Intro to LO, Lecture 11

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University of Wrocław, Winter 2023/2024

Complementary slackness

D(Slack): Suppose we have a system of linear inequalities (S) and, more specifically, the j-th inequality

$$a_{j1}x_1 + a_{j2}x_2 + a_{j3}x_3 + \ldots + a_{jn}x_n \leq b_j.$$

Suppose we are also given a vector x' that satisfies the j-th inequality. Then the *slack* of the j-th inequality and the solution x' is $s_j^{(S)} = b_j - \sum_{i=1}^n a_{ji}x_i'$.

Notice that it always holds that $s_j^{(S)} \geq 0$. If the inequality is $\geq$, we define the slack as $s_j^{(S)} = \sum_{i=1}^n a_{ji}x_i' - b_j$, so that again $s_j^{(S)} \geq 0$.

Take our sample LPs:

$$\begin{array}{rrcrcl}
\max & 2x_1 & + & 3x_2 & & \\
\text{s.t.} & 4x_1 & + & 8x_2 & \leq & 12 \\
 & 2x_1 & + & x_2 & \leq & 3 \\
 & 3x_1 & + & 2x_2 & \leq & 4 \\
 & x_1 & , & x_2 & \geq & 0
\end{array}$$

The dual is:

$$\begin{array}{rrcrcrcl}
\min & 12y_1 & + & 3y_2 & + & 4y_3 & & \\
\text{s.t.} & 4y_1 & + & 2y_2 & + & 3y_3 & \geq & 2 \\
 & 8y_1 & + & y_2 & + & 2y_3 & \geq & 3 \\
 & & & & & & & \\
 & y_1 & , & y_2 & , & y_3 & \geq & 0
\end{array}$$

The optimum of the primal has value 4.75 with $x = (0.5, 1.25)$. We also note down the slack vector $s^{(P)} = (0, 1.25, 0)$.

For the dual, the optimum is 4.75 (no surprise there) with $y = (0.3125, 0, 0.25)$ and the dual slack vector $s^{(D)} = (0, 0)$.

It is to be expected that the slack vector of the primal and the variable vector of the dual y have the same length, but we can see an additional pattern, if we put them one above each other: $s_i^{(P)}$ is non-zero if and only if y_i is zero. The same pattern holds for x and $s^{(D)}$.

This is actually a general property:

T(Complementary slackness): Assume we have a linear program (P) and its dual (D) of the following form.

$$\max c^T x, Ax \leq b, x \geq 0, \tag{P}$$
$$\min b^T y, A^T y \geq c, y \geq 0. \tag{D}$$

We are also given a pair of feasible solutions of the primal and dual (x', y'). Then the following holds:

The pair (x', y') is a pair of optimal solutions if and only if all the following conditions are satisfied:

$$\forall i \in \{1, \ldots, n\}\colon x_i' \cdot s_i^{(D)} = 0, \tag{1}$$
$$\forall j \in \{1, \ldots, m\}\colon s_j^{(P)} \cdot y_j' = 0. \tag{2}$$

P: From the conditions of the linear programs, we have that

$$x^T c \leq x^T(A^T y) \leq (x^T A^T) y \leq b^T y.$$

Since we assume that x', y' are two optimal solutions for (P) and (D), respectively, then it holds that $x^T c = b^T y$, so all inequalities hold, so we have

$$x^T c = x^T(A^T y) = (x^T A^T) y = b^T y.$$

Focusing on the first part, we can rewrite it as

$$x^T(c - A^T y) = 0.$$

We have that $x \geq 0$, in other words, each $x_i \geq 0$. On the other hand, the i-th coordinate of the vector $(c - A^T y)$ is ≤ 0, because $A^T y \geq c$. Product of a non-negative and a non-positive number is non-positive. Still, we know that $x^T(c - A^T y) = 0$. If a sum of non-positive numbers is 0, then they were all equal 0 – which means that each product $(c - A^T y)_i \cdot x_i = 0$, which is what we set up to show. The remaining equations follow from analogously from the last part of the equality chain.

Dual rounding techniques

D(Set cover): In the Set Cover problem, we are given a ground set of elements $E = \{e_1, \ldots, e_n\}$, some subsets of those elements $S_1, S_2 \ldots, S_m$ where each $S_j \subseteq E$, and a nonnegative weight $w_j \geq 0$ for each subset S_j. The goal is to find a minimum-weight collection of subsets that covers all of E; that is, we wish to find an $I \subseteq \{1, \ldots, m\}$ that minimizes $\sum_{j \in I} w_j$ subject to $\bigcup_{j \in I} S_j = E$.

D(Max frequency): For a given Set Cover instance, we define f to be the maximum number of times a single element e_i occurs in multiple sets. In other words, $f = \max_i |\{j | e_i \in S_j\}|$.

The integer LP would be:

$$\begin{aligned}
&\text{minimize} \sum_{j=1}^m w_j x_j \\
&\text{subject to} \sum_{j: e_i \in S_j} x_j \geq 1, \quad i = 1, \ldots, n, \\
&x_j \in \{0, 1\}, \quad j = 1, \ldots, m.
\end{aligned}$$

and the dual to the linear program (with $x_j \in [0, 1]$) is

$$\begin{aligned}
&\text{maximize} \sum_{i=1}^n y_i \\
&\text{subject to} \sum_{i: e_i \in S_j} y_i \leq w_j, \quad j = 1, \ldots, m, \\
&y_i \geq 0, \quad i = 1, \ldots, n.
\end{aligned}$$

For this session, we will consider *approximation algorithms* again, which we have also seen for Vertex Cover. Recall:

D(Approximation): A feasible solution x^{APP} to a minimization problem is a c-approximate solution if for the optimum solution x^{OPT} we have $\sum_j w_j x_j^{APP} \leq c \cdot \sum_j w_j x_j^{OPT}$.

In other words, a c-approximate solution can be bigger than the absolute minimal solution, but no bigger than c-times that absolute minimum solution.

Recall also:

T: If x^* is an optimum solution to a minimization linear program and x^{OPT} is the optimum solution to the corresponding integer linear program, then $\sum_j w_j x_j^* \leq \sum_j w_j x_j^{OPT}$.

Let y^* be the optimal solution to the dual, and s^D the vector of slacks of the dual. Complementarity tells us that if x^* is the optimal solution to the primal, then $x^* > 0 \Rightarrow s^D = 0$ and $s^D > 0 \Rightarrow x^* = 0$. We will take it as a guiding property for an algorithm, and use it "the wrong way" as an if and only if:

$$\text{If } s_j^D = 0, \text{ then set } x_j^{APP} = 1, \text{ otherwise set } x_j^{APP} = 0.$$

We observe the following:

O: If we take the sets for which $x^{APP} = 1$, then they form a valid set cover.

P: Suppose not. Then, one element e_k is not covered. This means all its $s^{(D)} > 0$, so no inequalities are tight. Then we can increase the value of y_k by ε and improve the dual solution, but we started with y^* which is the optimum dual solution – a contradiction.

T: The dual rounding approach produces a f-approximate solution for Set Cover.

P: Let I' denote the sets that we set to one as part of $x^{(APP)}$. Since we only took $x_j^{APP} = 1$ if $s_j^{(D)} = 0$, which in other words means $\sum_{i: e_i \in S_j} y_i^* = w_j$, we can rewrite

$$\sum_j w_j x_j^{APP} = \sum_{j \in I'} \sum_{i: e_i \in S_j} y_i^*.$$

The next trick is to switch the order of counting, instead counting over elements first and sets second

$$\sum_{j \in I'} \sum_{i: e_i \in S_j} y_i^* = \sum_{i=1}^n |\{j \in I' \mid e_i \in S_j\}| y_i^*.$$

The expression $|\{j \in I' \mid e_i \in S_j\}|$ counts how many sets in I' does a given element e_i participate in, and we know this is at most f by definition.

$$\sum_{i=1}^{n} |\{j \in I' \mid e_i \in S_j\}| y_i^* \leq \sum_{i=1}^{n} f y_i^*.$$

In other words, our objective value is at most $f \cdot \sum_{i=1}^{n} y_i^*$, which is the optimum dual value (times f). But by strong duality, we know the optimum value equals $\sum_j w_j x_j^*$ and we have already noted above that $\sum_j w_j x_j^* \leq \sum_j w_j x_j^{OPT}$.

Solving the LP via primal-dual

We know we can solve the dual LP in polynomial time using a linear programming solver, but we can actually proceed directly to the solution x^{APP}, without solving either the primal or the dual:

Algorithm Primal-Dual:

$y \leftarrow 0$

$I \leftarrow \emptyset$

while there exists $e_i \notin \bigcup_{j \in I} S_j$:

Increase the dual variable y_i until there is some ℓ with $e_i \in S_\ell$ such that

$$\sum_{j : e_j \in S_\ell} y_j = w_\ell$$

$I \leftarrow I \cup \{\ell\}$

For all $\ell \in I$ set $x_\ell^{APP} = 1$.

L: We observe:

1. The resulting set I is a feasible set cover.
2. Whenever $x_\ell^{APP} = 1$, then $\sum_{j : e_j \in S_\ell} y_j = w_\ell$.

Introduction to matchings

D(Matching)**:** A *matching* M is a subgraph of a graph $G = (V, E)$ such that the degree of each vertex in M is at most one. That means, each vertex sees at most one edge. A matching is *perfect* if its number of edges is $|V|/2$. A matching is *maximum* if there does not exist any other matching in the graph which is larger.

D: Given two subsets of edges E_1, E_2 of a graph, the symmetric difference operator $E_1 \triangle E_2$ equals $(E_1 \setminus E_2) \cup (E_2 \setminus E_1)$.

D: For a matching M on a graph G, an M-augmenting path is a path $(e_1, e_2, \ldots, e_k)$ of odd length from v_1 to v_2 such that v_1 and v_2 are not covered by M, $e_1, e_k \notin M$, and the edges e_i alternate membership in M.

T: For a matching M on a graph G, M is a maximum matching in $G \Leftrightarrow$ there is no M-augmenting path in G.

P: $\neg B \Rightarrow \neg A$: Easy, just extend the matching.

$\neg A \Rightarrow \neg B$: Take two matchings M_1, M_2, with $|M_2| > |M_1|$, and consider graph $M_1 \triangle M_2$. There must be a path where M_2 contributes more edges $\Rightarrow$ an augmenting path.

Exercises

Exercise one. During an optimization exam, Joseph K. copied from his neighbor the statement of a dual program and a feasible solution of the primal:

The dual is:

$$\begin{aligned} \max\ & 2x_1 + 3x_2 + 5x_3 + 4x_4 \\ & x_1 + 2x_2 + 3x_3 + x_4 \leq 5 \\ & x_1 + x_2 + 2x_3 + 3x_4 = 3 \\ & x_1 + x_2 + x_3 + x_4 \geq 1 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

The feasible primal solution is $y = (4, 0, 0)$. However, the goal of the exercise was to check if y is an optimum or not. Use complementary slackness to check this. (You do not even need to dualize.)

Exercise two. Josephine K. also cheated during her exam, but she copied a statement of the primal and an optimal solution of the dual:

$$\begin{aligned} \min\ & 10x_1 - 4x_2 \\ & x_1 + 0.6x_3 + 4x_4 \geq 43 \\ & x_1 - x_2 + 0.6x_3 + 10x_4 \geq 27 \\ & x_1 - x_2 - 0.4x_3 - x_4 \geq 24 \\ & x_1 - x_2 - 0.4x_3 - 2x_4 \geq 22 \\ & x_1 + 3.6x_3 - 3x_4 \geq 56 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

The optimal dual solution is $y = (3.36, 0, 0, 6.48, 0.16)$. However, the goal was to compute the optimal solution of the primal. Help Josephine using complementary slackness.

Exercise three. For the LP and the dual from the third exercise find a pair of vectors x and y such that:

$$\forall i \in \{1, \ldots, n\} \colon x_i \cdot s_i^{(D)} = 0, \tag{1}$$

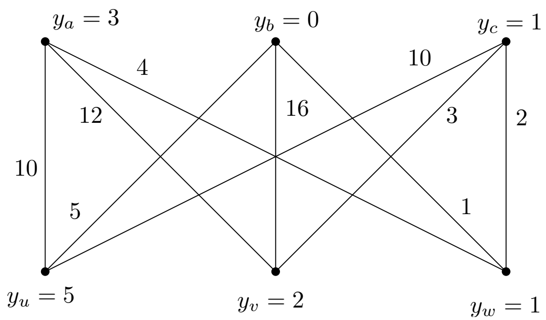
$$\forall j \in \{1, \ldots, m\} \colon s_j^{(P)} \cdot y_j = 0. \tag{2}$$

but x and y **are not** a pair of optimal solutions.

Tip: Find the difference between the statement of the complementary slackness theorem and this exercise.

Exercise four

1. Recall the LP for the bipartite matching problem, and dualize it.
2. Below is a bipartite graph with weights on the edges. Next to each vertex you see a value to a supposedly optimal dual solution for Perfect Bipartite Matching of Minimum Cost. Prove that this dual solution is optimal.



Exercise five. Formulate the problem of finding the shortest path from s to t in a graph with non-negative edge weights w_e as a $\{0,1\}$-integer program. In your program, you should have a constraint for every s,t-cut in the graph.

Dualize this problem as well.

Exercise six. Consider the following algorithm:

1. $y \leftarrow \vec{0}$, where y is the dual variable vector.
2. $F \leftarrow \emptyset$ – infeasible solution of the primal.
3. While there is no s,t-path in $G[F]$:
4. Consider the (unique) component C of the graph $G[F]$ containing s.
5. Increase y_C equally, until some constraint (corresponding to e) becomes tight.
6. Add e to F.
7. For every $e \in F$:
8. If $G[F \setminus \{e\}]$ contains a s,t-path, remove e from F.
9. Return F as a candidate for the shortest s,t-path.

Your tasks:

- Prove that this algorithm always returns a path.
- Prove the following observation: $y_S > 0 \Rightarrow |\delta(S) \cap F| \leq 1$.
- Finally, prove that F is the shortest path by arguing about the sum $\sum_{e \in F} c_e$.