Institute of Systems Science and Intelligent Control Technology 系统科学与智能控制技术研究所

鲁棒控制：建模、跟踪、抗扰、容错

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提纲

1. 古典控制基础
2. 鲁棒控制理论基础
3. 鲁棒控制在迟滞系统中应用
4. 高精度跟踪与抗扰控制
5. 故障诊断与容错控制
6. 教材2-16章

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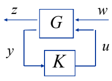
LQR, LQG, H_2, H_∞ 关系

- **LFT system**

$$G(s)=\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right].$$

$$\dot{x}=Ax+B_1w+B_2u$$
$$z=C_1x+D_{12}u$$
$$y=C_2x+D_{21}w$$



- **LQR问题**

$$\dot{x}=Ax+B_2u,\ J=\int_0^\infty (x'Qx+u'Ru)dt=\int_0^\infty \|z\|^2\,dt,\ Q=Q\ge 0,\ R=R'>0$$

$$x'Qx+u'Ru=\left\|Q^{\frac{1}{2}}x\right\|^2+\left\|R^{\frac{1}{2}}u\right\|^2=\left\|\begin{bmatrix} Q^{\frac{1}{2}}x \\ R^{\frac{1}{2}}u \end{bmatrix}\right\|^2=\left\|\begin{bmatrix} Q^{\frac{1}{2}} \\ 0 \end{bmatrix}x+\begin{bmatrix} 0 \\ R^{\frac{1}{2}} \end{bmatrix}u\right\|^2$$

定义 $z=\begin{bmatrix} Q^{\frac{1}{2}} \\ 0 \end{bmatrix}x+\begin{bmatrix} 0 \\ R^{\frac{1}{2}} \end{bmatrix}u,\ C_1=\begin{bmatrix} Q^{\frac{1}{2}} \\ 0 \end{bmatrix},\ D_{12}=\begin{bmatrix} 0 \\ R^{\frac{1}{2}} \end{bmatrix},\ D_{12}'C_1=0$

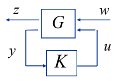
LQR, LQG, H_2, H_∞ 关系

- **滤波问题**

$$\dot{x}=Ax+v_1+B_2u$$
$$y=C_2x+v_2$$
$$E\{v_1v_1'\}=V_1,\ E\{v_2v_2'\}=V_2,\ E\{v_1v_2'\}=0$$

定义w: $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}=\begin{bmatrix} V_1^{1/2} & 0 \\ 0 & V_2^{1/2} \end{bmatrix}w,\ E\{ww'\}=I$

$$\dot{x}=Ax+\begin{bmatrix} V_1^{1/2} & 0 \end{bmatrix}w+B_2u=Ax+B_1w+B_2u,\ B_1=\begin{bmatrix} V_1^{1/2} & 0 \end{bmatrix}$$
$$y=C_2x+\begin{bmatrix} 0 & V_2^{1/2} \end{bmatrix}w=C_2x+D_{21}w,\ D_{21}=\begin{bmatrix} 0 & V_2^{1/2} \end{bmatrix}$$



- **一般LQR问题**

$$J=\int_0^\infty (x'Qx+2x'Su+u'Ru)dt=\int_0^\infty \begin{bmatrix} x \\ u \end{bmatrix}'\begin{bmatrix} Q & S \\ S' & R \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix}dt=\int_0^\infty \|z\|^2\,dt$$

$$\begin{bmatrix} Q & S \\ S' & R \end{bmatrix}=\begin{bmatrix} C_1 & D_{12} \end{bmatrix}'\begin{bmatrix} C_1 & D_{12} \end{bmatrix},\quad z=C_1x+D_{12}u$$

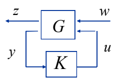
Chapter 15: Controller Reduction

- **Problem Formulation**
- **Additive Reduction**
- **Coprime Factor Reduction**

Problem Formulation

- **Consider a general LFT system**

$$G(s)=\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right].$$



All stabilizing controllers satisfying $\|T_{zw}\|_\infty<\gamma$: $K=F_l(M_\infty,Q)$, $Q\in RH_\infty$, $\|Q\|_\infty<\gamma$ where M_∞ is of the form

$$M_\infty=\begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{bmatrix}=\left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & \hat{D}_{22} \end{array}\right]$$

Such that $\hat{A}-\hat{B}_2\hat{D}_{12}^{-1}\hat{C}_1$ and $\hat{A}-\hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2$ are both stable, i.e., M_{12}^{-1} and M_{21}^{-1} are both stable.

- **Find a controller K with a minimal order such that $\|F_l(G,K)\|_\infty<\gamma$. $\Leftrightarrow$**
- **Find a stable Q such that $K=F_l(M_\infty,Q)$ has minimal order and $\|Q\|_\infty<\gamma$.**

Additive Reduction

Consider the class of (reduced order) controllers:

$$\hat{K} = K_0 + W_2 \Delta W_1, \quad \Delta \in RH_\infty, \; W_1,\, W_1^{-1},\, W_2,\, W_2^{-1} \in RH_\infty$$

such that $\|F_\ell(G, K_0)\|_\infty < \gamma$ where $\hat{K}$ and K_0 have the same right half plane poles.

Then $\|F_\ell(G,\hat{K})\|_\infty < \gamma \;\Leftrightarrow\; \exists Q \in RH_\infty$, with $\|Q\|_\infty < \gamma$ such that $\hat{K} = F_\ell(M_\infty, Q)$

$$\Downarrow$$

$$Q = F_\ell(\bar{K}_a^{-1}, \hat{K}), \qquad \bar{K}_a^{-1} := \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} M_\infty^{-1} \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

$$\|Q\|_\infty < \gamma \Leftrightarrow \left\|F_\ell(\bar{K}_a^{-1}, \hat{K})\right\|_\infty < \gamma \quad \Leftrightarrow \left\|F_\ell(\bar{K}_a^{-1}, K_0 + W_2 \Delta W_1)\right\|_\infty < \gamma$$

$$\Leftrightarrow \left\|F_\ell(\tilde{R}, \Delta)\right\|_\infty < 1$$

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where
$$\tilde{R} = \begin{bmatrix} \gamma^{-1/2} I & 0 \\ 0 & W_1 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} \gamma^{-1/2} I & 0 \\ 0 & W_2 \end{bmatrix}$$

$$\begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = S(\bar{K}_a^{-1}, \begin{bmatrix} K_0 & I \\ I & 0 \end{bmatrix}).$$

Redheffer's Lemma: $\|\tilde{R}\|_\infty \le 1$ and $\|\Delta\|_\infty < 1 \Rightarrow \|F_\ell(\tilde{R}, \Delta)\|_\infty < 1$

Theorem 15.2: Suppose W_1 and W_2 are stable, minimum phase and invertible transfer matrices such that $\tilde{R}$ is a contraction. Let K_0 be a stabilizing controller such that $\|F_\ell(G, K_0)\|_\infty < \gamma$. Then $\hat{K}$ is also a stabilizing controller such that $\|F_\ell(G, \hat{K})\|_\infty < \gamma$ if

$$\|\Delta\|_\infty = \left\|W_2^{-1}(\hat{K} - K_0) W_1^{-1}\right\|_\infty < 1.$$

$\tilde{R}$ can always be made contractive for sufficiently small W_1 and W_2. We would like to select the 'largest' W_1 and W_2.

Assume $\|R_{22}\|_\infty < \gamma$ and define

$$L = \begin{bmatrix} L_1 & L_2 \\ L_2^\sim & L_3 \end{bmatrix} = F_\ell(\left[\begin{array}{cc:cc} 0 & -R_{11} & 0 & R_{11} \\ -R_{11}^\sim & 0 & R_{21}^\sim & 0 \\ \hdashline 0 & R_{21} & 0 & -R_{22} \\ R_{12}^\sim & 0 & -R_{22}^\sim & 0 \end{array}\right], \gamma^{-1} I).$$

Then $\tilde{R}$ is a contraction if W_1 and W_2 satisfy

$$\begin{bmatrix} (W_1^\sim W_1)^{-1} & 0 \\ 0 & (W_2 W_2^\sim)^{-1} \end{bmatrix} \ge \begin{bmatrix} L_1 & L_2 \\ L_2^\sim & L_3 \end{bmatrix}$$

An algorithm that maximizes $\det(W_1^\sim W_1)\det(W_2 W_2^\sim)$ has been developed by Goddard and Glover[1993].

Coprime Factor Reduction

All controllers such that $\|T_{zw}\|_\infty < \gamma$ can also be written as

$$K = F_\ell(M_\infty, Q), = (\Theta_{11} Q + \Theta_{12})(\Theta_{21} Q + \Theta_{22})^{-1} := UV^{-1}$$
$$= (Q\tilde{\Theta}_{12} + \tilde{\Theta}_{22})^{-1}(Q\tilde{\Theta}_{11} + \tilde{\Theta}_{21}) := \tilde{V}^{-1}\tilde{U}$$

where $Q \in RH_\infty$, $\|Q\|_\infty < \gamma$, and UV^{-1} and $\tilde{V}^{-1}\tilde{U}$ are respectively right and left coprime factorizations over RH_∞, and

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} = \left[\begin{array}{c|cc} \hat{A} - \hat{B}_1 \hat{D}_{21}^{-1} \hat{C}_2 & \hat{B}_2 - \hat{B}_1 \hat{D}_{21}^{-1} \hat{D}_{22} & \hat{B}_1 \hat{D}_{21}^{-1} \\ \hline \hat{C}_1 - \hat{D}_{11} \hat{D}_{21}^{-1} \hat{C}_2 & \hat{D}_{12} - \hat{D}_{11} \hat{D}_{21}^{-1} \hat{D}_{22} & \hat{D}_{11} \hat{D}_{21}^{-1} \\ -\hat{D}_{21}^{-1} \hat{C}_2 & -\hat{D}_{21}^{-1} \hat{D}_{22} & \hat{D}_{21}^{-1} \end{array}\right]$$

$$\tilde{\Theta} = \begin{bmatrix} \tilde{\Theta}_{11} & \tilde{\Theta}_{12} \\ \tilde{\Theta}_{21} & \tilde{\Theta}_{22} \end{bmatrix} = \left[\begin{array}{c|cc} \hat{A} - \hat{B}_2 \hat{D}_{12}^{-1} \hat{C}_1 & \hat{B}_1 - \hat{B}_2 \hat{D}_{12}^{-1} \hat{D}_{11} & \hat{B}_2 \hat{D}_{12}^{-1} \\ \hline \hat{C}_2 - \hat{D}_{22} \hat{D}_{12}^{-1} \hat{C}_1 & \hat{D}_{21} - \hat{D}_{22} \hat{D}_{12}^{-1} \hat{D}_{11} & \hat{D}_{22} \hat{D}_{12}^{-1} \\ -\hat{D}_{12}^{-1} \hat{C}_1 & -\hat{D}_{12}^{-1} \hat{D}_{11} & \hat{D}_{12}^{-1} \end{array}\right]$$

$$\Theta^{-1} = \left[\begin{array}{c|cc} \hat{A} - \hat{B}_2 \hat{D}_{12}^{-1} \hat{C}_1 & \hat{B}_2 \hat{D}_{12}^{-1} & \hat{B}_1 - \hat{B}_2 \hat{D}_{12}^{-1} \hat{D}_{11} \\ \hline -\hat{D}_{12}^{-1} \hat{C}_1 & \hat{D}_{12}^{-1} & -\hat{D}_{12}^{-1} \hat{D}_{11} \\ \hat{C}_2 - \hat{D}_{22} \hat{D}_{12}^{-1} \hat{C}_1 & \hat{D}_{22} \hat{D}_{12}^{-1} & \hat{D}_{21} - \hat{D}_{22} \hat{D}_{12}^{-1} \hat{D}_{11} \end{array}\right]$$

$$\tilde{\Theta}^{-1} = \left[\begin{array}{c|cc} \hat{A} - \hat{B}_1 \hat{D}_{21}^{-1} \hat{C}_2 & \hat{B}_1 \hat{D}_{21}^{-1} & \hat{B}_2 - \hat{B}_1 \hat{D}_{21}^{-1} \hat{D}_{22} \\ \hline -\hat{D}_{21}^{-1} \hat{C}_2 & \hat{D}_{21}^{-1} & -\hat{D}_{21}^{-1} \hat{D}_{22} \\ \hat{C}_1 - \hat{D}_{11} \hat{D}_{21}^{-1} \hat{C}_2 & \hat{D}_{11} \hat{D}_{21}^{-1} & \hat{D}_{12} - \hat{D}_{11} \hat{D}_{21}^{-1} \hat{D}_{22} \end{array}\right].$$

Theorem 15.5: Let $K_0 = \Theta_{12}\Theta_{22}^{-1}$ be the central H_∞ controller:

$$\|F_\ell(G, K_0)\|_\infty < \gamma$$

and let $\hat{U}, \hat{V} \in RH_\infty$ with $\det \hat{V}(\infty) \ne 0$ be such that

$$\left\| \begin{bmatrix} \gamma^{-1/2} I & 0 \\ 0 & I \end{bmatrix} \Theta^{-1} \left(\begin{bmatrix} \Theta_{12} \\ \Theta_{22} \end{bmatrix} - \begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} \right) \right\|_\infty < 1/\sqrt{2}$$

Then $\hat{K} = \hat{U}\hat{V}^{-1}$ is also a stabilizing controller and $\|F_\ell(G, \hat{K})\|_\infty < \gamma$

Proof: Note that K is a stabilizing controller such that $\|T_{zw}\|_\infty < \gamma$ if and only if there exists a $Q \in RH_\infty$ with $\|Q\|_\infty < \gamma$ such that

$$\begin{bmatrix} U \\ V \end{bmatrix} := \begin{bmatrix} \Theta_{11} Q + \Theta_{12} \\ \Theta_{21} Q + \Theta_{22} \end{bmatrix} = \Theta \begin{bmatrix} Q \\ I \end{bmatrix}$$

and $K = UV^{-1}$.

Define $\Delta := \begin{bmatrix} \gamma^{-1/2} I & 0 \\ 0 & I \end{bmatrix} \Theta^{-1} \left(\begin{bmatrix} \Theta_{12} \\ \Theta_{22} \end{bmatrix} - \begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} \right)$

and partition Δ as $\Delta := \begin{bmatrix} \Delta_U \\ \Delta_V \end{bmatrix}$.

Then $\begin{bmatrix} \hat{U} \\ \hat{V} \end{bmatrix} = \begin{bmatrix} \Theta_{12} \\ \Theta_{22} \end{bmatrix} - \Theta \begin{bmatrix} \gamma I & 0 \\ 0 & I \end{bmatrix} \Delta = \Theta \begin{bmatrix} -\gamma \Delta_U \\ I - \Delta_V \end{bmatrix}$

and

$$\begin{bmatrix} \hat{U}(I - \Delta_V)^{-1} \\ \hat{V}(I - \Delta_V)^{-1} \end{bmatrix} = \Theta \begin{bmatrix} -\gamma \Delta_U (I - \Delta_V)^{-1} \\ I \end{bmatrix}.$$

Define

$$U := \hat{U}(I-\Delta_V)^{-1}, \quad V := \hat{V}(I-\Delta_V)^{-1}, \quad Q := -\gamma\Delta_U(I-\Delta_V)^{-1}$$

Then $\hat{K} = \hat{U}\hat{V}^{-1} = UV^{-1}$ and

$$Q := -\gamma\Delta_U(I-\Delta_V)^{-1} = -\begin{bmatrix} I & 0 \end{bmatrix}\Delta\left(I-\begin{bmatrix} 0 & I \end{bmatrix}\Delta\right)^{-1}$$

$$= -\gamma F_\ell\left(\begin{bmatrix} 0 & \begin{bmatrix} I & 0 \end{bmatrix} \\ I/\sqrt{2} & \begin{bmatrix} 0 & I/\sqrt{2} \end{bmatrix} \end{bmatrix}, \sqrt{2}\Delta\right)$$

Again by Redheffer's Lemma, $\left\|\Delta_U(I-\Delta_V)^{-1}\right\|_\infty < 1$ since

$$\begin{bmatrix} 0 & \begin{bmatrix} I & 0 \end{bmatrix} \\ I/\sqrt{2} & \begin{bmatrix} 0 & I/\sqrt{2} \end{bmatrix} \end{bmatrix}$$

is a contraction and $\left\|\sqrt{2}\Delta\right\|_\infty < 1$.

$$\Rightarrow \|Q\|_\infty = \left\|\gamma\Delta_U(I-\Delta_V)^{-1}\right\|_\infty < \gamma$$

Therefore $\left\|F_\ell(G,\hat{K})\right\|_\infty < \gamma$.

Similarly, we have

Theorem 15.6: Let $K_0 = \tilde{\Theta}_{22}^{-1}\tilde{\Theta}_{21}$ be the central H_∞ controller:

$$\|F_\ell(G,K_0)\|_\infty < \gamma$$

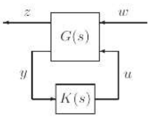
and let $\hat{\tilde{U}}, \hat{\tilde{V}} \in RH_\infty$ with $\det\hat{\tilde{V}}(\infty) \neq 0$ be such that

$$\left\|\left(\begin{bmatrix} \tilde{\Theta}_{21} & \tilde{\Theta}_{22} \end{bmatrix} - \begin{bmatrix} \hat{\tilde{U}} & \hat{\tilde{V}} \end{bmatrix}\right)\tilde{\Theta}^{-1}\begin{bmatrix} \gamma^{-1}I & 0 \\ 0 & I \end{bmatrix}\right\|_\infty < 1/\sqrt{2}.$$

Then $\hat{K} = \hat{\tilde{V}}^{-1}\hat{\tilde{U}}$ is also a stabilizing controller and $\left\|F_\ell(G,\hat{K})\right\|_\infty < \gamma$

Conclusion: H_∞ controller reduction $\Rightarrow$
frequency weighted H_∞ model reduction.

H_∞控制一般形式



$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

Find K(s) such that $\|T_{zw}\|_\infty < \gamma$

Assumptions:

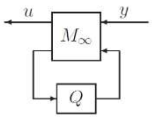
(A1) (A,B_2) stabilizable and (C_2,A) detectable;

(A2) $D_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix}$ and $D_{21} = \begin{bmatrix} 0 & I \end{bmatrix}$;

(A3) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ full column rank for all ω (G_{12} zeros)

(A4) $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ full row rank for all ω (G_{21} zeros).

H_∞控制器集



K(s) such that $\|T_{zw}\|_\infty < \gamma$

$$X_\infty A + A^* X_\infty + X_\infty (B_1 B_1^*/\gamma^2 - B_2 B_2^*) X_\infty + C_1^* C_1 = 0$$

$$A Y_\infty + Y_\infty A^* + Y_\infty (C_1^* C_1/\gamma^2 - C_2^* C_2) Y_\infty + B_1 B_1^* = 0$$

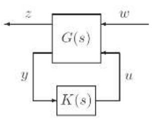
$$\rho(X_\infty Y_\infty) < \gamma^2$$

$$K = \mathcal{F}_\ell(M_\infty, Q), \quad Q \in \mathscr{RH}_\infty, \quad \|Q\|_\infty < \gamma$$

$$M_\infty = \begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & \hat{D}_{22} \end{array}\right]$$

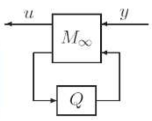
$\hat{A} - \hat{B}_2\hat{D}_{12}^{-1}\hat{C}_1$ *and* $\hat{A} - \hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2$ *are both stable, i.e.,* M_{12}^{-1} *and* M_{21}^{-1} *are both stable..*

H_∞控制器降阶



找低阶K(s) 满足 $\|T_{zw}\|_\infty < \gamma$

$$K = \mathcal{F}_\ell(M_\infty, Q), \quad Q \in \mathscr{RH}_\infty, \quad \|Q\|_\infty < \gamma$$



$$M_\infty = \begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & \hat{D}_{22} \end{array}\right]$$

- 控制器阶数可能非常高，可能不好用

$K_r = M_{11} + M_{12}Q(I - M_{22}Q)^{-1}M_{21}$

H_∞控制器参数化方法降阶

- 假设 K_r 是一个降价控制器满足 H_∞ 性能. 那么
  $K_r = F_l(M_\infty, Q)$ for some $Q \in RH_\infty$, $\|Q\|_\infty < \gamma$
  i.e., $K_r = M_{11} + M_{12}Q(I - M_{22}Q)^{-1}M_{21}$
- Note that $K_0 = M_{11}$. 定义 $\Delta := M_{12}^{-1}(K_r - K_0)M_{21}^{-1}$
- 那么解出 Q 得到 $Q = (I + \Delta M_{22})^{-1}\Delta$
  如果$\|\Delta\|_\infty < \gamma/(1 + \gamma\|M_{22}\|_\infty)$, 则 $\|Q\|_\infty < \gamma$.

Theorem: 命 K_r 为一个降价控制器并满足

$$\|M_{12}^{-1}(K_r - K_0)M_{21}^{-1}\|_\infty < \gamma/(1 + \gamma\|M_{22}\|_\infty)$$

则 K_r 满足 H_∞ 性能。

Example

four-disk control system studied by Enns [1984]

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= \begin{bmatrix} \sqrt{q_1} H \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ I \end{bmatrix} u \qquad q_1 = 1 \times 10^{-6}, q_2 = 1. \\ y &= C_2 x + \begin{bmatrix} 0 & I \end{bmatrix} w \end{aligned}$$

$$A = \begin{bmatrix} -0.161 & -6.004 & -0.58215 & -9.9835 & -0.40727 & -3.982 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 00 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} \sqrt{q_2} B_2 & 0 \end{bmatrix},$$

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0.55 & 11 & 1.32 & 18 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 0 & 0 & 6.4432 \times 10^{-3} & 2.3196 \times 10^{-3} & 0.071252 & 1 & 0.10455 & 0.99551 \end{bmatrix}$$

$\gamma_{opt} = 1.1272$

$\gamma = 1.2$

Order of $\hat{K}$	7	6	5	4	3	2
PWA	1.196	1.196	1.199	1.197	U	4.99
PWRCF	1.2	1.196	1.207	1.195	2.98	1.674
PWLCF	1.197	1.196	U	1.197	U	U
UWA	U	1.321	U	U	U	U
UWRCF	1.198	1.196	1.199	1.196	U	U
UWLCF	1.985	1.258	27.04	5.059	U	U
SWA	1.327	1.199	2.27	1.47	23.5	U
SWRCF	1.236	1.197	1.251	1.201	13.91	1.415
SWLCF	1.417	1.217	48.04	3.031	U	U
NU1	1.197	1.196	1.199	1.196	U	2.978
NU2	1.197	1.196	1.199	1.196	U	2.978
KZ3	1.197	1.196	U	1.197	U	U
KZ4	U	1.197	1.197	1.197	U	U
YH	1.197	1.196	U	1.197	U	U
YHx	1.197	1.196	1.199	1.196	U	3.112

HIMAT Examples

HIMAT Example $\gamma = 1$

Criterion A: Controller reduction via criterion $\left\| M_{12}^{-1}(\hat{K} - K_0) M_{21}^{-1} \right\|_\infty$;

Criterion B: Controller reduction via criterion $\left\| M_{12}^{-1}(\hat{K} - K_0) M_{21}^{-1} M_{22} \right\|_\infty$;

Criterion C: Controller reduction via criterion $\left\| M_{22} M_{12}^{-1}(\hat{K} - K_0) M_{21}^{-1} \right\|_\infty$;

Criterion D: Controller reduction via criterion $\left\| (\hat{K} - K_0) M_{21}^{-1} M_{22} M_{12}^{-1} \right\|_\infty$;

Criterion E: Controller reduction via criterion $\left\| M_{21}^{-1} M_{22} M_{12}^{-1}(\hat{K} - K_0) \right\|_\infty$.

central control order=30

Order of $\hat{K}$	Criterion A	Criterion B	Criterion C	Criterion D	Criterion E
14	0.9998	0.9998	0.9998	**0.9998**	0.9998
13	**0.9998**	**0.9998**	0.9998	1.0000	**0.9999**
12	1.0503	1.0010	0.9999	1.0191	1.0167
11	1.0579	1.0012	**0.9998**	1.0260	1.0254

HIMAT Examples

- **Remark: Truncating on weighting functions can impact controller reduction results. In addition, low order weighting functions contribute to simplify calculation.**

Order of $\hat{K}$	$\left\| M_{12}^{-1}(\hat{K} - K_0) M_{21}^{-1} \right\|_\infty$ No truncating	$\left\| M_{12}^{-1}(\hat{K} - K_0) M_{21}^{-1} \right\|_\infty$ W_o: 2 W_i: 2
14	0.9998	**0.9998**
13	**0.9998**	1.0015

- **Remark: It is possible to get better results by optimizing constant term and other parameters of the reduced controller. (only optimize B and D here.)**

Order of $\hat{K}$	Criterion A		Criterion B		Criterion C	
	opt	no opt	opt	no opt	opt	no opt
13	0.9957	**0.9998**	0.9971	**0.9998**	0.9987	0.9998
12	0.9965	1.0503	0.9968	1.0010	0.9982	0.9999
11	0.9966	1.0579	0.9964	1.0012	0.9976	**0.9998**
10	0.9960	1.0099	0.9990	1.0005	0.9988	1.0013
9	0.9985	1.0468	**0.9985**	1.0018	0.9969	1.0159
8	**0.9992**	0.9997	1.0157	1.0479	0.9980	1.0150
7	1.0031	1.0238	1.0018	1.0158	0.9985	1.0144
6	1.0052	1.0242	1.0014	1.0163	**0.9988**	1.0138

Chapter 16: H_∞ Loop Shaping

- **Robust stabilization of Coprime factors**
- **Robust Stabilization of Normalized Coprime Factors**
- **H_∞ Loop Shaping Design**
- **Weighted H_∞ Control Interpretation**
- **Optimal Stability Margin**
- **Further Guidelines for Loop Shaping**

Robust Stabilization of Coprime Factors

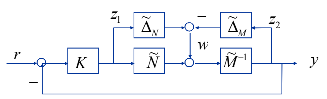
Let $P = \tilde{M}^{-1} \tilde{N}$ be the nominal model and

$$P_\Delta = \left(\tilde{M} + \tilde{\Delta}_M \right)^{-1} \left(\tilde{N} + \tilde{\Delta}_N \right)$$

with $\tilde{M}, \tilde{N}, \tilde{\Delta}_M, \tilde{\Delta}_N \in RH_\infty$ and $\left\| \begin{bmatrix} \tilde{\Delta}_N & \tilde{\Delta}_M \end{bmatrix} \right\|_\infty < \varepsilon$

The perturbed system is robustly stable iff

$$\left\| \begin{bmatrix} K \\ I \end{bmatrix} (I + PK)^{-1} \tilde{M}^{-1} \right\|_\infty \le 1/\varepsilon.$$



State Space Coprime Factorization:

Let $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and let L be such that $A+LC$ is stable. Then

$$P = \tilde{M}^{-1}\tilde{N}, \quad \begin{bmatrix}\tilde{N} & \tilde{M}\end{bmatrix} = \left[\begin{array}{c|cc} A+LC & B+LD & L \\ \hline C & D & I \end{array}\right]$$

A pair of left coprime factorization $(\tilde{M}, \tilde{N})$ is called **normalized left coprime factorization** if

$$\tilde{M}(j\omega)\tilde{M}^*(j\omega) + \tilde{N}(j\omega)\tilde{N}^*(j\omega) = I$$

which can be obtained as

$$\begin{bmatrix}\tilde{N} & \tilde{M}\end{bmatrix} = \left[\begin{array}{c|cc} A - YC^*C & B & -YC^* \\ \hline C & 0 & I \end{array}\right]$$

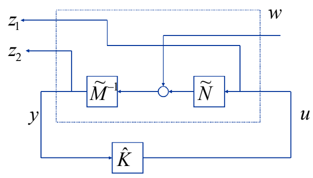
where $L = -YC^*$ and $Y \geq 0$ is the stabilizing solution to

$$AY + YA^* - YC^*CY + BB^* = 0$$

Denote $\hat{K} = -K$. Then the problem can be put in LFT form:

$$\begin{bmatrix} K \\ I \end{bmatrix}(I+PK)^{-1}\tilde{M}^{-1} = F_l(G, \hat{K})$$

$$G(s) = \begin{bmatrix} \begin{bmatrix} 0 \\ \tilde{M}^{-1} \end{bmatrix} & \begin{bmatrix} I \\ P \end{bmatrix} \\ \tilde{M}^{-1} & P \end{bmatrix} = \left[\begin{array}{c|cc} A & -L & B \\ \hline \begin{bmatrix} 0 \\ C \end{bmatrix} & \begin{bmatrix} 0 \\ I \end{bmatrix} & \begin{bmatrix} I \\ D \end{bmatrix} \\ C & I & D \end{array}\right] =: \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right].$$



Controller for a Special Case: $D = 0$.

Apply the standard H_∞ solution formula, we get

Theorem 16.1

$$\left\| \begin{bmatrix} K \\ I \end{bmatrix}(I+PK)^{-1}\tilde{M}^{-1} \right\|_\infty < \gamma$$

iff $\gamma > 1$ and there exists a stabilizing solution $X_\infty \geq 0$ solving

$$X_\infty\left(A - \frac{LC}{\gamma^2-1}\right) + \left(A - \frac{LC}{\gamma^2-1}\right)^* X_\infty - X_\infty\left(BB^* - \frac{LL^*}{\gamma^2-1}\right)X_\infty + \frac{\gamma^2 C^*C}{\gamma^2-1} = 0.$$

Furthermore, a central controller is given by

$$K = \left[\begin{array}{c|c} A - BB^*X_\infty + LC & L \\ \hline -B^*X_\infty & 0 \end{array}\right].$$

Normalized Coprime Factors

Corollary 16.2: Let $D = 0$ and let $P = \tilde{M}^{-1}\tilde{N}$ be normalized coprime factorization. Then

$$\gamma_{\min} := \inf_{K \text{ stabilizing}} \left\| \begin{bmatrix} K \\ I \end{bmatrix}(I+PK)^{-1}\tilde{M}^{-1} \right\|_\infty = \frac{1}{\sqrt{1-\lambda_{\max}(YQ)}}$$

$$\lambda_{\max}(YQ) = \left\| \begin{bmatrix}\tilde{N} & \tilde{M}\end{bmatrix} \right\|_H^2$$

where Y and Q are the solutions to

$$AY + YA^* - YC^*CY + BB^* = 0$$

$$Q(A - YC^*C) + (A - YC^*C)^*Q + C^*C = 0.$$

Moreover, for any $\gamma > \gamma_{\min}$ a controller achieving

$$\left\| \begin{bmatrix} K \\ I \end{bmatrix}(I+PK)^{-1}\tilde{M}^{-1} \right\|_\infty < \gamma$$

is given by

$$K(s) = \left[\begin{array}{c|c} A - BB^*X_\infty - YC^*C & -YC^* \\ \hline -B^*X_\infty & 0 \end{array}\right]$$

where $X_\infty = \frac{\gamma^2}{\gamma^2-1}Q\left(I - \frac{\gamma^2}{\gamma^2-1}YQ\right)^{-1}$.

- Let $P = \tilde{M}^{-1}\tilde{N}$ be a normalized left coprime factorization and

$$P_\Delta = \left(\tilde{M} + \tilde{\Delta}_M\right)^{-1}\left(\tilde{N} + \tilde{\Delta}_N\right)$$

with $\left\| \begin{bmatrix}\tilde{\Delta}_N & \tilde{\Delta}_M\end{bmatrix} \right\|_\infty < \varepsilon$.

Then there is a robustly stabilizing controller for P_Δ if and only if

$$\varepsilon \leq \sqrt{1-\lambda_{\max}(YQ)} = \sqrt{1 - \left\| \begin{bmatrix}\tilde{N} & \tilde{M}\end{bmatrix} \right\|_H^2} \quad (= b_{\text{opt}}(P))$$

- Let $X \geq 0$ be the stabilizing solution to

$$XA + A^*X - XBB^*X + C^*C = 0$$

then $Q = (I + XY)^{-1}X$ and

$$\gamma_{\min} = \frac{1}{\sqrt{1-\lambda_{\max}(YQ)}} = \left(1 - \left\| \begin{bmatrix}\tilde{N} & \tilde{M}\end{bmatrix} \right\|_H^2\right)^{-1/2} = \sqrt{1 + \lambda_{\max}(XY)}$$

- Let $P = \tilde{M}^{-1}\tilde{N}$ be a normalized left coprime factorization. Then

$$\left\| \begin{bmatrix} K \\ I \end{bmatrix}(I+PK)^{-1}\tilde{M}^{-1} \right\|_\infty = \left\| \begin{bmatrix} K \\ I \end{bmatrix}(I+PK)^{-1}\begin{bmatrix} I & P \end{bmatrix} \right\|_\infty$$

- $$\left\| \begin{bmatrix} I \\ K \end{bmatrix}(I+PK)^{-1}\begin{bmatrix} I & P \end{bmatrix} \right\| = \left\| \begin{bmatrix} I \\ P \end{bmatrix}(I+KP)^{-1}\begin{bmatrix} I & K \end{bmatrix} \right\|$$

- Let $P = \tilde{M}^{-1}\tilde{N} = NM^{-1}$ be respectively the normalized left and right coprime factorizations. Then

$$\left\| \begin{bmatrix} K \\ I \end{bmatrix}(I+PK)^{-1}\tilde{M}^{-1} \right\|_\infty = \left\| M^{-1}(I+KP)^{-1}\begin{bmatrix} I & K \end{bmatrix} \right\|_\infty.$$

H_∞ Loop Shaping Design

Given nominal model $P(s)$.

- **(1) Loop Shaping: Obtain a desired open-loop shape (singular values) by using a precompensator W_1 and/or a postcompensator W_2,**

$$P_s=W_2PW_1$$

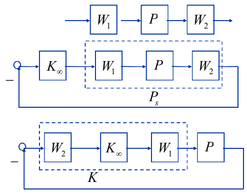
Assume that W_1 and W_2 are such that P_s contains no hidden modes.

- **(2) (a) Calculate robust stability margin $b_{opt}(P_s)$. If $b_{opt}(P_s)$ <<1, return to (1) and adjust W_1 and W_2. (b) Select $\varepsilon \le b_{opt}(P_s)$, then synthesize a stabilizing controller K_∞ which satisfies**

$$\left\| \begin{bmatrix} I \\ K_\infty \end{bmatrix} (I+P_sK_\infty)^{-1}\tilde{M}_s^{-1} \right\|_\infty \le \varepsilon^{-1}.$$

- **(3) The final controller $K=W_1K_\infty W_2$**

- **A typical design works as follows: the designer inspects the open-loop singular values of the nominal plant, and shapes these by pre- and/or postcompensation until nominal performance (and possibly robust stability) specifications are met. (Recall that the open-loop shape is related to closed-loop objectives.) A feedback controller K_∞ with associated stability margin (for the shaped plant) $\varepsilon \le b_{opt}(P_s)$ is then synthesized. If $b_{opt}(P_s)$ is small, then the specified loop shape is incompatible with robust stability requirements, and should be adjusted accordingly, then K_∞ is reevaluated.**



- **Note that the final controller is $K=W_1K_\infty W_2$, so it is necessary to check if the loop properties are significantly changed. It is helpful to choose W_1 and W_2 with small condition numbers.**
- **Only W_1 or W_2 is needed if P is SISO.**