Example 4

$X_1, \dots, X_n$ i.i.d. $X_i \sim \mathcal{U}(0,\theta)$, $f(x,\theta) = \frac{1}{\theta} \mathbb{1}_{[0,\theta]}(x)$

$$L(\theta) = \left(\frac{1}{\theta}\right)^n \prod_{i=1}^{n} \mathbb{1}_{[0,\theta]}(x_i) = \left(\frac{1}{\theta}\right)^n \mathbb{1}_{[0,+\infty)}(x_{(1)}) \mathbb{1}_{(-\infty,\theta]}(x_{(n)})$$

$\forall_i \quad 0 \leq x_i \leq \theta$

$\hat{\theta} = X_{(n)} \qquad 0 \leq x_{(1)}$ & $x_{(n)} \leq \theta$

Theorem 2

Let $X_1, \dots, X_n$ be i.i.d. with the pdf $f(x,\theta)$, $\theta \in \Theta$. For a specified function $g: \Theta \to \mathbb{R}$, let $\eta = g(\theta)$ be a parameter of interest. Suppose $\hat{\theta}$ is the mle of θ. Then $g(\hat{\theta})$ is the mle of $\eta = g(\theta)$.

2. Rao-Cramér Lower Bound and Efficiency

Let X be a random variable with pdf $f(x,\theta)$, $\theta \in \Theta$, where Θ is a open set.

Assumptions (Additional Regularity Conditions)

(R3) The pdf $f(x,\theta)$ is twice differentiable as a function of θ.

(R4) The integral $\int_{\mathbb{R}} f(x,\theta)\,dx$ can be differentiated twice under the ~~sig~~ integral sign as a function of θ.

We have

$$1 = \int_{-\infty}^{+\infty} f(x,\theta)\,dx \quad \Big/ \frac{\partial}{\partial\theta}$$

$$0 = \int_{-\infty}^{+\infty} \frac{\partial f(x,\theta)}{\partial\theta}\,dx$$

Equivalently,

$$0 = \int_{-\infty}^{+\infty} \frac{\frac{\partial f(x,\theta)}{\partial \theta}}{f(x,\theta)} f(x,\theta)\,dx = \int_{-\infty}^{+\infty} \frac{\partial \log f(x,\theta)}{\partial \theta} f(x,\theta)\,dx \qquad (*)$$

Thus

$$\mathbb{E}\left[\frac{\partial \log f(X,\theta)}{\partial \theta}\right] = 0.$$

Furthermore, differentiate once more $(*)$, we obtain

$$0 = \int_{-\infty}^{+\infty} \frac{\partial^2 \log f(x,\theta)}{\partial \theta^2} f(x,\theta)\,dx + \int_{-\infty}^{+\infty} \frac{\partial \log f(x,\theta)}{\partial \theta} \underbrace{\frac{\partial \log f(x,\theta)}{\partial \theta} f(x,\theta)}_{\frac{\partial f(x,\theta)}{\partial \theta}}\,dx,$$

Therefore

$$-\int_{-\infty}^{+\infty} \frac{\partial^2 \log f(x,\theta)}{\partial \theta^2} f(x,\theta)\,dx = \mathbb{E}\left[\frac{\partial \log f(X,\theta)}{\partial \theta}\right]^2.$$

<u>Definition 1</u>

The number

$$I(\theta) = \mathbb{E}\left[\left(\frac{\partial \log f(X,\theta)}{\partial \theta}\right)^2\right]$$

is called the Fisher information.

<u>Corollary 1</u>

Under the assumptions (R0) - (R4)

$$I(\theta) = -\int_{-\infty}^{\infty} \frac{\partial^2 \log f(x,\theta)}{\partial \theta^2} f(x,\theta)\,dx = Var\left[\frac{\partial \log f(X,\theta)}{\partial \theta}\right].$$

<u>Example 1</u>

$X \sim b(1,\theta)$, $f(x,\theta) = \theta^x (1-\theta)^{1-x}$

$$\log f(x,\theta) = x \log \theta + (1-x)\log(1-\theta)$$

$$\frac{\partial \log f(x,\theta)}{\partial \theta} = \frac{x}{\theta} - \frac{1-x}{1-\theta}$$

$$\frac{\partial^2 \log f(x,\theta)}{\partial \theta^2} = -\frac{x}{\theta^2} - \frac{1-x}{(1-\theta)^2}$$

Therefore,

$$I(\theta) = -\mathbb{E}\left[-\frac{X}{\theta^2} - \frac{1-X}{(1-\theta)^2}\right] = \frac{\theta}{\theta^2} + \frac{1-\theta}{(1-\theta)^2} = \frac{1}{\theta} + \frac{1}{1-\theta} = \frac{1}{\theta(1-\theta)}$$

Example 2,

$X_1, \dots, X_n$ i.i.d. such that

$$X_i = \theta + e_i, \quad i = 1, \dots, n, \qquad \text{(location model)}$$

where $e_1, \dots, e_n$ are i.i.d ~~with~~ $e_i \sim f(x)$.

Then $X_i \sim f_X(x, \theta) = f(x-\theta)$

Assume that f satisfies the regularity conditions. Then

$$I(\theta) = \int_{-\infty}^{+\infty} \left(\frac{f'(x-\theta)}{f(x-\theta)}\right)^2 f(x-\theta)dx = \begin{Bmatrix} z = x-\theta \\ dz = dx \end{Bmatrix} = \int_{-\infty}^{+\infty} \left(\frac{f'(z)}{f(z)}\right)^2 f(z) dz.$$

Hence, in the location model, the Fisher information does not depend on θ.

Suppose that X_i has the Laplace distribution, i.e., $f(x,\theta) = \frac{1}{2} e^{-|x_i - \theta|}$

Since

$$X_i = \theta + e_i,$$
$$e_i \sim f(z_i) = \tfrac{1}{2} e^{-|z_i|}.$$

Furthermore, $f'(z) = -\frac{1}{2} e^{-|z|} \operatorname{sgn}(z)$.

Therefore,

$$I(\theta) = \int_{-\infty}^{+\infty} \left(\frac{f'(z)}{f(z)}\right)^2 f(z) dz = \int_{-\infty}^{+\infty} f(z) dz = 1$$

Remark 2,

If $X_1, \dots, X_n$ are i.i.d, $X_i \sim f(x, \theta)$ and $I(\theta)$ is the Fisher information of X_1, then $nI(\theta)$ is the Fisher information of the sample.

Proof

$$\mathrm{Var}\left(\frac{\partial \log L(\theta, \underline{X})}{\partial \theta}\right) = \mathrm{Var}\left(\sum_{i=1}^{n} \frac{\partial \log f(X_i, \theta)}{\partial \theta}\right) = \sum_{i=1}^{n} \mathrm{Var}\left(\frac{\partial \log f(X_i, \theta)}{\partial \theta}\right) = nI(\theta)$$

Theorem 1 (Cramér - Rao inequality)

Let $X_1, \dots, X_n$ be i.i.d. with pdf $f(x, \theta)$, $\theta \in \Theta$.

Assume that the regularity conditions (R0) - (R4) hold.

Let $Y = u(X_1, \dots, X_n)$ be a statistic with mean $\mathbb{E}Y = \mathbb{E}[u(X_1, \dots, X_n)] = k(\theta)$. Then

$$\mathrm{Var}\, Y \geq \frac{[k'(\theta)]^2}{nI(\theta)}$$

Proof

I) (Consider the) continuous case

1. We have $k(\theta) = \mathbb{E}Y = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} u(x_1, \dots, x_n) f(x_1, \theta) \dots f(x_n, \theta)\, dx_1 \dots dx_n$

2. By the above

$$k'(\theta) = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} u(x_1, \dots, x_n) \left[\sum_{i=1}^{n} \frac{1}{f(x_i, \theta)} \frac{\partial f(x_i, \theta)}{\partial \theta}\right] f(x_1, \theta) \dots f(x_n, \theta)\, dx_1 \dots dx_n$$

$$= \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} u(x_1, \dots, x_n) \left[\sum_{i=1}^{n} \frac{\partial \log f(x_i, \theta)}{\partial \theta}\right] f(x_1, \theta) \dots f(x_n, \theta)\, dx_1 \dots dx_n$$

3. Define random variable $Z = \sum_{i=1}^{n} \frac{\partial \log f(X_i, \theta)}{\partial \theta}$

4. Then $\mathbb{E}Z = 0$, $\mathrm{Var}\, Z = nI(\theta)$

5. Moreover, $k'(\theta) = \mathbb{E}[Y \cdot Z] = \mathbb{E}Y \cdot \mathbb{E}Z + \rho\, \sigma_Y \sigma_Z \overset{(4)}{=} \rho\, \sigma_Y \sqrt{nI(\theta)}$,

   where $\rho = \mathrm{corr}(Y, Z)$

6. Thus

$$\rho = \frac{k'(\theta)}{\sigma_Y \sqrt{nI(\theta)}}.$$

7. Since $\rho^2 \leq 1$, we have

$$\frac{[k'(\theta)]^2}{\sigma_Y^2\, nI(\theta)} \leq 1 \iff \mathrm{Var}\, Y \geq \frac{[k'(\theta)]^2}{nI(\theta)}$$