEP-9

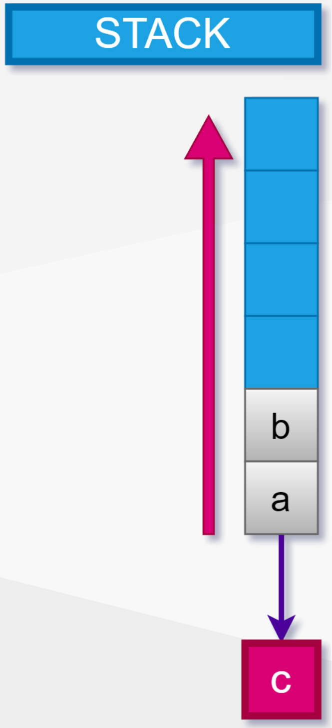


Graph Traversal

Depth-first search (DFS) Example-1

- STEP-10

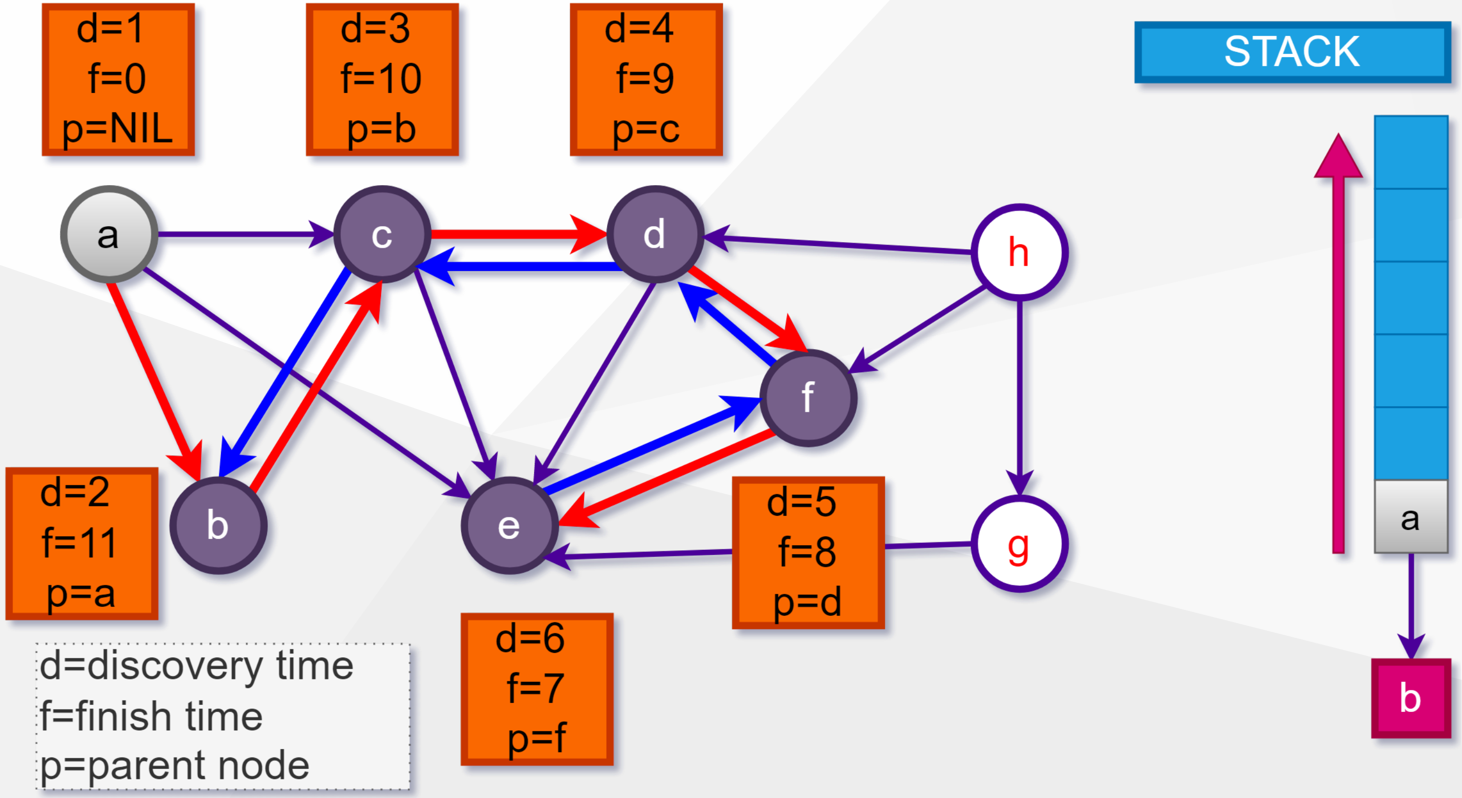




Graph Traversal

Depth-first search (DFS) Example-1

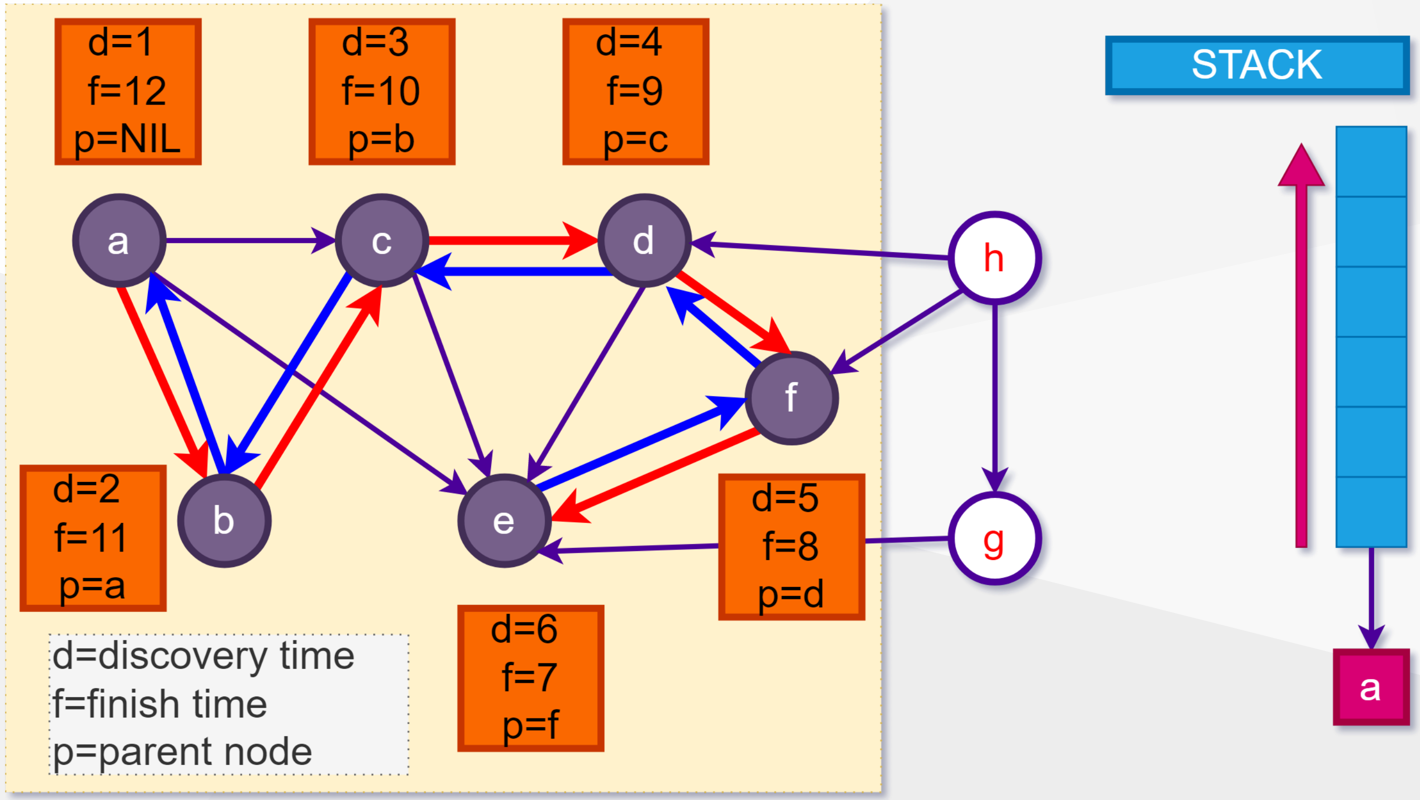
- STEP-11



Graph Traversal

Depth-first search (DFS) Example-1

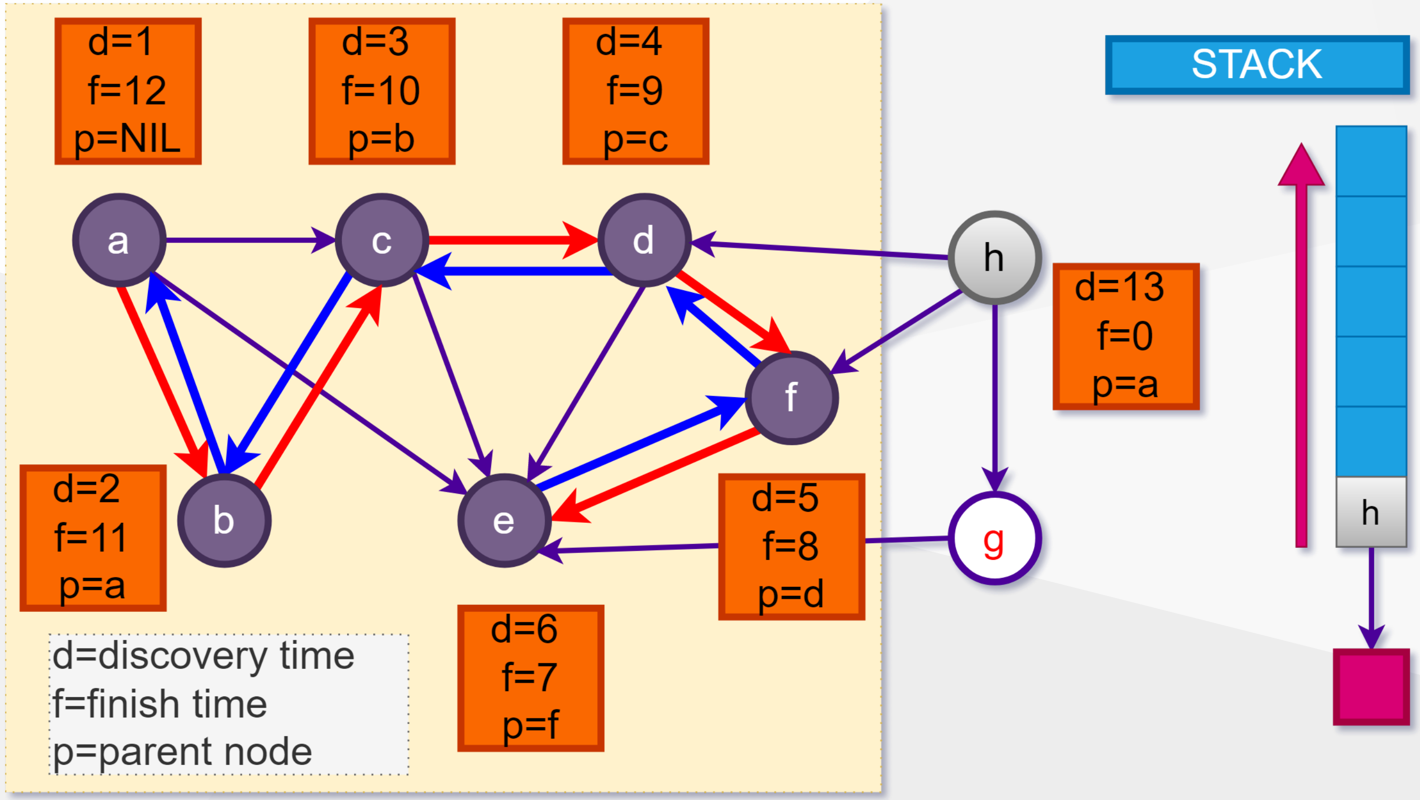
- STEP-12



Graph Traversal

Depth-first search (DFS) Example-1

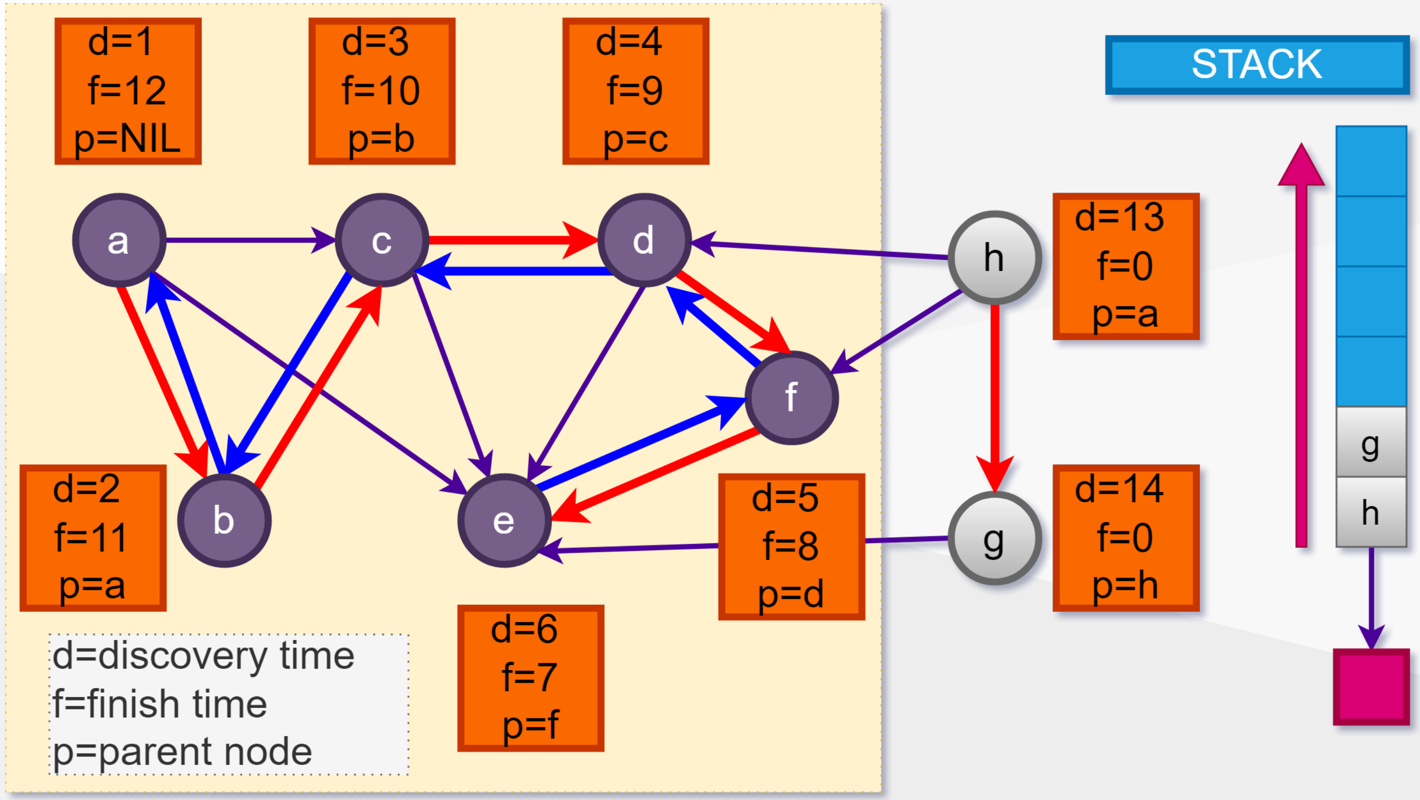
- STEP-13



Graph Traversal

Depth-first search (DFS) Example-1

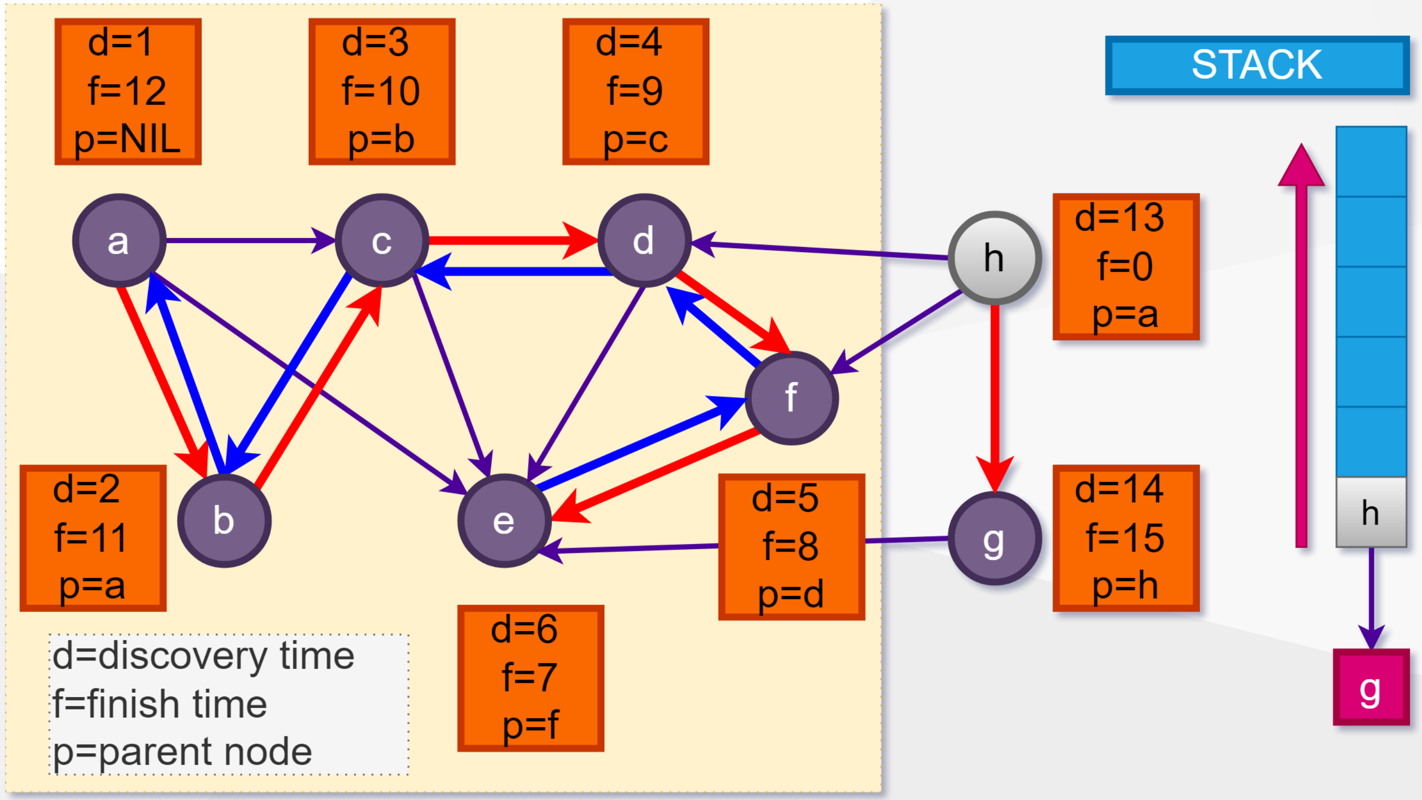
- STEP-14



Graph Traversal

Depth-first search (DFS) Example-1

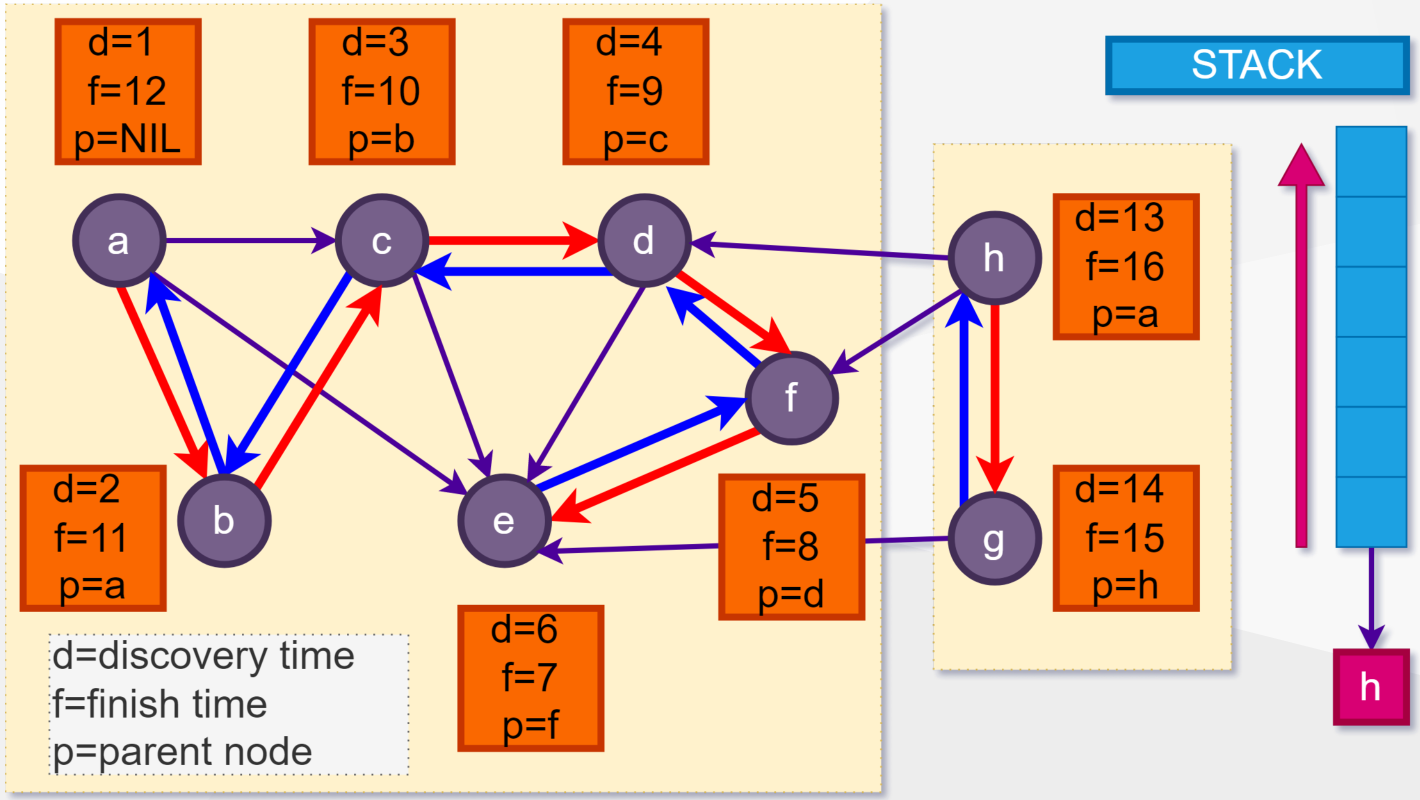
- STEP-15



Graph Traversal

Depth-first search (DFS) Example-1

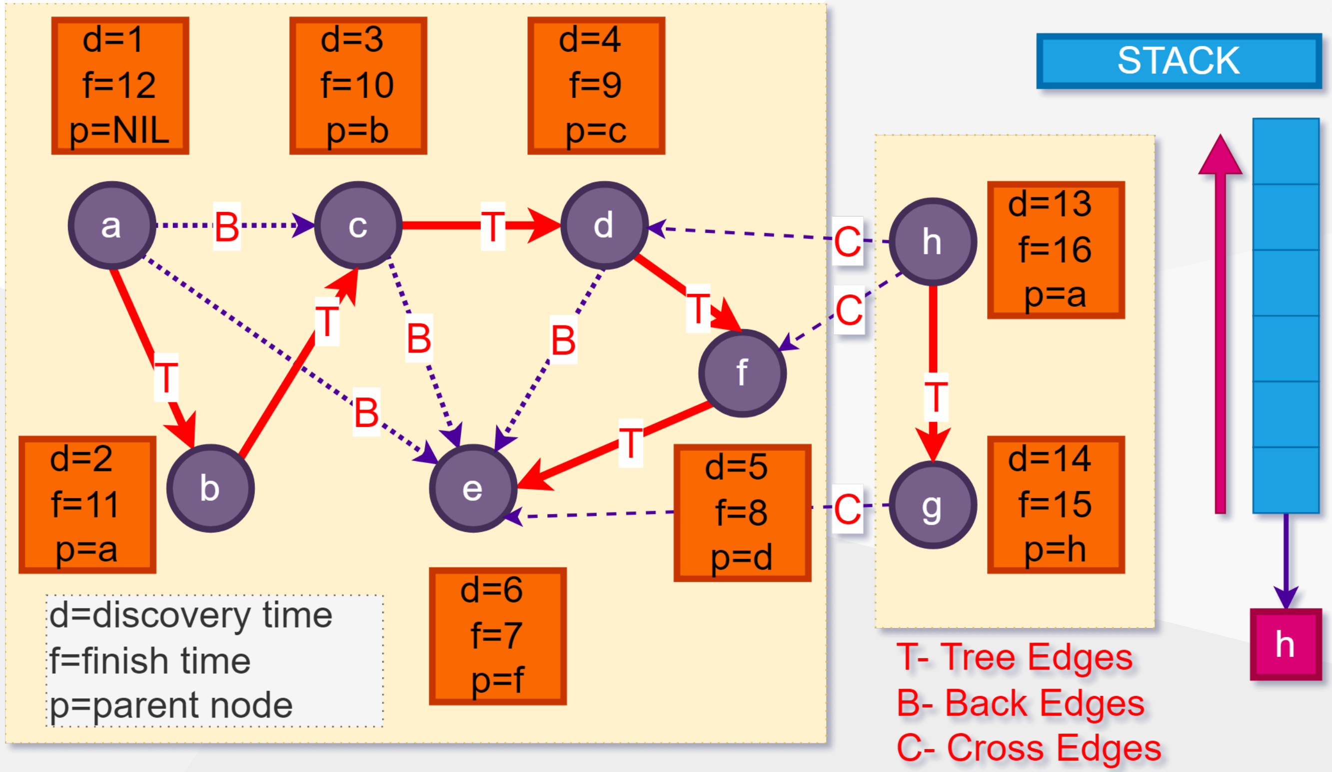
- FINAL STEP-16



Graph Traversal

Depth-first search (DFS) Example-1

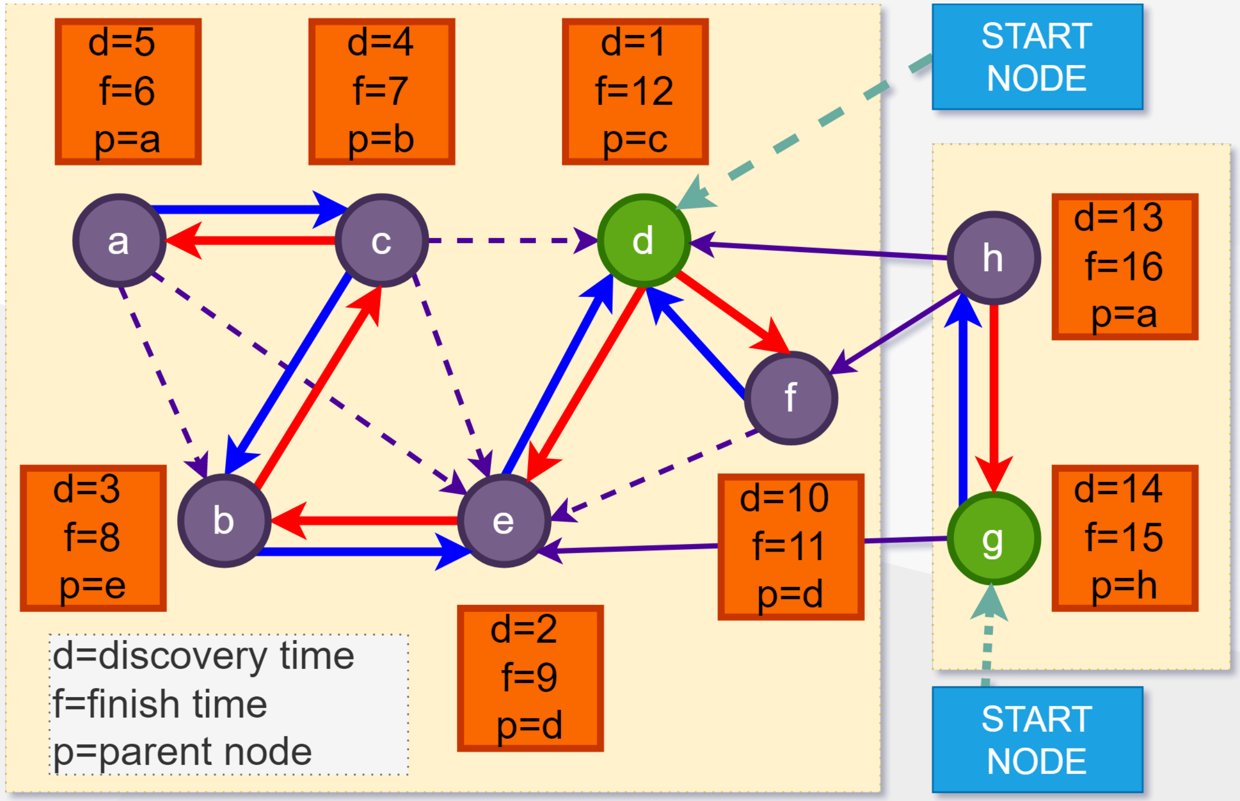
- Edges and Clusters after DFS



Graph Traversal

Depth-first search (DFS) Example-2

- Different Start Point and Different Graph



Elementary Graph Algorithms

Depth-first search (DFS) Algorithm Summary

- **Step 1** - Define a Stack of size total number of vertices in the graph.
- **Step 2** - Select any vertex as starting point for traversal. Visit that vertex and push it on to the Stack.
- **Step 3** - Visit any one of the non-visited adjacent vertices of a vertex which is at the top of stack and push it on to the stack.
- **Step 4** - Repeat step 3 until there is no new vertex to be visited from the vertex which is at the top of the stack.
- **Step 5** - When there is no new vertex to visit then use back tracking and pop one vertex from the stack.
- **Step 6** - Repeat steps 3, 4 and 5 until stack becomes Empty.
- **Step 7** - When stack becomes Empty, then produce final spanning tree by removing unused edges from the graph

$TODO - Add\ more\ examples$

Graph Segmentation

SCC (Strongly Connected Components)

- SCC Algorithm is used to find the connected components in a graph.

Graph Segmentation

SCC (Strongly Connected Components)

- $G^T = (V, E^T)$ can create $G^T \rightarrow \Theta(V + E)$ adjency list.
- $SCC(G)$ complexity is $O(V + E)$

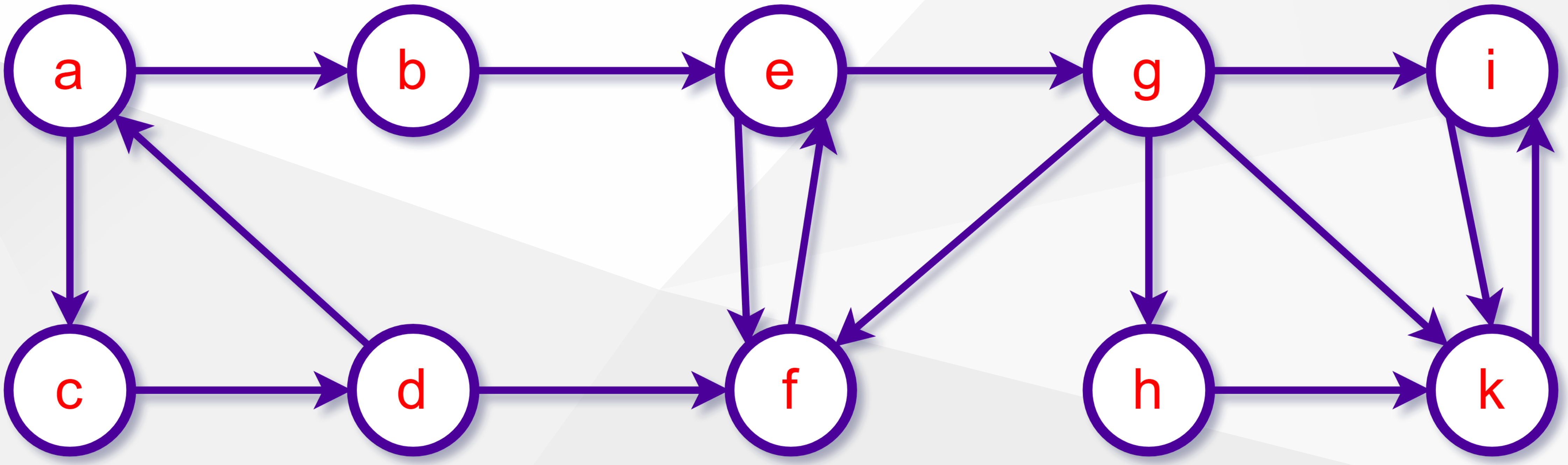
Graph Segmentation

SCC Algorithm

```
1- call DFS(G) compute all u.finishTime values
2- compute G^T = (V,E^T) and reverse edge directions
3- call DFS(G^T) but in the main loop,
consider vertices
in order of decreasing u.finishTime
(as computed in DFS)
```

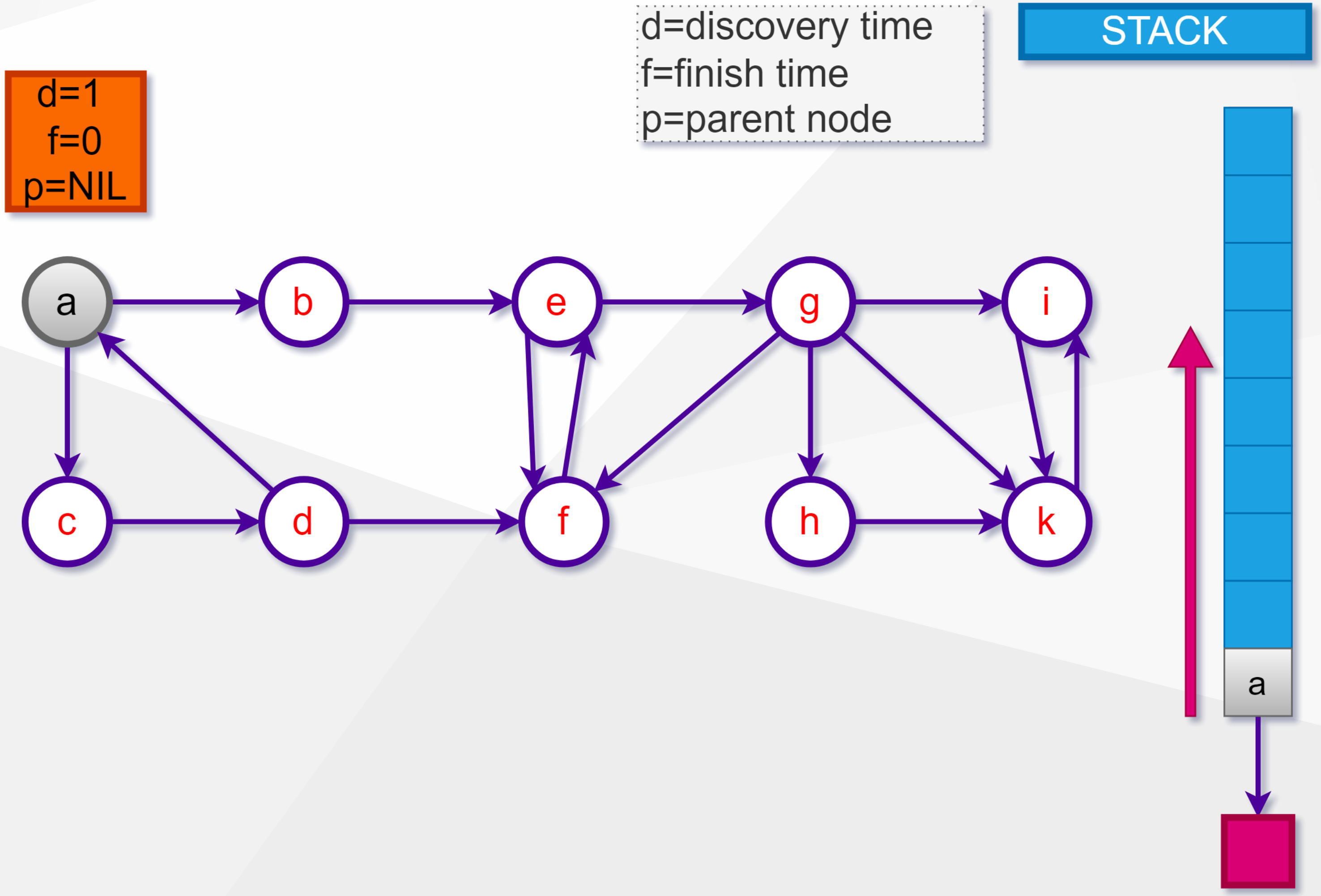
Graph Segmentation

SCC Algorithm - Example-1 Graph



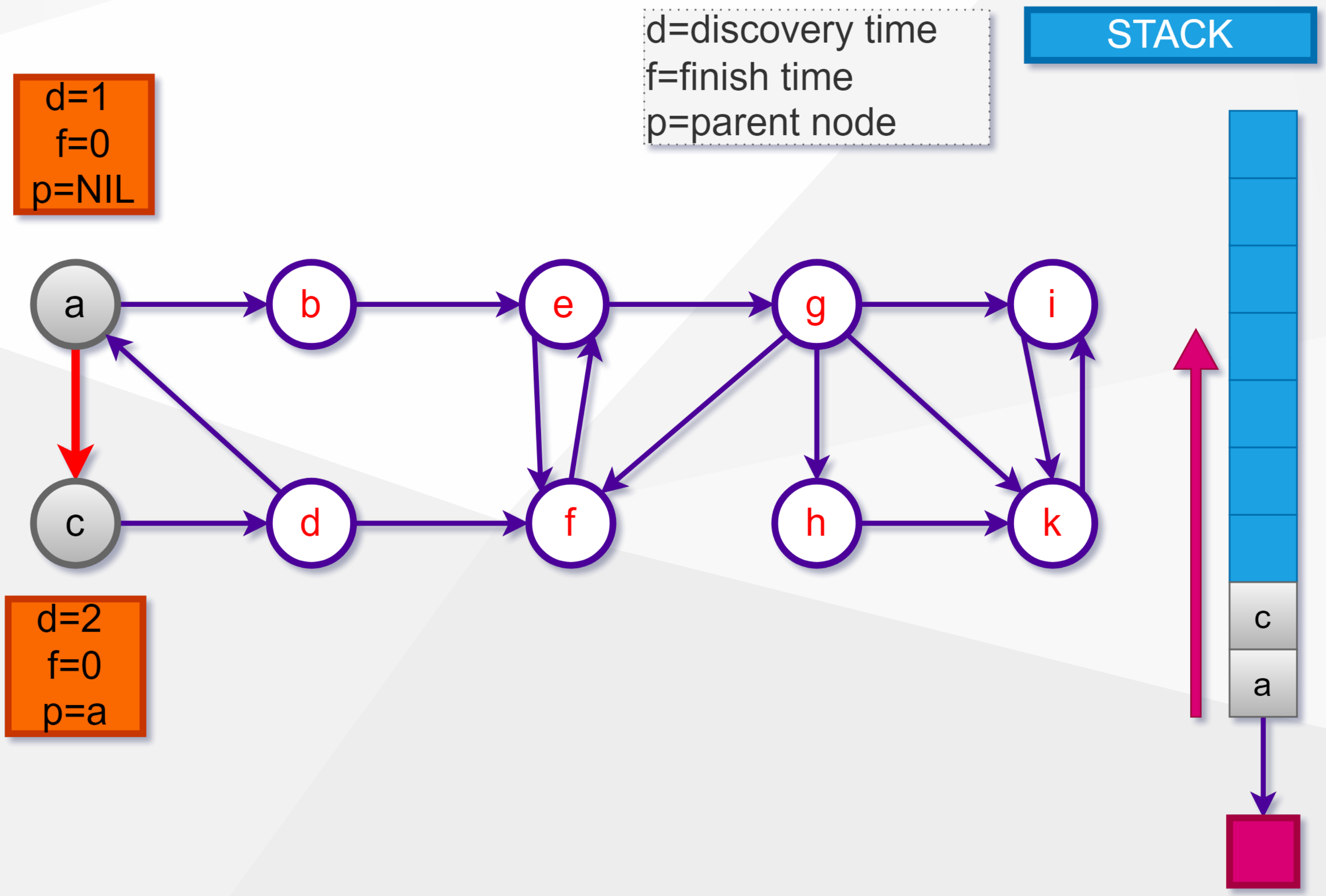
Graph Segmentation

SCC Algorithm - Example-1 / Step - 1



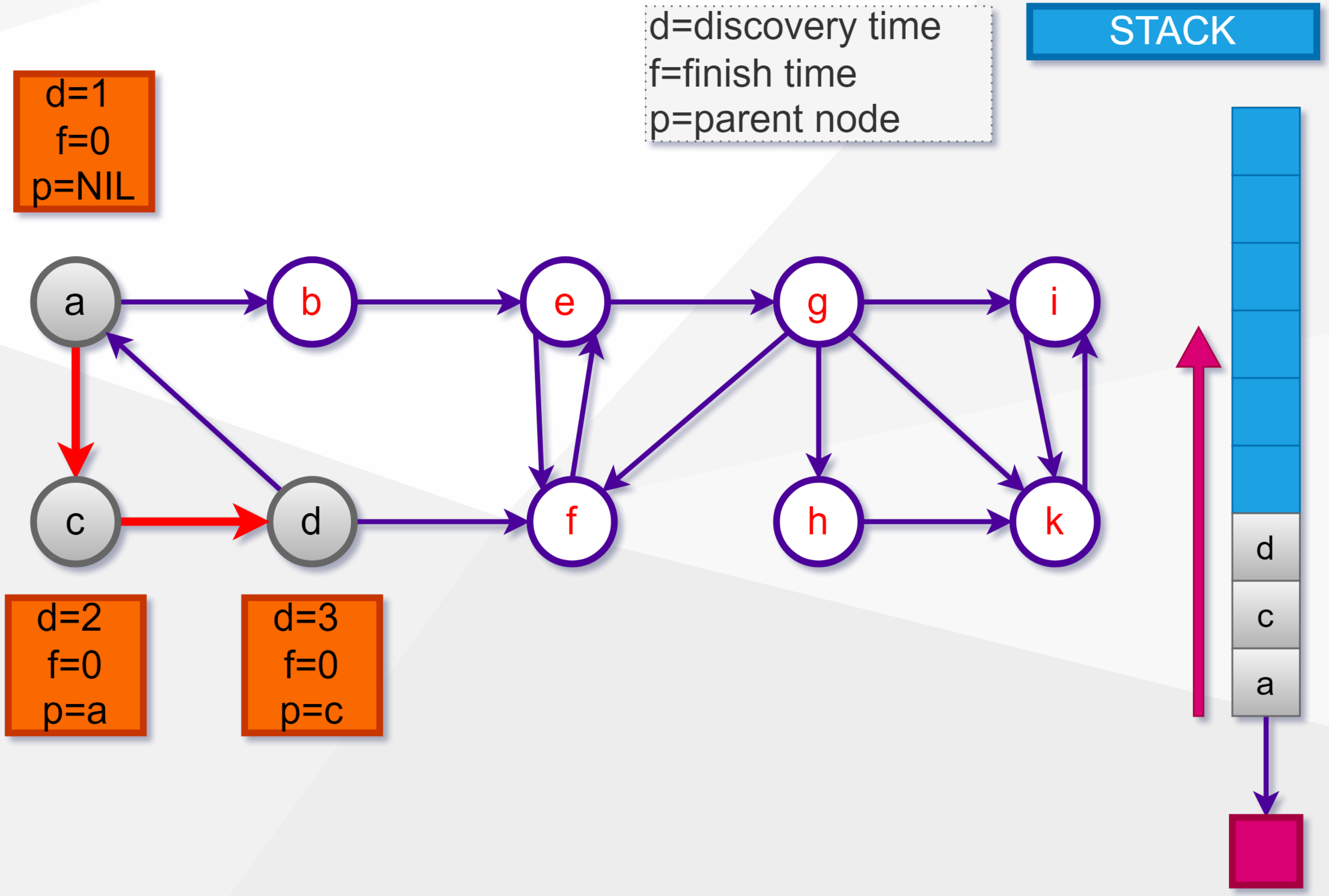
Graph Segmentation

SCC Algorithm - Example-1 / Step - 2



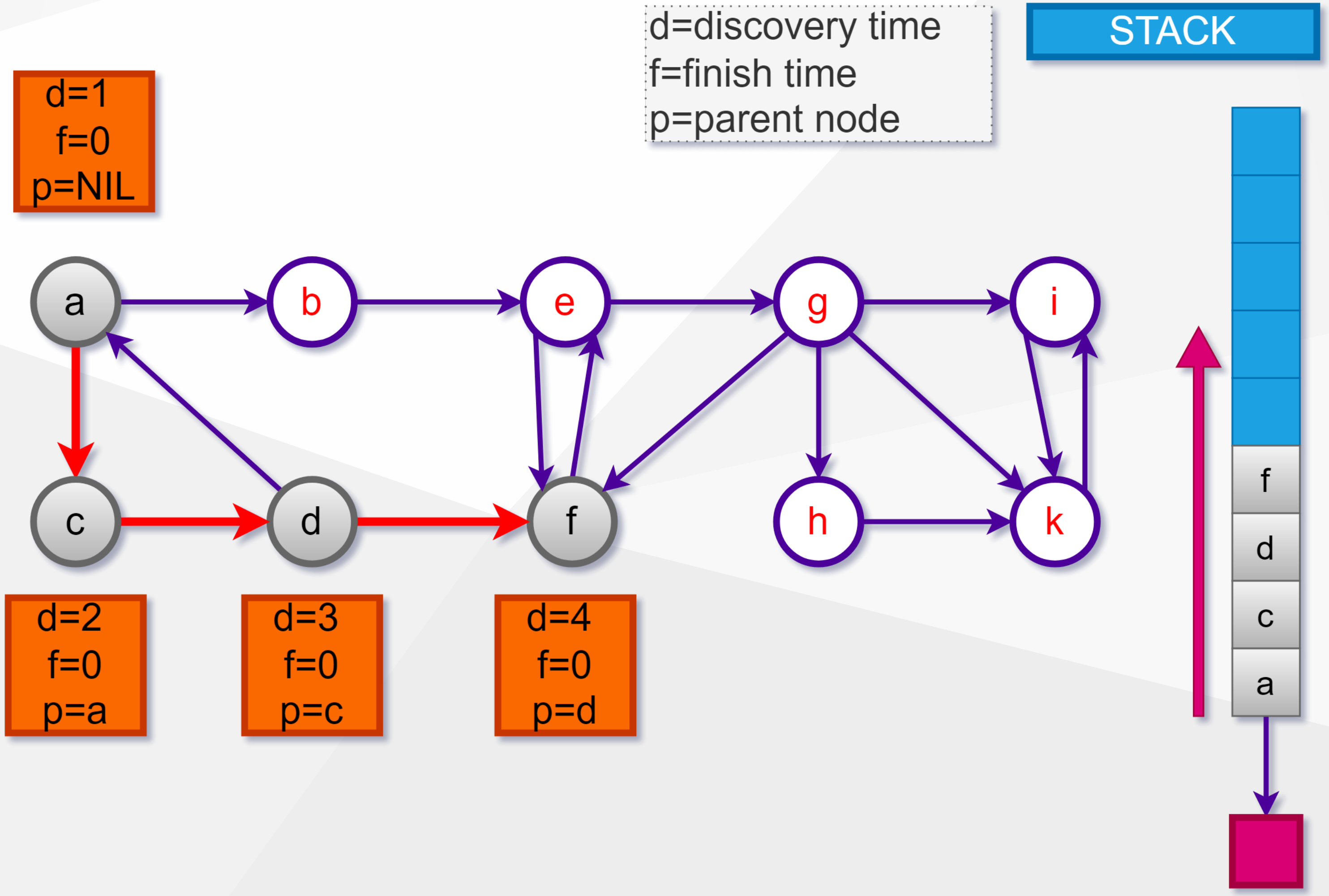
Graph Segmentation

SCC Algorithm - Example-1 / Step - 3



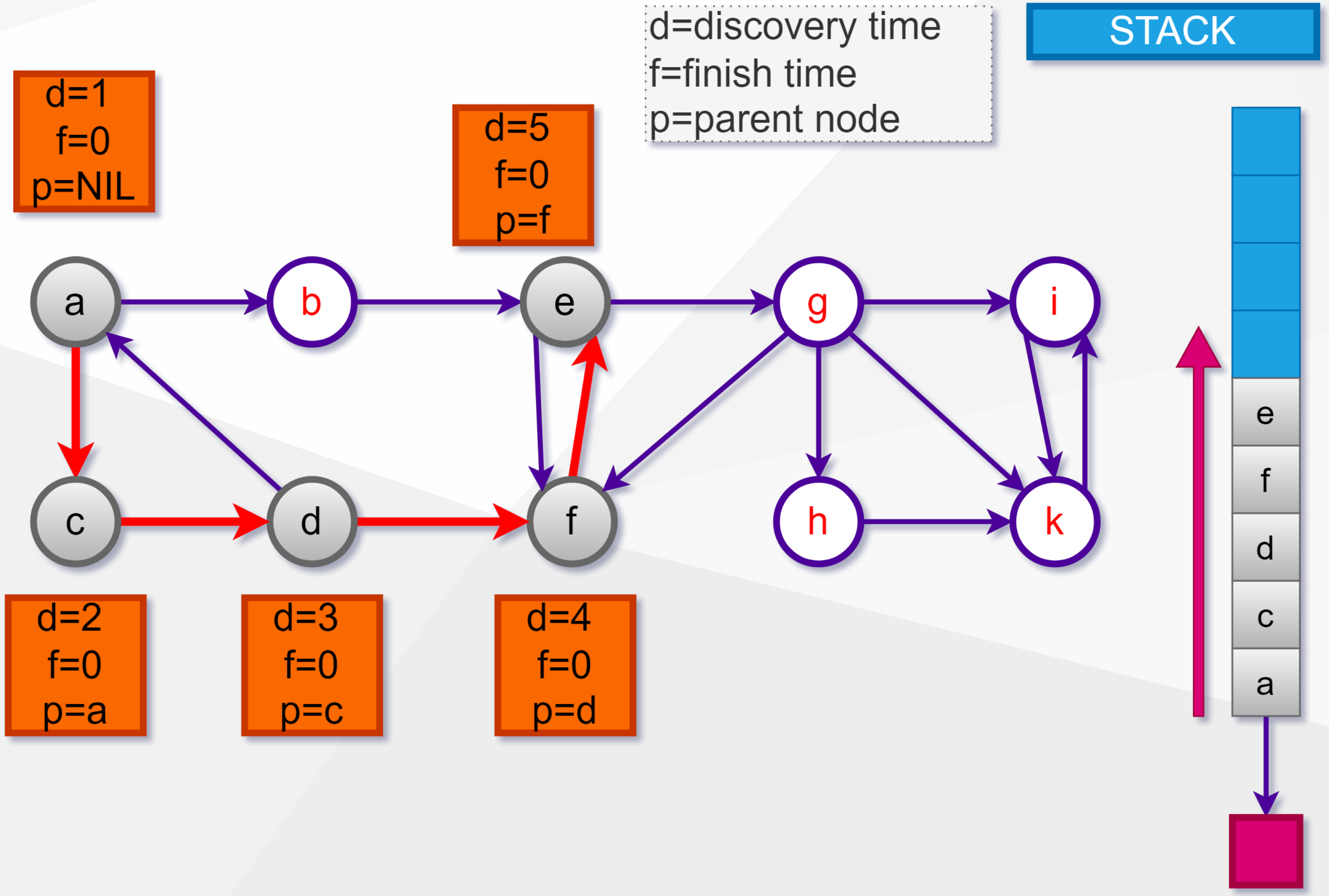
Graph Segmentation

SCC Algorithm - Example-1/ Step - 4



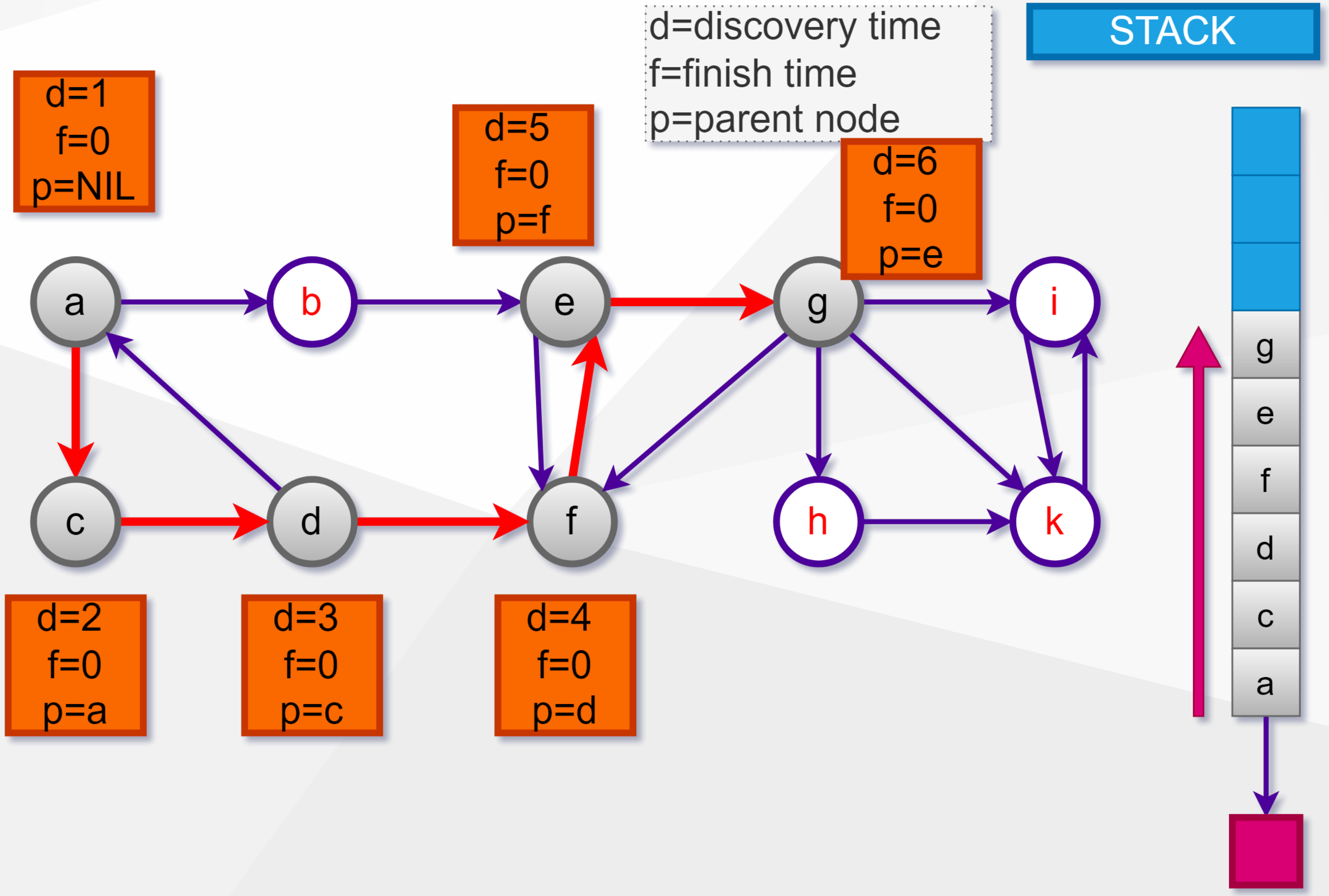
Graph Segmentation

SCC Algorithm - Example-1/ Step - 5



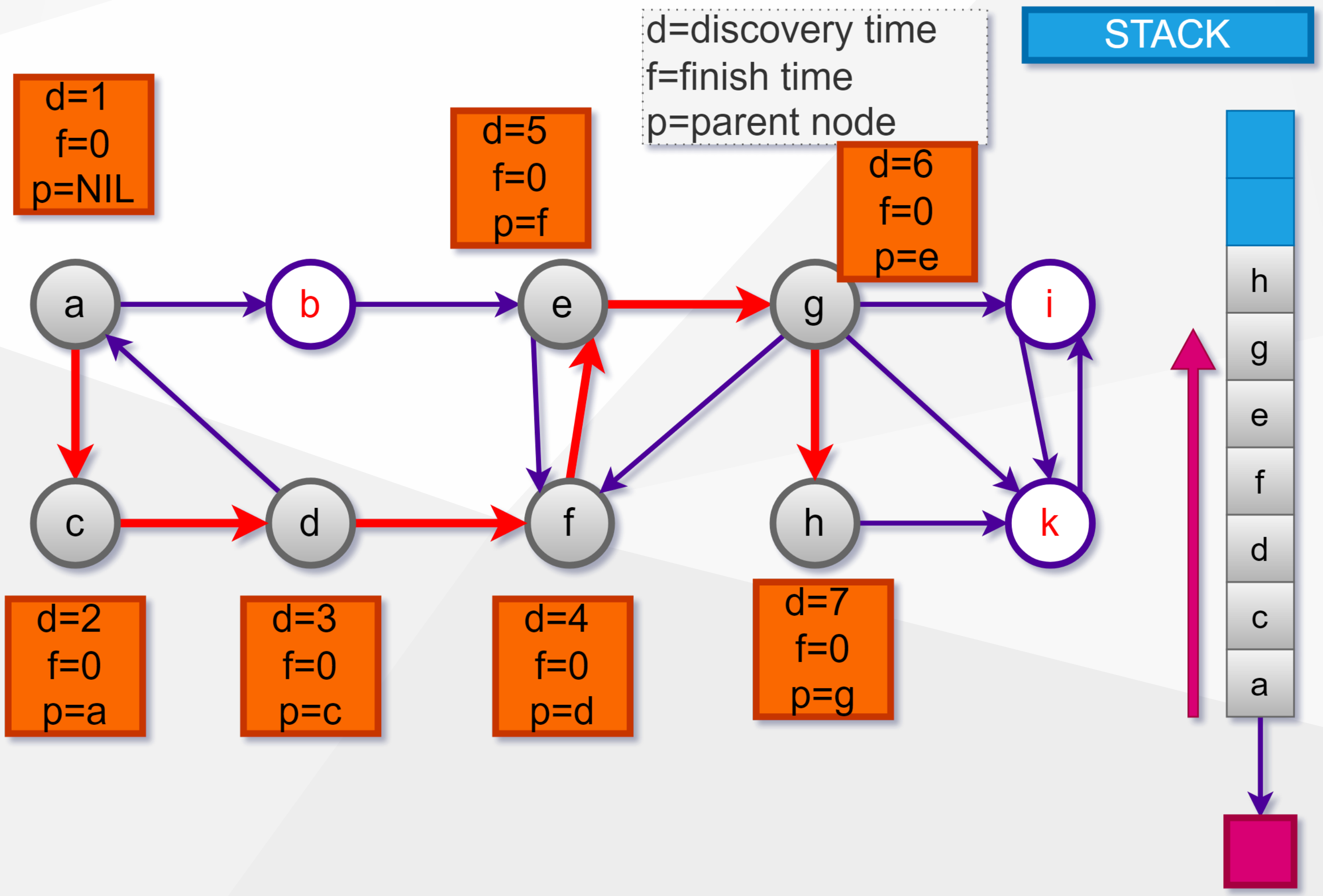
Graph Segmentation

SCC Algorithm - Example-1/ Step - 6



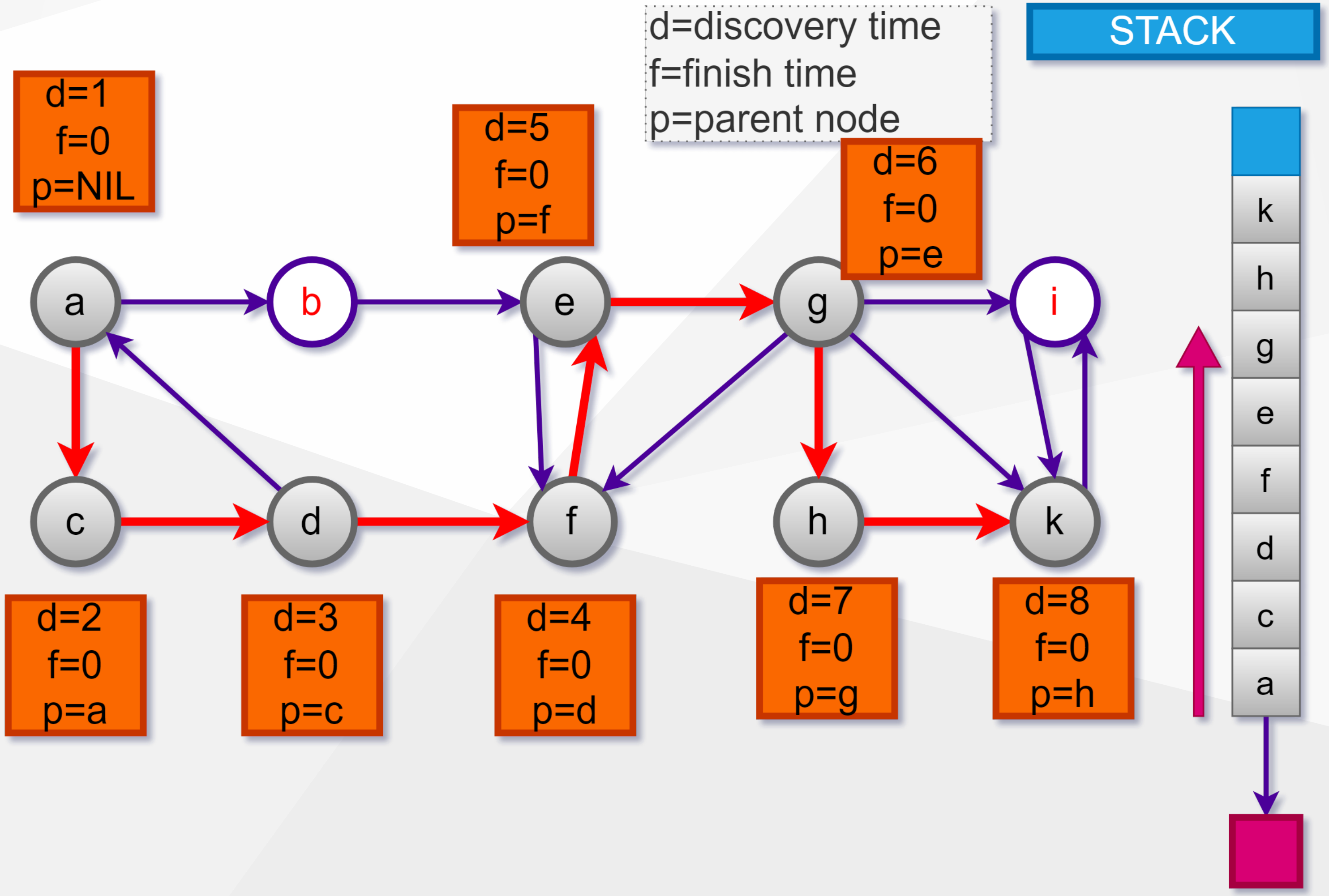
Graph Segmentation

SCC Algorithm - Example-1/ Step - 7



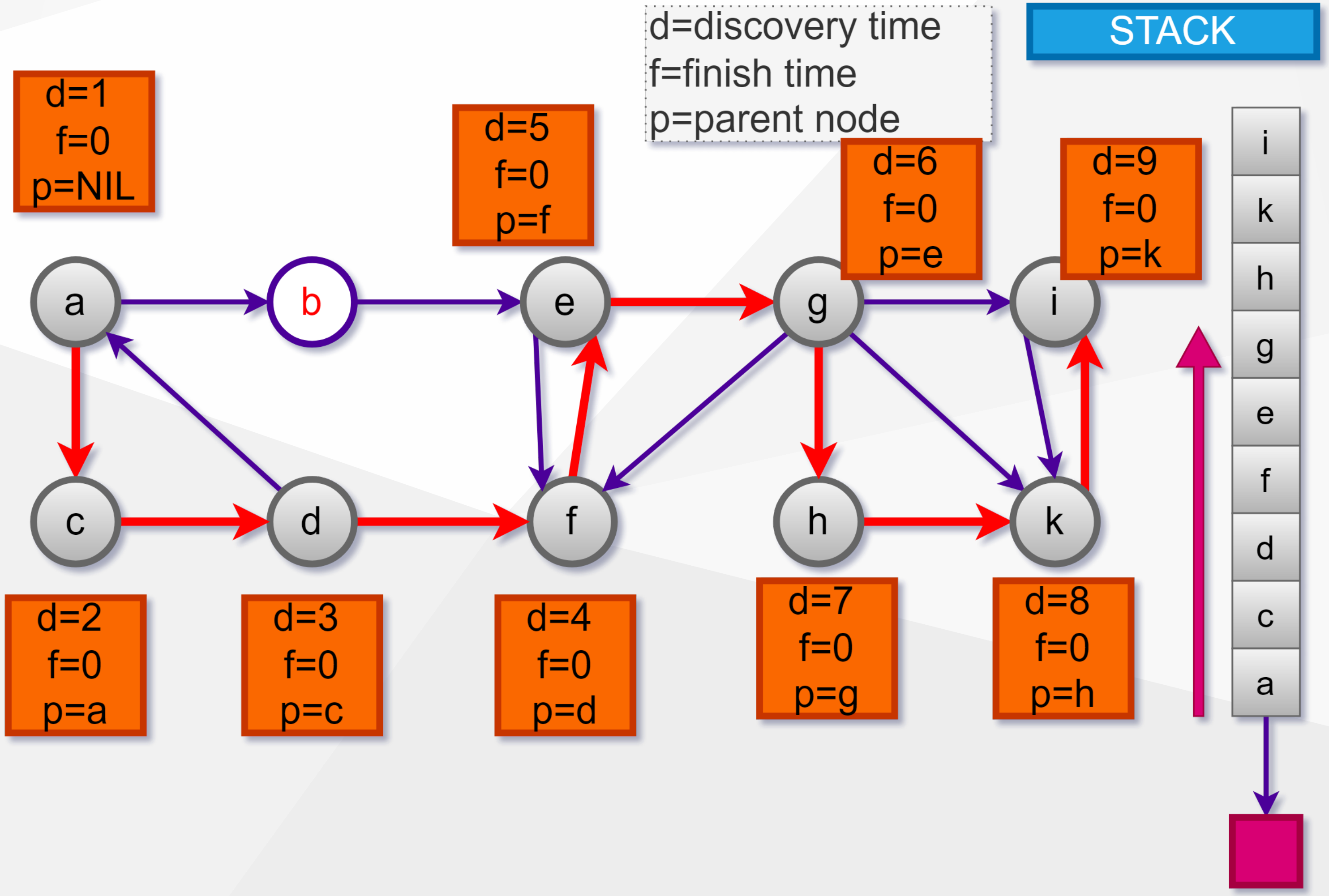
Graph Segmentation

SCC Algorithm - Example-1/ Step - 8



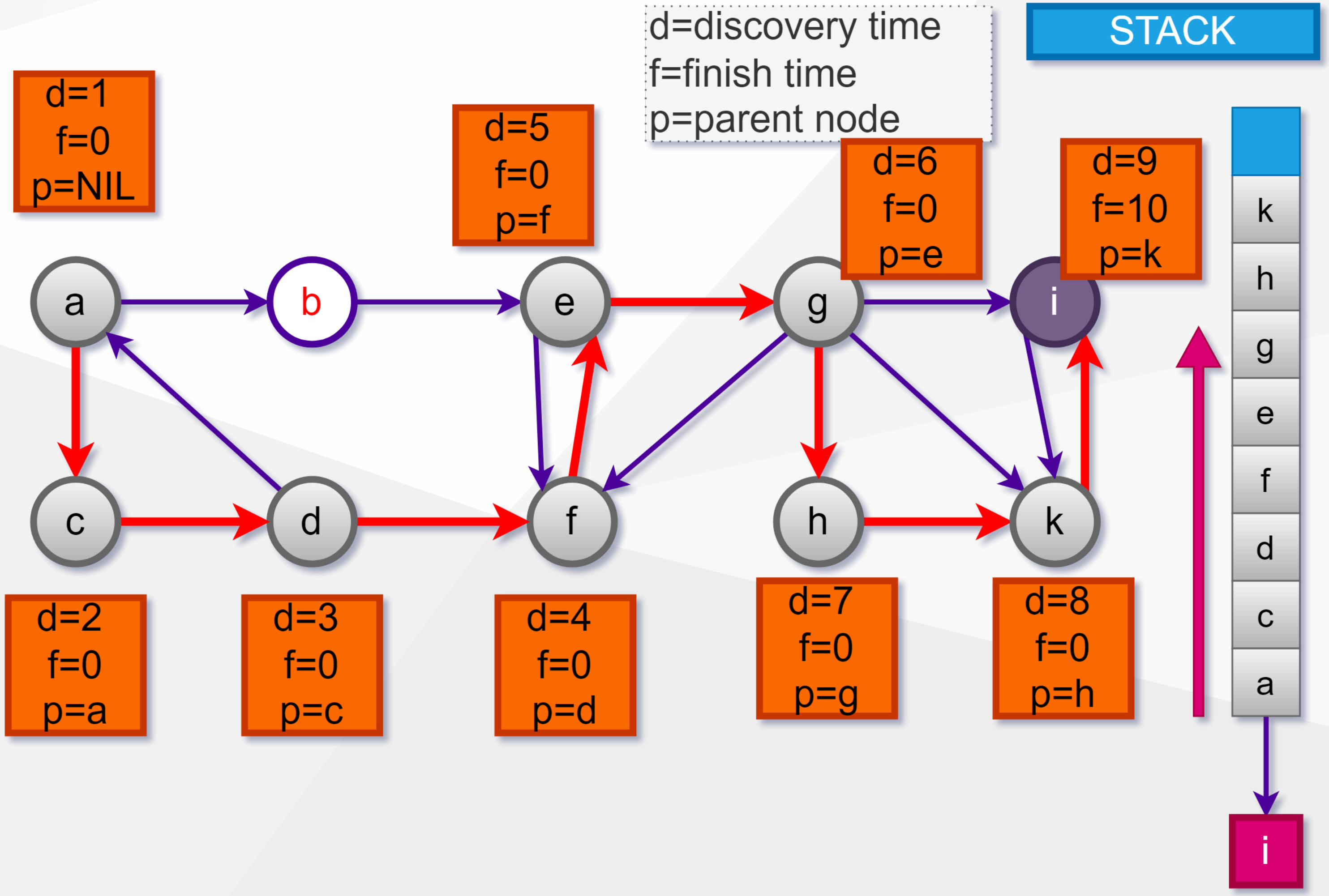
Graph Segmentation

SCC Algorithm - Example-1/ Step - 9



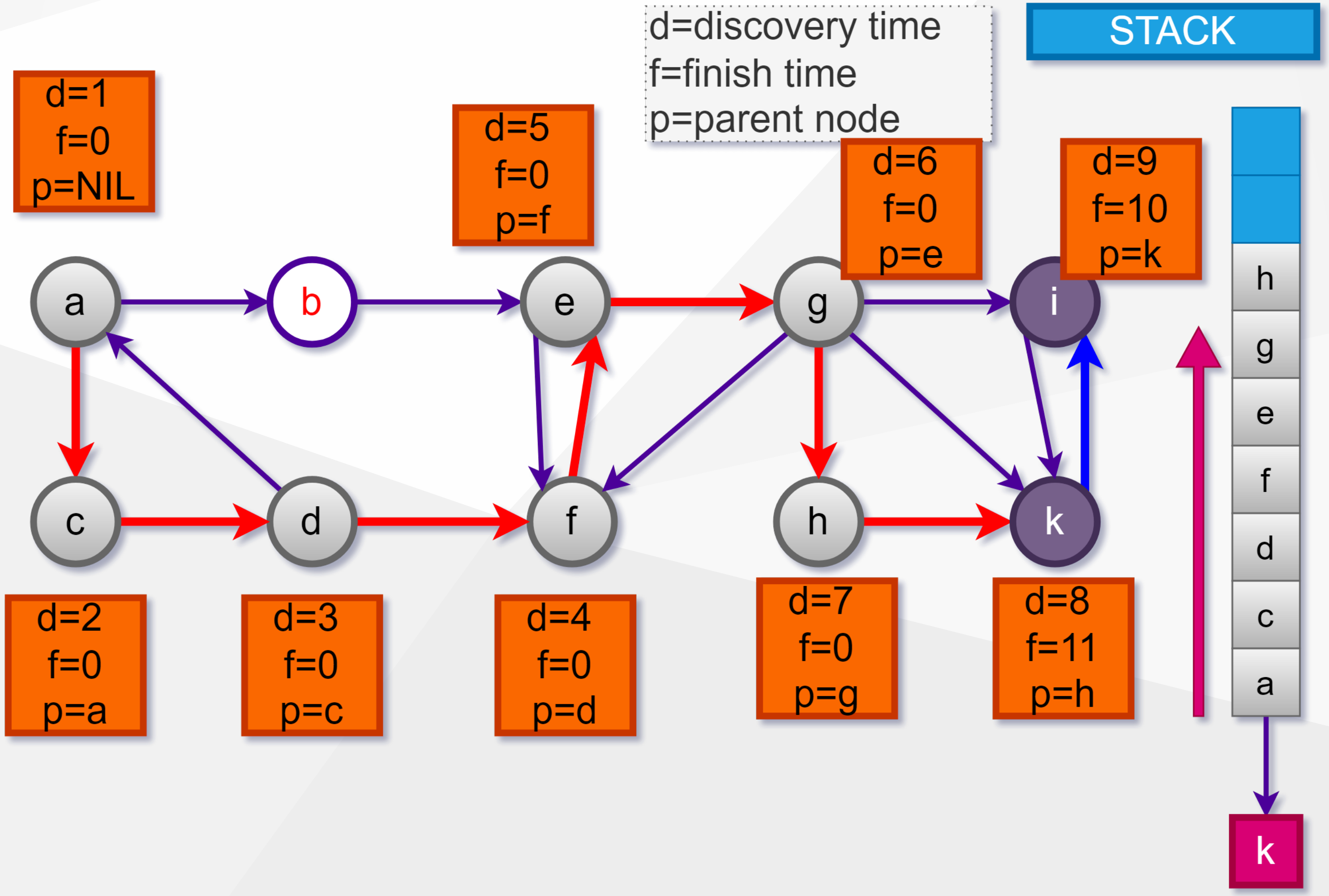
Graph Segmentation

SCC Algorithm - Example-1/ Step - 10



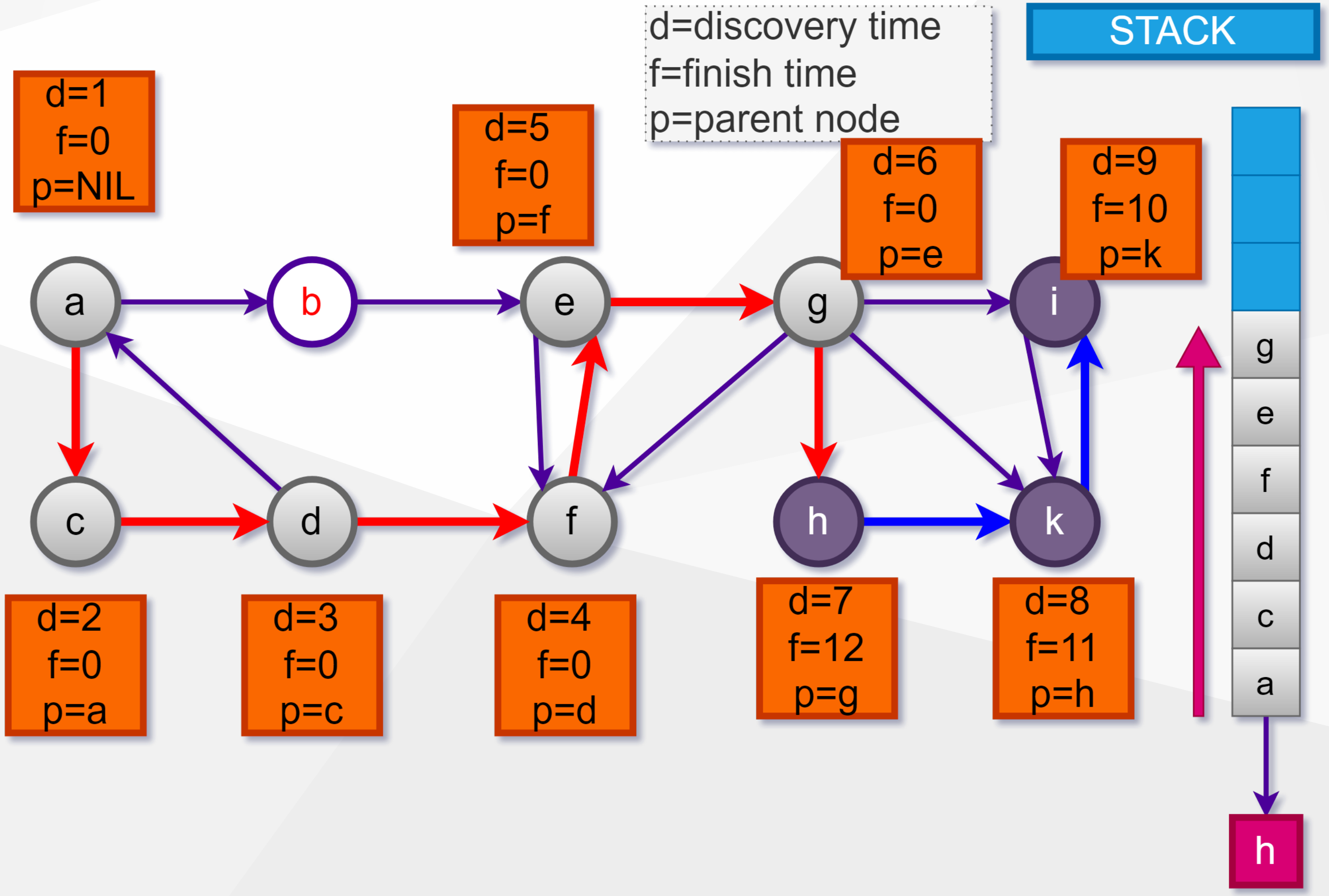
Graph Segmentation

SCC Algorithm - Example-1/ Step - 11



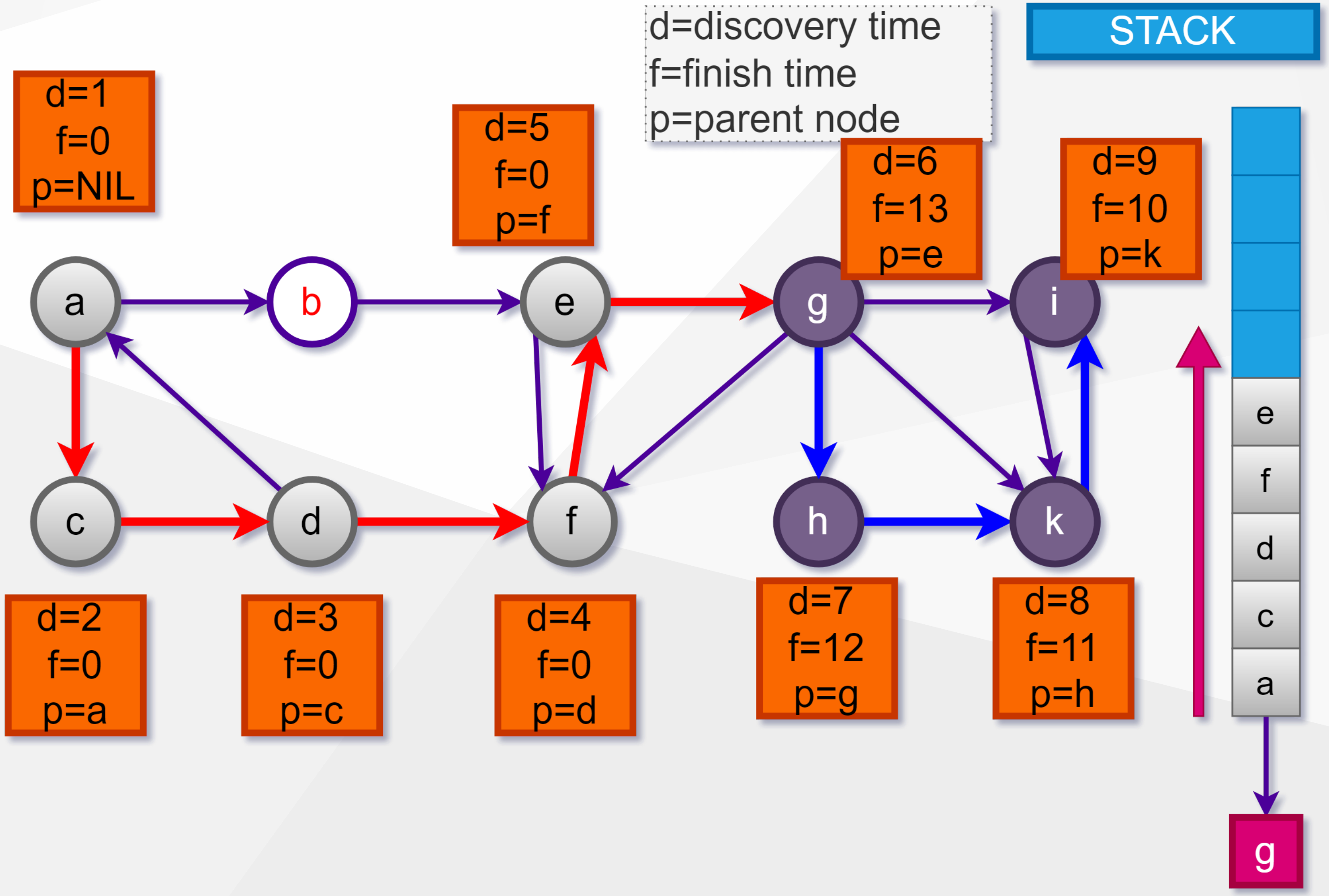
Graph Segmentation

SCC Algorithm - Example-1/ Step - 12



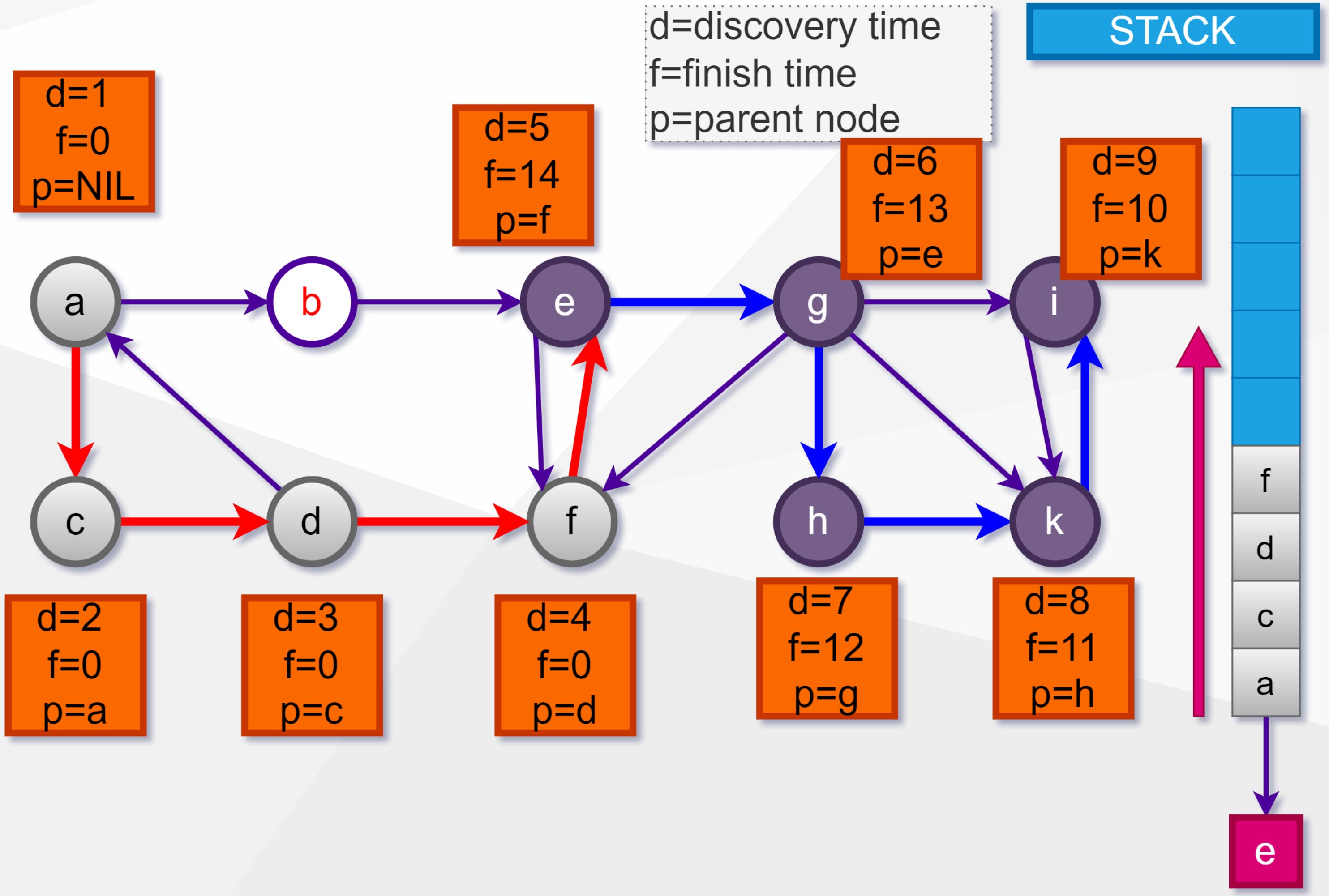
Graph Segmentation

SCC Algorithm - Example-1/ Step - 13



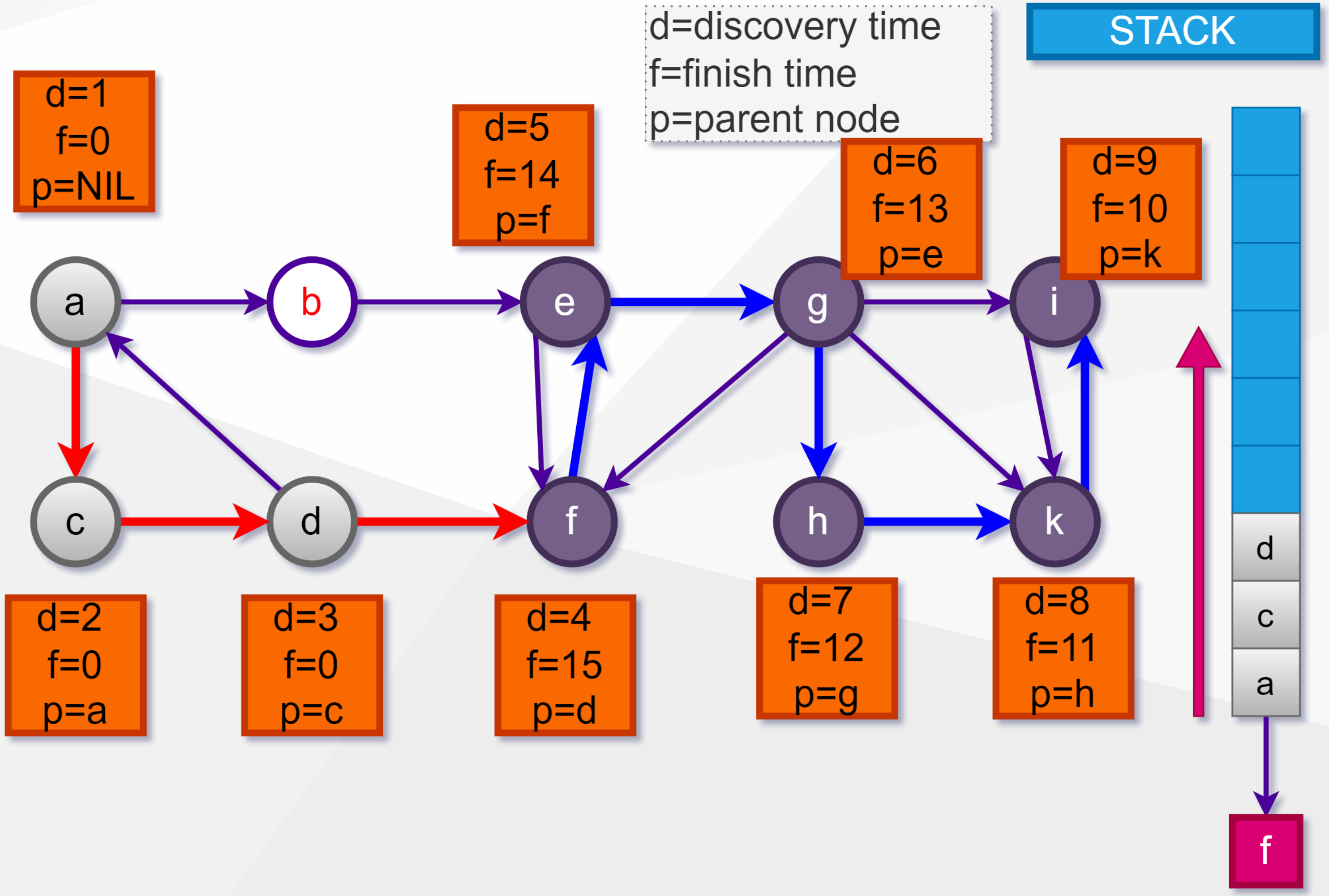
Graph Segmentation

SCC Algorithm - Example-1/ Step - 14



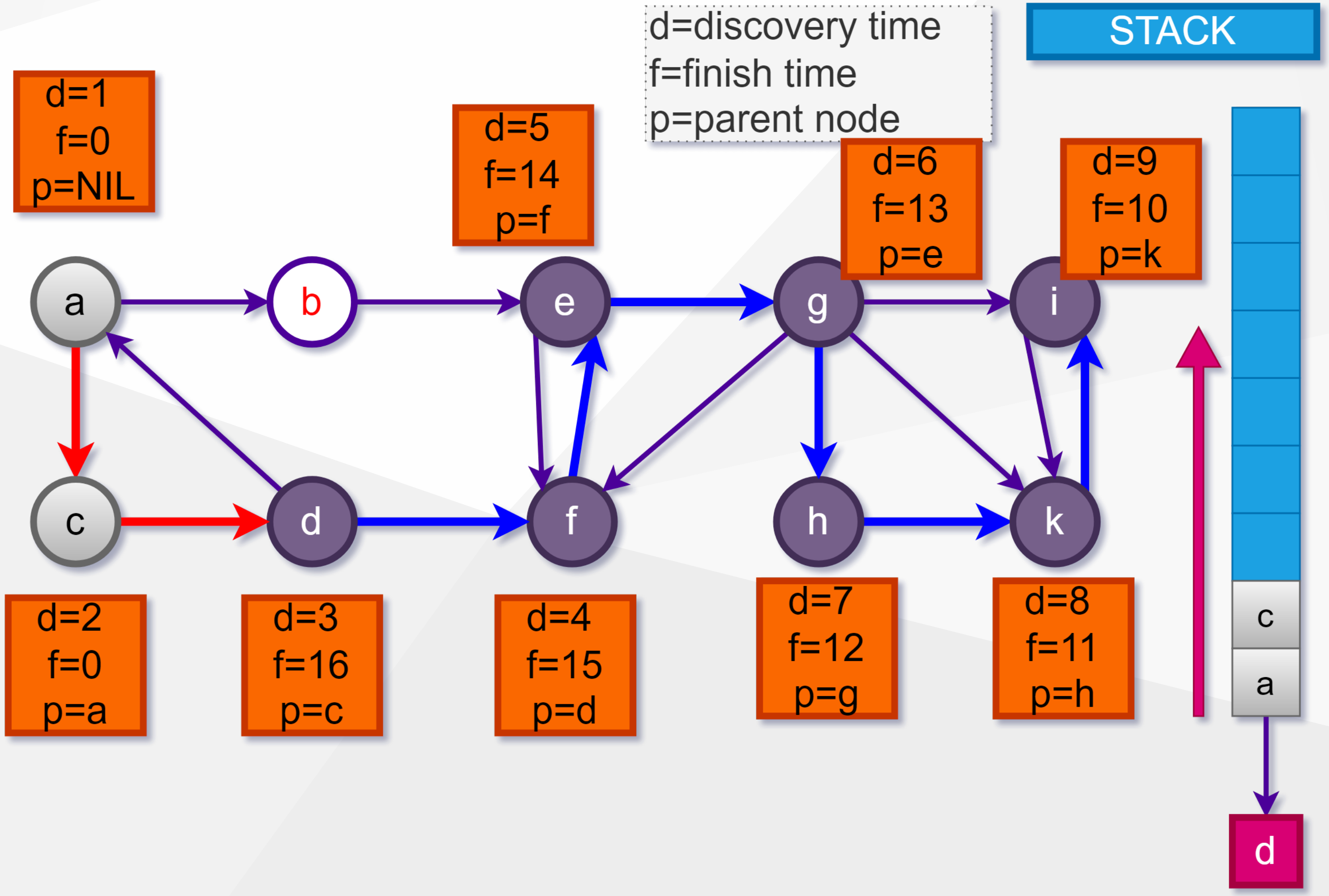
Graph Segmentation

SCC Algorithm - Example-1/ Step - 15



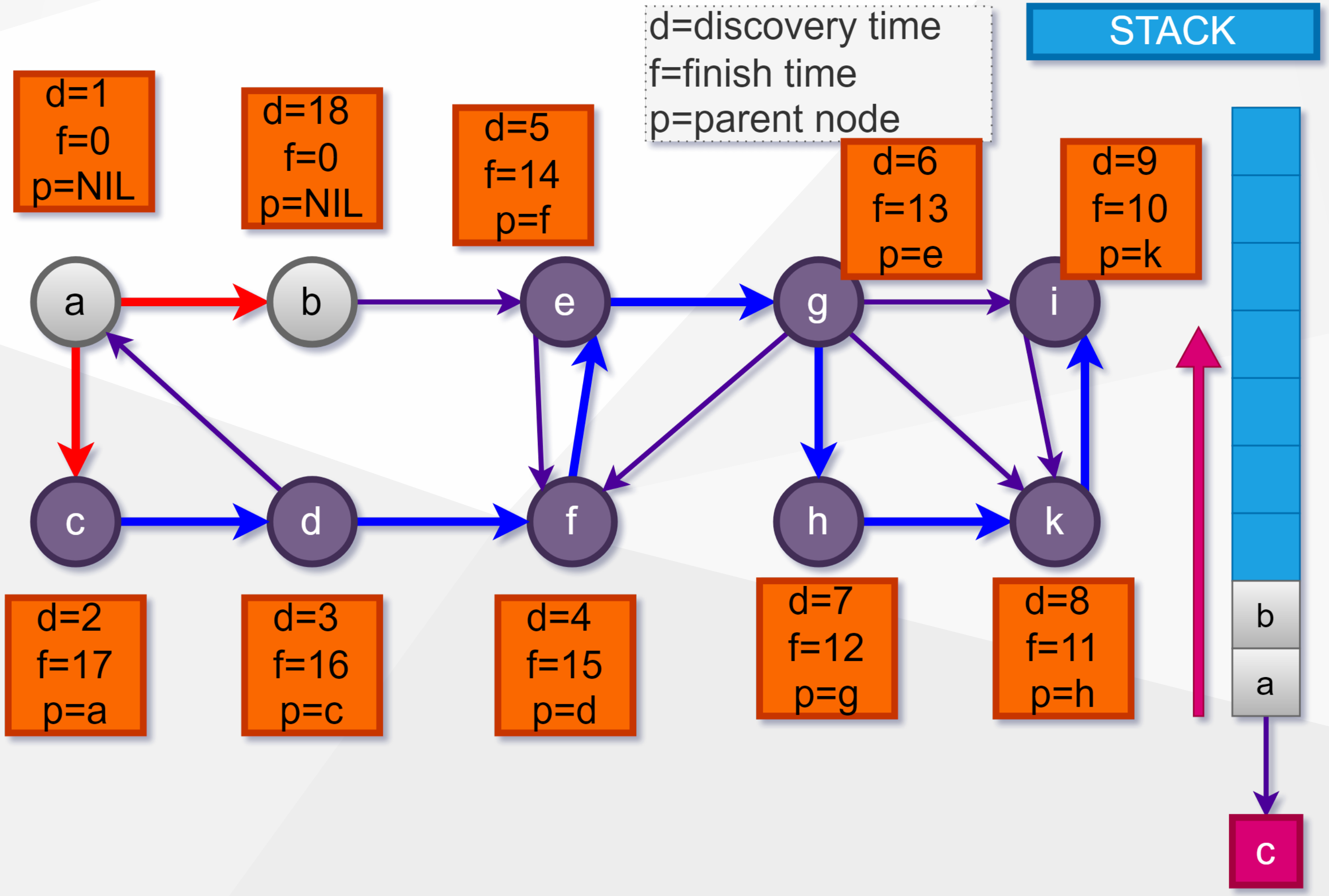
Graph Segmentation

SCC Algorithm - Example-1/ Step - 16



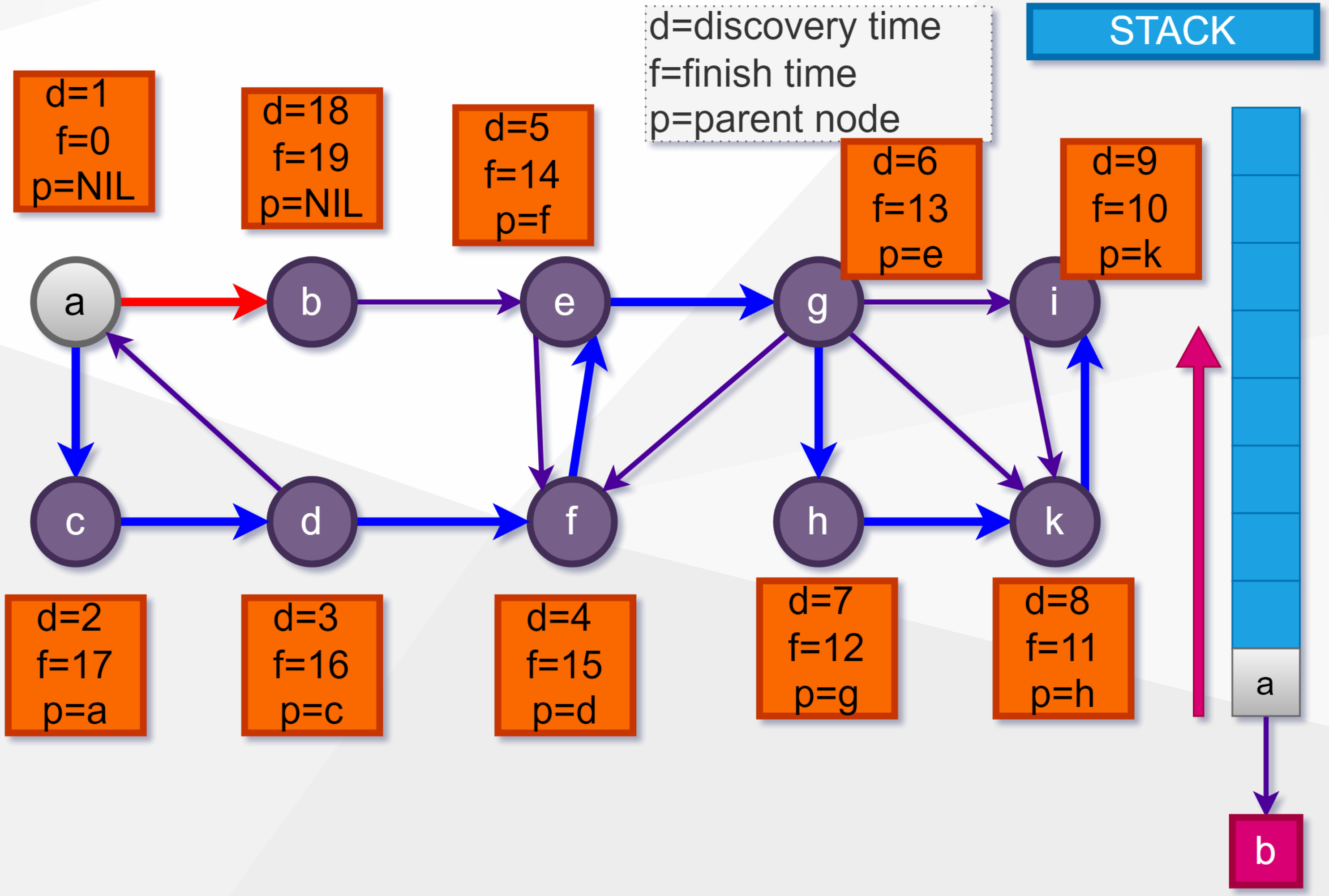
Graph Segmentation

SCC Algorithm - Example-1/ Step - 17



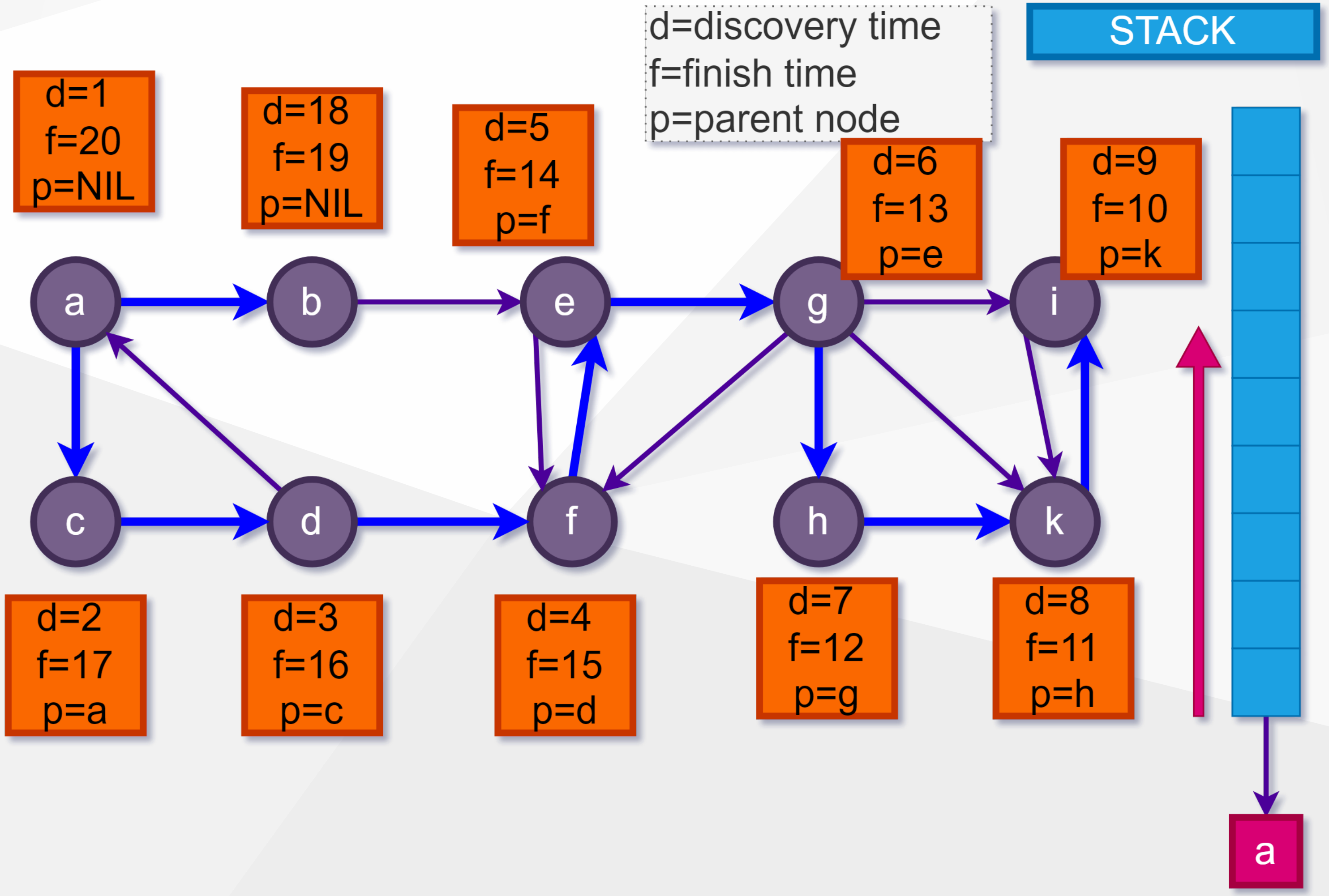
Graph Segmentation

SCC Algorithm - Example-1/ Step - 18



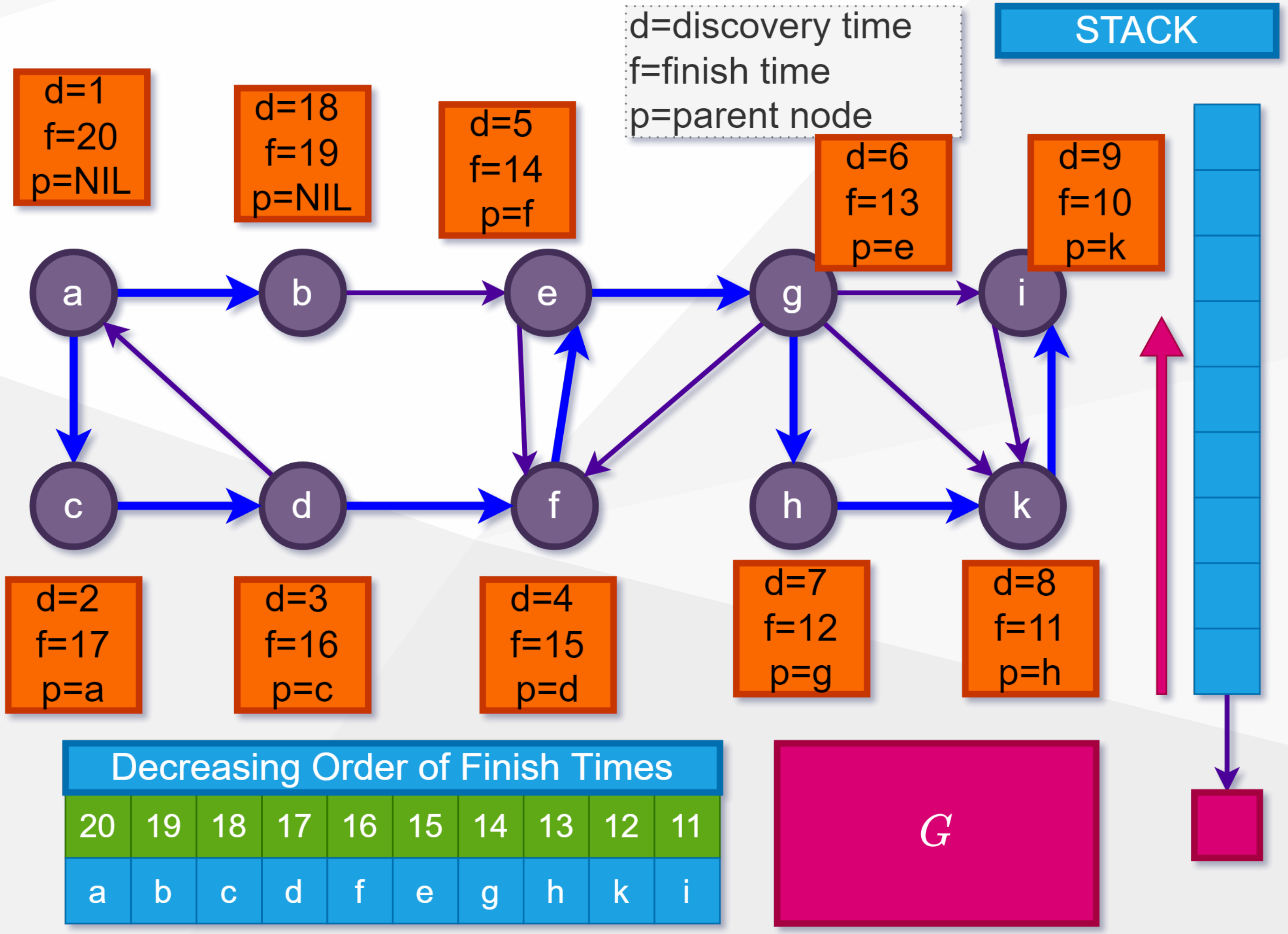
Graph Segmentation

SCC Algorithm - Example-1/ Step - 19



Graph Segmentation

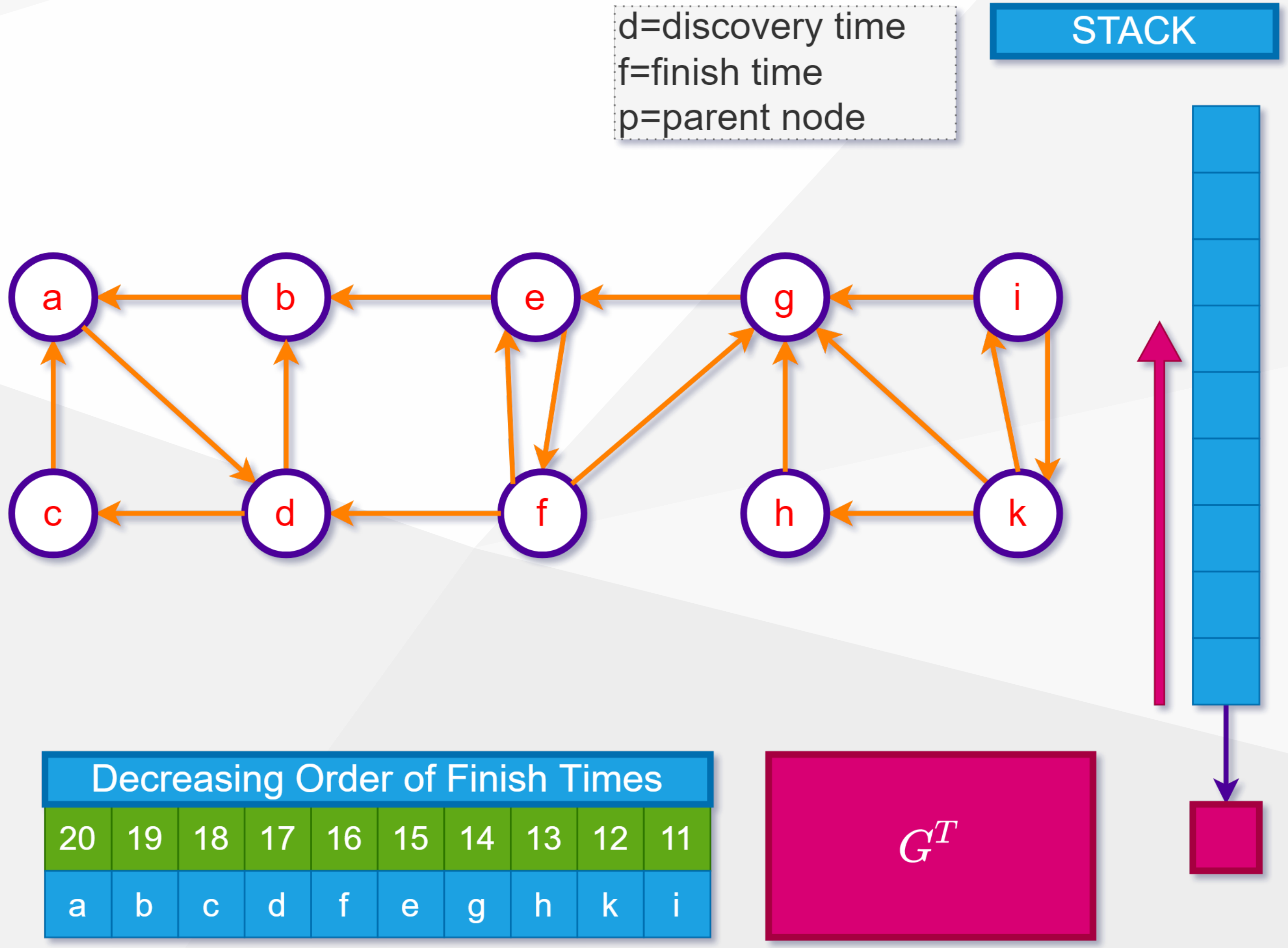
SCC Algorithm - Example-1/ Step - 20



Decreasing Order of Finish Times									
20	19	18	17	16	15	14	13	12	11
a	b	c	d	f	e	g	h	k	i

Graph Segmentation

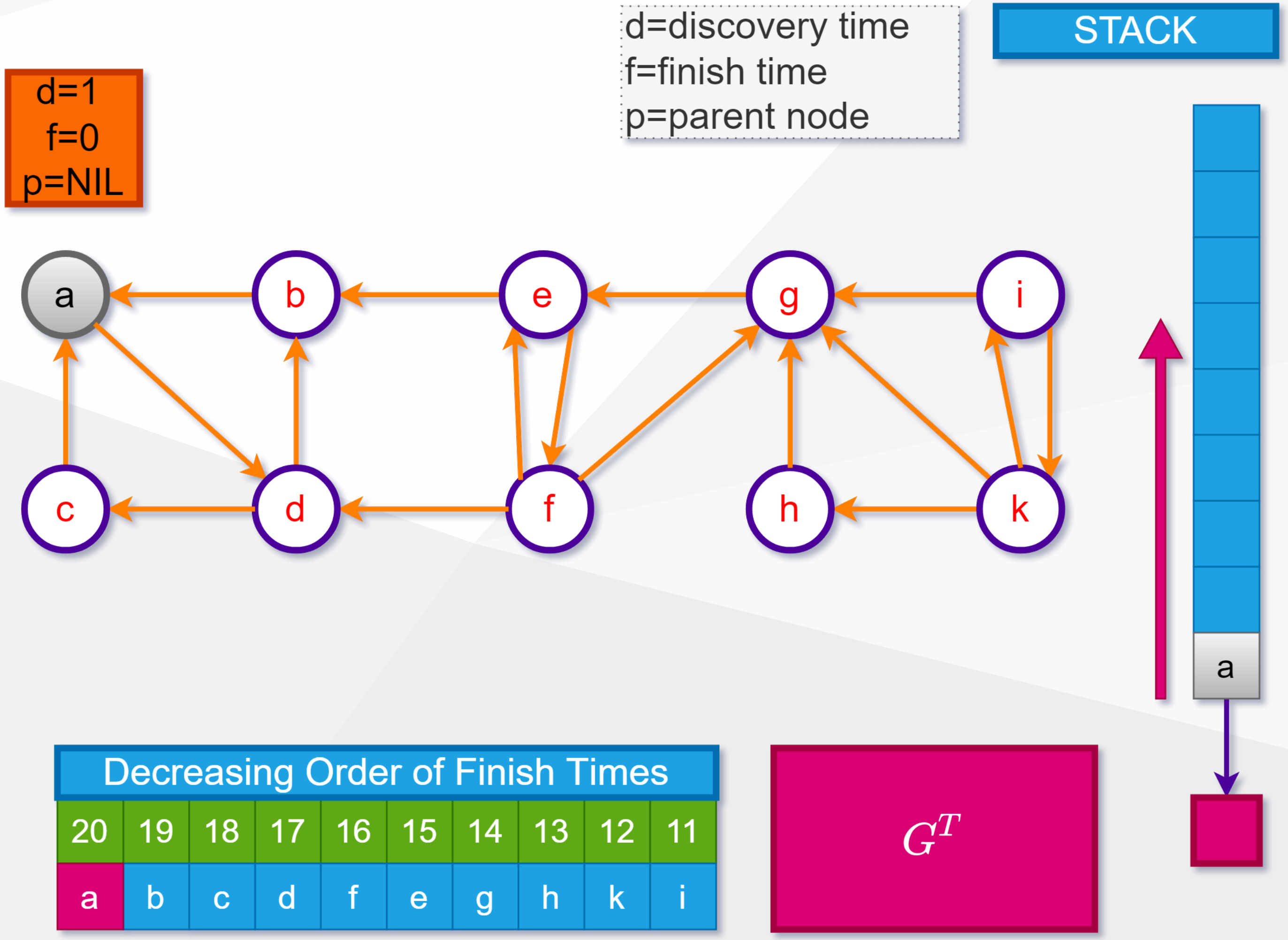
SCC Algorithm - Example-1/ Step - 21



Decreasing Order of Finish Times									
20	19	18	17	16	15	14	13	12	11
a	b	c	d	f	e	g	h	k	i

Graph Segmentation

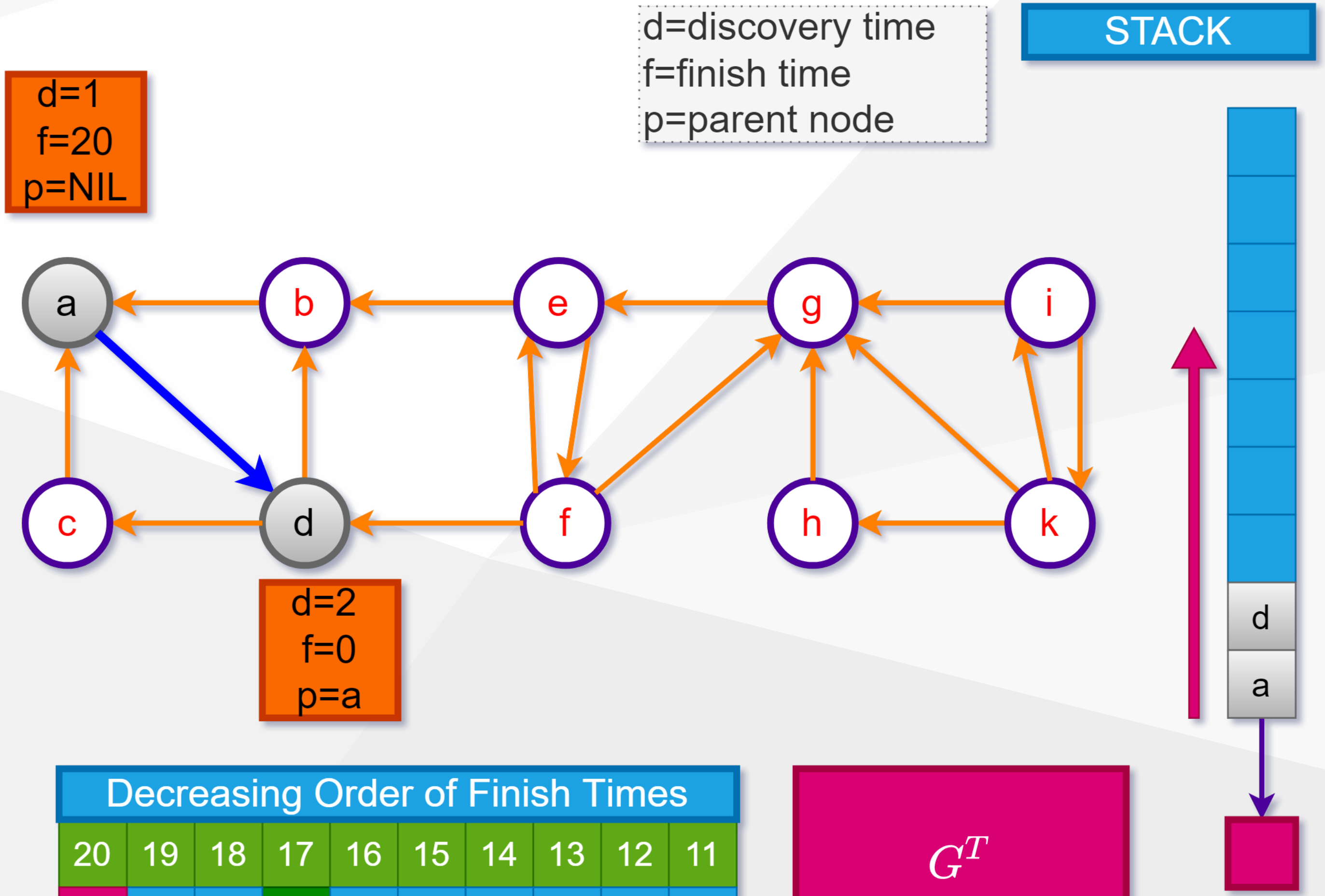
SCC Algorithm - Example-1/ Step - 22



Decreasing Order of Finish Times									
20	19	18	17	16	15	14	13	12	11
a	b	c	d	f	e	g	h	k	i

Graph Segmentation

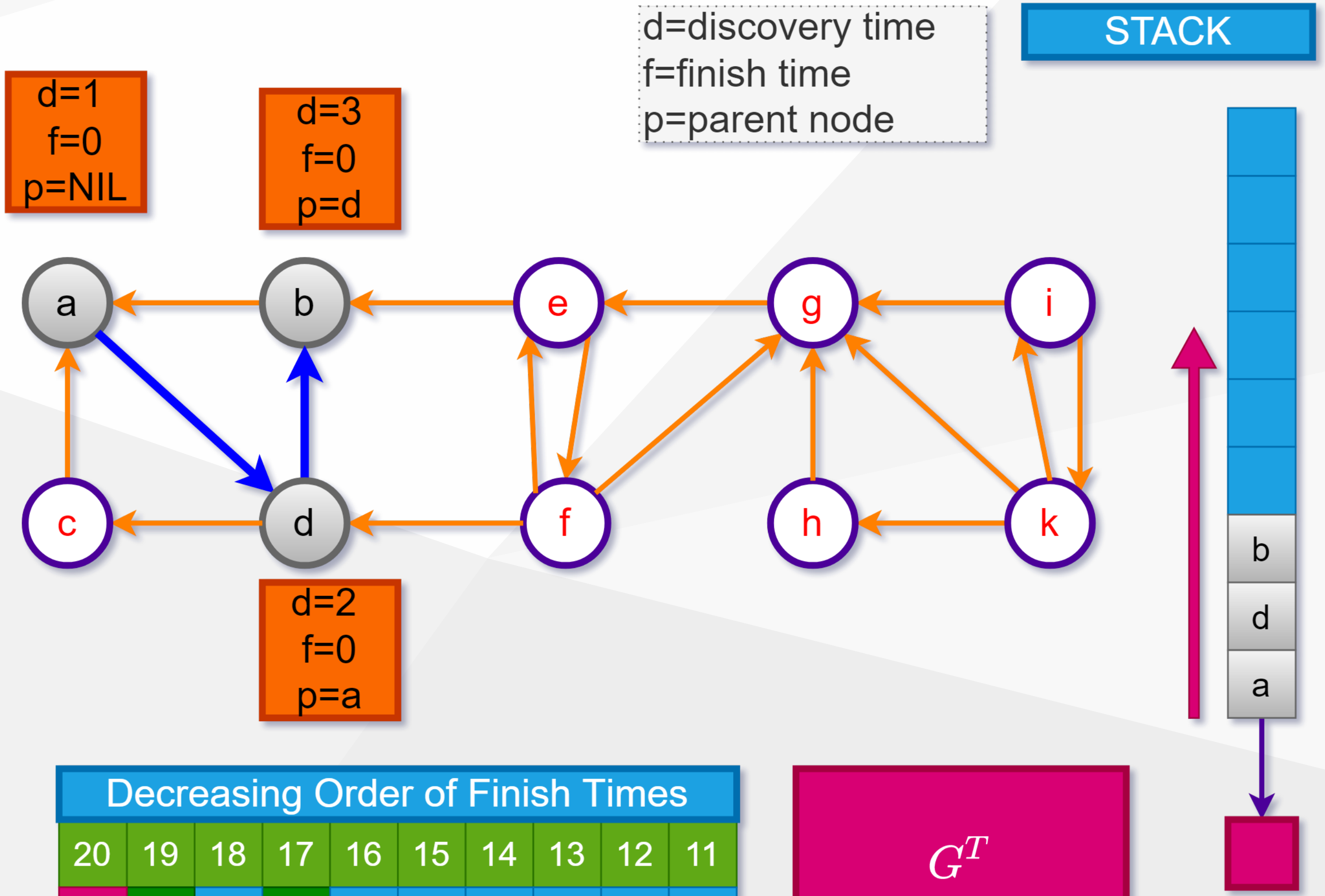
SCC Algorithm - Example-1/ Step - 23



Decreasing Order of Finish Times									
20	19	18	17	16	15	14	13	12	11
a	b	c	d	f	e	g	h	k	i

Graph Segmentation

SCC Algorithm - Example-1/ Step - 24

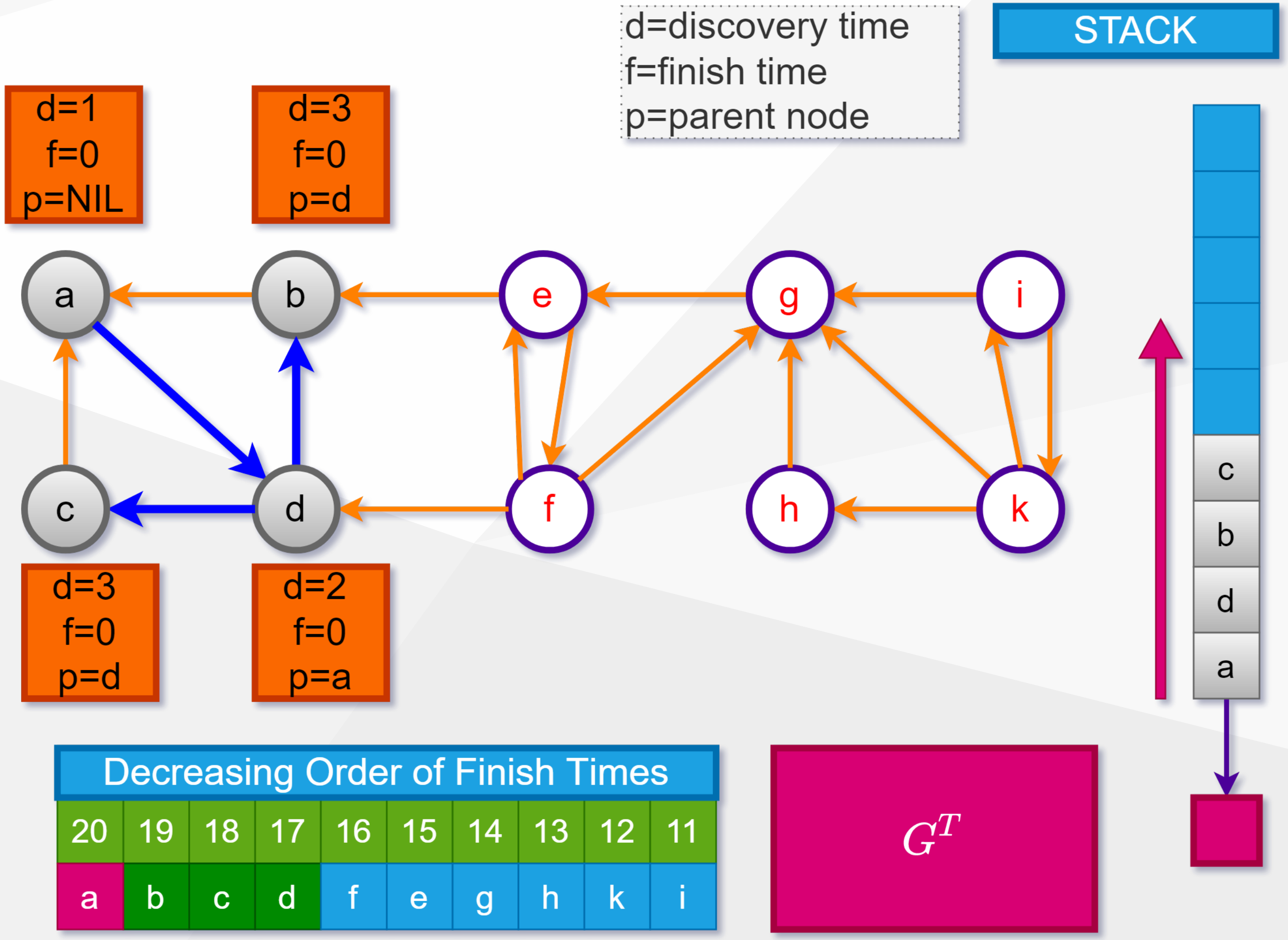


Decreasing Order of Finish Times									
20	19	18	17	16	15	14	13	12	11
a	b	c	d	f	e	g	h	k	i

G^T

Graph Segmentation

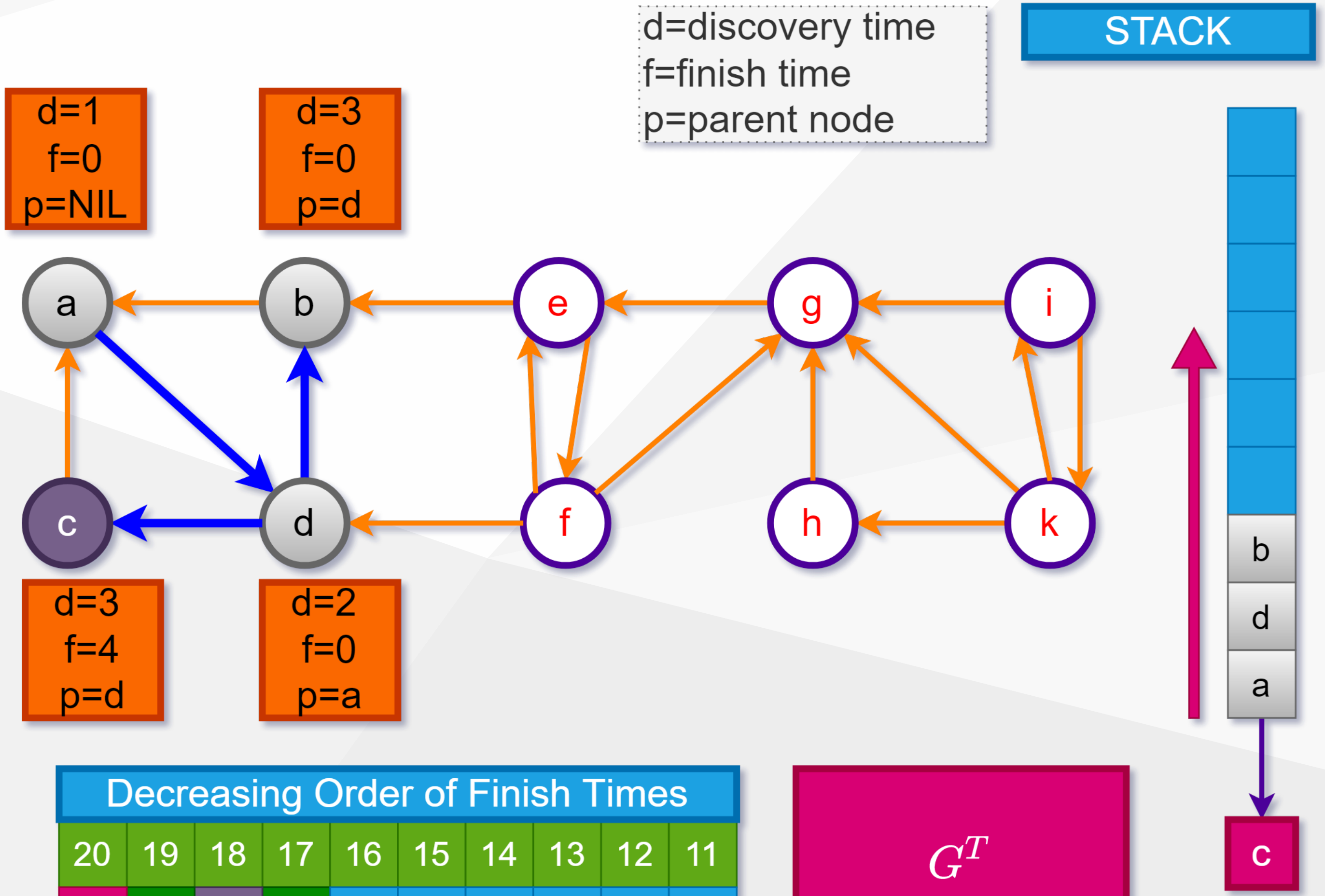
SCC Algorithm - Example-1/ Step - 25



Decreasing Order of Finish Times									
20	19	18	17	16	15	14	13	12	11
a	b	c	d	f	e	g	h	k	i

Graph Segmentation

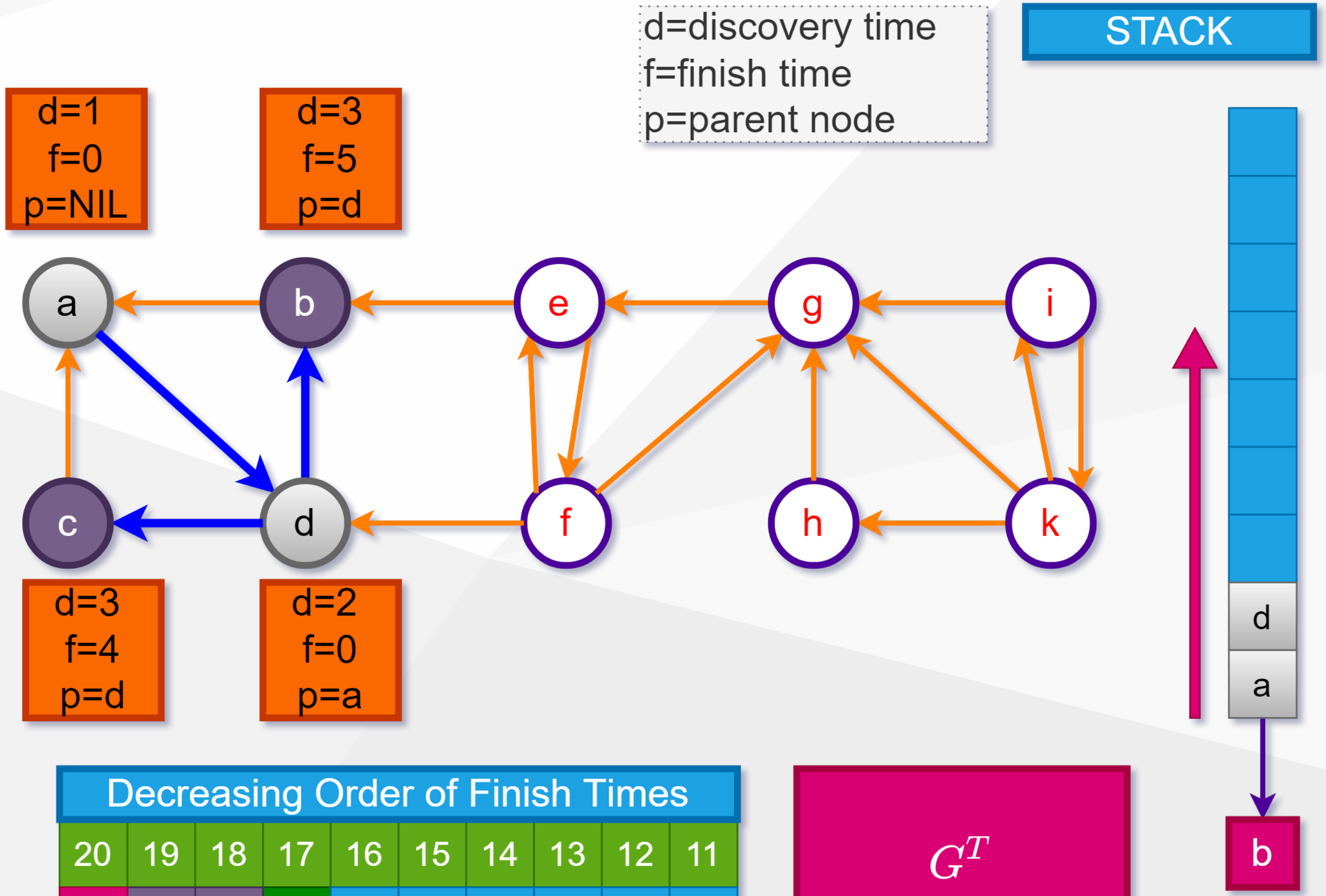
SCC Algorithm - Example-1/ Step - 26



Decreasing Order of Finish Times									
20	19	18	17	16	15	14	13	12	11
a	b	c	d	f	e	g	h	k	i

Graph Segmentation

SCC Algorithm - Example-1/ Step - 27

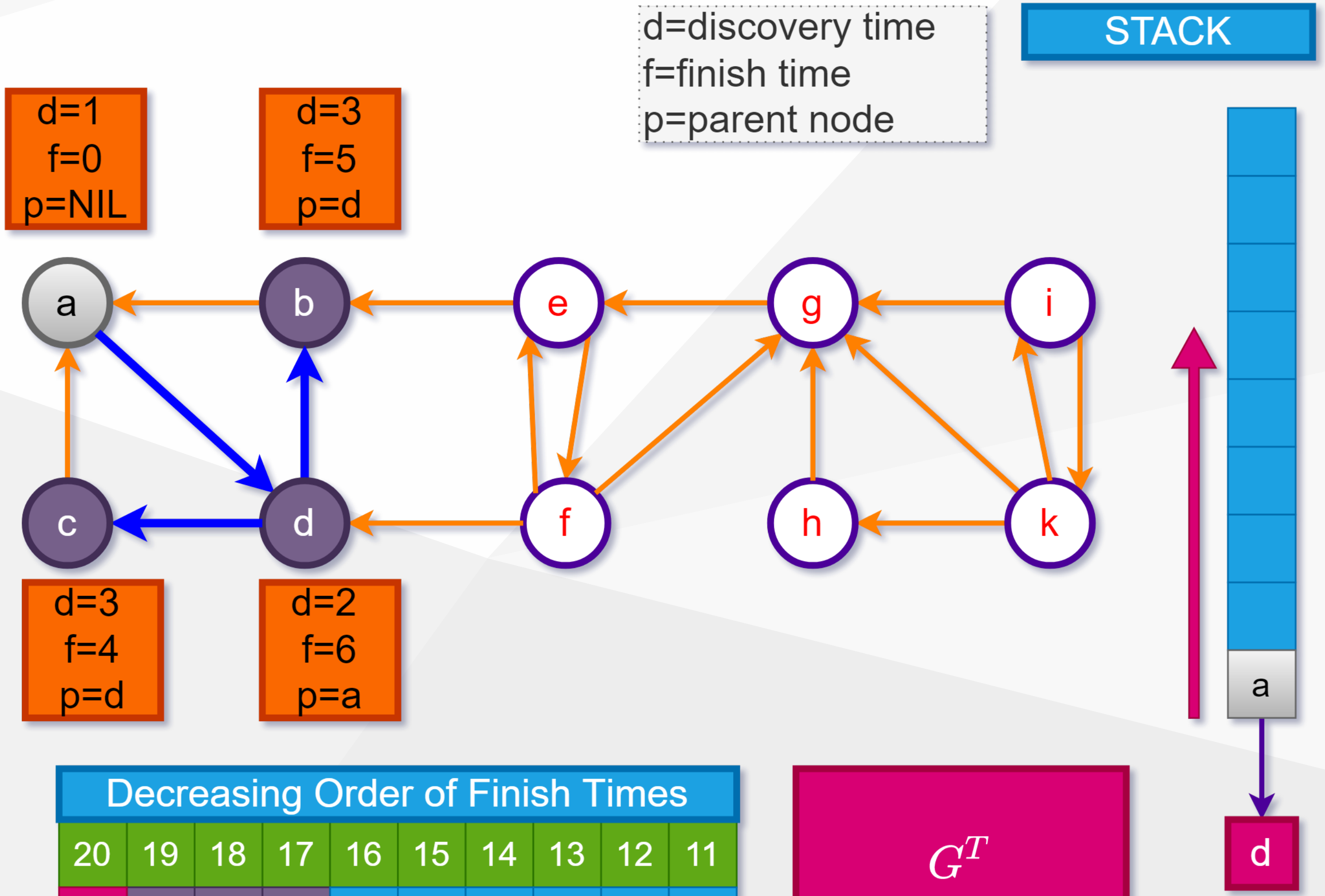


Decreasing Order of Finish Times									
20	19	18	17	16	15	14	13	12	11
a	b	c	d	f	e	g	h	k	i

G^T

Graph Segmentation

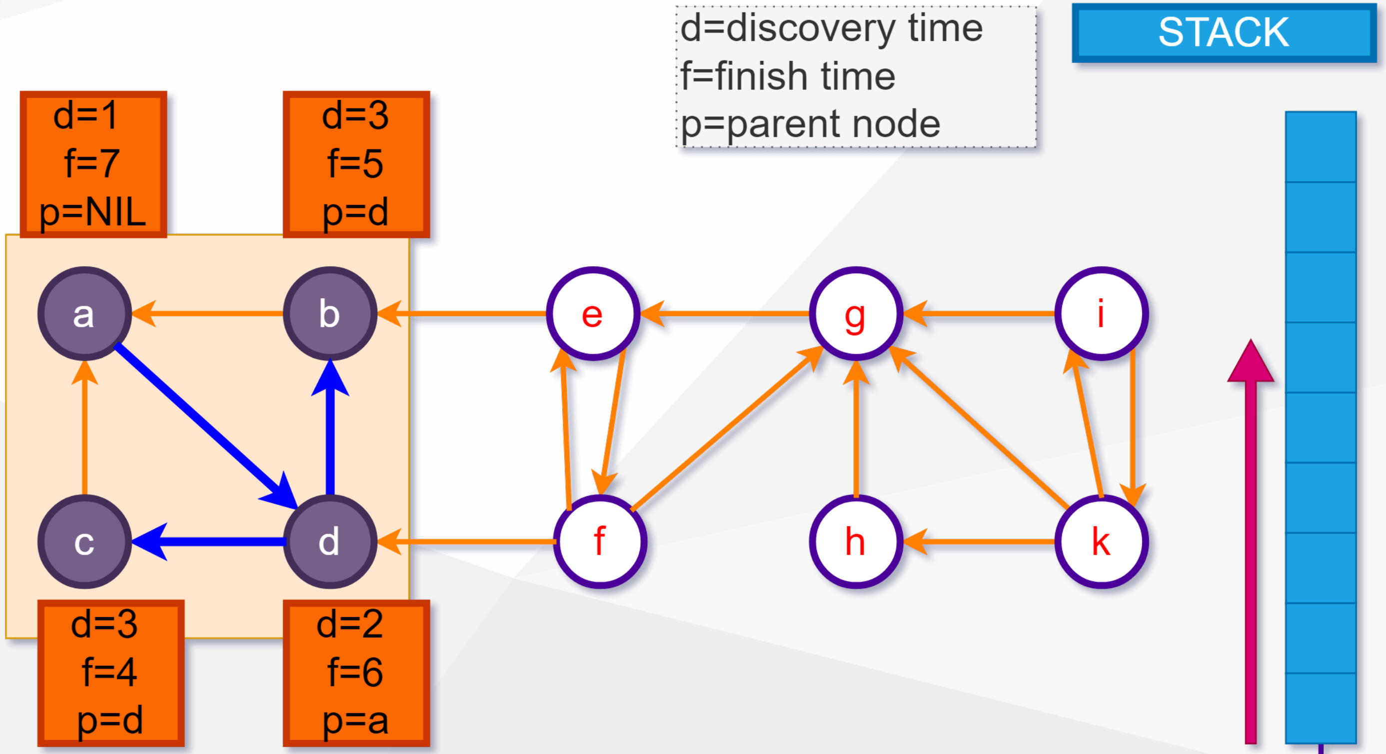
SCC Algorithm - Example-1/ Step - 28



Decreasing Order of Finish Times									
20	19	18	17	16	15	14	13	12	11
a	b	c	d	f	e	g	h	k	i

Graph Segmentation

SCC Algorithm - Example-1/ Step - 29



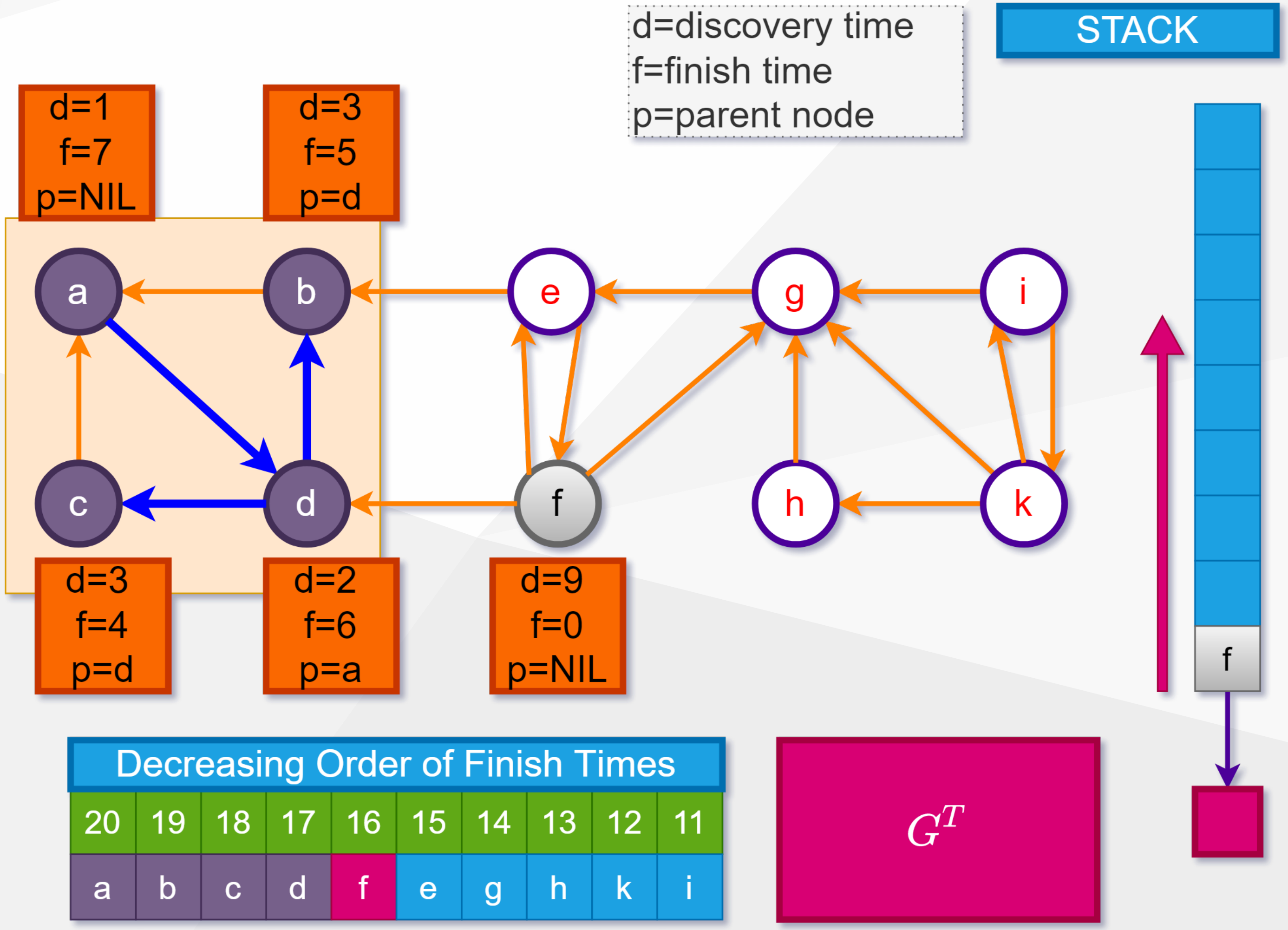
Decreasing Order of Finish Times									
20	19	18	17	16	15	14	13	12	11
a	b	c	d	f	e	g	h	k	i

G^T

a

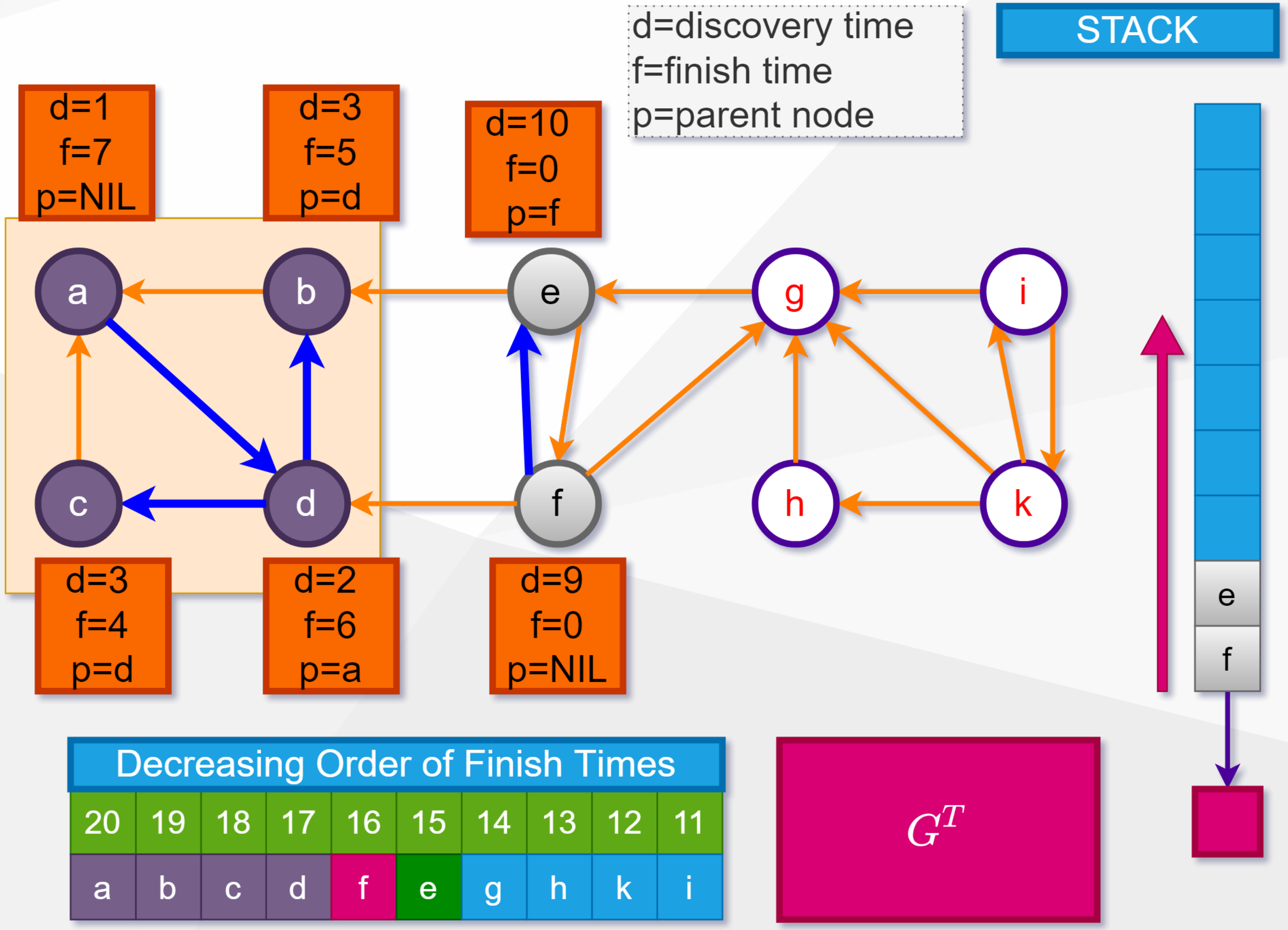
Graph Segmentation

SCC Algorithm - Example-1/ Step - 30



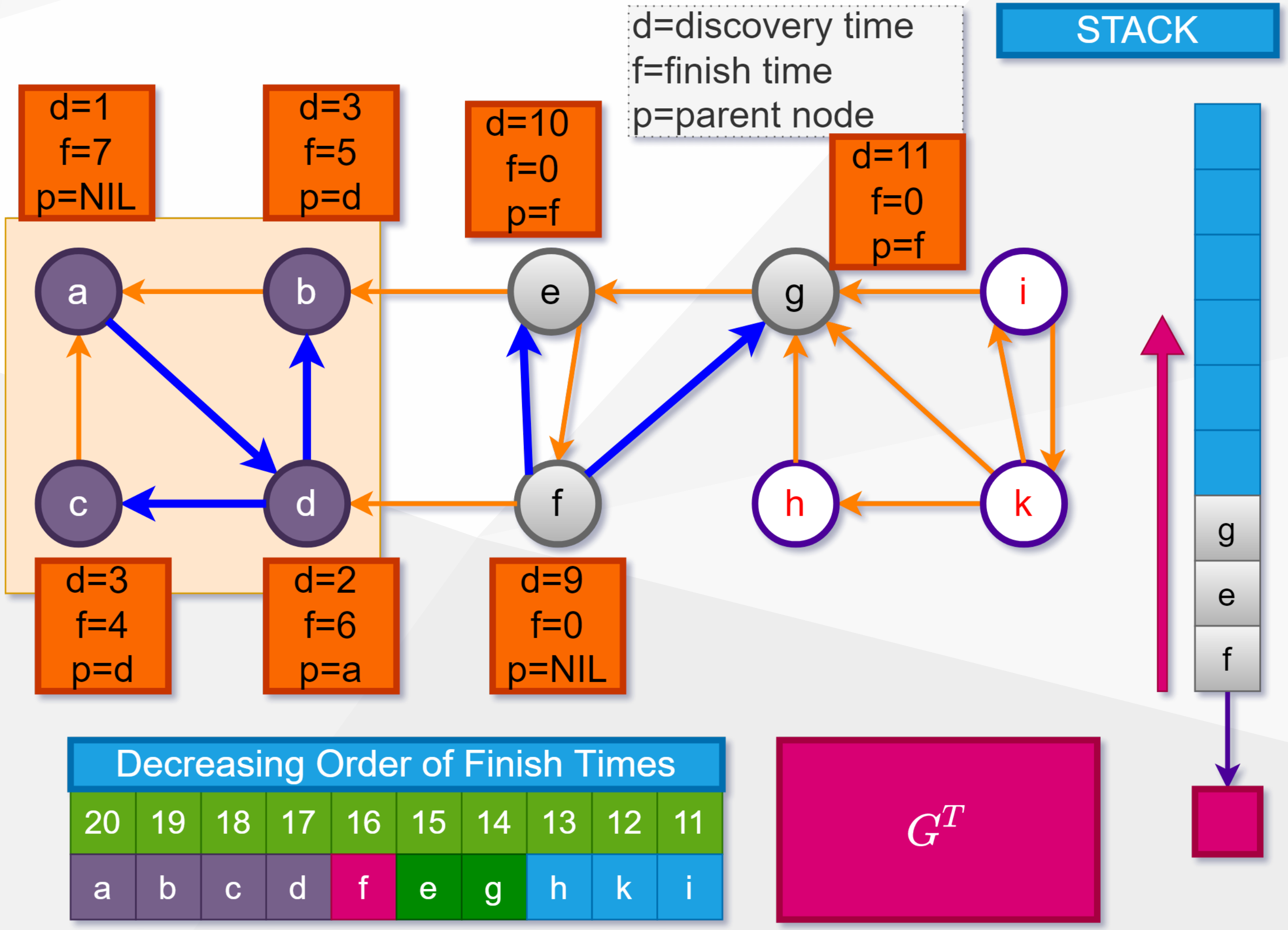
Graph Segmentation

SCC Algorithm - Example-1/ Step - 31



Graph Segmentation

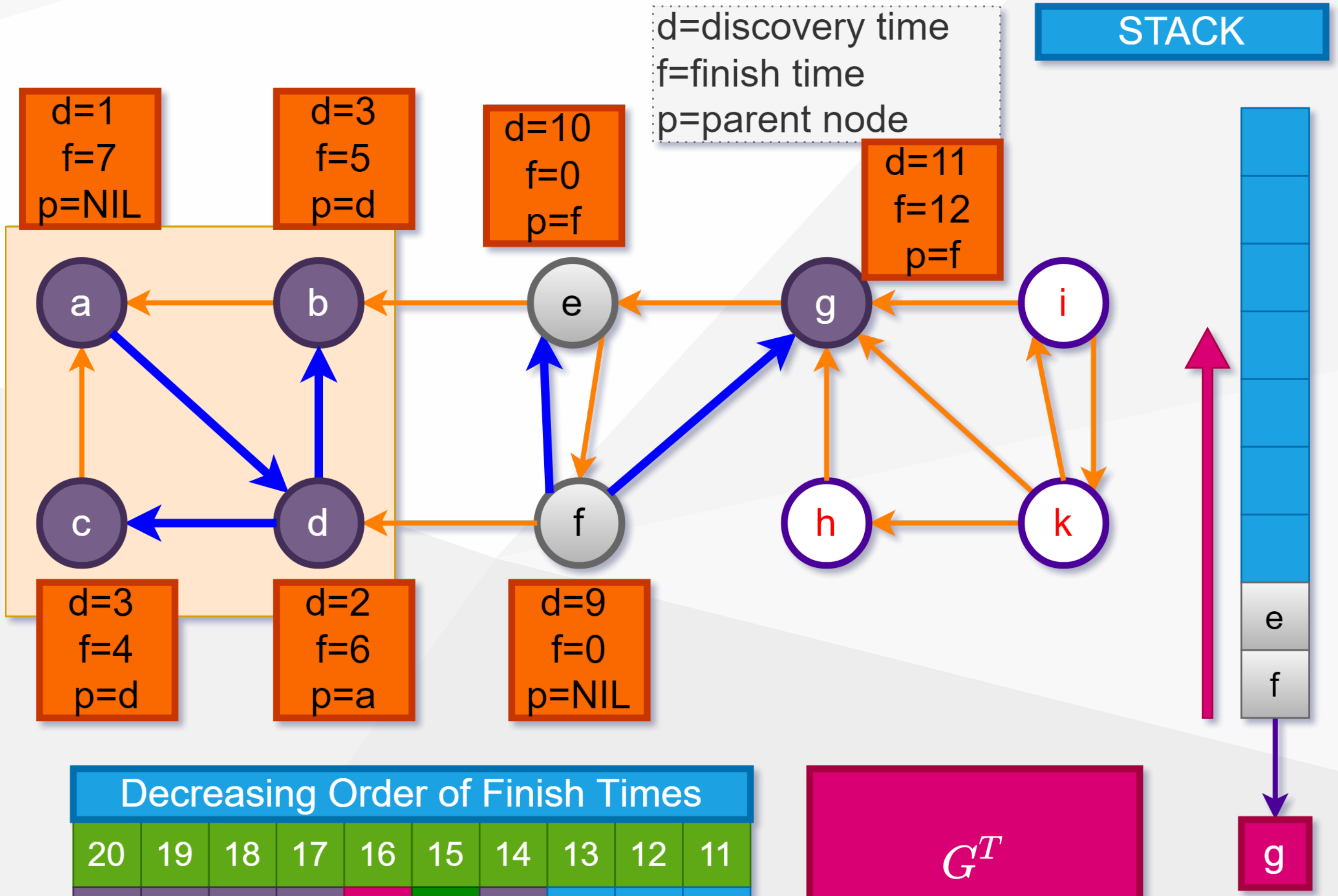
SCC Algorithm - Example-1/ Step - 32



20	19	18	17	16	15	14	13	12	11
a	b	c	d	f	e	g	h	k	i

Graph Segmentation

SCC Algorithm - Example-1/ Step - 33

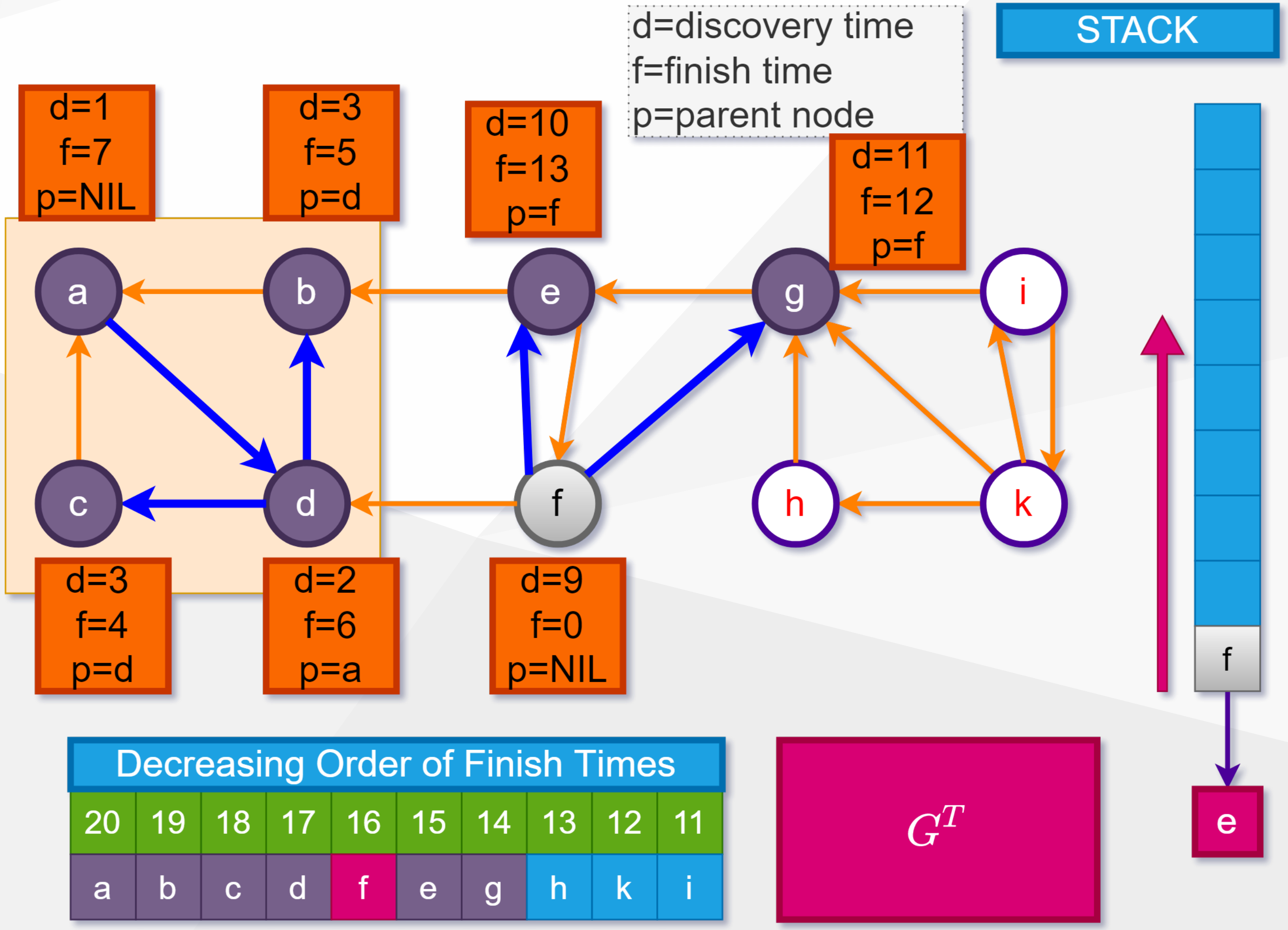


Decreasing Order of Finish Times									
20	19	18	17	16	15	14	13	12	11
a	b	c	d	f	e	g	h	k	i

G^T

Graph Segmentation

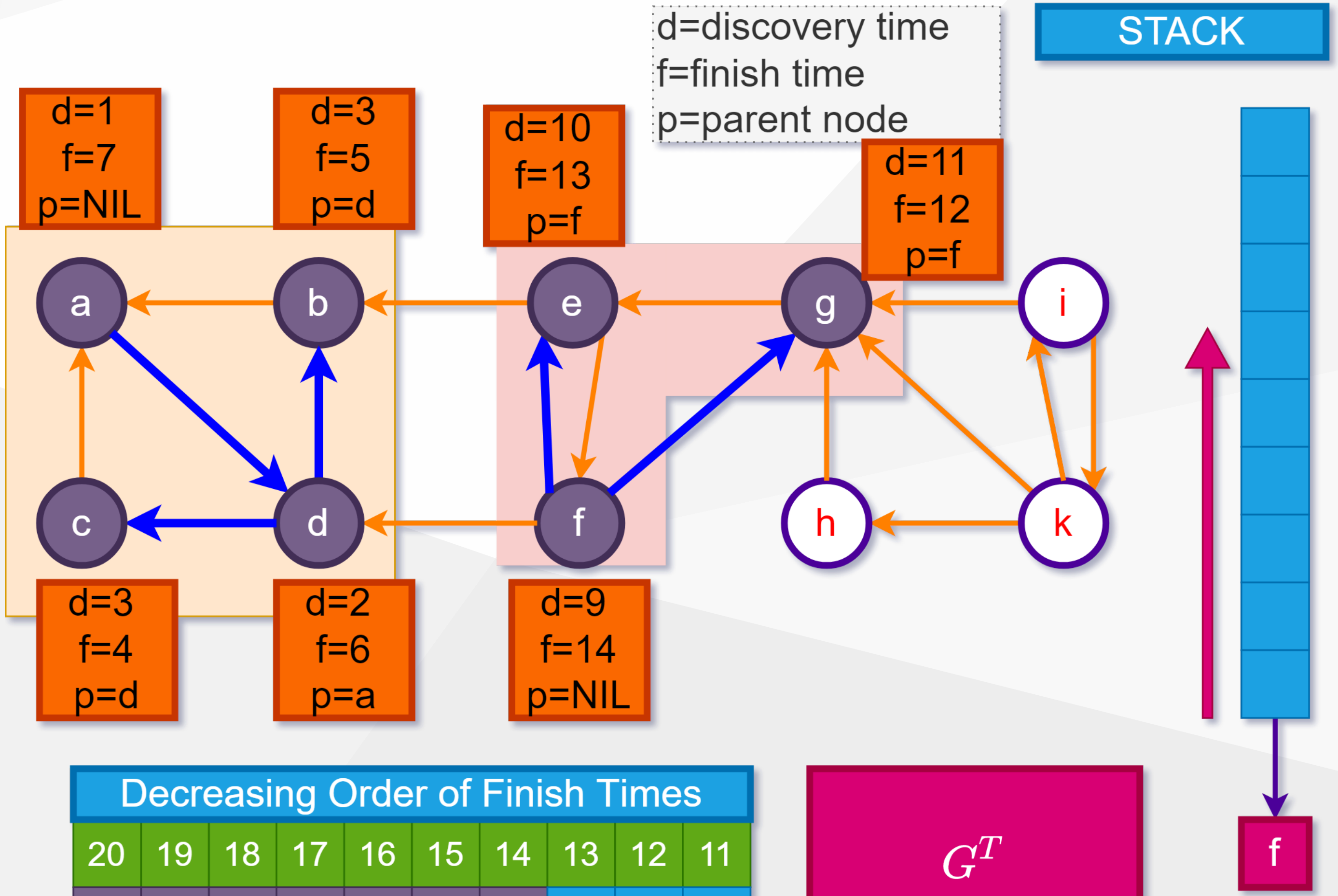
SCC Algorithm - Example-1/ Step - 34



Decreasing Order of Finish Times									
20	19	18	17	16	15	14	13	12	11
a	b	c	d	f	e	g	h	k	i

Graph Segmentation

SCC Algorithm - Example-1/ Step - 35



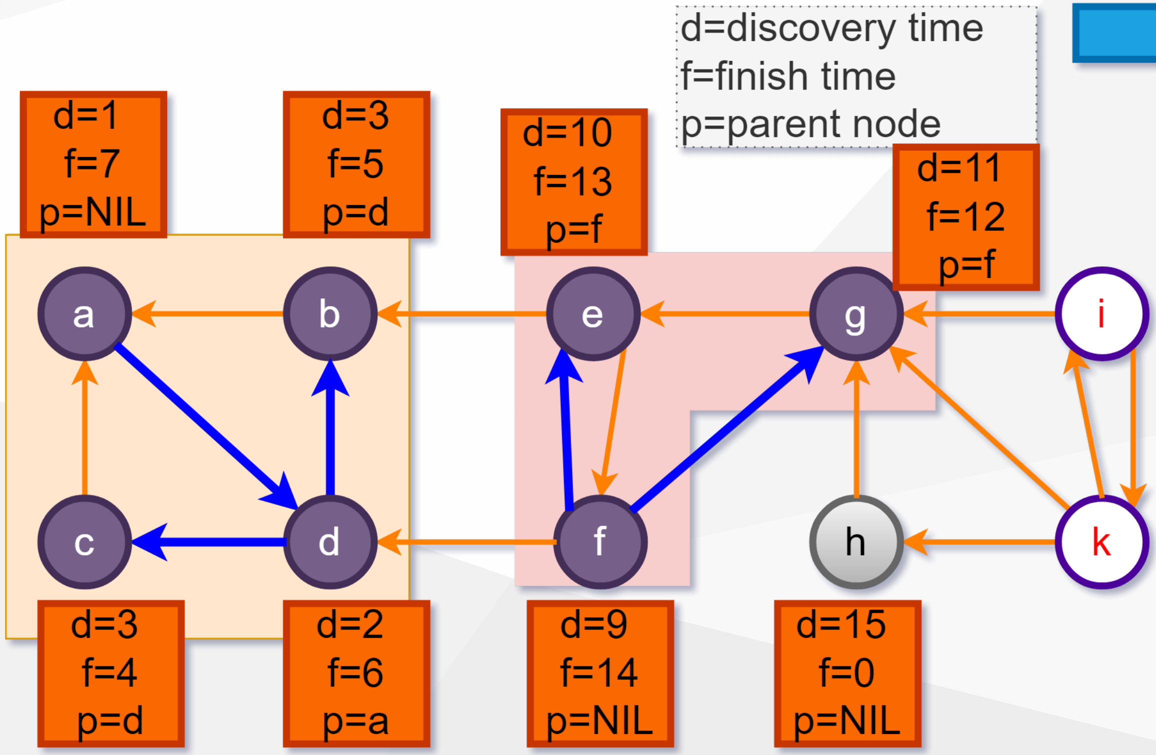
Decreasing Order of Finish Times

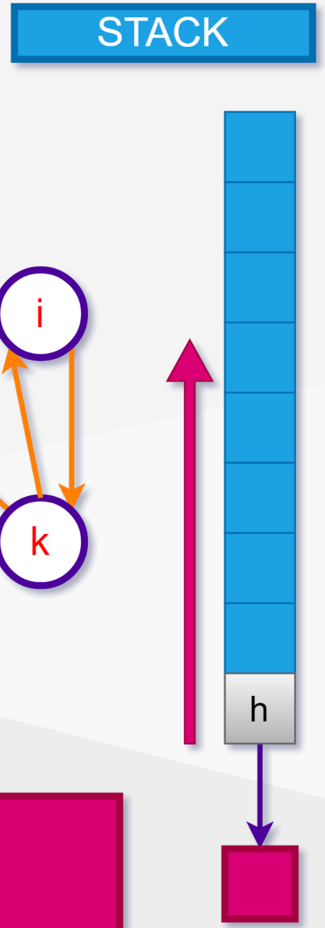
20	19	18	17	16	15	14	13	12	11
a	b	c	d	f	e	g	h	k	i

G^T

Graph Segmentation

SCC Algorithm - Example-1/ Step - 36



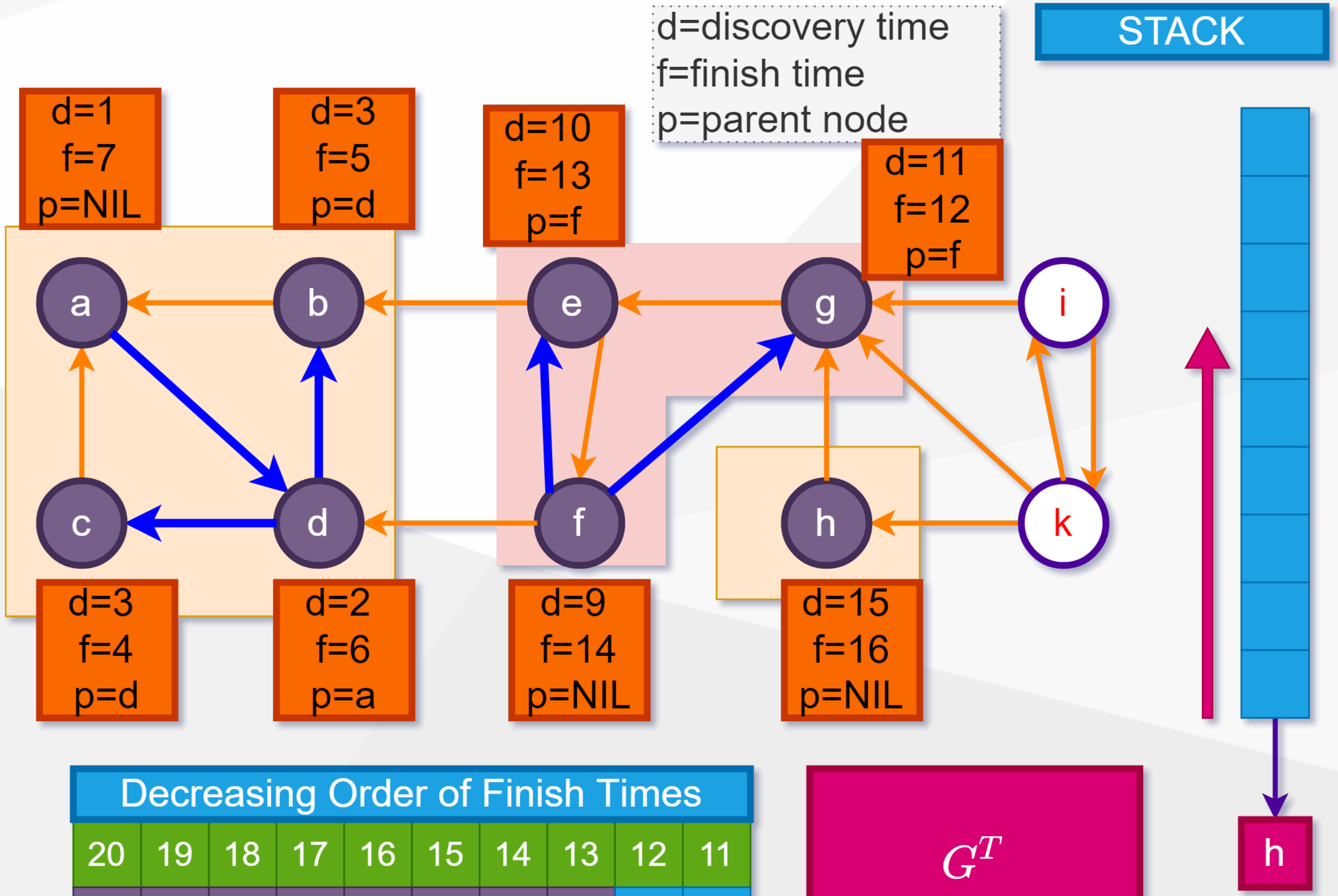


Decreasing Order of Finish Times									
20	19	18	17	16	15	14	13	12	11
a	b	c	d	f	e	g	h	k	i

G^T

Graph Segmentation

SCC Algorithm - Example-1/ Step - 37

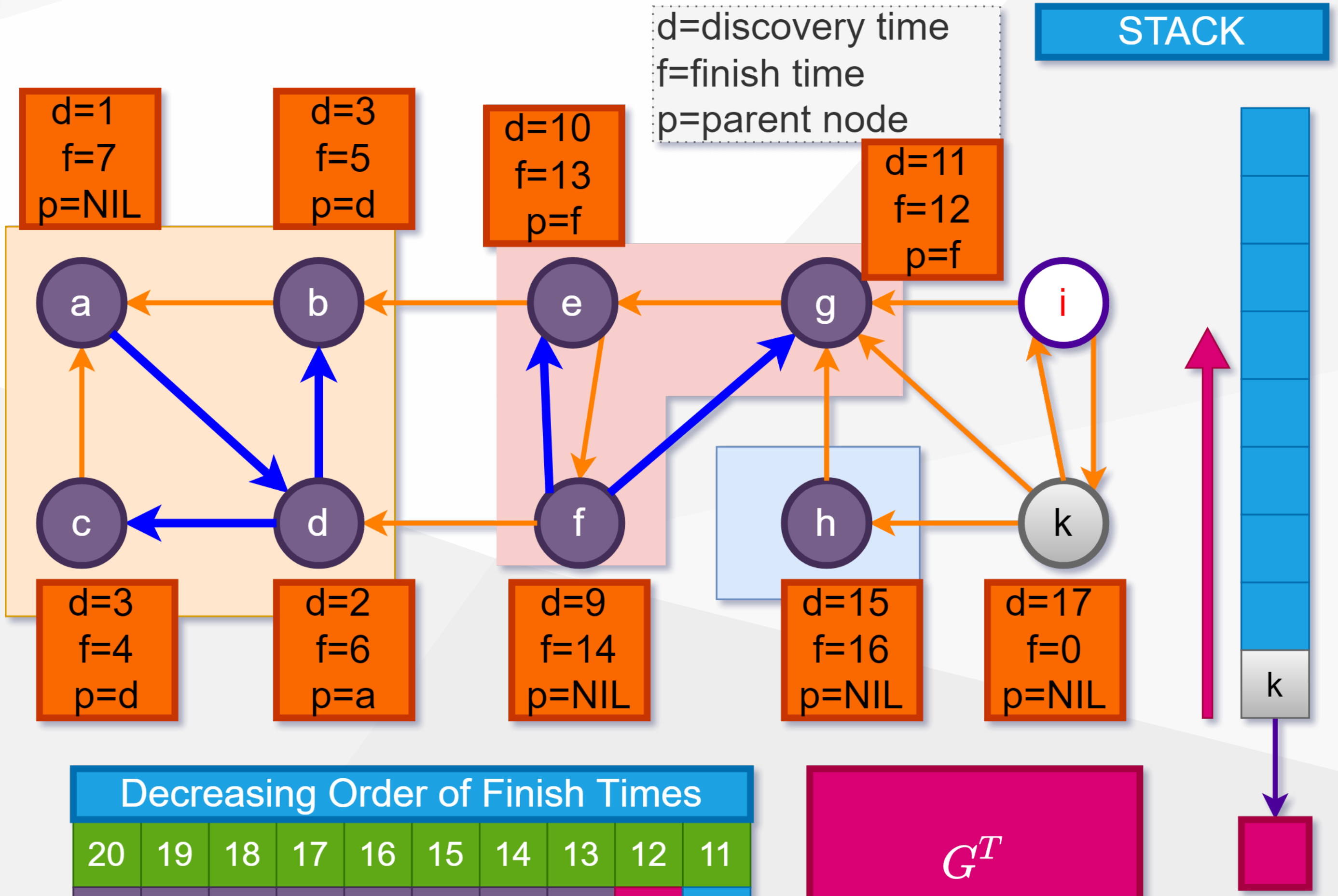


Decreasing Order of Finish Times									
20	19	18	17	16	15	14	13	12	11
a	b	c	d	f	e	g	h	k	i

G^T

Graph Segmentation

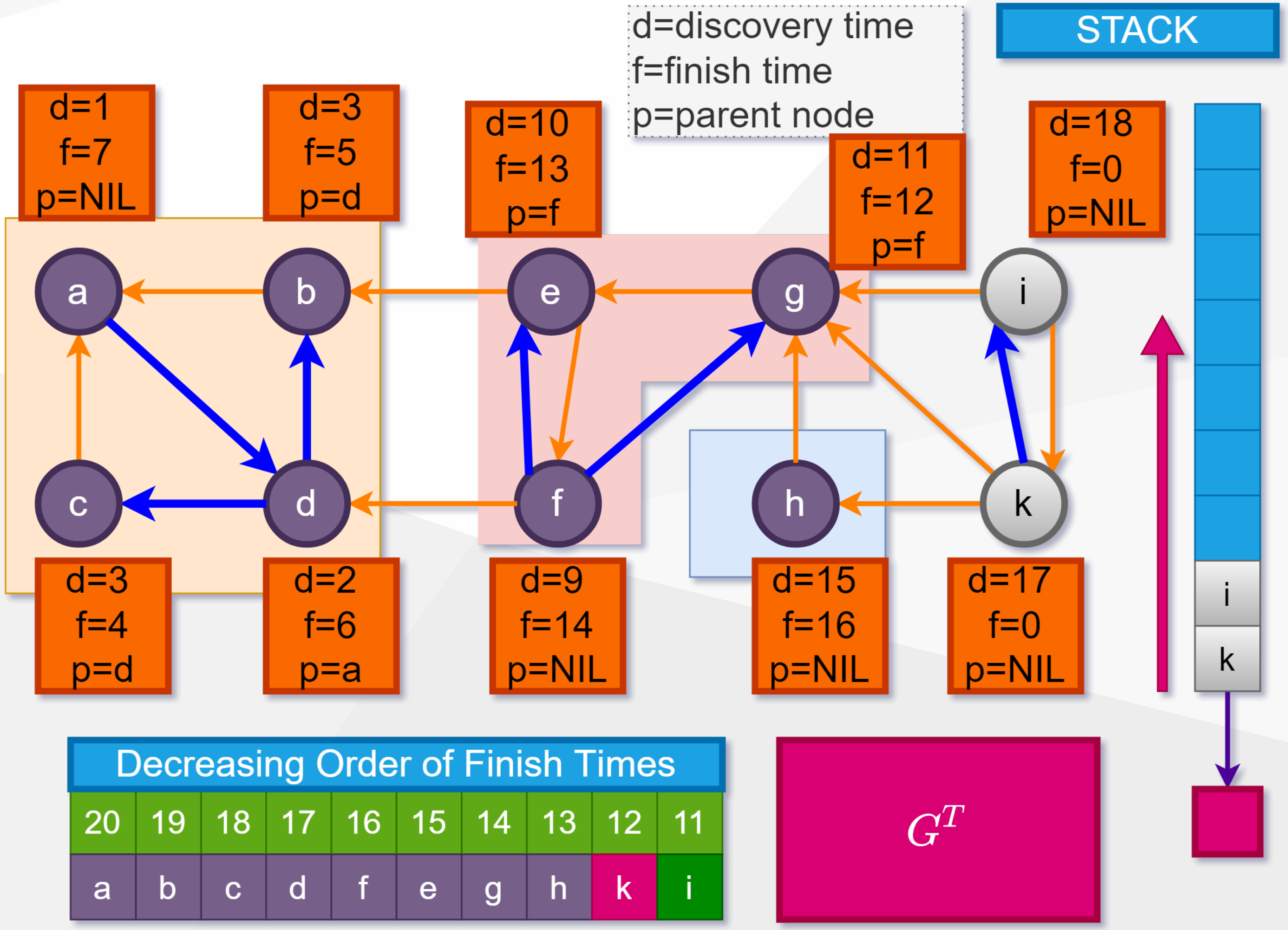
SCC Algorithm - Example-1/ Step - 38



Decreasing Order of Finish Times									
20	19	18	17	16	15	14	13	12	11
a	b	c	d	f	e	g	h	k	i

Graph Segmentation

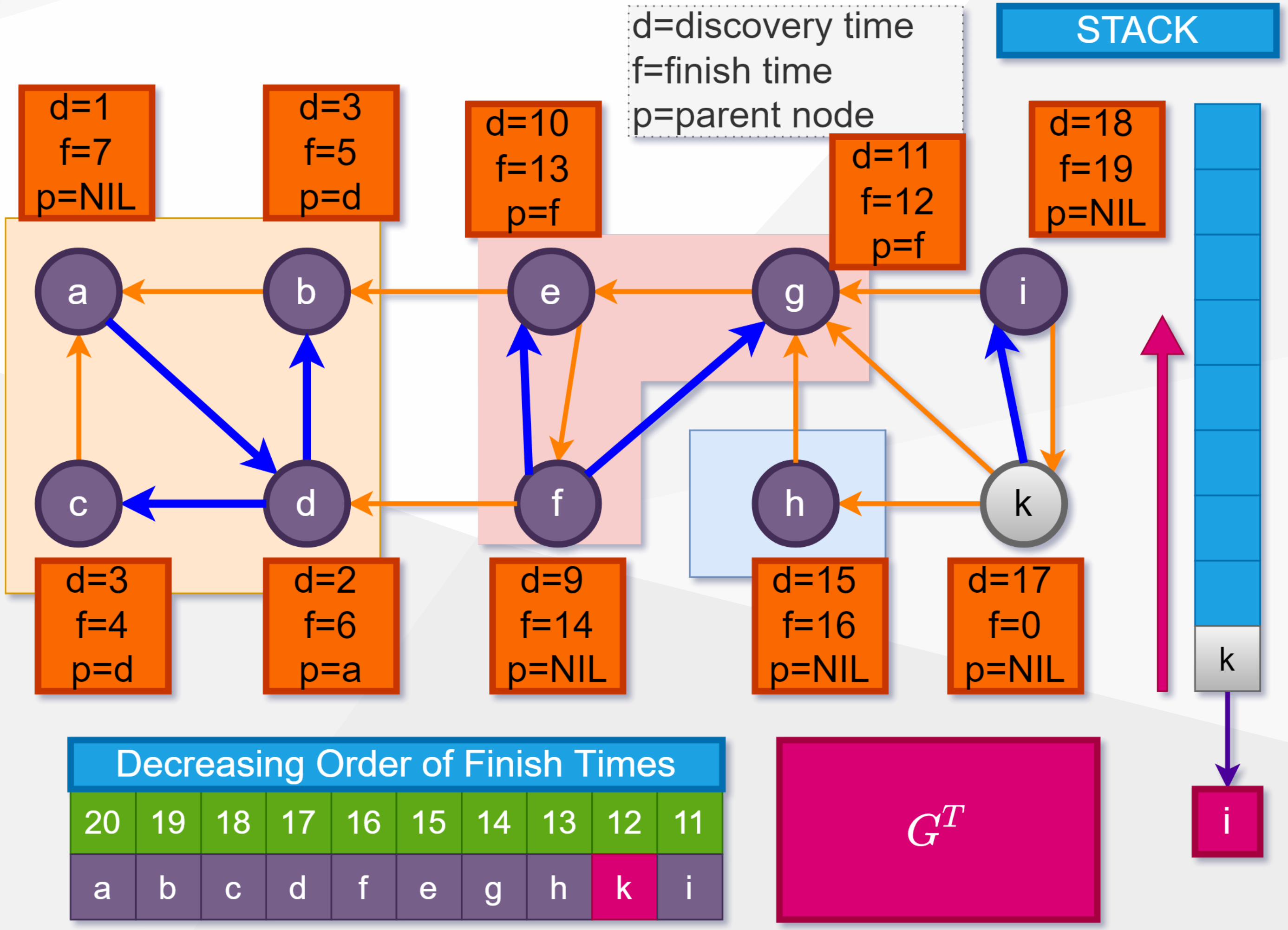
SCC Algorithm - Example-1/ Step - 39



Decreasing Order of Finish Times									
20	19	18	17	16	15	14	13	12	11
a	b	c	d	f	e	g	h	k	i

Graph Segmentation

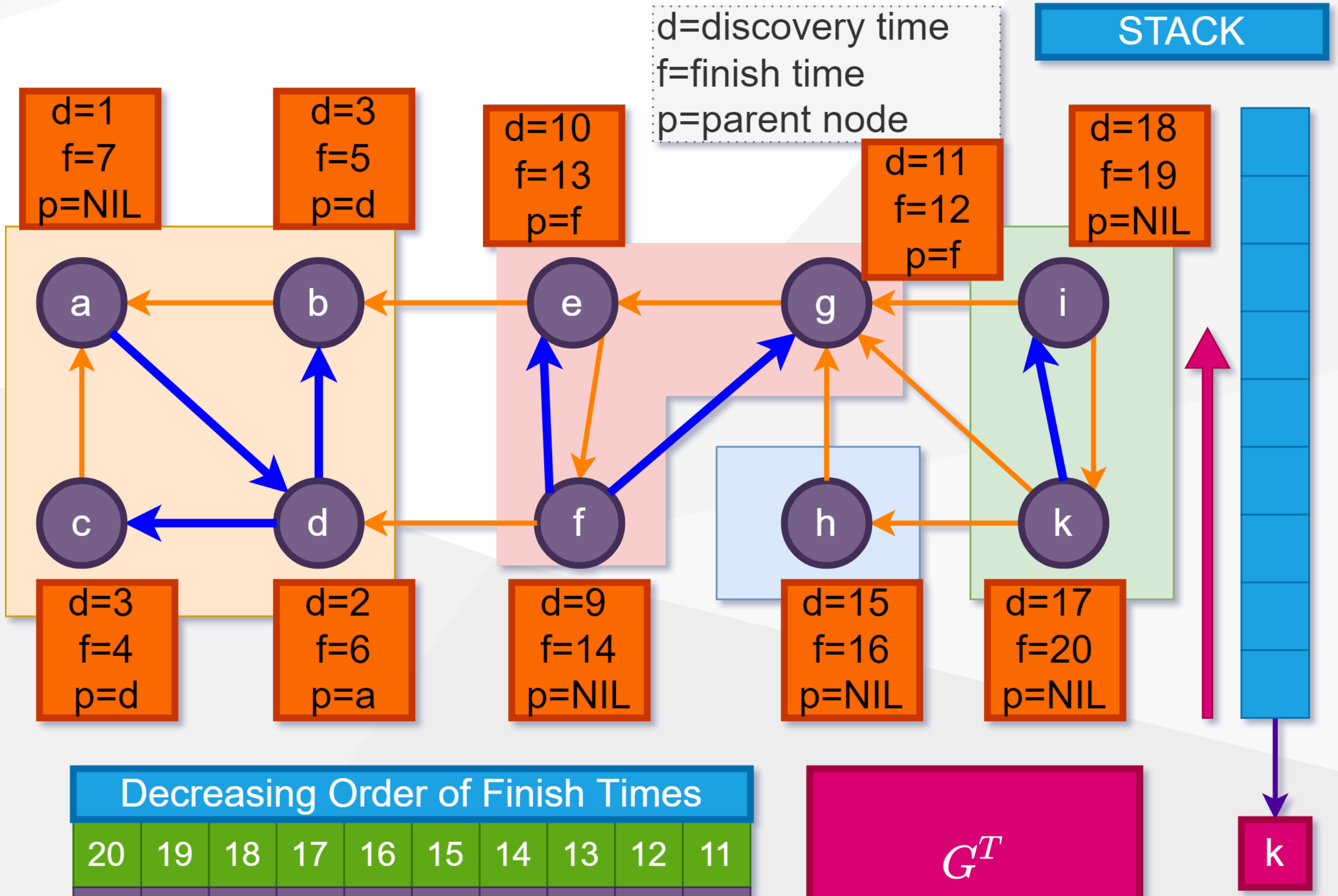
SCC Algorithm - Example-1/ Step - 40



20	19	18	17	16	15	14	13	12	11
a	b	c	d	f	e	g	h	k	i

Graph Segmentation

SCC Algorithm - Example-1/ Step - 41

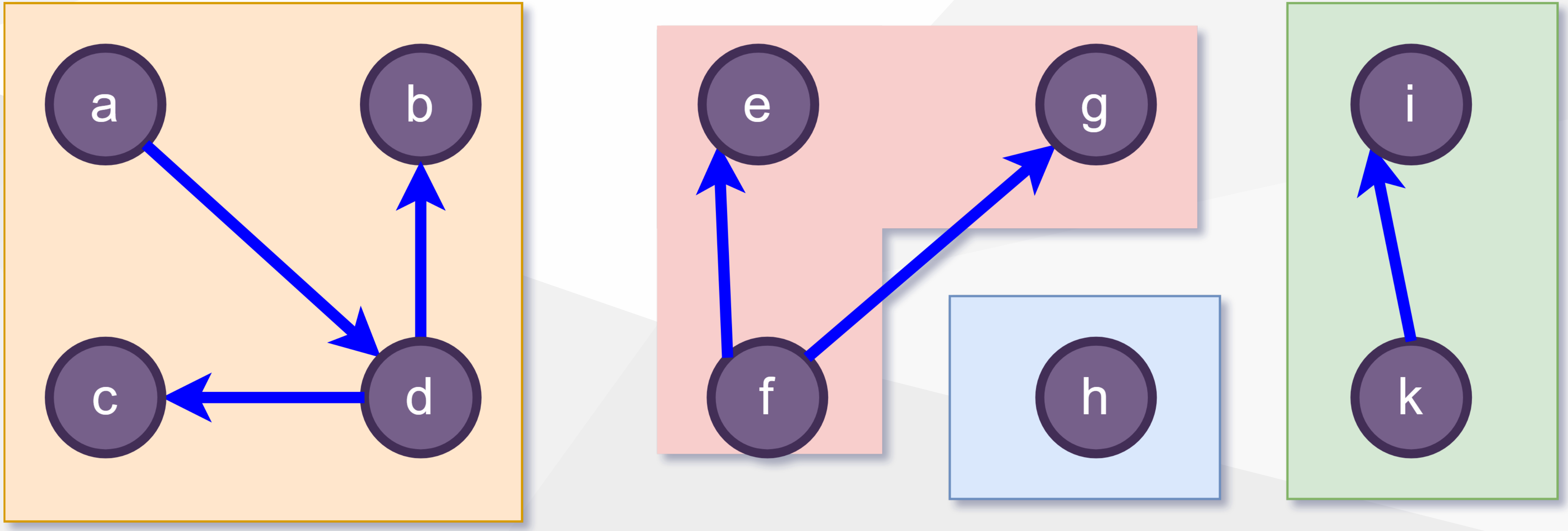


Decreasing Order of Finish Times									
20	19	18	17	16	15	14	13	12	11
a	b	c	d	f	e	g	h	k	i

G^T

Graph Segmentation

SCC Algorithm - Example-1/ Step - 42



Graph Traversal

Topological Sort

- When we are scheduling jobs or tasks, they may have dependencies.
- For example, before we finish task a, we have to finish b first.
  - In this case, given a set of tasks and their dependencies, how shall we arrange our schedules? There comes an interesting graph algorithm: Topological Sort.
- According to Introduction to Algorithms, given a directed acyclic graph (DAG),
- a topological sort is a linear ordering of all vertices such that for any edge (u, v), u comes before v.
- Another way to describe it is that when you put all vertices horizontally on a line, all of the edges are pointing from left to right.

Graph Traversal

Topological Sort

- Topological sort is a linear ordering of a directed acyclic graph.
- If a graph has a cycle, it is not a directed acyclic graph.
- A graph is acyclic if it has no cycles.

Graph Traversal

Topological Sort

DFS version

- The key observation is that, leaf nodes should always come after their parents and ancestors. Following this intuition we can apply DFS and output nodes from leaves to the root.
- We need to implement a boolean array visited so that visited[i] indicates if we have visited vertex i.
- For each unvisited node, we would first mark it as visited and call DFS() to start searching its neighbours.
- After finishing this, we can insert it to the front of a list. After visiting all nodes, we can return that list.

Graph Traversal

Topological Sort - DFS Version

```
run DFS(G)
when a vertex finished, output it
vertices output in reverse topologically sorted order
```

- Runs in O(V+E) time

Graph Traversal

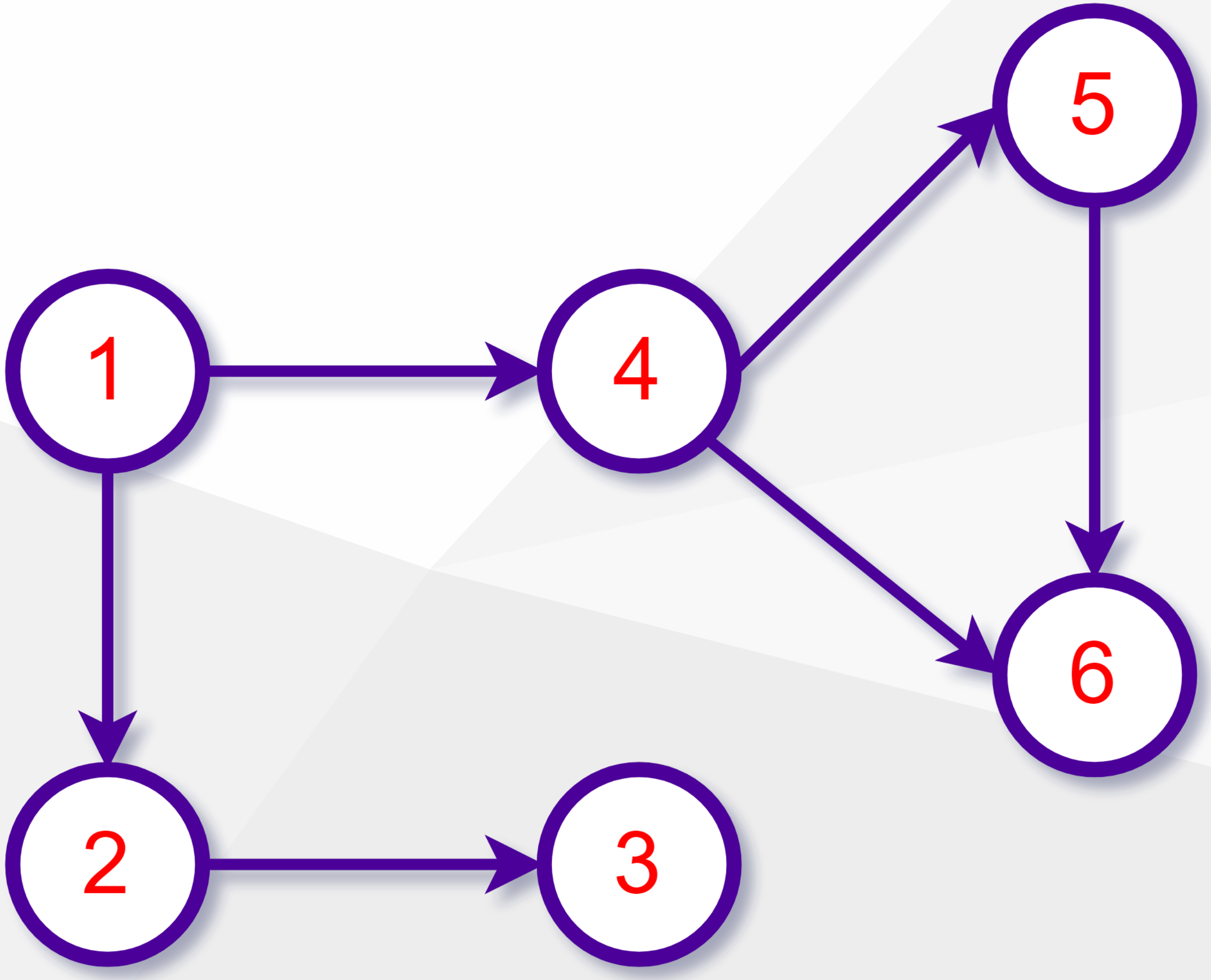
Topological Sort - DFS Version

```
def topological_sort():
    for each node:
        if visited[node] is False:
            dfs(node)

def dfs(node):
    visited[node] = True
    for nei in neighbours[node]:
        dfs(node)
        if visited(node) = false:
                ret.insert_at_the_front(node)
```

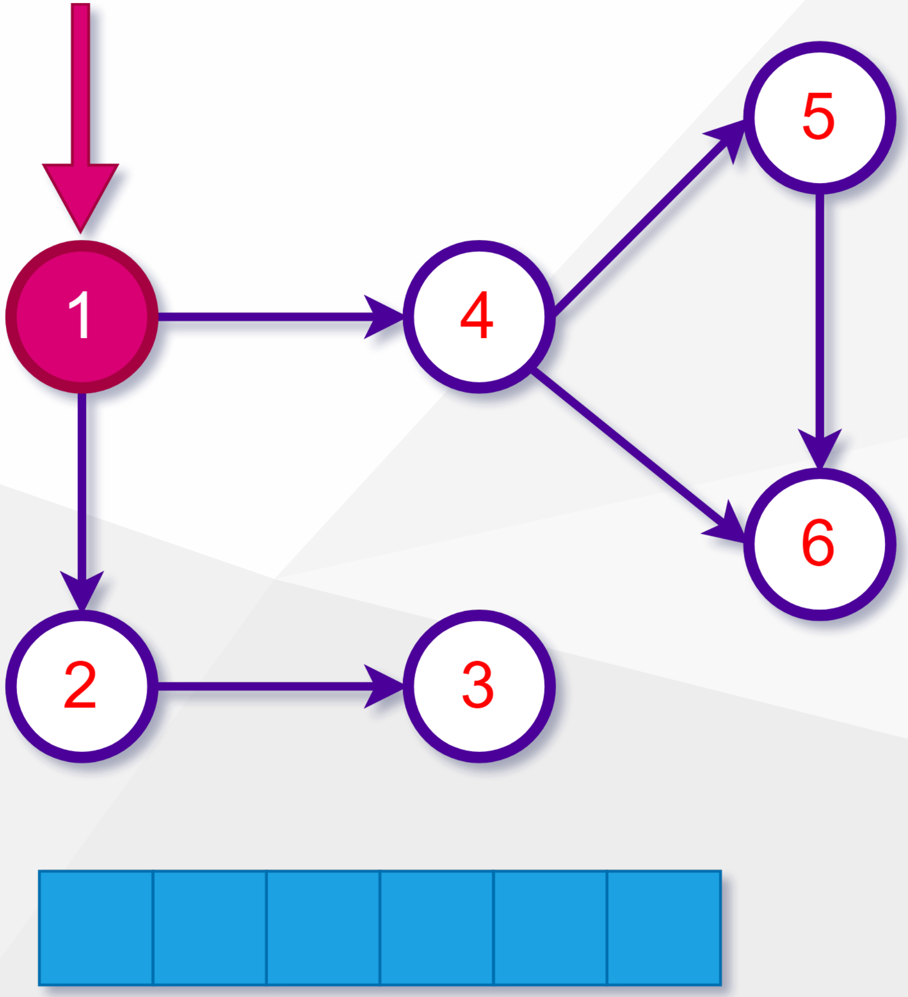
Graph Traversal

Topological Sort - DFS Version (Example)



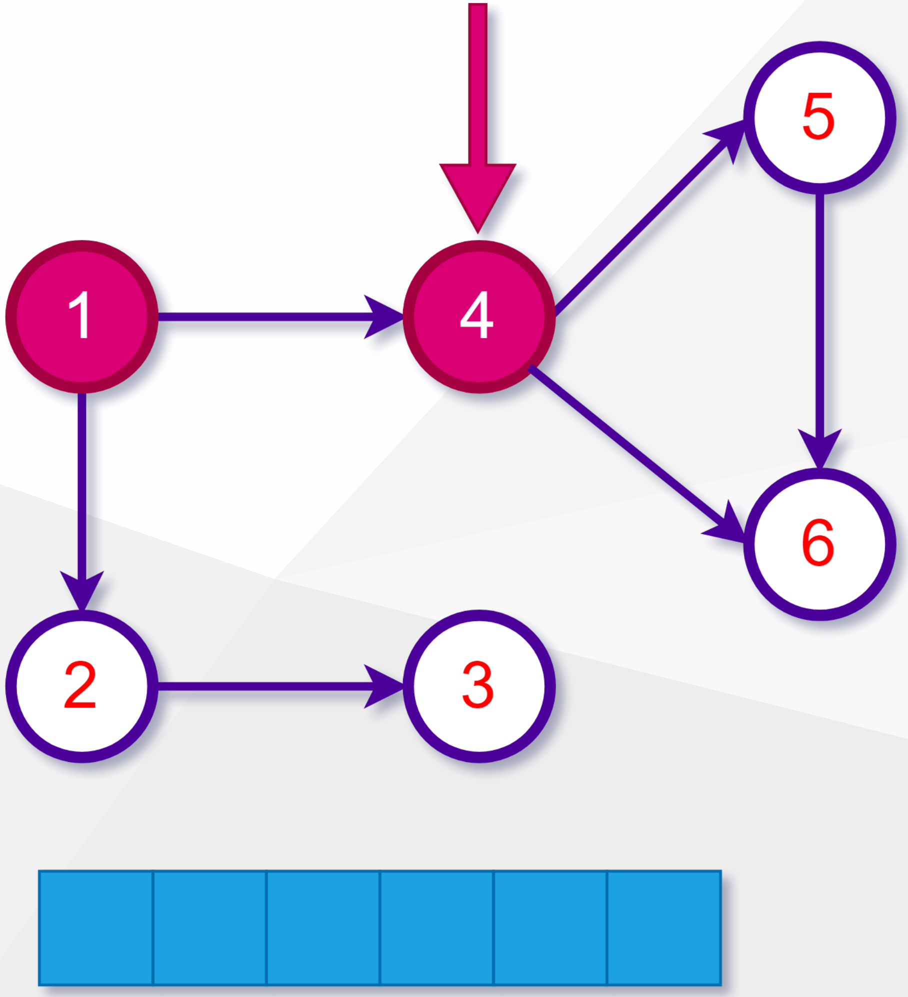
Graph Traversal

Topological Sort - DFS Version STEP-1



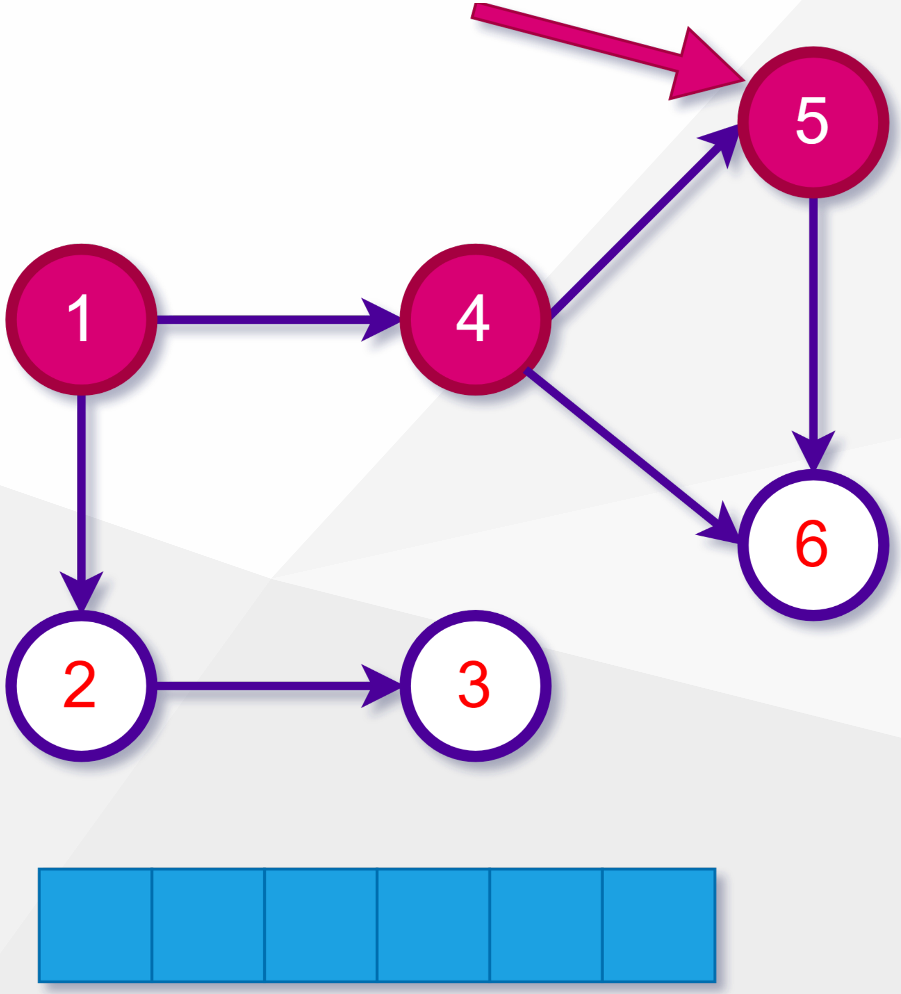
Graph Traversal

Topological Sort - DFS Version STEP-2



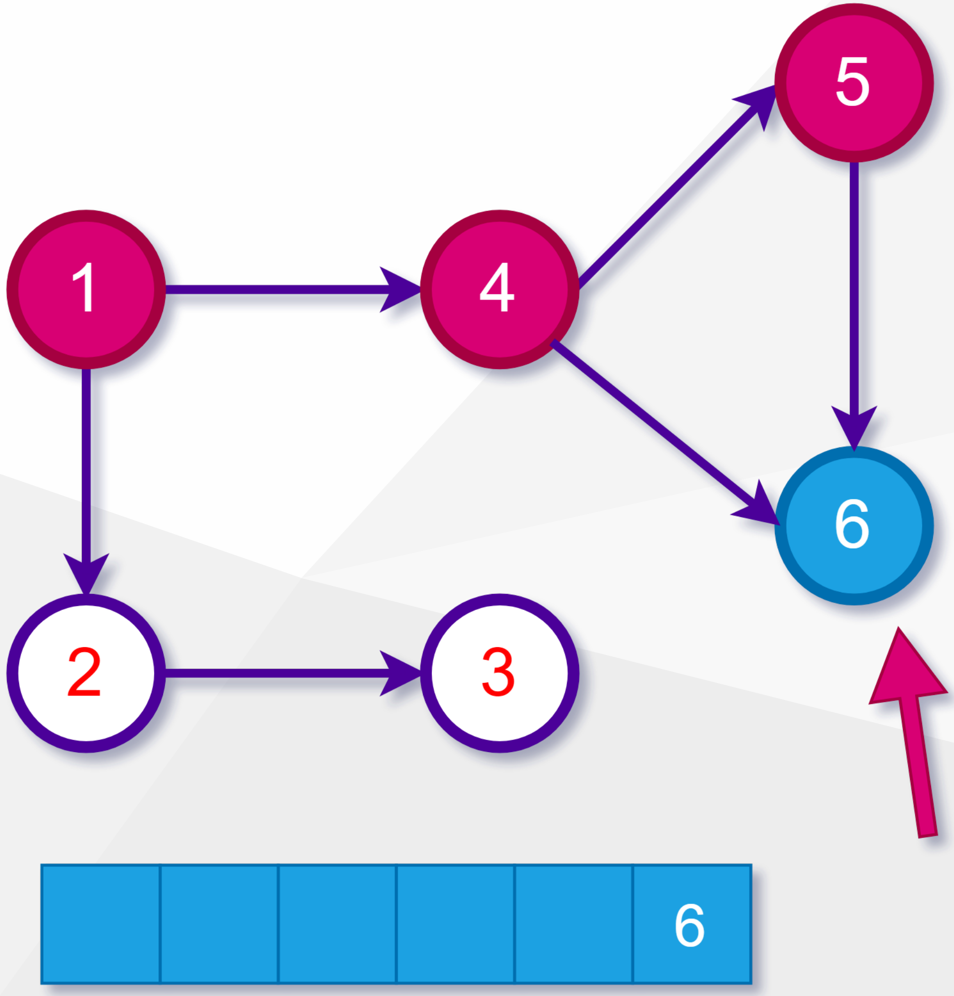
Graph Traversal

Topological Sort - DFS Version STEP-3



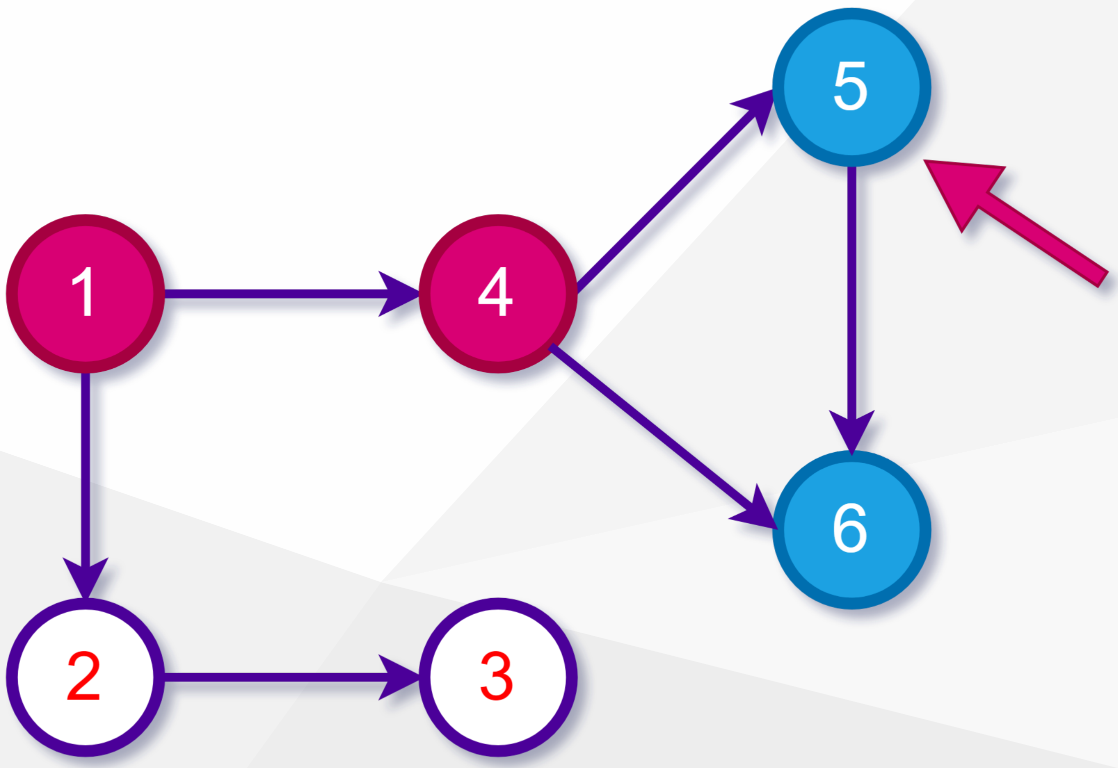
Graph Traversal

Topological Sort - DFS Version STEP-4



Graph Traversal

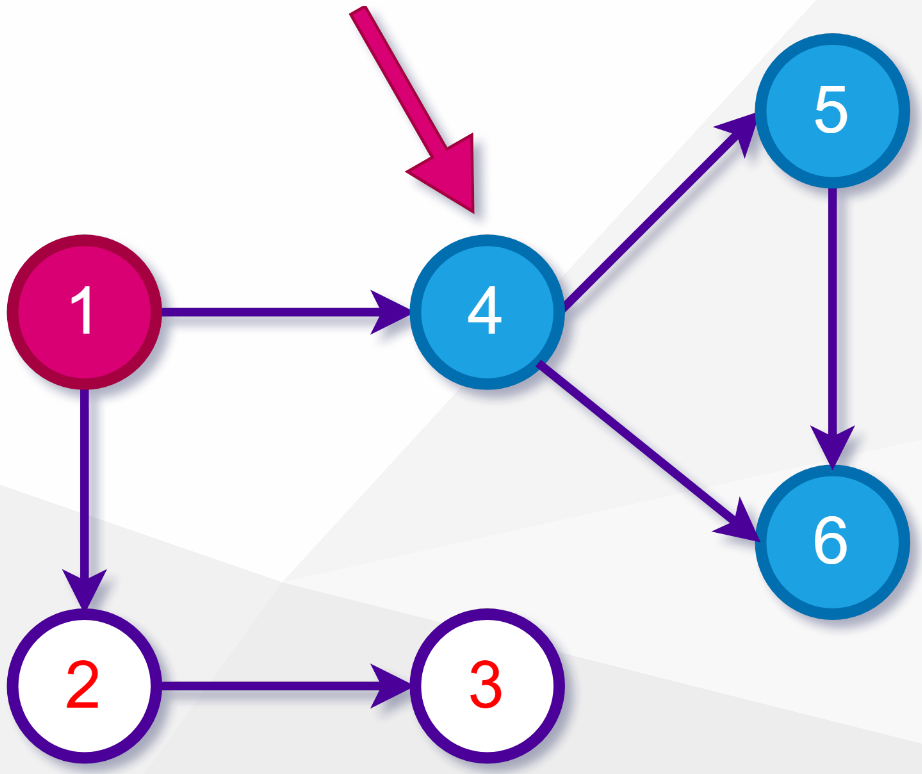
Topological Sort - DFS Version STEP-5



				5	6

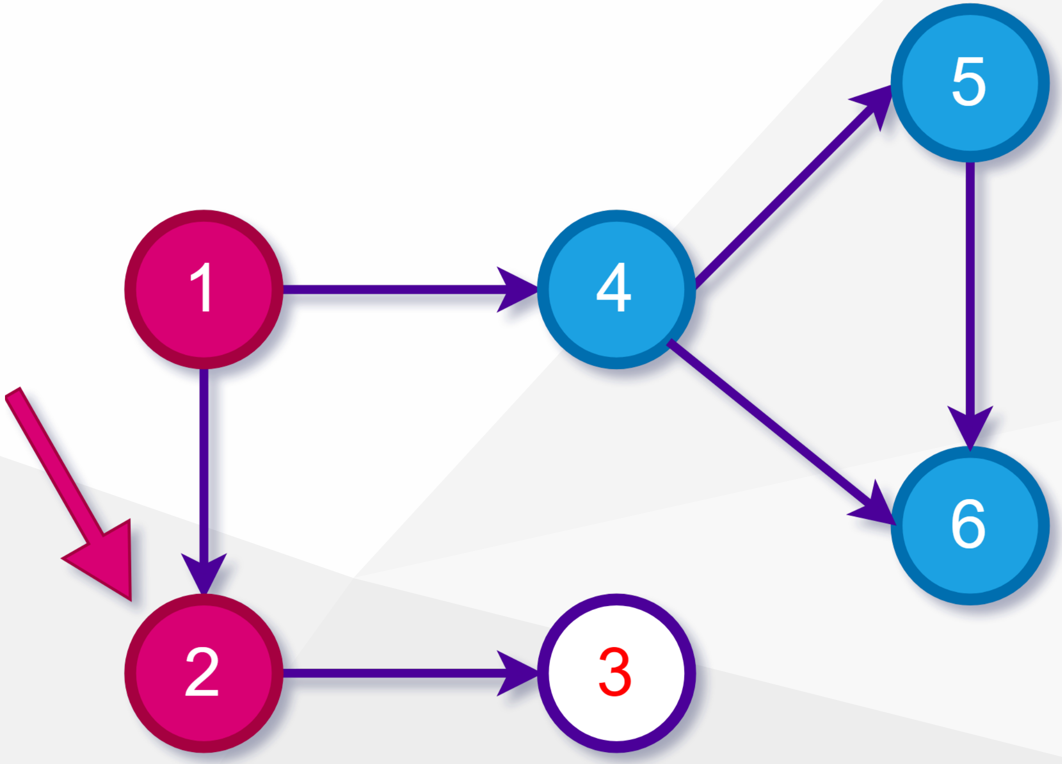
Graph Traversal

Topological Sort - DFS Version STEP-6



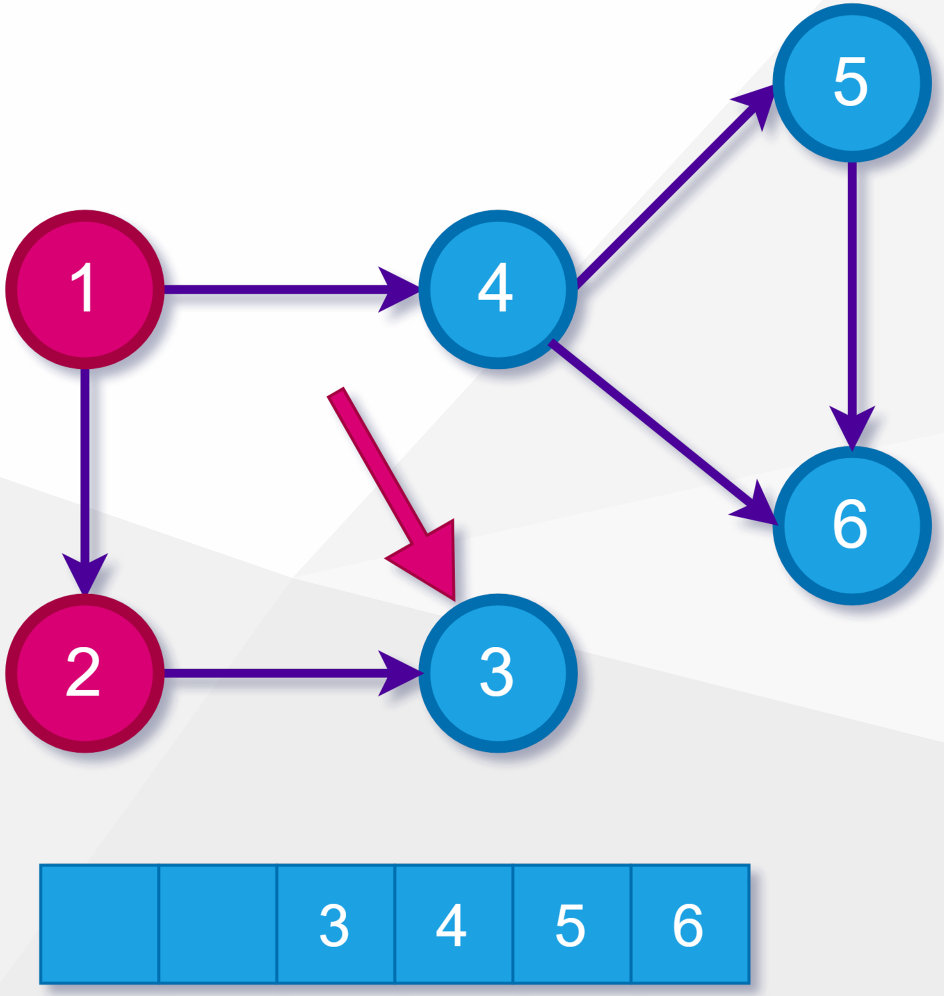
			4	5	6

Topological Sort - DFS Version STEP-7



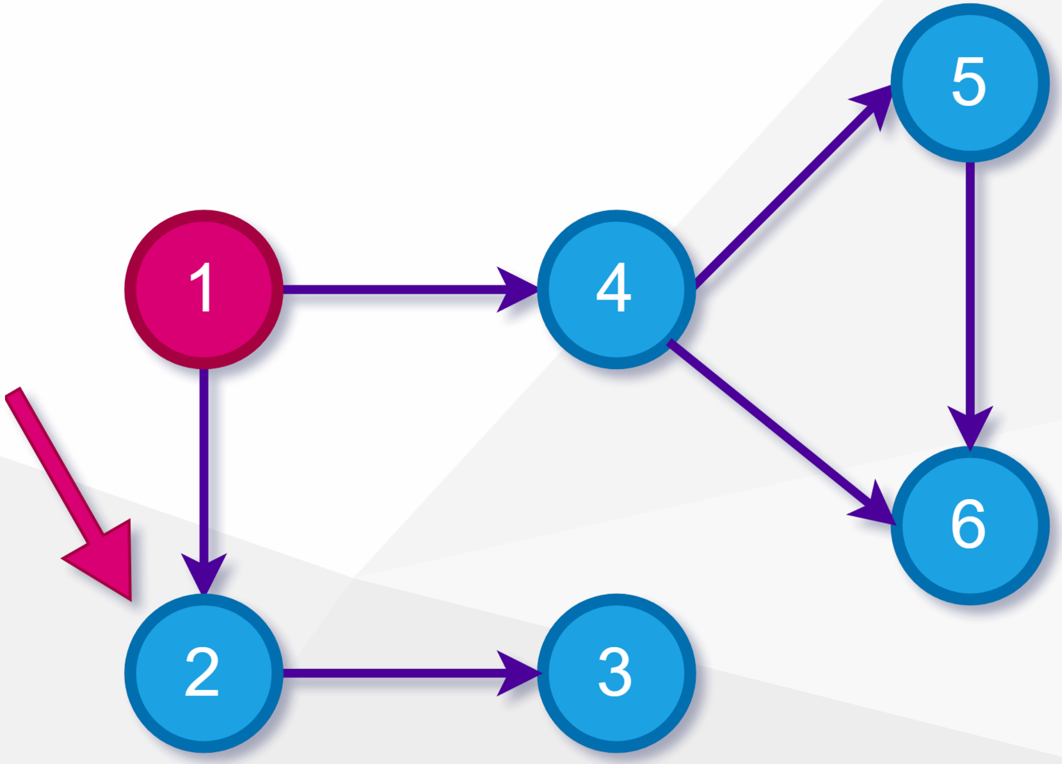
			4	5	6

Topological Sort - DFS Version STEP-8



Graph Traversal

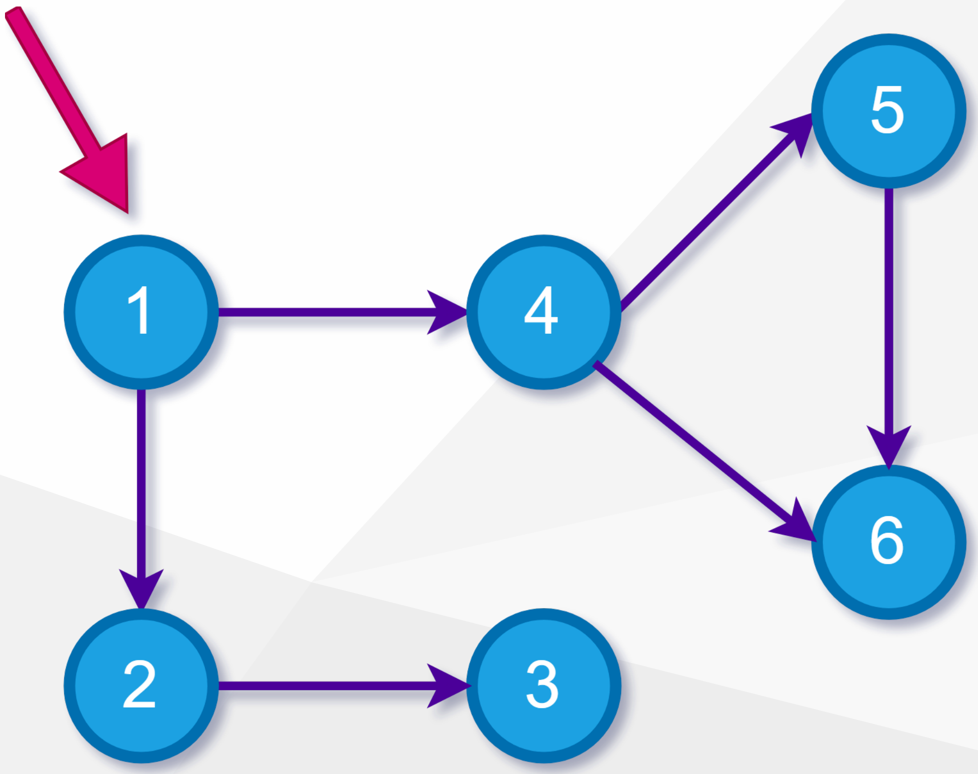
Topological Sort - DFS Version STEP-9



	2	3	4	5	6

Graph Traversal

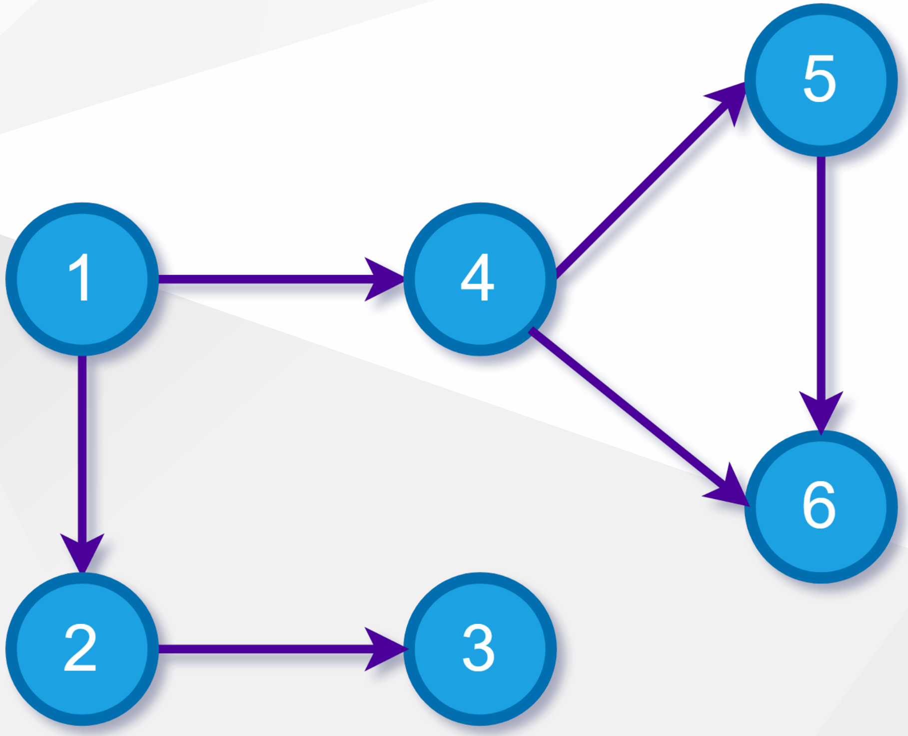
Topological Sort - DFS Version STEP-10

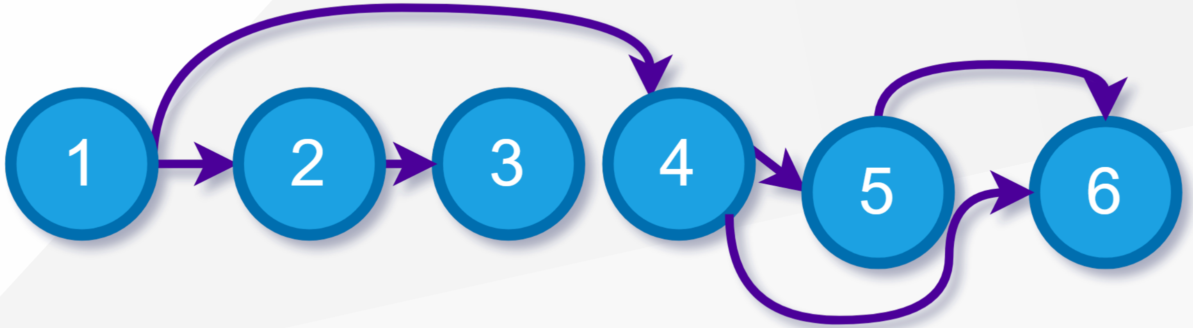


1	2	3	4	5	6

Graph Traversal

Topological Sort - DFS Version





1	2	3	4	5	6

Graph Traversal

Topological Sort

BFS version (Kahn's algorithm)

- For BFS, we need an array indegree to keep the track of indegrees. Then we will try to output all nodes with 0 indegree, and remove the edges coming out of them at the same time. Besides, remember to put the nodes that become 0 indegree in the queue.
- Then, we can keep doing this until all nodes are visited. To implement it, we can store the graph in an adjacent list (a hashmap or a dictionary in Python) and a queue to loop.

Graph Traversal

Topological Sort

BFS version (Kahn's algorithm)

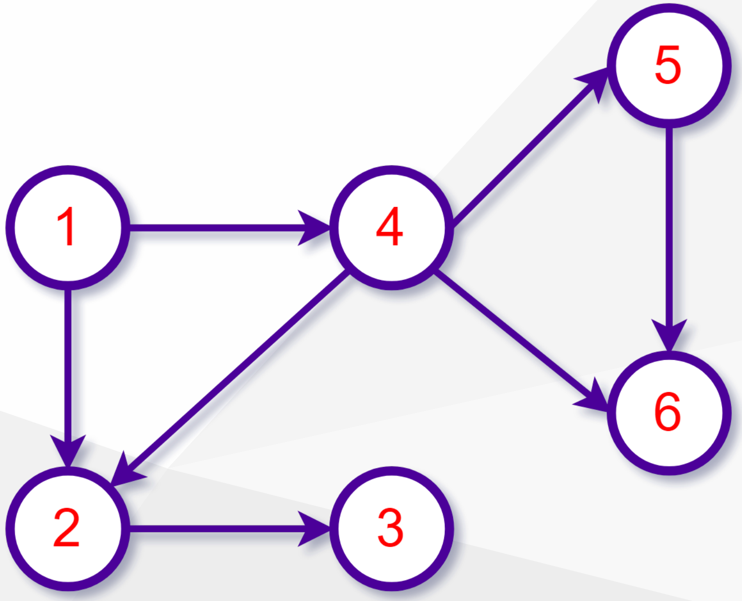
```
indegree = an array indicating indegrees for each node
neighbours = a HashMap recording neighbours of each node
queue = []
for i in indegree:
    if indegree[i] == 0:
        queue.append(i)

while !queue.empty():
    node = queue.dequeue()
    for neighbour in neighbours[node]:
        indegree[neighbour] -= 1
        if indegree[neighbour] == 0:
            queue.append(neighbour)
```

Graph Traversal

Topological Sort - BFS version (Kahn's algorithm)

- STEP-1

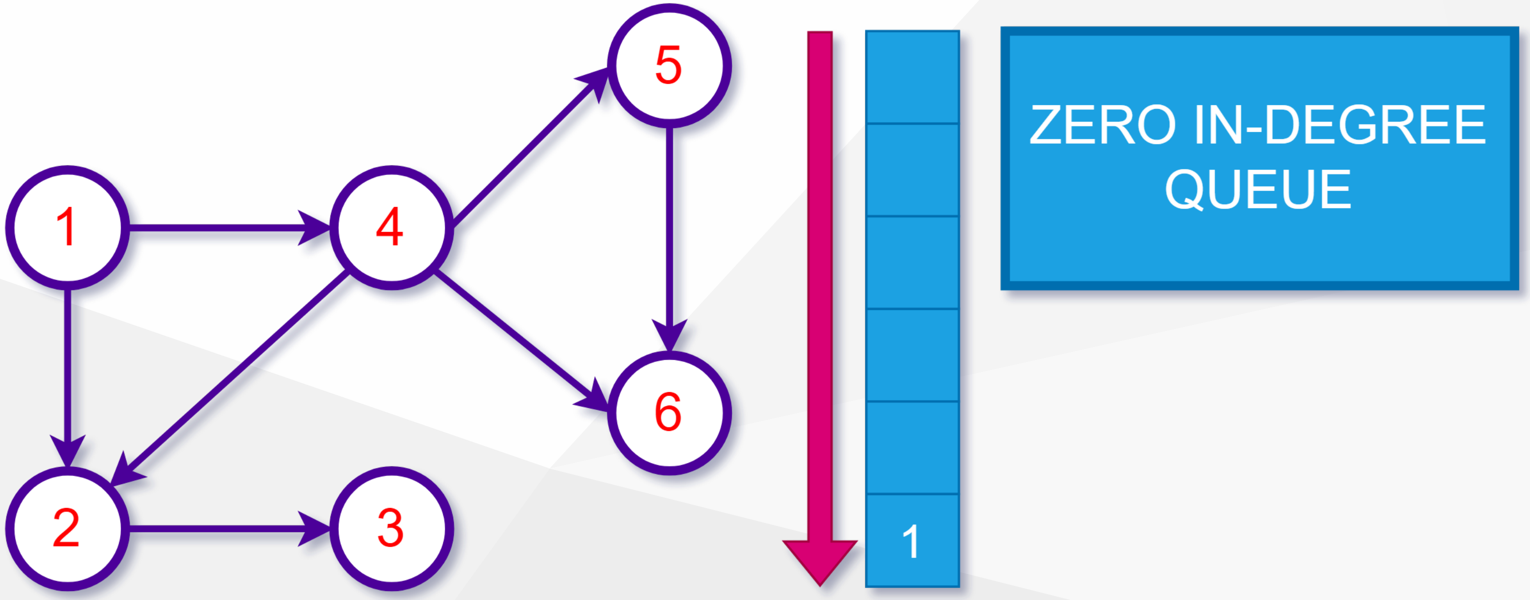


ARRAY for IN-DEGREE						
Node	1	2	3	4	5	6
In-Degree	0	1	1	1	1	2

Graph Traversal

Topological Sort - BFS version (Kahn's algorithm)

- STEP-2

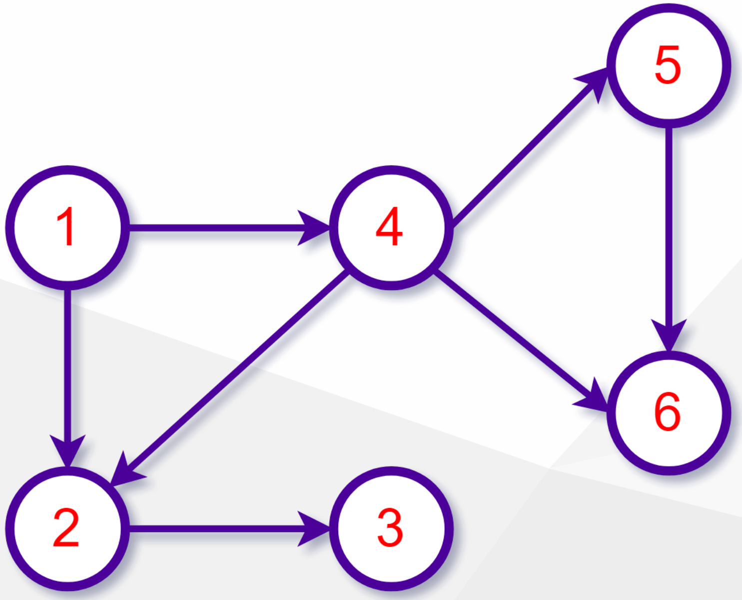


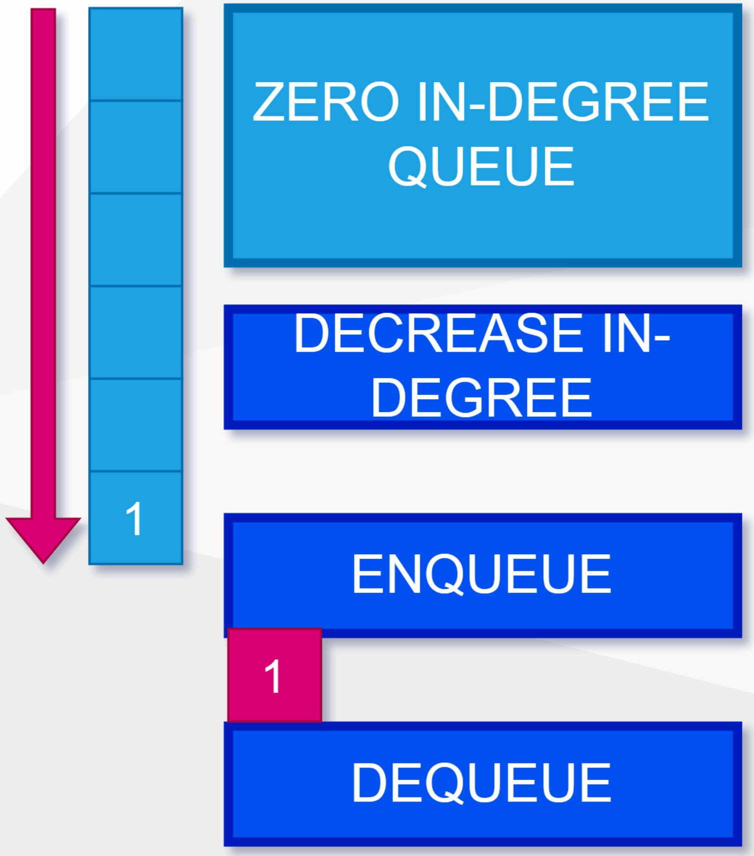
ARRAY for IN-DEGREE						
Node	1	2	3	4	5	6
In-Degree	0	1	1	1	1	2

Graph Traversal

Topological Sort - BFS version (Kahn's algorithm)

- STEP-3



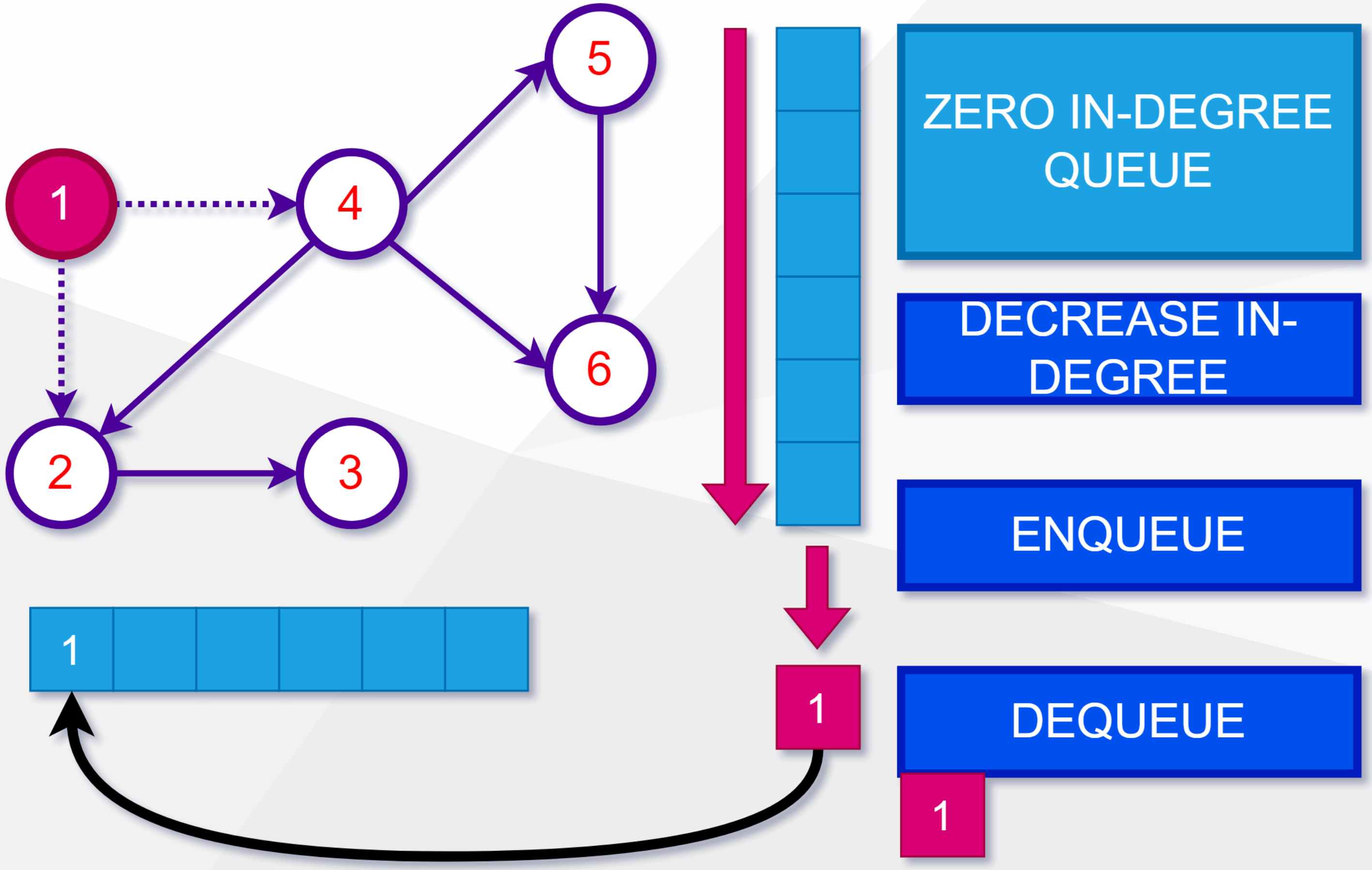


ARRAY for IN-DEGREE						
Node	1	2	3	4	5	6
In-Degree	0	1	1	1	1	2

Graph Traversal

Topological Sort - BFS version (Kahn's algorithm)

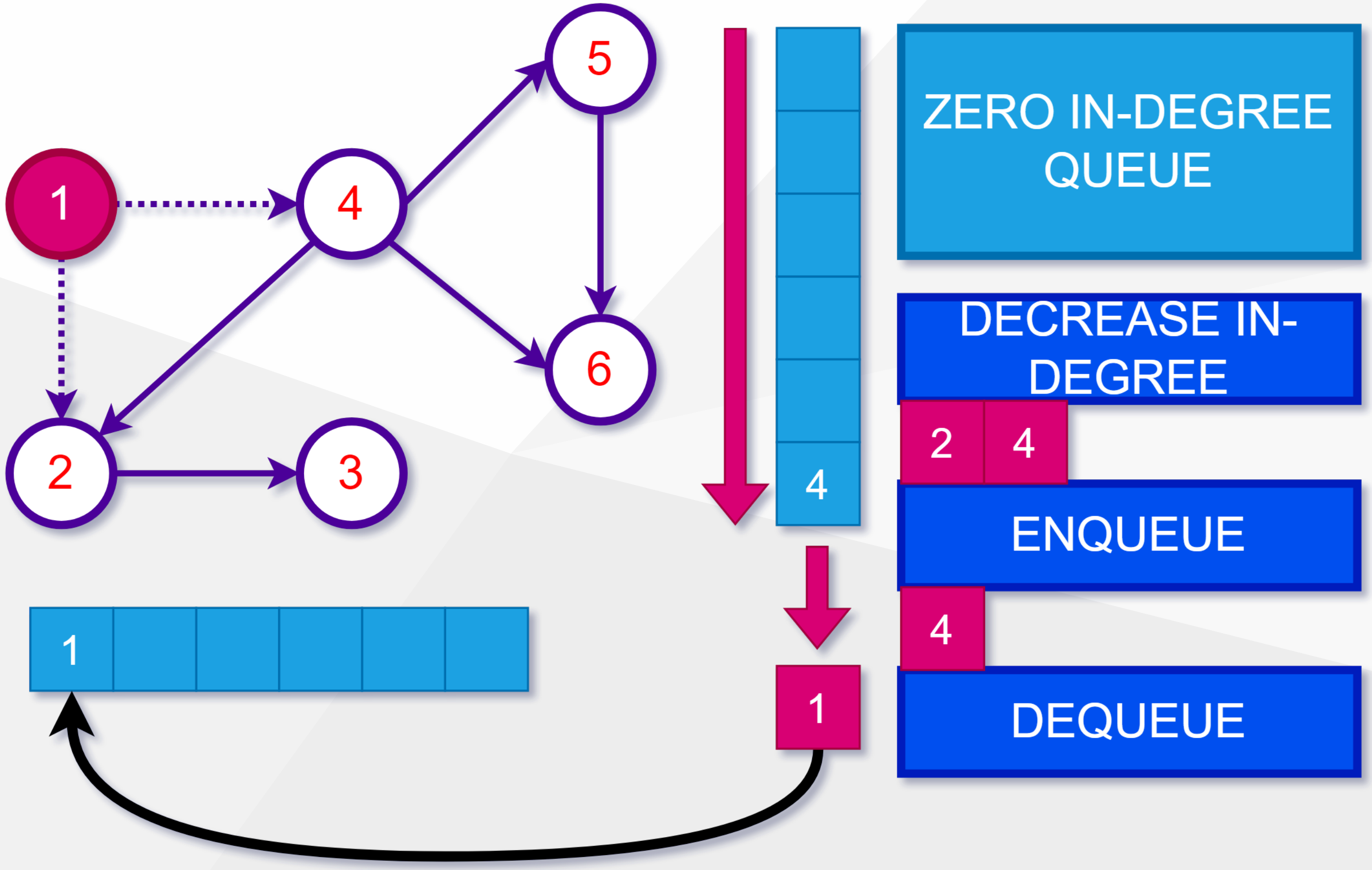
- STEP-4



Graph Traversal

Topological Sort - BFS version (Kahn's algorithm)

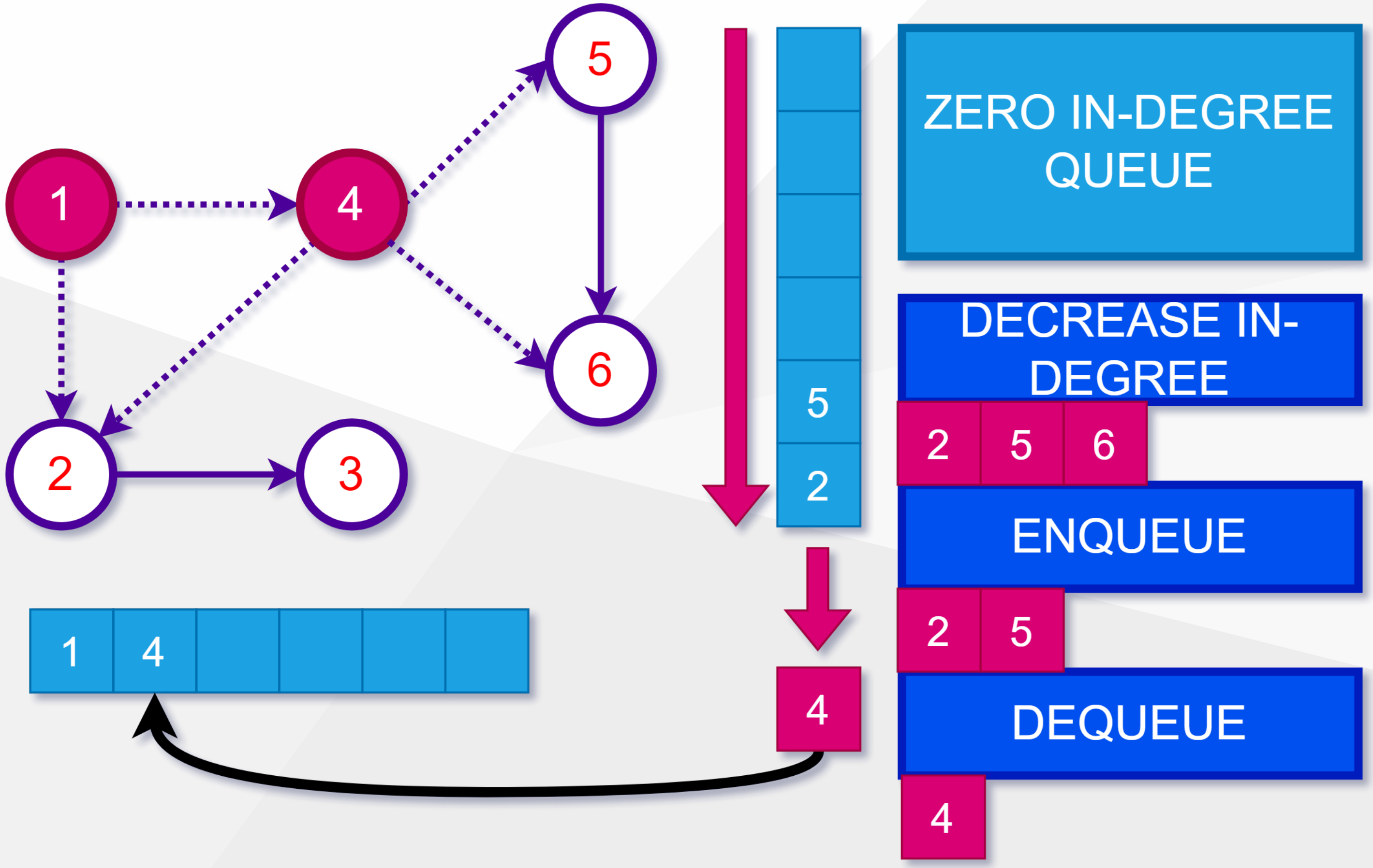
- STEP-5



Graph Traversal

Topological Sort - BFS version (Kahn's algorithm)

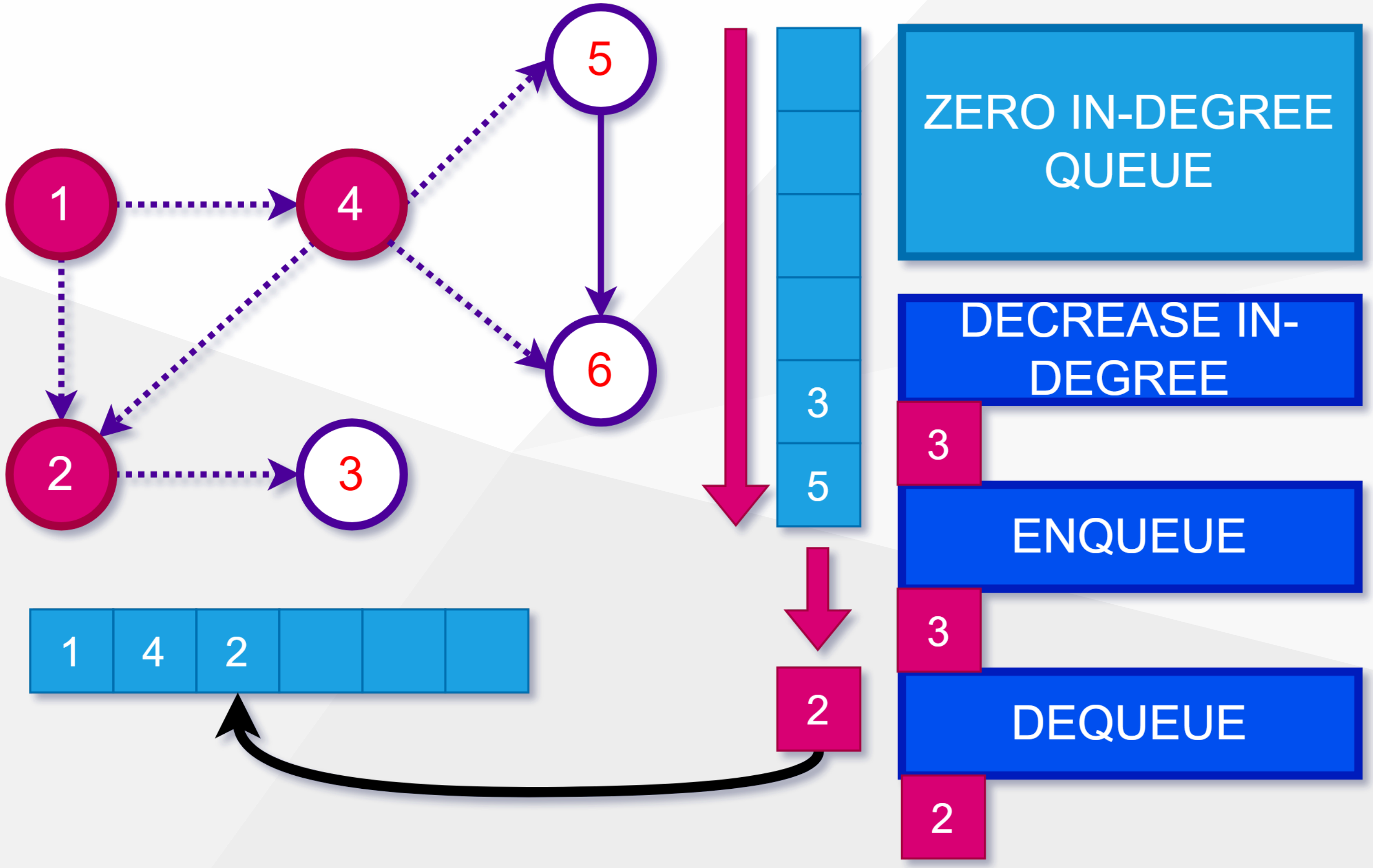
- STEP-6



Graph Traversal

Topological Sort - BFS version (Kahn's algorithm)

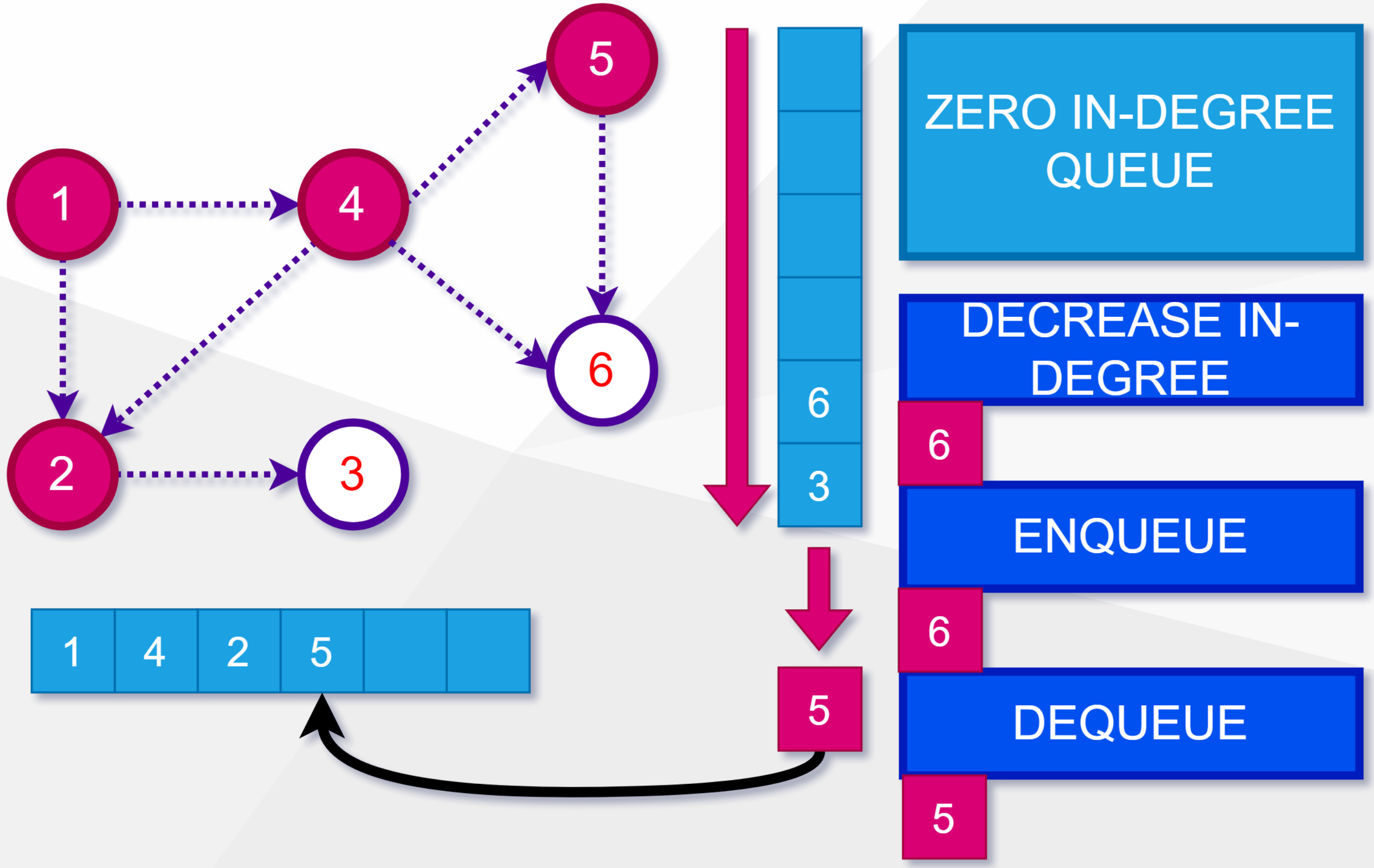
- STEP-7



Graph Traversal

Topological Sort - BFS version (Kahn's algorithm)

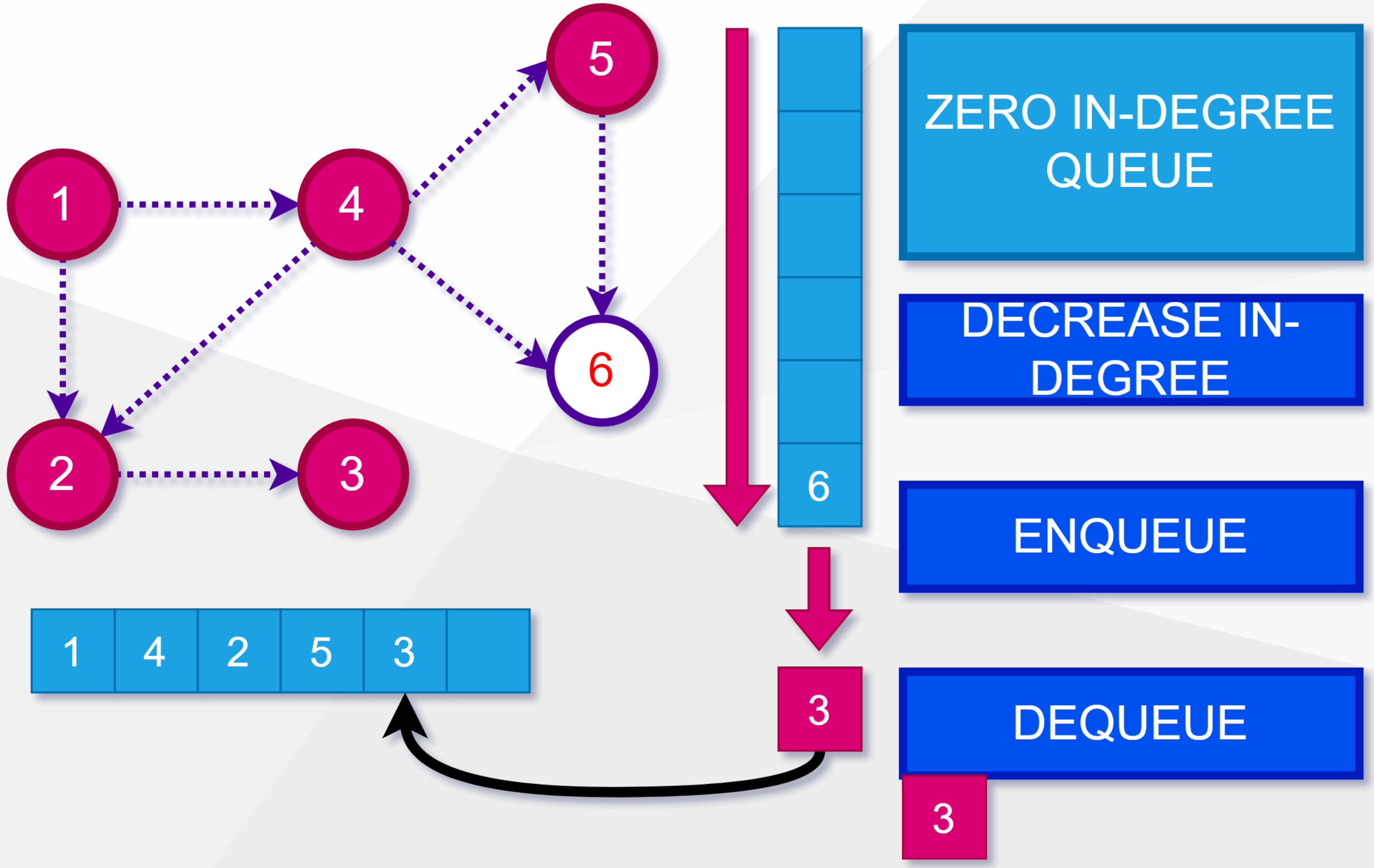
- STEP-8



Graph Traversal

Topological Sort - BFS version (Kahn's algorithm)

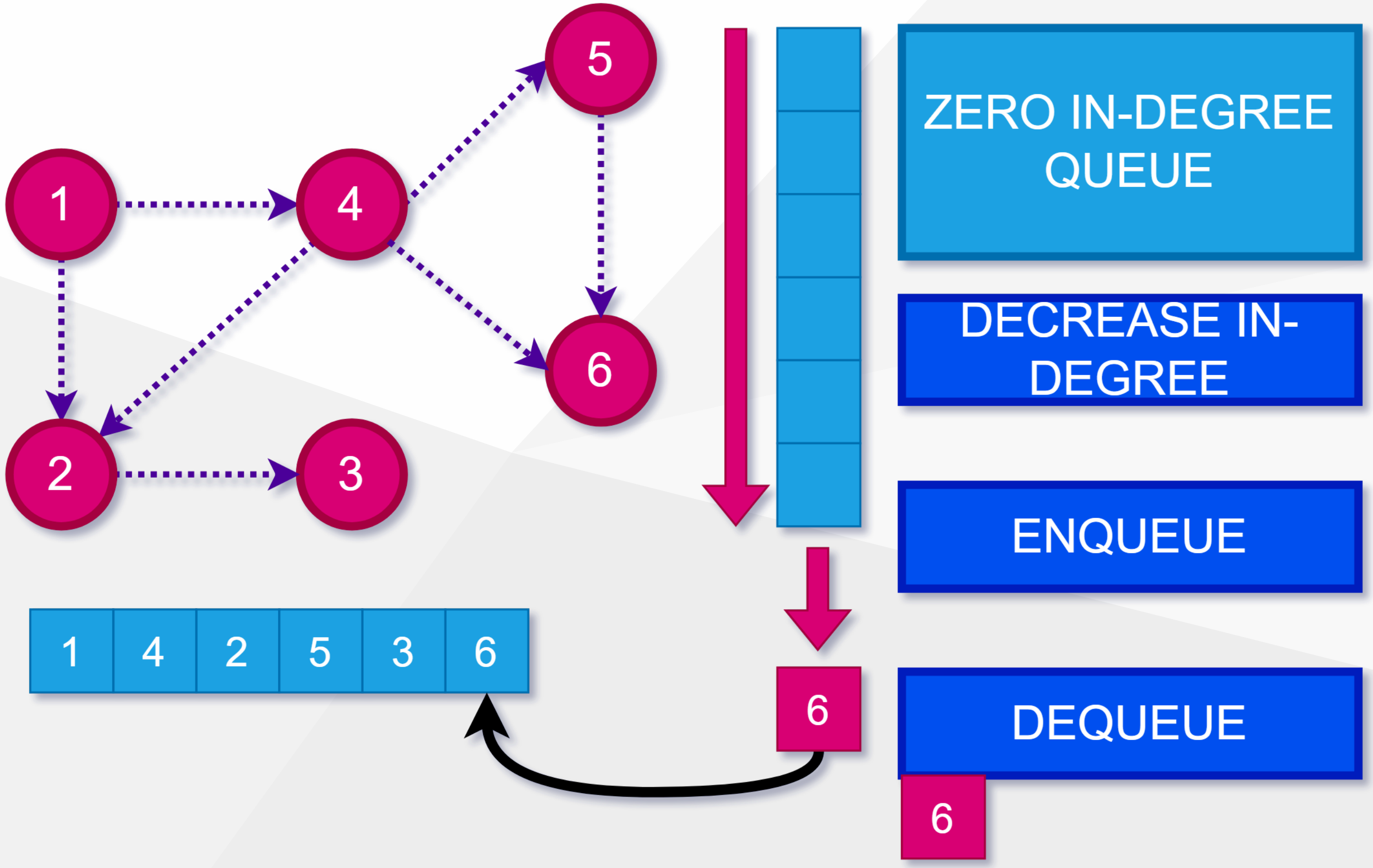
- STEP-9



Graph Traversal

Topological Sort - BFS version (Kahn's algorithm)

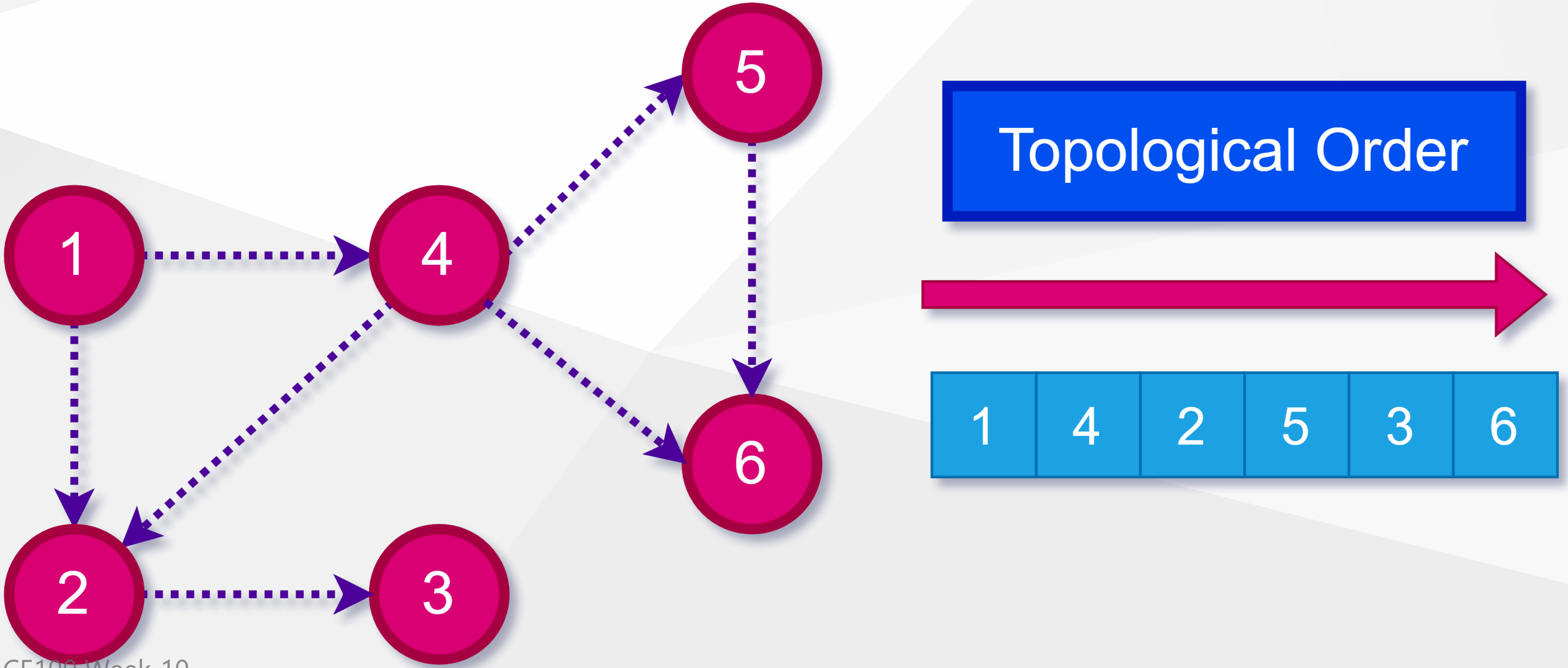
- STEP-10



Graph Traversal

Topological Sort - BFS version (Kahn's algorithm)

- STEP-11 (Final)



References

- BtechSmartClass-Introduction to Graphs
- BtechSmartClass-Graph Representations
- Leetcode - Topological Sort

$$End - Of - Week - 10 - Course - Notes$$