4. Complete statistics and uniqueness of MVUEs.

Let $X_1, \dots, X_n$ be a random sample from the Poisson distribution

$$f(x, \theta) = \frac{\theta^x e^{-\theta}}{x!}, \quad x = 0, 1, 2, \dots, \quad \theta > 0.$$

Recall that $Y_1 = \sum_{i=1}^{n} X_i$ is a sufficient statistic for θ and

$$g_1(y_1, \theta) = \frac{(n\theta)^{y_1} e^{-n\theta}}{y_1!}, \quad y_1 = 0, 1, \dots.$$

Consider the family $\{ g_1(y_1, \theta) : \theta > 0 \}$.

Suppose that the function $u(Y_1)$ of Y_1 is such that $E_\theta[u(Y_1)] = 0$ for every $\theta > 0$. We shall show that then

$$u(y_1) = 0 \quad \text{for } y_1 = 0, 1, 2, \dots$$

We have, for all $\theta > 0$,

$$0 = E[u(Y_1)] = \sum_{y_1=0}^{\infty} u(y_1) \frac{(n\theta)^{y_1} e^{-n\theta}}{y_1!} =$$

$$e^{-n\theta} \left[u(0) + u(1) \frac{n\theta}{1!} + u(2) \frac{(n\theta)^2}{2!} + \dots \right].$$

Since $e^{-n\theta} > 0$, ~~the~~ we have

$$u(0) + u(1) \frac{n\theta}{1!} + u(2) \frac{(n\theta)^2}{2!} + \dots = 0. \quad (\text{polynomial } (\theta))$$

(Polynomial vanishes, then coefficients as well)

$$u(0) = 0, \quad u(1) \frac{n\theta}{1!} = 0, \quad \frac{n^2\theta^2}{2!} u(2) = 0, \dots$$

Thereby

$$u(0) = 0, \quad u(1) = 0, \quad u(2) = 0, \dots$$

Definition 1

Let Z be a random variable with "the density" from the family $\{h(z, \theta) : \theta \in \Theta\}$.

If

$$\bigwedge_{\theta \in \Theta} \bigwedge_{u: \mathbb{R} \to \mathbb{R}} \mathbb{E}_\theta[u(Z)] = 0 \implies u(z) \equiv 0 \text{ almost surely},$$

then ~~the family~~ $\{h(z, \theta) : \theta \in \Theta\}$ is called a complete family while Z is called a complete statistic.

Let $X_1, \ldots, X_n$ be a sample with $f(x, \theta), \theta \in \Theta$.
Let $Y_1 = u_1(X_1, \ldots, X_n)$ be a sufficient statistic for θ, while $f_{Y_1}(y_1, \theta)$ is the density of Y_1. If Y_2 is ~~un~~ an unbiased estimator of θ which is not a function of Y_1, then $\varphi(Y_1) = \mathbb{E}[Y_2 | Y_1]$ is also the unbiased estimator of θ. Suppose, there is another function $\psi \neq \varphi$ of Y_1 such that $\mathbb{E}[\psi(Y_1)] = \theta$ for any $\theta \in \Theta$.

Thus

$$\mathbb{E}[\varphi(Y_1) - \psi(Y_1)] = 0 \quad \text{for } \theta \in \Theta.$$

If the ~~sa~~ family $\{f_{Y_1}(y_1, \theta) : \theta \in \Theta\}$ is complete, $\varphi(y_1) - \psi(y_1) = 0$ except on a set of points that has probability zero. Equivalently $\varphi(y_1) = \psi(y_1)$ a.s.
Therefore, $\varphi(Y_1)$ is the unique function of Y_1 such that $\mathbb{E}[\varphi(Y_1)] = \theta$. As a result, the Rao-Blackwell

implies that $\varphi(Y_1)$ is uniquely determined MVUE of θ.

Theorem 1 (Lehmann-Scheffé)

Let $X_1, \ldots, X_n$ be a sample for $f(x, \theta)$, $\theta \in \Theta$.
Let $Y_1 = u_1(X_1, \ldots, X_n)$ be a sufficient statistic for θ, and let the family $\{f_{Y_1}(y_1, \theta) : \theta \in \Theta\}$ be complete.
If there is a function φ of Y_1 that is $\varphi(Y_1)$, an unbiased estimator of θ, then ~~this function~~ $\varphi(Y_1)$ is uniquely determined MVUE of θ.

5. Exponential Class of Distributions

Consider a family of distributions $\{f(x, \theta) : \theta \in \Theta\}$, where $\Theta = \{\theta : \gamma < \theta < \delta\}$ while γ and δ are known constants (they may be $\pm\infty$), and

$$f(x, \theta) = \begin{cases} \exp[p(\theta)K(x) + S(x) + q(\theta)], & x \in S, \\ 0, & \text{elsewhere}, \end{cases}$$

where S is the support of X.

Definition 1

It is said that $f(x, \theta)$ is a member of the regular exponential class if

(i) S does not depend upon θ,

(ii) $p(\theta)$ is a nontrivial continuous function of θ

(iii) - if X is continuous, $K'(x) \not\equiv 0$ and $S(x)$ is a continuous function of $x \in S$
- if X is discrete, $K(x)$ is a nontrivial function of $x \in S$

Example 1

(i) The family $\{f(x|\theta) : 0<\theta<+\infty\}$, where

$$f(x|\theta) = \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{x^2}{2\theta}} = \exp\left[-\frac{1}{2\theta}x^2 - \log\sqrt{2\pi\theta}\right], \quad x\in\mathbb{R}$$

is a regular exponential class of the continuous type.

(ii) The family $\{f(x|\theta) : 0<\theta<+\infty\}$, where

$$f(x,\theta) = \frac{1}{\theta}\mathbb{1}_{(0,\theta)}(x) = \exp\{-\log\theta\}\mathbb{1}_{(0,\theta)}(x)$$

is not a regular exponential class.

Let $X_1,\dots,X_n$ denote a random sample from a distribution being a regular exponential class. The joint distribution has the form

$$\exp\left[p(\theta)\sum_{i=1}^{n}K(x_i) + \sum_{i=1}^{n}S(x_i) + nq(\theta)\right] \quad \text{for } x_i\in S, \; i=1,\dots,n.$$

Equivalently

$$\exp\left[p(\theta)\sum_{i=1}^{n}K(x_i) + nq(\theta)\right]\cdot\exp\left[\sum_{i=1}^{n}S(x_i)\right]$$

The factorization theorem implies that $Y_1 = \sum_{i=1}^{n}K(X_i)$ is a sufficient statistic for θ.

Theorem 2.

Let $f(x|\theta)$, $\gamma<\theta<\delta$ be the distribution of a random variable X being a member of a regular exponential class. If $X_1,\dots,X_n$ is a random sample from $f(x|\theta)$, the statistic $Y_1 = \sum_{i=1}^{n}K(X_i)$ is a complete sufficient stat for θ.

Corollary 1

If $X_1, \dots, X_n$, a random sample, comes from a regular exponential class and φ is a function of $Y_1 = \sum_{i=1}^{n} K(X_i)$ such that $\mathbb{E}[\varphi(Y_1)] = \theta$, then $\varphi(Y_1)$ is ~~an~~ uniquely determined MVUE of the parameter θ.

Example 2

$X_1, \dots, X_n$ i.i.d $X_1 \sim f(x, \theta) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{-\frac{(x-\theta)^2}{2\sigma^2}\right\}$, $x \in \mathbb{R}$,
$\theta \in \mathbb{R}$, $\sigma^2 > 0$.

Therefore

$$f(x, \theta) = \exp\left\{\frac{\theta}{\sigma^2} x - \frac{x^2}{2\sigma^2} - \log \sqrt{2\pi}\,\sigma - \frac{\theta^2}{2\sigma^2}\right\}$$

is a member of REC. Specifically,

$$p(\theta) = \frac{\theta}{\sigma^2},\ K(x) = x,\ S(x) = \frac{-x^2}{2\sigma^2} - \log\sqrt{2\pi\sigma^2},\ q(\theta) = -\frac{\theta^2}{2\sigma^2}.$$

Thus $Y_1 = \sum_{i=1}^{n} X_i$ is a complete sufficient statistic for θ.

Since $\varphi(Y_1) = \frac{Y_1}{n} = \bar{X}$ is an unbiased estimator of θ, $\varphi(Y_1)$ is ~~a~~ uniquely determined MVUE of θ.

$\bar{X}$ is also a complete sufficient statistic for θ because $Y_1 \xrightarrow{1-1} \bar{X}$.

Example 3

$X \sim Pois(\theta)$, $\theta \in (0, +\infty)$, $S = \{0, 1, 2, \dots\}$

$$f(x, \theta) = e^{-\theta} \frac{\theta^x}{x!} = \exp\{(\log\theta) x + \log\left(\frac{1}{x!}\right) + (-\theta)\} \quad \text{REC}$$

$p(\theta) = \log\theta$, $K(x) = x$, $S(x) = \log\left(\frac{1}{x!}\right)$, $q(\theta) = -\theta$.

$Y_1 = \sum_{i=1}^{n} X_i$ - complete and sufficient statistic for θ

$\mathbb{E} Y_1 = n\theta$ $\qquad \varphi(Y_1) = \bar{X}$ - UD MVUE ~~of~~ for θ.

III Theory of Testing Statistical Hypotheses

1. Introduction

Let X be a random variable with the distribution $f(x,\theta)$, $\theta \in \Theta$. Let Θ_0 and Θ_1 be (subsets of Θ) such that $\Theta_0 \cup \Theta_1 = \Theta$ and $\Theta_0 \cap \Theta_1 = \emptyset$.

Definition 1

Supposition $\theta \in \Theta_0$ is called the null hypothesis and is denoted by $H_0: \theta \in \Theta_0$, while supposition $\theta \in \Theta_1$ is called the alternative hypothesis and is denoted by $H_1: \theta \in \Theta_1$.

Definition 2

The testing formulation

$$H_0: \theta \in \Theta_0$$

against

$$H_1: \theta \in \Theta_1$$

is called the testing problem. Cheking statistal hypotheses is called testing (verifying) hypotheses.

Definition 3

If $\# \Theta_0 = 1$ $(\# \Theta_1 = 1)$ the hypothesis H_0 (H_1) is called simple. Otherwise, it is said that the hypothesis H_0 (H_1) is composite.

Let $X_1, \ldots, X_n$ be a sample with $f(x, \theta)$. Consider the testing problem

$$H_0: \theta \in \Theta_0,$$
$$H_1: \theta \in \Theta_1.$$

Let $\mathcal{X} = \{X_1(\omega), \ldots, X_n(\omega) : \omega \in \Omega\}$ be the sample space.

Definition 4

The statistic $T = T(X_1, \ldots, X_n)$ allowing one to assert in the above problem is called the test statistic.

Definition 5

The set $C = \{\underline{x} : \underline{x} = (x_1, \ldots, x_n), \underline{x} \in \mathcal{X}\}$ such that for $\underline{x} \in C$, $T(\underline{x})$ leads to rejection of the null hypothesis is called the critical region.

Remark 1

The critical region C of the form

(i) $\{\underline{x} : T(\underline{x}) > c_1\}$ for some $c_1 \in R$ is called the right-tailed critical region.

(ii) $\{\underline{x} : T(\underline{x}) < c_2\}$ for some $c_2 \in \mathbb{R}$ is called the left-tailed critical region.

(iii) $\{\underline{x} : T(\underline{x}) > c_3\} \cup \{\underline{x} : T(\underline{x}) < c_4\}$ for some $c_3, c_4 \in \mathbb{R}$ is called two-tailed critical region

(iv) In general the critical region (i) or (ii) is called a one-tailed critical region.