Definition 6

An error relying on rejection of a true null hypothesis H_0 is called the Type I error (the error of the first kind).

An error relying on acceptance of a false null hypothesis H_0 is called the Type II error (the error of the second kind).

Illustration

		Decision	
		H_0	H_1
Truth	H_0	X	Type I error
	H_1	Type II error	X

Definition 7

Let C be a critical region. The measurable function of the form $\mathbb{1}_C(\underline{x})$ is called a (non-randomized) test of the hypothesis H_0 against the alternative H_1 and is denoted by $\varphi(\underline{x})$ or φ, for short.

Definition 8

A number $\alpha \in (0,1)$ is called the significance level.

Remark 2

Usually, $\alpha = 0.01$, $\alpha = 0.05$ $\alpha = 0.1$.

Definition 9

Let $\alpha \in (0,1)$. It is said ~~(called)~~ that the test φ is at the significance level α, if (and only if)

$$\sup_{\theta \in \Theta_0} \mathbb{E}_\theta[\varphi(\underline{X})] = \sup_{\theta \in \Theta_0} \mathbb{P}_\theta(\underline{X} \in C) \leq \alpha .$$

If

$$\sup_{\theta \in \Theta_0} \mathbb{E}_\theta[\varphi(\underline{X})] = \alpha ,$$

it is said that the test φ has the size α.

Definition 10

The function $\gamma : \Theta \to [0,1]$ defined as follows:

$\gamma(\theta) = P_\theta(\underline{X} \in C) = \mathbb{E}_\theta[\varphi(\underline{X})]$ for $\theta \in \Theta$

is called ~~the~~ power function of the test φ.

The number $\gamma(\theta)$ for $\theta \in \Theta_1$ is called the power of the test φ under the alternative θ.

Remark 3

Statistical tests are constructed in such a manner in order to minimize the probability of making the Type II error under ~~give~~ fixed probability of making the Type I error equals α.

2. Neyman - Pearson Lemma

Definition 1

It is said that the test φ_0 is ~~the~~ uniformly most powerful at the significance level α, if for any another test φ at the same significance level

$$\mathbb{E}_\theta[\varphi(\underline{X})] \leq \mathbb{E}_\theta[\varphi_0(\underline{X})] \quad \text{for } \cancel{\text{each}} \text{ any } \theta \in \Theta_1.$$

Theorem 1 (Neyman Pearson Lemma)

Let $X_1, \dots, X_n$ be a sample with $f(x, \theta)$.
Consider the testing problem

$$H_0: \theta \in \Theta_0,$$
$$H_1: \theta = \theta_1,$$

and the α-size φ_0 test of the form

$$\varphi_0(\underline{x}) = \begin{cases} 1, & \text{if } \prod_{i=1}^n f(x_i, \theta_0) < k \prod_{i=1}^n f(x_i, \theta_1), \\ \gamma, & \text{if } \qquad\qquad = \qquad\qquad, \\ 0, & \text{if } \qquad\qquad > \qquad\qquad, \end{cases}$$

where the constants k and γ ~~are~~ satisfy the condition $\mathbb{E}_{\theta_0}[\varphi(\underline{X})] = \alpha$. Then, φ_0 is the UMP test in the problem (H_0, H_1).

Corollary 1

Under the conditions of Theorem 1, $\gamma_{\varphi_0}(\theta_1) \geq \alpha$.

Example 1

$X_1, \dots, X_n$ i.i.d $X_i \sim f(x, \theta) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(x-\theta)^2}{2}\right\}$, $x \in \mathbb{R}$.

We verify

$$H_0: \theta = 0,$$
$$H_1: \theta = 1.$$

We have

$$\frac{L(1, \underline{x})}{L(0, \underline{x})} = \frac{\left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left(-\frac{\sum_{i=1}^n (x_i - 1)^2}{2}\right)}{\left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left(-\frac{\sum_{i=1}^n x_i^2}{2}\right)} = \exp\left(\sum_{i=1}^n x_i - \frac{n}{2}\right) > k_1.$$

Thus

$$\sum_{i=1}^n x_i - \frac{n}{2} > \log k_1 \iff \sum_{i=1}^n x_i > \log k_1 + \frac{n}{2} = k.$$

Thereby, the critical region of the UMP test has the form

$$C = \left\{\underline{x} : \sum_{i=1}^n x_i > k\right\},$$

while the constant k satifies the condition $P_0(\underline{X} \in C) = \alpha$.

So, $P_0\left(\sum_{i=1}^n X_i > k\right) = \alpha$. Since, under H_0, $\sum_{i=1}^n X_i \sim N(0, n)$, we have $P_0\left(\sum_{i=1}^n X_i > k\right) = P_0\left(\frac{\sum_{i=1}^n X_i}{\sqrt{n}} > \frac{k}{\sqrt{n}}\right) =$

$1 - \Phi\left(\frac{k}{\sqrt{n}}\right) = \alpha$. As a result, $\frac{k}{\sqrt{n}} = \Phi^{-1}(1-\alpha)$ and $k = \sqrt{n}\,\Phi^{-1}(1-\alpha)$.

On the other hand, under H_1, $\sum_{i=1}^n X_i \sim N(n, n)$, and

$$\gamma(\theta_1) = \gamma(1) = P_1\left(\sum_{i=1}^n X_i > k\right) = P_1\left(\frac{\sum_{i=1}^n X_i - n}{\sqrt{n}} > \frac{k-n}{\sqrt{n}}\right) =$$

$$1 - \Phi\left(\frac{k-n}{\sqrt{n}}\right) = 1 - \Phi\left(\Phi^{-1}(1-\alpha) - \sqrt{n}\right).$$

3. The UMP for a composite ~~hyp~~ alternative

Example 1

$X_1, \dots, X_n$ i.i.d , $X_1 \sim N(0, \theta)$, $\theta > 0$.

We test

$$H_0: \theta = \theta_0,$$
$$A: \theta > \theta_0.$$

We find the UMP α-level test.

We have

$$L(\theta) = L(\theta, \underline{x}) = \left(\frac{1}{2\pi\theta}\right)^{n/2} \exp\left\{-\frac{1}{2\theta}\sum_{i=1}^{n} x_i^2\right\}.$$

Let $\theta_1 > \theta_0$, and $k_1 > 0$. Then,

$$\frac{L(\theta_1)}{L(\theta_0)} \geq k_1 \Leftrightarrow \left(\frac{\theta_0}{\theta_1}\right)^n \exp\left\{-\frac{\theta_1 - \theta_0}{2\theta_0\theta_1}\sum_{i=1}^{n} x_i^2\right\} \geq k_1 \Leftrightarrow$$

$$\sum_{i=1}^{n} x_i^2 \geq k$$

The critical region has a form

$$C = \left\{(x_1, \dots, x_n) : \sum_{i=1}^{n} x_i^2 \geq k\right\}$$

and correspods the the UMP test in the problem

$$H_0: \theta = \theta_0,$$
$$H_1: \theta = \theta_1,$$

where the constant k satisfies the condition $P_0\left(\sum_{i=1}^{n} X_i^2 \geq k\right) = \alpha$.

Since $\sum_{i=1}^{n} X_i^2 / \theta \sim \chi_n^2$, we have

$$P_0\left(\sum_{i=1}^{n} X_i^2 \geq k\right) = P_0\left(\sum_{i=1}^{n} X_i^2 / \theta_0 \geq \frac{k}{\theta_0}\right) = 1 - F_{\chi^2(n)}\left(\frac{k}{\theta_0}\right).$$

As a result,
$$\frac{k}{\theta_0} = q_{\chi^2(n)}(0.95)$$