

# Equation-Free function toolbox for Matlab/Octave: Full Developers Manual

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August 17, 2020

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## Abstract

This ‘equation-free toolbox’ empowers the computer-assisted analysis of complex, multiscale systems. Its aim is to enable you to use microscopic simulators to perform system level tasks and analysis, because microscale simulations are often the best available description of a system. The methodology bypasses the derivation of macroscopic evolution equations by computing only short bursts of the microscale simulator (Kevrekidis & Samaey 2009, Kevrekidis et al. 2004, 2003, e.g.), and often only computing on small patches of the spatial domain (Roberts et al. 2014, e.g.). This suite of functions empowers users to start implementing such methods in their own applications. Download via <https://github.com/uoa1184615/EquationFreeGit>

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## 1 Introduction

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This Developers Manual contains complete descriptions of the code in each function in the toolbox, and of each example. For concise descriptions of each function, quick start guides, and some basic examples, see the User Manual.

**Users** Download via <https://github.com/uaa1184615/EquationFreeGit>. Place the folder of this toolbox in a path searched by MATLAB/Octave. Then read the section(s) that documents the function of interest.

**Quick start** Maybe start by adapting one of the included examples. Many of the main functions include, at their beginning, example code of their use—code which is executed when the function is invoked without any arguments.

- To projectively integrate over time a multiscale, slow-fast, system of ODEs you could use `PIRK2()`, or `PIRK4()` for higher-order accuracy: adapt the Michaelis–Menten example at the beginning of `PIRK2.m` ([Section 2.2.2](#)).
- You may use forward bursts of simulation in order to simulate the slow dynamics backward in time, as in `egPIMM.m` ([Section 2.3](#)).
- To only resolve the slow dynamics in the projective integration, use lifting and restriction functions by adapting the singular perturbation ODE example at the beginning of `PIG.m` ([Section 2.4.2](#)).

**Space-time systems** Consider an evolving system over a large spatial domain when all you have is a microscale code. To efficiently simulate over the large domain, one can simulate in just small patches of the domain, appropriately coupled.

- In 1D space adapt the code at the beginning of `configPatches1.m` for Burgers' PDE ([Section 3.2.2](#)), or the staggered patches of 1D water wave equations in `waterWaveExample.m` ([Section 3.8](#)).
- In 2D space adapt the code at the beginning of `configPatches2.m` for nonlinear diffusion ([Section 3.11.2](#)), or the regular patches of the 2D wave PDE of `wave2D.m` ([Section 3.14](#)).
- In 3D space adapt the code at the beginning of `configPatches3.m` for wave propagation through a heterogeneous medium ([Section 3.16.2](#)), or the patches of the 3D heterogeneous diffusion of `homoDiffEdgy3.m` ([Section 3.19](#)).
- Other provided examples include cases of macroscale *computational homogenisation* of microscale heterogeneity.

**Verification** Most of these schemes have proven ‘accuracy’ when compared to the underlying specified microscale system. In the spatial patch schemes, we measure ‘accuracy’ by the order of consistency between macroscale dynamics and the specified microscale.

- [Roberts & Kevrekidis \(2007\)](#) and [Roberts et al. \(2014\)](#) proved reasonably general high-order consistency for the 1D and 2D patch schemes, respectively.
- In wave-like systems, [Cao & Roberts \(2016b\)](#) established high-order consistency for the 1D staggered patch scheme.
- A heterogeneous microscale is more difficult, but [Bunder et al. \(2017\)](#) showed good accuracy in a variety of circumstances, for appropriately chosen parameters. Further, [Bunder et al. \(2020\)](#) developed a new ‘edgy’ inter-patch interpolation that is proven to be good for simulating the macroscale homogenised dynamics of microscale heterogeneous systems—now coded in the toolbox.

**Blackbox scenarios** Suppose that you have a *detailed and trustworthy* computational simulation of some problem of interest. Let’s say the simulation is coded in terms of detailed (microscale) variable values  $\vec{u}(t)$ , in  $\mathbb{R}^p$  for some number  $p$  of field variables, and evolving in time  $t$ . The details  $\vec{u}$  could represent particles, agents, or states of a system. When the computation is too time consuming to simulate all the times of interest, then Projective Integration may be able to predict long-time dynamics, both forward and backward in time. In this case, provide your detailed computational simulation as a ‘black box’ to the Projective Integration functions of [Chapter 2](#).

In many scenarios, the problem of interest involves space or a ‘spatial’ lattice. Let’s say that indices  $i$  correspond to ‘spatial’ coordinates  $\vec{x}_i(t)$ , which are often fixed: in lattice problems the positions  $\vec{x}_i$  would be fixed in time (unless employing a moving mesh on the microscale); however, in particle problems the positions would evolve. And suppose your detailed and trustworthy simulation is coded also in terms of micro-field variable values  $\vec{u}_i(t) \in \mathbb{R}^p$  at time  $t$ . Often the detailed computational simulation is too expensive over all the desired spatial domain  $\vec{x} \in \mathbb{X} \subset \mathbb{R}^d$ . In this case, the toolbox functions of [Chapter 3](#) empower you to simulate on only small, well-separated, patches of space by appropriately coupling between patches your simulation code, as a ‘black box’, executing on each small patch. The computational savings may be enormous, especially if combined with projective integration.

**Contributors** The aim of this project is to collectively develop a MATLAB/Octave toolbox of equation-free algorithms. Initially the algorithms are basic, and the plan is to subsequently develop more and more capability.

MATLAB appears a good choice for a first version since it is widespread, efficient, supports various parallel modes, and development costs are reasonably low. Further it is built on BLAS and LAPACK so the cache and superscalar CPU are potentially well utilised. We aim to develop functions that work for MATLAB/Octave. [Appendix A](#) outlines some details for contributors.

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## 2 Projective integration of deterministic ODEs

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## 2.1 Introduction

This section provides some good projective integration functions ([Gear & Kevrekidis 2003b,c](#), [Givon et al. 2006](#), [Marschler et al. 2014](#), [Maclean & Gottwald 2015](#), [Sieber et al. 2018](#), e.g.). The goal is to enable computationally expensive multiscale dynamic simulations/integrations to efficiently compute over very long time scales.

**Quick start** [Section 2.2.2](#) shows the most basic use of a projective integration function. [Section 2.3](#) shows how to code more variations of the introductory example of a long time simulation of the Michaelis–Menton multiscale system of differential equations. Then see [Figures 2.1](#) and [2.2](#)

**Scenario** When you are interested in a complex system with many interacting parts or agents, you usually are primarily interested in the self-organised emergent macroscale characteristics. Projective integration empowers us to efficiently simulate such long-time emergent dynamics. We suppose you have coded some accurate, fine-scale, microscale simulation of the complex system, and call such code a microsolver.

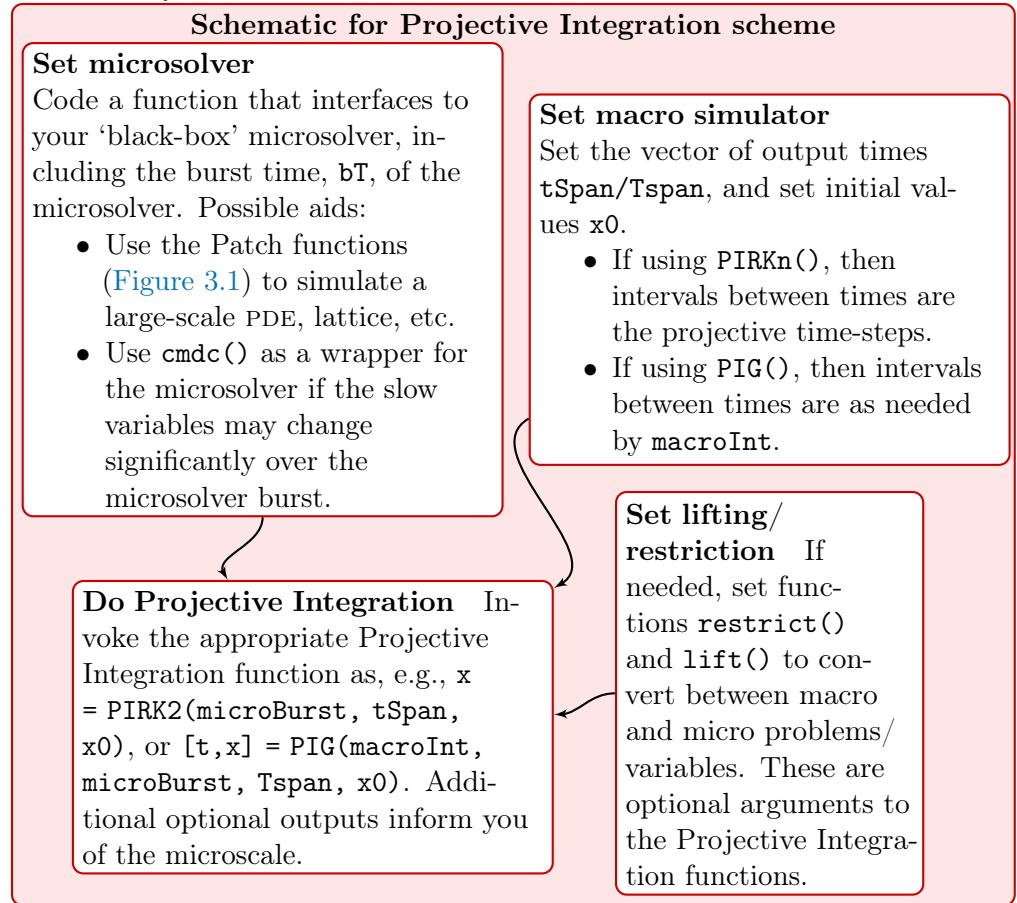
The Projective Integration section of this toolbox consists of several functions. Each function implements over a long-time scale a variant of a standard numerical method to simulate/integrate the emergent dynamics of the complex system. Each function has standardised inputs and outputs.

[Petersik \(2019–\)](#) is also developing, in python, some projective integration functions.

### Main functions

- Projective Integration by second or fourth-order Runge–Kutta is implemented by `PIRK2()` or `PIRK4()` respectively. These schemes are suitable for precise simulation of the slow dynamics, provided the time period spanned by an application of the microsolver is not too large.
- Projective Integration with a General method, `PIG()`. This function enables a Projective Integration implementation of any integration method over macroscale time-steps. It does not matter whether the method is a standard MATLAB/Octave algorithm, or one supplied by the user. `PIG()` should only be used directly in very stiff systems, less stiff systems additionally require `cdmc()`.
- *Constraint-defined manifold computing*, `cdmc()`, is a helper function, based on the method introduced in [Gear et al. \(2005a\)](#), that iteratively applies the microsolver and backward projection in time. The result is to project the fast variables close to the slow manifold, without advancing the current time by the burst time of the microsolver. This function reduces errors related to the simulation length of the microsolver in the `PIG` function. In particular, it enables `PIG()` to be used on problems that are not particularly stiff.

*Figure 2.1: The Projective Integration method greatly accelerates simulation/integration of a system exhibiting multiple time scales. The Projective Integration Chapter 2 presents several separate functions, as well as several optional wrapper functions that may be invoked. This chart overviews constructing a Projective Integration simulation, whereas Figure 2.2 roughly guides which top-level Projective Integration functions should be used. Chapter 2 fully details each function.*



The above functions share dependence on a user-specified *microsolver* that accurately simulates some problem of interest.

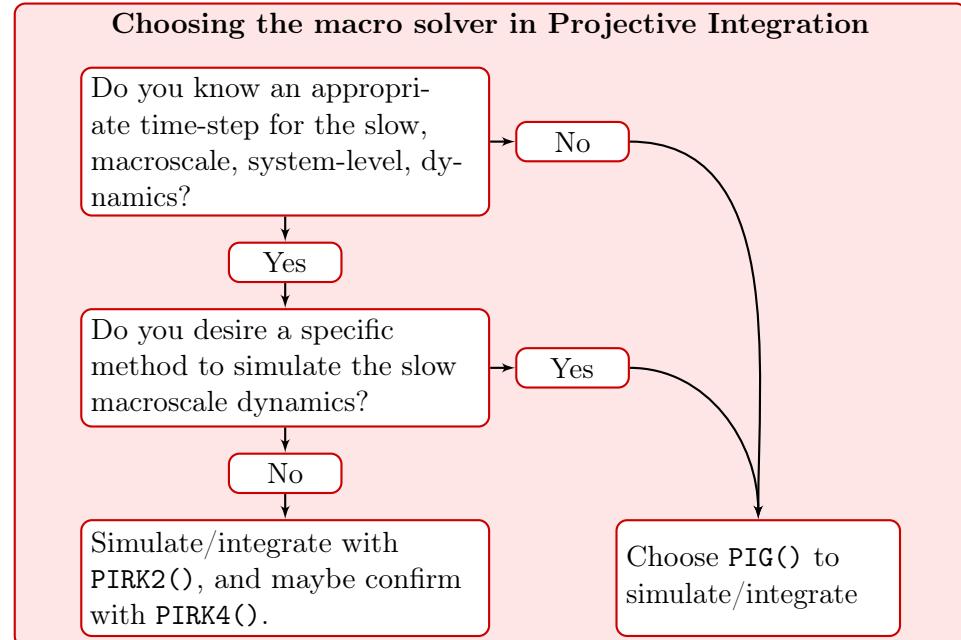
The following sections describe the `PIRK2()` and `PIG()` functions in detail, providing an example for each. The function `PIRK4()` is very similar to `PIRK2()`. Descriptions for the minor functions follow, and an example using `cdmc()`.

## 2.2 `PIRK2()`: projective integration of second-order accuracy

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*Figure 2.2: The Projective Integration method greatly accelerates simulation/integration of a system exhibiting multiple time scales. In conjunction with Figure 2.1, this chart roughly guides which top-level Projective Integration functions should be used. Chapter 2 fully details each function.*



#### 2.2.4 If no output specified, then plot the simulation . . . . 14

##### 2.2.1 Introduction

This Projective Integration scheme implements a macroscale scheme that is analogous to the second-order Runge–Kutta Improved Euler integration.

```
21 function [x, tms, xms, rm, svf] = PIRK2(microBurst, tSpan, x0, bT)
```

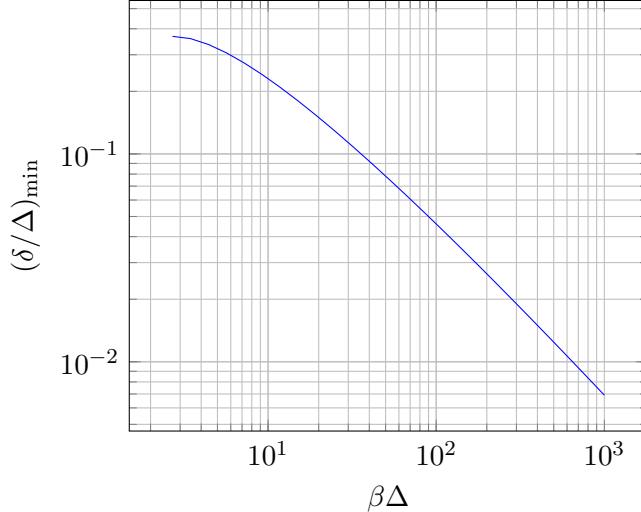
**Input** If there are no input arguments, then this function applies itself to the Michaelis–Menton example: see the code in Section 2.2.2 as a basic template of how to use.

- `microBurst()`, a user-coded function that computes a short-time burst of the microscale simulation.

```
[tOut, xOut] = microBurst(tStart, xStart, bT)
```

- Inputs: `tStart`, the start time of a burst of simulation; `xStart`, the row  $n$ -vector of the starting state; `bT`, optional, the total time to simulate in the burst—if your `microBurst()` determines the burst time, then replace `bT` in the argument list by `varargin`.
- Outputs: `tOut`, the column vector of solution times; and `xOut`, an array in which each row contains the system state at corresponding times.

Figure 2.3: Need macroscale step  $\Delta$  such that  $|\alpha\Delta| \lesssim \sqrt{6\varepsilon}$  for given relative error  $\varepsilon$  and slow rate  $\alpha$ , and then  $\delta/\Delta \gtrsim \frac{1}{\beta\Delta} \log |\beta\Delta|$  determines the minimum required burst length  $\delta$  for every given fast rate  $\beta$ .



- `tSpan` is an  $\ell$ -vector of times at which the user requests output, of which the first element is always the initial time. `PIRK2()` does not use adaptive time-stepping; the macroscale time-steps are (nearly) the steps between elements of `tSpan`.
- `x0` is an  $n$ -vector of initial values at the initial time `tSpan(1)`. Elements of `x0` may be `Nan`: such `Nans` are carried in the simulation through to the output, and often represent boundaries/edges in spatial fields.
- `bT`, *optional*, either missing, or empty (`[]`), or a scalar: if a given scalar, then it is the length of the micro-burst simulations—the minimum amount of time needed for the microscale simulation to relax to the slow manifold; else if missing or `[]`, then `microBurst()` must itself determine the length of a burst.

```
70 if nargin<4, bT=[]; end
```

**Choose a long enough burst length** Suppose: firstly, you have some desired relative accuracy  $\varepsilon$  that you wish to achieve (e.g.,  $\varepsilon \approx 0.01$  for two digit accuracy); secondly, the slow dynamics of your system occurs at rate/frequency of magnitude about  $\alpha$ ; and thirdly, the rate of *decay* of your fast modes are faster than the lower bound  $\beta$  (e.g., if three fast modes decay roughly like  $e^{-12t}, e^{-34t}, e^{-56t}$  then  $\beta \approx 12$ ). Then set

1. a macroscale time-step,  $\Delta = \text{diff}(\text{tSpan})$ , such that  $\alpha\Delta \approx \sqrt{6\varepsilon}$ , and
2. a microscale burst length,  $\delta = bT \gtrsim \frac{1}{\beta} \log |\beta\Delta|$ , see Figure 2.3.

**Output** If there are no output arguments specified, then a plot is drawn of the computed solution `x` versus `tSpan`.

- $\mathbf{x}$ , an  $\ell \times n$  array of the approximate solution vector. Each row is an estimated state at the corresponding time in `tSpan`. The simplest usage is then `x = PIRK2(microBurst,tSpan,x0,bT)`.

However, microscale details of the underlying Projective Integration computations may be helpful. `PIRK2()` provides up to four optional outputs of the microscale bursts.

- $\mathbf{tms}$ , optional, is an  $L$  dimensional column vector containing the microscale times within the burst simulations, each burst separated by `NaN`;
- $\mathbf{xms}$ , optional, is an  $L \times n$  array of the corresponding microscale states—each rows is an accurate estimate of the state at the corresponding time `tms` and helps visualise details of the solution.
- $\mathbf{rm}$ , optional, a struct containing the ‘remaining’ applications of the `microBurst` required by the Projective Integration method during the calculation of the macrostep:
  - `rm.t` is a column vector of microscale times; and
  - `rm.x` is the array of corresponding burst states.

The states `rm.x` do not have the same physical interpretation as those in `xms`; the `rm.x` are required in order to estimate the slow vector field during the calculation of the Runge–Kutta increments, and do *not* accurately approximate the macroscale dynamics.

- $\mathbf{svf}$ , optional, a struct containing the Projective Integration estimates of the slow vector field.
  - `svf.t` is a  $2\ell$  dimensional column vector containing all times at which the Projective Integration scheme is extrapolated along `microBurst` data to form a macrostep.
  - `svf.dx` is a  $2\ell \times n$  array containing the estimated slow vector field.

### 2.2.2 If no arguments, then execute an example

```
175 if nargin==0
```

**Example code for Michaelis–Menton dynamics** The Michaelis–Menton enzyme kinetics is expressed as a singularly perturbed system of differential equations for  $x(t)$  and  $y(t)$ :

$$\frac{dx}{dt} = -x + (x + \frac{1}{2})y \quad \text{and} \quad \frac{dy}{dt} = \frac{1}{\epsilon} [x - (x + 1)y]$$

(encoded in function `MMburst()` in the next paragraph). With initial conditions  $x(0) = 1$  and  $y(0) = 0$ , the following code computes and plots a solution over time  $0 \leq t \leq 6$  for parameter  $\epsilon = 0.05$ . Since the rate of decay is  $\beta \approx 1/\epsilon$  we choose a burst length  $\epsilon \log(\Delta/\epsilon)$  as here the macroscale time-step  $\Delta = 1$ .

```
196 global MMepsilon
197 MMepsilon = 0.05
```

```

198 ts = 0:6
199 bT = MMepsilon*log((ts(2)-ts(1))/MMepsilon)
200 [x,tms,xms] = PIRK2(@MMburst, ts, [1;0], bT);
201 figure, plot(ts,x,'o:',tms,xms)
202 title('Projective integration of Michaelis--Menten enzyme kinetics')
203 xlabel('time t'), legend('x(t)', 'y(t)')

```

Upon finishing execution of the example, exit this function.

```

209 return
210 end%if no arguments

```

**Code a burst of Michaelis–Menten enzyme kinetics** Integrate a burst of length  $bT$  of the ODES for the Michaelis–Menten enzyme kinetics at parameter  $\epsilon$  inherited from above. Code ODES in function `dMMdt` with variables  $x = x(1)$  and  $y = x(2)$ . Starting at time  $ti$ , and state  $xi$  (row), we here simply use MATLAB/Octave's `ode23/lode` to integrate a burst in time.

```

15 function [ts, xs] = MMburst(ti, xi, bT)
16     global MMepsilon
17     dMMdt = @(t,x) [-x(1)+(x(1)+0.5)*x(2)
18                     1/MMepsilon*( x(1)-(x(1)+1)*x(2) )];
19     if ~exist('OCTAVE_VERSION','builtin')
20         [ts, xs] = ode23(dMMdt, [ti ti+bT], xi);
21     else % octave version
22         [ts, xs] = odeOct(dMMdt, [ti ti+bT], xi);
23     end
24 end
25
26 function [ts,xs] = odeOct(dxdt,tSpan,x0)
27     if length(tSpan)>2, ts = tSpan;
28     else ts = linspace(tSpan(1),tSpan(end),21);
29     end
30     % mimic ode45 and ode23, but much slower for non-PI
31     lsode_options('integration method','non-stiff');
32     xs = lsode(@(x,t) dxdt(t,x),x0,ts);
33 end

```

### 2.2.3 The projective integration code

Determine the number of time-steps and preallocate storage for macroscale estimates.

```

229 nT=length(tSpan);
230 x=nan(nT,length(x0));

```

Get the number of expected outputs and set logical indices to flag what data should be saved.

```

238 nArgOut=nargout();
239 saveMicro = (nArgOut>1);

```

```

240 saveFullMicro = (nArgOut>3);
241 saveSvf = (nArgOut>4);

```

Run a preliminary application of the microBurst on the given initial state to help relax to the slow manifold. This is done in addition to the microBurst in the main loop, because the initial state is often far from the attracting slow manifold. Require the user to input and output rows of the system state.

```

254 x0 = reshape(x0,1,[]);
255 [relax_t,relax_x0] = microBurst(tSpan(1),x0,bT);

```

Use the end point of this preliminary microBurst as the initial state for the loop of macro-steps.

```

263 tSpan(1) = relax_t(end);
264 x(1,:)=relax_x0(end,:);

```

If saving information, then record the first application of the microBurst. Allocate cell arrays for times and states for outputs requested by the user, as concatenating cells is much faster than iteratively extending arrays.

```

274 if saveMicro
275     tms = cell(nT,1);
276     xms = cell(nT,1);
277     tms{1} = reshape(relax_t,[],1);
278     xms{1} = relax_x0;
279     if saveFullMicro
280         rm.t = cell(nT,1);
281         rm.x = cell(nT,1);
282         if saveSvf
283             svf.t = nan(2*nT-2,1);
284             svf.dx = nan(2*nT-2,length(x0));
285         end
286     end
287 end

```

### Loop over the macroscale time-steps

```

295 for jT = 2:nT
296     T = tSpan(jT-1);

```

If two applications of the microBurst would cover one entire macroscale time-step, then do so (setting some internal states to NaN); else proceed to projective step.

```

304     if ~isempty(bT) && 2*abs(bT)>=abs(tSpan(jT)-T) && bT*(tSpan(jT)-T)>0
305         [t1,xm1] = microBurst(T, x(jT-1,:), tSpan(jT)-T);
306         x(jT,:) = xm1(end,:);
307         t2=nan; xm2=nan(1,size(xm1,2));
308         dx1=xm2; dx2=xm2;
309     else

```

Run the first application of the microBurst; since this application directly follows from the initial conditions, or from the latest macrostep, this microscale information is physically meaningful as a simulation of the system. Extract the size of the final time-step.

```
320      [t1,xm1] = microBurst(T, x(jT-1,:), bT);
321      del = t1(end)-t1(end-1);
```

Check for round-off error.

```
327      xt=[reshape(t1(end-1:end),[],1) xm1(end-1:end,:)];
328      roundingTol=1e-8;
329      if norm(diff(xt))/norm(xt,'fro') < roundingTol
330      warning(['significant round-off error in 1st projection at T=' num2str(T))
331      end
```

Find the needed time-step to reach time  $tSpan(n+1)$  and form a first estimate  $dx1$  of the slow vector field.

```
340      Dt = tSpan(jT)-t1(end);
341      dx1 = (xm1(end,:)-xm1(end-1,:))/del;
```

Project along  $dx1$  to form an intermediate approximation of  $x$ ; run another application of the microBurst and form a second estimate of the slow vector field (assuming the burst length is the same, or nearly so).

```
351      xint = xm1(end,:)+ (Dt-(t1(end)-t1(1)))*dx1;
352      [t2,xm2] = microBurst(T+Dt, xint, bT);
353      del = t2(end)-t2(end-1);
354      dx2 = (xm2(end,:)-xm2(end-1,:))/del;
```

Check for round-off error.

```
360      xt=[reshape(t2(end-1:end),[],1) xm2(end-1:end,:)];
361      if norm(diff(xt))/norm(xt,'fro') < roundingTol
362      warning(['significant round-off error in 2nd projection at T=' num2str(T))
363      end
```

Use the weighted average of the estimates of the slow vector field to take a macro-step.

```
371      x(jT,:) = xm1(end,:)+ Dt*(dx1+dx2)/2;
```

Now end the if-statement that tests whether a projective step saves simulation time.

```
379      end
```

If saving trusted microscale data, then populate the cell arrays for the current loop iterate with the time-steps and output of the first application of the microBurst. Separate bursts by NaNs.

```
389      if saveMicro
390          tms{jT} = [reshape(t1,[],1); nan];
391          xms{jT} = [xm1; nan(1,size(xm1,2))];
```

If saving all microscale data, then repeat for the remaining applications of the microBurst.

```
399      if saveFullMicro
400          rm.t{jT} = [reshape(t2,[],1); nan];
401          rm.x{jT} = [xm2; nan(1,size(xm2,2))];
```

If saving Projective Integration estimates of the slow vector field, then populate the corresponding cells with times and estimates.

```
410      if saveSvf
411          svf.t(2*jT-3:2*jT-2) = [t1(end); t2(end)];
412          svf.dx(2*jT-3:2*jT-2,:) = [dx1; dx2];
413      end
414  end
415 end
```

End the main loop over all the macro-steps.

```
421 end
Overwrite x(1,:) with the specified initial condition tSpan(1).
430 x(1,:) = reshape(x0,1,[]);
```

For additional requested output, concatenate all the cells of time and state data into two arrays.

```
438 if saveMicro
439     tms = cell2mat(tms);
440     xms = cell2mat(xms);
441     if saveFullMicro
442         rm.t = cell2mat(rm.t);
443         rm.x = cell2mat(rm.x);
444     end
445 end
```

#### 2.2.4 If no output specified, then plot the simulation

```
453 if nArgOut==0
454     figure, plot(tSpan,x,'o:')
455     title('Projective Simulation with PIRK2')
456 end
```

This concludes PIRK2().

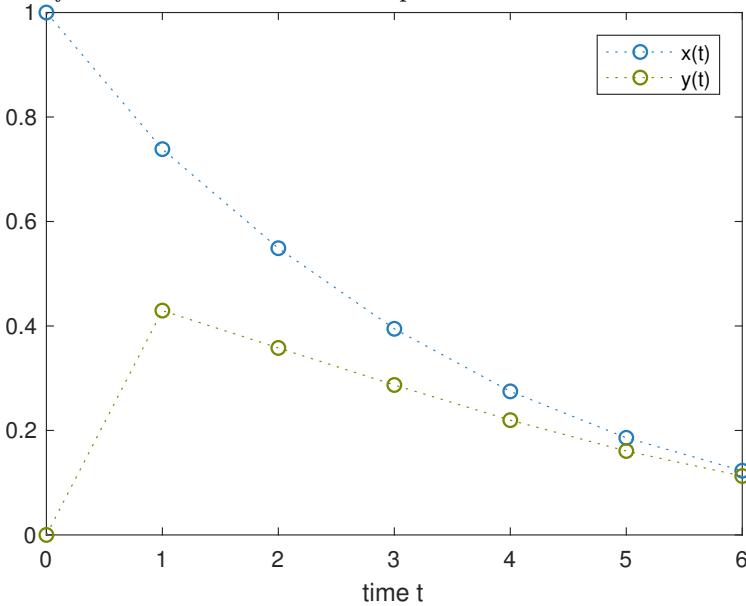
```
463 end
```

### 2.3 egPIMM: Example projective integration of Michaelis–Menton kinetics

The Michaelis–Menton enzyme kinetics is expressed as a singularly perturbed system of differential equations for  $x(t)$  and  $y(t)$ :

$$\frac{dx}{dt} = -x + (x + \frac{1}{2})y \quad \text{and} \quad \frac{dy}{dt} = \frac{1}{\epsilon} [x - (x + 1)y]$$

Figure 2.4: Michaelis–Menten enzyme kinetics simulated with the projective integration of PIRK2(): macroscale samples.



(encoded in function `MMburst()` below). As illustrated by Figure 2.5, the slow variable  $x(t)$  evolves on a time scale of one, whereas the fast variable  $y(t)$  evolves on a time scale of the small parameter  $\epsilon$ .

**Invoke projective integration** Clear, and set the scale separation parameter  $\epsilon$  to something small like 0.01. Here use  $\epsilon = 0.1$  for clearer graphs.

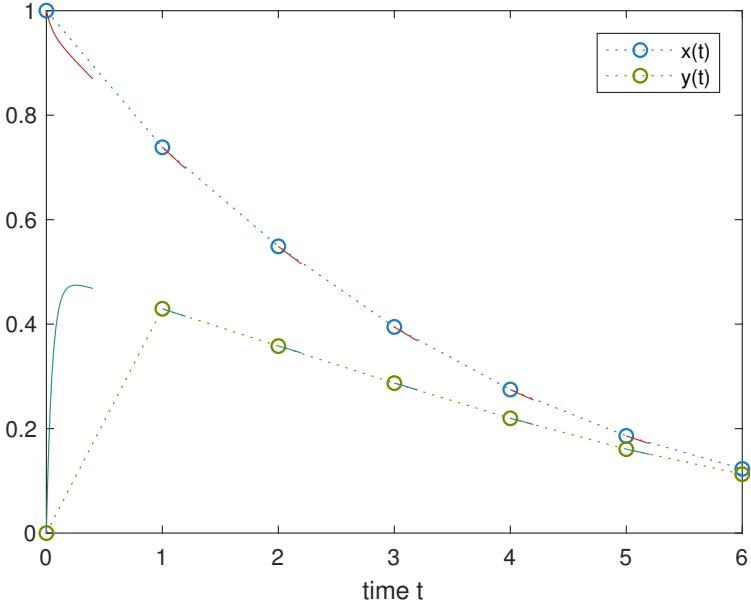
```
31 clear all, close all
32 global MMepsilon
33 MMepsilon = 0.1
```

First, the end of this section encodes the computation of bursts of the Michaelis–Menten system in a function `MMburst()`. Second, here set macroscale times of computation and interest into vector `ts`. Then, invoke Projective Integration with `PIRK2()` applied to the burst function, say using bursts of simulations of length  $2\epsilon$ , and starting from the initial condition for the Michaelis–Menten system, at time  $t = 0$ , of  $(x, y) = (1, 0)$  (off the slow manifold).

```
48 ts = 0:6
49 xs = PIRK2(@MMburst, ts, [1;0], 2*MMepsilon)
50 plot(ts,xs,'o:')
51 xlabel('time t'), legend('x(t)', 'y(t)')
52 pause(1)
```

Figure 2.4 plots the macroscale results showing the long time decay of the Michaelis–Menten system on the slow manifold. Sieber et al. (2018) [§4] used this system as an example of their analysis of the convergence of Projective Integration.

*Figure 2.5: Michaelis–Menten enzyme kinetics simulated with the projective integration of PIRK2(): the microscale bursts show the initial transients on a time scale of  $\epsilon = 0.1$ , and then the alignment along the slow manifold.*



**Request and plot the microscale bursts** Because the initial conditions of the simulation are off the slow manifold, the initial macroscale step appears to ‘jump’ (Figure 2.4). In order to see the initial transient attraction to the slow manifold we plot some microscale data in Figure 2.5. Two further output variables provide this microscale burst information.

```

78 [xs,tMicro,xMicro] = PIRK2(@MMburst, ts, [1;0], 2*MMepsilon);
79 figure, plot(ts,xs,'o:',tMicro,xMicro)
80 xlabel('time t'), legend('x(t)', 'y(t)')
81 pause(1)

```

Figure 2.5 plots the macroscale and microscale results—also showing that the initial burst is by default twice as long. Observe the slow variable  $x(t)$  is also affected by the initial transient (hence other schemes which ‘freeze’ slow variables are less accurate).

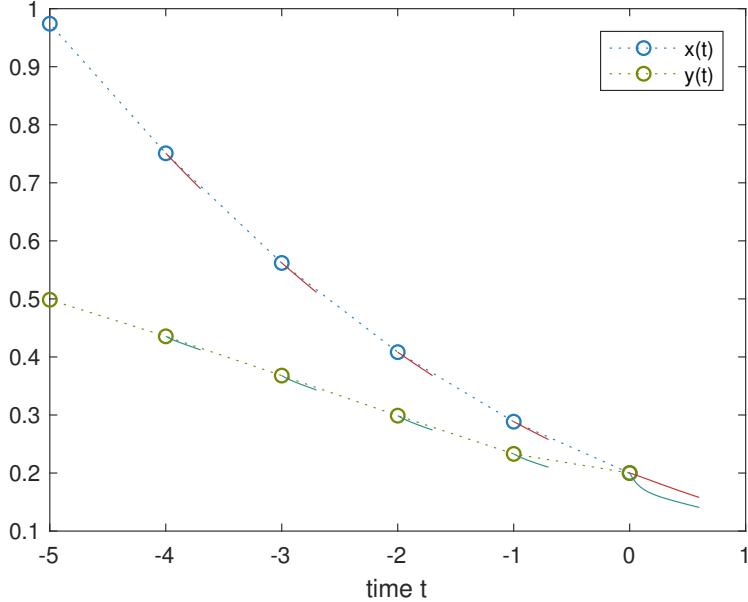
**Simulate backward in time** Figure 2.6 shows that projective integration even simulates backward in time along the slow manifold using short forward bursts (Gear & Kevrekidis 2003a, Frewen et al. 2009). Such backward macroscale simulations succeed despite the fast variable  $y(t)$ , when backward in time, being viciously unstable. However, backward integration appears to need longer bursts, here  $3\epsilon$ .

```

111 ts = 0:-1:-5
112 [xs,tMicro,xMicro] = PIRK2(@MMburst, ts, 0.2*[1;1], 3*MMepsilon);
113 figure, plot(ts,xs,'o:',tMicro,xMicro)
114 xlabel('time t'), legend('x(t)', 'y(t)')

```

*Figure 2.6: Michaelis–Menten enzyme kinetics at  $\epsilon = 0.1$  simulated backward with the projective integration of PIRK2(): the microscale bursts show the short forward simulations used to projectively integrate backward in time.*



**Code a burst of Michaelis–Menten enzyme kinetics** Integrate a burst of length  $bT$  of the ODEs for the Michaelis–Menten enzyme kinetics at parameter  $\epsilon$  inherited from above. Code ODEs in function `dMMdt` with variables  $x = \mathbf{x}(1)$  and  $y = \mathbf{x}(2)$ . Starting at time `ti`, and state `xi` (row), we here simply use MATLAB/Octave’s `ode23/lode` to integrate a burst in time.

```

15 function [ts, xs] = MMburst(ti, xi, bT)
16     global MMepsilon
17     dMMdt = @(t,x) [-x(1)+(x(1)+0.5)*x(2)
18                     1/MMepsilon*( x(1)-(x(1)+1)*x(2) ) ];
19     if ~exist('OCTAVE_VERSION','builtin')
20         [ts, xs] = ode23(dMMdt, [ti ti+bT], xi);
21     else % octave version
22         [ts, xs] = odeOct(dMMdt, [ti ti+bT], xi);
23     end
24 end
25
26 function [ts,xs] = odeOct(dxdt,tSpan,x0)
27     if length(tSpan)>2, ts = tSpan;
28     else ts = linspace(tSpan(1),tSpan(end),21);
29     end
30     % mimic ode45 and ode23, but much slower for non-PI
31     lsode_options('integration method','non-stiff');
32     xs = lsode(@(x,t) dxdt(t,x),x0,ts);
33 end
34
35

```

## 2.4 PIG(): Projective Integration via a General macroscale integrator

### Section contents

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### 2.4.1 Introduction

This is a Projective Integration scheme when the macroscale integrator is any specified coded method. The advantage is that one may use MATLAB/Octave's inbuilt integration functions, with all their sophisticated error control and adaptive time-stepping, to do the macroscale integration/simulation.

By default, for the microscale simulations `PIG()` uses 'constraint-defined manifold computing', `cdmc()` ([Section 2.6](#)). This algorithm, initiated by [Gear et al. \(2005b\)](#), uses a backward projection so that the simulation time is unchanged after running the microscale simulator.

```
30 function [T,X,tms,xms,svf] = PIG(macroInt,microBurst,Tspan,x0 ...
31 ,restrict,lift,cdmcFlag)
```

#### Inputs:

- `macroInt()`, the numerical method that the user wants to apply on a slow-time macroscale. Either specify a standard MATLAB/Octave integration function (such as '`ode23`' or '`ode45`'), or code your own integration function using standard arguments. That is, if you code your own, then it must be

$$[Ts, Xs] = \text{macroInt}(F, Tspan, X0)$$

where

- function  $F(T, X)$  notionally evaluates the time derivatives  $d\vec{X}/dt$  at any time;
- $Tspan$  is either the macro-time interval, or the vector of macroscale times at which macroscale values are to be returned; and
- $X0$  are the initial values of  $\vec{X}$  at time  $Tspan(1)$ .

Then the  $i$ th *row* of  $Xs$ ,  $Xs(i,:)$ , is to be the vector  $\vec{X}(t)$  at time  $t = Ts(i)$ . Remember that in `PIG()` the function  $F(T, X)$  is to be estimated by Projective Integration.

- `microBurst()` is a function that produces output from the user-specified code for a burst of microscale simulation. The function must internally specify/decide how long a burst it is to use. Usage

```
[tbs,xbs] = microBurst(tb0,xb0)
```

*Inputs:* `tb0` is the start time of a burst; `xb0` is the  $n$ -vector microscale state at the start of a burst.

*Outputs:* `tbs`, the vector of solution times; and `xbs`, the corresponding microscale states.

- `Tspan`, a vector of macroscale times at which the user requests output. The first element is always the initial time. If `macroInt` reports adaptively selected time steps (e.g., `ode45`), then `Tspan` consists of an initial and final time only.
- `x0`, the  $n$ -vector of initial microscale values at the initial time `Tspan(1)`.

**Optional Inputs:** `PIG()` allows for none, two or three additional inputs after `x0`. If you distinguish distinct microscale and macroscale states and your aim is to do Projective Integration on the macroscale only, then lifting and restriction functions must be provided to convert between them. Usage `PIG(...,restrict,lift)`:

- `restrict(x)`, a function that takes an input high-dimensional,  $n$ -D, microscale state  $\vec{x}$  and computes the corresponding low-dimensional,  $N$ -D, macroscale state  $\vec{X}$ ;
- `lift(X,xApprox)`, a function that converts an input low-dimensional,  $N$ -D, macroscale state  $\vec{X}$  to a corresponding high-dimensional,  $n$ -D, microscale state  $\vec{x}$ , given that `xApprox` is a recently computed microscale state on the slow manifold.

Either both `restrict()` and `lift()` are to be defined, or neither. If neither are defined, then they are assumed to be identity functions, so that `N=n` in the following.

If desired, the default constraint-defined manifold computing microsolver may be disabled, via `PIG(...,restrict,lift,cdmcFlag)`

- `cdmcFlag`, *any* seventh input to `PIG()`, will disable `cdmc()`, e.g., the string '`cdmc off`'.

If the `cdmcFlag` is to be set without using a `restrict()` or `lift()` function, then use empty matrices [] for the restrict and lift functions.

**Output** Between zero and five outputs may be requested. If there are no output arguments specified, then a plot is drawn of the computed solution `X` versus `T`. Most often you would store the first two output results of `PIG()`, via say `[T,X] = PIG(...)`.

- `T`, an  $L$ -vector of times at which `macroInt` produced results.

- $\mathbf{X}$ , an  $L \times N$  array of the computed solution: the  $i$ th row of  $\mathbf{X}$ ,  $\mathbf{X}(i, :)$ , is to be the macro-state vector  $\vec{X}(t)$  at time  $t = T(i)$ .

However, microscale details of the underlying Projective Integration computations may be helpful, and so `PIG()` provides some optional outputs of the microscale bursts, via `[T,X,tms,xms] = PIG(...)`

- $\mathbf{tms}$ , optional, is an  $\ell$ -dimensional column vector containing microscale times with bursts, each burst separated by `NaN`;
- $\mathbf{xms}$ , optional, is an  $\ell \times n$  array of the corresponding microscale states.

In some contexts it may be helpful to see directly how Projective Integration approximates a reduced slow vector field, via `[T,X,tms,xms,svf] = PIG(...)` in which

- $\mathbf{svf}$ , optional, a struct containing the Projective Integration estimates of the slow vector field.
  - $\mathbf{svf}.T$  is a  $\hat{L}$ -dimensional column vector containing all times at which the microscale simulation data is extrapolated to form an estimate of  $d\vec{x}/dt$  in `macroInt()`.
  - $\mathbf{svf}.dX$  is a  $\hat{L} \times N$  array containing the estimated slow vector field.

If `macroInt()` is, for example, the forward Euler method (or the Runge–Kutta method), then  $\hat{L} = L$  (or  $\hat{L} = 4L$ ).

#### 2.4.2 If no arguments, then execute an example

```
180 if nargin==0
```

As a basic example, consider a microscale system of the singularly perturbed system of differential equations

$$\frac{dx_1}{dt} = \cos(x_1) \sin(x_2) \cos(t) \quad \text{and} \quad \frac{dx_2}{dt} = \frac{1}{\epsilon} [\cos(x_1) - x_2]. \quad (2.1)$$

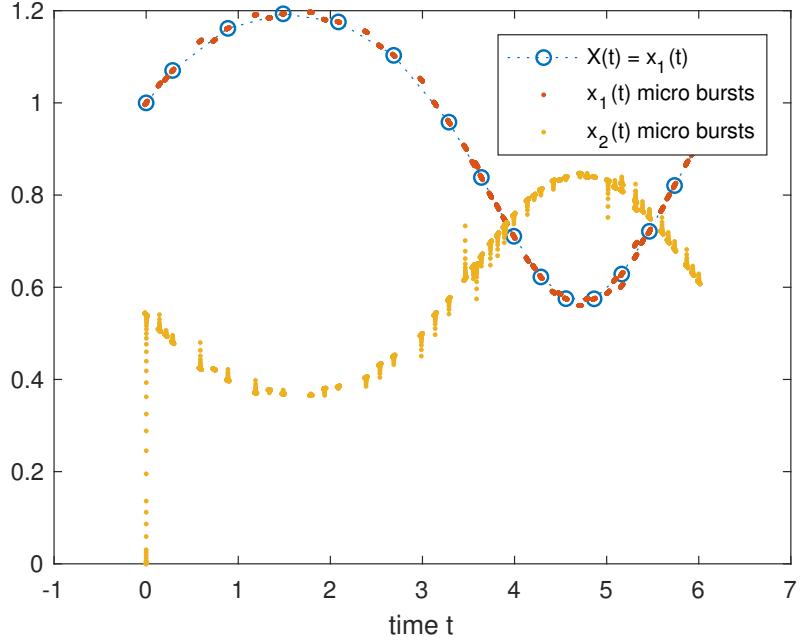
The macroscale variable is  $X(t) = x_1(t)$ , and the evolution  $dX/dt$  is unclear. With initial condition  $X(0) = 1$ , the following code computes and plots a solution of the system (2.1) over time  $0 \leq t \leq 6$  for parameter  $\epsilon = 10^{-3}$  (Figure 2.7). Whenever needed by `microBurst()`, the microscale system (2.1) is initialised ('lifted') using  $x_2(t) = x_2^{\text{approx}}$  (yellow dots in Figure 2.7).

First we code the right-hand side function of the microscale system (2.1) of ODEs.

```
214 epsilon = 1e-3;
215 dxdt=@(t,x) [ cos(x(1))*sin(x(2))*cos(t)
216           ( cos(x(1))-x(2) )/epsilon ];
```

Second, we code microscale bursts, here using the standard `ode45()`. We choose a burst length  $2\epsilon \log(1/\epsilon)$  as the rate of decay is  $\beta \approx 1/\epsilon$  but we do not know the macroscale time-step invoked by `macroInt()`, so blithely assume  $\Delta \leq 1$  and then double the usual formula for safety.

Figure 2.7: Projective Integration by PIG of the example system (2.1) with  $\epsilon = 10^{-3}$  (Section 2.4.2). The macroscale solution  $X(t)$  is represented by just the blue circles. The microscale bursts are the microscale states  $(x_1(t), x_2(t)) = (\text{red}, \text{yellow})$  dots.



```

227 bT = 2*epsilon*log(1/epsilon)
228 if ~exist('OCTAVE_VERSION','builtin')
229     micB='ode45'; else micB='rk2Int'; end
230 microBurst = @(tb0, xb0) feval(micB,dxdt,[tb0 tb0+bT],xb0);

```

Third, code functions to convert between macroscale and microscale states.

```

237 restrict = @(x) x(1);
238 lift = @(X,xApprox) [X; xApprox(2)];

```

Fourth, invoke PIG to use MATLAB/Octave's `ode23/lsode`, say, on the macroscale slow evolution. Integrate the micro-bursts over  $0 \leq t \leq 6$  from initial condition  $\vec{x}(0) = (1, 0)$ . You could set `Tspan=[0 -6]` to integrate backward in macroscale time with forward microscale bursts (Gear & Kevrekidis 2003a, Frewen et al. 2009).

```

250 Tspan = [0 6];
251 x0 = [1;0];
252 if ~exist('OCTAVE_VERSION','builtin')
253     macInt='ode23'; else macInt='odeOct'; end
254 [Ts,Xs,tms,xms] = PIG(macInt,microBurst,Tspan,x0,restrict,lift);

```

Plot output of this projective integration.

```

260 figure, plot(Ts,Xs,'o:',tms,xms,'.')
261 title('Projective integration of singularly perturbed ODE')
262 xlabel('time t')
263 legend('X(t) = x_1(t)', 'x_1(t) micro bursts', 'x_2(t) micro bursts')

```

Upon finishing execution of the example, exit this function.

```
269 return
270 end%if no arguments
```

### 2.4.3 The projective integration code

If no lifting/restriction functions are provided, then assign them to be the identity functions.

```
287 if nargin < 5 || isempty(restrict)
288     lift=@(X,xApprox) X;
289     restrict=@(x) x;
290 end
```

Get the number of expected outputs and set logical indices to flag what data should be saved.

```
298 nArgOut = nargin();
299 saveMicro = (nArgOut>2);
300 saveSvf = (nArgOut>4);
```

Find the number of time-steps at which output is expected, and the number of variables.

```
308 nT = length(Tspan)-1;
309 nx = length(x0);
310 nX = length(restrict(x0));
```

Reformulate the microsolver to use `cdmc()`, unless flagged otherwise. The result is that the solution from microBurst will terminate at the given initial time.

```
320 if nargin<7
321     microBurst = @(t,x) cdmc(microBurst,t,x);
322 else
323     warning(['A ' class(cdmFlag) ' seventh input to PIG()'...
324         ' PIG will not use constraint-defined manifold computing.'])
325 end
```

Execute a preliminary application of the microBurst on the initial state. This is done in addition to the microBurst in the main loop, because the initial state is often far from the attracting slow manifold.

```
337 [relaxT,x0MicroRelax] = microBurst(Tspan(1),x0);
338 xMicroLast = x0MicroRelax(end,:);
339 X0Relax = restrict(xMicroLast);
```

Update the initial time.

```
346 Tspan(1) = relaxT(end);
```

Allocate cell arrays for times and states for any of the outputs requested by the user. If saving information, then record the first application of the microBurst. It is unknown a priori how many applications of microBurst will

be required; this code may be run more efficiently if the correct number is used in place of  $nT+1$  as the dimension of the cell arrays.

```

358 if saveMicro
359     tms=cell(nT+1,1); xms=cell(nT+1,1);
360     n=1;
361     tms{n} = reshape(relaxT,[],1);
362     xms{n} = x0MicroRelax;
363
364     if saveSvf
365         svf.T = cell(nT+1,1);
366         svf.dX = cell(nT+1,1);
367     else
368         svf = [];
369     end
370 else
371     tms = [] ; xms = [] ; svf = [] ;
372 end

```

**Define a function of macro simulation** The idea of `PIG()` is to use the output from the `microBurst()` to approximate an unknown function  $F(t, X)$  that computes  $d\vec{X}/dt$ . This approximation is then used in the system/user-defined ‘coarse solver’ `macroInt()`. The approximation is computed in the function

```
385 function [dXdt]=PIFun(t,X)
```

Run a microBurst from the given macroscale initial values.

```

391 x = lift(X,xMicroLast);
392 [tTmp,xMicroTmp] = microBurst(t,reshape(x,[],1));
393 xMicroLast = xMicroTmp(end,:).';

```

Compute the standard Projective Integration approximation of the slow vector field.

```

400 X2 = restrict(xMicroTmp(end,:));
401 X1 = restrict(xMicroTmp(end-1,:));
402 dt = tTmp(end)-tTmp(end-1);
403 dXdt = (X2 - X1).'/dt;

```

Save the microscale data, and the Projective Integration slow vector field, if requested.

```

410 if saveMicro
411     n=n+1;
412     tms{n} = [reshape(tTmp,[],1); nan];
413     xms{n} = [xMicroTmp; nan(1,nx)];
414     if saveSvf
415         svf.T{n-1} = t;
416         svf.dX{n-1} = dXdt;
417     end

```

```

418     end
419 end% PIFun function

```

**Invoke the macroscale integration** Integrate PIF() with the user-specified simulator macroInt(). For some reason, in MATLAB/Octave we need to use a one-line function, PIF, that invokes the above macroscale function, PIFun. We also need to use feval because macroInt() has multiple outputs.

```

432 PIF = @t,x) PIFun(t,x);
433 [T,X] = feval(macroInt,PIF,Tspan,X0Relax.');

```

Overwrite X(1,:) and T(1), which a user expects to be X0 and Tspan(1) respectively, with the given initial conditions.

```

442 X(1,:) = restrict(x0);
443 T(1) = Tspan(1);

```

Concatenate all the additional requested outputs into arrays.

```

450 if saveMicro
451     tms = cell2mat(tms);
452     xms = cell2mat(xms);
453     if saveSvf
454         svf.T = cell2mat(svf.T);
455         svf.dX = cell2mat(svf.dX);
456     end
457 end

```

#### 2.4.4 If no output specified, then plot the simulation

```

465 if nArgOut==0
466     figure, plot(T,X,'o:')
467     title('Projective Simulation via PIG')
468 end

```

This concludes PIG().

```

476 end

```

### 2.5 PIRK4(): projective integration of fourth-order accuracy

*Section contents*

2.5.1	Introduction . . . . .	25
2.5.2	The projective integration code . . . . .	27
2.5.3	If no output specified, then plot the simulation . . . . .	31

### 2.5.1 Introduction

This Projective Integration scheme implements a macrosolver analogous to the fourth-order Runge–Kutta method.

```
19 function [x, tms, xms, rm, svf] = PIRK4(microBurst, tSpan, x0, bT)
```

See [Section 2.2](#) as the inputs and outputs are the same as `PIRK2()`.

**If no arguments, then execute an example**

```
29 if nargin==0
```

**Example of Michaelis–Menton backwards in time** The Michaelis–Menton enzyme kinetics is expressed as a singularly perturbed system of differential equations for  $x(t)$  and  $y(t)$  (encoded in function `MMburst`):

$$\frac{dx}{dt} = -x + (x + \frac{1}{2})y \quad \text{and} \quad \frac{dy}{dt} = \frac{1}{\epsilon} [x - (x + 1)y].$$

With initial conditions  $x(0) = y(0) = 0.2$ , the following code uses forward time bursts in order to integrate backwards in time to  $t = -5$  ([Frewen et al. 2009](#), e.g.). It plots the computed solution over time  $-5 \leq t \leq 0$  for parameter  $\epsilon = 0.1$ . Since the rate of decay is  $\beta \approx 1/\epsilon$  we choose a burst length  $\epsilon \log(|\Delta|/\epsilon)$  as here the macroscale time-step  $\Delta = -1$ .

```
50 global MMepsilon
51 MMepsilon = 0.1
52 ts = 0:-1:-5
53 bT = MMepsilon*log(abs(ts(2)-ts(1))/MMepsilon)
54 [x,tms,xms,rm,svf] = PIRK4(@MMburst, ts, 0.2*[1;1], bT);
55 figure, plot(ts,x,'o:',tms,xms)
56 xlabel('time t'), legend('x(t)', 'y(t)')
57 title('Backwards-time projective integration of Michaelis--Menton')

Upon finishing execution of the example, exit this function.

63 return
64 end%if no arguments
```

**Code a burst of Michaelis–Menton enzyme kinetics** Integrate a burst of length `bT` of the ODEs for the Michaelis–Menton enzyme kinetics at parameter  $\epsilon$  inherited from above. Code ODEs in function `dMMdt` with variables  $x = \mathbf{x}(1)$  and  $y = \mathbf{x}(2)$ . Starting at time `ti`, and state `xi` (row), we here simply use MATLAB/Octave’s `ode23/lsode` to integrate a burst in time.

```
15 function [ts, xs] = MMburst(ti, xi, bT)
16     global MMepsilon
17     dMMdt = @(t,x) [ -x(1)+(x(1)+0.5)*x(2)
18                     1/MMepsilon*( x(1)-(x(1)+1)*x(2) ) ];
19     if ~exist('OCTAVE_VERSION','builtin')
20         [ts, xs] = ode23(dMMdt, [ti ti+bT], xi);
```

```

21      else % octave version
22      [ts, xs] = odeOct(dMMdt, [ti ti+bT], xi);
23      end
24  end

8  function [ts,xs] = odeOct(dxdt,tSpan,x0)
9      if length(tSpan)>2, ts = tSpan;
10     else ts = linspace(tSpan(1),tSpan(end),21);
11     end
12     % mimic ode45 and ode23, but much slower for non-PI
13     lsode_options('integration method','non-stiff');
14     xs = lsode(@(x,t) dxdt(t,x),x0,ts);
15 end

```

**Input** If there are no input arguments, then this function applies itself to the Michaelis–Menton example: see the code in [Section 2.2.2](#) as a basic template of how to use.

- `microBurst()`, a user-coded function that computes a short-time burst of the microscale simulation.

```
[t0out, x0out] = microBurst(tStart, xStart, bT)
```

- Inputs: `tStart`, the start time of a burst of simulation; `xStart`, the row  $n$ -vector of the starting state; `bT`, *optional*, the total time to simulate in the burst—if your `microBurst()` determines the burst time, then replace `bT` in the argument list by `varargin`.
- Outputs: `t0out`, the column vector of solution times; and `x0out`, an array in which each *row* contains the system state at corresponding times.

- `tSpan` is an  $\ell$ -vector of times at which the user requests output, of which the first element is always the initial time. `PIRK4()` does not use adaptive time-stepping; the macroscale time-steps are (nearly) the steps between elements of `tSpan`.
- `x0` is an  $n$ -vector of initial values at the initial time `tSpan(1)`. Elements of `x0` may be `Nan`: such `Nans` are carried in the simulation through to the output, and often represent boundaries/edges in spatial fields.
- `bT`, *optional*, either missing, or empty (`[]`), or a scalar: if a given scalar, then it is the length of the micro-burst simulations—the minimum amount of time needed for the microscale simulation to relax to the slow manifold; else if missing or `[]`, then `microBurst()` must itself determine the length of a burst.

```
124 if nargin<4, bT=[]; end
```

**Output** If there are no output arguments specified, then a plot is drawn of the computed solution `x` versus `tSpan`.

- $\mathbf{x}$ , an  $\ell \times n$  array of the approximate solution vector. Each row is an estimated state at the corresponding time in  $\mathbf{tSpan}$ . The simplest usage is then  $\mathbf{x} = \text{PIRK4}(\text{microBurst}, \mathbf{tSpan}, \mathbf{x0}, \mathbf{bT})$ .

However, microscale details of the underlying Projective Integration computations may be helpful. `PIRK4()` provides up to four optional outputs of the microscale bursts.

- $\mathbf{tms}$ , optional, is an  $L$  dimensional column vector containing the microscale times within the burst simulations, each burst separated by NaN;
- $\mathbf{xms}$ , optional, is an  $L \times n$  array of the corresponding microscale states—each rows is an accurate estimate of the state at the corresponding time  $\mathbf{tms}$  and helps visualise details of the solution.
- $\mathbf{rm}$ , optional, a struct containing the ‘remaining’ applications of the `microBurst` required by the Projective Integration method during the calculation of the macrostep:
  - $\mathbf{rm.t}$  is a column vector of microscale times; and
  - $\mathbf{rm.x}$  is the array of corresponding burst states.

The states  $\mathbf{rm.x}$  do not have the same physical interpretation as those in  $\mathbf{xms}$ ; the  $\mathbf{rm.x}$  are required in order to estimate the slow vector field during the calculation of the Runge–Kutta increments, and do *not* accurately approximate the macroscale dynamics.

- $\mathbf{svf}$ , optional, a struct containing the Projective Integration estimates of the slow vector field.
  - $\mathbf{svf.t}$  is a  $4\ell$  dimensional column vector containing all times at which the Projective Integration scheme is extrapolated along `microBurst` data to form a macrostep.
  - $\mathbf{svf.dx}$  is a  $4\ell \times n$  array containing the estimated slow vector field.

### 2.5.2 The projective integration code

Determine the number of time-steps and preallocate storage for macroscale estimates.

```
194 nT = length(tSpan);
195 x = nan(nT,length(x0));
```

Get the number of expected outputs and set logical indices to flag what data should be saved.

```
203 nArgOut = nargin();
204 saveMicro = (nArgOut>1);
205 saveFullMicro = (nArgOut>3);
206 saveSvf = (nArgOut>4);
```

Run a preliminary application of the micro-burst on the initial state to help relax to the slow manifold. This is done in addition to the micro-burst in

the main loop, because the initial state is often far from the attracting slow manifold. Require the user to input and output rows of the system state.

```
219 x0 = reshape(x0,1,[]);
220 [relax_t,relax_x0] = microBurst(tSpan(1),x0,bT);
```

Use the end point of the micro-burst as the initial state for the macroscale time-steps.

```
228 tSpan(1) = relax_t(end);
229 x(1,:) = relax_x0(end,:);
```

If saving information, then record the first application of the micro-burst. Allocate cell arrays for times and states for outputs requested by the user, as concatenating cells is much faster than iteratively extending arrays.

```
239 if saveMicro
240     tms = cell(nT,1);
241     xms = cell(nT,1);
242     tms{1} = reshape(relax_t,[],1);
243     xms{1} = relax_x0;
244     if saveFullMicro
245         rm.t = cell(nT,1);
246         rm.x = cell(nT,1);
247         if saveSvf
248             svf.t = nan(4*nT-4,1);
249             svf.dx = nan(4*nT-4,length(x0));
250         end
251     end
252 end
```

### Loop over the macroscale time-steps

```
260 for jT = 2:nT
261     T = tSpan(jT-1);
```

If four applications of the micro-burst would cover the entire macroscale time-step, then do so (setting some internal states to NaN); else proceed to projective step.

```
270     if ~isempty(bT) && 4*abs(bT)>=abs(tSpan(jT)-T) && bT*(tSpan(jT)-T)>0
271         [t1,xm1] = microBurst(T, x(jT-1,:), tSpan(jT)-T);
272         x(jT,:) = xm1(end,:);
273         t2=nan; xm2=nan(1,size(xm1,2));
274         t3=nan; t4=nan; xm3=xm2; xm4 = xm2; dx1=xm2; dx2=xm2;
275     else
```

Run the first application of the micro-burst; since this application directly follows from the initial conditions, or from the latest macrostep, this microscale information is physically meaningful as a simulation of the system. Extract the size of the final time-step.

```

286      [t1,xm1] = microBurst(T, x(jT-1,:), bT);
287      del = t1(end)-t1(end-1);
```

Check for round-off error.

```

293      xt = [reshape(t1(end-1:end),[],1) xm1(end-1:end,:)];
294      roundingTol = 1e-8;
295      if norm(diff(xt))/norm(xt,'fro') < roundingTol
296      warning(['significant round-off error in 1st projection at T=' num2str(T)
297      end
```

Find the needed time-step to reach time  $tSpan(n+1)$  and form a first estimate  $dx1$  of the slow vector field.

```

306      Dt = tSpan(jT)-t1(end);
307      dx1 = (xm1(end,:)-xm1(end-1,:))/del;
```

Assume burst times are the same length for this macro-step, or effectively so (recall that  $bT$  may be empty as it may be only coded and known in `microBurst()`).

```
316      abT = t1(end)-t1(1);
```

Project along  $dx1$  to form an intermediate approximation of  $x$ ; run another application of the micro-burst and form a second estimate of the slow vector field.

```

327      xint = xm1(end,:)+ (Dt/2-abT)*dx1;
328      [t2,xm2] = microBurst(T+Dt/2, xint, bT);
329      del = t2(end)-t2(end-1);
330      dx2 = (xm2(end,:)-xm2(end-1,:))/del;
331
332      xint = xm1(end,:)+ (Dt/2-abT)*dx2;
333      [t3,xm3] = microBurst(T+Dt/2, xint, bT);
334      del = t3(end)-t3(end-1);
335      dx3 = (xm3(end,:)-xm3(end-1,:))/del;
336
337      xint = xm1(end,:)+ (Dt-abT)*dx3;
338      [t4,xm4] = microBurst(T+Dt, xint, bT);
339      del = t4(end)-t4(end-1);
340      dx4 = (xm4(end,:)-xm4(end-1,:))/del;
```

Check for round-off error.

```

346      xt = [reshape(t2(end-1:end),[],1) xm2(end-1:end,:)];
347      if norm(diff(xt))/norm(xt,'fro') < roundingTol
348      warning(['significant round-off error in 2nd projection at T=' num2str(T)
349      end
```

Use the weighted average of the estimates of the slow vector field to take a macrostep.

```
357      x(jT,:) = xm1(end,:)+ Dt*(dx1 + 2*dx2 + 2*dx3 + dx4)/6;
```

Now end the if-statement that tests whether a projective step saves simulation time.

```
365      end
```

If saving trusted microscale data, then populate the cell arrays for the current loop iterate with the time-steps and output of the first application of the micro-burst. Separate bursts by NaNs.

```
375      if saveMicro
376          tms{jT} = [reshape(t1,[],1); nan];
377          xms{jT} = [xm1; nan(1,size(xm1,2))];
```

If saving all microscale data, then repeat for the remaining applications of the micro-burst.

```
385      if saveFullMicro
386          rm.t{jT} = [reshape(t2,[],1); nan;...
387                      reshape(t3,[],1); nan;...
388                      reshape(t4,[],1); nan];
389          rm.x{jT} = [xm2; nan(1,size(xm2,2));...
390                      xm3; nan(1,size(xm2,2));...
391                      xm4; nan(1,size(xm2,2))];
```

If saving Projective Integration estimates of the slow vector field, then populate the corresponding cells with times and estimates.

```
400      if saveSvf
401          svf.t(4*jT-7:4*jT-4) = [t1(end); t2(end); t3(end); t4(end)];
402          svf.dx(4*jT-7:4*jT-4,:) = [dx1; dx2; dx3; dx4];
403      end
404  end
405 end
```

End of the main loop of all macro-steps.

```
411 end
```

Overwrite  $x(1,:)$  with the specified initial state  $tSpan(1)$ .

```
420 x(1,:) = reshape(x0,1,[]);
```

For additional requested output, concatenate all the cells of time and state data into two arrays.

```
428 if saveMicro
429     tms = cell2mat(tms);
430     xms = cell2mat(xms);
431     if saveFullMicro
432         rm.t = cell2mat(rm.t);
433         rm.x = cell2mat(rm.x);
434     end
435 end
```

### 2.5.3 If no output specified, then plot the simulation

```

443 if nArgOut==0
444     figure, plot(tSpan,x,'o:')
445     title('Projective Simulation with PIRK4')
446 end
453 end

```

This concludes PIRK4().

## 2.6 cdmc(): constraint defined manifold computing

The function `cdmc()` iteratively applies the given micro-burst and then projects backward to the initial time. The cumulative effect is to relax the variables to the attracting slow manifold, while keeping the ‘final’ time for the output the same as the input time.

```

17 function [ts, xs] = cdmc(microBurst, t0, x0)

```

### Input

- `microBurst()`, a black-box micro-burst function suitable for Projective Integration. See any of `PIRK2()`, `PIRK4()`, or `PIG()` for a description of `microBurst()`.
- `t0`, an initial time.
- `x0`, an initial state vector.

### Output

- `ts`, a vector of times.
- `xs`, an array of state estimates produced by `microBurst()`.

This function is a wrapper for the micro-burst. For instance if the problem of interest is a dynamical system that is not too stiff, and which is simulated by the micro-burst function `sol(t,x)`, one would invoke `cdmc()` by defining

```
cdmcSol = @(t,x) cdmc(sol,t,x)|
```

and thereafter use `cdmcSol()` in place of `sol()` as the microBurst in any Projective Integration scheme. The original microBurst `sol()` could create large errors if used in the `PIG()` scheme, but the output via `cdmc()` should not.

Begin with a standard application of the micro-burst. Need `feval` as `microBurst` has multiple outputs.

```

56 [t1,x1] = feval(microBurst,t0,x0);
57 bT = t1(end)-t1(1);

```

Project backwards to before the initial time, then simulate just one burst forward to obtain a simulation burst that ends at the original  $t_0$ .

```
66 dxdt = (x1(end,:) - x1(end-1,:))/(t1(end) - t1(end-1));
67 x0 = x1(end,:)-2*bT*dxdt;
68 t0 = t1(1)-bT;
69 [t2,x2] = feval(microBurst,t0,x0.');
```

Return both sets of output(?), although only  $(t_2, x_2)$  should be used in Projective Integration—maybe safer to return only  $(t_2, x_2)$ .

```
77 ts = [t1(:); t2(:)];
78 xs = [x1; x2];
```

## 2.7 Example: PI using Runge–Kutta macrosolvers

This script demonstrates the PIRK4() scheme that uses a Runge–Kutta macrosolver, applied to simple linear systems with some slow and fast directions.

Clear workspace and set a seed.

```
15 clear
16 rand('seed',1) % albeit discouraged in Matlab
17 global dxdt
```

The majority of this example involves setting up details for the microsolver. We use a simple function gen\_linear\_system() that outputs a function  $f(t, x) = A\vec{x} + \vec{b}$ , where matrix  $A$  has some eigenvalues with large negative real part, corresponding to fast variables, and some eigenvalues with real part close to zero, corresponding to slow variables. The function gen\_linear\_system() requires that we specify bounds on the real part of the strongly stable eigenvalues,

```
32 fastband = [-5e2; -1e2];
```

and bounds on the real part of the weakly stable/unstable eigenvalues,

```
39 slowband = [-0.002; 0.002];
```

We now generate a random linear system with seven fast and three slow variables.

```
46 dxdt = gen_linear_system(7,3,fastband,slowband);
```

Set the macroscale times at which we request output from the PI scheme and the initial state.

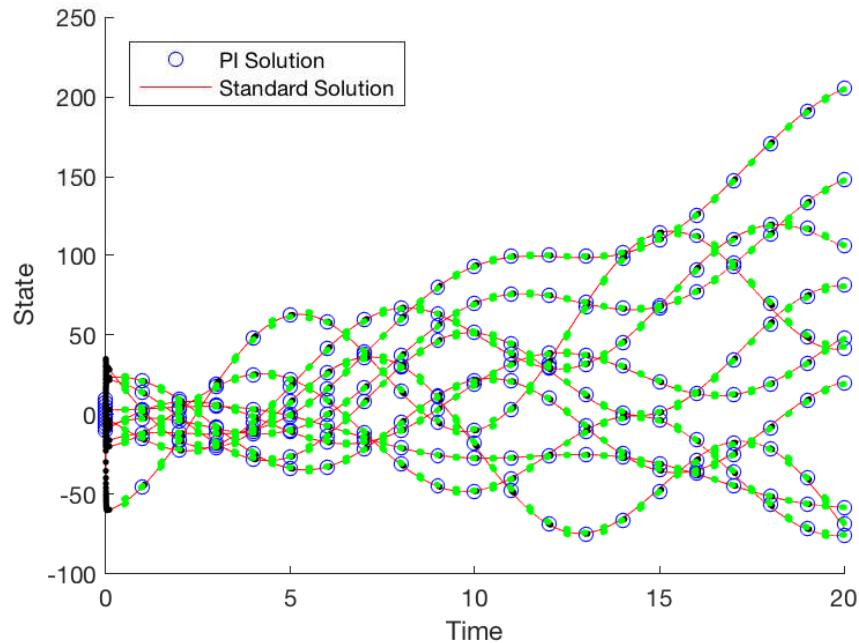
```
56 tSpan = 0:1:20;
57 x0 = linspace(-10,10,10)';
```

We implement the PI scheme, saving the coarse states in  $x$ , the ‘trusted’ applications of the microsolver in  $tms$  and  $xms$ , and the additional applications of the microsolver in  $rm$  (the second, third and fourth outputs are optional).

```
70 [x, tms, xms, rm] = PIRK4(@linearBurst, tSpan, x0);
```

To verify, we also compute the trajectories using a standard integrator.

*Figure 2.8: Demonstration of PIRK4(). From initial conditions, the system rapidly transitions to an attracting invariant manifold. The PI solution accurately tracks the evolution of the variables over time while requiring only a fraction of the computations of the standard solver.*



```

77 if ~exist('OCTAVE_VERSION','builtin')
78     [tt,xode] = ode45(dxdt,tSpan([1,end]),x0);
79 else % octave version
80     tt = linspace(tSpan(1),tSpan(end),101);
81     xode = lsode(@(x,t) dxdt(t,x),x0,tt);
82 end

```

Figure 2.8 plots the output.

```

98 clf()
99 hold on
100 PI_sol=plot(tSpan,x,'bo');
101 std_sol=plot(tt,xode,'r');
102 plot(tms,xms,'k.', rm.t,rm.x,'g.');
103 legend([PI_sol(1),std_sol(1)],'PI Solution',...
104 'Standard Solution','Location','NorthWest')
105 xlabel('Time'), ylabel('State')

Save plot to a file.

111 if ~exist('OCTAVE_VERSION','builtin')
112 set(gcf,'PaperUnits','centimeters','PaperPosition',[0 0 14 10])
113 print('-depsc2','PIRKexample')
114 end

```

**A micro-burst simulation** Used by `PIRKexample.m`. Code the micro-burst function using simple Euler steps. As a rule of thumb, the time-steps  $dt$  should satisfy  $dt \leq 1/|\text{fastband}(1)|$  and the time to simulate with each application of the microsolver,  $bT$ , should be larger than or equal to  $1/|\text{fastband}(2)|$ . We set the integration scheme to be used in the microsolver. Since the time-steps are so small, we just use the forward Euler scheme

```

17 function [ts, xs] = linearBurst(ti, xi, varargin)
18 global dxdt
19 dt = 0.001;
20 ts = ti+(0:dt:0.05)';
21 nts = length(ts);
22 xs = NaN(nts,length(xi));
23 xs(1,:)=xi;
24 for k=2:nts
25     xi = xi + dt*dxdt(ts(k),xi.')';
26     xs(k,:)=xi;
27 end
28 end

```

## 2.8 Example: Projective Integration using General macrosolvers

In this example the Projective Integration-General scheme is applied to a singularly perturbed ordinary differential equation. The aim is to use a standard non-stiff numerical integrator, such as `ode45()`, on the slow, long-time macroscale. For this stiff system, `PIG()` is an order of magnitude faster than ordinary use of `ode45`.

```
18 clear all, close all
```

Set time scale separation and the underlying ODES:

$$\frac{dx_1}{dt} = \cos x_1 \sin x_2 \cos t, \quad \frac{dx_2}{dt} = \frac{1}{\epsilon}(-x_2 + \cos x_1).$$

```

30 epsilon = 1e-4;
31 dxdt=@(t,x) [ cos(x(1))*sin(x(2))*cos(t)
32             (cos(x(1))-x(2))/epsilon ];

```

Set the ‘black-box’ microsolver to be an integration using a standard solver, and set the standard time of simulation for the microsolver.

```

41 bT = epsilon*log(1/epsilon);
42 if ~exist('OCTAVE_VERSION','builtin')
43     micB='ode45'; else micB='rk2Int'; end
44 microBurst = @(tb0, xb0) feval(micB,dxdt,[tb0 tb0+bT],xb0);

```

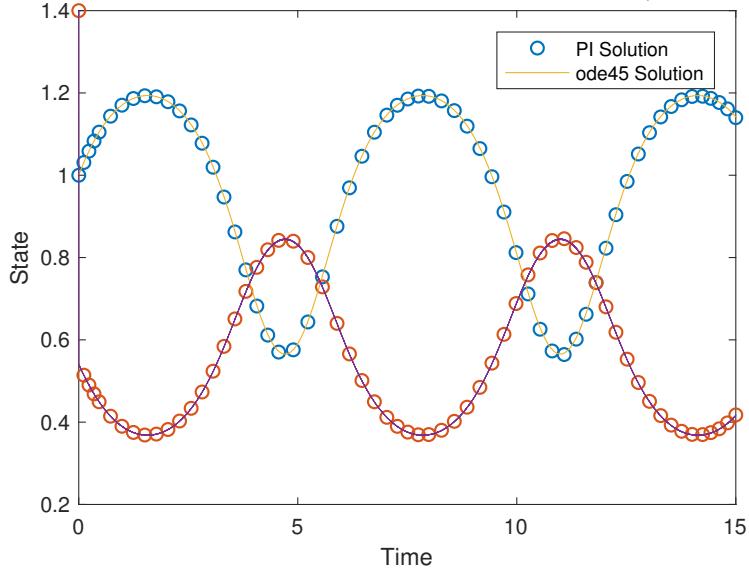
Set initial conditions, and the time to be covered by the macrosolver.

```

52 x0 = [1 0.9];
53 tSpan = [0 5];

```

*Figure 2.9: Accurate simulation of a stiff nonautonomous system by PIG(). The microsolver is called on-the-fly by the macrosolver ode45/lode.*



Now time and integrate the above system over `tSpan` using `PIG()` and, for comparison, a brute force implementation of `ode45/lode`. Report the time taken by each method (in seconds).

```

62 if ~exist('OCTAVE_VERSION','builtin')
63     macInt='ode45'; else macInt='odeOct'; end
64 tic
65 [ts,xs,tms,xms] = PIG(macInt,microBurst,tSpan,x0);
66 secsPIGusingODEasMacro = toc
67 tic
68 [tClassic,xClassic] = feval(macInt,dxdt,tSpan,x0);
69 secsODEalone = toc

```

Plot the output on two figures, showing the truth and macrosteps on both, and all applications of the microsolver on the first figure.

```

79 figure
80 h = plot(ts,xs,'o', tClassic,xClassic,'-', tms,xms,'.');
81 legend(h(1:2:5),'Pro Int method','classic method','PI microsolver')
82 xlabel('Time'), ylabel('State')
83
84 figure
85 h = plot(ts,xs,'o', tClassic,xClassic,'-');
86 legend(h([1 3]),'Pro Int method','classic method')
87 xlabel('Time'), ylabel('State')
88 if ~exist('OCTAVE_VERSION','builtin')
89 set(gcf,'PaperUnits','centimeters','PaperPosition',[0 0 14 10])
90 %print('-depsc2','PIGExample')
91 end

```

`Figure 2.9` plots the output.

- The problem may be made more stiff or less stiff by changing the time-scale separation parameter  $\epsilon = \text{epsilon}$ . The compute time of `PIG()` is almost independent of  $\epsilon$ , whereas that of `ode45()` is proportional to  $1/\epsilon$ .

If the problem is ‘semi-stiff’ (larger  $\epsilon$ ), then `PIG()`’s default of using `cdmc()` avoids nonsense ([Section 2.9](#)).

- The stiff but low dimensional problem in this example may be solved efficiently by a standard stiff solver (e.g., `ode15s()`). The real advantage of the Projective Integration schemes is in high dimensional stiff problems, that are not efficiently solved by most standard methods.

## 2.9 Explore: Projective Integration using constraint-defined manifold computing

In this example the Projective Integration-General scheme is applied to a singularly perturbed ordinary differential equation in which the time scale separation is not large. The results demonstrate the value of the default `cdmc()` wrapper for the microsolver.

```
16 clear all, close all
```

Set a weak time scale separation, and the underlying ODES:

$$\frac{dx_1}{dt} = \cos x_1 \sin x_2 \cos t, \quad \frac{dx_2}{dt} = \frac{1}{\epsilon}(-x_2 + \cos x_1).$$

```
28 epsilon = 0.01;
29 dxdt=@(t,x) [ cos(x(1))*sin(x(2))*cos(t)
30             (cos(x(1))-x(2))/epsilon ];
```

Set the microsolver to be an integration using a standard solver, and set the standard time of simulation for the microsolver.

```
39 bT = epsilon*log(1/epsilon);
40 if ~exist('OCTAVE_VERSION','builtin')
41     micB='ode45'; else micB='rk2Int'; end
42 microBurst = @(tb0, xb0) feval(micB,dxdt,[tb0 tb0+bT],xb0);
```

Set initial conditions, and the time to be covered by the macrosolver.

```
50 x0 = [1 0];
51 tSpan=0:0.5:15;
```

Simulate using `PIG()`, first without the default treatment of `cdmc` for the microsolver and second with. Generate a trusted solution using standard numerical methods.

```
62 if ~exist('OCTAVE_VERSION','builtin')
63     macInt='ode45'; else macInt='odeOct'; end
64 [nt,nx] = PIG(macInt,microBurst,tSpan,x0,[],[],'no cdm');
65 [ct,cx] = PIG(macInt,microBurst,tSpan,x0);
66 [tClassic,xClassic] = feval(macInt,dxdt,tSpan,x0);
```

Figure 2.10: Accurate simulation of a weakly stiff non-autonomous system by PIG() using `cdmc()`, and an inaccurate solution using a naive application of PIG().

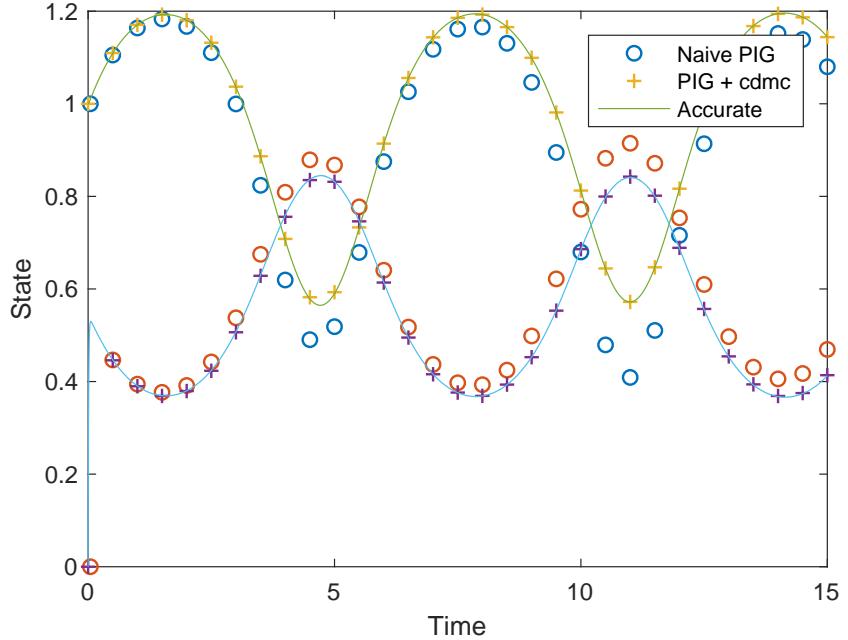


Figure 2.10 plots the output.

```

83 figure
84 h = plot(nt,nx,'rx', ct,cx,'bo', tClassic,xClassic,'-');
85 legend(h(1:2:5), 'Naive PIG', 'PIG + cdmc', 'Accurate')
86 xlabel('Time'), ylabel('State')
87 if ~exist('OCTAVE_VERSION','builtin')
88 set(gcf,'PaperUnits','centimeters','PaperPosition',[0 0 14 10])
89 %print('-depsc2','PIGExplore')
90 end

```

A source of error in the standard PIG() scheme is the finite length of each burst,  $bT$ . This computes a time derivative at a time that is significantly different to that requested by standard coded schemes. Set  $bT$  to  $20*\epsilon$  or  $50*\epsilon$ <sup>1</sup> to worsen the error in both schemes. This example reflects a general principle: most Projective Integration schemes incur a global error term proportional to the burst time of the microsolver and independent of the order of the microsolver. The PIRKn() schemes are written to eliminate this error, but PIG() works with any user-defined macrosolver and cannot reduce this error, except by using the function `cdmc()`, its default.

## 2.10 To do/discuss

- Implement lifting and restriction for PIRKn() functions.

<sup>1</sup> This example is quite extreme: at  $bT=50*\epsilon$ , it would be computationally much cheaper to simulate the entire length of  $tSpan$  using the microsolver alone.

- Could implement Projective Integration by ‘arbitrary’ Runge–Kutta scheme; that is, by having the user input a particular Butcher table—surely only specialists would be interested.
- Can maybe implement microsolvers that terminate a burst when the fast dynamics have settled using, for example, the ‘Events’ function handle in `ode23`.
- Need projective integration of systems with fast oscillations, perhaps by DMD.
- Need projective integration for stochastic systems.

---

### 3 Patch scheme for given microscale discrete space system

---

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### 3.1 Introduction

Consider spatio-temporal multiscale systems where the spatial domain is so large that a given microscale code cannot be computed in a reasonable time. The *patch scheme* computes the microscale details only on small patches of the space-time domain, and produce correct macroscale predictions by craftily coupling the patches across unsimulated space (Hyman 2005, Samaey et al. 2005, 2006, Roberts & Kevrekidis 2007, Liu et al. 2015, e.g.). The resulting macroscale predictions were generally proved to be consistent with the microscale dynamics, to some specified order of accuracy, in a series of papers: 1D-space dissipative systems (Roberts & Kevrekidis 2007, Bunder et al. 2017); 2D-space dissipative systems (Roberts et al. 2014, Bunder et al. 2020); and 1D-space wave-like systems (Cao & Roberts 2016b).

The microscale spatial structure is to be on a lattice such as obtained from finite difference/element/volume approximation of a PDE. The microscale is either continuous or discrete in time.

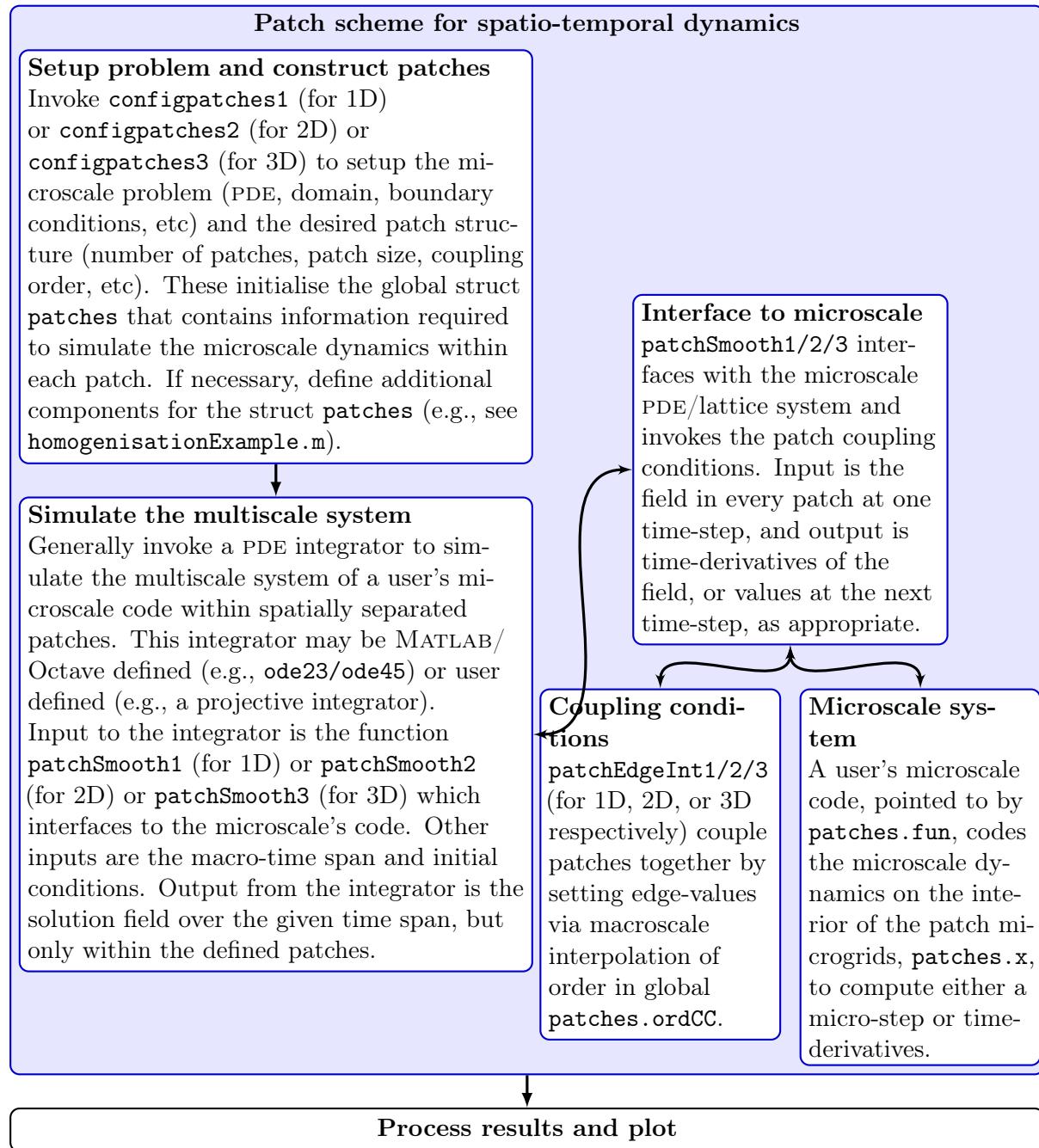
**Quick start** See Sections 3.2.2 and 3.11.2 which respectively list example basic code that uses the provided functions to simulate the 1D Burgers' PDE, and a 2D nonlinear ‘diffusion’ PDE. Then see Figure 3.1.

### 3.2 configPatches1(): configures spatial patches in 1D

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*Figure 3.1: The Patch methods, Chapter 3, accelerate simulation/integration of multiscale systems with interesting spatial/network structure/patterns. The methods use your given microsimulators whether coded from PDEs, lattice systems, or agent/particle microscale simulators. The patch functions require that a user configure the patches, and interface the coupled patches with a time integrator/simulator. This chart overviews the main functional recursion involved.*



### 3.2.1 Introduction

Makes the struct `patches` for use by the patch/gap-tooth time derivative/step function `patchSmooth1()`. [Section 3.2.2](#) lists an example of its use.

```
19 function configPatches1(fun,Xlim,BCs,nPatch,ordCC,ratio,nSubP ...
20                               ,varargin)
21 global patches
```

**Input** If invoked with no input arguments, then executes an example of simulating Burgers' PDE—see [Section 3.2.2](#) for the example code.

- `fun` is the name of the user function, `fun(t,u,x)`, that computes time derivatives (or time-steps) of quantities on the patches.
- `Xlim` give the macro-space spatial domain of the computation: patches are equi-spaced over the interior of the interval `[Xlim(1),Xlim(2)]`.
- `BCs` somehow will define the macroscale boundary conditions. Currently, `BCs` is ignored and the system is assumed macro-periodic in the spatial domain.
- `nPatch` is the number of equi-spaced spaced patches.
- `ordCC`, must be  $\geq -1$ , is the ‘order’ of interpolation across empty space of the macroscale patch values to the edge of the patches for inter-patch coupling: where `ordCC` of 0 or  $-1$  gives spectral interpolation; and `ordCC` being odd is for staggered spatial grids.
- `ratio` (real) is the ratio of (depending upon `EdgyInt`) either the half-width or full-width of a patch to the spacing of the patch mid-points. So either `ratio = 1/2` means the patches abut and `ratio = 1` is overlapping patches as in holistic discretisation, or `ratio = 1` means the patches abut. Small `ratio` should greatly reduce computational time.
- `nSubP` is the number of equi-spaced microscale lattice points in each patch. If not using `EdgyInt`, then must be odd so that there is a central lattice point.
- ‘`nEdge`’ (not yet implemented), *optional*, default=1, for each patch, the number of edge values set by interpolation at the edge regions of each patch. The default is one (suitable for microscale lattices with only nearest neighbour interactions).
- ‘`EdgyInt`’, true/false, *optional*, default=false. If true, then interpolate to left/right edge-values from right/left next-to-edge values.
- ‘`nEnsem`’, *optional-experimental*, default one, but if more, then an ensemble over this number of realisations.
- ‘`hetCoeffs`’, *optional*, default empty. Supply an array of microscale heterogeneous coefficients to be used by the given microscale `fun` in each patch. Say the given array `cs` is of size  $m_x \times n_c$ , where  $n_c$  is the number of different sets of coefficients. The coefficients are to be the same for

each and every patch; however, macroscale variations are catered for by the  $n_c$  coefficients being  $n_c$  parameters in some macroscale formula.

- If  $\text{nEnsem} = 1$ , then the array of coefficients is just tiled across the patch size to fill up each patch, starting from the first element.
- If  $\text{nEnsem} > 1$  (value immaterial), then reset  $\text{nEnsem} := m_x$  and construct an ensemble of all  $m_x$  phase-shifts of the coefficients. In this scenario, the inter-patch coupling couples different members in the ensemble. When **EdgyInt** is true, and when the coefficients are diffusivities/elasticities, then this coupling cunningly preserves symmetry .
- ‘**nCore**’, *optional-experimental*, default one, but if more, and only for non-**EdgyInt**, then interpolates from an average over the core of a patch, a core of size  $??$ . Then edge values are set according to interpolation of the averages $??$  or so that average at edges is the interpolant $??$

**Output** The *global* struct **patches** is created and set with the following components.

- **.fun** is the name of the user’s function **fun(t,u,x)** that computes the time derivatives (or steps) on the patchy lattice.
- **.ordCC** is the specified order of inter-patch coupling.
- **.stag** is true for interpolation using only odd neighbouring patches as for staggered grids, and false for the usual case of all neighbour coupling.
- **.Cwtsr** and **.Cwtsl** are the **ordCC**-vector of weights for the inter-patch interpolation onto the right and left edges (respectively) with patch:macroscale ratio as specified.
- **.x** (4D) is  $\text{nSubP} \times 1 \times 1 \times \text{nPatch}$  array of the regular spatial locations  $x_{ij}$  of the  $i$ th microscale grid point in the  $j$ th patch.
- **.nEdge** is, for each patch, the number of edge values set by interpolation at the edge regions of each patch.
- **.le**, **.ri** determine inter-patch coupling of members in an ensemble. Each a column vector of length **nEnsem**.
- **.cs** either
  - [] 0D, or
  - if  $\text{nEnsem} = 1$ ,  $(\text{nSubP}(1) - 1) \times n_c$  2D array of microscale heterogeneous coefficients, or
  - if  $\text{nEnsem} > 1$ ,  $(\text{nSubP}(1) - 1) \times n_c \times m_x$  3D array of  $m_x$  ensemble of phase-shifts of the microscale heterogeneous coefficients.

### 3.2.2 If no arguments, then execute an example

```
165  if nargin==0
```

The code here shows one way to get started: a user's script may have the following three steps (left-right arrows denote function recursion).

1. configPatches1
2. ode15s integrator  $\leftrightarrow$  patchSmooth1  $\leftrightarrow$  user's PDE
3. process results

Establish global patch data struct to point to and interface with a function coding Burgers' PDE: to be solved on  $2\pi$ -periodic domain, with eight patches, spectral interpolation couples the patches, each patch of half-size ratio 0.2, and with seven microscale points forming each patch.

```
184 configPatches1(@BurgersPDE, [0 2*pi], nan, 8, 0, 0.2, 7);
Set an initial condition, with some microscale randomness.
190 u0=0.3*(1+sin(patches.x))+0.1*randn(size(patches.x));
```

Simulate in time using a standard stiff integrator and the interface function `patchsmooth1()` ([Section 3.3](#)).

```
198 if ~exist('OCTAVE_VERSION','builtin')
199 [ts,us] = ode15s( @patchSmooth1,[0 0.5],u0(:));
200 else % octave version
201 [ts,us] = odeOcts(@patchSmooth1,[0 0.5],u0(:));
202 end
```

Plot the simulation using only the microscale values interior to the patches: either set  $x$ -edges to `nan` to leave the gaps; or use `patchEdgyInt1` to re-interpolate correct patch edge values and thereby join the patches. [Figure 3.2](#) illustrates an example simulation in time generated by the patch scheme applied to Burgers' PDE.

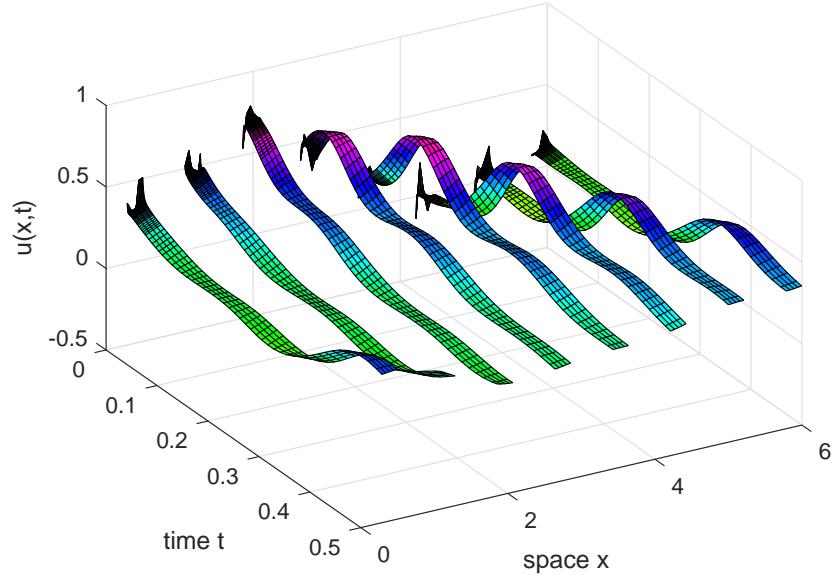
```
214 figure(1),clf
215 if 1, patches.x([1 end],:,:,:) = nan; us=us.';
216 else us=reshape(patchEdgyInt1(us.'),[],length(ts));
217 end
218 surf(ts,patches.x(:,us)
219 view(60,40), colormap(hsv)
220 title('Burgers PDE: patches in space, continuous time')
221 xlabel('time t'), ylabel('space x'), zlabel('u(x,t)')
222 if0urCf2eps(mfilename)
```

Upon finishing execution of the example, exit this function.

```
233 return
234 end%if no arguments
```

**Example of Burgers PDE inside patches** As a microscale discretisation of Burgers' PDE  $u_t = u_{xx} - 30uu_x$ , here code  $\dot{u}_{ij} = \frac{1}{\delta x^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) - 30u_{ij}\frac{1}{2\delta x}(u_{i+1,j} - u_{i-1,j})$ . Here there is only one field variable, and one in the ensemble, so for simpler coding of the PDE we squeeze them out (with no need to reshape when via `patchSmooth1()`).

*Figure 3.2: field  $u(x, t)$  of the patch scheme applied to Burgers' PDE.  
Burgers PDE: patches in space, continuous time*



```

15 function ut=BurgersPDE(t,u,x)
16     u=squeeze(u); % omit singleton dimensions
17     dx=diff(x(1:2)); % microscale spacing
18     i=2:size(u,1)-1; % interior points in patches
19     ut=nan(size(u)); % preallocate storage
20     ut(i,:)=diff(u,2)/dx^2 ...
21         -30*u(i,:).*(u(i+1,:)-u(i-1,:))/(2*dx);
22 end

10 function [ts,xs] = ode0cts(dxdt,tSpan,x0)
11     if length(tSpan)>2, ts = tSpan;
12     else ts = linspace(tSpan(1),tSpan(end),21)';
13     end
14     lsode_options('integration method','non-stiff');
15     xs = lsode(@(x,t) dxdt(t,x),x0,ts);
16 end

```

### 3.2.3 Parse input arguments and defaults

```

250 p = inputParser;
251 fnValidation = @(f) isa(f, 'function_handle'); %test for a function name
252 addRequired(p,'fun',fnValidation);
253 addRequired(p,'Xlim',@isnumeric);
254 addRequired(p,'BCs'); % nothing yet decided
255 addRequired(p,'nPatch',@isnumeric);
256 addRequired(p,'ordCC',@isnumeric);
257 addRequired(p,'ratio',@isnumeric);
258 addRequired(p,'nSubP',@isnumeric);
259 addParameter(p,'nEdge',1,@isnumeric);

```

```

260 addParameter(p,'EdgyInt',false,@islogical);
261 addParameter(p,'nEnsem',1,@isnumeric);
262 addParameter(p,'nCore',1,@isnumeric);
263 addParameter(p,'hetCoeffs',[],@isnumeric);
264 parse(p,fun,Xlim,BCs,nPatch,ordCC,ratio,nSubP,varargin{:});
```

Set the optional parameters.

```

270 patches.nEdge = p.Results.nEdge;
271 patches.EdgyInt = p.Results.EdgyInt;
272 patches.nEnsem = p.Results.nEnsem;
273 cs = p.Results.hetCoeffs;
274 patches.nCore = p.Results.nCore;
```

Check parameters.

```

280 assert(patches.nEdge==1 ...
281     , 'multi-edge-value interp not yet implemented')
282 assert(2*patches.nEdge+1<=nSubP ...
283     , 'too many edge values requested')
284 if patches.nCore>1
285     warning('nCore>1 not yet tested in this version')
286 end
```

### 3.2.4 The code to make patches and interpolation

First, store the pointer to the time derivative function in the struct.

```
298 patches.fun=fun;
```

Second, store the order of interpolation that is to provide the values for the inter-patch coupling conditions. Spectral coupling is ordCC of 0 and  $-1$ .

```

307 assert((ordCC>=-1) & (floor(ordCC)==ordCC), ...
308     'ordCC out of allowed range integer>=-1')
```

For odd ordCC, interpolate based upon odd neighbouring patches as is useful for staggered grids.

```

315 patches.stag=mod(ordCC,2);
316 ordCC=ordCC+patches.stag;
317 patches.ordCC=ordCC;
```

Check for staggered grid and periodic case.

```

323 if patches.stag, assert(mod(nPatch,2)==0, ...
324     'Require an even number of patches for staggered grid')
325 end
```

Might as well precompute the weightings for the interpolation of field values for coupling. (Could sometime extend to coupling via derivative values.)

```
333 if ordCC>0, patchCwts(ratio,ordCC,patches.stag), end
```

Third, set the centre of the patches in a the macroscale grid of patches assuming periodic macroscale domain.

```

342 X=linspace(Xlim(1),Xlim(2),nPatch+1);
343 X=X(1:nPatch)+diff(X)/2;
344 DX=X(2)-X(1);
```

Construct the microscale grid in each patch, assuming Dirichlet patch edges, and a half-patch length of `ratio·DX`, unless `patches.EdgyInt` is set in which case the patches are of length `ratio*DX+dx`. Reshape the grid to be 4D to suit dimensions (micro,Vars,Ens,macro).

```

355 assert(patches.EdgyInt | mod(nSubP,2)==1, ...
356     'configPatches1: nSubP must be odd')
357 i0=(nSubP+1)/2;
358 if ~patches.EdgyInt, dx = ratio*DX/(i0-1);
359 else dx = ratio*DX/(nSubP-2);
360 end
361 patches.x = dx*(-i0+1:i0-1)'*X; % micro-grid
362 patches.x = reshape(patches.x,nSubP,1,1,nPatch);
```

### 3.2.5 Set ensemble inter-patch communication

For `EdgyInt` or centre interpolation respectively, the right-edge/centre realisations `1:nEnsem` are to interpolate to left-edge `le`, and the left-edge/centre realisations `1:nEnsem` are to interpolate to `re`. `re` and `li` are ‘transposes’ of each other as `re(li)=le(ri)` are both `1:nEnsem`. Alternatively, one may use the statement

```
c=hankel(c(1:nSubP-1),c([nSubP 1:nSubP-2]));
```

to correspondingly generates all phase shifted copies of microscale heterogeneity (see `homoDiffEdgy1` of [Section 3.6](#)).

The default is nothing shifty. This setting reduces the number of if-statements in function `patchEdgeInt2()`.

```

388 nE = patches.nEnsem;
389 patches.le = 1:nE;
390 patches.ri = 1:nE;
```

However, if heterogeneous coefficients are supplied via `hetCoeffs`, then do some non-trivial replications. First, get microscale periods, patch size, and replicate many times in order to subsequently sub-sample: `nSubP` times should be enough.

```

400 if ~isempty(cs)
401 [mx,nc] = size(cs);
402 nx = nSubP(1);
403 cs = repmat(cs,nSubP,1);
```

If only one member of the ensemble is required, then sub-sample to patch size, and store coefficients in `patches` as is.

```

411 if nE==1, patches.cs = cs(1:nx-1,:); else
```

But for `nEnsem` > 1 an ensemble of  $m_x$  phase-shifts of the coefficients is constructed from the over-supply. Here code phase-shifts over the periods—the phase shifts are like Hankel-matrices.

```

420     patches.nEnsem = mx;
421     patches.cs = nan(nx-1,nc,mx);
422     for i = 1:mx
423         is = (i:i+nx-2);
424         patches.cs(:,:,i) = cs(is,:);
425     end
426     patches.cs = reshape(patches.cs,nx-1,nc,[]);

```

Further, set a cunning left/right realisation of inter-patch coupling. The aim is to preserve symmetry in the system when also invoking `EdgyInt`. What this coupling does without `EdgyInt` is unknown. Use auto-replication.

```

436     patches.le = mod((0:mx-1)' + mod(nx-2,mx),mx)+1;
437     patches.ri = mod((0:mx-1)' - mod(nx-2,mx),mx)+1;

```

Issue warning if the ensemble is likely to be affected by lack of scale separation. Need to justify this and the arbitrary threshold more carefully??

```

445 if ratio*patches.nEnsem>0.9, warning( ...
446 'Probably poor scale separation in ensemble of coupled phase-shifts')
447 scaleSeparationParameter=ratio*patches.nEnsem
448 end

End the two if-statements.

454 end%if nEnsem=1 or >1
455 end%if not-empty(cs)

```

**Fin**

```
463 end% function
```

### 3.3 patchSmooth1(): interface 1D space to time integrators

To simulate in time with 1D spatial patches we often need to interface a user's time derivative function with time integration routines such as `ode15s` or `PIRK2`. This function provides an interface. It mostly assumes that the sub-patch structure is *smooth* so that the patch centre-values are sensible macroscale variables, and patch edge values are determined by macroscale interpolation of the patch-centre values. However, we have found that microscale heterogeneous systems may be accurately simulated with this function via appropriate interpolation. Communicate patch-design variables to this function using the previously established global struct `patches` ([Section 3.2](#)).

```

28 function dudt=patchSmooth1(t,u)
29 global patches

```

### Input

- $\mathbf{u}$  is a vector of length  $n_{SubP} \cdot n_{Patch} \cdot n_{Vars}$  where there are  $n_{Vars}$  field values at each of the points in the  $n_{SubP} \times n_{Patch}$  grid.
- $t$  is the current time to be passed to the user's time derivative function.
- **patches** a struct set by `configPatches1()` with the following information used here.
  - `.fun` is the name of the user's function `fun(t,u,x)` that computes the time derivatives on the patchy lattice. The array  $\mathbf{u}$  has size  $n_{SubP} \times n_{Patch} \times n_{Vars}$ . Time derivatives must be computed into the same sized array, although herein the patch edge values are overwritten by zeros.
  - `.x` is  $n_{SubP} \times n_{Patch}$  array of the spatial locations  $x_{ij}$  of the microscale grid points in every patch. Currently it *must* be an equi-spaced lattice on both macro- and microscales.

### Output

- $dudt$  is  $n_{SubP} \cdot n_{Patch} \cdot n_{Vars}$  vector of time derivatives, but with patch edge values set to zero.

Reshape the fields  $\mathbf{u}$  as a 2/3D-array, and sets the edge values from macroscale interpolation of centre-patch values. [Section 3.4](#) describes `patchEdgeInt1()`.

79 `u=patchEdgeInt1(u);`

Ask the user function for the time derivatives computed in the array, overwrite its edge values with the dummy value of zero, then return to a to the user/integrator as column vector.

89 `dudt=patches.fun(t,u,patches.x);`  
 90 `dudt([1 end],:,:, :)=0;`  
 91 `dudt=reshape(dudt,[],1);`

Fin.

## 3.4 `patchEdgeInt1()`: sets edge values from interpolation over the macroscale

Couples 1D patches across 1D space by computing their edge values from macroscale interpolation of either the mid-patch value, or the patch-core average, or the opposite next-to-edge values (this last choice often maintains symmetry). This function is primarily used by `patchSmooth1()` but is also useful for user graphics. A spatially discrete system could be integrated in time via the patch or gap-tooth scheme ([Roberts & Kevrekidis 2007](#)). When using core averages (not yet implemented in this version??), assumes they are in some sense *smooth* so that these averages are sensible macroscale variables: then patch edge values are determined by macroscale interpolation of the core averages ([Bunder et al. 2017](#)). Communicates patch-design variables via the global struct **patches**.

```
28 function u=patchEdgeInt1(u)
29 global patches
```

### Input

- **u** is a vector (or indeed any dim array) of length **nSubP · nVars · nEnsem · nPatch** where there are **nVars · nEnsem** field values at each of the points in the **nSubP × nPatch** grid.
- **patches** a struct largely set by **configPatches1()**, and which includes the following.
  - **.x** is **nSubP × 1 × 1 × nPatch** array of the spatial locations  $x_{ij}$  of the microscale grid points in every patch. Currently it *must* be an equi-spaced lattice on both macro- and microscales.
  - **.ordCC** is order of interpolation integer  $\geq -1$ .
  - **.stag** in  $\{0, 1\}$  is one for staggered grid (alternating) interpolation.
  - **.Cwtsr** and **.Cwts1** define the coupling coefficients for finite width interpolation.
  - **.EdgyInt** true/false is true for interpolating patch-edge values from opposite next-to-edge values (often preserves symmetry).
  - **.nEnsem** the number of realisations in the ensemble.

### Output

- **u** is **nSubP × nVars × nEnsem × nPatch** 4D array of the fields with edge values set by interpolation.

Determine the sizes of things. Any error arising in the reshape indicates **u** has the wrong size.

```
86 [nSubP,~,~,nPatch] = size(patches.x);
87 nEnsem = patches.nEnsem;
88 nVars = round(numel(u)/numel(patches.x)/nEnsem);
89 assert(numel(u) == nSubP*nVars*nEnsem*nPatch ...
90 , 'patchEdgeInt1: input u has wrong size for parameters')
91 u = reshape(u,nSubP,nVars,nEnsem,nPatch);
```

Compute lattice sizes from inside the patches as the edge values may be NaNs, etc.

```
98 dx = patches.x(3,1,1,1)-patches.x(2,1,1,1);
99 DX = patches.x(2,1,1,2)-patches.x(2,1,1,1);
```

If the user has not defined the patch core, then we assume it to be a single point in the middle of the patch, unless we are interpolating from next-to-edge values. For **patches.nCore**  $\neq 1$  the half width ratio is reduced, as described by [Bunder et al. \(2017\)](#).

```
109 if ~patches.EdgyInt
110     r = dx*(nSubP-1)/2/DX*(nSubP-patches.nCore)/(nSubP-1);
```

```

111 else r = dx*(nSubP-2)/DX;
112 end

```

For the moment assume the physical domain is macroscale periodic so that the coupling formulas are simplest. Should eventually cater for periodic, odd-mid-gap, even-mid-gap, even-mid-patch, Dirichlet, Neumann, ???. These index vectors point to patches and their two immediate neighbours.

```

123 j = 1:nPatch; jp = mod(j,nPatch)+1; jm = mod(j-2,nPatch)+1;

```

Calculate centre of each patch and the surrounding core (`nSubP` and `nCore` are both odd).

```

130 i0 = round((nSubP+1)/2);
131 c = round((patches.nCore-1)/2);

```

**Lagrange interpolation gives patch-edge values** Consequently, compute centred differences of the patch core averages for the macro-interpolation of all fields. Assumes the domain is macro-periodic.

```

141 if patches.ordCC>0 % then non-spectral interpolation
142   if patches.EdgeyInt % next-to-edge as double nVars, interleaved
143     uCore = reshape(u([2 end-1],:,:,j),2*nVars,nEnsem,nPatch);
144     dmu = zeros(patches.ordCC,2*nVars,nEnsem,nPatch);
145   else % interpolate mid-patch values
146     uCore = reshape(sum(u((i0-c):(i0+c),:,:,j),1),nVars,nEnsem,nPatch);
147     dmu = zeros(patches.ordCC,nVars,nEnsem,nPatch);
148   end;
149   if patches.stag % use only odd numbered neighbours
150     dmu(1,:,:,:) = (uCore(:,:,jp)+uCore(:,:,jm))/2; % \mu
151     dmu(2,:,:,:) = (uCore(:,:,jp)-uCore(:,:,jm)); % \delta
152     jp = jp(jp); jm = jm(jm); % increase shifts to \pm2
153   else % standard
154     dmu(1,:,:,:) = (uCore(:,:,jp)-uCore(:,:,jm))/2; % \mu\delta
155     dmu(2,:,:,:) = (uCore(:,:,jp)-2*uCore(:,:,j)+uCore(:,:,jm)); % \delta^2
156   end% if odd/even

```

Recursively take  $\delta^2$  of these to form higher order centred differences (could unroll a little to cater for two in parallel).

```

164 for k = 3:patches.ordCC
165   dmu(k,:,:,:) = dmu(k-2,:,:,:,:) - 2*dmu(k-2,:,:,:,:) + dmu(k-2,:,:,:,:) ;
166 end

```

Interpolate macro-values to be Dirichlet edge values for each patch ([Roberts & Kevrekidis 2007](#), [Bunder et al. 2017](#)), using weights computed in `configPatches1()`. Here interpolate to specified order.

For the case point values interpolate to point edge-values. When we have an ensemble of configurations, different realisations are coupled to each other as specified by `patches.le` and `patches.ri`.

```

180    if patches.nCore==1
181        if patches.EdgyInt
182            u(nSubP,:,:atches.ri,j) ...
183            = shiftdim(uCore(1:2:end,:,:j),-1)*(1-patches.stag) ...
184                +sum( patches.Cwtsr.*dmu(:,1:2:end,:,:j) );
185            u( 1 ,:,atches.le,j) ...
186            = shiftdim(uCore(2:2:end,:,:j),-1)*(1-patches.stag) ...
187                +sum( patches.Cwtsl.*dmu(:,2:2:end,:,:j) );
188        else % mid-patch interpolation
189            u(nSubP,:,:atches.ri,j) ...
190            = shiftdim(uCore(:,:,j),-1)*(1-patches.stag) ...
191                +sum( patches.Cwtsr.*dmu(:,:,j) );
192            u( 1 ,:,atches.le,j) ...
193            = shiftdim(uCore(:,:,j),-1)*(1-patches.stag) ...
194                +sum( patches.Cwtsl.*dmu(:,:,j) );
195    end

```

For a non-trivial core then more needs doing: the core (one or more) of each patch interpolates to the edge action regions. When more than one in the core, the edge is set depending upon near edge values so the average near the edge is correct.

```

205    else error('not yet considered, july 2020 ??')
206        u(nSubP,:,:,:,j) = uCore(:,:,j)*(1-patches.stag) ...
207            + reshape(-sum(u((nSubP-patches.nCore+1):(nSubP-1),:,:,j),1) ...
208                +sum( patches.Cwtsr.*dmu ),nPatch,nVars);
209        u(1,:,:,:,j) = uCore(:,:,j)*(1-patches.stag) ...
210            +reshape(-sum(u(2:patches.nCore,:,:,:,j),1) ...
211                +sum( patches.Cwtsl.*dmu ),nPatch,nVars);
212    end;

```

**Case of spectral interpolation** Assumes the domain is macro-periodic.

```
222 else% spectral interpolation
```

As the macroscale fields are  $N$ -periodic, the macroscale Fourier transform writes the centre-patch values as  $U_j = \sum_k C_k e^{ik2\pi j/N}$ . Then the edge-patch values  $U_{j\pm r} = \sum_k C_k e^{ik2\pi/N(j\pm r)} = \sum_k C'_k e^{ik2\pi j/N}$  where  $C'_k = C_k e^{ikr2\pi/N}$ . For  $nPatch$  patches we resolve ‘wavenumbers’  $|k| < nPatch/2$ , so set row vector  $\mathbf{ks} = k2\pi/N$  for ‘wavenumbers’  $k = (0, 1, \dots, k_{\max}, -k_{\max}, \dots, -1)$  for odd  $N$ , and  $k = (0, 1, \dots, k_{\max}, (k_{\max} + 1), -k_{\max}, \dots, -1)$  for even  $N$ .

Deal with staggered grid by doubling the number of fields and halving the number of patches (`configPatches1()` tests that there are an even number of patches). Then the patch-ratio is effectively halved. The patch edges are near the middle of the gaps and swapped. Have not yet tested whether works for Edgy Interpolation??

```

245    if patches.stag % transform by doubling the number of fields
246        v = nan(size(u)); % currently to restore the shape of u
247        u = [u(:,:,:,1:2:nPatch) u(:,:,:,:,2:2:nPatch)];

```

```

248     stagShift = 0.5*[ones(1,nVars) -ones(1,nVars)];
249     iV = [nVars+1:2*nVars 1:nVars]; % scatter interp to alternate field
250     r = r/2; % ratio effectively halved
251     nPatch = nPatch/2; % halve the number of patches
252     nVars = nVars*2; % double the number of fields
253 else % the values for standard spectral
254     stagShift = 0;
255     iV = 1:nVars;
256 end

```

Now set wavenumbers (when nPatch is even then highest wavenumber is  $\pi$ ).

```

263     kMax = floor((nPatch-1)/2);
264     ks = shiftdim( ...
265         2*pi/nPatch*(mod((0:nPatch-1)+kMax,nPatch)-kMax) ...
266         ,-2);

```

Test for reality of the field values, and define a function accordingly. Could be problematic if some variables are real and some are complex, or if variables are of quite different sizes.

```

275     if max(abs(imag(u(:))))<1e-9*max(abs(u(:)))
276         uclean=@(u) real(u);
277     else uclean=@(u) u;
278 end

```

Compute the Fourier transform across patches of the patch centre-values for all the fields. If there are an even number of points, then if complex, treat as positive wavenumber, but if real, treat as cosine. When using an ensemble of configurations, different configurations might be coupled to each other, as specified by `patches.le` and `patches.ri`.

```

290 if ~patches.EdgyInt
291     Cleft = fft(u( i0    ,:,:, :) ,[],4);
292     Cright = Cleft;
293 else
294     Cleft = fft(u( 2    ,:,:, :) ,[],4);
295     Cright= fft(u(nSubP-1,:,:,:) ,[],4);
296 end

```

The inverse Fourier transform gives the edge values via a shift a fraction  $r$  to the next macroscale grid point. Enforce reality when appropriate.

```

304     u(nSubP,iV,patches.ri,:) = uclean(ifft( ...
305         Cleft.*exp(1i*ks.*(stagShift+r)) ,[],4));
306     u( 1 ,iV,patches.le,:) = uclean(ifft( ...
307         Cright.*exp(1i*ks.*(stagShift-r)) ,[],4));

```

Restore staggered grid when appropriate. This dimensional shifting appears to work.

```

315 if patches.stag
316     nVars = nVars/2;
317     u=reshape(u,nSubP,nVars,2,nEnsem,nPatch);

```

```

318     nPatch = 2*nPatch;
319     v(:,:,1:2:nPatch) = u(:,:,1,:,:);
320     v(:,:,2:2:nPatch) = u(:,:,2,:,:);
321     u = v;
322 end
323 end% if spectral

```

Fin, returning the 4D array of field values.

### 3.5 homogenisationExample: simulate heterogeneous diffusion in 1D on patches

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Figure 3.3 shows an example simulation in time generated by the patch scheme applied to heterogeneous diffusion. That such simulations of heterogeneous diffusion makes valid predictions was established by [Bunder et al. \(2017\)](#) who proved that the scheme is accurate when the number of points in a patch is one more than a multiple of the periodic of the microscale heterogeneity.

The first part of the script implements the following gap-tooth scheme (left-right arrows denote function recursion).

1. configPatches1
2. ode15s  $\leftrightarrow$  patchSmooth1  $\leftrightarrow$  heteroDiff
3. process results

Consider a lattice of values  $u_i(t)$ , with lattice spacing  $dx$ , and governed by the heterogeneous diffusion

$$\dot{u}_i = [c_{i-1/2}(u_{i-1} - u_i) + c_{i+1/2}(u_{i+1} - u_i)]/dx^2. \quad (3.1)$$

In this 1D space, the macroscale, homogenised, effective diffusion should be the harmonic mean of these coefficients.

#### 3.5.1 Script to simulate via stiff or projective integration

Set the desired microscale periodicity, and correspondingly choose random microscale diffusion coefficients (with subscripts shifted by a half).

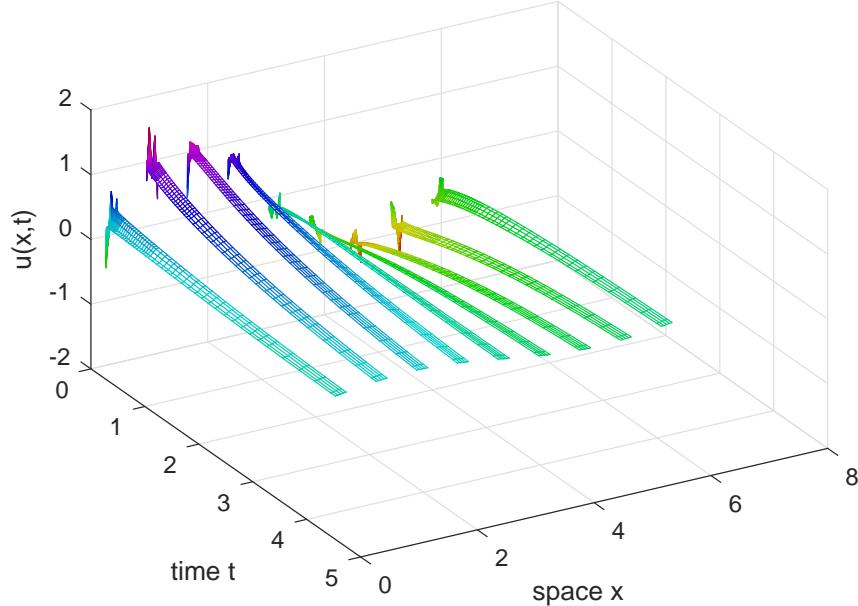
```

53 mPeriod = 3
54 cDiff = exp(randn(mPeriod,1))
55 cHomo = 1/mean(1./cDiff)

```

Establish global data struct `patches` for heterogeneous diffusion on  $2\pi$ -periodic domain. Use nine patches, each patch of half-size ratio 0.2. Quartic (fourth-order) interpolation `ordCC = 4` provides values for the inter-patch

Figure 3.3: the diffusing field  $u(x, t)$  in the patch (gap-tooth) scheme applied to microscale heterogeneous diffusion (Section 3.5).



coupling conditions. Here include the diffusivity coefficients, repeated to fill up a patch.

```

67 global patches
68 nPatch = 9
69 ratio = 0.2
70 nSubP = 2*mPeriod+1
71 Len = 2*pi;
72 ordCC = 4;
73 configPatches1(@heteroDiff,[0 Len],nan,nPatch ...
74 ,ordCC,ratio,nSubP,'hetCoeffs',cDiff);
```

**For comparison: conventional integration in time** Set an initial condition, and here integrate forward in time using a standard method for stiff systems—because of the simplicity of linear problems this method works quite efficiently here. Integrate the interface `patchSmooth1` (Section 3.3) to the microscale differential equations.

```

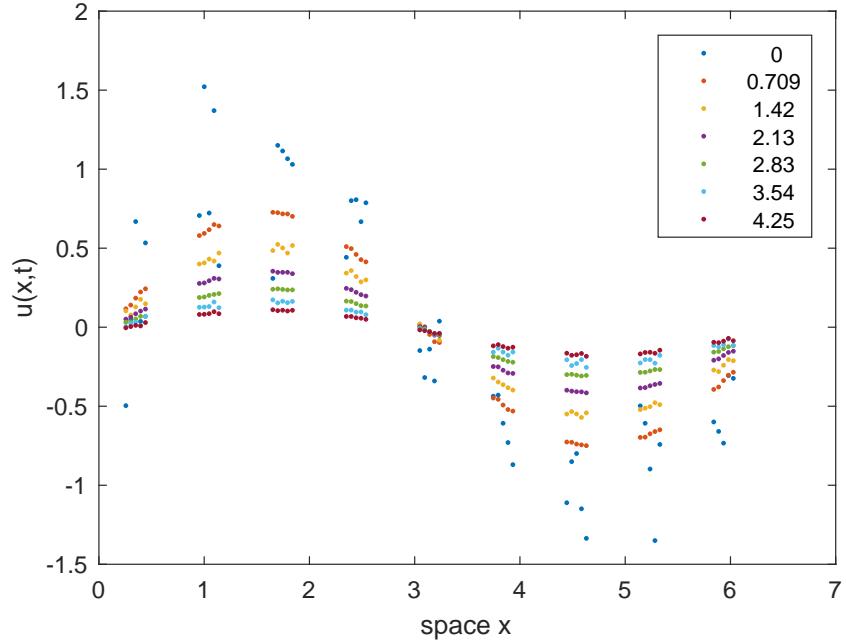
88 u0 = sin(patches.x)+0.3*randn(nSubP,1,1,nPatch);
89 if ~exist('OCTAVE_VERSION','builtin')
90 [ts,ucts] = ode15s(@patchSmooth1, [0 2/cHomo], u0(:));
91 else % octave version
92 [ts,ucts] = odeOcts(@patchSmooth1, [0 2/cHomo], u0(:));
93 end
94 ucts = reshape(ucts,length(ts),length(patches.x(:)),[]);
```

Plot the simulation in Figure 3.3.

```

101 figure(1),clf
102 xs = patches.x; xs([1 end],:) = nan;
```

*Figure 3.4: field  $u(x, t)$  shows basic projective integration of patches of heterogeneous diffusion: different colours correspond to the times in the legend. This field solution displays some fine scale heterogeneity due to the heterogeneous diffusion.*



```

103 mesh(ts, xs(:, 1), ucts'), view(60, 40)
104 xlabel('time t'), ylabel('space x'), zlabel('u(x, t)')
105 if0urCf2eps([mfilename 'CtsU'])

```

The code may invoke this integration interface.

```

10 function [ts, xs] = odeOcts(dxdt, tSpan, x0)
11     if length(tSpan)>2, ts = tSpan;
12     else ts = linspace(tSpan(1), tSpan(end), 21)';
13     end
14     lsode_options('integration method', 'non-stiff');
15     xs = lsode(@(x, t) dxdt(t, x), x0, ts);
16 end

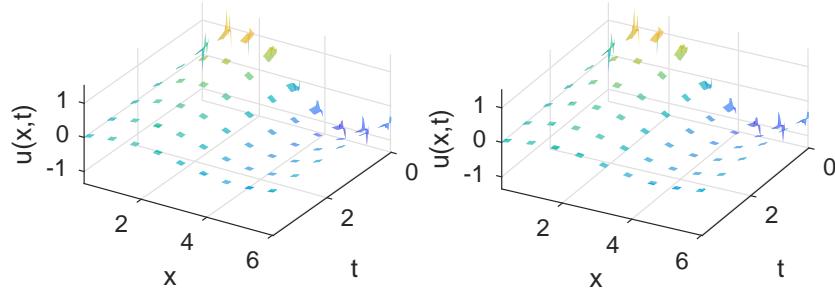
```

**Use projective integration in time** Now take `patchSmooth1`, the interface to the time derivatives, and wrap around it the projective integration PIRK2 (Section 2.2), of bursts of simulation from `heteroBurst` (Section 3.5.3), as illustrated by Figure 3.4.

This second part of the script implements the following design, where the micro-integrator could be, for example, `ode45` or `rk2int`.

1. configPatches1 (done in first part)
2. PIRK2  $\leftrightarrow$  heteroBurst  $\leftrightarrow$  micro-integrator  $\leftrightarrow$  patchSmooth1  $\leftrightarrow$  heteroDiff
3. process results

Figure 3.5: cross-eyed stereo pair of the field  $u(x, t)$  during each of the microscale bursts used in the projective integration of heterogeneous diffusion.



Mark that edge of patches are not to be used in the projective extrapolation by setting initial values to NaN.

```
141 u0([1 end], :) = nan;
```

Set the desired macro- and microscale time-steps over the time domain: the macroscale step is in proportion to the effective mean diffusion time on the macroscale; the burst time is proportional to the intra-patch effective diffusion time; and lastly, the microscale time-step is proportional to the diffusion time between adjacent points in the microscale lattice.

```
153 ts = linspace(0,2/cHomo,7)
154 bT = 3*( ratio*Len/nPatch )^2/cHomo
155 addpath('..../ProjInt')
156 [us,tss,uss] = PIRK2(@heteroBurst, ts, u0(:), bT);
```

Plot the macroscale predictions to draw Figure 3.4.

```
163 figure(2),clf
164 plot(xs(:,us,'.')
165 ylabel('u(x,t)'), xlabel('space x')
166 legend(num2str(ts',3))
167 ifOurCf2eps([mfilename 'U'])
```

Also plot a surface detailing the microscale bursts as shown in the stereo Figure 3.5.

```
182 figure(3),clf
183 for k = 1:2, subplot(2,2,k)
184 surf(tss,xs(:,uss', 'EdgeColor','none')
185 ylabel('x'), xlabel('t'), zlabel('u(x,t)')
186 axis tight, view(126-4*k,45)
187 end
188 ifOurCf2eps([mfilename 'Micro'])
```

End of this example script.

### 3.5.2 heteroDiff(): heterogeneous diffusion

This function codes the lattice heterogeneous diffusion inside the patches. For 2D input arrays  $u$  and  $x$  (via edge-value interpolation of patchSmooth1,

[Section 3.3](#)), computes the time derivative (3.1) at each point in the interior of a patch, output in `ut`. The column vector, or possibly array of phase-shifts, of diffusivities  $c_i$  have previously been stored in struct `patches.cs`.

```

22 function ut = heteroDiff(t,u,x)
23     global patches
24     dx = diff(x(2:3));    % space step
25     i = 2:size(u,1)-1;    % interior points in a patch
26     ut = nan(size(u)); % preallocate output array
27     ut(i,:,:,:) = diff(patches.cs.*diff(u))/dx^2;
28     % if set, include heterogeneous Burgers' advection
29     if isfield(patches,'b') % check for advection coeffs
30         buu = patches.b.*u.^2;
31         ut(i,:) = ut(i,:)-(buu(i+1,:)-buu(i-1,:))/(dx*2);
32     end
33 end% function

```

### 3.5.3 `heteroBurst()`: a burst of heterogeneous diffusion

This code integrates in time the derivatives computed by `heteroDiff` from within the patch coupling of `patchSmooth1`. Try `ode23` or `rk2Int`, although `ode45` may give smoother results.

```

15 function [ts, ucts] = heteroBurst(ti, ui, bT)
16     if ~exist('OCTAVE_VERSION','builtin')
17         [ts,ucts] = ode23( @patchSmooth1,[ti ti+bT],ui(:));
18     else % octave version
19         [ts,ucts] = rk2Int(@patchSmooth1,[ti ti+bT],ui(:));
20     end
21 end

```

Fin.

## 3.6 `homoDiffEdgy1`: computational homogenisation of a 1D diffusion by simulation on small patches

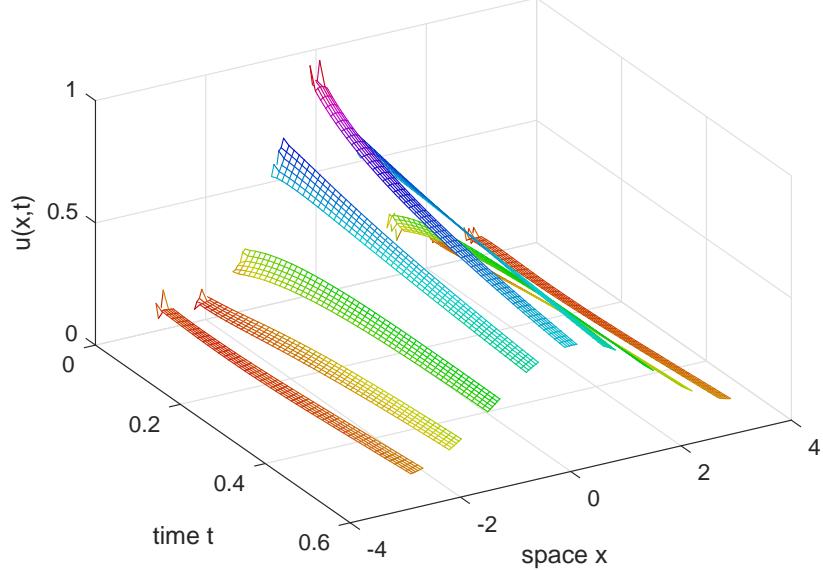
[Figure 3.6](#) shows an example simulation in time generated by the patch scheme applied to macroscale diffusion propagation through a medium with microscale heterogeneity. The inter-patch coupling is realised by quartic interpolation of the patch's next-to-edge values to the patch opposite edges. Such coupling preserves symmetry in many systems, and quartic appears to be the lowest order that generally gives good accuracy.

Suppose the spatial microscale lattice is at points  $x_i$ , with constant spacing  $dx$ . With dependent variables  $u_i(t)$ , simulate the microscale lattice diffusion system

$$\frac{\partial u_i}{\partial t} = \frac{1}{dx^2} \delta[c_{i-1/2} \delta u_i], \quad (3.2)$$

in terms of the centred difference operator  $\delta$ . The system has a microscale heterogeneity via the coefficients  $c_{i+1/2}$  which we assume to have some given

*Figure 3.6: diffusion field  $u(x,t)$  of the gap-tooth scheme applied to the diffusion (3.2). The microscale random component to the initial condition, the sub-patch fluctuations, decays, leaving the emergent macroscale diffusion. This simulation uses nine patches of ‘large’ size ratio 0.25 for visibility.*



known periodicity. [Figure 3.6](#) shows one patch simulation of this system: observe the effects of the heterogeneity within each patch.

### 3.6.1 Script code to simulate heterogeneous diffusion systems

This example script implements the following patch/gap-tooth scheme (left-right arrows denote function recursion).

1. configPatches1, and add micro-information
2. ode15s  $\leftrightarrow$  patchSmooth1  $\leftrightarrow$  heteroDiff
3. plot the simulation
4. use patchSmooth1 to explore the Jacobian

First establish the microscale heterogeneity has micro-period `mPeriod` on the lattice, and random log-normal values, albeit normalised to have harmonic mean one. This normalisation then means that macroscale diffusion on a domain of length  $2\pi$  should have near integer decay rates, the squares of  $0, 1, 2, \dots$ . Then the heterogeneity is repeated to fill each patch, and phase-shifted for an ensemble.

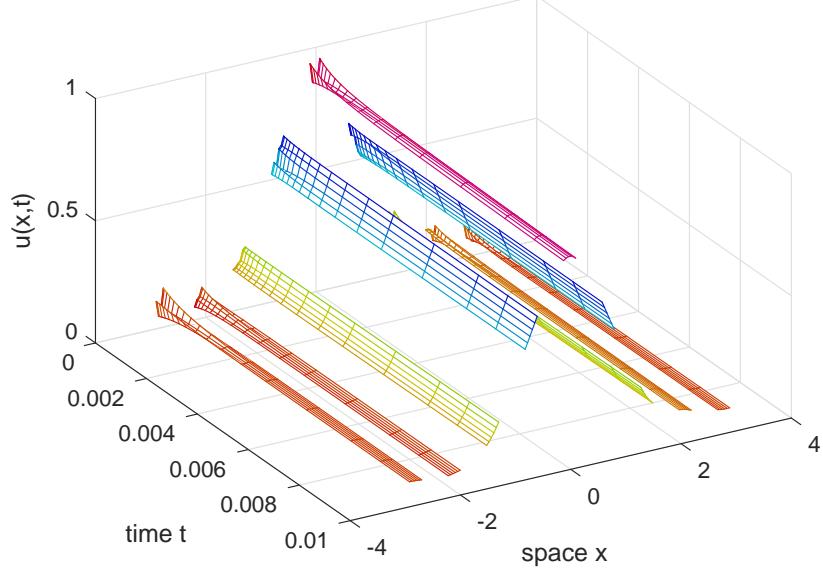
```

88 mPeriod = 3%randi([2 5])
89 % set random diffusion coefficients
90 cHetr=exp(0*randn(mPeriod,1));
91 %cHetr = [3.966;2.531;0.838;0.331;7.276];
92 cHetr = cHetr*mean(1./cHetr) % normalise

```

Establish the global data struct `patches` for the microscale heterogeneous lattice diffusion system (3.2) solved on  $2\pi$ -periodic domain, with seven patches, here each patch of size ratio 0.25 from one side to the other, with five micro-grid

*Figure 3.7: diffusion field  $u(x,t)$  of the gap-tooth scheme applied to the diffusive (3.2). Over this short meso-time we see the macroscale diffusion emerging from the damped sub-patch fast quasi-equilibration.*



points in each patch, and quartic interpolation (4) to provide the edge-values of the inter-patch coupling conditions. Setting `patches.EdgeInt` to one means the edge-values come from interpolating the opposite next-to-edge values of the patches (not the mid-patch values). In this case we appear to need at least fourth order (quartic) interpolation to get reasonable decay rate for heterogeneous diffusion. When simulating an ensemble of configurations, `nSubP` (the number of points in a patch) need not be dependent on the period of the heterogeneous diffusion.

```

114 global patches
115 nPatch = 9
116 ratio = 0.25;
117 nSubP = mPeriod+3 %randi([mPeriod+1 2*mPeriod+2])
118 nEnsem = mPeriod % number realisations in ensemble
119 if mod(nSubP,mPeriod)==2, nEnsem=1, end
120 configPatches1(@heteroDiff, [-pi pi], nan, nPatch ...
121 ,4, ratio, nSubP, 'EdgeInt', true, 'nEnsem', nEnsem ...
122 , 'hetCoeffs', cHetr);

```

**Simulate** Set the initial conditions of a simulation to be that of a lump perturbed by significant random microscale noise, via `randn`.

```

133 u0 = 0.8*exp(-patches.x.^2)+0.2*rand(nSubP,1,nEnsem,nPatch);
134 du0dt = patchSmooth1(0,u0(:));

```

Integrate using standard integrators.

```

140 if ~exist('OCTAVE_VERSION', 'builtin')
141 [ts,us] = ode23(@patchSmooth1, [0 0.6], u0(:));

```

```

142 else % octave version
143     [ts,us] = odeOcts(@patchSmooth1, 0.6*linspace(0,1).^2, u0(:));
144 end

```

**Plot space-time surface of the simulation** We want to see the edge values of the patches, so we adjoin a row of `nans` in between patches. For the field values (which are rows in `us`) we need to reshape, permute, interpolate to get edge values, pad with `nans`, and reshape again. In the case of an ensemble of phase-shifts, we plot the mean over the ensemble.

```

157 xs = squeeze(patches.x);
158 us = patchEdgeInt1( permute( reshape(us ...
159     ,length(ts),nSubP,nEnsem,nPatch) ,[2 1 3 4]) );
160 usstd = squeeze(std(us,0,3));
161 us = squeeze(mean(us,3));
162 if 0, % omit interpolated edges
163     us([1 end],:,:) = nan;
164     usstd([1 end],:,:) = nan;
165 else % insert nans between patches
166     xs(end+1,:)=nan;
167     us(end+1,:,:) = nan;
168     usstd(end+1,:,:) = nan;
169 end
170 us=reshape(permute(us,[1 3 2]),[],length(ts));
171 usstd=reshape(permute(usstd,[1 3 2]),[],length(ts));

```

Now plot two space-time graphs. The first is every time step over a meso-time to see the oscillation and decay of the fast sub-patch diffusions. The second is subsampled surface over the macroscale duration of the simulation to show the propagation of the macroscale diffusion over the heterogeneous lattice.

```

183 for p=1:2
184     switch p
185         case 1, j=find(ts<0.01);
186         case 2, [~,j]=min(abs(ts(:)-linspace(ts(1),ts(end),50)));
187     end
188     figure(p),clf
189     mesh(ts(j),xs(:,us(:,j)))
190     view(60,40), colormap(0.8*hsv)
191     xlabel('time t'), ylabel('space x'), zlabel('u(x,t)')
192     if0urCf2eps([mfilename 'U' num2str(p)])
193 end
194 pause(3)

```

**Compute Jacobian and its spectrum** Let's explore the Jacobian dynamics for a range of orders of interpolation, all for the same patch design and heterogeneity. Here use a smaller ratio, and more patches, as we do not plot.

```

207 nPatch = 13
208 ratio = 0.01;
209
210 leadingEvals=[];
211 for ord=0:2:8
212     ordInterp=ord
213     configPatches1(@heteroDiff,[-pi pi],nan,nPatch ...
214         ,ord,ratio,nSubP,'EdgyInt',true,'nEnsem',nEnsem ...
215         , 'hetCoeffs',cHetr);

```

Form the Jacobian matrix, linear operator, by numerical construction about a zero field. Use *i* to store the indices of the micro-grid points that are interior to the patches and hence are the system variables.

```

225 u0 = zeros(nSubP,1,nEnsem,nPatch);
226 u0([1 end],:,:,:) = nan; u0=u0(:);
227 i=find(~isnan(u0));
228 nJ=length(i);
229 Jac=nan(nJ);
230 for j=1:nJ
231     u0(i)=((1:nJ)==j);
232     dudt=patchSmooth1(0,u0);
233     Jac(:,j)=dudt(i);
234 end
235 nonSymmetric=norm(Jac-Jac')
236 assert(nonSymmetric<5e-9,'failed symmetry')
237 Jac(abs(Jac)<1e-12)=0;

```

Find the eigenvalues of the Jacobian, and list for inspection in [Table 3.1](#): the spectral interpolation is effectively exact for the macroscale; quadratic interpolation is usually quantitatively in error; quartic interpolation appears to be the lowest order for reliable quantitative accuracy.

The number of zero eigenvalues, *nZeroEv*, indicates the number of decoupled systems in this patch configuration.

```

279 [evecs,evals]=eig(Jac);
280 eval=-sort(-diag(real(evals)));
281 nZeroEv=sum(eval(:)>-1e-5)
282 leadingEvals=[leadingEvals eval(1:3*nPatch)];
283 % leadingEvals=[leadingEvals eval([1, (nZeroEv+1):2:(nZeroEv*nPatch+4)])];

```

End of the for-loop over orders of interpolation, and output the tables of eigenvalues.

```

290 end
291 disp('    spectral    quadratic    quartic    sixth-order ...')
292 leadingEvals=leadingEvals

```

End of the main script.

Table 3.1: example parameters and list of eigenvalues (every fourth one listed is sufficient due to symmetry): `nPatch = 19`, `ratio = 0.1`, `nSubP = 5`. The columns are for various `ordCC`, in order: 0, spectral interpolation; 2, quadratic; 4, quartic; and 6, sixth order.

```
cHetr =
    6.9617
    0.4217
    2.0624
leadingEvals =
    2e-11      -2e-12      4e-12      -2e-11
    -0.9999     -1.5195     -1.0127     -1.0003
    -3.9992     -11.861     -4.7785     -4.0738
    -8.9960     -45.239     -17.164     -10.703
    -15.987     -116.27     -56.220     -30.402
    -24.969     -230.63     -151.74     -92.830
    -35.936     -378.80     -327.36     -247.37
    -48.882     -535.89     -570.87     -521.89
    -63.799     -668.21     -818.33     -855.72
    -80.678     -743.96     -976.57     -1093.4
    -29129      -29233      -29227      -29222
    -29151      -29234      -29229      -29223
```

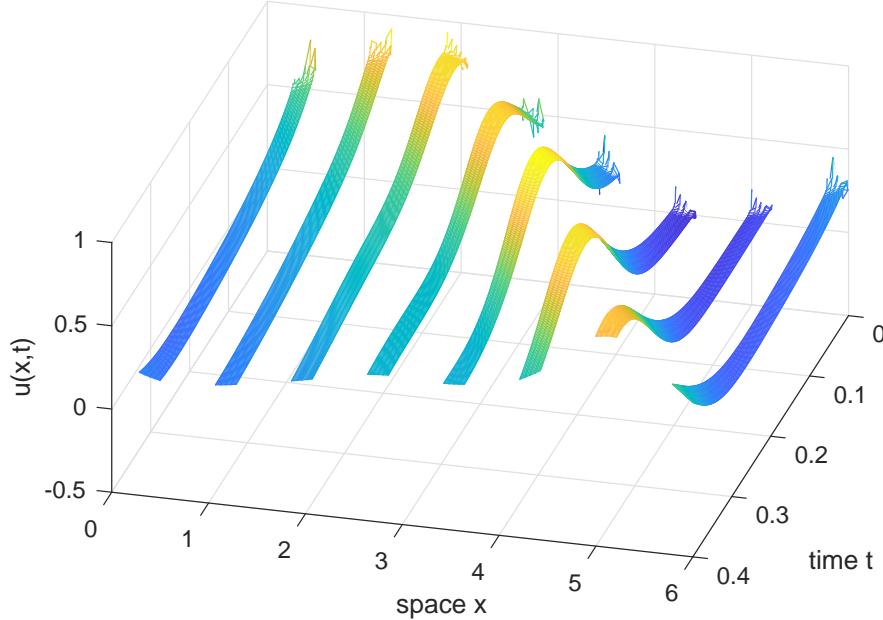
### 3.6.2 `heteroDiff()`: heterogeneous diffusion

This function codes the lattice heterogeneous diffusion inside the patches. For 2D input arrays `u` and `x` (via edge-value interpolation of `patchSmooth1`, [Section 3.3](#)), computes the time derivative (3.1) at each point in the interior of a patch, output in `ut`. The column vector, or possibly array of phase-shifts, of diffusivities  $c_i$  have previously been stored in struct `patches.cs`.

```
22 function ut = heteroDiff(t,u,x)
23     global patches
24     dx = diff(x(2:3)); % space step
25     i = 2:size(u,1)-1; % interior points in a patch
26     ut = nan(size(u)); % preallocate output array
27     ut(:,:, :, :) = diff(patches.cs.*diff(u))/dx^2;
28     % if set, include heterogeneous Burgers' advection
29     if isfield(patches,'b') % check for advection coeffs
30         buu = patches.b.*u.^2;
31         ut(i,:,:) = ut(i,:)-(buu(i+1,:)-buu(i-1,:))/(dx*2);
32     end
33 end% function
```

Fin.

*Figure 3.8: a short time simulation of the Burgers' map (Section 3.7.3) on patches in space. It requires many very small time-steps only just visible in this mesh.*



## 3.7 BurgersExample: simulate Burgers' PDE on patches

### Section contents

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3.7.2	Alternatively use projective integration . . . . .	66
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3.7.4	burgerBurst(): code a burst of the patch map . . . . .	68

[Figure 3.2](#) shows a previous example simulation in time generated by the patch scheme applied to Burgers' PDE. The code in the example of this section similarly applies the patch scheme to a microscale space-time map ([Figure 3.8](#)), a map derived as a microscale space-time discretisation of Burgers' PDE. Then this example applies projective integration to simulate further in time.

### 3.7.1 Script code to simulate a microscale space-time map

This first part of the script implements the following patch/gap-tooth scheme (left-right arrows denote function recursion).

1. configPatches1
2. burgerBurst  $\leftrightarrow$  patchSmooth1  $\leftrightarrow$  burgersMap
3. process results

Establish global data struct for the microscale Burgers' map ([Section 3.7.3](#)) solved on  $2\pi$ -periodic domain, with eight patches, each patch of half-size ratio 0.2, with seven points within each patch, and say fourth-order interpolation provides edge-values that couple the patches.

```

50 global patches
51 nPatch = 8
52 ratio = 0.2
53 nSubP = 7
54 interpOrd = 4
55 Len = 2*pi
56 configPatches1(@burgersMap, [0 Len], nan, nPatch, interpOrd, ratio, nSubP);

```

Set an initial condition, and simulate a burst of the microscale space-time map over a time 0.2 using the function `burgerBurst()` ([Section 3.7.4](#)).

```

64 u0 = 0.4*(1+sin(patches.x))+0.1*randn(size(patches.x));
65 [ts,us] = burgersBurst(0,u0,0.4);

```

Plot the simulation. Use only the microscale values interior to the patches by setting the edges to `nan` in order to leave gaps.

```

73 figure(1),clf
74 xs = patches.x; xs([1 end],:) = nan;
75 mesh(ts,xs(:,us'))
76 xlabel('time t'), ylabel('space x'), zlabel('u(x,t)')
77 view(105,45)

```

Save the plot to file to form [Figure 3.8](#).

```
83 if0urCf2eps([mfilename 'MapU'])
```

### 3.7.2 Alternatively use projective integration

Around the microscale burst `burgerBurst()`, wrap the projective integration function `PIRK2()` of [Section 2.2](#). [Figure 3.9](#) shows the resultant macroscale prediction of the patch centre values on macroscale time-steps.

This second part of the script implements the following design.

1. `configPatches1` (done in [Section 3.7.1](#))
2. `PIRK2`  $\leftrightarrow$  `burgerBurst`  $\leftrightarrow$  `patchSmooth1`  $\leftrightarrow$  `burgersMap`
3. process results

Mark that edge-values of patches are not to be used in the projective extrapolation by setting initial values to NaN.

```
115 u0([1 end],:) = nan;
```

Set the desired macroscale time-steps, and microscale burst length over the time domain. Then projectively integrate in time using `PIRK2()` which is second-order accurate in the macroscale time-step.

```

124 ts = linspace(0,0.5,11);
125 bT = 3*(ratio*Len/nPatch/(nSubP/2-1))^2

```

Figure 3.9: macroscale space-time field  $u(x, t)$  in a basic projective integration of the patch scheme applied to the microscale Burgers' map.

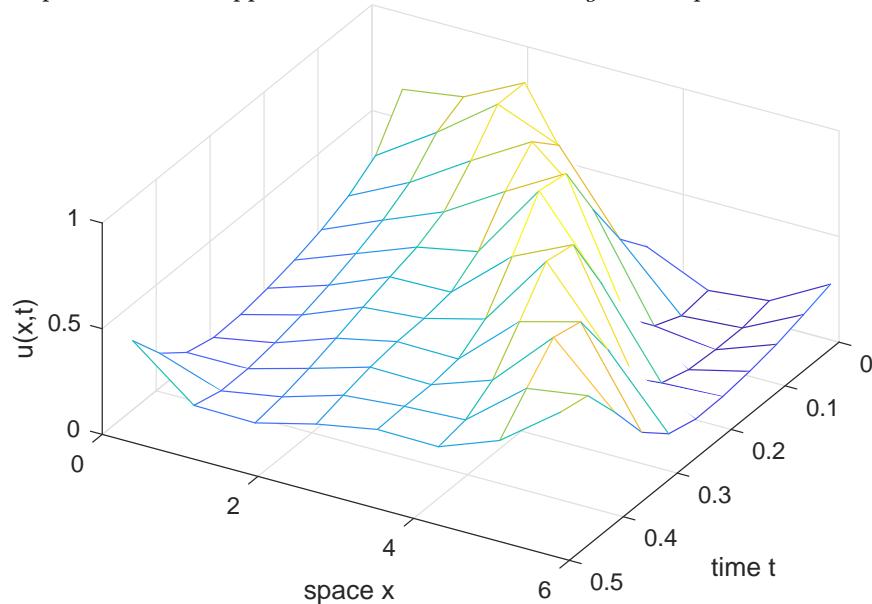
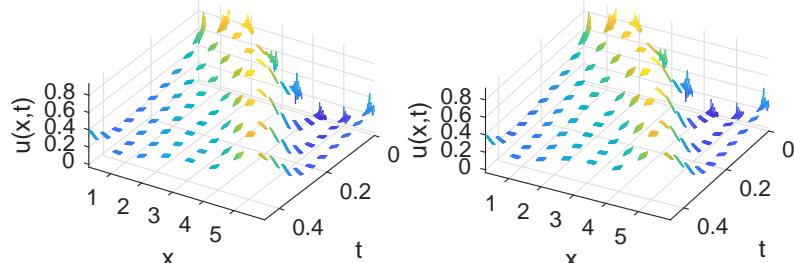


Figure 3.10: the microscale field  $u(x, t)$  during each of the microscale bursts used in the projective integration. View this stereo pair cross-eyed.



```

126 addpath('..../ProjInt')
127 [us,tss,uss] = PIRK2(@burgersBurst,ts,u0(:,bT));

```

Plot and save the macroscale predictions of the mid-patch values to give the macroscale mesh-surface of Figure 3.9 that shows a progressing wave solution.

```

135 figure(2),clf
136 midP = (nSubP+1)/2;
137 mesh(ts,xs(midP,:),us(:,midP:nSubP:end)')
138 xlabel('time t'), ylabel('space x'), zlabel('u(x,t)')
139 view(120,50)
140 ifUrCf2eps([mfilename 'U'])

```

Then plot and save the microscale mesh of the microscale bursts shown in Figure 3.10 (a stereo pair). The details of the fine microscale mesh are almost invisible.

```

155 figure(3),clf
156 for k = 1:2, subplot(2,2,k)

```

```

157     mesh(tss, xs(:,uss'))
158     ylabel('x'), xlabel('t'), zlabel('u(x,t)')
159     axis tight, view(126-4*k,50)
160 end
161 ifOutCf2eps([mfilename 'Micro'])

```

### 3.7.3 burgersMap(): discretise the PDE microscale

This function codes the microscale Euler integration map of the lattice differential equations inside the patches. Only the patch-interior values are mapped (`patchSmooth1()` overrides the edge-values anyway).

```

14 function u = burgersMap(t,u,x)
15 u = squeeze(u);
16 dx = diff(x(2:3));
17 dt = dx^2/2;
18 i = 2:size(u,1)-1;
19 u(i,:) = u(i,:)+dt*( diff(u,2)/dx^2 ...
20 -20*u(i,:).*(u(i+1,:)-u(i-1,:))/(2*dx) );
21 end

```

### 3.7.4 burgerBurst(): code a burst of the patch map

```
10 function [ts, us] = burgersBurst(ti, ui, bT)
```

First find and set the number of microscale time-steps.

```

16 global patches
17 dt = diff(patches.x(2:3))^2/2;
18 ndt = ceil(bT/dt -0.2);
19 ts = ti+(0:ndt)*dt;

```

Use `patchSmooth1()` (Section 3.3) to apply the microscale map over all time-steps in the burst. The `patchSmooth1()` interface provides the interpolated edge-values of each patch. Store the results in rows to be consistent with ODE and projective integrators.

```

29 us = nan(ndt+1,numel(ui));
30 us(1,:) = reshape(ui,1,[]);
31 for j = 1:ndt
32     ui = patchSmooth1(ts(j),ui);
33     us(j+1,:) = reshape(ui,1,[]);
34 end

```

Linearly interpolate (extrapolate) to get the field values at the precise final time of the burst. Then return.

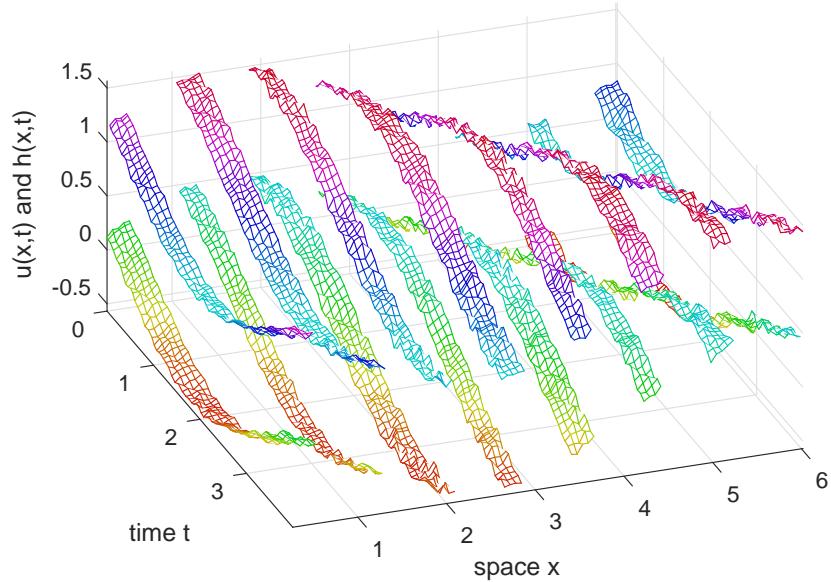
```

41 ts(ndt+1) = ti+bT;
42 us(ndt+1,:) = us(ndt,:)... + diff(ts(ndt:ndt+1))/dt*diff(us(ndt:ndt+1,:));
43
44 end

```

Fin.

*Figure 3.11: water depth  $h(x, t)$  (above) and velocity field  $u(x, t)$  (below) of the gap-tooth scheme applied to the ideal linear wave PDE (3.3) with  $f_1 = f_2 = 0$ . The microscale random component to the initial condition persists in the simulation—but the macroscale wave still propagates.*



### 3.8 waterWaveExample: simulate a water wave PDE on patches

#### Section contents

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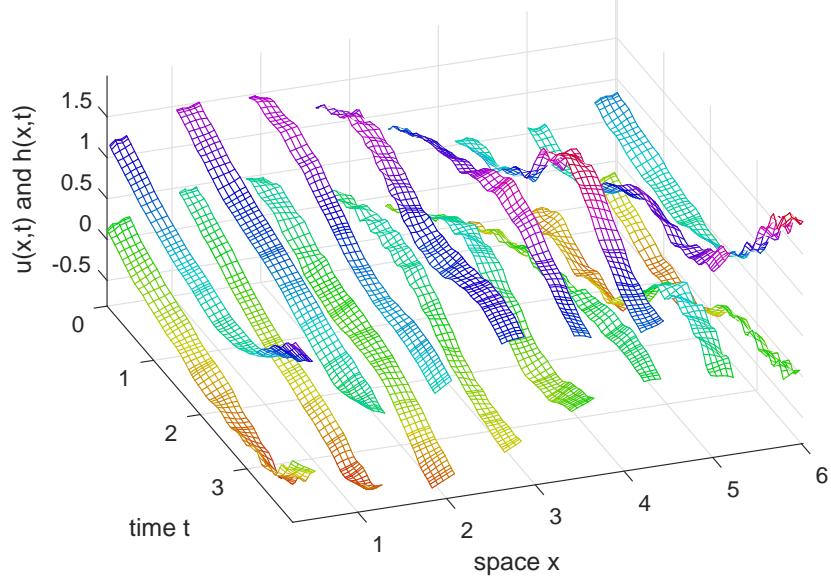
Figure 3.11 shows an example simulation in time generated by the patch scheme applied to an ideal wave PDE (Cao & Roberts 2013). The inter-patch coupling is realised by spectral interpolation of the mid-patch values to the patch edges.

This approach, based upon the differential equations coded in Section 3.8.2, may be adapted by a user to a wide variety of 1D wave and wave-like systems. For example, the differential equations of Section 3.8.3 that describe the nonlinear microscale simulator of the nonlinear shallow water wave PDE derived from the Smagorinski model of turbulent flow (Cao & Roberts 2012, 2016a).

Often, wave-like systems are written in terms of two conjugate variables, for example, position and momentum density, electric and magnetic fields, and water depth  $h(x, t)$  and mean longitudinal velocity  $u(x, t)$  as herein. The approach developed in this section applies to any wave-like system in the form

$$\frac{\partial h}{\partial t} = -c_1 \frac{\partial u}{\partial x} + f_1[h, u] \quad \text{and} \quad \frac{\partial u}{\partial t} = -c_2 \frac{\partial h}{\partial x} + f_2[h, u], \quad (3.3)$$

Figure 3.12: water depth  $h(x, t)$  (above) and velocity field  $u(x, t)$  (below) of the gap-tooth scheme applied to the Smagorinski shallow water wave PDEs (3.4). The microscale random initial component decays where the water speed is non-zero due to ‘turbulent’ dissipation.



where the brackets indicate that the two nonlinear functions  $f_1$  and  $f_2$  may involve various spatial derivatives of the fields  $h(x, t)$  and  $u(x, t)$ . For example, Section 3.8.3 encodes a nonlinear Smagorinski model of turbulent shallow water (Cao & Roberts 2012, 2016a, e.g.) along an inclined flat bed: let  $x$  measure position along the bed and in terms of fluid depth  $h(x, t)$  and depth-averaged longitudinal velocity  $u(x, t)$  the model PDEs are

$$\frac{\partial h}{\partial t} = -\frac{\partial(hu)}{\partial x}, \quad (3.4a)$$

$$\frac{\partial u}{\partial t} = 0.985 \left( \tan \theta - \frac{\partial h}{\partial x} \right) - 0.003 \frac{u|u|}{h} - 1.045u \frac{\partial u}{\partial x} + 0.26h|u| \frac{\partial^2 u}{\partial x^2}, \quad (3.4b)$$

where  $\tan \theta$  is the slope of the bed. The PDE (3.4a) represents conservation of the fluid. The momentum PDE (3.4b) represents the effects of turbulent bed drag  $u|u|/h$ , self-advection  $u\partial u/\partial x$ , nonlinear turbulent dispersion  $h|u|\partial^2 u/\partial x^2$ , and gravitational hydrostatic forcing ( $\tan \theta - \partial h/\partial x$ ). Figure 3.12 shows one simulation of this system—for the same initial condition as Figure 3.11.

For such wave-like systems, let’s implement both a staggered microscale grid and also staggered macroscale patches, as introduced by Cao & Roberts (2016b) in their Figures 3 and 4, respectively.

### 3.8.1 Script code to simulate wave systems

This example script implements the following patch/gap-tooth scheme (left-right arrows denote function recursion).

1. configPatches1, and add micro-information

2. `ode15s` ↔ `patchSmooth1` ↔ `idealWavePDE`
3. process results
4. `ode15s` ↔ `patchSmooth1` ↔ `waterWavePDE`
5. process results

Establish the global data struct `patches` for the PDES (3.3) (linearised) solved on  $2\pi$ -periodic domain, with eight patches, each patch of half-size ratio 0.2, with eleven micro-grid points within each patch, and spectral interpolation (-1) of ‘staggered’ macroscale patches to provide the edge-values of the inter-patch coupling conditions.

```

119 global patches
120 nPatch = 8
121 ratio = 0.2
122 nSubP = 11 %of the form 4*n-1
123 Len = 2*pi;
124 configPatches1(@idealWavePDE,[0 Len],nan,nPatch,-1,ratio,nSubP);

```

Identify which micro-grid points are  $h$  or  $u$  values on the staggered micro-grid. Also store the information in the struct `patches` for use by the time derivative function.

```

134 uPts = mod( (1:nSubP)' + (1:nPatch) ,2);
135 hPts = find(uPts==0);
136 uPts = find(uPts==1);
137 patches.hPts = hPts; patches.uPts = uPts;

```

Set an initial condition of a progressive wave, and check evaluation of the time derivative. The capital letter U denotes an array of values merged from both  $u$  and  $h$  fields on the staggered grids (here with some optional microscale wave noise).

```

148 U0 = nan(nSubP,nPatch);
149 U0(hPts) = 1+0.5*sin(patches.x(hPts));
150 U0(uPts) = 0+0.5*sin(patches.x(uPts));
151 U0 = U0+0.02*randn(nSubP,nPatch);

```

**Conventional integration in time** Integrate in time using standard MATLAB/Octave stiff integrators. Here do the two cases of the ideal wave and the water wave equations in the one loop.

```
161 for k = 1:2
```

When using `ode15s`/`lsode` we subsample the results because micro-grid scale waves do not dissipate and so the integrator takes very small time-steps for all time.

```

169 if ~exist('OCTAVE_VERSION','builtin')
170     [ts,Ucts] = ode15s( @patchSmooth1,[0 4],U0(:));
171     ts = ts(1:5:end);
172     Ucts = Ucts(1:5:end,:);
173 else % octave version is slower

```

```

174     [ts,Ucts] = odeOcts(@patchSmooth1,[0 4],U0(:));
175 end

```

Plot the simulation.

```

181 figure(k),clf
182 xs = squeeze(patches.x); xs([1 end],:) = nan;
183 mesh(ts,xs(hPts),Ucts(:,hPts)'),hold on
184 mesh(ts,xs(uPts),Ucts(:,uPts)'),hold off
185 xlabel('time t'), ylabel('space x'), zlabel('u(x,t) and h(x,t)')
186 axis tight, view(70,45)

```

Optionally save the plot to file.

```

192 if0urCf2eps([mfilename num2str(k) 'CtsUH'])

```

For the second time through the loop, change to the Smagorinski turbulence model (3.4) of shallow water flow, keeping other parameters and the initial condition the same.

```

202 patches.fun = @waterWavePDE;
203 end

```

**Could use projective integration** As yet a simple implementation appears to fail, so it needs more exploration and thought. End of the main script.

### 3.8.2 idealWavePDE(): ideal wave PDE

This function codes the staggered lattice equation inside the patches for the ideal wave PDE system  $h_t = -u_x$  and  $u_t = -h_x$ . Here code for a staggered micro-grid, index  $i$ , of staggered macroscale patches, index  $j$ : the array

$$U_{ij} = \begin{cases} u_{ij} & i + j \text{ even}, \\ h_{ij} & i + j \text{ odd}. \end{cases}$$

The output  $\mathbf{Ut}$  contains the merged time derivatives of the two staggered fields. So set the micro-grid spacing and reserve space for time derivatives.

```

24 function Ut = idealWavePDE(t,U,x)
25 global patches
26 dx = diff(x(2:3));
27 U = squeeze(U);
28 Ut = nan(size(U)); ht = Ut;

```

Compute the PDE derivatives only at interior micro-grid points of the patches.

```

35 i = 2:size(U,1)-1;

```

Here ‘wastefully’ compute time derivatives for both PDEs at all grid points—for simplicity—and then merge the staggered results. Since  $\dot{h}_{ij} \approx -(u_{i+1,j} - u_{i-1,j})/(2 \cdot dx) = -(U_{i+1,j} - U_{i-1,j})/(2 \cdot dx)$  as adding/subtracting one from the index of a  $h$ -value is the location of the neighbouring  $u$ -value on the staggered micro-grid.

```
47     ht(i,:) = -(U(i+1,:)-U(i-1,:))/(2*dx);
```

Since  $\dot{u}_{ij} \approx -(h_{i+1,j} - h_{i-1,j})/(2 \cdot dx) = -(U_{i+1,j} - U_{i-1,j})/(2 \cdot dx)$  as adding/subtracting one from the index of a  $u$ -value is the location of the neighbouring  $h$ -value on the staggered micro-grid.

```
57     Ut(i,:) = -(U(i+1,:)-U(i-1,:))/(2*dx);
```

Then overwrite the unwanted  $\dot{u}_{ij}$  with the corresponding wanted  $\dot{h}_{ij}$ .

```
64     Ut(patches.hPts) = ht(patches.hPts);
65 end
```

### 3.8.3 waterWavePDE(): water wave PDE

This function codes the staggered lattice equation inside the patches for the nonlinear wave-like PDE system (3.4). Also, regularise the absolute value appearing the the PDEs via the one-line function `rabs()`.

```
16 function Ut = waterWavePDE(t,U,x)
17 global patches
18 rabs = @(u) sqrt(1e-4 + u.^2);
```

As before, set the micro-grid spacing, reserve space for time derivatives, and index the patch-interior points of the micro-grid.

```
26 dx = diff(x(2:3));
27 U = squeeze(U);
28 Ut = nan(size(U)); ht = Ut;
29 i = 2:size(U,1)-1;
```

Need to estimate  $h$  at all the  $u$ -points, so into  $V$  use averages, and linear extrapolation to patch-edges.

```
37 ii = i(2:end-1);
38 V = Ut;
39 V(ii,:) = (U(ii+1,:)+U(ii-1,:))/2;
40 V(1:2,:) = 2*U(2:3,:)-V(3:4,:);
41 V(end-1:end,:) = 2*U(end-2:end-1,:)-V(end-3:end-2,:);
```

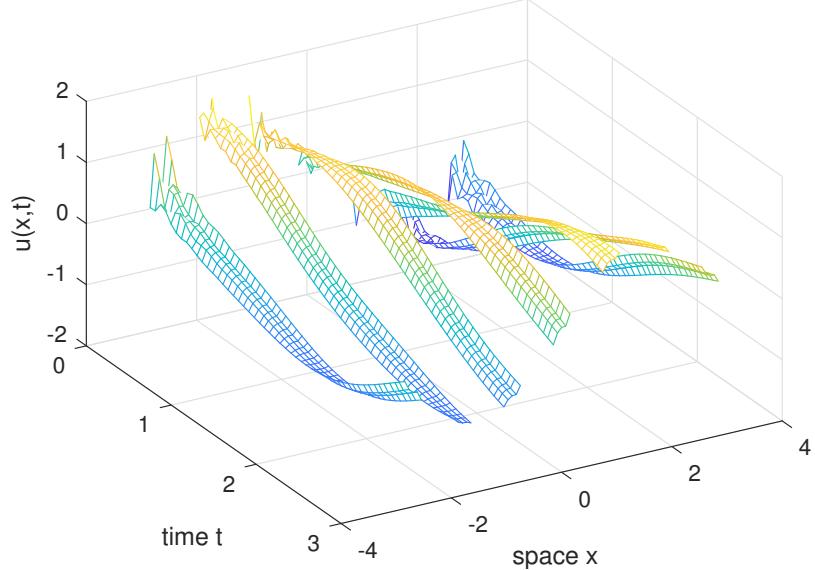
Then estimate  $\partial(hu)/\partial x$  from  $u$  and the interpolated  $h$  at the neighbouring micro-grid points.

```
48 ht(i,:) = -(U(i+1,:).*V(i+1,:)-U(i-1,:).*V(i-1,:))/(2*dx);
```

Correspondingly estimate the terms in the momentum PDE:  $u$ -values in  $U_i$  and  $V_{i\pm 1}$ ; and  $h$ -values in  $V_i$  and  $U_{i\pm 1}$ .

```
56 Ut(i,:) = -0.985*(U(i+1,:)-U(i-1,:))/(2*dx) ...
57 -0.003*U(i,:).*rabs(U(i,:)./V(i,:)) ...
58 -1.045*U(i,:).*(V(i+1,:)-V(i-1,:))/(2*dx) ...
59 +0.26*rabs(V(i,:).*U(i,:)).*(V(i+1,:)-2*U(i,:)+V(i-1,:))/dx^2/2;
```

*Figure 3.13: wave field  $u(x, t)$  of the gap-tooth scheme applied to the weakly damped wave (3.5). The microscale random component to the initial condition persists in the simulation until the weak damping smooths the sub-patch fluctuations—but the macroscale wave still propagates.*



where the mysterious division by two in the second derivative is due to using the averaged values of  $u$  in the estimate:

$$\begin{aligned} u_{xx} &\approx \frac{1}{4\delta^2}(u_{i-2} - 2u_i + u_{i+2}) \\ &= \frac{1}{4\delta^2}(u_{i-2} + u_i - 4u_i + u_i + u_{i+2}) \\ &= \frac{1}{2\delta^2} \left( \frac{u_{i-2} + u_i}{2} - 2u_i + \frac{u_i + u_{i+2}}{2} \right) \\ &= \frac{1}{2\delta^2} (\bar{u}_{i-1} - 2u_i + \bar{u}_{i+1}). \end{aligned}$$

Then overwrite the unwanted  $\dot{u}_{ij}$  with the corresponding wanted  $h_{ij}$ .

75        `Ut(patches.hPts) = ht(patches.hPts);`

76        `end`

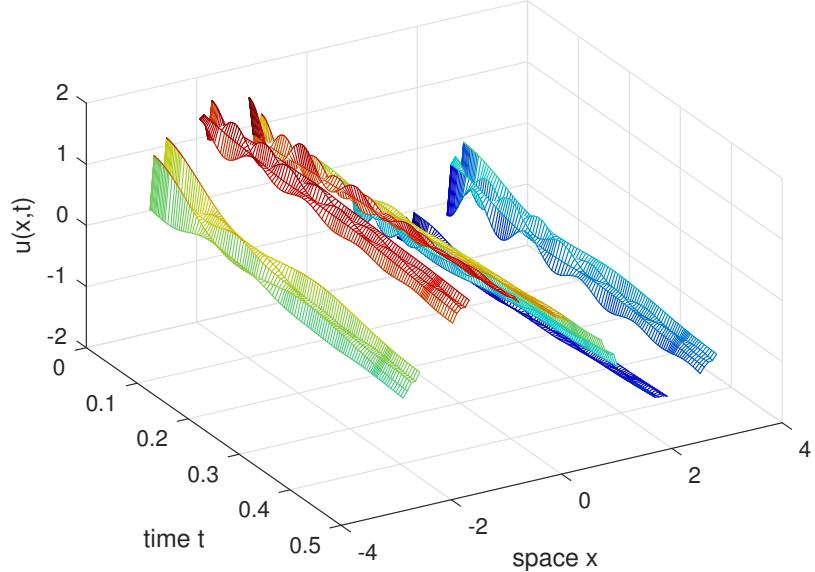
Fin.

### 3.9 homoWaveEdgy1: computational homogenisation of a 1D wave by simulation on small patches

**Figure 3.13** shows an example simulation in time generated by the patch scheme applied to macroscale wave propagation through a medium with microscale heterogeneity. The inter-patch coupling is realised by spectral interpolation of the patch's next-to-edge values to the patch opposite edges. This coupling preserves symmetry in many systems.

Often, wave-like systems are written in terms of two conjugate variables, for example, position and momentum density, electric and magnetic fields,

Figure 3.14: wave field  $u(x, t)$  of the gap-tooth scheme applied to the weakly damped wave (3.5). Over this shorter meso-time we see the macroscale wave emerging from the damped sub-patch fast waves.



and water depth and mean longitudinal velocity. Here suppose the spatial microscale lattice is at points  $x_i$ , with constant spacing  $dx$ . With dependent variables  $u_i(t)$  and  $v_i(t)$ , simulate the microscale lattice, weakly damped, wave system

$$\frac{\partial u_i}{\partial t} = v_i, \quad \frac{\partial v_i}{\partial t} = \frac{1}{dx^2} \delta[c_{i-1/2} \delta u_i] + \frac{0.02}{dx^2} \delta^2 v_i, \quad (3.5)$$

in terms of the centred difference operator  $\delta$ . The system has a microscale heterogeneity via the coefficients  $c_{i+1/2}$  which we assume to have some given known periodicity. Figure 3.13 shows one patch simulation of this system: observe the effects of the heterogeneity within each patch.

### 3.9.1 Script code to simulate heterogeneous wave systems

This example script implements the following patch/gap-tooth scheme (left-right arrows denote function recursion).

1. configPatches1, and add micro-information
2. ode15s  $\leftrightarrow$  patchSmooth1  $\leftrightarrow$  heteroWave
3. plot the simulation
4. use patchSmooth1 to check the Jacobian

First establish the microscale heterogeneity has micro-period `mPeriod` on the lattice, and random log-normal values, albeit normalised to have harmonic mean one. This normalisation then means that macroscale waves on a domain of length  $2\pi$  have near integer frequencies, 1, 2, 3, .... Then the heterogeneity is to be repeated `nPeriodsPatch` times within each patch.

```
91 mPeriod = 3
92 cHetr = exp(1*randn(mPeriod,1));
```

```

93 cHetr = cHetr*mean(1./cHetr) % normalise
94 nPeriodsPatch=1

```

Establish the global data struct `patches` for the microscale heterogeneous lattice wave system (3.5) solved on  $2\pi$ -periodic domain, with seven patches, here each patch of size ratio 0.25 from one side to the other, with five micro-grid points in each patch, and spectral interpolation (0) to provide the edge-values of the inter-patch coupling conditions. Setting `patches.EdgeInt` to one means the edge-values come from interpolating the opposite next-to-edge values of the patches (not the mid-patch values).

```

111 global patches
112 nPatch = 7
113 ratio = 0.25
114 nSubP = nPeriodsPatch*mPeriod+2
115 configPatches1(@heteroWave, [-pi pi], nan, nPatch ...
116     , 0, ratio, nSubP, 'EdgeInt', true, 'hetCoeffs', cHetr);

```

**Simulate** Set the initial conditions of a simulation to be that of a macroscopic progressive wave, via sin / cos, perturbed by significant random microscale noise, via `randn`.

```

128 uv0(:,1,1,:) = -sin(patches.x)+0.3*randn(nSubP,1,1,nPatch);
129 uv0(:,2,1,:) = +cos(patches.x)+0.3*randn(nSubP,1,1,nPatch);

```

Integrate for about half a wave period using standard stiff integrators (which do not work efficiently until after the fast waves have decayed).

```

137 if ~exist('OCTAVE_VERSION', 'builtin')
138     [ts,us] = ode15s(@patchSmooth1, [0 3], uv0(:));
139 else % octave version
140     [ts,us] = ode0cts(@patchSmooth1, [0 3], uv0(:));
141 end

```

**Plot space-time surface of the simulation** We want to see the edge values of the patches, so we adjoin a row of `nans` in between patches. For the field values (which are rows in `us`) we need to reshape, permute, interpolate to get edge values, pad with `nans`, and reshape again.

```

153 xs = squeeze(patches.x);
154 us = patchEdgeInt1( permute( reshape(us,length(ts) ...
155     ,nSubP,2,nPatch) ,[2 3 1 4]) );
156 xs(end+1,:) = nan; us(end+1,:,:,:) = nan;
157 us = reshape(permute(us,[1 4 2 3]),length(xs(:)),2,[]);

```

Now plot two space-time graphs. The first is every time step over a meso-time to see the oscillation and decay of the fast sub-patch waves. The second is subsampled surface over the macroscale duration of the simulation to show the propagation of the macroscale wave over the heterogeneous lattice.

```

169 for p=1:2
170     switch p

```

Table 3.2: example parameters and list of eigenvalues (every fourth one listed is sufficient due to symmetry): `nPatch = 7`, `ratio = 0.25`, `nSubP = 5`. The spectrum is satisfactory for weakly damped macroscale waves, and medium-damped microscale sub-patch fast waves.

```
cHetr =
    0.58459
    1.0026
    3.4253
eval =
    2.2701e-16 + 1.4225e-07i
    -0.013349 + 0.99941i
    -0.053324 + 1.9952i
    -0.11971 + 2.9838i
    -5.1527 + 19.554i
    -5.2679 + 19.695i
    -5.3383 + 19.779i
    -5.3619 + 36.632i
    -5.3722 + 36.632i
    -5.4026 + 36.631i
    -5.4514 + 36.63i

171 case 1, j=find(ts<0.5);
172 case 2, [~,j]=min(abs(ts-linspace(ts(1),ts(end),50)));
173 end
174 figure(p),clf
175 mesh(ts(j),xs(:,1),squeeze(us(:,1,j))), view(60,40)
176 xlabel('time t'), ylabel('space x'), zlabel('u(x,t)')
177 ifOurCf2eps([mfilename 'U' num2str(p)])
178 end
```

**Compute Jacobian and its spectrum** Form the Jacobian matrix, linear operator, by numerical construction about a zero field. Use `i` to store the indices of the micro-grid points that are interior to the patches and hence are the systems variables.

```
190 u0=repmat(0*patches.x,1,2); u0([1 end],:)=nan; u0=u0(:);
191 i=find(~isnan(u0));
192 nJ=length(i);
193 Jac=nan(nJ);
194 for j=1:nJ
195     u0(i)=((1:nJ)==j);
196     dudt=patchSmooth1(0,u0);
197     Jac(:,j)=dudt(i);
198 end
199 Jac(abs(Jac)<1e-12)=0;
```

Find the eigenvalues of the Jacobian, and list for inspection in Table 3.2.

```

231 [evecs,evals]=eig(Jac);
232 eval=sort(diag(evals));
233 slowestEvals=eval(2:4:4*nPatch)

```

End of the main script.

### 3.9.2 heteroWave(): wave in heterogeneous media with weak viscous damping

This function codes the lattice heterogeneous wave equation, with weak viscosity, inside the patches. For 3D input array  $u$  ( $u_{ij} = u(i,1,j)$  and  $v_{ij} = u(i,2,j)$ ) and 2D array  $x$  (obtained in full via edge-value interpolation of patchSmooth1, Section 3.3), computes the time derivatives at each point in the interior of a patch, output in  $ut$ :

$$\frac{\partial u_{ij}}{\partial t} = v_{ij}, \quad \frac{\partial v_{ij}}{\partial t} = \frac{1}{dx^2} \delta[c_{i-1/2} \delta u_{ij}] + \frac{0.02}{dx^2} \delta^2 v_{ij}.$$

The column vector (or possibly array) of diffusion coefficients  $c_i$  have previously been stored in struct `patches`.

```

27 function ut = heteroWave(t,u,x)
28   global patches
29   u = squeeze(u);
30   dx = diff(x(2:3)); % space step
31   i = 2:size(u,1)-1; % interior points in a patch
32   ut = nan(size(u)); % preallocate output array
33   ut(i,1,:) = u(i,2,:); % du/dt=v then dvdt=
34   ut(i,2,:) = diff(patches.cs.*diff(u(:,1,:))/dx^2 ...
35     +0.02*diff(u(:,2,:),2)/dx^2;
36 end% function

```

Fin.

## 3.10 waveEdgy1: simulate a 1D, first-order, wave PDE on small patches

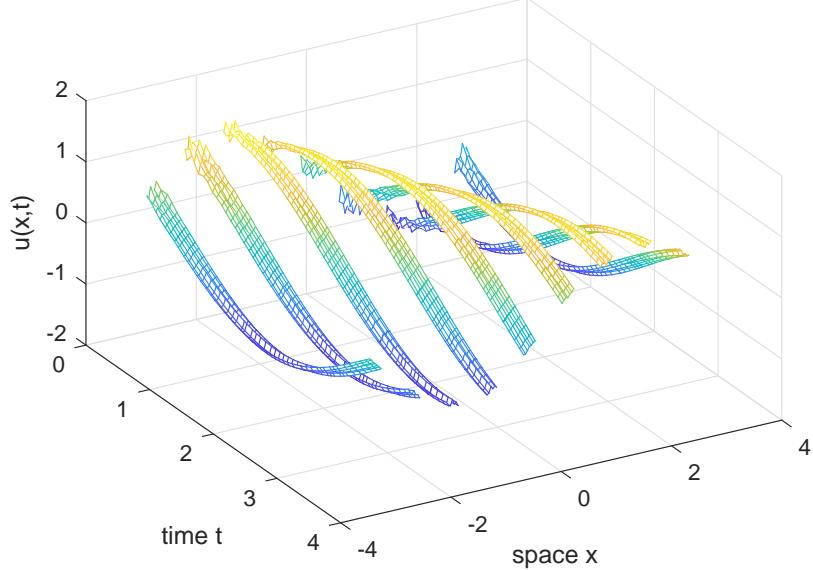
### *Section contents*

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Figure 3.15 shows an example simulation in time generated by the patch scheme applied to macroscale diffusion propagation through a medium with microscale heterogeneity. The inter-patch coupling is realised by spectral interpolation of the patch's next-to-edge values to the patch opposite edges. Such coupling preserves symmetry in many systems, and in this first-order wave PDE preserves skew-symmetry.

The first-order wave-like PDE is  $u_t = -\frac{1}{2}(cu)_x - \frac{1}{2}cu_x$ , which when  $c$  is constant becomes the canonical first-order wave PDE  $u_t = -cu_x$ . The differential operator on the right-hand side is skew-symmetric: letting  $\mathcal{D} = -\frac{1}{2}(c\cdot)_x -$

*Figure 3.15: wave field  $u(x, t)$  of the gap-tooth scheme applied to the wave (3.6). The microscale random component to the initial condition, the sub-patch fluctuations, decays, leaving the emergent macroscale wave in the heterogeneous media. This simulation uses nine patches of ‘large’ size ratio 0.25 for visibility.*



$\frac{1}{2}c\partial_x$  then  $\int v\mathcal{D}u \, dx = \int -v(cu)_x - v\frac{1}{2}cu_x \, dx = -\int \frac{1}{2}v_x cu + \frac{1}{2}(vc)_x u \, dx = -\int u\mathcal{D}v \, dx$ .

To discretise in space, suppose the spatial microscale lattice is at points  $x_i$ , with constant spacing  $d$ . With dependent variables  $u_i(t)$ , simulate the microscale lattice, in terms of the centred difference  $\delta$  and mean  $\mu$ , wave system

$$\frac{du_i}{dt} = -\frac{1}{2d} [\delta(c_i\mu u_i) + \mu(c_i\delta u_i)] = -\frac{1}{2d} \left[ c_{i+\frac{1}{2}} u_{i+1} - c_{i-\frac{1}{2}} u_{i-1} \right]. \quad (3.6)$$

**Figure 3.15** shows one patch simulation of this space-time system, except it also includes a  $\nu = 0.001$  small ‘viscous’ dissipation,  $\nu\delta^2 u_i/d^2$ , to weakly damp the microscale, sub-patch, fast waves.

### 3.10.1 Script code to simulate heterogeneous wave systems

This example script implements the following patch/gap-tooth scheme (left-right arrows denote function recursion).

1. configPatches1, and add micro-information
2. ode15s  $\leftrightarrow$  patchSmooth1  $\leftrightarrow$  waveFirst
3. plot the simulation
4. use patchSmooth1 to explore the Jacobian

First establish the microscale heterogeneity has (odd-valued) micro-period `mPeriod` on the lattice, and random log-normal values, normalised. This normalisation means that macroscale wave on a domain of length  $2\pi$  should have nearly integer frequencies,  $0, 1, 2, \dots$ —except that the normalisation is exact only for periods 3 and 5. Then the heterogeneity is repeated `nPeriodsPatch` times within each patch.

```

89 mPeriod = 5 % needs to be odd for a wave
90 cHetr = exp(0.1*randn(mPeriod,1)); % 0.3 appears max reasonable
91 if mPeriod==3,
92     cHetr=cHetr*mean(cHetr.^2)/prod(cHetr) % normalise
93 elseif mPeriod==5,
94     cHetr=cHetr*mean(cHetr.^2.*cHetr([3 4 5 1 2]).^2)/prod(cHetr)
95 else cHetr=cHetr*mean(1./cHetr) % roughly normalise
96 end
97 nPeriodsPatch=1 % also needs to be odd

```

Establish the global data struct `patches` for the microscale heterogeneous lattice wave system (3.6) solved on  $2\pi$ -periodic domain, with nine patches, here each patch of size ratio 0.25 from one side to the other, with five micro-grid points in each patch, and quartic interpolation (4) to provide the edge-values via the inter-patch coupling conditions. Setting `EdgyInt` to

- true means the edge-values come from interpolating the opposite next-to-edge values of the patches (not the mid-patch values); whereas
- false means the time integration appears OK, but the Jacobian is, correctly, not skew-symmetric for this case of interpolating mid-patch values.

```

120 global patches
121 nPatch = 9
122 ratio = 0.25
123 EdgyInt=true
124 nPeriodsPatch = (2-EdgyInt)*nPeriodsPatch;
125 nSubP = nPeriodsPatch*mPeriod+1+EdgyInt
126 configPatches1(@waveFirst, [-pi pi], nan, nPatch, 4 ...
127     , ratio, nSubP, 'EdgyInt', EdgyInt, 'hetCoeffs', cHetr);

```

Specify the weak damping of the sub-patch, fast, microscale waves.

```
135 patches.nu=0.003;
```

**Simulate** Set the initial conditions of a simulation to be that of a sine wave perturbed by significant random microscale noise, via `randn`.

```
145 xs=squeeze(patches.x);
146 u0 = -sin(xs)+0.1*randn(nSubP,nPatch);
```

Integrate using standard stiff integrators.

```
152 if ~exist('OCTAVE_VERSION', 'builtin')
153     [ts,us] = ode23(@patchSmooth1, [0 3.5], u0(:));
154 else % octave version
155     [ts,us] = odeOcts(@patchSmooth1, [0 0.5], u0(:));
156 end
```

**Plot space-time surface of the simulation** Let's see the edge values of the patches. For the field values (which are rows in `us`) we need to reshape,

permute, interpolate with `patchEdgeInt1` to get edge values, pad with `nans`, and reshape again.

```

167 xs(end+1,:) = nan;
168 us = patchEdgeInt1( permute( reshape(us ...
169     ,length(ts),nSubP,nPatch) ,[2 1 3]) );
170 us(end+1,:,:,:)=nan;
171 us=reshape(permute(squeeze(us),[1 3 2]),[],length(ts));

```

Now plot a space-time graph. Subsample the data over the macroscale duration of the simulation to show the propagation of the macroscale wave over the heterogeneous lattice.

```

181 [~,j]=min(abs(ts-linspace(ts(1),ts(end),50)));
182 figure(1),clf
183 mesh(ts(j),xs(:,us(:,j)), view(60,40)
184 xlabel('time t'), ylabel('space x'), zlabel('u(x,t)')
185 if0urCf2eps([mfilename 'U' num2str(2)])

```

**Compute Jacobian and its spectrum** Let's explore the Jacobian dynamics for a range of orders of interpolation, all for the same patch design and heterogeneity. Here use a smaller ratio, and more patches, as we do not plot. Set the weak damping to zero so we explore the ideal case of the wave system (3.6).

```

198 ratio=0.01
199 nPatch=19
200 leadingFreqs=[];
201 for ord=0:2:8
202     ordInterp=ord
203     configPatches1(@waveFirst,[-pi pi],nan,nPatch,ord ...
204         ,ratio,nSubP,'EdgyInt',EdgyInt,'hetCoeffs',cHetr);
205     patches.nu=0;

```

Form the Jacobian matrix, linear operator, by numerical construction about a zero field. Use `i` to store the indices of the micro-grid points that are interior to the patches and hence are the system variables.

```

215 u0=0*patches.x; u0([1 end],:)=nan; u0=u0(:);
216 i=find(~isnan(u0));
217 nJ=length(i);
218 Jac=nan(nJ);
219 for j=1:nJ
220     u0(i)=((1:nJ)==j);
221     dudt=patchSmooth1(0,u0);
222     Jac(:,j)=dudt(i);
223 end
224 nonSkewSymmetric=norm(Jac+Jac')
225 assert(nonSkewSymmetric<1e-10,'failed skew-symmetry')
226 Jac(abs(Jac)<1e-12)=0;

```

Find the eigenvalues of the Jacobian, and list for inspection in [Table 3.3](#):

Table 3.3: example parameters and list of eigenvalues (every second one listed is sufficient due to symmetry): `nPatch = 19`, `ratio = 0.03`, `nSubP = 7`. The columns are for various `ordCC`, in order: 0, spectral interpolation; 2, quadratic; 4, quartic; and 6, sixth order. Rows are ordered in the effective wavenumber of the corresponding eigenvector (the number of zero crossings).

```
cHetr =
    0.6614
    1.5758
    1.8645
    1.4600
    0.8486
leadingFreqs =
    0      0      0      0
    1.0000  0.9819  0.9996  1.0000
    2.0000  1.8574  1.9879  1.9989
    2.9999  2.5318  2.9138  2.9830
    3.9997  2.9320  3.6688  3.8910
    4.9995  3.0146  4.1015  4.5720
    5.9991  2.7705  4.0640  4.7890
    6.9985  2.2261  3.4699  4.3042
    7.9978  1.4402  2.3421  3.0200
    8.9969  0.4981  0.8278  1.0897
    698.0262 698.0852 698.0370 698.0283
    698.1548 698.1728 698.1563 698.1549
```

the spectral interpolation is effectively exact for the macroscale; quadratic interpolation is usually qualitatively good; quartic interpolation appears to be the lowest order for quantitative accuracy.

```
268     [evecs,evals]=eig(Jac);
269     maxRealPartEvals=max(abs(real(diag(evals))))
270     assert(maxRealPartEvals<1e-10,'failed real-part zero')
271     freqs=imag(diag(evals));
```

Use a count of zero crossings in the corresponding eigenvector in order to try to sort on the spatial wavenumber.

```
279     [~,j]=sort(sum(abs(diff(sign(real(evecs))))));
280     leadingFreqs=[leadingFreqs -freqs(j(1:2:nPatch+4))];
```

End of the for-loop over orders of interpolation, and display the spectra.

```
287 end
288 disp(' spectral   quadratic   quartic   sixth-order ...')
289 leadingFreqs = leadingFreqs
```

End of the main script.

### 3.10.2 waveFirst(): first-order wave PDE

This function codes a lattice, first-order, heterogeneous, wave PDE inside patches. Optionally adds some viscous dissipation. For 2D input arrays  $u$  and  $x$  (via edge-value interpolation of `patchSmooth1`, [Section 3.3](#)), computes the time derivative [\(3.6\)](#) at each point in the interior of a patch, output in  $ut$ .

```

17 function ut = waveFirst(t,u,x)
18     global patches
19     u=squeeze(u);
20     dx = diff(x(2:3)); % space step
21     i = 2:size(u,1)-1; % interior points in a patch
22     ut = nan(size(u)); % preallocate output array
23     ut(i,:) = -(patches.cs(i).*u(i+1,:)) ...
24             -patches.cs(i-1).*u(i-1,:))/(2*dx) ...
25             +patches.nu*diff(u,2)/dx^2;
26 end% function

```

Fin.

## 3.11 configPatches2(): configures spatial patches in 2D

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### 3.11.1 Introduction

Makes the struct `patches` for use by the patch/gap-tooth time derivative/step function `patchSmooth2()`. [Section 3.11.2](#) lists an example of its use.

```

23 function configPatches2(fun,Xlim,BCs,nPatch,ordCC,ratio,nSubP ...
24     ,varargin)
25     global patches

```

**Input** If invoked with no input arguments, then executes an example of simulating a nonlinear diffusion PDE relevant to the lubrication flow of a thin layer of fluid—see [Section 3.11.2](#) for the example code.

- `fun` is the name of the user function, `fun(t,u,x,y)`, that computes time-derivatives (or time-steps) of quantities on the 2D micro-grid within all the 2D patches.

- `Xlim` array/vector giving the macro-space domain of the computation: patches are distributed equi-spaced over the interior of the rectangle  $[Xlim(1), Xlim(2)] \times [Xlim(3), Xlim(4)]$ : if `Xlim` is of length two, then the domain is the square domain of the same interval in both directions.
- `BCs` eventually and somehow will define the macroscale boundary conditions. Currently, `BCs` is ignored and the system is assumed macro-periodic in the domain.
- `nPatch` determines the number of equi-spaced spaced patches: if scalar, then use the same number of patches in both directions, otherwise `nPatch(1:2)` gives the number of patches in each direction.
- `ordCC` is the ‘order’ of interpolation for inter-patch coupling across empty space of the macroscale patch values to the edge-values of the patches: currently must be 0, 2, 4, . . . ; where 0 gives spectral interpolation.
- `ratio` (real) is the ratio of (depending upon `EdgyInt`) either the half-width or full-width of a patch to the spacing of the patch mid-points. So either  $\text{ratio} = \frac{1}{2}$  means the patches abut and `ratio` = 1 is overlapping patches as in holistic discretisation, or `ratio` = 1 means the patches abut. Small `ratio` should greatly reduce computational time. If scalar, then use the same ratio in both directions, otherwise `ratio(1:2)` gives the ratio in each direction.
- `nSubP` is the number of equi-spaced microscale lattice points in each patch: if scalar, then use the same number in both directions, otherwise `nSubP(1:2)` gives the number in each direction. If not using `EdgyInt`, then must be odd so that there is a central micro-grid point in each patch.
- ‘`nEdge`’ (not yet implemented), *optional*, default=1, for each patch, the number of edge values set by interpolation at the edge regions of each patch. The default is one (suitable for microscale lattices with only nearest neighbour interactions).
- ‘`EdgyInt`’, true/false, *optional*, default=false. If true, then interpolate to left/right/top/bottom edge-values from right/left/bottom/top next-to-edge values.
- ‘`nEnsem`’, *optional-experimental*, default one, but if more, then an ensemble over this number of realisations.
- ‘`hetCoeffs`’, *optional*, default empty. Supply an array of microscale heterogeneous coefficients to be used by the given microscale `fun` in each patch. Say the given array `cs` is of size  $m_x \times m_y \times n_c$ , where  $n_c$  is the number of different sets of coefficients. For example, in heterogeneous diffusion,  $n_c = 2$  for the diffusivities in the *two* different spatial directions. The coefficients are to be the same for each and every patch; however, macroscale variations are catered for by the  $n_c$  coefficients being  $n_c$  parameters in some macroscale formula.
  - If `nEnsem` = 1, then the array of coefficients is just tiled across the patch size to fill up each patch, starting from the (1, 1)-element.

- If  $nEnsem > 1$  (value immaterial), then reset  $nEnsem := m_x \cdot m_y$  and construct an ensemble of all  $m_x \cdot m_y$  phase-shifts of the coefficients. In this scenario, the inter-patch coupling couples different members in the ensemble. When `EdgyInt` is true, and when the coefficients are diffusivities/elasticities in  $x$  and  $y$  directions, respectively, then this coupling cunningly preserves symmetry.

**Output** The *global* struct `patches` is created and set with the following components.

- `.fun` is the name of the user's function `fun(t,u,x,y)` that computes the time derivatives (or steps) on the patchy lattice.
- `.ordCC` is the specified order of inter-patch coupling.
- `.stag` is true for interpolation using only odd neighbouring patches as for staggered grids, and false for the usual case of all neighbour coupling—not yet implemented.
- `.Cwtsr` and `.Cwtsl` are the  $\text{ordCC} \times 2$ -array of weights for the inter-patch interpolation onto the right/top and left/bottom edges (respectively) with patch:macroscale ratio as specified.
- `.x` (6D) is  $nSubP(1) \times 1 \times 1 \times 1 \times nPatch(1) \times 1$  array of the regular spatial locations  $x_{ij}$  of the microscale grid points in every patch.
- `.y` (6D) is  $1 \times nSubP(2) \times 1 \times 1 \times 1 \times nPatch(2)$  array of the regular spatial locations  $y_{ij}$  of the microscale grid points in every patch.
- `.nEdge` is, for each patch, the number of edge values set by interpolation at the edge regions of each patch.
- `.le`, `.ri`, `.bo`, `.to` determine inter-patch coupling of members in an ensemble. Each a column vector of length `nEnsem`.
- `.cs` either
  - [] 0D, or
  - if  $nEnsem = 1$ ,  $(nSubP(1) - 1) \times (nSubP(2) - 1) \times n_c$  3D array of microscale heterogeneous coefficients, or
  - if  $nEnsem > 1$ ,  $(nSubP(1) - 1) \times (nSubP(2) - 1) \times n_c \times m_x m_y$  4D array of  $m_x m_y$  ensemble of phase-shifts of the microscale heterogeneous coefficients.

### 3.11.2 If no arguments, then execute an example

```
193  if nargin==0
```

The code here shows one way to get started: a user's script may have the following three steps (arrows indicate function recursion).

1. configPatches2
2. ode23 integrator  $\leftrightarrow$  patchSmooth2  $\leftrightarrow$  user's PDE

### 3. process results

Establish global patch data struct to interface with a function coding a nonlinear ‘diffusion’ PDE: to be solved on  $6 \times 4$ -periodic domain, with  $9 \times 7$  patches, spectral interpolation (0) couples the patches, each patch of half-size ratio 0.4 (relatively large for visualisation), and with  $5 \times 5$  points forming each patch. [Roberts et al. \(2014\)](#) established that this scheme is consistent with the PDE (as the patch spacing decreases).

```
215 nSubP = 5;
216 configPatches2(@nonDiffPDE, [-3 3 -2 2], nan, [9 7] ...
217 , 0, 0.4, nSubP , 'EdgyInt', false);
```

Set a perturbed-Gaussian initial condition using auto-replication of the spatial grid.

```
224 u0 = exp(-patches.x.^2-patches.y.^2);
225 u0 = u0.* (0.9+0.1*rand(size(u0)));
```

Initiate a plot of the simulation using only the microscale values interior to the patches: set  $x$  and  $y$ -edges to `nan` to leave the gaps between patches.

```
233 figure(1), clf, colormap(hsv)
234 x = squeeze(patches.x); y = squeeze(patches.y);
235 if 1, x([1 end], :) = nan; y([1 end], :) = nan; end
```

Start by showing the initial conditions of [Figure 3.16](#) while the simulation computes.

```
242 u = reshape(permute(squeeze(u0) ...
243 ,[1 3 2 4]), [numel(x) numel(y)]);
244 hsurf = surf(x(:, ),y(:, ),u');
245 axis([-3 3 -3 3 -0.03 1]), view(60,40)
246 legend('time = 0.00', 'Location', 'north')
247 xlabel('space x'), ylabel('space y'), zlabel('u(x,y)')
248 colormap(hsv)
```

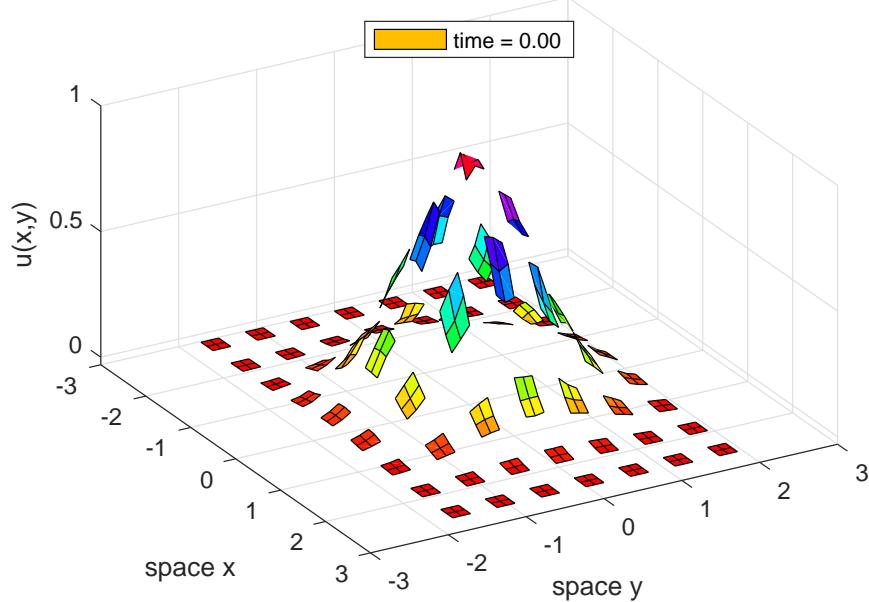
Optionally save the initial condition to graphic file for [Figure 3.16](#).

```
255 ifOurCf2eps([mfilename 'ic'])
```

Integrate in time using standard functions. In Matlab `ode15s` would be natural as the patch scheme is naturally stiff, but `ode23` is quicker. Ask for output at non-uniform times as the diffusion slows.

```
272 disp('Wait to simulate nonlinear diffusion h_t=(h^3)_xx+(h^3)_yy')
273 drawnow
274 if ~exist('OCTAVE_VERSION', 'builtin')
275     [ts,us] = ode23(@patchSmooth2,linspace(0,2).^2,u0(:));
276 else % octave version is quite slow for me
277     lsode_options('absolute tolerance',1e-4);
278     lsode_options('relative tolerance',1e-4);
279     [ts,us] = odeOctave(@patchSmooth2,[0 1],u0(:));
280 end
```

Figure 3.16: initial field  $u(x, y, t)$  at time  $t = 0$  of the patch scheme applied to a nonlinear ‘diffusion’ PDE: Figure 3.17 plots the computed field at time  $t = 3$ .



Animate the computed simulation to end with Figure 3.17. Use `patchEdgeInt2` to interpolate patch-edge values (even if not drawn).

```

288 for i = 1:length(ts)
289     u = patchEdgeInt2(us(i,:));
290     u = reshape(permute(squeeze(u) ...
291                 ,[1 3 2 4]), [numel(x) numel(y)]);
292     set(hsurf,'ZData', u');
293     legend(['time = ' num2str(ts(i),'%4.2f')])
294     pause(0.1)
295 end
296 ifOrCf2eps([mfilename 't3'])

```

Upon finishing execution of the example, exit this function.

```

311 return
312 end%if no arguments

```

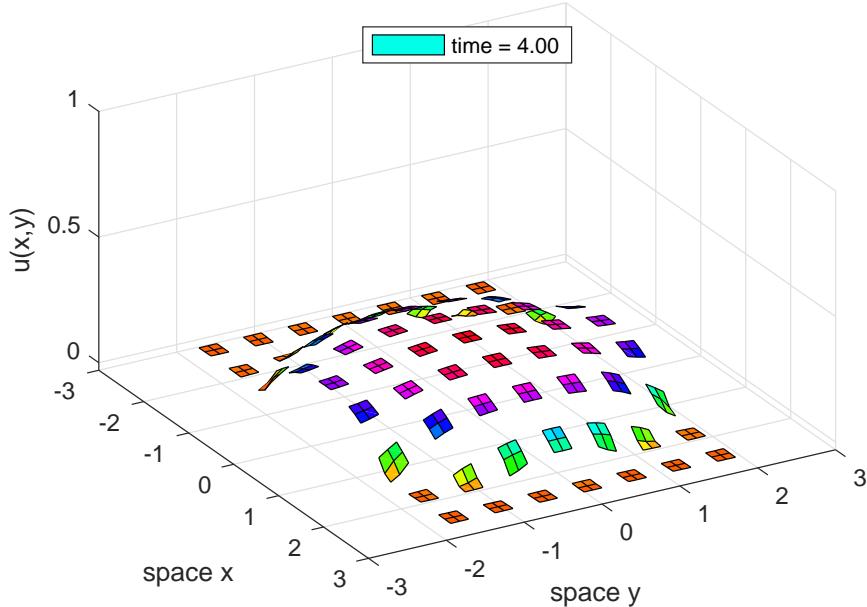
**Example of nonlinear diffusion PDE inside patches** As a microscale discretisation of  $u_t = \nabla^2(u^3)$ , code  $\dot{u}_{ijkl} = \frac{1}{\delta x^2}(u_{i+1,j,k,l}^3 - 2u_{i,j,k,l}^3 + u_{i-1,j,k,l}^3) + \frac{1}{\delta y^2}(u_{i,j+1,k,l}^3 - 2u_{i,j,k,l}^3 + u_{i,j-1,k,l}^3)$ .

```

13 function ut = nonDiffPDE(t,u,x,y)
14     u = squeeze(u); % reduce to 4D
15     dx = diff(x(1:2)); dy = diff(y(1:2)); % microgrid spacing
16     i = 2:size(u,1)-1; j = 2:size(u,2)-1; % interior patch points
17     ut = nan(size(u)); % preallocate storage
18     ut(i,j,:,:,:) = diff(u(:,:,j,:,:).^3,2,1)/dx.^2 ...

```

Figure 3.17: field  $u(x, y, t)$  at time  $t = 3$  of the patch scheme applied to a nonlinear ‘diffusion’ PDE with initial condition in Figure 3.16.



```

19           +diff(u(i,:,:,:).^3,2,2)/dy^2;
20   end

```

### 3.11.3 Parse input arguments and defaults

```

326 p = inputParser;
327 fnValidation = @(f) isa(f, 'function_handle');
328 addRequired(p, 'fun', fnValidation); % is there test for a function name??
329 addRequired(p, 'Xlim', @isnumeric);
330 addRequired(p, 'BCs'); % nothing yet decided
331 addRequired(p, 'nPatch', @isnumeric);
332 addRequired(p, 'ordCC', @isnumeric);
333 addRequired(p, 'ratio', @isnumeric);
334 addRequired(p, 'nSubP', @isnumeric);
335 addParameter(p, 'nEdge', 1, @isnumeric);
336 addParameter(p, 'EdgyInt', false, @islogical);
337 addParameter(p, 'nEnsem', 1, @isnumeric);
338 addParameter(p, 'hetCoeffs', [], @isnumeric);
339 %addParameter(p, 'nCore', 1, @isnumeric); % not yet implemented
340 parse(p, fun, Xlim, BCs, nPatch, ordCC, ratio, nSubP, varargin{:});

```

Set the optional parameters.

```

346 patches.nEdge = p.Results.nEdge;
347 patches.EdgyInt = p.Results.EdgyInt;
348 patches.nEnsem = p.Results.nEnsem;
349 cs = p.Results.hetCoeffs;
350 %patches.nCore = p.Results.nCore;

```

Initially duplicate parameters for both space dimensions as needed.

```
358 if numel(Xlim)==2, Xlim = repmat(Xlim,1,2); end
359 if numel(nPatch)==1, nPatch = repmat(nPatch,1,2); end
360 if numel(ratio)==1, ratio = repmat(ratio,1,2); end
361 if numel(nSubP)==1, nSubP = repmat(nSubP,1,2); end
```

Check parameters.

```
368 assert(patches.nEdge==1 ...
369     , 'multi-edge-value interp not yet implemented')
370 assert(all(2*patches.nEdge<nSubP) ...
371     , 'too many edge values requested')
372 %if patches.nCore>1
373 %    warning('nCore>1 not yet tested in this version')
374 %    end
```

### 3.11.4 The code to make patches

First, store the pointer to the time derivative function in the struct.

```
388 patches.fun = fun;
```

Second, store the order of interpolation that is to provide the values for the inter-patch coupling conditions. Spectral coupling is `ordCC` of 0 or  $-1$ .

```
397 assert((ordCC>=-1) & (floor(ordCC)==ordCC), ...
398     'ordCC out of allowed range integer>=-1')
```

For odd `ordCC` do interpolation based upon odd neighbouring patches as is useful for staggered grids.

```
405 patches.stag = mod(ordCC,2);
406 assert(patches.stag==0,'staggered not yet implemented??')
407 ordCC = ordCC+patches.stag;
408 patches.ordCC = ordCC;
```

Check for staggered grid and periodic case.

```
414 if patches.stag, assert(mod(nPatch,2)==0, ...
415     'Require an even number of patches for staggered grid')
416 end
```

Might as well precompute the weightings for the interpolation of field values for coupling. (Could sometime extend to coupling via derivative values.)

```
425 ratio = reshape(ratio,1,2); % force to be row vector
426 if ordCC>0, patchCwts(ratio,ordCC,patches.stag), end
```

Third, set the centre of the patches in a the macroscale grid of patches assuming periodic macroscale domain.

```
435 X = linspace(Xlim(1),Xlim(2),nPatch(1)+1);
436 X = X(1:nPatch(1))+diff(X)/2;
437 DX = X(2)-X(1);
438 Y = linspace(Xlim(3),Xlim(4),nPatch(2)+1);
```

```

439 Y = Y(1:nPatch(2))+diff(Y)/2;
440 DY = Y(2)-Y(1);

Construct the microscale in each patch, assuming Dirichlet patch edges, and a
half-patch length of ratio(1)·DX and ratio(2)·DY, unless patches.EdgyInt
is set in which case the patches are of length ratio*DX+dx and ratio*DY+dy.

451 nSubP = reshape(nSubP,1,2); % force to be row vector
452 assert(patches.EdgyInt | all(mod(nSubP,2)==1), ...
453     'configPatches2: nSubP must be odd')
454 i0 = (nSubP(1)+1)/2;
455 if ~patches.EdgyInt, dx = ratio(1)*DX/(i0-1);
456 else
457     dx = ratio(1)*DX/(nSubP(1)-2);
458 patches.x = dx*(-i0+1:i0-1)'+X; % micro-grid
459 patches.x = reshape(patches.x,nSubP(1),1,1,1,nPatch(1),1);
460 i0 = (nSubP(2)+1)/2;
461 if ~patches.EdgyInt, dy = ratio(2)*DY/(i0-1);
462 else
463     dy = ratio(2)*DY/(nSubP(2)-2);
464 patches.y = dy*(-i0+1:i0-1)'+Y; % micro-grid
465 patches.y = reshape(patches.y,1,nSubP(2),1,1,1,nPatch(2));

```

### 3.11.5 Set ensemble inter-patch communication

For `EdgyInt` or centre interpolation respectively, the right-edge/centre realisations `1:nEnsem` are to interpolate to left-edge `le`, and the left-edge/centre realisations `1:nEnsem` are to interpolate to `re`. `re` and `li` are ‘transposes’ of each other as `re(li)=le(ri)` are both `1:nEnsem`. Similarly for bottom-edge/centre interpolation to top-edge via `to` and top-edge/centre interpolation to bottom-edge via `bo`.

The default is nothing shifty. This setting reduces the number of if-statements in function `patchEdgeInt2()`.

```

488 nE = patches.nEnsem;
489 patches.le = 1:nE; patches.ri = 1:nE;
490 patches.bo = 1:nE; patches.to = 1:nE;

```

However, if heterogeneous coefficients are supplied via `hetCoeffs`, then do some non-trivial replications. First, get microscale periods, patch size, and replicate many times in order to subsequently sub-sample: `nSubP` times should be enough.

```

500 if ~isempty(cs)
501     [mx,my,nc] = size(cs);
502     nx = nSubP(1); ny = nSubP(2);
503     cs = repmat(cs,nSubP);

```

If only one member of the ensemble is required, then sub-sample to patch size, and store coefficients in `patches` as is.

```

511     if nE==1, patches.cs = cs(1:mx-1,1:my-1,:); else

```

But for `nEnsem` > 1 an ensemble of  $m_x m_y$  phase-shifts of the coefficients is constructed from the over-supply. Here code phase-shifts over the periods—the phase shifts are like Hankel-matrices.

```

520     patches.nEnsem = mx*my;
521     patches.cs = nan(nx-1,ny-1,nc,rx,ry);
522     for j = 1:my
523         js = (j:j+ny-2);
524         for i = 1:mx
525             is = (i:i+nx-2);
526             patches.cs(:,:,,:,i,j) = cs(is,js,:);
527         end
528     end
529     patches.cs = reshape(patches.cs,nx-1,ny-1,nc,[]);

```

Further, set a cunning left/right/bottom/top realisation of inter-patch coupling. The aim is to preserve symmetry in the system when also invoking `EdgyInt`. What this coupling does without `EdgyInt` is unknown. Use auto-replication.

```

539     le = mod((0:mx-1)+mod(nx-2,mx),mx)+1;
540     patches.le = reshape( le'+mx*(0:my-1) ,[],1);
541     ri = mod((0:mx-1)-mod(nx-2,mx),mx)+1;
542     patches.ri = reshape( ri'+mx*(0:my-1) ,[],1);
543     bo = mod((0:my-1)+mod(ny-2,my),my)+1;
544     patches.bo = reshape( (1:mx)'+mx*(bo-1) ,[],1);
545     to = mod((0:my-1)-mod(ny-2,my),my)+1;
546     patches.to = reshape( (1:mx)'+mx*(to-1) ,[],1);

```

Issue warning if the ensemble is likely to be affected by lack of scale separation.  
Need to justify this and the arbitrary threshold more carefully??

```

554     if prod(ratio)*patches.nEnsem>0.9, warning( ...
555     'Probably poor scale separation in ensemble of coupled phase-shifts')
556     scaleSeparationParameter=ratio*patches.nEnsem
557     end

```

End the two if-statements.

```

563     end%if nEnsem=1 or >1
564     end%if not-empty(cs)

```

**Fin**

```
572 end% function
```

### 3.12 patchSmooth2(): interface 2D space to time integrators

To simulate in time with 2D spatial patches we often need to interface a users time derivative function with time integration routines such as `ode15s` or `PIRK2`. This function provides an interface. It assumes that the sub-patch structure is *smooth* so that the patch centre-values are sensible macroscale

variables, and patch edge-values are determined by macroscale interpolation of the patch-centre values. Communicate patch-design variables to this function via the previously established global struct `patches`.

```
24 function dudt = patchSmooth2(t,u)
25 global patches
```

### Input

- `u` is a vector of length `prod(nSubP) · nVars · nEnsem · prod(nPatch)` where there are `nVars · nEnsem` field values at each of the points in the `nSubP(1) × nSubP(2) × nPatch(1) × nPatch(2)` grid.
- `t` is the current time to be passed to the user's time derivative function.
- `patches` a struct set by `configPatches2()` with the following information used here.
  - `.fun` is the name of the user's function `fun(t,u,x,y)` that computes the time derivatives on the patchy lattice. The array `u` has size `nSubP(1) × nSubP(2) × nVars × nEsem × nPatch(1) × nPatch(2)`. Time derivatives must be computed into the same sized array, but herein the patch edge-values are overwritten by zeros.
  - `.x` is `nSubP(1) × 1 × 1 × 1 × 1 × nPatch(1) × 1` array of the spatial locations  $x_{ij}$  of the microscale grid points in every patch. Currently it *must* be an equi-spaced lattice on both macro- and microscales.
  - `.y` is similarly `1 × nSubP(2) × 1 × 1 × 1 × nPatch(2)` array of the spatial locations  $y_{ij}$  of the microscale grid points in every patch. Currently it *must* be an equi-spaced lattice on both macro- and microscales.

### Output

- `dudt` is `prod(nSubP) · nVars · nEnsem · prod(nPatch)` vector of time derivatives, but with patch edge-values set to zero.

Reshape the fields `u` as a 6D-array, and sets the edge values from macroscale interpolation of centre-patch values. [Section 3.13](#) describes `patchEdgeInt2()`.

```
84 u = patchEdgeInt2(u);
```

Ask the user function for the time derivatives computed in the array, overwrite its edge values with the dummy value of zero, then return to a to the user/integrator as column vector.

```
94 dudt = patches.fun(t,u,patches.x,patches.y);
95 dudt([1 end],:,:,:,:,:) = 0;
96 dudt(:,[1 end],:,:,:,:,:) = 0;
97 dudt = reshape(dudt,[],1);
```

Fin.

### 3.13 patchEdgeInt2(): sets 2D patch edge values from 2D macroscale interpolation

Couples 2D patches across 2D space by computing their edge values via macroscale interpolation. Assumes that the patch centre-values are sensible macroscale variables, and patch edge values are determined by macroscale interpolation of the patch-centre values?? Communicate patch-design variables via the global struct `patches`.

```
20 function u = patchEdgeInt2(u)
21 global patches
```

#### Input

- `u` is a vector of length  $n_{SubP1} \cdot n_{SubP2} \cdot n_{Vars} \cdot n_{Ensem} \cdot n_{Patch1} \cdot n_{Patch2}$  where there are `nVars` field values at each of the points in the  $n_{SubP1} \cdot n_{SubP2} \cdot n_{Patch1} \cdot n_{Patch2}$  grid on the `nPatch1` $\cdot$ `nPatch2` array of patches.
- `patches` a struct set by `configPatches2()` which includes the following information.
  - `.x` is  $n_{SubP1} \times 1 \times 1 \times 1 \times n_{Patch1} \times 1$  array of the spatial locations  $x_{ij}$  of the microscale grid points in every patch. Currently it *must* be an equi-spaced lattice on both macro- and microscales.
  - `.y` is similarly  $\times 1 n_{SubP1} \times 1 \times 1 \times 1 \times n_{Patch1}$  array of the spatial locations  $y_{ij}$  of the microscale grid points in every patch. Currently it *must* be an equi-spaced lattice on both macro- and microscales.
  - `.ordCC` is order of interpolation, currently (July 2020) only  $\{0, 2, 4, \dots\}$
  - `.stag` in  $\{0, 1\}$  is one for staggered grid (alternating) interpolation.
  - `.Cwtsr` and `.Cwtsl` define the coupling coefficients for finite width interpolation.
  - `.EdgyInt` true/false is true for interpolating patch-edge values from opposite next-to-edge values (often preserves symmetry).

#### Output

- `u` is  $n_{SubP1} \cdot n_{SubP2} \cdot n_{Vars} \cdot n_{Ensem} \cdot n_{Patch1} \cdot n_{Patch2}$  6D array of the fields with edge values set by interpolation.

Determine the sizes of things. Any error arising in the reshape indicates `u` has the wrong size.

```
88 [~,ny,~,~,~,Ny] = size(patches.y);
89 [nx,~,~,~,Nx,~] = size(patches.x);
90 nEnsem = patches.nEnsem;
91 nVars = round(numel(u)/numel(patches.x)/numel(patches.y)/nEnsem);
92 assert(numel(u) == nx*ny*Nx*Ny*nVars*nEnsem ...
93 , 'patchEdgeInt2: input u has wrong size for parameters')
```

```

94 % nSubP=[nx ny], nPatch=[Nx Ny], nVars=nVars, sizeu=size(u)
95 u = reshape(u,[nx ny nVars nEnsem Nx Ny ]);
```

With Dirichlet patches, the half-length of a patch is  $h = dx(n_\mu - 1)/2$  (or  $-2$  for specified flux), unless we are interpolating from next-to-edge values, and the ratio needed for interpolation is then  $r = h/\Delta X$ . Compute lattice sizes from inside the patches as the edge values may be NaNs, etc.

```

106 dx = patches.x(3,1,1,1,1,1)-patches.x(2,1,1,1,1,1);
107 DX = patches.x(2,1,1,1,2,1)-patches.x(2,1,1,1,1,1);
108 if patches.EdgyInt==0, rx = dx*(nx-1)/2/DX;
109     else                      rx = dx*(nx-2)/DX;
110 end
111 dy = patches.y(1,3,1,1,1,1)-patches.y(1,2,1,1,1,1);
112 DY = patches.y(1,2,1,1,1,2)-patches.y(1,2,1,1,1,1);
113 if patches.EdgyInt==0, ry = dy*(ny-1)/2/DY;
114     else                      ry = dy*(ny-2)/DY;
115 end
116
```

For the moment assume the physical domain is macroscale periodic so that the coupling formulas are simplest. Should eventually cater for periodic, odd-mid-gap, even-mid-gap, even-mid-patch, Dirichlet, Neumann, Robin?? These index vectors point to patches and their two immediate neighbours—currently not needed.

```

128 I=1:Nx; Ip=mod(I,Nx)+1; Im=mod(I-2,Nx)+1;
129 J=1:Ny; Jp=mod(J,Ny)+1; Jm=mod(J-2,Ny)+1;
```

The centre of each patch (as `nx` and `ny` are odd for centre-patch interpolation) is at indices

```

136 i0 = round((nx+1)/2);
137 j0 = round((ny+1)/2);
```

**Lagrange interpolation gives patch-edge values** Compute centred differences of the mid-patch values for the macro-interpolation, of all fields. Assumes the domain is macro-periodic. Currently, only next-to-edge interpolation is implemented.

```

150 ordCC=patches.ordCC;
151 if ordCC>0 % then finite-width polynomial interpolation
152     if patches.EdgyInt % next-to-edge values
153         uCorex = u([2 nx-1],2:(ny-1),:,:,I,J);
154         uCorey = u(2:(nx-1),[2 ny-1],:,:,I,J);
155     else
156         %disp('polynomial interpolation, currently couple from the centre-cross??')
157         uCorex = u(i0,2:(ny-1),:,:,I,J);
158         uCorey = u(2:(nx-1),j0,:,:,I,J);
159     end;
160     dmux = zeros([ordCC,size(uCorex)]); % 7D
161     dmuy = zeros([ordCC,size(uCorey)]); % 7D
```

```

162 if patches.stag % use only odd numbered neighbours
163 error('polynomial interpolation not yet for staggered patch coupling')
164 else %disp('starting standard interpolation')
165 dmux(1,:,:,:, :,I,: ) = (uCorex(:, :, :, :,Ip,: ) - uCorex(:, :, :, :,Im,: ))/2; % \
166 dmux(2,:,:,:, :,I,: ) = (uCorex(:, :, :, :,Ip,: ) - 2*uCorex(:, :, :, :,I,: ) +uCorex(:, :, :, :,J,: ))/2; % \
167 dmuy(1,:,:,:, :,J,: ) = (uCorey(:, :, :, :,Jp,: ) - uCorey(:, :, :, :,Jm,: ))/2; % \
168 dmuy(2,:,:,:, :,J,: ) = (uCorey(:, :, :, :,Jp,: ) - 2*uCorey(:, :, :, :,J,: ) +uCorey(:, :, :, :,Jm,: ))/2; % \
169 end% if odd/even

```

Recursively take  $\delta^2$  of these to form higher order centred differences.

```

176 for k = 3:ordCC
177 dmux(k,:,:,:, :,I,: ) = dmux(k-2,:,:,:, :,Ip,: ) ...
178 -2*dmux(k-2,:,:,:, :,I,: ) +dmux(k-2,:,:,:, :,Im,: );
179 dmuy(k,:,:,:, :,J,: ) = dmuy(k-2,:,:,:, :,Jp,: ) ...
180 -2*dmuy(k-2,:,:,:, :,J,: ) +dmuy(k-2,:,:,:, :,Jm,: );
181 end
182 %dmux33=dmux(:, :, :, :, 3,3), sizedmux33=size(dmx33)

```

Interpolate macro-values to be Dirichlet edge values for each patch ([Roberts & Kevrekidis 2007](#), [Bunder et al. 2017](#)), using weights computed in `configPatches2()`. Here interpolate to specified order.

Where next-to-edge values interpolate to the opposite edge-values. When we have an ensemble of configurations, different configurations might be coupled to each other, as specified by `patches.le`, `patches.ri`, `patches.to` and `patches.bo`.

```

197 k=1+patches.EdgyInt; % use centre or two edges
198 u(nx,2:(ny-1),:, patches.ri,I,: ) ...
199 = uCorex(1,:,:,:, :)*(1-patches.stag) ...
200 +shiftdim(sum( patches.Cwtsr(:,1).*dmux(:,1,:,:,:, :) ),1);
201 u(1 ,2:(ny-1),:, patches.le,I,: ) ...
202 = uCorex(k,:,:,:, :)*(1-patches.stag) ...
203 +shiftdim(sum( patches.Cwtsl(:,1).*dmux(:,k,:,:,:, :) ),1);
204 u(2:(nx-1),ny, :, patches.to,:,J,...)
205 = uCorey(:,1,:,:,:, :)*(1-patches.stag) ...
206 +shiftdim(sum( patches.Cwtsr(:,2).*dmuy(:, :,1,:,:,:, :)),1);
207 u(2:(nx-1),1 ,:, patches.bo,:,J,...)
208 = uCorey(:,k,:,:,:, :)*(1-patches.stag) ...
209 +shiftdim(sum( patches.Cwtsl(:,2).*dmuy(:, :,k,:,:,:, :)),1);
210 u([1 nx],[1 ny], :, :, :, :)=nan; % remove corner values

```

**Case of spectral interpolation** Assumes the domain is macro-periodic. We interpolate in terms of the patch index  $j$ , say, not directly in space. As the macroscale fields are  $N$ -periodic in the patch index  $j$ , the macroscale Fourier transform writes the centre-patch values as  $U_j = \sum_k C_k e^{ik2\pi j/N}$ . Then the edge-patch values  $U_{j\pm r} = \sum_k C_k e^{ik2\pi/N(j\pm r)} = \sum_k C'_k e^{ik2\pi j/N}$  where  $C'_k = C_k e^{ikr2\pi/N}$ . For  $N$  patches we resolve ‘wavenumbers’  $|k| < N/2$ , so set row vector  $\mathbf{ks} = k2\pi/N$  for ‘wavenumbers’  $k = (0, 1, \dots, k_{\max}, -k_{\max}, \dots, -1)$  for odd  $N$ , and  $k = (0, 1, \dots, k_{\max}, \pm(k_{\max} + 1) - k_{\max}, \dots, -1)$  for even  $N$ .

---

```
233 else% spectral interpolation
```

Deal with staggered grid by doubling the number of fields and halving the number of patches (`configPatches2` tests there are an even number of patches). Then the patch-ratio is effectively halved. The patch edges are near the middle of the gaps and swapped.

```
243 if patches.stag % transform by doubling the number of fields
244 error('staggered grid not yet implemented??')
245 v=nan(size(u)); % currently to restore the shape of u
246 u=cat(3,u(:,1:2:nPatch,:),u(:,2:2:nPatch,:));
247 stagShift=reshape(0.5*[ones(nVars,1);-ones(nVars,1)],1,1,[]);
248 iV=[nVars+1:2*nVars 1:nVars]; % scatter interp to alternate field
249 r=r/2; % ratio effectively halved
250 nPatch=nPatch/2; % halve the number of patches
251 nVars=nVars*2; % double the number of fields
252 else % the values for standard spectral
253     stagShift = 0;
254     iV = 1:nVars;
255 end
```

Now set wavenumbers in the two directions into two vectors at the correct dimension. In the case of even  $N$  these compute the + -case for the highest wavenumber zig-zag mode,  $k = (0, 1, \dots, k_{\max}, +(k_{\max} + 1) - k_{\max}, \dots, -1)$ .

```
265 kMax = floor((Nx-1)/2);
266 krx = shiftdim( rx*2*pi/Nx*(mod((0:Nx-1)+kMax,Nx)-kMax) ,-3);
267 kMay = floor((Ny-1)/2);
268 kry = shiftdim( ry*2*pi/Ny*(mod((0:Ny-1)+kMay,Ny)-kMay) ,-4);
```

Test for reality of the field values, and define a function accordingly. Could be problematic if some variables are real and some are complex, or if variables are of quite different sizes.

```
277 if max(abs(imag(u(:))))<1e-9*max(abs(u(:)))
278     uclean=@(u) real(u);
279 else uclean=@(u) u;
280 end
```

Compute the Fourier transform of the patch centre-values for all the fields. Unless doing patch-edgy interpolation when FT the next-to-edge values. If there are an even number of points, then if complex, treat as positive wavenumber, but if real, treat as cosine. When using an ensemble of configurations, different configurations might be coupled to each other, as specified by `patches.le`, `patches.ri`, `patches.to` and `patches.bo`.

```
293 ix=(2:nx-1)'; iy=2:ny-1; % indices of interior
294 if ~patches.EdgyInt
295     Cle = fft(fft(u(i0,j0,:,:,:, :) ,[],5),[],6);
296     Cbo = Cle;
297 else
298     Cle = fft(fft(u( 2,iy ,:,patches.le,:,:),[],5),[],6);
299     Cri = fft(fft(u(nx-1,iy ,:,patches.ri,:,:),[],5),[],6);
```

```

300      Cbo = fft(fft(u(ix,2      ,:,patches.bo,:,:),[],5),[],6);
301      Cto = fft(fft(u(ix,ny-1 ,:,patches.to,:,:),[],5),[],6);
302  end
303 % fill in the cross of Fourier-shifted mid-values
304 if ~patches.EdgyInt
305     % y-fraction of kry along left/right edges
306     ks = (iy-j0)*2/(ny-1).*kry;
307     Cle = Cle.*exp(1i*ks);
308     Cri = Cle;
309     % x-fraction of krx along bottom/top edges
310     ks = (ix-i0)*2/(nx-1).*krx;
311     Cbo = Cbo.*exp(1i*ks);
312     Cto = Cbo;
313 end
314 u(nx, iy, :, :, :, :) = uclean( ifft(ifft( ...
315     Cle.*exp(1i*(stagShift+krx)) ,[],5),[],6) );
316 u( 1, iy, :, :, :, :) = uclean( ifft(ifft( ...
317     Cri.*exp(1i*(stagShift-krx)) ,[],5),[],6) );
318 u(ix, ny, :, :, :, :) = uclean( ifft(ifft( ...
319     Cbo.*exp(1i*(stagShift+kry)) ,[],5),[],6) );
320 u(ix, 1, :, :, :, :) = uclean( ifft(ifft( ...
321     Cto.*exp(1i*(stagShift-kry)) ,[],5),[],6) );
322 u([1 nx],[1 ny],:,:,:,:)=nan; % remove corner values
323
324 end% if spectral
325 end% function patchEdgeInt2

```

Fin, returning the 6D array of field values with interpolated edges.

### 3.14 wave2D: example of a wave on patches in 2D

#### *Section contents*

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3.14.3 wavePDE(): Example of simple wave PDE inside patches	100

For  $u(x, y, t)$ , test and simulate the simple wave PDE in 2D space:

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u.$$

This script shows one way to get started: a user's script may have the following three steps (left-right arrows denote function recursion).

1. configPatches2
2. ode15s integrator  $\leftrightarrow$  patchSmooth2  $\leftrightarrow$  wavePDE
3. process results

Establish the global data struct `patches` to interface with a function coding the wave PDE: to be solved on  $2\pi$ -periodic domain, with  $9 \times 9$  patches, spectral interpolation (0) couples the patches, each patch of half-size ratio 0.25 (big enough for visualisation), and with a  $5 \times 5$  micro-grid within each patch.

```
35 global patches
36 nSubP = 5;
37 nPatch = 9;
38 configPatches2(@wavePDE, [-pi pi], nan, nPatch, 0, 0.25, nSubP);
```

### 3.14.1 Check on the linear stability of the wave PDE

Construct the systems Jacobian via numerical differentiation. Set a zero equilibrium as basis. Then find the indices of patch-interior points as the only ones to vary in order to construct the Jacobian.

```
51 disp('Check linear stability of the wave scheme')
52 uv0 = zeros(nSubP,nSubP,2,1,nPatch,nPatch);
53 uv0([1 end],:,:,:,:,:) = nan;
54 uv0(:,[1 end],:,:,:,:,:) = nan;
55 i = find(~isnan(uv0));
```

Now construct the Jacobian. Since this is a *linear* wave PDE, use large perturbations.

```
62 small = 1;
63 jac = nan(length(i));
64 sizeJacobian = size(jac)
65 for j = 1:length(i)
66     uv = uv0(:);
67     uv(i(j)) = uv(i(j))+small;
68     tmp = patchSmooth2(0,uv)/small;
69     jac(:,j) = tmp(i);
70 end
```

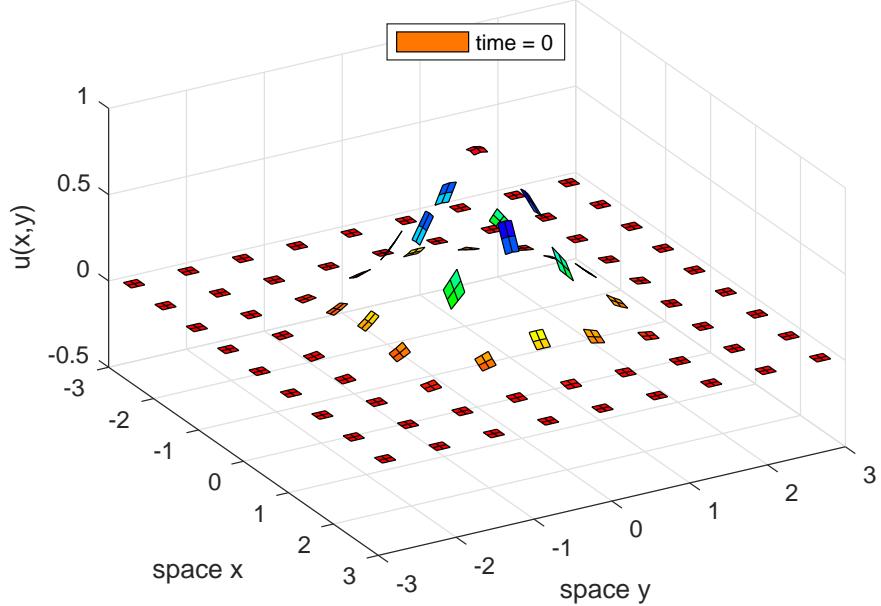
Now explore the eigenvalues a little: find the ten with the biggest real-part; if these are small enough, then the method may be good.

```
78 evals = eig(jac);
79 nEvals = length(evals)
80 [~,k] = sort(-abs(real(evals)));
81 evalsWithBiggestRealPart = evals(k(1:10))
82 if abs(real(evals(k(1))))>1e-4
83     warning('eigenvalue failure: real-part > 1e-4')
84     return, end
```

Check that the eigenvalues are close to true waves of the PDE (not yet the micro-discretised equations).

```
91 kwave = 0:(nPatch-1)/2;
92 freq = sort(reshape(sqrt(kwave.^2+kwave.^2),1,[]));
93 freq = freq(diff([-1 freq])>1e-9);
94 freqerr = [freq; min(abs(imag(evals)-freq))]
```

Figure 3.18: initial field  $u(x, y, t)$  at time  $t = 0$  of the patch scheme applied to the simple wave PDE: Figure 3.19 plots the computed field at time  $t = 2$ .



### 3.14.2 Execute a simulation

Set a Gaussian initial condition using auto-replication of the spatial grid: here  $u_0$  and  $v_0$  are in the form required for computation:  $n_x \times n_y \times 1 \times 1 \times N_x \times N_y$ .

```
109 u0 = exp(-patches.x.^2-patches.y.^2);
110 v0 = zeros(size(u0));
```

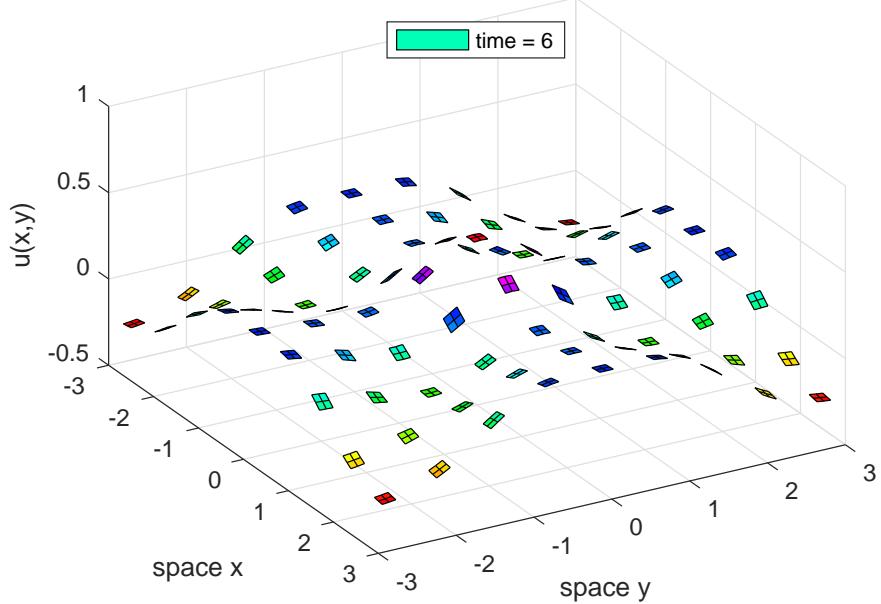
Initiate a plot of the simulation using only the microscale values interior to the patches: set  $x$  and  $y$ -edges to `nan` to leave the gaps. Start by showing the initial conditions of Figure 3.16 while the simulation computes. To mesh/surf plot we need to ?? ‘transpose’ to size  $n_x \times N_x \times n_y \times N_y$ , then reshape to size  $n_x \cdot N_x \times n_y \cdot N_y$ .

```
122 x = squeeze(patches.x); y = squeeze(patches.y);
123 x([1 end], :) = nan; y([1 end], :) = nan;
124 u = reshape(permute(squeeze(u0), [1 3 2 4]), [numel(x) numel(y)]);
125 usurf = surf(x(:, ), y(:, ), u');
126 axis([-3 3 -3 3 -0.5 1]), view(60,40)
127 xlabel('space x'), ylabel('space y'), zlabel('u(x,y)')
128 legend('time = 0', 'Location', 'north')
129 colormap(hsv)
130 drawnow
131 if0urCf2eps([mfilename 'ic'])
```

Integrate in time using standard functions.

```
144 disp('Wait while we simulate u_t=v, v_t=u_xx+u_yy')
145 uv0 = cat(3,u0,v0);
146 if ~exist('OCTAVE_VERSION', 'builtin')
147 [ts,uv] = ode23( @patchSmooth2, [0 6], uv0(:));
```

Figure 3.19: field  $u(x, y, t)$  at time  $t = 6$  of the patch scheme applied to the simple wave PDE with initial condition in Figure 3.18.



```

148 else % octave version is slower
149 [ts,uv] = odeOctave(@patchSmooth2,linspace(0,6),uv0(:));
150 end

```

Animate the computed simulation to end with Figure 3.19. Because of the very small time-steps, subsample to plot at most 100 times.

```

158 di = ceil(length(ts)/100);
159 for i = [1:di:length(ts)-1 length(ts)]
160   uv = patchEdgeInt2(uvs(i,:));
161   uv = reshape(permute(uv,[1 5 2 6 3 4]), [numel(x) numel(y) 2]);
162   set(usurf,'ZData', uv(:,:,1));
163   legend(['time = ' num2str(ts(i),2)])
164   pause(0.1)
165 end
166 if0urCf2eps([mfilename 't' num2str(ts(end))])

```

### 3.14.3 wavePDE(): Example of simple wave PDE inside patches

As a microscale discretisation of  $u_{tt} = \nabla^2(u)$ , so code  $\dot{u}_{ijkl} = v_{ijkl}$  and  $\dot{v}_{ijkl} = \frac{1}{\delta x^2}(u_{i+1,j,k,l} - 2u_{i,j,k,l} + u_{i-1,j,k,l}) + \frac{1}{\delta y^2}(u_{i,j+1,k,l} - 2u_{i,j,k,l} + u_{i,j-1,k,l})$ .

```

14 function uvt = wavePDE(t,uv,x,y)
15   dx = diff(x(1:2)); dy = diff(y(1:2)); % microscale spacing
16   i = 2:size(uv,1)-1; j = 2:size(uv,2)-1; % interior patch-points
17   uvt = nan(size(uv)); % preallocate storage
18   uvt(i,j,1,:) = uv(i,j,2,:);
19   uvt(i,j,2,:) = diff(uv(:,j,1,:),2,1)/dx^2 ...
20           +diff(uv(:,1,:),2,2)/dy^2;
21 end

```

```

10 function [ts,xs] = odeOcts(dxdt,tSpan,x0)
11     if length(tSpan)>2, ts = tSpan;
12     else ts = linspace(tSpan(1),tSpan(end),21)';
13     end
14     lsode_options('integration method','non-stiff');
15     xs = lsode(@(x,t) dxdt(t,x),x0,ts);
16 end

```

### 3.15 homoDiffEdgy2: computational homogenisation of a 2D diffusion via simulation on small patches

This section extends to 2D the 1D code discussed in [Section 3.6](#). First set random heterogeneous diffusivities of random period in each of the two directions. Crudely normalise by the harmonic mean so the decay time scale is roughly one.

```

23 mPeriod = randi([2 3],1,2)
24 cHetr = exp(1*randn([mPeriod 2]));
25 cHetr = cHetr*mean(1./cHetr(:))

```

Configure the patch scheme with some arbitrary choices of domain, patches, size ratios. Use spectral interpolation as we test other orders subsequently. In 2D we appear to get only real eigenvalues by using edgy interpolation. What happens for non-edgy interpolation is unknown.

```

36 edgyInt = true;
37 nEnsem = prod(mPeriod) % or just set one
38 if nEnsem==1% use more patches
39     nPatch = [9 9]
40     nSubP = (2-edgyInt)*mPeriod+1+edgyInt
41 else % when nEnsem>1 use fewer patches
42     nPatch = [5 5]
43     nSubP = mPeriod+randi([1 4],1,2) % +2 is decoupled
44 end
45 ratio = 0.2+0.2*rand(1,2)
46 configPatches2(@heteroDiff2,[-pi pi -pi pi],nan,nPatch ...
47 ,0,ratio,nSubP,'EdgyInt',edgyInt,'nEnsem',nEnsem ...
48 , 'hetCoeffs',cHetr );

```

**Simulate** Set initial conditions of a simulation, replicated for each in the ensemble.

```

58 global patches
59 u0 = 0.8*cos(patches.x).*sin(patches.y) ...
60     +0.1*randn([nSubP,1,1,nPatch]);
61 u0 = repmat(u0,1,1,1,nEnsem,1,1);

```

Integrate using standard integrators, unevenly spaced in time to better display transients.

```

68 if ~exist('OCTAVE_VERSION','builtin')
69     [ts,us] = ode23(@patchSmooth2, 0.3*linspace(0,1).^2, u0(:));

```

```

70    else % octave version
71        [ts,us] = odeOcts(@patchSmooth2, 0.3*linspace(0,1).^2, u0(:));
72    end

    Plot the solution as an animation over time.

79    if ts(end)>0.099, disp('plot animation of solution field')
80    figure(1), clf, colormap(hsv)

    Get spatial coordinates and pad them with NaNs to separate patches.

87    x = squeeze(patches.x); y = squeeze(patches.y);
88    x(end+1,:)=nan; y(end+1,:)=nan; % pad with nans

    For every time step draw the surface and pause for a short display.

95    for i = 1:length(ts)

    Get the row vector of data, form into the 6D array via the interpolation to
    the edges, then pad with Nans between patches, and reshape to suit the surf
    function.

103    u = squeeze( mean(patchEdgeInt2(us(i,:)) ,4));
104    u(end+1,:,:,:)=nan; u(:,end+1,:,:)=nan;
105    u = reshape(permute(u,[1 3 2 4]), [numel(x) numel(y)]);

    If the initial time then draw the surface with labels, otherwise just update
    the surface data.

112    if i==1
113        hsurf = surf(x(:,y(:,u')); view(60,40)
114        axis([-pi pi -pi pi -1 1]), caxis([-1 1])
115        xlabel('x'), ylabel('y'), zlabel('u(x,y)')
116    else set(hsurf,'ZData', u');
117    end
118    legend(['time = ', num2str(ts(i),2)],'Location','north')
119    pause(0.05)

    finish the animation loop and if-plot.

125    end%for over time
126    end%if-plot

```

### 3.15.1 Compute Jacobian and its spectrum

Let's explore the Jacobian dynamics for a range of orders of interpolation, all for the same patch design and heterogeneity. Except here use a small ratio as we do not plot.

```

142 ratio = [0.1 0.1]
143 nLeadEvals=prod(nPatch)+max(nPatch);
144 leadingEvals=[];

```

Evaluate eigenvalues for spectral as the base case for polynomial interpolation of order 2,4,....

```

152 maxords=10;
153 for ord=0:2:maxords
154     ord=ord

```

Configure with same parameters, then because they are reset by this configuration, restore coupling.

```

161     configPatches2(@heteroDiff2,[-pi pi -pi pi],nan,nPatch ...
162         ,ord,ratio,nSubP,'EdgyInt',edgyInt,'nEnsem',nEnsem ...
163         , 'hetCoeffs',cHetr);

```

Find which elements of the 6D array are interior micro-grid points and hence correspond to dynamical variables.

```

170     u0 = zeros([nSubP,1,nEnsem,nPatch]);
171     u0([1 end],:,:) = nan;
172     u0(:,[1 end],:) = nan;
173     i = find(~isnan(u0));

```

Construct the Jacobian of the scheme as the matrix of the linear transformation, obtained by transforming the standard unit vectors.

```

181     jac = nan(length(i));
182     sizeJacobian = size(jac)
183     for j = 1:length(i)
184         u = u0(:)+(i(j)==(1:numel(u0))');
185         tmp = patchSmooth2(0,u);
186         jac(:,j) = tmp(i);
187     end

```

Test for symmetry, with error if we know it should be symmetric.

```

194     notSymmetric=norm(jac-jac')
195     if edgyInt, assert(notSymmetric<1e-7,'failed symmetry')
196     elseif notSymmetric>1e-7, disp('failed symmetry')
197     end

```

Find all the eigenvalues (as `eigs` is unreliable).

```

203     if edgyInt, [evecs,evals] = eig((jac+jac')/2,'vector');
204     else evals = eig(jac);
205     end
206     biggestImag=max(abs(imag(evals)));
207     if biggestImag>0, biggestImag=biggestImag, end

```

Sort eigenvalues on their real-part with most positive first, and most negative last. Store the leading eigenvalues in `egs`, and write out when computed all orders. The number of zero eigenvalues, `nZeroEv`, gives the number of decoupled systems in this patch configuration.

```

217     [~,k] = sort(-real(evals));
218     evals=evals(k); evecs=evecs(:,k);
219     if ord==0, nZeroEv=sum(abs(evals(:))<1e-5), end
220     if ord==0, evec0=evecs(:,1:nZeroEv*nLeadEvals);
221     else % find evec closest to that of each leading spectral

```

```

222      [~,k]=max(abs(evecs'*evec0));
223      evals=evals(k); % sort in corresponding order
224    end
225    leadingEvals=[leadingEvals evals(nZeroEv*(1:nLeadEvals))];
226  end
227  disp('    spectral    quadratic    quartic    sixth-order ...')
228  leadingEvals=leadingEvals

```

Plot the errors in the eigenvalues using the spectral ones as accurate. Only plot every second, iEv, as all are repeated eigenvalues.

```

237 if maxords>2
238   iEv=2:2:12;
239   figure(2);
240   err=abs(leadingEvals-leadingEvals(:,1)) ...
241     ./(1e-7+abs(leadingEvals(:,1)));
242   semilogy(2:2:maxords,err(iEv,2:end)', 'o:');
243   xlabel('coupling order')
244   ylabel('eigenvalue relative error')
245   leg=legend( ...
246     strcat('$',num2str(real(leadingEvals(iEv,1)), '%.4f'), '$') ...
247     , 'Location', 'northeastoutside');
248   if ~exist('OCTAVE_VERSION','builtin')
249     title(leg,'eigenvalues'), end
250   legend boxoff
251 end%if-plot

```

### 3.15.2 heteroDiff2(): heterogeneous diffusion

This function codes the lattice heterogeneous diffusion inside the patches. For 6D input arrays u, x, and y (via edge-value interpolation of `patchSmooth2`, Section 3.12), computes the time derivative (3.1) at each point in the interior of a patch, output in ut. The two 2D array of diffusivities,  $c_{ij}^x$  and  $c_{ij}^y$ , have previously been stored in `patches.cs` (3D).

```

19 function ut = heteroDiff2(t,u,x,y)
20   global patches
21   dx = diff(x(2:3)); % x space step
22   dy = diff(y(2:3)); % y space step
23   ix = 2:size(u,1)-1; % x interior points in a patch
24   iy = 2:size(u,2)-1; % y interior points in a patch
25   ut = nan(size(u)); % preallocate output array
26   ut(ix,iy,:,:,:,:) ...
27   = diff(patches.cs(:,iy,1,:).*diff(u(:,iy,:,:,:,:),1,1)/dx^2 ...
28     +diff(patches.cs(ix,:,2,:).*diff(u(ix,:,:,:,:),1,2),1,2)/dy^2;
29 end% function

```

Fin.

### 3.16 configPatches3(): configures spatial patches in 3D

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#### 3.16.1 Introduction

Makes the struct `patches` for use by the patch/gap-tooth time derivative/step function `patchSmooth3()`. Sections 3.16.2 and 3.19 lists an example of its use.

```
24 function configPatches3(fun,Xlim,BCs,nPatch,ordCC,ratio,nSubP ...
25     ,varargin)
26 global patches
```

**Input** If invoked with no input arguments, then executes an example of simulating a heterogeneous wave PDE.

- `fun` is the name of the user function, `fun(t,u,x,y,z)`, that computes time-derivatives (or time-steps) of quantities on the 3D micro-grid within all the 3D patches.
- `Xlim` array/vector giving the macro-space domain of the computation: patches are distributed equi-spaced over the interior of the rectangle  $[Xlim(1), Xlim(2)] \times [Xlim(3), Xlim(4)] \times [Xlim(5), Xlim(6)]$ : if `Xlim` is of length two, then the domain is the cubic domain of the same interval in all three directions.
- `BCs` eventually and somehow will define the macroscale boundary conditions. Currently, `BCs` is ignored and the system is assumed macro-periodic in the domain.
- `nPatch` determines the number of equi-spaced spaced patches: if scalar, then use the same number of patches in both directions, otherwise `nPatch(1:3)` gives the number of patches in each direction.
- `ordCC` is the ‘order’ of interpolation for inter-patch coupling across empty space of the macroscale patch values to the edge-values of the patches: currently must be 0, 2, 4, . . . ; where 0 gives spectral interpolation.
- `ratio` (real) is the ratio of (depending upon `EdgyInt`) either the half-width or full-width of a patch to the spacing of the patch mid-points. So either `ratio = 1/2` means the patches abut and `ratio = 1` is overlapping

patches as in holistic discretisation, or `ratio = 1` means the patches abut. Small `ratio` should greatly reduce computational time. If scalar, then use the same ratio in both directions, otherwise `ratio(1:3)` gives the ratio in each direction.

- `nSubP` is the number of equi-spaced microscale lattice points in each patch: if scalar, then use the same number in both directions, otherwise `nSubP(1:3)` gives the number in each direction. If not using `EdgyInt`, then must be odd so that there is a central micro-grid point/planes in each patch.
- ‘`nEdge`’ (not yet implemented), *optional*, default=1, for each patch, the number of edge values set by interpolation at the edge regions of each patch. The default is one (suitable for microscale lattices with only nearest neighbour interactions).
- ‘`EdgyInt`’, true/false, *optional*, default=false. If true, then interpolate to left/right/top/bottom/front/back face-values from right/left/bottom/top/back/front next-to-face values.
- ‘`nEnsem`’, *optional-experimental*, default one, but if more, then an ensemble over this number of realisations.
- ‘`hetCoeffs`’, *optional*, default empty. Supply a 3/4D array of microscale heterogeneous coefficients to be used by the given microscale `fun` in each patch. Say the given array `cs` is of size  $m_x \times m_y \times m_z \times n_c$ , where  $n_c$  is the number of different arrays of coefficients. For example, in heterogeneous diffusion,  $n_c = 3$  for the diffusivities in the *three* different spatial directions. The coefficients are to be the same for each and every patch. However, macroscale variations are catered for by the  $n_c$  coefficients being  $n_c$  parameters in some macroscale formula.
  - If `nEnsem` = 1, then the array of coefficients is just tiled across the patch size to fill up each patch, starting from the (1, 1)-element.
  - If `nEnsem` > 1 (value immaterial), then reset `nEnsem := m_x · m_y · m_z` and construct an ensemble of all  $m_x \cdot m_y \cdot m_z$  phase-shifts of the coefficients. In this scenario, the inter-patch coupling couples different members in the ensemble. When `EdgyInt` is true, and when the coefficients are diffusivities/elasticities in  $x, y, z$  directions, respectively, then this coupling cunningly preserves symmetry.

**Output** The *global* struct `patches` is created and set with the following components.

- `.fun` is the name of the user’s function `fun(t,u,x,y,z)` that computes the time derivatives (or steps) on the patchy lattice.
- `.ordCC` is the specified order of inter-patch coupling.
- `.stag` is true for interpolation using only odd neighbouring patches as for staggered grids, and false for the usual case of all neighbour coupling—not yet implemented.

- `.Cwtsr` and `.Cwtsl` are the  $\text{ordCC} \times 3$ -array of weights for the inter-patch interpolation onto the right/top and left/bottom edges (respectively) with patch:macroscale ratio as specified.
- `.x` (8D) is  $\text{nSubP}(1) \times 1 \times 1 \times 1 \times 1 \times \text{nPatch}(1) \times 1 \times 1$  array of the regular spatial locations  $x_{ijk}$  of the microscale grid points in every patch.
- `.y` (8D) is  $1 \times \text{nSubP}(2) \times 1 \times 1 \times 1 \times 1 \times \text{nPatch}(2) \times 1$  array of the regular spatial locations  $y_{ijk}$  of the microscale grid points in every patch.
- `.z` (8D) is  $1 \times 1 \times \text{nSubP}(3) \times 1 \times 1 \times 1 \times 1 \times \text{nPatch}(3)$  array of the regular spatial locations  $z_{ijk}$  of the microscale grid points in every patch.
- `.nEdge` is, for each patch, the number of edge values set by interpolation at the edge regions of each patch.
- `.le`, `.ri`, `.bo`, `.to`, `.ba`, `.fr` determine inter-patch coupling of members in an ensemble. Each a column vector of length `nEnsem`.
- `.cs` either
  - [] 0D, or
  - if `nEnsem = 1`,  $(\text{nSubP}(1) - 1) \times (\text{nSubP}(2) - 1) \times (\text{nSubP}(3) - 1) \times n_c$  4D array of microscale heterogeneous coefficients, or
  - if `nEnsem > 1`,  $(\text{nSubP}(1) - 1) \times (\text{nSubP}(2) - 1) \times (\text{nSubP}(3) - 1) \times n_c \times m_x m_y$  5D array of  $m_x m_y$  ensemble of phase-shifts of the microscale heterogeneous coefficients.

### 3.16.2 If no arguments, then execute an example

```
202 if nargin==0
```

The code here shows one way to get started: a user's script may have the following three steps (arrows indicate function recursion).

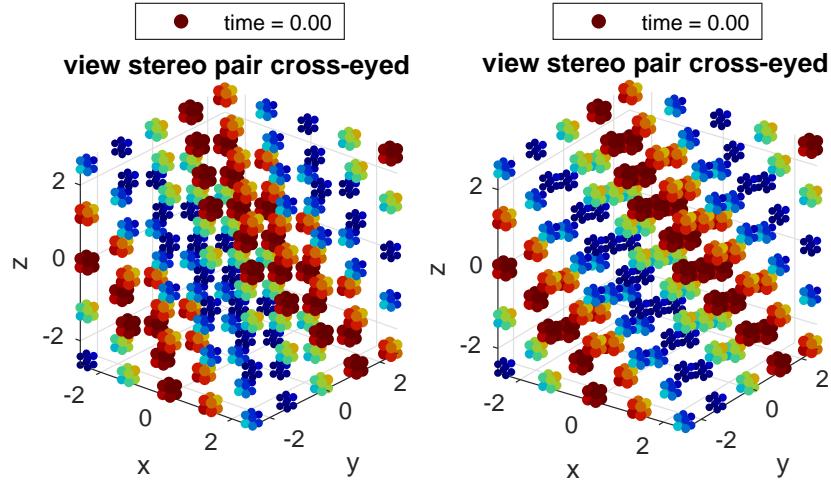
1. configPatches3
2. ode23 integrator  $\leftrightarrow$  patchSmooth3  $\leftrightarrow$  user's PDE
3. process results

Set random heterogeneous coefficients of period two in each of the three directions. Crudely normalise by the harmonic mean so the decay time scale is roughly one.

```
220 mPeriod = [2 2 2];
221 cHetr = exp(0.3*randn([mPeriod 3]));
222 cHetr = cHetr*mean(1./cHetr(:))
```

Establish global patch data struct to interface with a function coding a nonlinear 'diffusion' PDE: to be solved on  $[-\pi, \pi]^3$ -periodic domain, with  $5^3$  patches, spectral interpolation (0) couples the patches, each patch of half-size

*Figure 3.20: initial field  $u(x, y, z, t)$  at time  $t = 0$  of the patch scheme applied to a heterogeneous wave PDE: [Figure 3.21](#) plots the computed field at time  $t = 6$ .*



ratio 0.4 (relatively large for visualisation), and with  $4^3$  points forming each patch.

```
234 configPatches3(@heteroWave3, [-pi pi], nan, 5 ...
235 , 0, 0.35, mPeriod+2, 'EdgyInt', true ...
236 , 'hetCoeffs', cHetr);
```

Set a wave initial state using auto-replication of the spatial grid, and as [Figure 3.20](#) shows.

```
243 u0 = 0.5+0.5*sin(patches.x+patches.y+patches.z);
244 v0 = -0.5*cos(patches.x+patches.y+patches.z)*sqrt(3);
245 uv0 = cat(4,u0,v0);
```

Integrate in time using standard functions. In Matlab `ode15s` would be natural as the patch scheme is naturally stiff, but `ode23` is much quicker.

```
261 disp('Simulate heterogeneous wave u_tt=div[C*grad(u)]')
262 if ~exist('OCTAVE_VERSION', 'builtin')
263 [ts,us] = ode23(@patchSmooth3,linspace(0,6),uv0(:));
264 else %disp('octave version is very slow for me')
265 lsode_options('absolute tolerance',1e-4);
266 lsode_options('relative tolerance',1e-4);
267 [ts,us] = ode0cts(@patchSmooth3,[0 1 2],uv0(:));
268 end
```

Animate the computed simulation to end with [Figure 3.21](#). Use `patchEdgeInt3` to interpolate patch-edge values (even if not drawn) in order to most easily reconstruct the array data structure.

Replicate  $x$ ,  $y$ , and  $z$  arrays to get individual spatial coordinates of every data point. Then, optionally, set faces to `nan` so the plot just shows patch-interior data.

```

282 figure(1), clf, colormap(0.8*jet)
283 xs = patches.x+0*patches.y+0*patches.z;
284 ys = patches.y+0*patches.x+0*patches.z;
285 zs = patches.z+0*patches.y+0*patches.x;
286 xs([1 end],:,:,:) = nan;
287 xs(:,:,1 end,:,:) = nan;
288 xs(:,:,1 end,:,:) = nan;
289 j = find(~isnan(xs));

```

The functions `pix()` and `col()` map the  $u$ -data values to the size of the dots and to the colour of the dots, respectively.

```

297 pix = @(u) 15*abs(u)+7;
298 col = @(u) sign(u).*abs(u);

```

Loop to plot at each and every time step.

```

304 for i = 1:length(ts)
305     uv = patchEdgeInt3(us(i,:,:));
306     u = uv(:,:,1,:,:);
307     for p=1:2
308         subplot(1,2,p)
309         if (i==1) | exist('OCTAVE_VERSION','builtin')
310             scat(p) = scatter3(xs(j),ys(j),zs(j) ...
311                 ,pix(u(j)),col(u(j)), 'filled');
312             axis equal, caxis(col([0 1])), view(45-5*p,25)
313             xlabel('x'), ylabel('y'), zlabel('z')
314             title('view stereo pair cross-eyed')
315         else % in matlab just update values
316             set(scat(p),'CData',col(u(j)), 'SizeData',pix(u(j)));
317         end
318         legend(['time = ' num2str(ts(i), '%4.2f')],'Location','north')
319     end

```

Optionally save the initial condition to graphic file for [Figure 3.16](#), and optionally save the last plot.

```

327 if i==1,
328     ifOurCf2eps([mfilename 'ic'])
329         disp('Type space character to animate simulation')
330         pause
331     else pause(0.05)
332     end
333 end% i-loop over all times
334 ifOurCf2eps([mfilename 'fin'])

```

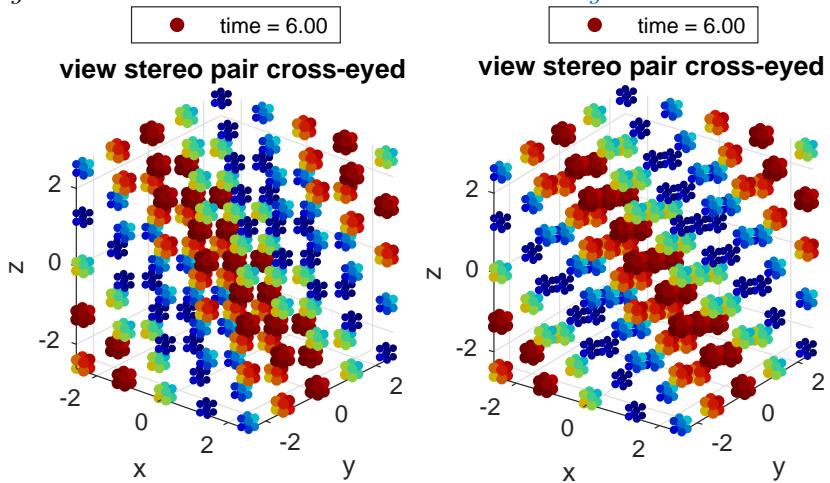
Upon finishing execution of the example, exit this function.

```

349 return
350 end%if no arguments

```

Figure 3.21: field  $u(x, y, z, t)$  at time  $t = 6$  of the patch scheme applied to the heterogeneous wave PDE with initial condition in Figure 3.20.



### 3.16.3 heteroWave3(): heterogeneous Waves

This function codes the lattice heterogeneous waves inside the patches. The wave PDE is

$$u_t = v, \quad v_t = \vec{\nabla}(C\vec{\nabla} \cdot u)$$

for diagonal matrix  $C$  which has microscale variations. For 8D input arrays  $u$ ,  $x$ ,  $y$ , and  $z$  (via edge-value interpolation of patchSmooth3, Section 3.17), computes the time derivative at each point in the interior of a patch, output in  $ut$ . The three 3D array of heterogeneous coefficients,  $c_{ijk}^x$ ,  $c_{ijk}^y$  and  $c_{ijk}^z$ , have previously been stored in patches.cs (4D).

```

23 function ut = heteroWave3(t,u,x,y,z)
24     global patches
25     dx = diff(x(2:3)); % x space step
26     dy = diff(y(2:3)); % y space step
27     dz = diff(z(2:3)); % z space step
28     ix = 2:size(u,1)-1; % x interior points in a patch
29     iy = 2:size(u,2)-1; % y interior points in a patch
30     iz = 2:size(u,3)-1; % z interior points in a patch
31     ut = nan(size(u)); % preallocate output array
32     ut(ix,iy,iz,1,:) = u(ix,iy,iz,2,:);
33     ut(ix,iy,iz,2,:) ...
34     =diff(patches.cs(:,iy,iz,1,:).*diff(u(:,iy,iz,1,:),1),1)/dx^2 ...
35     +diff(patches.cs(ix,:,:iz,2,:).*diff(u(ix,:,:iz,1,:),1,2),1,2)/dy^2 ...
36     +diff(patches.cs(ix,iy,:,:3,:).*diff(u(ix,iy,:,:1,:),1,3),1,3)/dz^2;
37 end% function

```

### 3.16.4 Parse input arguments and defaults

```

367 p = inputParser;
368 fnValidation = @(f) isa(f, 'function_handle'); %test for a function name
369 addRequired(p,'fun',fnValidation);

```

---

```

370 addRequired(p,'Xlim',@isnumeric);
371 addRequired(p,'BCs'); % nothing yet decided
372 addRequired(p,'nPatch',@isnumeric);
373 addRequired(p,'ordCC',@isnumeric);
374 addRequired(p,'ratio',@isnumeric);
375 addRequired(p,'nSubP',@isnumeric);
376 addParameter(p,'nEdge',1,@isnumeric);
377 addParameter(p,'EdgyInt',false,@islogical);
378 addParameter(p,'nEnsem',1,@isnumeric);
379 addParameter(p,'hetCoeffs',[],@isnumeric);
380 %addParameter(p,'nCore',1,@isnumeric); % not yet implemented
381 parse(p,fun,Xlim,BCs,nPatch,ordCC,ratio,nSubP,varargin{:});

```

Set the optional parameters.

```

387 patches.nEdge = p.Results.nEdge;
388 patches.EdgyInt = p.Results.EdgyInt;
389 patches.nEnsem = p.Results.nEnsem;
390 cs = p.Results.hetCoeffs;
391 %patches.nCore = p.Results.nCore;

```

Initially duplicate parameters for both space dimensions as needed.

```

399 if numel(Xlim)==2, Xlim = repmat(Xlim,1,3); end
400 if numel(nPatch)==1, nPatch = repmat(nPatch,1,3); end
401 if numel(ratio)==1, ratio = repmat(ratio,1,3); end
402 if numel(nSubP)==1, nSubP = repmat(nSubP,1,3); end

```

Check parameters.

```

409 assert(patches.nEdge==1 ...
410     , 'multi-edge-value interp not yet implemented')
411 assert(all(2*patches.nEdge<nSubP) ...
412     , 'too many edge values requested')
413 %if patches.nCore>1
414 %    warning('nCore>1 not yet tested in this version')
415 %    end

```

### 3.16.5 The code to make patches

First, store the pointer to the time derivative function in the struct.

```

429 patches.fun = fun;

```

Second, store the order of interpolation that is to provide the values for the inter-patch coupling conditions. Spectral coupling is ordCC of 0 or -1.

```

438 assert((ordCC>=-1) & (floor(ordCC)==ordCC), ...
439     'ordCC out of allowed range integer>=-1')

```

For odd ordCC do interpolation based upon odd neighbouring patches as is useful for staggered grids.

```

446 patches.stag = mod(ordCC,2);
447 assert(patches.stag==0,'staggered not yet implemented??')
448 ordCC = ordCC+patches.stag;
449 patches.ordCC = ordCC;

Check for staggered grid and periodic case.

455 if patches.stag, assert(mod(nPatch,2)==0, ...
456     'Require an even number of patches for staggered grid')
457 end

```

Might as well precompute the weightings for the interpolation of field values for coupling. (Could sometime extend to coupling via derivative values.)

```

466 ratio = reshape(ratio,1,3); % force to be row vector
467 if ordCC>0, patchCwts(ratio,ordCC,patches.stag), end

```

Third, set the centre of the patches in a the macroscale grid of patches assuming periodic macroscale domain.

```

476 X = linspace(Xlim(1),Xlim(2),nPatch(1)+1);
477 X = X(1:nPatch(1))+diff(X)/2;
478 DX = X(2)-X(1);
479 Y = linspace(Xlim(3),Xlim(4),nPatch(2)+1);
480 Y = Y(1:nPatch(2))+diff(Y)/2;
481 DY = Y(2)-Y(1);
482 Z = linspace(Xlim(5),Xlim(6),nPatch(3)+1);
483 Z = Z(1:nPatch(3))+diff(Z)/2;
484 DZ = Z(2)-Z(1);

```

Construct the microscale in each patch, assuming Dirichlet patch edges, and a half-patch length of  $\text{ratio}(1) \cdot DX$ ,  $\text{ratio}(2) \cdot DY$  and  $\text{ratio}(3) \cdot DZ$ , unless `patches.EdgyInt` is set in which case the patches are of length  $\text{ratio} \cdot DX + dx$ ,  $\text{ratio} \cdot DY + dy$  and  $\text{ratio} \cdot DZ + dz$ .

```

496 nSubP = reshape(nSubP,1,3); % force to be row vector
497 assert(patches.EdgyInt | all(mod(nSubP,2)==1), ...
498     'configPatches3: nSubP must be odd')
499 i0 = (nSubP(1)+1)/2;
500 if ~patches.EdgyInt, dx = ratio(1)*DX/(i0-1);
501 else
502     dx = ratio(1)*DX/(nSubP(1)-2);
503 end
504 patches.x = dx*(-i0+1:i0-1)' + X; % micro-grid
505 patches.x = reshape(patches.x,nSubP(1),1,1,1,1,nPatch(1),1,1);
506 i0 = (nSubP(2)+1)/2;
507 if ~patches.EdgyInt, dy = ratio(2)*DY/(i0-1);
508 else
509     dy = ratio(2)*DY/(nSubP(2)-2);
510 end
511 patches.y = dy*(-i0+1:i0-1)' + Y; % micro-grid
512 patches.y = reshape(patches.y,1,nSubP(2),1,1,1,1,nPatch(2),1);
513 i0 = (nSubP(3)+1)/2;
514 if ~patches.EdgyInt, dz = ratio(3)*DZ/(i0-1);
515 else
516     dz = ratio(3)*DZ/(nSubP(3)-2);

```

```

514 end
515 patches.z = dz*(-i0+1:i0-1)'+Z; % micro-grid
516 patches.z = reshape(patches.z,1,1,nSubP(3),1,1,1,1,nPatch(3));

```

### 3.16.6 Set ensemble inter-patch communication

For EdgyInt or centre interpolation respectively, the right-edge/centre realisations  $1:n_{\text{Ensem}}$  are to interpolate to left-edge  $\text{le}$ , and the left-edge/centre realisations  $1:n_{\text{Ensem}}$  are to interpolate to  $\text{re}$ .  $\text{re}$  and  $\text{li}$  are ‘transposes’ of each other as  $\text{re}(\text{li})=\text{le}(\text{ri})$  are both  $1:n_{\text{Ensem}}$ . Similarly for bottom-edge/centre interpolation to top-edge via  $\text{to}$  and top-edge/centre interpolation to bottom-edge via  $\text{bo}$ .

The default is nothing shifty. This setting reduces the number of if-statements in function `patchEdgeInt3()`.

```

539 nE = patches.nEnsem;
540 patches.le = 1:nE; patches.ri = 1:nE;
541 patches.bo = 1:nE; patches.to = 1:nE;
542 patches.ba = 1:nE; patches.fr = 1:nE;

```

However, if heterogeneous coefficients are supplied via `hetCoeffs`, then do some non-trivial replications. First, get microscale periods, patch size, and replicate many times in order to subsequently sub-sample: `nSubP` times should be enough.

```

552 if ~isempty(cs)
553 [mx,my,mz,nc] = size(cs);
554 nx = nSubP(1); ny = nSubP(2); nz = nSubP(3);
555 cs = repmat(cs,nSubP);

```

If only one member of the ensemble is required, then sub-sample to patch size, and store coefficients in `patches` as is.

```
563 if nE==1, patches.cs = cs(1:nx-1,1:ny-1,1:nz-1,:); else
```

But for  $n_{\text{Ensem}} > 1$  an ensemble of  $m_x m_y m_z$  phase-shifts of the coefficients is constructed from the over-supply. Here code phase-shifts over the periods—the phase shifts are like Hankel-matrices.

```

573 patches.nEnsem = mx*my*mz;
574 patches.cs = nan(nx-1,ny-1,nz-1,nc,mx,my,mz);
575 for k = 1:mz
576     ks = (k:k+nz-2);
577     for j = 1:my
578         js = (j:j+ny-2);
579         for i = 1:mx
580             is = (i:i+nx-2);
581             patches.cs(:,:, :, i, j, k) = cs(is,js,ks,:);
582         end
583     end
584 end
585 patches.cs = reshape(patches.cs,nx-1,ny-1,nz-1,nc,[]);

```

Further, set a cunning left/right/bottom/top realisation of inter-patch coupling. The aim is to preserve symmetry in the system when also invoking `EdgyInt`. What this coupling does without `EdgyInt` is unknown. Use auto-replication.

```

595 mmx=(0:mx-1)'; mmy=0:my-1; mmz=shiftdim(0:mz-1,-1);
596 le = mod(mmx+mod(nx-2,mx),mx)+1;
597 patches.le = reshape( le+mx*(mmy+my*mmz) ,[],1);
598 ri = mod(mmx-mod(nx-2,mx),mx)+1;
599 patches.ri = reshape( ri+mx*(mmy+my*mmz) ,[],1);
600 bo = mod(mmy+mod(ny-2,my),my)+1;
601 patches.bo = reshape( 1+mmx+mx*(bo-1+my*mmz) ,[],1);
602 to = mod(mmy-mod(ny-2,my),my)+1;
603 patches.to = reshape( 1+mmx+mx*(to-1+my*mmz) ,[],1);
604 ba = mod(mmz+mod(nz-2,mz),mz)+1;
605 patches.ba = reshape( 1+mmx+mx*(mmy+my*(ba-1)) ,[],1);
606 fr = mod(mmz-mod(nz-2,mz),mz)+1;
607 patches.fr = reshape( 1+mmx+mx*(mmy+my*(fr-1)) ,[],1);

```

Issue warning if the ensemble is likely to be affected by lack of scale separation.  
Need to justify this and the arbitrary threshold more carefully??

```

615 if prod(ratio)*patches.nEnsem>0.9, warning( ...
616 'Probably poor scale separation in ensemble of coupled phase-shifts')
617 scaleSeparationParameter=ratio*patches.nEnsem
618 end

```

End the two if-statements.

```

624 end%if nEnsem=1 or >1
625 end%if not-empty(cs)

```

**Fin**

```
633 end% function
```

### 3.17 patchSmooth3(): interface 3D space to time integrators

To simulate in time with 3D spatial patches we often need to interface a users time derivative function with time integration routines such as `ode15s` or `PIRK2`. This function provides an interface. It assumes that the sub-patch structure is *smooth enough* so that the patch centre-values are sensible macroscale variables, and patch edge-values are determined by macroscale interpolation of the patch-centre values. Communicate patch-design variables to this function via the previously established global struct `patches`.

```

24 function dudt = patchSmooth3(t,u)
25 global patches

```

**Input**

- $\mathbf{u}$  is a vector of length  $\text{prod}(\text{nSubP}) \cdot \text{nVars} \cdot \text{nEnsem} \cdot \text{prod}(\text{nPatch})$  where there are  $\text{nVars} \cdot \text{nEnsem}$  field values at each of the points in the  $\text{nSubP}(1) \times \text{nSubP}(2) \times \text{nSubP}(3) \times \text{nPatch}(1) \times \text{nPatch}(2) \times \text{nPatch}(3)$  grid.
- $t$  is the current time to be passed to the user's time derivative function.
- **patches** a struct set by `configPatches3()` with the following information used here.
  - `.fun` is the name of the user's function `fun(t,u,x,y,z)` that computes the time derivatives on the patchy lattice. The array  $\mathbf{u}$  has size  $\text{nSubP}(1) \times \text{nSubP}(2) \times \text{nSubP}(3) \times \text{nVars} \times \text{nEnsem} \times \text{nPatch}(1) \times \text{nPatch}(2) \times \text{nPatch}(3)$ . Time derivatives must be computed into the same sized array, but herein the patch edge-values are overwritten by zeros.
  - `.x` is  $\text{nSubP}(1) \times 1 \times 1 \times 1 \times 1 \times 1 \times \text{nPatch}(1) \times 1 \times 1$  array of the spatial locations  $x_{ij}$  of the microscale grid points in every patch. Currently it *must* be an equi-spaced lattice on both macro- and microscales.
  - `.y` is similarly  $1 \times \text{nSubP}(2) \times 1 \times 1 \times 1 \times 1 \times \text{nPatch}(2) \times 1$  array of the spatial locations  $y_{ij}$  of the microscale grid points in every patch. Currently it *must* be an equi-spaced lattice on both macro- and microscales.
  - `.z` is similarly  $1 \times 1 \times \text{nSubP}(3) \times 1 \times 1 \times 1 \times 1 \times \text{nPatch}(3)$  array of the spatial locations  $z_{ij}$  of the microscale grid points in every patch. Currently it *must* be an equi-spaced lattice on both macro- and microscales.

## Output

- `dudt` is  $\text{prod}(\text{nSubP}) \cdot \text{nVars} \cdot \text{nEnsem} \cdot \text{prod}(\text{nPatch})$  vector of time derivatives, but with patch edge-values set to zero.

Reshape the fields  $\mathbf{u}$  as a 8D-array, and sets the edge values from macroscale interpolation of centre-patch values. [Section 3.18](#) describes `patchEdgeInt3()`.

```
94 u = patchEdgeInt3(u);
```

Ask the user function for the time derivatives computed in the array, overwrite its edge values with the dummy value of zero, then return to a to the user/integrator as column vector.

```
104 dudt = patches.fun(t,u,patches.x,patches.y,patches.z);
105 dudt([1 end],:,:,:,:) = 0;
106 dudt(:,[1 end],:,:,:) = 0;
107 dudt(:,:,1 end,:,:) = 0;
108 dudt = reshape(dudt,[],1);
```

Fin.

### 3.18 patchEdgeInt3(): sets 3D patch face values from 3D macroscale interpolation

Couples 3D patches across 3D space by computing their face values via macroscale interpolation. Assumes that the patch centre-values are sensible macroscale variables, and patch face values are determined by macroscale interpolation of the patch-centre values?? Communicate patch-design variables via the global struct `patches`.

```
19 function u = patchEdgeInt3(u)
20 global patches
```

#### Input

- `u` is a vector of length `nSubP1·nSubP2·nSubP3·nVars·nEnsem·nPatch1·nPatch2 · nPatch3` where there are `nVars` field values at each of the points in the `nSubP1·nSubP2·nSubP3·nPatch1·nPatch2·nPatch3` grid on the `nPatch1·nPatch2·nPatch3` array of patches.
- `patches` a struct set by `configPatches3()` which includes the following information.
  - `.x` is  $nSubP1 \times 1 \times 1 \times 1 \times 1 \times nPatch1 \times 1 \times 1$  array of the spatial locations  $x_{ijk}$  of the microscale grid points in every patch. Currently it *must* be an equi-spaced lattice on both macro- and micro-scales.
  - `.y` is similarly  $\times 1 nSubP2 \times 1 \times 1 \times 1 \times 1 \times nPatch2 \times 1$  array of the spatial locations  $y_{ijk}$  of the microscale grid points in every patch. Currently it *must* be an equi-spaced lattice on both macro- and micro-scales.
  - `.z` is similarly  $\times 1 \times 1 nSubP3 \times 1 \times 1 \times 1 \times 1 \times nPatch3$  array of the spatial locations  $z_{ijk}$  of the microscale grid points in every patch. Currently it *must* be an equi-spaced lattice on both macro- and micro-scales.
  - `.ordCC` is order of interpolation, currently (Aug 2020) only  $\{0, 2, 4, \dots\}$
  - `.stag` in  $\{0, 1\}$  is one for staggered grid (alternating) interpolation.
  - `.Cwtsr` and `.Cwtsl` define the coupling coefficients for finite width interpolation.
  - `.EdgyInt` true/false is true for interpolating patch-face values from opposite next-to-face values (often preserves symmetry).

#### Output

- `u` is `nSubP1·nSubP2·nSubP3·nVars·nEnsem·nPatch1·nPatch2·nPatch3` 8D array of the fields with face values set by interpolation.

Determine the sizes of things. Any error arising in the reshape indicates `u` has the wrong size.

```

96 [~,~,nz,~,~,~,~,Nz] = size(patches.z);
97 [~,ny,~,~,~,~,Ny,~] = size(patches.y);
98 [nx,~,~,~,Nx,~,~] = size(patches.x);
99 nEnsem = patches.nEnsem;
100 nVars = round( numel(u)/numel(patches.x) ...
101             /numel(patches.y)/numel(patches.z)/nEnsem );
102 numel= numel(u);
103 assert(numel(u) == nx*ny*nz*Nx*Ny*Nz*nVars*nEnsem ...
104       , 'patchEdgeInt3: input u has wrong size for parameters')
105 u = reshape(u,[nx ny nz nVars nEnsem Nx Ny Nz]);

```

With Dirichlet patches, the half-length of a patch is  $h = dx(n_\mu - 1)/2$  (or  $-2$  for specified flux), unless we are interpolating from next-to-face values, and the ratio needed for interpolation is then  $r = h/\Delta X$ . Compute lattice sizes from inside the patches as the face values may be NaNs, etc.

```

116 dx = patches.x(3,1,1,1,1,1,1)-patches.x(2,1,1,1,1,1,1);
117 DX = patches.x(2,1,1,1,1,2,1)-patches.x(2,1,1,1,1,1,1);
118 if patches.EdgyInt==0, rx = dx*(nx-1)/2/DX;
119   else
120     rx = dx*(nx-2)/DX;
121   end
122 dy = patches.y(1,3,1,1,1,1,1)-patches.y(1,2,1,1,1,1,1);
123 DY = patches.y(1,2,1,1,1,2,1)-patches.y(1,2,1,1,1,1,1);
124 if patches.EdgyInt==0, ry = dy*(ny-1)/2/DY;
125   else
126     ry = dy*(ny-2)/DY;
127   end
128 dz = patches.z(1,1,3,1,1,1,1)-patches.z(1,1,2,1,1,1,1);
129 DZ = patches.z(1,1,2,1,1,1,2)-patches.z(1,1,2,1,1,1,1);
130 if patches.EdgyInt==0, rz = dz*(nz-1)/2/DZ;
131   else
132     rz = dz*(nz-2)/DZ;
133   end

```

For the moment assume the physical domain is macroscale periodic so that the coupling formulas are simplest. Should eventually cater for periodic, odd-mid-gap, even-mid-gap, even-mid-patch, Dirichlet, Neumann, Robin?? These index vectors point to patches and their two immediate neighbours—currently not needed.

```

142 I=1:Nx; Ip=mod(I,Nx)+1; Im=mod(I-2,Nx)+1;
143 J=1:Ny; Jp=mod(J,Ny)+1; Jm=mod(J-2,Ny)+1;
144 K=1:Nz; Kp=mod(K,Nz)+1; Km=mod(K-2,Nz)+1;

```

The centre of each patch (as  $nx$  and  $ny$  are odd for centre-patch interpolation) is at indices

```

151 i0 = round((nx+1)/2);
152 j0 = round((ny+1)/2);
153 k0 = round((nz+1)/2);

```

**Lagrange interpolation gives patch-face values** Compute centred differences of the mid-patch values for the macro-interpolation, of all fields.

Assumes the domain is macro-periodic. Currently, only next-to-face interpolation is implemented.

```

165 ordCC=patches.ordCC;
166 if ordCC>0 % then finite-width polynomial interpolation
167   if patches.EdgeyInt % next-to-face values
168     uCorex = u([2 nx-1],2:(ny-1),2:(nz-1),:,:,I,J,K);
169     uCorey = u(2:(nx-1),[2 ny-1],2:(nz-1),:,:,I,J,K);
170     uCorez = u(2:(nx-1),2:(ny-1),[2 nz-1],:,:,I,J,K);
171   else
172     %disp('currently couple from the mid-planes??')
173     uCorex = u(i0,2:(ny-1),2:(nz-1),:,:,I,J,K);
174     uCorey = u(2:(nx-1),j0,2:(nz-1),:,:,I,J,K);
175     uCorez = u(2:(nx-1),2:(ny-1),k0,:,:,I,J,K);
176   end;
177   dmux = zeros([ordCC,size(uCorex)]); % 9D
178   dmuy = zeros([ordCC,size(uCorey)]); % 9D
179   dmuz = zeros([ordCC,size(uCorez)]); % 9D
180   if patches.stag % use only odd numbered neighbours
181     error('polynomial interpolation not yet for staggered patch coupling')
182   else %disp('starting standard interpolation')
183     dmux(1,:,:,:,:,:,I,:,:,:) = (uCorex(:,:,:,:, :,Ip,:,:)) ...
184     -uCorex(:,:,:,:, :,Im,:,:))/2; % \mu\delta
185     dmux(2,:,:,:,:,:,I,:,:,:) = (uCorex(:,:,:,:, :,Ip,:,:)) ...
186     -2*uCorex(:,:,:,:, :,I,:,:)+uCorex(:,:,:,:, :,Im,:,:)); % \delta^2
187     dmuy(1,:,:,:,:,:,J,:,:) = (uCorey(:,:,:,:, :,Jp,:,:)) ...
188     -uCorey(:,:,:,:, :,Jm,:,:))/2; % \mu\delta
189     dmuy(2,:,:,:,:,:,J,:,:) = (uCorey(:,:,:,:, :,Jp,:,:)) ...
190     -2*uCorey(:,:,:,:, :,J,:,:)+uCorey(:,:,:,:, :,Jm,:,:)); % \delta^2
191     dmuz(1,:,:,:,:,:,K) = (uCorez(:,:,:,:, :,Kp,:,:)) ...
192     -uCorez(:,:,:,:, :,Km,:,:))/2; % \mu\delta
193     dmuz(2,:,:,:,:,:,K) = (uCorez(:,:,:,:, :,Kp,:,:)) ...
194     -2*uCorez(:,:,:,:, :,K,:,:)+uCorez(:,:,:,:, :,Km,:,:)); % \delta^2
195   end% if odd/even

```

Recursively take  $\delta^2$  of these to form higher order centred differences.

```

202 for k = 3:ordCC
203   dmux(k,:,:,:,:,:,I,:,:,:) = dmux(k-2,:,:,:,:,:,Ip,:,:) ...
204   -2*dmux(k-2,:,:,:,:,:,I,:,:)+dmux(k-2,:,:,:,:,:,Im,:,:);
205   dmuy(k,:,:,:,:,:,J,:,:) = dmuy(k-2,:,:,:,:,:,Jp,:,:) ...
206   -2*dmuy(k-2,:,:,:,:,:,J,:,:)+dmuy(k-2,:,:,:,:,:,Jm,:,:);
207   dmuz(k,:,:,:,:,:,K) = dmuz(k-2,:,:,:,:,:,Kp,:,:) ...
208   -2*dmuz(k-2,:,:,:,:,:,K,:,:)+dmuz(k-2,:,:,:,:,:,Km,:,:);
209 end

```

Interpolate macro-values to be Dirichlet face values for each patch ([Roberts & Kevrekidis 2007](#), [Bunder et al. 2017](#)), using weights computed in `configPatches2()`. Here interpolate to specified order.

Where next-to-face values interpolate to the opposite face-values. When

we have an ensemble of configurations, different configurations might be coupled to each other, as specified by `patches.le`, `patches.ri`, `patches.to`, `patches.bo`, `patches.fr` and `patches.ba`.

```

225 k=1+patches.EdgyInt; % use centre or two faces
226 u(nx,2:(ny-1),2:(nz-1),:,patches.ri,I,:,:,:) ...
227 = uCorex(1,:,:,:,::,:)*(1-patches.stag) ...
228 +shiftdim(sum( patches.Cwtsr(:,1).*dmux(:,1,:,:,:,::,:,:) ),1);
229 u(1,2:(ny-1),2:(nz-1),:,patches.le,I,:,:,:) ...
230 = uCorex(k,:,:,:,::,:)*(1-patches.stag) ...
231 +shiftdim(sum( patches.Cwtsl(:,1).*dmux(:,k,:,:,:,::,:,:) ),1);
232 u(2:(nx-1),ny,2:(nz-1),:,patches.to,:,J,:,:) ...
233 = uCorey(:,1,:,:,:,::,:)*(1-patches.stag) ...
234 +shiftdim(sum( patches.Cwtsr(:,2).*dmuy(:, :,1,:,:,:,::,:,:) ),1);
235 u(2:(nx-1),1,2:(nz-1),:,patches.bo,:,J,:,:) ...
236 = uCorey(:,k,:,:,:,::,:)*(1-patches.stag) ...
237 +shiftdim(sum( patches.Cwtsl(:,2).*dmuy(:, :,k,:,:,:,::,:,:) ),1);
238 u(2:(nx-1),2:(ny-1),nz,:,patches.fr,:,:,:,K) ...
239 = uCorez(:, :,1,:,:,:,::,:)*(1-patches.stag) ...
240 +shiftdim(sum( patches.Cwtsr(:,3).*dmuz(:, :, :,1,:,:,:,::,:,:) ),1);
241 u(2:(nx-1),2:(ny-1),1,::,patches.ba,:,:,:,K) ...
242 = uCorez(:, :,k,:,:,:,::,:)*(1-patches.stag) ...
243 +shiftdim(sum( patches.Cwtsl(:,3).*dmuz(:, :, :,k,:,:,:,::,:,:) ),1);

```

**Case of spectral interpolation** Assumes the domain is macro-periodic. We interpolate in terms of the patch index  $j$ , say, not directly in space. As the macroscale fields are  $N$ -periodic in the patch index  $j$ , the macroscale Fourier transform writes the centre-patch values as  $U_j = \sum_k C_k e^{ik2\pi j/N}$ . Then the face-patch values  $U_{j\pm r} = \sum_k C_k e^{ik2\pi N(j\pm r)} = \sum_k C'_k e^{ik2\pi j/N}$  where  $C'_k = C_k e^{ikr2\pi/N}$ . For  $N$  patches we resolve ‘wavenumbers’  $|k| < N/2$ , so set row vector  $\mathbf{ks} = k2\pi/N$  for ‘wavenumbers’  $k = (0, 1, \dots, k_{\max}, -k_{\max}, \dots, -1)$  for odd  $N$ , and  $k = (0, 1, \dots, k_{\max}, \pm(k_{\max} + 1) - k_{\max}, \dots, -1)$  for even  $N$ .

```
266 else% spectral interpolation
```

Deal with staggered grid by doubling the number of fields and halving the number of patches (`configPatches2` tests there are an even number of patches). Then the patch-ratio is effectively halved. The patch faces are near the middle of the gaps and swapped.

```

276 if patches.stag % transform by doubling the number of fields
277 error('staggered grid not yet implemented??')
278 v=nan(size(u)); % currently to restore the shape of u
279 u=cat(3,u(:,1:2:nPatch,:),u(:,2:2:nPatch,:));
280 stagShift=reshape(0.5*[ones(nVars,1);-ones(nVars,1)],1,1,[]);
281 iV=[nVars+1:2*nVars 1:nVars]; % scatter interp to alternate field
282 r=r/2; % ratio effectively halved
283 nPatch=nPatch/2; % halve the number of patches
284 nVars=nVars*2; % double the number of fields
285 else % the values for standard spectral

```

```

286     stagShift = 0;
287     iV = 1:nVars;
288 end

```

Now set wavenumbers in the two directions into two vectors at the correct dimension. In the case of even  $N$  these compute the + -case for the highest wavenumber zig-zag mode,  $k = (0, 1, \dots, k_{\max}, +(k_{\max} + 1) - k_{\max}, \dots, -1)$ .

```

298 kMax = floor((Nx-1)/2);
299 krx = shiftdim( rx*2*pi/Nx*(mod((0:Nx-1)+kMax,Nx)-kMax) ,-4);
300 kMay = floor((Ny-1)/2);
301 kry = shiftdim( ry*2*pi/Ny*(mod((0:Ny-1)+kMay,Ny)-kMay) ,-5);
302 kMaz = floor((Nz-1)/2);
303 krz = shiftdim( rz*2*pi/Nz*(mod((0:Nz-1)+kMaz,Nz)-kMaz) ,-6);

```

Test for reality of the field values, and define a function accordingly. Could be problematic if some variables are real and some are complex, or if variables are of quite different sizes.

```

312 if max(abs(imag(u(:))))<1e-9*max(abs(u(:)))
313     uclean=@(u) real(u);
314 else uclean=@(u) u;
315 end

```

Compute the Fourier transform of the patch centre-values for all the fields. Unless doing patch-edgy interpolation when FT the next-to-face values. If there are an even number of points, then if complex, treat as positive wavenumber, but if real, treat as cosine. When using an ensemble of configurations, different configurations might be coupled to each other, as specified by `patches.le`, `patches.ri`, `patches.to`, `patches.bo`, `patches.fr` and `patches.ba`.

```

329 % indices of interior
330 ix=(2:nx-1)'; iy=2:ny-1; iz=shiftdim(2:nz-1,-1);
331 if ~patches.EdgyInt
332     Cle = fft(fft(fft( u(i0,j0,k0,:,:,:, :, :) ...
333         ,[],6),[],7),[],8);
334     Cbo = Cle; Cba = Cle;
335 else
336     Cle = fft(fft(fft( u(    2,iy,iz ,:, patches.le,:,:,:) ...
337         ,[],6),[],7),[],8);
338     Cri = fft(fft(fft( u(nx-1,iy,iz ,:, patches.ri,:,:,:) ...
339         ,[],6),[],7),[],8);
340     Cbo = fft(fft(fft( u(ix,2    ,iz ,:, patches.bo,:,:,:) ...
341         ,[],6),[],7),[],8);
342     Cto = fft(fft(fft( u(ix,ny-1,iz ,:, patches.to,:,:,:) ...
343         ,[],6),[],7),[],8);
344     Cba = fft(fft(fft( u(ix,iy,2    ,:, patches.ba,:,:,:) ...
345         ,[],6),[],7),[],8);
346     Cfr = fft(fft(fft( u(ix,iy,nz-1 ,:, patches.fr,:,:,:) ...
347         ,[],6),[],7),[],8);
348 end

```

Fill in the cross of Fourier-shifted mid-values

```

354 if ~patches.EdgeyInt
355 % yz-fraction of kry/z along left/right faces
356 ks = (iy-j0)*2/(ny-1).*kry+(iz-k0)*2/(nz-1).*krz;
357 Cle = Cle.*exp(1i*ks);
358 Cri = Cle;
359 % xz-fraction of krx/z along bottom/top faces
360 ks = (ix-i0)*2/(nx-1).*krx+(iz-k0)*2/(nz-1).*krz;
361 Cbo = Cbo.*exp(1i*ks);
362 Cto = Cbo;
363 % xy-fraction of krx/y along back/front faces
364 ks = (ix-i0)*2/(nx-1).*krx+(iy-j0)*2/(ny-1).*kry;
365 Cba = Cba.*exp(1i*ks);
366 Cfr = Cba;
367 end

```

Now invert the triple Fourier transforms to complete interpolation.

```

374 u(nx, iy, iz, :, :, :, :, :) = uclean( ifft(ifft(ifft( ...
375 Cle.*exp(1i*(stagShift+krx)) ,[],6),[],7),[],8) );
376 u( 1, iy, iz, :, :, :, :, :) = uclean( ifft(ifft(ifft( ...
377 Cri.*exp(1i*(stagShift-krx)) ,[],6),[],7),[],8) );
378 u(ix, ny, iz, :, :, :, :, :) = uclean( ifft(ifft(ifft( ...
379 Cbo.*exp(1i*(stagShift+kry)) ,[],6),[],7),[],8) );
380 u(ix, 1, iz, :, :, :, :, :) = uclean( ifft(ifft(ifft( ...
381 Cto.*exp(1i*(stagShift-kry)) ,[],6),[],7),[],8) );
382 u(ix, iy, nz, :, :, :, :, :) = uclean( ifft(ifft(ifft( ...
383 Cba.*exp(1i*(stagShift+krz)) ,[],6),[],7),[],8) );
384 u(ix, iy, 1, :, :, :, :, :) = uclean( ifft(ifft(ifft( ...
385 Cfr.*exp(1i*(stagShift-krz)) ,[],6),[],7),[],8) );
386 end% if spectral

```

Nan the values in corners and edges, of every 3D patch.

```

392 u(:, [1 ny], [1 nz], :)=nan;
393 u([1 nx], :, [1 nz], :)=nan;
394 u([1 nx], [1 ny], :, :)=nan;
395 end% function patchEdgeInt2

```

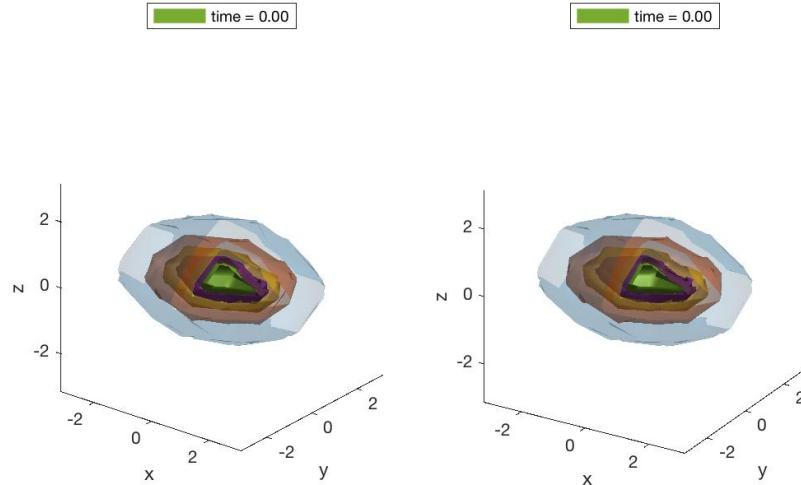
Fin, returning the 8D array of field values with interpolated faces.

### 3.19 homoDiffEdgy3: computational homogenisation of a 3D diffusion via simulation on small patches

Simulate heterogeneous diffusion in 3D space on 3D patches as an example application. Then compute macroscale eigenvalues of the patch scheme applied to this heterogeneous diffusion to validate and to compare various orders of inter-patch interpolation.

This code extends to 3D the 2D code discussed in [Section 3.15](#). First set random heterogeneous diffusivities of random (small) period in each of the

*Figure 3.22: initial field  $u(x, y, z, 0)$  of the patch scheme applied to a heterogeneous diffusion PDE. Plotted are the isosurfaces at field values  $u = 0.1, 0.3, \dots, 0.9$ , with the front quadrant omitted so you can see inside. Figure 3.23 plots the isosurfaces of the computed field at time  $t = 0.3$ .*



three directions. Crudely normalise by the harmonic mean so the decay time scale is roughly one.

```

29 mPeriod = randi([2 3],1,3)
30 cHetr = exp(0.3*randn([mPeriod 3]));
31 cHetr = cHetr*mean(1./cHetr(:))

```

Configure the patch scheme with some arbitrary choices of domain, patches, size ratios. Use spectral interpolation as we test other orders subsequently. In 3D we appear to get only real eigenvalues by using edgy interpolation. What happens for non-edgy interpolation is unknown.

```

42 nSubP=mPeriod+2;
43 nPatch=[5 5 5];
44 configPatches3(@heteroDiff3, [-pi pi], nan, nPatch ...
45 ,0, 0.3, nSubP, 'EdgyInt',true ...
46 , 'hetCoeffs',cHetr );

```

### 3.19.1 Simulate heterogeneous diffusion

Set initial conditions of a simulation as shown in Figure 3.22.

```

56 global patches
57 u0 = exp(-patches.x.^2/4-patches.y.^2/2-patches.z.^2);
58 u0 = u0.*((1+0.3*rand(size(u0)));

```

Integrate using standard integrators, unevenly spaced in time to better display transients.

```

76 if ~exist('OCTAVE_VERSION','builtin')
77     [ts,us] = ode23(@patchSmooth3, 0.3*linspace(0,1,50).^2, u0(:));
78 else % octave version
79     [ts,us] = ode0cts(@patchSmooth3, 0.3*linspace(0,1).^2, u0(:));
80 end

```

**Plot the solution** as an animation over time.

```

88 figure(1), clf
89 rgb=get(gca,'defaultAxesColorOrder');
90 colormap(0.8*hsv)

```

Get spatial coordinates of patch interiors.

```

96 x = reshape( patches.x([2:end-1],:,:, :) ,[],1);
97 y = reshape( patches.y(:,[2:end-1],:,:) ,[],1);
98 z = reshape( patches.z(:,:, [2:end-1],:) ,[],1);

```

For every time step draw the surface and pause for a short display.

```
105 for i = 1:length(ts)
```

Get the row vector of data, form into a 6D array, then omit patch faces, and reshape to suit the isosurface function. We do not use interpolation to get face values as the interpolation omits the corner edges and so breaks up the isosurfaces.

```

115 u = reshape( us(i,:), [nSubP nPatch]);
116 u = u([2:end-1],[2:end-1],[2:end-1],:,:,:);
117 u = reshape( permute(u,[1 4 2 5 3 6]) ...
118 , [numel(x) numel(y) numel(z)]);

```

Optionally cut-out the front corner so we can see inside.

```
124 u( (x>0) & (y<0) & (shiftdim(z,-2)>0) ) = nan;
```

The `isosurface` function requires us to transpose  $x$  and  $y$ .

```
131 v = permute(u,[2 1 3]);
```

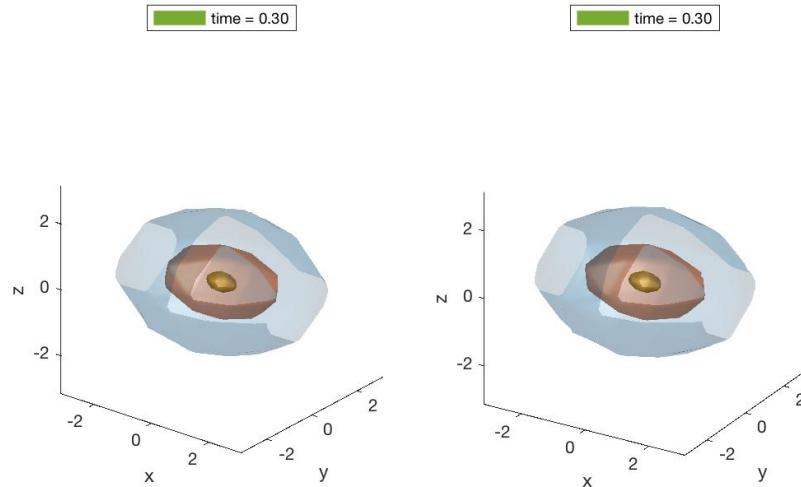
Draw cross-eyed stereo view of some isosurfaces.

```

137 clf;
138 for p=1:2
139     subplot(1,2,p)
140     for iso=5:-1:1
141         isov=(iso-0.5)/5;
142         hsurf(iso) = patch(isosurface(x,y,z,v,isov));
143         isonormals(x,y,z,v,hsurf(iso))
144         set(hsurf(iso) , 'FaceColor',rgb(iso,:)) ...
145             , 'EdgeColor', 'none' ...
146             , 'FaceAlpha',iso/5);

```

*Figure 3.23: final field  $u(x, y, z, 0.3)$  of the patch scheme applied to a heterogeneous diffusion PDE. Plotted are the isosurfaces at field values  $u = 0.1, 0.3, \dots, 0.9$ , with the front quadrant omitted so you can see inside.*



```

147      hold on
148  end
149  axis equal, view(45-7*p,25)
150  axis(pi*[-1 1 -1 1 -1 1])
151  xlabel('x'), ylabel('y'), zlabel('z')
152  legend(['time = ' num2str(ts(i),'%4.2f')], 'Location', 'north')
153  camlight, lighting gouraud
154  hold off
155  end% each p
156  if i==1 % pause for the viewer
157    makeJpeg=false;
158    if makeJpeg, print(['Figs/' mfilename 't0'], '-djpeg'), end
159    disp('Press any key to start animation')
160    pause
161  else pause(0.05)
162  end
178  end%for over time
179  if makeJpeg, print(['Figs/' mfilename 'tFin'], '-djpeg'), end

```

Finish the animation loop, and optionally output the isosurfaces of the final field, [Figure 3.23](#).

### 3.19.2 Compute Jacobian and its spectrum

Let's explore the Jacobian dynamics for a range of orders of interpolation, all for the same random patch design and heterogeneity. Except here use a small ratio as we do not plot and then the scale separation is clearest.

```
195 ratio = 0.025*(1+rand(1,3))
196 nSubP=randi([3 5],1,3)
197 nPatch=[3 3 3]
198 nEnsem = prod(mPeriod) % or just set one
```

Find which elements of the 8D array are interior micro-grid points and hence correspond to dynamical variables.

```
205 u0 = zeros([nSubP,1,nEnsem,nPatch]);
206 u0([1 end],:,:, :) = nan;
207 u0(:,[1 end],:,:, :) = nan;
208 u0(:,:, [1 end],:) = nan;
209 i = find(~isnan(u0));
210 sizeJacobian = length(i)
211 assert(sizeJacobian<4000 ...
212 , 'Jacobian is too big to quickly generate and analyse')
```

Store this many eigenvalues in array across different orders of interpolation.

```
219 nLeadEvals=prod(nPatch)+max(nPatch);
220 leadingEvals=[];
```

Evaluate eigenvalues for spectral as the base case for polynomial interpolation of order 2, 4, ....

```
228 maxords=6;
229 for ord=0:2:maxords
230     ord=ord
```

Configure with same heterogeneity.

```
236 configPatches3(@heteroDiff3,[-pi pi],nan,nPatch ...
237 ,ord,ratio,nSubP,'EdgyInt',true,'nEnsem',nEnsem ...
238 , 'hetCoeffs',cHetr);
```

Construct the Jacobian of the scheme as the matrix of the linear transformation, obtained by transforming the standard unit vectors.

```
246 jac = nan(length(i));
247 for j = 1:length(i)
248     u = u0(:)+(i(j)==(1:numel(u0))');
249     tmp = patchSmooth3(0,u);
250     jac(:,j) = tmp(i);
251 end
```

Test for symmetry, with error if we know it should be symmetric.

```
258 notSymmetric=norm(jac-jac')
259 %      if notSymmetric>1e-7, spy(abs(jac-jac'))>1e-7), end%??
```

---

```
260     assert(notSymmetric<1e-7,'failed symmetry')
```

Find all the eigenvalues (as `eigs` is unreliable), and put eigenvalues in a vector.

```
267     [evecs,evals] = eig((jac+jac')/2,'vector');
268     biggestImag=max(abs(imag(evals)));
269     if biggestImag>0, biggestImag=biggestImag, end
```

Sort eigenvalues on their real-part with most positive first, and most negative last. Store the leading eigenvalues in `egs`, and write out when computed all orders. The number of zero eigenvalues, `nZeroEv`, gives the number of decoupled systems in this patch configuration.

```
279     [~,k] = sort(-real(evals));
280     evals=evals(k); evecs=evecs(:,k);
281     if ord==0, nZeroEv=sum(abs(evals(:))<1e-5), end
282     if ord==0, evec0=evecs(:,1:nZeroEv*nLeadEvals);
283     else % find evec closest to that of each leading spectral
284         [~,k]=max(abs(evecs'*evec0));
285         evals=evals(k); % re-sort in corresponding order
286     end
287     leadingEvals=[leadingEvals evals(nZeroEv*(1:nLeadEvals))];
288 end
289 disp('      spectral      quadratic      quartic      sixth-order ...')
290 leadingEvals=leadingEvals
```

### 3.19.3 `heteroDiff3()`: heterogeneous diffusion

This function codes the lattice heterogeneous diffusion inside the patches. For 8D input arrays `u`, `x`, `y`, and `z` (via edge-value interpolation of `patchSmooth3`, Section 3.17), computes the time derivative (3.1) at each point in the interior of a patch, output in `ut`. The three 3D array of diffusivities,  $c_{ijk}^x$ ,  $c_{ijk}^y$  and  $c_{ijk}^z$ , have previously been stored in `patches.cs` (4D).

```
20 function ut = heteroDiff3(t,u,x,y,z)
21     global patches
22     dx = diff(x(2:3)); % x space step
23     dy = diff(y(2:3)); % y space step
24     dz = diff(z(2:3)); % z space step
25     ix = 2:size(u,1)-1; % x interior points in a patch
26     iy = 2:size(u,2)-1; % y interior points in a patch
27     iz = 2:size(u,3)-1; % z interior points in a patch
28     ut = nan(size(u)); % preallocate output array
29     ut(ix,iy,iz,:,:,:,:,:) ...
30     = diff(patches.cs(:,iy,iz,1,:).*diff(u(:,iy,iz,:,:,:,:,:),1),1)/dx^2 ...
31     +diff(patches.cs(ix,:,iz,2,:).*diff(u(ix,:,iz,:,:,:,:,:),1,2),1,2)/dy^2 ...
32     +diff(patches.cs(ix,iy,:,3,:).*diff(u(ix,iy,:,:,:,:,:),1,3),1,3)/dz^2;
33 end% function
```

Fin.

## 3.20 To do

- Some users will have microscale that has a fixed microscale lattice spacing, in which case we should code the scale ratio  $r$  to follow from the choice of the number of lattice points in a patch.
- Core averages code.
- Parallel processing versions. Matlab makes it easy with `distributed`, `codistributed`, et al.
- Adapt to maps in micro-time? Surely easy, just an example.

## 3.21 Miscellaneous tests

### 3.21.1 patchEdgeInt1test: test the 1D patch coupling

A script to test the spectral and finite-order polynomial interpolation of function `patchEdgeInt1()`. Tests one or several variables, normal and staggered grids, and also tests centre and edge interpolation. But does not yet test core averaging, etc.

Start by establishing global data struct for the range of various cases.

```
22 clear all
23 global patches
24 nSubP=5
```

Test standard spectral interpolation

Test over various numbers of patches, random domain lengths and random ratios. Say do three realisations for each number of patches.

```
41 nReal=3
42 for nPatch=repmat(5:10,1,nReal)
43 nPatch=nPatch
44 Len=10*rand
45 ratio=0.5*rand
46 configPatches1(@sin,[0,Len],nan,nPatch,0,ratio,nSubP ...
47 , 'EdgyInt',rand<0.5); % random Edgy or not
48 if mod(nPatch,2)==0, disp('Avoiding highest wavenumber'), end
49 kMax=floor((nPatch-1)/2);
50 if patches.EdgyInt, i0=[2 nSubP-1], else i0=(nSubP+1)/2, end
```

**Test single field** Set a profile, and evaluate the interpolation.

```
58 for k=-kMax:kMax
59 u0=exp(1i*k*patches.x*2*pi/Len);
60 ui=patchEdgeInt1(u0(:));
61 normError=rms(ui(:)-u0(:));
62 if abs(normError)>5e-14
63 normError=normError, k=k
64 error(['failed single var interpolation k=' num2str(k)])
```

```

65     end
66 end

```

**Test multiple fields** Use this to measure some of the errors in order to omit singleton dimensions, and also squish any errors if the second argument is essential zero (to cater for cosine aliasing errors).

```

77 normDiff=@(u,v) ...
78     norm(squeeze(u)-squeeze(v))*norm(squeeze(v(i0,:,:,:)));

```

Set a profile, and evaluate the interpolation. For the case of the highest wavenumber, squash the error when the centre-patch values are all zero by multiplying by result norm. Not yet working for edgy interpolation.

```

88 for k=1:(nPatch-1)/2 % not checking the highest wavenumber
89     u0=sin(k*patches.x*2*pi/Len);
90     v0=cos(k*patches.x*2*pi/Len);
91     uvi=patchEdgeInt1( reshape([u0 v0],[],1) );
92     normuError=normDiff(uvi(:,1,:,:),u0);
93     normvError=normDiff(uvi(:,2,:,:),v0);
94     if abs(normuError)+abs(normvError)>2e-13
95         normuError=normuError, normvError=normvError
96         error(['failed double field interpolation k=' num2str(k)])
97     end
98 end

```

End the for-loop over various geometries.

```

105 end
106 disp('Passed standard spectral interpolation tests')

```

Now test spectral interpolation on staggered grid

Must have even number of patches for a staggered grid.

```

122 disp('*** spectral interpolation on staggered grid')
123 for nPatch=repmat(6:2:20,1,nReal)
124     nPatch=nPatch
125     ratio=0.5*rand
126     nSubP=7; % of form 4*N-1
127     Len=10*rand
128     configPatches1(@simpleWavePde,[0,Len],nan,nPatch,-1,ratio,nSubP ...
129                 , 'EdgyInt',rand<0.5);
130     if mod(nPatch,4)==0, disp('Avoiding highest wavenumber'), end
131     kMax=floor((nPatch/2-1)/2)
132     if patches.EdgyInt, i0=[2 nSubP-1], else i0=(nSubP+1)/2, end%??

```

Identify which microscale grid points are  $h$  or  $u$  values.

```

138 uPts=mod( (1:nSubP)+(1:nPatch) ,2);
139 hPts=find(1-uPts);
140 uPts=find(uPts);

```

Set a profile for various wavenumbers. The capital letter U denotes an array of values merged from both  $u$  and  $h$  fields on the staggered grids.

```

148 fprintf('Staggered: single field-pair test.\n')
149 for k=-kMax:kMax
150     U0=nan(nSubP,nPatch);
151     U0(hPts)=rand*exp(+1i*k*patches.x(hPts)*2*pi/Len);
152     U0(uPts)=rand*exp(-1i*k*patches.x(uPts)*2*pi/Len);
153     Ui=patchEdgeInt1(U0(:));
154     normError=norm(Ui(:)-U0(:));
155     if abs(normError)>5e-14
156         normError=normError
157         patches=patches
158         error(['staggered: failed single sys interpolation k=' num2str(k)])
159     end
160 end

```

**Test multiple fields** Use this to measure some of the errors in order to omit singleton dimensions, and also squish any errors if the third argument is essential zero (to cater for cosine aliasing errors).

```

171     normDiff=@(u,v,w) ...
172     norm(squeeze(u)-squeeze(v))*norm(squeeze(w(i0,:,:,:)));

```

Set a profile, and evaluate the interpolation. For the case of the highest wavenumber zig-zag, squash the error when the alternate centre-patch values are all zero. First shift the  $x$ -coordinates so that the zig-zag mode is centred on a patch.

```

182 fprintf('Staggered: Two field-pairs test.\n')
183 x0=patches.x((nSubP+1)/2,1);
184 patches.x=patches.x-x0;
185 oddP=1:2:nPatch; evnP=2:2:nPatch;
186 for k=1:kMax
187     U0=nan(nSubP,1,1,nPatch); V0=U0;
188     U0(hPts)=rand*sin(k*patches.x(hPts)*2*pi/Len);
189     U0(uPts)=rand*sin(k*patches.x(uPts)*2*pi/Len);
190     V0(hPts)=rand*cos(k*patches.x(hPts)*2*pi/Len);
191     V0(uPts)=rand*cos(k*patches.x(uPts)*2*pi/Len);
192     UVi=patchEdgeInt1([U0 V0]);
193 %     U0i0odd=squeeze(U0(i0,:,:,:,oddP)), U0i0evn=squeeze(U0(i0,:,:,:,evnP))
194 %     V0i0odd=squeeze(V0(i0,:,:,:,oddP)), V0i0evn=squeeze(V0(i0,:,:,:,evnP))
195     normuError=[normDiff(UVi(:,1,:,:,:,oddP),U0(:,:, :,oddP),U0(:,:, :,evnP))
196                 normDiff(UVi(:,1,:,:,:,evnP),U0(:,:, :,evnP),U0(:,:, :,oddP))];
197     normvError=[normDiff(UVi(:,2,:,:,:,oddP),V0(:,:, :,oddP),V0(:,:, :,evnP))
198                 normDiff(UVi(:,2,:,:,:,evnP),V0(:,:, :,evnP),V0(:,:, :,oddP))];
199     if norm(normuError)+norm(normvError)>2e-13
200         normuError=normuError, normvError=normvError
201     %     [U0 UVi(:,1,:,:,:) V0 UVi(:,2,:,:,:)]
202     patches=patches

```

```

203     error(['staggered: failed double field interpolation k=' num2str(k)])
204 end
205 end

    End for-loop over patches

212 end

```

Check standard finite width interpolation

Check over various types and orders of interpolation, numbers of patches, random domain lengths and random ratios. (The `@sin` is a dummy.)

```

229 for edgyInt=[mod(nSubP,2)==0 true]
230 for ordCC=2:2:8
231 for nPatch=ordCC+(2:4)
232     edgyInt=edgyInt
233     ordCC=ordCC
234     nPatch=nPatch
235     Domain=5*[-rand rand]
236     ratio=0.5*rand
237     configPatches1(@sin,Domain,nan,nPatch,ordCC,ratio,nSubP ...
238         , 'EdgyInt',edgyInt);

```

**Check multiple fields simultaneously** Set profiles to be various powers of  $x$ ,  $ps$ , and store as different ‘variables’ at each point.

```

247 ps=1:ordCC
248 cs=randn(size(ps))
249 u0=patches.x.^ps.*cs+randn;

```

Then evaluate the interpolation and squeeze the singleton dimension of an ‘ensemble’.

```

256 ui=patchEdgeInt1(u0(:));
257 ui=squeeze(ui);

```

The interior patches should have zero error.

```

263 j=ordCC/2+1:nPatch-ordCC/2;
264 iError=ui(:,:,j)-u0(:,:,j);
265 normError=norm(iError(:))
266 assert(normError<5e-12 ...
267     , 'failed finite stencil interpolation')

```

End the for-loops over various parameters.

```

274 end,end,end
275 disp('Passed all standard polynomial interpolation')

```

### 3.21.2 patchEdgeInt2test: tests 2D patch coupling

A script to test the spectral and finite-order polynomial interpolation of function `patchEdgeInt2()`. Tests one or several variables, normal grids, and

also tests centre and edge interpolation. But does not yet test staggered grids, core averaging, etc.

Start by establishing global data struct for the range of various cases.

```
18 clear all, close all
19 global patches
```

Test standard spectral interpolation

Test over various numbers of patches, random domain lengths and random ratios. Try 99 realisations of random tests.

```
31 for realisation=1:99
```

Choose and configure random sized domains, random sub-patch resolution, random size-ratios, random number of periodic-patches, randomly edge or mid-patch interpolation.

```
39 Lx = 1+3*rand, Ly = 1+3*rand
40 nSubP = 1+2*randi(3,1,2)
41 ratios = rand(1,2)/2
42 nPatch = randi([3 6],1,2)
43 edgyInt = (rand>0.5)
44 configPatches2(@sin,[0 Lx 0 Ly],nan,nPatch,0,ratios,nSubP ...
45 , 'EdgyInt',edgyInt)
```

Choose a random number of fields, then generate trigonometric shape with random wavenumber and random phase shift. But if an even number of patches in either direction, then do not test the highest wavenumber because of aliasing problem.

```
55 nV=randi(3)
56 [nx,Nx]=size(squeeze(patches.x));
57 [ny,Ny]=size(squeeze(patches.y));
58 u0=nan(nx,ny,nV,1,Nx,Ny);
59 for iV=1:nV
60     kx=randi([0 floor((nPatch(1)-1)/2)])
61     ky=randi([0 floor((nPatch(2)-1)/2)])
62     phix=pi*rand*(2*kx~=nPatch(1))
63     phiy=pi*rand*(2*ky~=nPatch(2))
64     % generate 6D array via auto-replication
65     u0(:,:,:,iV,1,:,:)=sin(2*pi*kx*patches.x/Lx+phix) ...
66             .*sin(2*pi*ky*patches.y/Ly+phiy);
67 end
```

Copy and NaN the edges, then interpolate

```
73 u=u0; u([1 end],:,:)=nan; u(:,[1 end],:)=nan;
74 u=patchEdgeInt2(u(:));
```

Compute difference, ignoring the nans which should only be in the corners. If there is an error in the interpolation, then abort the script for checking: please record parameter values and inform us.

```

83  err=u-u0;
84  normerr=norm(err(~isnan(err)))
85  assert(normerr<1e-12, '2D interpolation failed')

    End the for-loop over realisations

92  disp('***** This test passed')
93  end

```

Check standard finite width interpolation

Check over various types and orders of interpolation, numbers of patches, random domain lengths and random ratios. (The @sin is a dummy.)

```

111 for ordCC=2:2:8
112 for realisations=1:9
113     ordCC=ordCC
114     nPatch=ordCC+randi([2 4],1,2)
115     edgyInt = (rand>0.5)
116     nSubP = 1+2*randi(3,1,2)
117     Domain=5*[-rand rand -rand rand]
118     ratios=0.5*rand(1,2)
119     configPatches2(@sin,Domain,nan,nPatch,ordCC,ratios,nSubP ...
120         , 'EdgyInt',edgyInt);

```

**Check multiple fields simultaneously** Set profiles to be various powers of  $x$ ,  $ps$ , and store as different ‘variables’ at each point.

```

129 [ps,qs]=meshgrid(0:ordCC);
130 ps=reshape(ps,1,1,[]); qs=reshape(qs,1,1,[]);
131 u0=ones(size(ps)).*patches.x.^ps.*patches.y.^qs;

```

Then evaluate the interpolation.

```
137 ui=patchEdgeInt2(u0(:));
```

The interior patches should have zero error. Appear to need error tolerance of  $10^{-8}$  because of the size of the domain and the high order of interpolation.

```

145 I=ordCC/2+1:nPatch(1)-ordCC/2;
146 J=ordCC/2+1:nPatch(2)-ordCC/2;
147 iError=ui(:,:, :, :, I, J)-u0(:,:, :, :, I, J);
148 normError=norm(iError(~isnan(iError)))
149 assert(normError<5e-8 ...
150     , 'failed finite stencil interpolation')

```

End the for-loops over various parameters.

```

157 end,end %order and realisations
158 disp('Passed all standard polynomial interpolation')

```

Finished

If no error messages, then all OK.

```
172 disp('**** If you read this, then all the tests were successful')
```

### 3.21.3 patchEdgeInt3test: tests 3D patch coupling

A script to test the spectral and finite-order polynomial interpolation of function `patchEdgeInt3()`. Tests one or several variables, normal grids, and also tests centre and edge interpolation. But does not yet test staggered grids, core averaging, etc.

Start by establishing global data struct for the range of various cases.

```
18 clear all, close all
19 global patches
```

Test standard spectral interpolation

Test over various numbers of patches, random domain lengths and random ratios. Try 99 realisations of random tests.

```
31 for realisation=1:99
```

Choose and configure random sized domains, random sub-patch resolution, random size-ratios, random number of periodic-patches, randomly edge or mid-patch interpolation.

```
39 Lx = 1+3*rand, Ly = 1+3*rand, Lz = 1+3*rand
40 nSubP = 1+2*randi(3,1,3)
41 ratios = 0.5*rand(1,3)
42 nPatch = randi([3 6],1,3)
43 edgyInt = (rand>0.5)
44 configPatches3(@sin,[0 Lx 0 Ly 0 Lz],nan,nPatch,0,ratios,nSubP ...
45 , 'EdgyInt',edgyInt)
```

Choose a random number of fields, then generate trigonometric shape with random wavenumber and random phase shift. But if an even number of patches in either direction, then do not test the highest wavenumber because of aliasing problem.

```
55 nV=randi(3)
56 [nx,Nx]=size(squeeze(patches.x));
57 [ny,Ny]=size(squeeze(patches.y));
58 [nz,Nz]=size(squeeze(patches.z));
59 u0=nan(nx,ny,nz,nV,1,Nx,Ny,Nz);
60 for iV=1:nV
61     ks=floor( rand(1,3).*floor(([Nx Ny Nz]+1)/2) )
62     phis = pi*rand(1,3).*(2*ks-[Nx Ny Nz])
63     % generate 8D array via auto-replication
64     u0(:,:,:,:,iV,1,:,:,:)=sin(2*pi*ks(1)*patches.x/Lx+phis(1)) ...
65             .*sin(2*pi*ks(2)*patches.y/Ly+phis(2)) ...
66             .*sin(2*pi*ks(3)*patches.z/Lz+phis(3));
67 end
```

Copy and NaN the edges, then interpolate

```

73  u=u0;
74  u([1 end],:,:,:) =nan; u(:,[1 end],:,:)=nan; u(:,:,,[1 end],:) =nan;
75  u=patchEdgeInt3(u(:));

```

Compute difference, ignoring the nans which should only be in the corners. If there is an error in the interpolation, then abort the script for checking: please record parameter values and inform us.

```

84  err=u-u0;
85  normerr=norm(err(~isnan(err)))
86  assert(normerr<1e-12, '3D interpolation failed')

```

End the for-loop over realisations

```

93  disp('***** This test passed')
94  end
95  disp('If you read this, then all spectral tests passed')

```

Check polynomial finite width interpolation

Check over various types and orders of interpolation, numbers of patches, random domain lengths and random ratios. (The `@sin` is a dummy.)

```

113 for ordCC=2:2:6
114 for realisations=1:9
115     ordCC=ordCC
116     nPatch=ordCC+randi([1 4],1,3)
117     edgyInt = (rand>0.5)
118     nSubP = 1+2*randi(3,1,3)
119     Domain=5*[-rand rand -rand rand -rand rand]
120     ratios=0.5*rand(1,3)
121     configPatches3(@sin,Domain,nan,nPatch,ordCC,ratios,nSubP ...
122         , 'EdgyInt',edgyInt);

```

**Check multiple fields simultaneously** Set profiles to be various powers of  $x, y, z$ , namely `ps`, `qs`, `rs`, and store as different ‘variables’ at each point.

```

132 [ps,qs,rs]=meshgrid(0:ordCC);
133 ps=reshape(ps,1,1,1,[]);
134 qs=reshape(qs,1,1,1,[]);
135 rs=reshape(rs,1,1,1,[]);
136 u0=ones(size(ps)).*patches.x.^ps ...
137     .*patches.y.^qs.*patches.z.^rs;

```

Then evaluate the interpolation.

```
143 ui=patchEdgeInt3(u0(:));
```

The interior patches should have zero error. Appear to need error tolerance of  $10^{-8}$  because of the size of the domain and the high order of interpolation.

```

151 I=ordCC/2+1:nPatch(1)-ordCC/2;
152 J=ordCC/2+1:nPatch(2)-ordCC/2;
153 K=ordCC/2+1:nPatch(3)-ordCC/2;

```

```
154     iError=ui(:,:,(:,:,I,J,K))-u0(:,:,(:,:,I,J,K));
155     normError=norm(iError(~isnan(iError)));
156     assert(normError<5e-8 ...
157         ,’failed finite stencil polynomial interpolation’)
```

End the for-loops over various parameters.

```
164 end,end %order and realisations
165 disp(’Passed all polynomial interpolation tests’)
```

Finished

If no error messages, then all OK.

```
179 disp(’**** If you read this, then all the tests were successful’)
```

---

## Appendix A Create, document and test algorithms

---

- Upon ‘finalising’ a version of the toolbox:
  1. pdflatex and bibtex `Doc/eqnFreeDevMan.tex` to ensure all is documented properly;
  2. execute `bibexport eqnFreeDevMan` to update the local bibliographic data-file;
  3. pdflatex `Doc/eqnFreeUserMan.tex`, several times, to get a shorter and more user friendly version;
  4. replace the root file `eqnFreeUserMan-newest.pdf` by a renamed copy of the new `Doc/eqnFreeUserMan.pdf`
- To create and document the various functions, we adapt an idea due to Neil D. Lawrence of the University of Sheffield in order to interleave MATLAB/Octave code, and its documentation in LaTeX ([Table A.2](#)).
- Each class of toolbox functions is located in separate folders in the repository, say `Dir`.
- Create a LaTeX file `Dir/funs.tex`: establish as one LaTeX chapter that `\input{../Dir/*.m}`s the files of the functions in the class, example scripts of use, and possibly test scripts, [Table A.1](#).
- Each such `Dir/funs.tex` file is to be included from the main LaTeX file `Doc/docBody.tex` so that people can most easily work on one chapter at a time:
  - create a ‘link’ file `Doc/funs.tex` whose only active content is the command `\input{../Dir/funs.tex}`;
  - put `\include{funs}` into `Doc/docBody.tex`;
  - in `Doc/docBody.tex` modify the `\graphicspath` command to include `{../Dir/Figs}`.
- Each toolbox function is documented as a separate section, within its chapter, with tests and examples as separate sections.
- Each function-section and test-section is to be created as a MATLAB/Octave `Dir/*.m` file, say `Dir/fun1.m`, so that users simply invoke the function in MATLAB/Octave as usual by `fun1(...)`.

Some editors may need to be told that `fun1.m` is a LaTeX file. For example, TexShop on the Mac requires one to execute (once) in a Terminal

```
defaults write TeXShop OtherTeXExtensions -array-add "m"
```

- [Table A.2](#) gives the template for the `Dir/*.m` function-sections. The format for a example/test-section is similar.

- Any figures from examples should be generated and then saved for later inclusion with the following (which finally works properly for MATLAB 2017+)

```
set(gcf, 'PaperUnits', 'centimeters', 'PaperPosition', [0 0 14 10]);% cm
print('-depsc2', 'filename')
```

If it is a suitable replacement for an existing graphic, then move it into the `Dir/Figs` folder. Include such a graphic into the LaTeX document with (do *not* postfix with `.eps` or `.pdf`)

```
\includegraphics[scale=0.9]{filename}
```

- In figures and other graphics, do *not* resize/scale fixed width constructs: instead use `\linewidth` to configure large-scale layout, `em` for small-widths, and `ex` for small-heights.
- For every function, generally include at the start of the function a simple example of its use. The example is only to be executed when the function is invoked with no input arguments (`if nargin==0`).

When appropriate, if a function is invoked with no output arguments (`if nargout==0`), then draw some reasonable graph of the results.

- In all MATLAB/Octave code, prefer camel case for variable names (not underscores).
- When a function is ‘finalised’, wrap (most) of the lines to be no more than 60 characters so that readers looking at the source can read the plain text reasonably.
- In the documentation (e.g., [Higham 1998](#), Ch. 4): write actively, not passively (e.g., avoid “-tion” words, and avoid “is/are verbed” phrases); avoid wishy-washy “can”; use the present tense; cross-reference precisely; avoid useless padding such as “note that”; and so on.

Table A.1: example `Dir/*.tex` file to typeset in the master document a function-section, say `fun.m`, and maybe the test/example-sections.

```
1 % input *.m files for ... Author, date
2 %!TEX root = ../Doc/eqnFreeDevMan.tex
3 \chapter{...}
4 \label{ch:...}
5 \localtableofcontents
6 \section{Introduction}
7 introduction...
8 \input{../Dir/fun.m} % prefix associated files with 'fun'
9 \input{../Dir/funExample.m}
10 ...
11 \begin{devMan}
12 \section{To do}
13 ...
14 \section{Miscellaneous tests}
15 \input{../Dir/funTest.m}
16 ...
17 \end{devMan}
```

Table A.2: template for a function-section Dir/\*.m file.

```
1 % Short explanation for users typing "help fun"
2 % Author, date
3 %!TEX root = ../Doc/eqnFreeDevMan.tex
4 %{
5 \section{\texttt{...}: ...}
6 \label{sec:...}
7 \localtableofcontents
8 \subsection{Introduction}
9 Overview LaTeX explanation.
10 \begin{matlab}
11 %}
12 function ...
13 %{
14 \end{matlab}
15 \paragraph{Input} ...
16 \paragraph{Output} ...
17 \begin{devMan}
18 Repeated as desired:
19 LaTeX between end-matlab and begin-matlab
20 \begin{matlab}
21 %}
22 Matlab code between \%} and \%{
23 %{
24 \end{matlab}
25 Concluding LaTeX before following final lines.
26 \end{devMan}
27 %}
```

---

## Appendix B Aspects of developing a ‘toolbox’ for patch dynamics

---

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This appendix documents sketchy further thoughts on aspects of the development.

### B.1 Macroscale grid

The patches are to be distributed on a macroscale grid: the  $j$ th patch ‘centred’ at position  $\vec{X}_j \in \mathbb{X}$ . In principle the patches could move, but let’s keep them fixed in the first version. The simplest macroscale grid will be rectangular (`meshgrid`), but we plan to allow a deformed grid to secondly cater for boundary fitting to quite general domain shapes  $\mathbb{X}$ . And plan to later allow for more general interconnect networks for more topologies in application.

### B.2 Macroscale field variables

The researcher/user has to know an appropriate set of macroscale field variables  $\vec{U}(t) \in \mathbb{R}^{d_U}$  for each patch. For example, first they might be a simple average over a core of a patch of all of the micro-field variables; second, they might be a subset of the average micro-field variables; and third in general the macro-variables might be a nonlinear function of the micro-field variables (such as temperature is the average speed squared). The core might be just one point, or a sizeable fraction of the patch.

The mapping from microscale variable to macroscale variables is often termed the restriction.

In practice, users may not choose an appropriate set of macro-variables, so will eventually need to code some diagnostic to indicate a failure of the assumed closure.

### B.3 Boundary and coupling conditions

The physical domain boundary conditions are distinct from the conditions coupling the patches together. Start with physical boundary conditions of periodicity in the macroscale.

Second, assume the physical boundary conditions are that the macro-variables are known at macroscale grid points around the boundary. Then the issue is to adjust the interpolation to cater for the boundary presence and shape. The coupling conditions for the patches should cater for the range of Robin-like boundary conditions, from Dirichlet to Neumann. Two possibilities arise: direct imposition of the coupling action ([Roberts & Kevrekidis 2007](#)), or control by the action.

Third, assume that some of the patches have some edges coincident with the boundary of the macroscale domain  $\mathbb{X}$ , and it is on these edges that macroscale physical boundary conditions are applied. Then the interpolation from the core of these edge patches is the same as the second case of prescribed boundary macro-variables. An issue is that each boundary patch should be big enough to cater for any spatial boundary layers transitioning from the applied boundary condition to the interior slow evolution.

Alternatively, we might have the physical boundary condition constrain the interpolation between patches.

Often microscale simulations are easiest to write when ‘periodic’ in microscale space. To cater for this we should also allow a control at perhaps the quartiles of a micro-periodic simulator.

### B.4 Mesotime communication

Since communication limits large scale parallelism, a first step in reducing communication will be to implement only updating the coupling conditions when necessary. Error analysis indicates that updating on times longer the microscale times and shorter than the macroscale times can be effective ([Bunder et al. 2016](#)). Implementations can communicate one or more derivatives in time, as well as macroscale variables.

At this stage we can effectively parallelise over patches: first by simply using Matlab’s `parfor`. Probably not using a GPU as we probably want to leave GPUs for the black-box to utilise within each patch.

### B.5 Projective integration

Have coded several schemes.

Should not need an implicit scheme as the fast dynamics are meant to be only in the micro variables, and the slow dynamics only in the macroscale variables.

However, it could be that the macroscale variables have fast oscillations and it is only the amplitude of the oscillations that are slow. Perhaps need to detect and then fix or advise, perhaps via DMD.

A further stage is to implement a projective integration scheme for stochastic macroscale variables: this is important because the averaging over a core of microscale roughness will almost invariably have at least some stochastic legacy effect. [Calderon \(2007\)](#) did some useful research on stochastic projective integration.

## B.6 Lift to many internal modes

In most problems the number of macroscale variables at any given position in space,  $d_{\vec{U}}$ , is less than the number of microscale variables at a position,  $d_{\vec{u}}$ ; often much less ([Kevrekidis & Samaey 2009](#), e.g.). In this case, every time we start a patch simulation we need to provide  $d_{\vec{u}} - d_{\vec{U}}$  data at each position in the patch: this is lifting. The first methodology is to first guess, then run repeated short bursts with reinitialisation, until the simulation reaches a slow manifold (e.g., `cdmc()`). Then run the real simulation.

If the time taken to reach a local quasi-equilibrium is too long, then it is likely that the macroscale closure is bad and the macroscale variables need to be extended.

A second step is to cater for cases where the slow manifold is stochastic or is surrounded by fast waves: when it is hard to detect the slow manifold, or the slow manifold is not attractive.

## B.7 Macroscale closure

In some circumstances a researcher/user will not code in a restriction the appropriately set of macroscale variables for a complete closure of the macroscale. For example, in thin film fluid dynamics at low Reynolds number the only macroscale variable is the fluid depth; however, at higher Reynolds number, circa ten, the inertia of the fluid becomes important and the macroscale variables must additionally include a measure of the mean lateral velocity/momentum ([Roberts & Li 2006](#), e.g.).

At some stage we need to detect any flaw in the closure implied by a restriction, and perhaps suggest additional appropriate macroscale variables, or at least their characteristics. Indeed, a poor closure and a stochastic slow manifold are really two faces of the same problem: the problem is that the chosen macroscale variables do not have a unique evolution in terms of themselves. A good resolution of the issue will account for both faces.

## B.8 Exascale fault tolerance

MATLAB/Octave is probably not an appropriate vehicle to deal with real exascale faults. However, we should cater by coding procedures for fault tolerance and testing them at least synthetically. Eventually provide hooks to a user routine to be invoked under various potential scenarios. The nature of

fault tolerant algorithms will vary depending upon the scenario, even assuming that each patch burst is executed on one CPU (or closely coupled CPUs): if there are many more CPUs than patches, then maybe simply duplicate all patch simulations; if many fewer CPUs than patches, then an asynchronous scheduling of patch bursts should effectively cater for recomputation of failed bursts; if comparable CPUs to patches, then more subtle action is needed.

Once mesotime communication and projective integration is provided, a recomputation approach to intermittent hardware faults should be effective because we then have the tools to restart a burst from available macroscale data. Should also explore proceeding with a lower order interpolation that misses the faulty burst—because an isolated lower order interpolation probably will not affect the global order of error (it does not in approximating some boundary conditions ([Gustafsson 1975, Svard & Nordstrom 2006](#))).

## B.9 Link to established packages

Several molecular/particle/agent based codes are well developed and used by a wide community of researchers. Plan to develop hooks to use some such codes as the microscale simulators on patches. First, may connect to LAMMPS ([Plimpton et al. 2016](#)). Second, will evaluate performance, issues, and then consider what other established packages are most promising.

---

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---

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