MLE

$X_1, \dots, X_n$ - RANDOM SAMPLE FROM DISTRIBUTION WITH PDF $f(x, \theta)$, + REGULARITY conditions for f $\theta \in \Omega$

Likelihood function:

$$L(\theta, \mathbf{x}) = \prod_{i=1}^{n} f(x_i, \theta)$$

$\mathbf{x} = (x_1 \dots x_n)$

MAXIMUM LIKELIHOOD ESTIMATOR (MLE) for θ is

$$\hat{\theta} = \underset{\theta \in \Omega}{\operatorname{argmax}}\, L(\theta, x) \iff \max_{\theta \in \Omega} L(\theta, x) = L(\hat{\theta}, x)$$

$$\ell(\theta, \mathbf{x}) = \log L(\theta, \mathbf{x})$$

$$\frac{\partial \ell}{\partial \theta} = 0 \qquad (1)$$

Let $\hat{\theta} = \hat{\theta}(x_1 \dots x_n)$ be a solution to (1)

$$\left.\frac{\partial^2 \ell}{\partial \theta^2}\right|_{\theta = \hat{\theta}} < 0.$$

The CORRESPONDING STATISTIC

$$\hat{\theta} = \hat{\theta}(X_1 \dots X_n)$$

is CALLED the maximum likelihood estimator of θ.

EXAMPLE:

LET $X_1 \dots X_n$ BE A RANDOM SAMPLE FROM DISTRIBUTION WITH PDF

$$f(x,\theta) = \begin{cases} \frac{1}{2\theta^3} x^2 e^{-\frac{x}{\theta}}, & x>0 \\ 0, & x \le 0, \end{cases}$$

here $\theta > 0$. Find MLE of θ:

$$L(\theta, x) = \prod_{i=1}^{n} f(x_i, \theta) =$$

$$= \frac{1}{2^n \theta^{3n}} (x_1 \cdots x_n)^2 e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i}$$

$$\ell(\theta, x) = \log L(\theta, x) =$$

$$= -n\log 2 - 3n\log\theta + 2\sum_{i=1}^{n}\log x_i - \frac{1}{\theta}\sum_{i=1}^{n} x_i$$

$$\frac{\partial \ell}{\partial \theta} = -\frac{3n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i = 0 \Rightarrow$$

$$\Rightarrow \hat{\theta} = \frac{1}{3n}\sum_{i=1}^{n} x_i = \frac{\bar{X}}{3},$$

where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$ is the SAMPLE MEAN.

$$\frac{\partial^2 \ell}{\partial \theta^2} = \frac{3n}{\theta^2} - \frac{2}{\theta^3}\underbrace{\left(\sum_{i=1}^{n} x_i\right)}_{= 3n\hat{\theta}}\Bigg|_{\theta=\hat{\theta}} = \frac{3n}{\hat{\theta}^2} - \frac{6n}{\hat{\theta}^2} < 0$$

The CORRESPONDING statistic

$$\hat{\theta} = \frac{\bar{X}}{3}$$

is MLE of θ.

FISHER INFORMATION

+ regularity conditions for f

$$I(\theta) = VAR\left(\frac{\partial \log f(x,\theta)}{\partial \theta}\right) \qquad / E\frac{\partial \log f}{\partial \theta} = 0 /$$

$$I(\theta) = E\left(\frac{\partial \log f}{\partial \theta}\right)^2 = -E\,\frac{\partial^2 \log f(x,\theta)}{\partial \theta^2}$$

EXAMPLE:

$$f(x,\theta) = \begin{cases} \frac{1}{2\theta^3}\, x^2 e^{-\frac{x}{\theta}} & , x > 0 \\ 0 & , x \le 0 \end{cases}$$

$\theta > 0.$

Find $I(\theta)$.

$\ulcorner$ $\log f(x,\theta) = -\log 2 - 3\log\theta + 2\log x - \frac{x}{\theta}$

$$\frac{\partial \log f(x,\theta)}{\partial \theta} = -\frac{3}{\theta} + \frac{x}{\theta^2}$$

$$\frac{\partial^2 \log f(x,\theta)}{\partial \theta^2} = \frac{3}{\theta^2} - \frac{2x}{\theta^3}$$

$$I(\theta) = -E\,\frac{\partial^2 \log f(x,\theta)}{\partial \theta^2} = -\frac{3}{\theta^2} + \frac{6}{\theta^2} = \frac{3}{\theta^2}$$

$EX = 3\theta.$ $\lrcorner$

RAO-CRAMER LOWER BOUND

$X_1 \dots X_n$ - RANDOM SAMPLE FROM distribution with pdf $f(x,\theta)$, $\theta \in \Omega$

Let $Y = u(X_1 \dots X_n)$ be a statistic.
$EY = k(\theta)$

$$\Rightarrow \quad \mathrm{VAR}\, Y \geq \frac{[k'(\theta)]^2}{nI(\theta)}$$

This inequality gives a lower bound on the variance

Corollary: If Y is unbiased estimator of θ / $EY = \theta \Leftrightarrow k(\theta) = \theta$ /, then

$$\mathrm{VAR}\, Y \geq \frac{1}{nI(\theta)}.$$

Theorem (6.2.2)

$$\sqrt{n}\,(\hat{\theta}_n - \theta_0) \xrightarrow{D} N\left(0, \frac{1}{I(\theta_0)}\right)$$

consistent sequence of solutions of MLE eq.

Example:

$$f(x,\theta)=\begin{cases}\frac{1}{2\theta^3}x^2e^{-x/\theta}, & x>0\\ 0, & x\le 0\end{cases},$$

$\theta>0.$

Let $X_1 \dots X_n$ be a random sample from distribution with pdf $f(x,\theta)$

$\sim$ Gamma$(3,\theta)$

$EX_1 = 3\theta$, $Var X_1 = 3\theta^2$

$\hat{\theta} = \hat{\theta}(X_1 \dots X_n) = \frac{\bar{X}}{3}$ (MLE of θ)

$\hat{\theta}$ - unbiased?

$$E\hat{\theta} = \frac{1}{3}\frac{1}{n}\sum_{i=1}^{n}\underbrace{EX_i}_{3\theta} = \theta.$$

$$Var\,\hat{\theta} = \frac{1}{9n^2}\sum_{i=1}^{n}\underbrace{Var\,X_i}_{3\theta^2} = \frac{\theta^2}{3n}$$

$\Rightarrow$ Rao-Cramer LB:

$Var\,\hat{\theta} = \frac{1}{nI(\theta)} \Rightarrow \hat{\theta}$ is efficient.

$$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{D}} N(0, \tfrac{\theta^2}{3})$$

$$\frac{\sqrt{3}\sqrt{n}(\hat{\theta}_n - \theta)}{\theta} \sim N(0,1)$$

$\theta \approx \hat{\theta}_n$

$\Rightarrow (1-\alpha)100\%$ confidence interval for θ:

$$\left(\hat{\theta}_n - z_{\alpha/2}\frac{\hat{\theta}_n}{\sqrt{3n}},\ \hat{\theta}_n + z_{\alpha/2}\frac{\hat{\theta}_n}{\sqrt{3n}}\right)$$

Method of moments

Example:

$X_1, \dots, X_n$ - RANDOM SAMPLE
$\sim$ GAMMA (α, β).

$$\mu_1 = EX_1 = \alpha\beta$$
$$\mu_2 = EX_1^2 = \alpha\beta^2 + \alpha^2\beta^2$$

$$\mu_1 \to \hat{\mu}_1 = \frac{1}{n}\sum_{i=1}^{n} X_i$$
$$\mu_2 \to \hat{\mu}_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

$\Rightarrow$ (method of moments)

$$\hat{\mu}_1 = \hat{\alpha}\hat{\beta}$$
$$\hat{\mu}_2 = \hat{\alpha}\hat{\beta}^2 + \hat{\alpha}^2\hat{\beta}^2$$

$$\hat{\mu}_2 = \hat{\mu}_1\hat{\beta} + \hat{\mu}_1^2 \implies \hat{\beta} = \frac{\hat{\mu}_2 - \hat{\mu}_1^2}{\hat{\mu}_1}$$
$$\hat{\alpha} = \frac{\hat{\mu}_1}{\hat{\beta}} = \frac{\hat{\mu}_1^2}{\hat{\mu}_2 - \hat{\mu}_1^2}$$