Intro to LO, Lecture 3

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Concepts from metric spaces

All of the definitions below work for general topological spaces, but we are only interested in the Euclidean space $\mathbb{R}^n$.

D(Open set): A set $S \subseteq \mathbb{R}^n$ is called open if for each point $p \in S$, there exists a radius r such that all points from a ball $B(p, r)$ are present in S.

D(Closed set): A set S is closed if its complement $(\mathbb{R}^n \setminus S)$ is open.

D(Bounded set): A set S is bounded if S fits into a ball of some finite diameter d.

D(Compact set): A set S is compact if it is closed and bounded.

T: Consider a function $f : \mathbb{R}^n \to \mathbb{R}$. If S is a compact set and f is continuous on S, then there exists points in S where f attains its infimum and supremum value over S.

Concepts from linear algebra

D: A set $A \subseteq \mathbb{R}^d$ is an *affine space*, if A is of the form $L + v$ for some linear space L and a shift vector $v \in \mathbb{R}^d$. By "A is of the form $L + v$" we mean a bijection between vectors of L and vectors of A given as $b(u) = u + v$. Each affine space has a *dimension*, defined as the dimension of its associated linear space L.

D: A vector x is an *affine combination* of a finite set of vectors $a_1, a_2, \dots a_n$ if $x = \sum_{i=1}^n \alpha_i a_i$, where α_i are real number satisfying $\sum_{i=1}^n \alpha_i = 1$.

A set of vectors $V \subseteq \mathbb{R}^d$ is *affinely independent* if it holds that no vector $v \in V$ is an affine combination of the rest.

C: A set of vectors $\{v_1, \dots, v_n\} \in \mathbb{R}^d$ is affinely *dependent* if there exists a not-all-zero set of coefficients α_i such that $\sum_i \alpha_i v_i = 0$ and $\sum_i \alpha_i = 0$.

O: A set of vectors $\{v_0, v_1, \dots, v_n\}$ is affinely indepedent if and only if the set $\{v_1 - v_0, \dots, v_n - v_0\}$ is linearly independent.

D: Given a set of vectors $V \subseteq \mathbb{R}^d$, we can think of its *affine span/affine hull*, which is a set of vectors A that are all possible affine combinations of any finite subset of V.

Similar to the linear spaces, affine spaces have a finite basis, so we do not need to consider all finite subsets of V, but we can generate the affine span as affine combinations of the base.

D(Dimension): The dimension of a set $X \subseteq \mathbb{R}^n, X \neq \emptyset$ is the dimension of the affine span of X.

O: The dimension of the set $X \neq \emptyset$ is the maximal d such that in X there exist affine independent points $\boldsymbol{a}_0, \dots, \boldsymbol{a}_d$.

T: Every linear space of dimension k contains a basis of k vectors. We can find a special basis that is *orthogonal* or even *orthonormal*). And for any basis (even a non-orthogonal one) we can compute its *orthogonal complement*. (How?)

Convexity

D: A set $K \subseteq \mathbb{R}^d$ is a *convex set*, if $\forall x, y \in K, \forall t \in [0, 1] : tx + (1-t)y \in K$. In other words, if you take two points inside the convex set K, the entire line segment between those two points must belong to K.

D: A vector x is a *convex combination* of a set of vectors $a_1, a_2, \dots a_n$ if $x = \sum_{i=1}^n \alpha_i a_i$, where α_i are real numbers satisfying $\sum_{i=1}^n \alpha_i = 1$ and also $\forall i : \alpha_i \in [0, 1]$.

A set of vectors/points $V \subseteq \mathbb{R}^d$ is *in a convex position*, if it holds that no vector $v \in V$ is a convex combination of the rest.

O: Let Y be the set of convex combinations of points from X. Then every convex combination of points of Y is a convex combination of points of X.

O: If all points $\boldsymbol{x} \in X$ satisfy the inequality $\boldsymbol{a}^T \boldsymbol{x} \leq b$, then any convex combination of points from X satisfies this inequality.

D: As with linearity and affinity, for convexity we also define a span/hull:

If we have a set of vectors $V \subseteq \mathbb{R}^d$, its *convex hull* is a set of all vectors C, which are convex combinations of any finite subset of the vectors in V.

Here, we really need to consider any finite subset of V, because convex sets in general do not have a finite basis.

T: The convex hull of $X, \operatorname{conv}(X)$, is equal to the set of all convex combinations

$$Y = \left\{ \alpha_0 \boldsymbol{a}_0 + \ldots + \alpha_k \boldsymbol{a}_k \mid k \in \mathbb{N}, \boldsymbol{a}_i \in X, \alpha_i \geq 0, \sum_{i=0}^k \alpha_i = 1 \right\}.$$

T(Caratheodory): Let $X \subseteq \mathbb{R}^n, \dim(X) = d$. Then,

$$\operatorname{conv}(X) = \left\{ \alpha_0 \boldsymbol{a}_0 + \ldots + \alpha_d \boldsymbol{a}_d \mid \boldsymbol{a}_i \in X, \alpha_i \geq 0, \sum_{i=0}^d \alpha_i = 1 \right\}.$$

D: A *hyperplane* is any affine space in $\mathbb{R}^d$ of dimension $d - 1$. Thus, on a 2D plane, any line is a hyperplane. In the 3D space, any plane is a hyperplane, and so on.

A hyperplane splits the space $\mathbb{R}^d$ into two *halfspaces*. We count the hyperplane itself as a part of both halfspaces.

D: A *convex polyhedron* (sometimes also called H-polytope) is any object in $\mathbb{R}^d$ that is an intersection of finitely many halfspaces. Alternatively, we can say that a convex polytope is any set of points of the form $\{x | Ax \leq b\}$ for some real matrix A and some real vector b.

T(Separation theorem): Let $C, D \subseteq \mathbb{R}^n$ be nonempty, closed, convex and disjoint. and let C be bounded. Then there is a hyperplane $\{\boldsymbol{x} \mid \boldsymbol{a}^T \boldsymbol{x} = b\}$, which strongly separates C and D, i.e., one that $C \subseteq \{\boldsymbol{x} \mid \boldsymbol{a}^T \boldsymbol{x} < b\}$ a $D \subseteq \{\boldsymbol{x} \mid \boldsymbol{a}^T \boldsymbol{x} > b\}$.

Exercises

Exercise one. Alice and Bob play a game. Alice will think of a linear inequality in $\mathbb{R}^3$ but it will not describe it to Bob. She will only tell Bob three points b_1, b_2, b_3 in $\mathbb{R}^3$, which satisfy the inequality.

Bob now must call out new points $b_4, b_5, b_6 \dots$ which also satisfy the inequality – until Alice gets bored of the game and they both go play hopscotch.

Suggest a strategy for Bob to win.

Exercise two.

1. Can two 2D planes intersect in exactly one point, if we place them in $\mathbb{R}^4$?
2. Can two $3D$ spaces (affine subspaces of dimension 3) intersect in exactly one point in $\mathbb{R}^5$?

Exercise three. We know that a set K is convex if the set contains all line segments with endpoints in K. Prove a very similar description for affinity:

A set A is an affine subspace of $\mathbb{R}^n$ if and only if for each two points $a, b \in A$ the entire *line* defined by a, b is contained in A.

Exercise four. Let $C \subseteq \mathbb{R}^n$ be a convex set and A be any $m \times n$ matrix. Show that the set $\{Ax \mid x \in C\}$ is also convex.

Exercise five. You might recall the definition of convexity for functions:

D(Convexity of a function): A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if for any $t \in [0, 1]$ and any two points x, y we have

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

Let us define the *epigraph* as "points above the curve of the function", formally:

$$\operatorname{epi} f \subseteq \mathbb{R}^{n+1}; \operatorname{epi} f = \{(x, v) \in \mathbb{R}^{n+1} \mid v \geq f(x)\}.$$

Prove that a function f is convex if and only if its epigraph is a convex set.

Exercise six. Prove the following: Let $C \subseteq \mathbb{R}^n$. Then, C is a closed convex set if and only if it can be expressed as $C = \bigcap_{F \in \mathcal{F}} F$ for some family of halfspaces $\mathcal{F}$. (A family here means just a set, not necessarily finite.)

Hints: One thing that might be useful here is that an arbitrary, even infinite, intersection of closed sets is again a closed set. Another thing that might be useful is some theorem from the lecture.