Since the above holds for any $\theta_1 > \theta_0$, the UMP test in the problem $(H_0, A)$ has the form $\Pi_C$.

Example 2

$X_1, \dots, X_n$ i.i.d, $X_i \sim N(\theta, 1)$

We verify

$$H_0: \theta = \theta_0,$$
$$H_1: \theta \neq \theta_0.$$

It will be shown that there is no UMP test in the above problem. We have $\Theta = \mathbb{R}$. Let $\theta_1 \in \Theta$ and $\theta_1 \neq \theta_0$. Consider

$$\frac{L(\theta_1)}{L(\theta_0)} = \frac{\left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \exp\left[-\frac{1}{2}\sum_{i=1}^{n}(x_i - \theta_1)^2\right]}{\left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \exp\left[-\frac{1}{2}\sum_{i=1}^{n}(x_i - \theta_0)^2\right]} \geq k.$$

Equivalently,

$$\exp\left[(\theta_1 - \theta_0)\sum_{i=1}^{n} x_i - \frac{n}{2}(\theta_1^2 - \theta_0^2)\right] \geq k$$

or

$$(\theta_1 - \theta_0)\sum x_i \geq \frac{n}{2}(\theta_1^2 - \theta_0^2) + \log k,$$

so,

$$\sum_{i=1}^{n} x_i \geq \frac{n}{2}(\theta_1^2 + \theta_0^2) + \frac{\log k}{\theta_1 - \theta_0} \quad \text{provided that } \theta_1 > \theta_0$$

or

$$\sum_{i=1}^{n} x_i \leq \frac{n}{2}(\theta_1 + \theta_0) + \frac{\log k}{\theta_1 - \theta_0} \quad \text{provided that } \theta_1 > \theta_0.$$

The first expression defines the critical region of the UMP test in the problem of testing $H_0: \theta = \theta_0$ against $H_1^+: \theta = \theta_1$ provided that $\theta_1 > \theta_0$, while the second expression defines the critical region of the UMP test ... $H_0: \theta = \theta_0$ ~ $H_1^-: \theta = \theta_1$ in $\theta_1 < \theta_0$.

The UMP test does not exist because these regions are different.

Remark 1

Under the assumptions of Example 2, there are UMP tests in the problems

$$H_0: \theta = \theta_0, \quad H_1^+: \theta > \theta_0, \qquad \text{and} \qquad H_0: \theta = \theta_0, \quad H_1^-: \theta < \theta_0.$$

Let $X_1, \dots, X_n$ be ~~th~~ sample from $f(x, \theta)$, $\theta \in \Theta$, and $Y = u(X_1, \dots, X_n)$ a sufficient statistic for $\theta$.

Factorization theorem implies

$L(\theta, \underline{x}) = k_1[u(\underline{x}), \theta] \cdot k_2(\underline{x})$, where $k(\underline{x})$ does not depend on $\theta$. Thereby, for $\theta_0, \theta_1 \in \Theta$

$$\frac{L(\theta_1, \underline{x})}{L(\theta_0, \underline{x})} = \frac{k_1[u(\underline{x}), \theta_1]}{k_1[u(\underline{x}), \theta_0]}$$

depends on $x_1, \dots, x_n$ only by $u(\underline{x})$.

Corollary 1

The UMP test is a function of a sufficient statistic.

Definition 1

It is said that the family of distributions ~~likelihood~~ $\{L(\theta, x): \theta \in \Theta\}$ has monotone likelihood ratio with respect to a statistic $Y = u(\underline{X})$ if for all $\theta_0, \theta_1 \in \Theta$ such that $\theta_0 < \theta_1$ the ratio

$$\frac{L(\theta_1, \underline{x})}{L(\theta_0, \underline{x})}$$

is a nondecreasing function of $y = u(\underline{x})$.

## Theorem 1 (Karlin-Rubin) (continuous version)

Consider the testing problem

$$H_0: \theta \leq \theta_0,$$
$$H_1: \theta > \theta_0.$$

If the family of distributions $\{L(\theta, \underline{x}) : \theta \in \Theta\}$ has monotone likelihood ratio with respect to a statistic $Y = u(\underline{X})$, then, in the problem $(H_0, H_1)$, there is UMP test with the critical region

$$C = \{\underline{x} : u(\underline{x}) > k\},$$

where the constant satisfies the condition $P_{\theta_0}(Y > k) = P_{\theta_0}(u(\underline{X}) > k) = \alpha$.

## Example 3

$X_1, \dots, X_n$ i.i.d $X_i \sim b(1, p)$, $p = \theta$, $0 < \theta < 1$. Let $\theta_1 > \theta_0$ and consider the ratio

$$\frac{L(\theta_1, \underline{x})}{L(\theta_0, \underline{x})} = \frac{\theta_1^{\sum_{i=1}^n x_i}(1-\theta_1)^{n - \sum x_i}}{\theta_0^{\sum x_i}(1-\theta_0)^{n - \sum x_i}} = \left[\frac{\theta_1(1-\theta_0)}{\theta_0(1-\theta_1)}\right]^{\sum_{i=1}^n x_i} \left(\frac{1-\theta_1}{1-\theta_0}\right)^n$$

which is an increasing function of $y = u(\underline{x}) = \sum_{i=1}^n x_i$.

As a result, the family of distributions $\{L(\theta, \underline{x}) : \theta \in (0,1)\}$ has monotone likelihood ratio with respect to $Y = \sum_{i=1}^n X_i$.

Consider the testing problem

$$H_0: \theta \leq \theta_0$$

against

$$H_1: \theta > \theta_0.$$

The test $\Delta_C$, where $C = \{\underline{x} : \sum_{i=1}^n x_i \geq k\}$, is the UMP test in the problem $(H_0, H_1)$ at the significance level $\alpha$ if $P_{\theta_0}(\sum X_i \geq k) = \alpha$.

Let $X_1, \dots, X_n$ be the sample from $f(x,\theta), \theta \in \Theta$,

where $f(x,\theta) = \exp[p(\theta)K(x) + S(x) + q(\theta)]$, $x \in \mathcal{S}$, while

$\mathcal{S}$ is the support of $X_1$ independent from $\theta$.

Assume that $p(\theta)$ is an increasing function of $\theta$.

Then, for $\theta_1 > \theta_0$, the ratio

$$\frac{L(\theta_1)}{L(\theta_0)} = \exp\left\{[p(\theta_1) - p(\theta_0)]\sum_{i=1}^{n} K(x_i) + n[q(\theta_1) - q(\theta_0)]\right\}$$

is a nondecreasing function of $Y = \sum_{i=1}^{n} K(X_i)$.

Thus, in the testing problem

$H_0: \theta \leq \theta_0$,
$H_1: \theta > \theta_0$,

there exists the UMP test of the form $\mathbb{1}_C$,

where $C = \{\underline{x} : \sum_{i=1}^{n} K(x_i) \geq k\}$ while $k$ satisfies

the condition $\alpha = P_{\theta_0}(\underline{X} \in C)$

## 4. Likelihood Ratio Tests

Let $X_1, \dots, X_n$ be a sample with $f(x,\theta), \theta \in \Theta$.

We test

$H_0: \theta = \theta_0$ against $H_1: \theta \neq \theta_0$,

where $\theta_0$ is a fixed constant.

Let $\hat{\theta}$ be MLE of the parameter $\theta$.

**Def 1**

The test with a critical region $C = \{\Lambda \leq c\}$, where

$$\Lambda = \frac{L(\theta_0)}{L(\hat{\theta})} = \frac{\sup_{\theta \in \Theta_0} L(\theta)}{\sup_{\theta \in \Theta} L(\theta)}$$

is called the likelihood ratio test in the problem $(H_0, H_1)$. If $P_{\theta_0}(\Lambda \leq c) = \alpha$, the test has a size $\alpha$. 

<u>Example 1</u>

$X_1, \dots, X_n$ i.i.d $X_i \sim f(x, \theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}$, $x, \theta > 0$.

Then

$$L(\theta) = \theta^{-n} \exp\{-\frac{n}{\theta} \bar{X}\}.$$

Furthermore, $\hat{\theta} = \bar{X}$ is the MLE of $\theta$. We have

$$\Lambda = \frac{L(\theta_0)}{L(\hat{\theta})} = \frac{\theta_0^{-n} \exp\{-\frac{n}{\theta_0} \hat{\theta}\}}{\hat{\theta}^{-n} \exp\{-\frac{n}{\hat{\theta}} \hat{\theta}\}} = \left(\frac{\hat{\theta}}{\theta_0}\right)^n \exp\{-\frac{n}{\theta_0} \hat{\theta} + n\}.$$

A critical region has the form

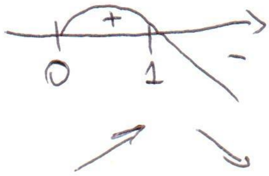
$$\Lambda \leq c_\alpha,$$

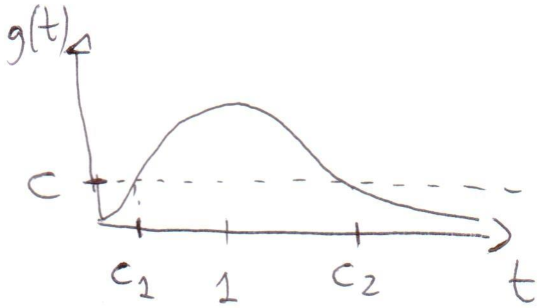
where $\Lambda = g(t) = t^n e^{-nt}$, while $t = \frac{\bar{X}}{\theta_0}$.

Moreover, $g'(t) = n t^{n-1} e^{-nt} + t^n e^{-nt}(-n) = n e^{-nt}(t^{n-1} - t^n) = 0$

$n e^{-t} \cdot t^{n-1}(1-t)$

$\iff t = 1, \ t = 0$




Thereby,

$\Lambda \leq c \iff g(t) \leq c \iff t \leq c_1$ or $t \geq c_2$

So,

$$\frac{\bar{X}}{\theta_0} \leq c_1 \text{ or } \frac{\bar{X}}{\theta_0} \geq c_2.$$

Under $H_0$, the statistic $\frac{2}{\theta_0} \sum_{i=1}^{n} X_i \sim \chi^2(2n)$. As a result, the critical region of the $\alpha$-size LRT has the form

$$C = \{\underline{x} : \frac{2}{\theta_0} \sum_{i=1}^{n} x_i \leq q_{\chi^2_{2n}}(\tfrac{\alpha}{2})\} \cup \{\underline{x} : \frac{2}{\theta_0} \sum_{i=1}^{n} x_i \geq q_{\chi^2_{2n}}(1-\tfrac{\alpha}{2})\},$$

where $q_{\chi^2_{2n}}(\alpha)$ is the $\alpha$-quantile of the chi-square distr. with 2n d.of.f.