Under H_0, the statistic T has the t-Student distribution with $n-1$ degrees of freedom. Taking $c'' = q_{t(n-1)}\left(1-\frac{\alpha}{2}\right)$, we obtain the α-size test.

Remark 1

If σ^2 is known, the LRT statistic has the form
$$U = \frac{\sqrt{n}\,(\bar{X}-\mu_0)}{\sigma}.$$

Under H_0, $U \sim N(0,1)$. We reject H_0 when $|U| > \Phi^{-1}\left(1-\frac{\alpha}{2}\right)$.

Remark 2

Let $X_1, \dots, X_m$ be a sample from $N(\mu_1, \sigma_1^2)$ distribution, and $Y_1, \dots, Y_n$ be an independent from X_i's sample from $N(\mu_2, \sigma_2^2)$ distribution. We test

$$H_0: \sigma_1^2 = \sigma_2^2,$$
$$H_1: \sigma_1^2 \neq \sigma_2^2.$$

(i) If μ_1, μ_2 are not known, the LRT statistic has the form

$$F = \frac{\frac{1}{m-1}\sum_{i=1}^{m}(X_i-\bar{X})^2}{\frac{1}{n-1}\sum_{j=1}^{n}(Y_j-\bar{Y})^2}.$$

Under H_0, $F \sim F(m-1, n-1)$. We reject H_0 when

$$F < q_{F(m-1,n-1)}\left(\frac{\alpha}{2}\right) \quad \text{or} \quad F > q_{F(m-1,n-1)}\left(1-\frac{\alpha}{2}\right).$$

(ii) If μ_1, μ_2 are known, the LRT statistic has the form

$$F = \frac{\frac{1}{m}\sum_{i=1}^{m}(X_i-\mu_1)^2}{\frac{1}{n}\sum_{j=1}^{n}(Y_j-\mu_2)^2}.$$

Under H_0, $F \sim F(m,n)$. We reject H_0 when

$$F < q_{F(m,n)}\left(\frac{\alpha}{2}\right) \quad \text{or} \quad F > q_{F(m,n)}\left(1-\frac{\alpha}{2}\right).$$

Remark 3

Let $X_1, \dots, X_m$ be a sample from $N(\mu_1, \sigma^2)$ distribution.
Let $Y_1, \dots, Y_n$ be — " — — " — $N(\mu_2, \sigma^2)$ — " —.
~~Both~~ The samples are independent, while σ^2 is unknown.

We verify

$$H_0: \mu_1 = \mu_2,$$
$$H_1: \mu \neq \mu_2.$$

The LRT statistic has the form

$$T = \sqrt{\frac{mn}{m+n}} \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{1}{m+n-2}\left[\sum_{i=1}^{m}(X_i - \bar{X})^2 + \sum_{j=1}^{n}(Y_j - \bar{Y})^2\right]}}.$$

Under H_0, ~~the~~ T statistic has the t-Student distribution with $m+n-2$ degrees of freedom. We reject H_0 when

$$|T| > q_{t(m+n-2)}\left(1-\frac{\alpha}{2}\right).$$

IV Inferences about Normal Models

1. Quadratic Forms

Definition 1

A homogeneous polynomial of degree 2 in n variables is called a quadratic form in those variables. If both the variables and the coefficients are real, the form is called a real quadratic form.

Example 1

$X_1^2 + X_1X_2 + X_2^2$ is a quadratic form in the two variables X_1 and X_2

$X_1^2 + X_2^2 + X_3^2 - 2X_1X_2$ is a quadratic form in the three variables X_1, X_2 and X_3.

$(X_1-1)^2 + (X_2-2)^2 = X_1^2 + X_2^2 - 2X_1 - 4X_2 + 5$ is not quadratic form in X_1 and X_2, although it is a quadratic form in the variables $X_1 - 1$ and $X_2 - 2$.

$$(n-1)S^2 = \sum_{i=1}^{n}(X_i - \overline{X})^2 = \sum_{i=1}^{n}\left(X_i - \frac{X_1 + \ldots + X_n}{n}\right)^2 =$$

$$\frac{n-1}{n}\left(X_1^2 + X_2^2 + \ldots + X_n^2\right) - \frac{2}{n}\left(X_1X_2 + \ldots + X_1X_n + \ldots + X_{n-1}X_n\right)$$

is a quadratic form in the n variables $X_1, X_2, \ldots, X_n$.

Theorem 1

Let $Q = Q_1 + Q_2 + \ldots + Q_{k-1} + Q_k$, where $Q, Q_1, \ldots, Q_k$ are $k+1$ random variables that are real quadratic forms in n independent random variables which are normally distributed with common mean and variance $\mu^{=0}$ and σ^2, respectively. Let $Q/\sigma^2, Q_1/\sigma^2, \ldots, Q_{k-1}/\sigma^2$ have chi-square distributions with degrees of freedom $r, r_1, \ldots, r_{k-1}$, respectively. Let Q_k be nonnegative. Then:

a) $Q_1, \ldots, Q_k$ are independent, and hence

b) Q_k/σ^2 has a chi-square distribution with $r-(r_1+\ldots+r_{k-1}) = r_k$ degrees of freedom.

Proof (later)

Example 2

Let the random variable X have a distribution that is $N(\mu, \sigma^2)$. Let a and b denote positive integers greater than 1 and let $n = ab$. Consider a random sample of size n from this normal distribution

$$
\begin{array}{l}
X_{11}, X_{12}, \ldots, X_{1j}, \ldots, X_{1b} \\
X_{21}, X_{22}, \ldots, X_{2j}, \ldots, X_{2b} \\
\vdots \\
X_{i1}, X_{i2}, \ldots, X_{ij}, \ldots, X_{ib} \\
\\
X_{a1}, X_{a2}, \ldots, X_{aj}, \ldots, X_{ab}.
\end{array}
$$

We now define $a+b+1$ statistics. They are

$$\overline{X}_{..} = \frac{X_{11}+\ldots+X_{1b}+\ldots X_{a1}+\ldots+X_{ab}}{ab} = \frac{\sum_{i=1}^{a}\sum_{j=1}^{b} X_{ij}}{ab},$$

$$\overline{X}_{i.} = \quad \ldots \quad = \frac{\sum_{j=1}^{b} X_{ij}}{b}, \quad i = 1, 2, \ldots, a,$$

and

$$\overline{X}_{.j} = \quad \ldots \quad = \frac{\sum_{i=1}^{a} X_{ij}}{a}, \quad j = 1, 2, \ldots, b.$$

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3

Consider the variance S^2 of the random sample of size $n = ab$. We have the algebraic identity

$$(ab-1)S^2 = \sum_{i=1}^{a}\sum_{j=1}^{b}\left(X_{ij} - \bar{X}_{..}\right)^2$$

$$= \sum_{i=1}^{a}\sum_{j=1}^{b}\left[\left(X_{ij} - \bar{X}_{i.}\right) + \left(\bar{X}_{i.} - \bar{X}_{..}\right)\right]^2$$

$$= \sum_{i=1}^{a}\sum_{j=1}^{b}\left(X_{ij} - \bar{X}_{i.}\right)^2 + \sum_{i=1}^{a}\sum_{j=1}^{b}\left(\bar{X}_{i.} - \bar{X}_{..}\right)^2$$

$$+ 2\sum_{i=1}^{a}\sum_{j=1}^{b}\left(X_{ij} - \bar{X}_{i.}\right)\left(\bar{X}_{i.} - \bar{X}_{..}\right).$$

Furthermore,

$$2\sum_{i=1}^{a}\left[\left(\bar{X}_{i.} - \bar{X}_{..}\right)\sum_{j=1}^{b}\left(X_{ij} - \bar{X}_{i.}\right)\right] = 2\sum_{i=1}^{a}\left[\left(\bar{X}_{i.} - \bar{X}_{..}\right)\left(b\bar{X}_{i.} - b\bar{X}_{i.}\right)\right] = 0,$$

and

$$\sum_{i=1}^{a}\sum_{j=1}^{b}\left(\bar{X}_{i.} - \bar{X}_{..}\right)^2 = b\sum_{i=1}^{a}\left(\bar{X}_{i.} - \bar{X}_{..}\right)^2.$$

Thus

$$(ab-1)S^2 = \sum_{i=1}^{a}\sum_{j=1}^{b}\left(X_{ij} - \bar{X}_{i.}\right)^2 + b\sum_{i=1}^{a}\left(\bar{X}_{i.} - \bar{X}_{..}\right)^2,$$

or, for brevity,

$$Q = Q_1 + Q_2.$$

Since $X_{ij} \sim N(\mu, \sigma^2)$, we have $(ab-1)S^2/\sigma^2 = Q/\sigma^2 \sim \chi^2(ab-1)$. For each fixed value of i, $\sum_{j=1}^{b}\left(X_{ij} - \bar{X}_{i.}\right)^2/\sigma^2 \sim \chi^2(b-1)$ and $\sum_{j=1}^{b}\left(X_{1j} - \bar{X}_{1.}\right)^2/\sigma^2, \ldots, \sum_{j=1}^{b}\left(X_{aj} - \bar{X}_{a.}\right)/\sigma^2$ are independent. Therefore,

$$\sum_{i=1}^{a}\left[\sum_{j=1}^{b}\left(X_{ij} - \bar{X}_{i.}\right)^2/\sigma^2\right] = Q_1/\sigma^2 \sim \chi^2(a[b-1]).$$ Now $Q_2 = b\sum_{i=1}^{a}\left(\bar{X}_{i.} - \bar{X}_{..}\right)^2 \geq 0$.

Theorem 1 implies that Q_1 and Q_2 are independent and $\frac{Q_2}{\sigma^2} \sim \chi^2(\cancel{a(b-1)}\ ab - 1 - a(b-1) = a-1)$

Similarly, replacing $X_{ij} - \bar{X}_{..}$ by $(X_{ij} - \bar{X}_{.j}) + (\bar{X}_{.j} - \bar{X}_{..})$, in $(ab-1)S^2$, we ~~obtain~~ get

$$(ab-1)S^2 = \sum_{j=1}^{b}\sum_{i=1}^{a}(X_{ij} - \bar{X}_{.j})^2 + a\sum_{j=1}^{b}(\bar{X}_{.j} - \bar{X}_{..})^2,$$

or, for ~~brevity~~ short,

$$Q = Q_3 + Q_4.$$

As a result $Q_3/\sigma^2 \sim \chi^2(b[a-1])$ and $Q_4/\sigma^2 \sim \chi^2(b-1)$ and are independent.

In like manner, in $(ab-1)S^2$, replacing $X_{ij} - \bar{X}_{..}$ by $(\bar{X}_{i.} - \bar{X}_{..}) + (\bar{X}_{.j} - \bar{X}_{..}) + (X_{ij} - \bar{X}_{i.} - \bar{X}_{.j} + \bar{X}_{..})$, we ~~get~~ obtain

$$(ab-1)S^2 = b\sum_{i=1}^{a}(\bar{X}_{i.} - \bar{X}_{..})^2 + a\sum_{j=1}^{b}(\bar{X}_{.j} - \bar{X}_{..})^2 + \sum_{j=1}^{b}\sum_{i=1}^{a}(X_{ij} - \bar{X}_{i.} - \bar{X}_{.j} + \bar{X}_{..})^2,$$

or, for clarity,

$$Q = Q_2 + Q_4 + Q_5.$$

Moreover,

$$\frac{Q}{\sigma^2} \sim \chi^2(ab-1), \quad \frac{Q_2}{\sigma^2} \sim \chi^2(a-1), \quad \frac{Q_4}{\sigma^2} \sim \chi^2(b-1).$$

Since $Q_5 \geqslant 0$, the theorem asserts that, Q_2, Q_4, and Q_5 are independent and that $\frac{Q_5}{\sigma^2} \sim \chi^2([a-1][b-1])$.

Finally,

$$\frac{\dfrac{Q_4}{\sigma^2(b-1)}}{\dfrac{Q_3}{\sigma^2 b(a-1)}} \sim F(b-1, a-1), \text{ and}$$

$$\frac{\dfrac{Q_4}{\sigma^2(b-1)}}{\dfrac{Q_5}{\sigma^2(a-1)(b-1)}} \sim F(b-1, [a-1][b-1]).$$

2. One-way ANOVA

Consider

$$\begin{matrix} X_{11}, \dots, X_{1b}; \\ \vdots \quad \ddots \quad \vdots \\ X_{a1}, \dots, X_{ab}, \end{matrix}$$

~~a ra~~ independent indentically distributed (iid) random variables, where $X_{ij} \sim N(\mu_j, \sigma^2)$, $i = 1, \dots, a$, $j = 1, \dots, b$, and all parameters are unknown.

The appropriate model for the observations is as follows

$$X_{ij} = \mu_j + e_{ij}; \quad i = 1, \dots, a, \ j = 1, \dots, b,$$

where e_{ij} are iid $N(0, \sigma^2)$.

Suppose that it is desired to test the composite hypothesis

$$H_0: \mu_1 = \mu_2 = \dots = \mu_b = \mu,$$

μ unspecified, against

$$H_1: \sim H_0.$$

A likelihood ratio test will be used.

Remark 1.

The problem is often summarized that we have one factor at b levels. The model is called a one-way model. As we will see, the likelihood ratio test can be thought of in terms of estimates of variance. Hence, this is an example of an analysis of variance (ANOVA). In short, we say that this example is a one-way ANOVA problem.