~~Let~~ Recall that $\alpha_i = \bar{\mu}_{i\cdot} - \bar{\mu}$, $i = 1, \ldots, a$; $\beta_j = \bar{\mu}_{\cdot j} - \bar{\mu}$, $j = 1, \ldots, b$, and $\mu = \bar{\mu}$. 4.03.14

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Then

$$\mu_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij},$$

where $\sum_{i=1}^{a} \alpha_i = 0$, $\sum_{j=1}^{b} \beta_j = 0$, and $\sum_{i=1}^{a} \gamma_{ij} = \sum_{j=1}^{b} \gamma_{ij} = 0$.

This model is called a <u>two-way model with interactions</u>.

<u>Example 2</u>

For $a=2$, $b=3$, $\mu=5$, $\alpha_1 = 1$, $\alpha_2 = -1$, $\beta_1 = 1$, $\beta_2 = 0$, $\beta_3 = -1$,

$\gamma_{11} = 1$, $\gamma_{12} = 1$, $\gamma_{13} = -2$, $\gamma_{21} = -1$, $\gamma_{22} = -1$, and $\gamma_{23} = 2$,

we have

		Factor B		
		1	2	3
Factor A	1	$\mu_{11} = 8$	$\mu_{12} = 7$	$\mu_{13} = 3$
	2	$\mu_{21} = 4$	$\mu_{22} = 3$	$\mu_{23} = 5$

First, we consider the testing problem

$$H_{0AB}: \gamma_{ij} = 0 \text{ for all } i,j \quad \text{versus} \quad H_{1AB}: \gamma_{ij} \neq 0, \text{ for some } i,j.$$

We have

$$\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c} (X_{ijk} - \bar{X}_{\ldots})^2 = \underbrace{bc \sum_{i=1}^{a} (\bar{X}_{i\cdot\cdot} - \bar{X}_{\ldots})^2}_{\text{row differences}} + \underbrace{ac \sum_{j=1}^{b} (\bar{X}_{\cdot j \cdot} - \bar{X}_{\ldots})^2}_{\text{column differences}} +$$

$\cancel{+ c \sum}$

$$+ c \sum_{i=1}^{a}\sum_{j=1}^{b} (\bar{X}_{ij\cdot} - \bar{X}_{i\cdot\cdot} - \bar{X}_{\cdot j\cdot} + \bar{X}_{\ldots})^2 + \quad \Big\} \text{ interaction}$$

$$+ \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c} (X_{ijk} - \bar{X}_{ij\cdot})^2 \quad \Big\} \text{ within cells}$$

$a-1$, $b-1$, $(a-1)(b-1)$ $\quad ab(c-1)$

$(a-1)(b-1)$ $ab - \cancel{b} - \cancel{a} + 1$ $\quad abc - ab$ $\qquad abc - 1 - (ab-1)$

$ab - 1$ $\qquad abc - ab = ab(c-1)$

The test statistic

$$F = \frac{\dfrac{c\sum_{i=1}^{a}\sum_{j=1}^{b}\left(\bar{X}_{ij\cdot}-\bar{X}_{i\cdot\cdot}-\bar{X}_{\cdot j\cdot}+\bar{X}_{\cdot\cdot\cdot}\right)^2}{(a-1)(b-1)}}{\dfrac{\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\left(X_{ijk}-\bar{X}_{ij\cdot}\right)^2}{ab(c-1)}}$$

has, under H_{0AB}, an F-distribution with $(a-1)(b-1)$ and $ab(c-1)$ degrees of freedom.

If H_{0AB} is accepted, ~~we~~ then, we test $c\sum^{b}\sum^{a}\frac{\gamma_{ij}^2}{\sigma^2}$

(i) $H_{0A}: \alpha_1 = \dots = \alpha_a = 0$ versus $H_{1A}: \alpha_i \neq 0$, for some i,

on the basis of

$$F = \frac{bc\sum_{i=1}^{a}\left(\bar{X}_{i\cdot\cdot}-\bar{X}_{\cdot\cdot\cdot}\right)^2 / [a-1]}{\dfrac{\sum\sum\sum\left(X_{ijk}-\bar{X}_{ij\cdot}\right)^2}{ab(c-1)}},$$

which, under H_{0A}, has an F-distribution with $(a-1)$ and $ab(c-1)$ degrees of freedom.

(ii) $H_{0B}: \beta_1 = \dots = \beta_b = 0$ versus $H_{1B}: \beta_j \neq 0$, for some j,

on the basis of

$$F = \frac{ac\sum_{j=1}^{b}\left(\bar{X}_{\cdot j\cdot}-\bar{X}_{\cdot\cdot\cdot}\right)^2 / [b-1]}{\sum\sum\sum\left(X_{ijk}-\bar{X}_{ij\cdot}\right)^2 / [ab(c-1)]},$$

which, under H_{0B}, has an F-distribution with $b-1$ and $ab(c-1)$ degrees of freedom.

$$(a-1)+(b-1)+(a-1)(b-1)+ab(c-1) = \cancel{a}-\cancel{1}+\cancel{b}-\cancel{1}+\cancel{ab}-\cancel{b}-\cancel{a}+\cancel{1}+abc-\cancel{ab}$$
$$abc-1$$

4. LS Estimation for Linear Model

6.05.14

We have p predictors $x_1, \dots, x_p$ and a response variable Y. We consider the model of the form

$$Y = h(x_1, \dots, x_p) + \varepsilon,$$

where ε is a random variable (a random error), and h is a specified function. We will restrict our attention to the case where h is linear in the β-coefficients. Our data consists of n vectors of the form $(Y_i, x_{i1}, \dots, x_{ip})$, for $i = 1, \dots, n$. We will center the x's, i.e. $x_{c_{ij}} = x_{ij} - \bar{x}_j$, where $\bar{x}_j = \frac{1}{n}\sum_{i=1}^{n} x_{ij}$. The linear model is

$$Y_i = \alpha + x_{ci1}\beta_1 + x_{ci2}\beta_2 + \dots + x_{cip}\beta_p + \varepsilon_i \quad , i = 1, \dots, n,$$

where $\alpha, \beta_1, \dots, \beta_p$ are unknown parameters (regression coefficients). We assume that the random errors $\varepsilon_1, \dots, \varepsilon_n$ are iid.

The matrix formulation of the model is as follows.

$$\begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \alpha + \begin{bmatrix} x_{c11} & \dots & x_{c1p} \\ \vdots & & \\ x_{cn1} & \dots & x_{cnp} \end{bmatrix} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_p \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{bmatrix},$$

or equivalently as

$$\underline{Y} = \mathbb{1}\alpha + \underline{X}_c \underline{\beta} + \underline{\varepsilon},$$

or in a more compact form

$$\underline{Y} = \underline{X}\,\underline{b} + \underline{\varepsilon}, \text{ where } \underline{X} = [\mathbb{1}\ \underline{X}_c] \text{ and } \underline{b} = (\alpha, \underline{\beta}')'.$$

We will assume that the $n \times (p+1)$ matrix $\underline{X}$ has full column rank $p+1$!

Let V be the space spanned by the columns of $\underline{X}$.

Then V is $(p+1)$-dimensional vector space of $\mathbb{R}^n$.

Put $\underline{\eta} = \underline{X}\,\underline{b}$. Then

$$\underline{Y} = \underline{\eta} + \underline{\varepsilon}, \quad \text{for } \underline{\eta} \in V.$$

Reading: Except for random error, $\underline{Y}$ would lie in the subspace V. So, to estimate η, find a vector in V which lies "closest" to Y (in terms of a given norm)

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4.1 Least Squares

The LS estimator of η has a form

$$\hat{\eta} = \arg\min_{\eta \in V} \| Y - \eta \|^2, \quad \text{where} \quad \|v\|^2 = \sum_{i=1}^{n} v_i^2 .$$

Let $V^\perp$ be the subspace which consists of all vectors in $\mathbb{R}^n$ which are orthogonal to all vectors in V, that is,

$$V^\perp = \{ w \in \mathbb{R}^n : w'v = 0, \text{ for all } v \in V \}.$$

The dimension of $V^\perp$ is $n-(p+1)$.

Definition 1

Let v be a vector in $\mathbb{R}^n$ and let V be a subspace of $\mathbb{R}^n$. We say that $\hat{v}$ is the projection of v onto V if

(i) $\hat{v} \in V$,

(ii) $v - \hat{v} \in V^\perp$.

Theorem 1

Projections are unique.

Proof

1. Let $\hat{v}_1$ and $\hat{v}_2$ be projections of v onto V.
2. Since V is a subspace, from (i) $\hat{v}_1 - \hat{v}_2 \in V$.
3. But $\hat{v}_1 - \hat{v}_2 = (v - \hat{v}_2) - (v - \hat{v}_1) \in V^\perp$.
4. Thus $\| \hat{v}_1 - \hat{v}_2 \|^2 = 0$.
5. Finally $\hat{v}_1 = \hat{v}_2$. □

The columns of $\underline{X}$ form a basis for the subspace V. Therefore, we will say that $\underline{X}$ is a basis matrix for V and that $\underline{X}$ has full column rank, which implies that $(X'X)^{-1}$ exists

Theorem 2

Let $\underline{X}$ be a basis matrix for a subspace V, let $\underline{H} = \underline{X}(\underline{X}'\underline{X})^{-1}\underline{X}'$, and let $\underline{v}$ be a vector in $\mathbb{R}^n$. Then the projection of $\underline{v}$ onto V is $\underline{Hv}$