Intro to LO, Lecture 6

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Duality of LPs

Main idea: Create good bounds on the optimal solution $\boldsymbol{x}^*$.

For example, if the LP is:

$$\begin{array}{rrcrcl} \max & 2x_1 & + & 3x_2 & & \\ \text{s.t.} & 4x_1 & + & 8x_2 & \leq & 12 \\ & 2x_1 & + & x_2 & \leq & 3 \\ & 3x_1 & + & 2x_2 & \leq & 4 \\ & x_1 & , & x_2 & \geq & 0 \end{array}$$

The dual would be:

$$\begin{array}{rrcrcrcl} \min & 12y_1 & + & 3y_2 & + & 4y_3 & & \\ \text{s.t.} & 4y_1 & + & 2y_2 & + & 3y_3 & \geq & 2 \\ & 8y_1 & + & y_2 & + & 2y_3 & \geq & 3 \\ \\ & y_1 & , & y_2 & , & y_3 & \geq & 0 \end{array}$$

T(Weak duality theorem): Let $P = \max\{c^T x \mid Ax \leq b\}$ and $D = \min\{b^T y \mid A^T y = c,\ y \geq 0\}$. If x_f is a feasible solution for P and y_f is a feasible solution for D, then $c^T x_f \leq b^T y_f$.

P: The theorem is restating our basic arguments from above. Formally, it is one line:

$$y_f^T b \geq y_f^T (Ax_f) = (y_f^T A)x_f = c^T x_f.$$

Reminder: Convex cones

Convex cones are special cases of convex sets. A subset C of $\mathbb{R}^n$ is called a *convex cone* if for any $x, y \in C$ and any $\lambda, \mu \geq 0$ one has $\lambda x + \mu y \in C$.

For any $X \subseteq \mathbb{R}^n$, cone (X) is the smallest cone containing X. One easily checks:

$$\text{cone}(X) = \{\lambda_1 x_1 + \cdots \lambda_t x_t \mid x_1, \ldots, x_t \in X; \lambda_1, \ldots, \lambda_t \geq 0\}$$

L(Exercise six from Lecture 4): Let $C \subseteq \mathbb{R}^n$. Then C is a closed convex cone if and only if $C = \bigcap \mathcal{F}$ for some collection $\mathcal{F}$ of linear halfspaces. (Notice the halfspaces are linear, i.e., going through 0.)

The Farkas Lemma

The Farkas lemma can be understood as a geometric observation on behavior of convex cones that is at the heart of the duality of linear programming.

L(Farkas): The system $Ax = b$ has a nonnegative solution if and only if there is no vector y satisfying $y^T A \geq 0$ and $y^T b < 0$.

P: $L \Rightarrow R$: Proving by contradiction, we consider $L \wedge \neg R$. If a nonnegative solution x_0 exists and a solution y_0 exists, it is basically a weak duality argument:

$$0 > y_0^T b = y_0^T (Ax_0) = (y_0^T A)x_0 \geq 0.$$

$R \Rightarrow L$: We prove the equivalent $\neg L \Rightarrow \neg R$. If $Ax = b$ has no solution $x \geq 0$, then $b \notin C = \text{cone}(a_1, \ldots, a_n)$. By Exercise six, we can separate b from C by a linear halfspace $H = \{x | c^T x \geq 0\}$. Choose $y := c$, and y is our vector satisfying $y^T A \geq 0$ and $y^T b < 0$.

Two corollaries

There are many variants of the lemma, for example:

C: The system $Ax \leq b$ has a solution x if and only if there is no vector y satisfying $y \geq 0$, $y^T A = 0$ and $y^T b < 0$.

C: Suppose that the system $Ax \leq b$ has at least one solution. Then the inequality $c^T x \leq \delta$ holds for every solution x of $Ax \leq b$ if and only if there exists a vector $y \geq 0$ such that $y^T A = c^T$ and $y^T b \leq \delta$.

P: $B \Rightarrow A$: Similar to weak duality. Formally, if the vector y exists, then for any x:

$$Ax \leq b \Rightarrow y^T Ax \leq y^T b \Rightarrow c^T x \leq y^T b \Rightarrow c^T x \leq \delta.$$

$\neg B \Rightarrow \neg A$: Proven by contradiction, i.e., $\neg B \wedge A$ is contradictory.

If y does not exist, then there exists no $\lambda \geq 0$ such that $y^T b + \lambda = \delta$. We can add that equality to the system so our variable vector extends to (y, λ).

Using Farkas Lemma: The system $Ax = b$ – for us the system with variables (y, λ) – has no nonnegative solution, so there exists a vector (z, μ) – named y in the Farkas lemma – such that $A^T z + \mu b \geq 0, \mu \geq 0$ and $c^T z + \delta\mu < 0$.

Distinguish two cases based on $\mu = 0$ or $\mu \neq 0$.

Case 1. $\mu = 0$. We have $Az \geq 0$, $c^T z < 0$. By contradiction assumption A, we know $Ax \leq b$ has a solution. But then subtract z from the solution x and keep the inequalities valid, in other words if we increase τ, we get

$$A(x - \tau z) \leq b, c^T(x - \tau z) > \text{ anything, including } > \delta.$$

This is a contradiction with *the inequality $c^T x \leq \delta$ holds for every solution x of $Ax \leq b$.*

Case 2. $\mu \neq 0$. Since the right-hand sides are zero, we can rescale to $\mu = 1$. This would mean that $A^T z \geq -b, c^T z < -\delta$. By relabeling $x = -z$, we would get $A^T x \leq b, c^T x > \delta$.

Again, a contradiction.

Strong Duality

T(Strong duality theorem): If P and D are a primal-dual pair of LPs, then exactly one of the following holds:

1. Both P and D are infeasible.
2. P is unbounded and D is infeasible.
3. D is unbounded and P is infeasible.
4. Both P and D are feasible and there are optimal solutions x^*, y^* such that $c^T x^* = b^T y^*$.

Original program:	*In the dual:*
maximum	minimum
$\max c^T x$	$\min b^T y$
m constraints n variables	m variables n constraints
the i-th constraint is $\leq$ the i-the constraint is $\geq$ the i-th constraint is $=$	$y_i \geq 0$ $y_i \leq 0$ $y_i \in \mathbb{R}$
$x_j \geq 0$ $x_j \leq 0$ $x_j \in \mathbb{R}$	the j-th constraint is $\geq$ the j-th constraint is $\leq$ the j-th constraint is $=$

P(Strong duality proof): We focus only on the case P feasible and D feasible $\Rightarrow$ optima are the same. (Other cases left as an exercise.)

Let $P = \{\max c^T x, Ax \leq b\}$ and $D = \{\min b^T y, y \geq 0, y^T A = c^T\}$. Since D is feasible by assumption, by weak duality we know that P is also bounded. Let $\delta = c^T x*$ be the optimum value of P. So, for every feasible solution of $Ax \leq b$ we have $c^T x \leq \delta$.

The corrolary of Farkas states: *The inequality $c^T x \leq \delta$ holds for every solution x of $Ax \leq b$ if and only if there exists a vector $y \geq 0$ such that $y^T A = c^T$ and $y^T b \leq \delta$.*

The existence of this vector y means that D has optimal objective value at most δ, which (by weak duality) is also its lower bound.

Exercises

Exercise one — Dualize the following LP:

$$\begin{aligned} \max x_1 - 2x_2 + 3x_4 \\ x_2 \leq 0 \\ x_4 \geq 0 \\ x_2 - 6x_3 + x_4 \leq 4 \\ -x_1 + 3x_2 - 3x_3 = 0 \\ 6x_1 - 2x_2 + 2x_3 - 4x_4 \geq 5 \end{aligned}$$

Exercise two — Recall the Graph 3-Coloring from Exercise session 2. There, we created an integer linear program for it.

It is a decision problem: given a graph G, decide if it its vertices can be colored with three colors such that, if we look at each edge in G, this edge is *not* monochromatic, thus it sees two colors on its endpoints.

Your task is to *relax* the integer linear program into a linear program – by replacing integer constraints such as $x \in \{0,1\}$ with linear constraints such as $x \geq 0, x \leq 1$ – and then dualize it.

EXERCISE THREE Recall the problem MAXIMUM FLOW from Lecture 1:

Input: An edge-weighted, directed graph $G = (V, E, c)$ (a network). Every directed edge has an associated capacity $c_e \geq 0$. Additionally, there are two special vertices – source s and sink t.

Output: Any flow – function $f : E \to \mathbb{R}_0^+$ which *obeys capacities* ($f(e) \leq c_e$) and *follows Kirchhoff's law* – in every non-special vertex, fluid coming in = fluid coming out.

Goal: Find a flow that sends as much fluid as possible from s to t.

$$\begin{aligned} & & \max \sum_{\vec{si}} x_{\vec{si}} \\ \text{s.t.} \quad & \forall \vec{ij} \in E(G): & x_{\vec{ij}} \leq c_{\vec{ij}} \\ & \forall v \in V(G) \setminus \{s,t\}: & \sum_{\vec{vi}} x_{\vec{vi}} - \sum_{\vec{jv}} x_{\vec{jv}} = 0 \\ & \forall \vec{ij} \in E(G): & x_{\vec{ij}} \geq 0, x_{\vec{ij}} \in \mathbb{R} \end{aligned}$$

Your task is to dualize this LP.

EXERCISE FOUR Suppose we are given a directed graph G and two vertices s and f and we wish to find the SHORTEST PATH between s and f, counting the number of edges on the path.

This problem is of course solvable in polynomial time by classical algorithms such as DFS/BFS. We wish to solve it using linear programming.

1. Design an LP inspired by the MAXIMUM FLOW above that solves this problem.
2. Then, dualize this LP.
3. *Bonus question:* Can you give meaning to the dual LP? What the variables describe, what do the constraints mean and so on?

EXERCISE FIVE We are given some linear program (P) which has some optimum solution, but we do not know it yet. The program (P) is of this form:

$$\max c^T x, Ax \leq b, x \geq 0.$$

Using duality, formulate a new LP that satisfies the following:

- it has no objective function (so it is just a polytope),
- if somebody gives us any feasible solution x of the polytope, we can read from its coordinates the optimum solution of the program (P).

EXERCISE SIX Prove or disprove the following:

If a linear program has an optimum solution with integral coefficients, then its dual also has an optimum solution with integral coefficients.