Intro to LO, Lecture 5

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The Simplex Method

Starting example:

$$\begin{array}{rl} \text{maximize} & 2x_1 + x_2 \\ \text{subject to} & -x_1 + x_2 \leq 1 \\ & x_1 + x_2 \leq 3 \\ \text{with variables} & x_1, x_2 \geq 0 \end{array}$$

Equational form:

$$\begin{array}{rl} \text{maximize} & 2x_1 + x_2 \\ \text{subject to} & -x_1 + x_2 + x_3 = 1 \\ & x_1 + x_2 + x_4 = 3 \\ \text{with variables} & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

We also performed *cleanup* to get a linearly independent set of equations. We can do this using e.g. Gaussian elimination.

Basic step represented by a *tableau*. We need at least one *basic feasible solution* to start the process.

Tableau with $B = \{3, 4\}$ **:**

$$\begin{array}{rccccc} x_3 = & 1 & + & x_1 & - & x_2 \\ x_4 = & 3 & - & x_1 & - & x_2 \\ z = & & & 2x_1 & + & x_2 \end{array}$$

Tableau has *basic variables* B on the left, non-basic variables below, and all non-basic variables are set to zero. We can easily read the current position in the space, $(0, 0, 1, 3)$, as well as the value of the objective function z (currently 0).

We can choose a variable to *enter the basis* if it has a positive coefficient in the objective function. In this case, we can choose either x_1 or x_2. If we choose x_1 to enter the basis, we imagine it continuously increasing, which may decrease x_3 and x_4 until one of them becomes zero, after which we can make it *leave the basis*.

Increasing x_1 forces x_4 to zero when $x_1 = 3$, which means that we reorder

$$x_4 = 3 - x_1 - x_2$$

to

$$x_1 = 3 - x_2 - x_4.$$

We now plug this equation in to the objective function and also express x_3, which stays in the basis, using x_2 and x_4.

Tableau with $B = \{1, 3\}$ **:**

$$\begin{array}{rccccc} x_1 = & 3 & - & x_2 & - & x_4 \\ x_3 = & 4 & - & 2x_2 & - & x_4 \\ z = & 6 & - & x_2 & - & 2x_4 \end{array}$$

Since both of the coefficients are negative, the simplex method terminates.

All four possible tableaus

Tableau with $B = \{3, 4\}$ **:**

$$\begin{array}{rccccc} x_3 = & 1 & + & x_1 & - & x_2 \\ x_4 = & 3 & - & x_1 & - & x_2 \\ z = & & & 2x_1 & + & x_2 \end{array}$$

Tableau with $B = \{2, 4\}$ **:**

$$\begin{array}{rccccc} x_2 = & 1 & + & x_1 & - & x_3 \\ x_4 = & 2 & - & 2x_1 & + & x_3 \\ z = & 1 & + & 3x_1 & - & x_3 \end{array}$$

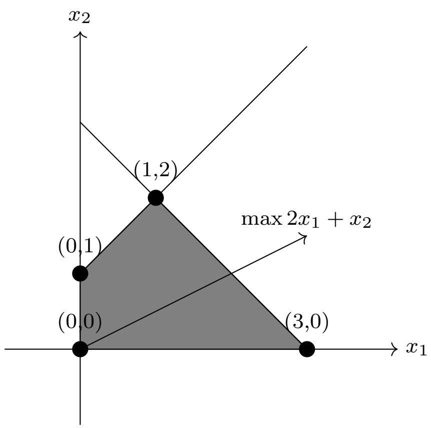
Tableau with $B = \{1, 3\}$ **:**

$$\begin{array}{rccccc} x_1 = & 3 & - & x_2 & - & x_4 \\ x_3 = & 4 & - & 2x_2 & - & x_4 \\ z = & 6 & - & x_2 & - & 2x_4 \end{array}$$

Tableau with $B = \{1, 2\}$ **:**

$$\begin{array}{rccccc} x_1 = & 1 & + & 0.5x_3 & - & 0.5x_4 \\ x_2 = & 2 & - & 0.5x_3 & - & 0.5x_4 \\ z = & 4 & + & 0.5x_3 & - & 0.5x_4 \end{array}$$

Graphical representation



Basic properties

- Any tableau describes one basic feasible solution.
- A basic feasible solution can have multiple tableaus (degeneracy, see below).
- Each tableau has m linearly-independent equalities.
- From each tableau we can read an equivalent LP to the original problem.
- Only variables with positive coefficients can *enter the basis.*
- If there are no variable that can enter the basis, we are at an optimal solution.
- Multiple variables could enter the basis, and there are many *rules* that can be used.
- Similarly, there may be multiple variables that can *leave the basis*, and only one should be selected based on the rules.
- Some *rules* lead to cycling (see below). Choosing a set of rules may affect performance.

Unboundedness

If the objective is unbounded, we will be able to increase the objective function without any variable needing to leave the basis.

Degeneracy

Important: The stopping rule is *no variable can enter the basis because all candidates have non-positive coefficients.* Notice that it is **not** *all possible variables that enter the basis have zero increase in the objective function.*

Sometimes it is necessary for a variable to enter the basis even with zero increase, so that we can find the right tableau (the right reordering of variables) to get a positive increase!

Cycling

Some natural rules:

- Among candidates, choose the variable to enter with the largest coefficient.
- Choose an improving variable which increases the objective function by the largest amount.
- Choose an improving variable whose entering into the basis moves the current basic feasible solution in a direction closest to the maximization direction c.

These three rules may lead to *cycling*, where we can just run around in the degenerate case without ever improving the objective function.

One rule that does not cycle is *Bland's rule*, which uses the indices of the variables (for example x_3 has index 3, and so on).

- Choose the improving variable with the smallest index, and if there are several possibilities for the leaving variable, also take the one with the smallest index.

Bootstrapping

- To start the simplex method, we need to have a basic feasible solution.
- However, finding a single basic feasible solution is computationally equivalent to solving a linear program. So we cannot have a simple algorithm that finds one!
- Solution: Create an *auxiliary* LP which has a very simple basic feasible solution, and whose optimal solution is a basic feasible solution of our original problem.

Exercise session

Exercise one

Suppose that we are given the following problem:

$$\begin{aligned} \max x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 \\ x_1 - x_5 + x_6 = 20 \\ x_1 + x_3 + x_7 = 30 \\ x_1 + x_2 + x_4 + x_8 = 10 \\ x_2 - x_3 - x_4 + x_5 + x_9 = 1 \\ x_1, x_2, \dots, x_9 \geq 0 \end{aligned}$$

and an initial basic solution $(0, 0, 0, 0, 0, 20, 30, 10, 1)$. Execute one step of the simplex algorithm. Which variable did you pick for your step and why?

Exercise two

Solve the following problem by the simplex method, executing all the steps:

$$\begin{aligned} \max 3x_1 + 2x_2 + 4x_3 \\ x_1 + x_2 + 2x_3 \leq 4 \\ 2x_1 + 3x_3 \leq 5 \\ 2x_1 + x_2 + 3x_3 \leq 7 \\ x_1, x_2, x_3 \geq 0 \end{aligned}$$

Exercise three

Find any basic feasible solution (a starting tableau for the simplex method) for the following linear program:

$$\begin{aligned} \max 4x_2 - x_4 \\ 3x_1 + x_2 - 2x_4 = 5 \\ -x_2 + x_3 = -2 \\ -2x_1 + 8x_2 + x_3 = 2 \\ x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Note: You can apply the direct simplex method approach or apply some math tricks to get the solution more directly. The choice is up to you.

Exercise four

Solve the following linear program using the simplex method:

$$\begin{aligned} \max 9x_1 + 5x_2 + 4x_3 + x_4 \\ 2x_1 + x_2 + x_3 + 2x_4 \leq 2 \\ 8x_1 + 4x_2 - 2x_3 - x_4 \geq 10 \\ 4x_1 + 7x_2 + 2x_3 + x_4 \leq 4 \\ x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

Please do not use any "human" shortcuts in your solution. Try to behave like the simplex method implemented by a computer, and write down every step of the process. The pivot selection rule is up to you.

Exercise five

In the following two exercises, we will play with the *Klee-Minty cube.* In three dimensions, this is the following LP:

$$\begin{aligned} \max 9x_1 + 3x_2 + x_3 \\ x_1 \leq 1 \\ 6x_1 + x_2 \leq 9 \\ 18x_1 + 6x_2 + x_3 \leq 81 \\ x_1, x_2, x_3 \geq 0 \end{aligned}$$

Starting from the basic feasible solution $(0, 0, 0, 1, 9, 81)$, can you find the smallest number of pivot steps (steps of the simplex method) that lead to the optimum solution?

Exercise six

Taking the linear program from the last exercise, and starting again at $(0, 0, 0, 1, 9, 81)$, what is the *largest* number of pivot steps (steps of the simplex method) – without any cycling – that lead to the optimum solution?