Example 2.

$X_1, \dots, X_n$ i.i.d., $X_1 \sim N(\theta, \sigma^2)$, $\theta \in \mathbb{R}$, $\sigma^2 > 0$ and known.

We verify

$$H_0 : \theta = \theta_0 \quad \text{against} \quad H_1 : \theta \neq \theta_0 ,$$

where θ_0 is fixed. We have

$$L(\theta) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta)^2\right\}$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\bar{x})^2\right\} \exp\left\{-\frac{1}{2\sigma^2} n(\bar{x}-\theta)^2\right\}$$

Furthermore, $\hat{\theta} = \bar{X}$ is the MLE of θ. Thus,

$$\Lambda = \frac{L(\theta_0)}{L(\hat{\theta})} = \exp\left\{-\frac{1}{2\sigma^2} n(\bar{X}-\theta_0)^2\right\},$$

and $\Lambda \leq c$ is equivalent to $-2\log\Lambda \geq -2\log c$.

Under H_0, $-2\log\Lambda = \left(\frac{\bar{X}-\theta_0}{\frac{\sigma}{\sqrt{n}}}\right)^2 \sim \chi^2(1)$.

We reject H_0 in favour of H_1, if

$$-2\log\Lambda \geq q_{\chi^2(1)}(1-\alpha).$$

Theorem 1

Let $X_1, \dots, X_n$ be a sample with $f(x, \theta)$, $\theta_0 \in \Theta \subseteq \mathbb{R}$ satisfying the regularity conditions (R0) - (R5).

Under $H_0 : \theta = \theta_0$,

$$-2\log\Lambda \xrightarrow{D} \chi^2(1).$$

Remark 1

If there is a problem with finding an exact form of the statistic Λ, we can apply the test based on the statistic $\chi^2_L = -2\log\Lambda$ at the asymptotic significance level α rejecting H_0 in favour of H_1 when

$$\chi^2_L \geq q_{\chi^2(1)}(1-\alpha).$$

Definition 2

In the testing problem $H_0: \theta = \theta_0$ against $H_1: \theta \neq \theta_0$ the test based on the statistic

$$\chi^2_W = \left\{ \sqrt{nI(\hat{\theta})}\,(\hat{\theta} - \theta_0) \right\}^2$$

is called the Wald test. We reject H_0, at the asymptotic significance level α, when

$$\chi^2_W \geq q_{\chi^2(1)}(1-\alpha).$$

Definition 3

In the problem of verifying $H_0: \theta = \theta_0$ against $H_1: \theta \neq \theta_0$ the test based on the statistic

$$\chi^2_R = \left\{ \frac{\ell'(\theta_0)}{\sqrt{nI(\theta_0)}} \right\}^2$$

is called the Rao-score test. We reject H_0, at the asymptotic significance level α, when

$$\chi^2_R \geq q_{\chi^2(1)}(1-\alpha).$$

Example 3

$X_1, \dots, X_n$ i.i.d , $X_i \sim B(\theta, 1)$

We test

$$H_0 : \theta = 1 \quad \text{against} \quad H_1 : \theta \neq 1.$$

Under H_0, $X_i \sim \mathcal{U}(0,1)$,

Moreover, $\hat{\theta} = \dfrac{-n}{\sum_{i=1}^{n} \log X_i}$ - EN MLE of θ.

We have,

$$f(x,\theta) = \frac{\Gamma(\theta+1)}{\Gamma(\theta)\Gamma(1)} x^{\theta-1}(1-x)^{1-1} = \theta x^{\theta-1} \mathbb{1}_{(0,1)}(x)$$

$$L(\theta) = \prod_{i=1}^{n} f(x_i,\theta) = \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta-1}, \quad L(1) = 1$$

$$L(\hat{\theta}) = \left(\frac{-n}{\sum \log X_i}\right)^n \left(\prod_{i=1}^{n} x_i\right)^{-\frac{n}{\sum \log X_i} - 1} =$$

$$n^n \left(-\sum \log X_i\right)^{-n} \exp\left(\log\left(\prod_{i=1}^{n} x_i\right)^{-\frac{n}{\sum \log X_i} - 1}\right) =$$

$$n^n \left(-\sum \log X_i\right)^{-n} \exp\left(\left[-\frac{n}{\sum \log X_i} - 1\right] \log \prod_{i=1}^{n} X_i\right) =$$

$$\cancel{\exp[n(\log n)]}\ \exp(n\log n)\left(-\sum \log X_i\right)^{-n} \exp\left[-n - \sum \log X_i\right] =$$

$$\left(-\sum_{i=1}^{n} \log X_i\right)^{-n} \exp\left(-\sum_{i=1}^{n} \log X_i\right) \exp\left[n(\log n - 1)\right]$$

Thus

$$\Lambda = \frac{L(\theta_0)}{L(\hat{\theta})} = \frac{1}{L(\hat{\theta})}. \quad \text{therefore,}$$

$$\chi_L^2 = -2\log\Lambda = -2\left\{-n\log\left(-\sum_{i=1}^{n}\log X_i\right) - \sum_{i=1}^{n}\log X_i + n(\log n - 1)\right\}$$

Recall that $I(\theta) = \theta^{-2}$. As a result

$$\chi^2_W = \left\{\sqrt{nI(\hat\theta)}\,(\hat\theta - \theta_0)\right\}^2 = \left\{\sqrt{\frac{n}{\hat\theta^2}}\,(\hat\theta - 1)\right\}^2 = n\left(1 - \frac{1}{\hat\theta}\right)^2 =$$

$$= n\left(1 + \frac{\sum_{i=1}^n \log X_i}{n}\right)^2.$$

Furthermore,

$$\ell'(1) = \ell'(\theta_0) = \sum_{i=1}^n \frac{\partial \log f(X_i, \theta)}{\partial\theta}\bigg|_{\theta=\theta_0} = \sum_{i=1}^n \frac{\partial \log(\theta X_i^{\theta-1})}{\partial\theta}\bigg|_{\theta=\theta_0}$$

$$= \sum_{i=1}^n \frac{\partial[\log\theta + (\theta-1)\log X_i]}{\partial\theta}\bigg|_{\theta=\theta_0} = \sum_{i=1}^n (1 + \log X_i) = n + \sum_{i=1}^n \log X_i$$

Finally,

$$\chi^2_R = \left\{\frac{\ell'(\theta_0)}{\sqrt{nI(\theta_0)}}\right\}^2 = \left(\frac{\sum \log X_i + n}{\sqrt{n}}\right)^2 = n\left(1 + \frac{\sum_{i=1}^n \log X_i}{n}\right)^2$$

<u>Example 4</u>

Consider the shift model

$$X_i = \theta + e_i, \quad i = 1, \ldots, n,$$

where e_i ~ $f(x) = \frac{1}{2} e^{-|x|}$.

We test

$$H_0: \theta = \theta_0,$$
$$H_1: \theta \neq \theta_0.$$

MLE of θ is $\hat\theta = \mathrm{med}\{X_1, \ldots, X_n\}$, $X_i \sim f(x, \theta) = \frac{1}{2}\exp\{-|x-\theta|\}$

$$L(\theta_0) = 2^{-n}\exp\left\{-\sum_{i=1}^n |X_i - \theta_0|\right\}$$

$$L(\hat\theta) = 2^{-n}\exp\left\{-\sum_{i=1}^n |X_i - \hat\theta|\right\}$$

So

$$-2\log\Lambda = -2\log\frac{L(\theta_0)}{L(\hat\theta)} = 2\left[\sum_{i=1}^n |X_i - \theta_0| - \sum_{i=1}^n |X_i - \hat\theta|\right]$$

We reject H_0, at the asymptotic significance level α when

$$-2\log\Lambda \geqslant q_{\chi^2(1)}(1-\alpha).$$

Since $I(\theta)=1$, $\chi^2_W = \{\sqrt{n}(\hat{\theta}-\theta_0)\}^2$.

Furthermore,

$$\frac{\partial \log f(x_i-\theta)}{\partial\theta} = \frac{\partial}{\partial\theta}\left[\log\tfrac{1}{2} - |x_i-\theta|\right] = \operatorname{sgn}(x_i-\theta)$$

Finally,

$$\chi^2_R = \left\{\frac{1}{\sqrt{n}}\sum_{i=1}^n \operatorname{sgn}(X_i-\theta_0)\right\}^2.$$

5. Likelihood Ratio Tests: multidimensional case

Let $X_1,\dots,X_n$ be a sample from $f(x,\underline{\theta})$, $\underline{\theta}=(\theta_1,\dots,\theta_p)\in \Theta \subset R^p$, $p>1$

The likelihood function has a form

$$L(\underline{\theta}) = \prod_{i=1}^n f(x_i,\underline{\theta}),$$

the log-likelihood

$$\ell(\underline{\theta}) = \log L(\underline{\theta}) = \sum_{i=1}^n \log f(x_i,\underline{\theta}).$$

Let $\underline{\theta}_0$ be ~~the~~ three value of the parameter θ.

Impose (additional) regularity conditions

(R6) There is an open subset $\Theta_0 \subseteq \Theta$, such that $\underline{\theta}_0\in\Theta_0$ and all third partial derivatives of $f(x,\underline{\theta})$ exist for all $\underline{\theta}\in\Theta_0$.

(R7) $\mathbb{E}_{\underline{\theta}}\left[\frac{\partial}{\partial\theta_j}\log f(x,\underline{\theta})\right]=0$ for $j=1,\dots,p$.

$$I_{jk}(\underline{\theta}) = \operatorname{Cov}\left(\frac{\partial\log f(X,\underline{\theta})}{\partial\theta_j}, \frac{\partial \log f(X,\underline{\theta})}{\partial\theta_k}\right) =$$

$$= -\mathbb{E}_{\underline{\theta}}\left[\frac{\partial^2 \log f(X,\underline{\theta})}{\partial\theta_j\partial\theta_k}\right] \text{ for } j,k=1,\dots,p.$$

(R8) For all $\theta \in \Theta_0$,

$$I(\underline{\theta}) = \left[I_{jk}(\underline{\theta}) \right]_{j,k=1}^{p}$$

is positive definite.

(R9) There exist ~~functio~~ functions $M_{jkl}(x)$, such that

$$\left| \frac{\partial^3 \log f(x, \underline{\theta})}{\partial \theta_j \, \partial \theta_k \, \partial \theta_l} \right| \leqslant M_{jkl}(x) \quad \text{for all } \underline{\theta} \in \Theta_0$$

and

$$\mathbb{E}_{\underline{\theta}_0} \left[M_{jkl}(X) \right] < +\infty \quad \text{for all } j,k,l = 1, \ldots, p.$$

Definition 1

The quantity

$$\hat{\underline{\theta}} = \underset{\underline{\theta} \in \Theta}{\operatorname{argmax}} \, L(\underline{\theta})$$

is called the maximum likelihood estimator of the parameter $\underline{\theta}$.

Remark 1

If $\hat{\underline{\theta}}$ is the MLE of $\underline{\theta}$, then $g(\hat{\underline{\theta}})$ is the MLE of $g(\underline{\theta})$

Example 1

$X_1, \ldots, X_n$ i.i.d $X_1 \sim N(\mu, \sigma^2)$. Let $\underline{\theta} = (\mu, \sigma^2)$ and $\Theta = \mathbb{R} \times (0, +\infty)$. We have

$$L(\underline{\theta}) = L(\mu, \sigma^2) = \prod_{i=1}^{n} f(x_i, \mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{ -\frac{1}{2} \frac{(x_i - \mu)^2}{\sigma^2} \right\}$$

$$= \left(\frac{1}{2\pi\sigma^2} \right)^{n/2} \exp\left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2 \right\},$$

$$l(\underline{\theta}) = \log L(\underline{\theta}) = -\frac{n}{2} \log(2\pi) - \frac{n}{2} \log \sigma^2 - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2.$$

Moreover,

$$\frac{\partial \ell(\mu,\sigma^2)}{\partial \mu} = \frac{1}{2\sigma^2} 2\sum_{i=1}^{n}(x_i-\mu) = \frac{\sum_{i=1}^{n}(x_i-\mu)}{\sigma^2} = 0$$

$$\frac{\partial \ell(\mu,\sigma^2)}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i-\mu)^2 = 0$$

$$\Leftrightarrow -\frac{n}{2\sigma^4}\left[\sigma^2 - \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2\right] = 0$$

$$\hat{\mu} = \bar{X}, \quad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2.$$

$$\frac{\partial^2 \ell(\mu,\sigma^2)}{\partial \mu^2} = -\frac{n}{\sigma^2}\Big|_{\sigma=\hat{\sigma}} = -\frac{n}{\hat{\sigma}^2}$$

$$\frac{\partial^2 \ell(\mu,\sigma^2)}{\partial \mu\,\partial \sigma^2} = -\frac{1}{\sigma^4}\sum_{i=1}^{n}(x_i-\mu) = \frac{\partial^2 \ell(\mu,\sigma^2)}{\partial \sigma^2\,\partial \mu}\Big|_{\substack{\mu=\hat{\mu}\\ \sigma^2=\hat{\sigma}^2}} = 0$$

$$\frac{\partial^2 \ell(\mu,\sigma^2)}{\partial^2 \sigma^4} = \frac{n}{2\sigma^4} - \frac{1}{\sigma^6}\sum_{i=1}^{n}(x_i-\mu)^2\Big|_{\substack{\mu=\hat{\mu}\\ \sigma^2=\hat{\sigma}^2}} =$$

$$= \frac{n}{2\hat{\sigma}^4} - \frac{1}{\hat{\sigma}^6} n\hat{\sigma}^2 = \frac{n}{2\hat{\sigma}^4} - \frac{n}{\hat{\sigma}^4} = -\frac{n}{2\hat{\sigma}^4}.$$

To sum up,

$$\frac{\partial^2 \ell(\mu,\sigma^2)}{\partial\cdot\,\partial\cdot} = \begin{bmatrix} -\frac{n}{\hat{\sigma}^2} & 0 \\ 0 & -\frac{n}{2\hat{\sigma}^4} \end{bmatrix} \qquad -\frac{n}{\hat{\sigma}^2} < 0 \ \& \ \underbrace{\det|\ |}_{= \frac{n^2}{2\hat{\sigma}^6}} > 0$$

Indeed, $\cancel{\hat{\mu} = \bar{X}}$

$$\hat{\Theta} = (\hat{\mu}, \hat{\sigma}^2) = \left(\bar{X}, \frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2\right)$$

is the MLE of $\underline{\Theta} = (\mu, \sigma^2)$.