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CHEBYSHEV PSEUDOSPECTRAL METHOD COMPUTING EIGENVALUES FOR ORDINARY DIFFERENTIAL EQUATIONS WITH HOMOGENEOUS DIRICHLET BOUNDARY CONDITION

Keywords: pseudospectral method; differential matrix; eigenvalue problems; Chebyshev–Gauss–Lobatto points; Dirichlet condition; ordinary differential equations.

Abstract: Calculation of eigenvalue problems for the second-order ordinary differential equations is relevant for the physics problems. The second-order ordinary differential equation with homogeneous Dirichlet boundary condition was considered. The Chebyshev pseudospectral method (**CPM**) was used for the problem of eigenvalues basing on the Chebyshev–Gauss–Lobatto points to create the differential matrices. The Mathematica version 10.4 to write computing programs was used. In the applications, the Chebyshev pseudospectral method was used to find eigenvalues that were approxi-mated gradually to the exact eigenvalues of the problem.

Introduction

The basic forms of eigenvalue problem for the second-order ordinary differential equation with homogeneous Dirichlet boundary condition are:

$$\frac{d^2}{dx^2}u(x)+\lambda f(x)u(x)=0,\ a\le x\le b, \tag{1}$$

$$\frac{d^2}{dx^2}u(x)+q(x)u(x)+\lambda u(x)=0,\ a\le x\le b \tag{2}$$

and

$$p(x)\frac{d^2}{dx^2}u(x)+g(x)\frac{d}{dx}u(x)+\lambda u(x)=0,\ a\le x\le b, \tag{3}$$

where by the functions $f(x)$, $q(x)$, $p(x)$ and $g(x)$ are dependent x; $a,b\in\lambda$ and $u(a)=0$, $u(b)=0$. We have to find eigenvalues λ in those problems.

Nowadays there are many articles that were numerical solutions for differential eigenvalue problems [1–8]. We are shown some research on many articles. Such as: the Chebyshev polynomial spectral method [1]; the collocation method [2]; the Newton-based methods [3]; the finite differences method [4]; the Chebyshev collocation method [5]; the functional-discrete method [6]; the method of external excitation and the backward substitution method [7]; the linear multistep method, the shooting method [8], and others.

Hereafter, the pseudospectral method using the differentiation matrix by the Chebyshev–Gauss–Lobatto points to solve the second-order differential eigenvalue problem will be presented.

Chebyshev differentiation matrix

It is supposed that we have $p(x)$ polynomial degree N, and then we can know about values at the points $p(x_0)$, $p(x_1)$, ..., $p(x_N)$ and the first and second derivatives $p(x)$ at the same points in expressing matrix form:

$$\begin{pmatrix} p'(x_0) \\ p'(x_1) \\ \vdots \\ p'(x_N) \end{pmatrix} = D \begin{pmatrix} p(x_0) \\ p(x_1) \\ \vdots \\ p(x_N) \end{pmatrix}, \quad \begin{pmatrix} p''(x_0) \\ p''(x_1) \\ \vdots \\ p''(x_N) \end{pmatrix} = D^2 \begin{pmatrix} p(x_0) \\ p(x_1) \\ \vdots \\ p(x_N) \end{pmatrix}, \tag{4}$$

where $D = \left\{ d_{i,j}^{(1)} \right\}$ is an $(N+1)\times(N+1)$ differentiation matrix [9–13].

A grid function $p(x)$ is defined on the Chebyshev–Gauss–Lobatto points $x = \left\{ x_0, x_1, ..., x_N \right\}$ such that $x_k = \cos\left(k\pi / N\right)$, $k = \overline{0, N}$. They are the extrema of the N-th order in the Chebyshev polynomial $T_N(x) = \cos(N\cos^{-1} x)$. The differential matrix at the quadrature points $\left\{ d_{i,j}^{(1)} \right\}$ is given by:

$$\begin{gathered} d_{0,0}^{(1)} = -d_{N,N}^{(1)} = \frac{2N^2+1}{6}; \quad d_{i,i}^{(1)} = -\frac{x_i}{2(1-x_i^2)}, \quad i = \overline{1, N-1}; \\ d_{i,j}^{(1)} = \frac{c_i}{c_j}\frac{(-1)^{i+j}}{x_i - x_j}, \quad i \neq j, \quad i, j = \overline{1, N-1}, \end{gathered} \tag{5}$$

where

$$c_j = \begin{cases} 2, & j = 0 \text{ or } N, \\ 1, & \text{otherwise.} \end{cases} \tag{6}$$

Pseudospectral method using the Chebyshev differentiation matrix

Suppose that

$$\frac{d^2}{dx^2}u(x) = t(x), \quad u(-1) = \alpha, \quad u(1) = \beta, \tag{7}$$

and the collocation points $\{x_i\}$ so that $1 = x_0 > x_1 > ... > x_N = -1$.

We know that

$$\frac{d^2}{dx^2}u_N(x_i) = \sum_{k=0}^{N} (D^2)_{i,k} u_N(x_k). \tag{8}$$

Therefore, equation (7) becomes

$$\sum_{k=0}^{N} (D^2)_{i,k} u_N(x_k) = t(x_i), \quad i = \overline{1, N-1}, \quad u_N(x_N) = \alpha, \; u_N(x_0) = \beta. \tag{9}$$

Alternately, we partition the matrix D into matrices:

$$E^{(1)} = \begin{pmatrix} d_{1,1}^{(1)} & d_{1,2}^{(1)} & \cdots & d_{1,N-1}^{(1)} \\ d_{2,1}^{(1)} & d_{2,2}^{(1)} & \cdots & d_{2,N-1}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ d_{N-1,1}^{(1)} & d_{N-1,2}^{(1)} & \cdots & d_{N-1,N-1}^{(1)} \end{pmatrix}, \; e_0^{(1)} = \begin{pmatrix} d_{1,0}^{(1)} \\ d_{2,0}^{(1)} \\ \vdots \\ d_{N-1,0}^{(1)} \end{pmatrix}, \; e_n^{(1)} = \begin{pmatrix} d_{1,N}^{(1)} \\ d_{2,N}^{(1)} \\ \vdots \\ d_{N-1,N}^{(1)} \end{pmatrix}. \tag{10}$$

Table 1. The first ten eigenvalues of equation (13) with $N = 16$ and $N = 64$

i	λ^*	CPM	
		$N = 16$	$N = 64$
1	2.46740110	–2.46740110	–2.46740110
2	9.86960440	–9.86960440	–9.86960440
3	22.20660990	–22.20660991	–22.20660990
4	39.47841760	–39.47841626	–39.47841760
5	61.68502751	–61.68499578	–61.68502751
6	88.82643961	–88.82764218	–88.82643961
7	120.90265391	–120.88717288	–120.90265391
8	157.91367042	–158.29435581	–157.91367042
9	199.85948912	–195.58665225	–199.85948912
10	246.74011003	–267.58675788	–246.74011003

Remark: We see that when $N = 16$ in terms eigenvalues λ_7, λ_8, λ_9, λ_{10} compared the exact eigenvalues have error increases. But when $N = 64$, the values will not happen the error

Table 2. The first ten eigenvalues of equation (15) with $N = 32$ and $N = 71$

i	λ^*	CPM	
		$N = 32$	$N = 71$
1	20.79228845	–20.79233828	–20.79229525
2	82.41915382	–82.41946925	–82.41917369
3	185.13059610	–185.13100922	–185.13065669
4	328.92661528	–328.92778836	–328.92669324
5	513.80721138	–513.80815828	–513.80737669
6	739.77238439	–739.77468166	–739.77255405
7	1006.82213431	–1006.82337559	–1006.82244938
8	1314.95646113	–1314.95965178	–1314.95674829
9	1664.17536487	–1664.17597083	–1664.17586580
10	2054.47884552	–2054.48188985	–2054.47926472

Or we can rewrite the same short form [14]:

$$e_0^{(1)} = \{d_{i,0}^{(1)}\},\ E^{(1)} = \{d_{i,j}^{(1)}\},\ e_N^{(1)} = \{d_{i,N}^{(1)}\},\ i, j = \overline{1, N-1}. \tag{11}$$

Similarly, we partition the matrix D^2 into matrices: $e_0^{(2)}$, $E^{(2)}$ and $e_N^{(2)}$.
So the equation (7) can then be written in the form matrix:

$$\beta e_0^{(2)} + E^{(2)} u + \alpha e_N^{(2)} = t, \tag{12}$$

where u, t denote the vectors:

$$u = \begin{pmatrix} u_N(x_1) \\ u_N(x_2) \\ \vdots \\ u_N(x_{N-1}) \end{pmatrix},\ t = \begin{pmatrix} t(x_1) \\ t(x_2) \\ \vdots \\ t(x_{N-1}) \end{pmatrix}.$$

Table 3. The first ten eigenvalues of equation (17) with $N = 24$ and $N = 64$

i	λ^*	CPM	
		$N = 24$	$N = 64$
1	–2.44986759	–2.44986759	–2.44986759
2	–9.87481776	–9.87481776	–9.87481776
3	–22.20972813	–22.20972813	–22.20972813
4	–39.48032793	–39.48032793	–39.48032793
5	–61.68629633	–61.68629633	–61.68629633
6	–88.82733817	–88.82733817	–88.82733817
7	–120.90332180	–120.90332190	–120.90332180
8	–157.91418560	–157.91418144	–157.91418560
9	–199.85989826	–199.85985380	–199.85989826
10	–246.74044263	–246.74114132	–246.74044263

Table 4. The first ten eigenvalues of equation (19) with $N = 64$ and $N = 100$

i	λ^*	CPM	
		$N = 64$	$N = 100$
1	21.54228846	–21.54229215	–21.54228944
2	83.16915382	–83.16917131	–83.16915831
3	185.88059610	–185.88062921	–185.88060491
4	329.67661528	–329.67668480	–329.67663321
5	514.55721138	–514.55730232	–514.55723576
6	740.52238439	–740.52253906	–740.52242454
7	1007.57213431	–1007.57230948	–1007.57218178
8	1315.70646113	–1315.70673175	–1315.70653206
9	1664.92536487	–1664.92564758	–1664.92544264
10	2055.22884552	–2055.22925947	–2055.22895543

Remark: When N increases, the Chebyshev pseudospectral method determines eigenvalues approximate gradually to the exact eigenvalues of the problem. If we have to define multiple eigenvalues then we need only increase N

Applications

1. For the equation (1) $a < x < b$ and homogeneous Dirichlet boundary conditions $u(a) = 0$ and $u(b) = 0$.

a. If $f(x) = 1$, equation (1) becomes the simplest eigenvalue problems in second-order linear ordinary differential equations are:

$$\frac{d^2}{dx^2}u(x) + \lambda u(x) = 0, \quad u(-1) = 0,\ u(1) = 0, \tag{13}$$

and since its solutions $\lambda^* = (k\pi/2)^2$, $u(x) = \sin\left[k\pi(x+1)/2\right]$, $k = 1,2,\ldots$.

When the equation (13) applied CPM using the differentiation matrix and we have eigenvalue equation:

$$E^{(2)}u + \lambda u = 0, \tag{14}$$

the problem (13) becomes find eigenvalues of the matrix $E(2)$, the results are symmetrical with λ. Table 1 shows the computed eigenvalues of CPM with the cases $N = 16$ and $N = 64$.

b. If $f(x) \neq 1$, we transform (1) into form $E^{(2)}u = -\lambda Fu$, here F denotes a diagonal matrix with elements $f(x_i)$, $1 \leq i \leq N-1$ and become a form $Bu = -\lambda u$ where $B = F^{-1}E^{(2)}$, the problems return to form (14).

For example, we consider eigenvalue problem [15]:

$$\frac{d^2}{dx^2}u(x) + \frac{\lambda}{(1+x)^2}u(x) = 0, \quad u(-1) = 0, \; u(1) = 0, \tag{15}$$

since its solutions $\lambda^* = 1/4 + \left(k\pi/\ln 2\right)^2$ and

$$u\left(x\right) = const\sqrt{1+x}\sin\left(k\pi\ln\left(1+x\right)/\ln 2\right), \; k = 1,2,\ldots.$$

Table 2 shows the first ten eigenvalues of CPM with the cases $N = 32$ and $N = 71$.

2. For the equation (2) $a < x < b$ and homogeneous Dirichlet boundary conditions $u(a) = 0$ and $u(b) = 0$. When the equation (2) applied CPM, equation (2) can be written as follows:

$$(-E^{(2)} + Q)u + \lambda u = 0, \tag{16}$$

here Q denotes a diagonal matrix with elements $q(x_i)$, $i = \overline{1, N-1}$.

For example, we consider eigenvalue problem:

$$\frac{d^2}{dx^2}u(x) + xu(x) = -\lambda u(x), \quad u(-1) = 0, \; u(1) = 0. \tag{17}$$

In table 3, the numerical result at λ^* column, we used the method to find the eigenvalues of the Mathematica [16]. The numerical result of CPM was shown the first ten eigenvalues with $N = 24$ and $N = 64$.

3. For the equation (3) with $a < x < b$ and homogeneous Dirichlet boundary conditions $u(a) = 0$ and $u(b) = 0$. Apply CPM to the equation (3), we can be written as follows:

$$(-PE^{(2)} + GE^{(1)})u + \lambda u = 0, \tag{18}$$

here P and G are the diagonal matrices with elements in turn are $p(x_i)$ and $g(x_i)$ with $i = \overline{1, N-1}$.

For example, consider eigenvalue problem [17]:

$$x^2\frac{d^2}{dx^2}u(x) + 3x\frac{d}{dx}u(x) = -\lambda u(x), \quad u(1) = 0, \; u(2) = 0, \tag{19}$$

since its solutions $\lambda^* = 1 + \left(k\pi/\ln 2\right)^2$, $u\left(x\right) = \sin\left[k\pi\ln x/\ln 2\right]/x$, $k = 1,2,\ldots$. Table 4 shows the first ten eigenvalues with the cases $N = 64$ and $N = 100$.

Conclusions

The eigenvalues of differential eigenvalue problems are found by pseudospectral Chebyshev method for the accurately approximate. But the numerical results show that the errors of eigenvalues with $\lambda_{\lfloor N/2\rfloor},\ldots,\lambda_{\lfloor N-1\rfloor}$ are large.

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Чебышевский псевдоспектральный метод вычисления собственных значений для обычных дифференциальных уравнений с однородным граничным условием Дирихле

Ключевые слова: дифференциальныы матрицы; псевдоспектральный метод; обыкновенные дифференциальные уравнения; проблемы собственных значений.

Аннотация: Вычисление собственных значений в задачах на собственные значения для обыкновенных дифференциальных уравнений второго порядка представляет важность для задачи физики. Рассмотрено обыкновенное дифференциальное уравнение второго порядка с однородным граничным условием Дирихле. Собственные значения задачи были использованы Чебышевским псевдоспектральным методом (CPM) на основе точек Чебышева–Гаусса–Лобатто для создания дифференциальных матриц. Была использована Mathematica версии 10.4 для написания компьютерных программ. В приложениях псевдоспектральным методом Чебышева были найдены собственные значения, постепенно приближающиеся к точным собственным значениям задачи.