Coherence centroids

March 31, 2023

1 Problem statement

We have N signals $x_i(t)$. We wish to find a centroid signal $\mu(t)$ such that, in some sense, the sum of the coherences $\sum_i \mathrm{coh}(x_i, \mu)$ is maximal.

Coherence is a property of a joint distribution over signals, not of specific signals. Thus, we intepret coh() as the windowed (i.e. Welch's method) magnitude-squared coherence estimator for stationary signals. That is, we form K windowed or tapered subsequences from each $x_i(t)$ and $\mu(t)$, find their Fourier transforms $X_{ik}(\omega)$ and $M_k(\omega)$, and define

$$\mathrm{coh}(x_i, \mu)(\omega) = \frac{|\sum_k X^*_{ik}(\omega) M_k(\omega)|^2}{(\sum_k X^*_{ik}(\omega) X_{ik}(\omega))(\sum_k M^*_k(\omega) M_k(\omega))}$$

This quantity is invariant to an arbitrary real scaling of $M_k(\omega)$, so we add the constraint that $(\sum_k M^*_k(\omega) M_k(\omega)) = 1$.

2 Result

Dropping the explicit ω indexing, and writing $\sigma_i^{-1} = \sum_k X^*_{ik} X_{ik}$ the objective is

$$\begin{aligned}
\mathcal{L} &= \sum_i \sigma_i \Big| \sum_k X^*_{ik} M_k \Big|^2 \\
&= \sum_i \sigma_i \sum_{kk'} X^*_{ik'} M_{k'} X_{ik} M^*_k \\
&= \sum_{kk'} M^*_k \underbrace{\left(\sum_i \sigma_i X_{ik} X^*_{ik'} \right)}_{P_{kk'}} M_{k'}
\end{aligned}$$

Note that $P_{kk'} = P^*_{k'k}$, so the elements form a Hermitian matrix, $\mathsf{P}(\omega)$. Writing $\mathbf{m}(\omega) = [M_k(\omega)]$ for the vector of window Fourier components at frequency ω in μ we can write the objective as

$$\operatorname{argmax} \mathbf{m}(\omega)^\dagger \mathsf{P}(\omega) \mathbf{m}(\omega) \text{ subject to } \mathbf{m}(\omega)^\dagger \mathbf{m}(\omega) = 1$$

from which it follows that $\mathbf{m}(\omega)$ is the leading eigenvector of $\mathsf{P}(\omega)$.

One possible problem with this approach is that if the windows that define the coherence estimator overlap (as they should) then it isn't immediately clear that the window Fourier transforms found this way are consistent (although they might be, as the X_{ik} are).

One way to avoid this issue is to write $M_k(\omega) = \mathbf{f}_k(\omega)^\dagger \boldsymbol{\mu}$, where $\mathbf{f}_k(\omega)$ is a (complex) signal that computes the ω component of the DFT in the kth window (basically the plane wave $e^{i\omega t}$ with the kth window applied).

Then the cost function is

$$\begin{aligned}
\mathcal{L}(\omega) &= \sum_{kk'} \boldsymbol{\mu}^\mathsf{T} \mathbf{f}_k(\omega) P_{kk'}(\omega) \mathbf{f}_{k'}(\omega)^\dagger \boldsymbol{\mu} \\
&= \boldsymbol{\mu}^\mathsf{T} \left(\sum_{kk'} \mathbf{f}_k(\omega) P_{kk'}(\omega) \mathbf{f}_{k'}(\omega)^\dagger \right) \boldsymbol{\mu}
\end{aligned}$$

and the optimal $\boldsymbol{\mu}$ is the leading eigenvector of a $T \times T$ real matrix obtained by summing the term in parentheses over ω. (Note that each ω constrains μ within a rank-K orthogonal subspace.)