Von Neumann algorithm.

For given pdf f we need to generate r. sample

$X_1 \dots X_n$ from distribution with pdf f.

Condition

$\exists g$ - known pdf (we know how to generate r. sample from distr. with pdf g)

$\exists M > 0$: $f(s) \le M g(s)$, $\forall s$.

Algorithm

1) $Y \sim g$, $U \sim [0,1]$ - independent.
2) if $U \le \frac{f(Y)}{Mg(Y)} \Rightarrow X := Y$,

   else $\Rightarrow$ 1).

$\Rightarrow X \sim F$, $F(x) = \int_{-\infty}^{x} f(y)\,dy$

$$\ulcorner\ P(X \le x) = P\left(Y \le x \,/\, U \le \frac{f(Y)}{Mg(Y)}\right) =$$

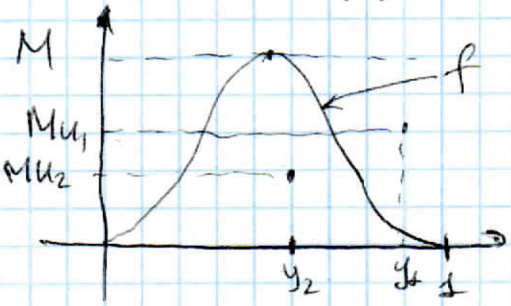
$$= \frac{P\left(Y \le x,\ U \le \frac{f(Y)}{Mg(Y)}\right)}{P\left(U \le \frac{f(Y)}{Mg(Y)}\right)} = \frac{\int_{-\infty}^{x} g(y)\left(\int_0^{f(y)/Mg(y)} 1 \cdot du\right)dy}{\int_{-\infty}^{+\infty} g(y)\left(\int_0^{f(y)/Mg(y)} 1\, du\right)dy} =$$

$$= \frac{\int_{-\infty}^{x} g(y) \cdot \frac{f(y)}{Mg(y)}\,dy}{\int_{-\infty}^{+\infty} g(y)\,\frac{f(y)}{Mg(y)}\,dy} = \frac{\frac{1}{M}\int_{-\infty}^{x} f(y)\,dy}{\frac{1}{M}\underbrace{\int_{-\infty}^{+\infty} f(y)\,dy}_{=1}} = \int_{-\infty}^{x} f(y)\,dy =$$

$$= F(x)\ \lrcorner$$

Example

Let $\operatorname{supp} f = [0,1]$ $\quad f \le M$.



$\Rightarrow \quad g(x) = \mathbb{1}_{[0,1]}(x)$

$Y \sim \mathcal{U}[0,1]$

Algorithm

1) $Y, U \sim \mathcal{U}[0,1]$, independ.
2) $(Y, MU) \sim \mathcal{U}([0,1] \times [0,M])$
   $Y: \quad MU < f(Y)$

Remark: Number of generated points Y, U is connected to the question, how good $M g(x)$ estimates $f(x)$. / the better upper estimation $\Rightarrow$ $\Rightarrow$ the average number Y, U needed to obtain sample will be smaller/