Institute of Systems Science and Intelligent Control Technology 系统科学与智能控制技术研究所

鲁棒控制：建模、跟踪、抗扰、容错

周克敏
山东科技大学
电气与自动化工程学院
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提纲

Chapter 2: Linear Algebra

- linear subspaces
- eigenvalues and eigenvectors
- matrix inversion formulas
- invariant subspaces
- vector norms and matrix norms
- singular value decomposition
- generalized inverses
- semidefinite matrices

Linear Subspaces

- *linear combination*: $\alpha_1 x_1+\alpha_2 x_2+\ldots+\alpha_k x_k$, $x_i\in F^n$, $\alpha_i\in F$
  $\text{span}\{x_1,\ldots,x_k\}:=\{x=\alpha_1 x_1+\ldots+\alpha_k x_k:\ \alpha_i\in F\}$
- $x_1,\ldots,x_k\in F^n$ *linearly dependent* if there exists $\alpha_1,\ldots,\alpha_k\in F$ not all zero such that $\alpha_1 x_1+\ldots+\alpha_k x_k=0$; otherwise they are *linearly independent*.
- $\{x_1,\ldots,x_k\}\in S$ is a *basis* for S if $x_1,\ldots,x_k$ are linearly independent and $S=\text{span}\{x_1,\ldots,x_k\}$.
- $\{x_1,\ldots,x_k\}$ in F^n are mutually orthogonal if $x_i^*x_j=0$ forall $i\neq j$ and *orthonormal* if $x_i^*x_j=\delta_{ij}$
- *orthogonal complement* of a subspace $S\subset F^n$

$$S^\perp:=\{y\in F^n:\ y^*x=0 \text{ for all } x\in S\}$$

- linear transformation $A: F^n\to F^m$.
  kernel or *null space*: $\text{Ker}A=N(A):=\{x\in F^n:\ Ax=0\}$
  image or *range* of A: $\text{Im}A=R(A):=\{y\in F^m:\ y=Ax,\ x\in F^n\}$
- Let a_i, $i=1,2,\ldots,n$ denote the columns of a matrix $A\in F^{m\times n}$. Then
  $\text{Im}\ A=\text{span}\{a_1, a_2,\ldots, a_n\}$.
- The *rank* of a matrix A is defined by $\text{rank}(A)=\dim(\text{Im}A)$.
  $\text{rank}(A)=\text{rank}(A^*)$.
  $A\in F^{m\times n}$ is *full row rank* if $m\le n$ and $\text{rank}(A)=m$.
  A is *full column rank* if $n\le m$ and $\text{rank}(A)=n$.
- *unitary matrix* $U^*U=I=UU^*$.
- Let $D\in F^{n\times k}$ $(n<k)$ be such that $D^*D=I$. Then there exists a matrix $D_\perp\in F^{n\times(n-k)}$ such that $[D\ \ D_\perp]$ is a unitary matrix

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- *Sylvester equation*

$$AX+XB=C$$

  with $A\in F^{n\times n}$, $B\in F^{m\times m}$, and $C\in F^{n\times m}$ has a unique solution $X\in F^{n\times m}$, if and only if $\lambda_i(A)+\lambda_j(B)\neq 0$, $\forall i=1,2,\ldots,n$ and $j=1,2,\ldots,m$.
- "*Lyapunov Equation*": $B=A^*$.
- Let $A\in F^{m\times n}$ and $B\in F^{n\times k}$. Then
  $\text{rank}(A)+\text{rank}(B)-n\le\text{rank}(AB)\le\min\{\text{rank}(A),\ \text{rank}(B)\}$.
- the *trace* of $A=[a_{ij}]\in C^{n\times n}$: $\text{Trace}(A):=\Sigma\, a_{ii}$
  trace has the following properties:
  $\text{Trace}(\alpha A)=\alpha\,\text{Trace}(A)$, $\forall\,\alpha\in C$, $A\in C^{n\times n}$
  $\text{Trace}(A+B)=\text{Trace}(A)+\text{Trace}(B)$, $\forall A, B\in C^{n\times n}$.
  $\text{Trace}(AB)=\text{Trace}(BA)$, $\forall A\in C^{n\times m}$, $B\in C^{m\times n}$.

Eigenvalues and Eigenvectors

- The eigenvalues and eigenvectors of $A\in \mathrm{C}^{n\times n}$: $\lambda \in \mathrm{C}$, $x \in \mathrm{C}^n$

  $Ax = \lambda x$ x is a *right eigenvector*

  y is a *left eigenvector*: $y^*A = \lambda y^*$
- eigenvalues: the roots of $\det(\lambda I\text{-}A)$.
- *spectral radius*: $\rho(A)$:=max $|\lambda_i|$
- Jordan canonical form: $A\in \mathrm{C}^{n\times n}$, $\exists$ T such that $A=TJT^{-1}$.

$$J = \mathrm{diag}\{J_1, J_2, \cdots, J_l\} \qquad J_{ij} = \begin{bmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_i & 1 \\ & & & & \lambda_i \end{bmatrix} \in \mathbf{C}^{n_{ij}\times n_{ij}}$$

$$J_i = \mathrm{diag}\{J_{i1}, J_{i2}, \cdots, J_{im_i}\}$$

The transformation T has the following form:

$$T = [T_1 \quad T_2 \quad \ldots \quad T_l], T_i = [T_{i1} \quad T_{i2} \quad \ldots \quad T_{im_i}], T_{ij} = [t_{ij1} \quad t_{ij2} \quad \ldots \quad t_{ijn_{ij}}]$$

where t_{ij1} are the eigenvectors of A: $A\, t_{ij1} = \lambda_i t_{ij1}$ and $t_{ijk} \neq 0$ defined by the following linear equations for $k \geq 2$

$$(A\text{-}\lambda_i I)\, t_{ijk} = t_{ij(k\text{-}1)}$$

are called the *generalized eigenvectors* of A.

$A\in \mathrm{R}^{n\times n}$ with distinct eigenvalues can be diagonalized:

$$A[x_1 \quad x_2 \quad \cdots \quad x_n] = [x_1 \quad x_2 \quad \cdots \quad x_n]\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}.$$

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and has the following spectral decomposition:

$$A = \Sigma\, \lambda_i x_i y_i^*$$

where $y_i \in \mathrm{C}^n$ is given by

$$\begin{bmatrix} y_1^* \\ y_2^* \\ \vdots \\ y_n^* \end{bmatrix} = [x_1 \quad x_2 \quad \cdots \quad x_n]^{-1}.$$

- $A\in \mathrm{R}^{n\times n}$ with real eigenvalue $\lambda\in \mathrm{R} \Rightarrow$ real eigenvector $x\in \mathrm{R}^n$
- A is *Hermitian*,i.e., $A=A^* \Rightarrow$ $\exists$ unitary U such that $A=U\Lambda U^*$ and Λ=diag$\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is real.

Matrix Inversion Formulas

- $$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ A_{21}A_{11}^{-1} & I \end{bmatrix}\begin{bmatrix} A_{11} & 0 \\ 0 & \Delta \end{bmatrix}\begin{bmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{bmatrix}$$

$$\Delta := A_{22} - A_{21}A_{11}^{-1}A_{12}$$

- $$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} I & A_{12}A_{22}^{-1} \\ 0 & I \end{bmatrix}\begin{bmatrix} \hat{\Delta} & 0 \\ 0 & A_{22} \end{bmatrix}\begin{bmatrix} I & 0 \\ A_{22}^{-1}A_{21} & I \end{bmatrix}$$

$$\hat{\Delta} := A_{22} - A_{21}A_{11}^{-1}A_{12}$$

- $$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}\Delta^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}\Delta^{-1} \\ -\Delta^{-1}A_{21}A_{11}^{-1} & \Delta^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} \hat{\Delta}^{-1} & -\hat{\Delta}^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}\hat{\Delta}^{-1} & A_{22}^{-1} + A_{22}^{-1}A_{21}\hat{\Delta}^{-1}A_{21}A_{22}^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ -A_{22}^{-1}A_{21}A_{11}^{-1} & A_{22}^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}.$$

$$\det A = \det A_{11}\det(A_{22} - A_{21}A_{11}^{-1}A_{12}) = \det A_{22}\det(A_{11} - A_{12}A_{22}^{-1}A_{21}).$$

In particular, for any $B \in \mathbf{C}^{m\times n}$ and $C \in \mathbf{C}^{n\times m}$, we have

$$\det\begin{bmatrix} I_m & B \\ -C & I_n \end{bmatrix} = \det(I_n + CB) = \det(I_m + BC)$$

and for $x, y \in \mathbf{C}^n$ $\det(I_n + xy^*) = 1 + y^*x.$

matrix inversion lemma :

$$(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} = A_{11}^{-1} + A_{11}^{-1}A_{12}(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1}.$$

Invariant Subspaces

- A subspace $S \subset \mathrm{C}^n$ is an *A-invariant subspace* if $Ax\in S$ for every $x\in S$.

  For example, $\{0\}$, C^n , Ker A, and Im A are all A-invariant subspaces.

  Let λ and x be an eigenvalue and a corresponding eigenvector of $A\in\mathrm{C}^{n\times n}$. Then S := span$\{x\}$ is an A-invariant subspace since

  $Ax=\lambda x\in S$.

  In general, let $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ (not necessarily distinct) and x_i be a set of eigenvalues and a set of corresponding eigenvectors and the generalized eigenvectors. Then S=span$\{x_1,\ldots,x_k\}$ is an invariant subspace provided that all the lower rank generalized eigenvectors are included.

- An A-invariant subspace $S \subset \mathbb{C}^n$ is called a ***stable invariant subspace*** if all the eigenvalues of A constrained to S have negative real parts.

  Stable invariant subspaces are used to compute the stabilizing solutions of the algebraic Riccati equations.
- Example: Let A be such that

$$A\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}\begin{bmatrix} \lambda_1 & 1 & & \\ & \lambda_1 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{bmatrix}.$$

  with Re $\lambda_1<0$, $\lambda_3<0$, and $\lambda_4>0$. Then it is easy to verify that

  S_1=span$\{x_1\}$, S_{12}=span$\{x_1, x_2\}$, S_{123}=span$\{x_1, x_2, x_3\}$, S_3=span$\{x_3\}$, S_{13}=span$\{x_1, x_3\}$, S_{124}=span$\{x_1, x_2, x_4\}$, S_4=span$\{x_4\}$, S_{14}=span$\{x_1, x_4\}$, S_{34}=span$\{x_3, x_4\}$

  are all A-invariant subspaces. Moreover, S_1, S_3, S_{12}, S_{13}, and S_{123} are stable A-invariant subspaces.

However, the subspaces S_2=span$\{x_2\}$, S_{23}=span$\{x_2, x_3\}$, S_{24}=span$\{x_2, x_4\}$, S_{234}=span$\{x_2, x_3, x_4\}$ are not A-invariant subspaces since the lower rank generalized eigenvector x_1 of x_2 is not in these subspaces.

To illustrate, consider the subspace S_{23}. It is an A-invariant subspace if $Ax_2 \in S_{23}$. Since $Ax_2=\lambda_1 x_2 + x_1$, $Ax_2 \in S_{23}$ would require that x_1 be a linear combination of x_2 and x_3, but this is impossible since x_1 is independent

Vector Norms and Matrix Norms

- Norm: Let X be a vector space. $\|\bullet\|$ is a norm if

  (i) $\|x\|\geq 0$ (positivity);

  (ii) $\|x\|=0$ if and only if $x = 0$ (positive definiteness);

  (iii) $\|\alpha x\|=|\alpha|\ \|x\|$ for any scalar α (homogeneity);

  (iv) $\|x+y\|\leq\|x\|+\|y\|$ (triangle inequality)

  for any $x\in X$ and $y\in X$.

  Let $x\in \mathbb{C}^n$. Then we define the vector

$$\|x\|_p := \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}, \text{for } 1\leq p\leq\infty.$$

In particular, when p =1, 2, ∞, we have

$$\|x\|_1 := \sum_{i=1}^n |x_i|; \qquad \|x\|_2 := \sqrt{\sum_{i=1}^n |x_i|^2}; \qquad \|x\|_\infty := \max_{1\leq i\leq n} |x_i|.$$

- Induced Matrix Norm: the matrix norm induced by a vector p-norm is defined as

$$\|A\|_p := \sup_{x\neq 0} \frac{\|Ax\|_p}{\|x\|_p}.$$

  In particular, for p = 1, 2, ∞, the corresponding induced matrix norm can be computed as

$$\|A\|_1 := \max_{1\leq i\leq n} \sum_{i=1}^m |a_{ij}| \quad \text{(column sum)};$$

$$\|A\|_2 = \sqrt{\lambda_{\max}(A^* A)};$$

$$\|A\|_\infty := \max_{1\leq i\leq m} \sum_{j=1}^n |a_{ij}| \quad \text{(row sum)}.$$

- Properties of Euclidean Norm: The Euclidean 2-norm has some very nice properties:

  Let $x\in \mathbb{F}^n$ and $y\in \mathbb{F}^m$

  1. Suppose $n \geq m$. Then $\|x\|=\|y\|$ iff there is a matrix $U\in \mathbb{F}^{n\times m}$ such that $x = Uy$ and $U^*U = I$.

  2. Suppose $n = m$. Then $\|x^*y\| \leq \|x\|\ \|y\|$. Moreover, the equality holds iff $x=\alpha y$ for some $\alpha\in$ F or $y = 0$.

  3. $\|x\|\leq\|y\|$ iff there is a matrix $\Delta\in \mathbb{F}^{n\times m}$ with $\|\Delta\|\leq 1$ such that $x=\Delta y$. Furthermore, $\|x\|<\|y\|$ iff $\|\Delta\| < 1$.

  4. $\|Ux\|=\|x\|$ for any appropriately dimensioned unitary matrices U.

- Properties of Matrix Norm:

  Frobenius norm

$$\|A\|_F := \sqrt{\text{Trace}\,(A^* A)} = \sqrt{\sum_{i=1}^m\sum_{j=1}^n |a_{ij}|^2}.$$

  Let A and B be any matrices with appropriate dimensions. Then

  1. $\rho(A) \leq \|A\|$ (This is also true for F norm and any induced matrix norm).

  2. $\|AB\|\leq\|A\|\ \|B\|$. In particular, this gives $\|A^{-1}\|\geq\|A\|^{-1}$ if A is invertible. (This is also true for any induced matrix norm.)

  3. $\|UAV\|=\|A\|$ and $\|UAV\|_F=\|A\|_F$, for any appropriately dimensioned unitary matrices U and V.

  4. $\|AB\|_F\leq\|A\|\ \|B\|_F$, and $\|AB\|_F\leq\|B\|\ \|A\|_F$

Singular Value Decomposition

- Let $A\in F^{m\times n}$. There exist unitary matrices

  $U=[u_1, u_2,\dots,u_m]\in F^{m\times m}$, $V=[v_1, v_2,\dots,v_n]\in F^{n\times n}$

  such that $A=U\Sigma V^*$ where

  with $\Sigma_1=\mathrm{diag}\{\sigma_1, \sigma_2, \dots ,\sigma_p\}$ and $\Sigma=\begin{bmatrix}\Sigma_1 & 0\\ 0 & 0\end{bmatrix}$,

  $\sigma_1\geq \sigma_2\geq \dots \geq \sigma_p\geq 0$, $p=\min\{m, n\}$.

  $=\sigma_{max}(A)=\sigma_1(A)=$ the largest singular value of A;

  $\underline{\sigma}(A)=\sigma_{min}(A)=\sigma_p(A)=$ the smallest singular value of A.

  Note that $\bar{\sigma}(A)$

  $Av_i=\sigma_i u_i$, $A^*u_i=\sigma_i v_i$

  $A^*Av_i=\sigma_i^2 v_i$, $AA^*u_i=\sigma_i^2 u_i$

- Singular values are good measures of the "size" of the matrix singular vectors are good indications of strong/weak input or output directions.

  Geometrically, the singular values of a matrix A are precisely the lengths of the semi-axes of the hyperellipsoid E defined by

  $E=\{y:\ y=Ax,\ x\in C^n,\ \|x\|=1\}$.

  Thus v_1 is the direction in which $\|y\|$ is largest for all $\|x\|=1$, while v_n is the direction in which $\|y\|$ is smallest for all $\|x\|=1$

  $v_1(v_n)$ is the *highest (lowest) gain input direction*

  $u_1(u_m)$ is the *highest (lowest) gain observing direction*

  e.g.,

  $$A=\begin{bmatrix}\cos\theta_1 & -\sin\theta_1\\ \sin\theta_1 & \cos\theta_1\end{bmatrix}\begin{bmatrix}\sigma_1 & \\ & \sigma_2\end{bmatrix}\begin{bmatrix}\cos\theta_2 & -\sin\theta_2\\ \sin\theta_2 & \cos\theta_2\end{bmatrix}.$$

  A maps a unit disk to an ellipsoid with semi-axes of

- Alternative definitions:

  $$\bar{\sigma}(A):=\max_{\|x\|=1}\|Ax\|$$

  and for the smallest singular value $\underline{\sigma}$ of a tall matrix:

  $$\underline{\sigma}(A):=\min_{\|x\|=1}\|Ax\|.$$

  Suppose A and Δ are square matrices. Then

  $$(i)\ |\underline{\sigma}(A+\Delta)-\underline{\sigma}(A)|\leq\bar{\sigma}(\Delta);$$
  $$(ii)\ \underline{\sigma}(A\Delta)\geq\underline{\sigma}(A)\underline{\sigma}(\Delta);$$
  $$(iii)\ \bar{\sigma}(A^{-1})=\frac{1}{\underline{\sigma}(A)}\ \text{if } A \text{ is invertible}.$$

- Some useful properties

  Let $A\in F^{m\times n}$ and $\sigma_1\geq \sigma_2\geq \dots \geq \sigma_r > \sigma_{r+1}=\dots=0$, $r\leq\min\{m, n\}$. Then

  1. rank(A)=r;

  2. Ker $A=\mathrm{span}\{v_{r+1}, v_{r+2},\dots,v_n\}$ and $(\mathrm{Ker}\ A)^{\perp}=\mathrm{span}\{v_1, v_2,\dots,v_r\}$;

  3. Im $A=\mathrm{span}\{u_1, u_2,\dots,u_r\}$ and $(\mathrm{Im}\ A)^{\perp}=\mathrm{span}\{u_{r+1}, u_{r+2},\dots,u_m\}$;

  4. $A\in F^{m\times n}$ has a dyadic expansion:

  $$A=\sum_{i=1}^{r}\sigma_i u_i v_i^*=U_r\Sigma_r V_r^*$$

  where $U_r=[u_1, u_2,\dots,u_r]$, $V_r=[v_1, v_2,\dots,v_r]$, and $\Sigma_r=\mathrm{diag}\{\sigma_1, \sigma_2,\dots, \sigma_r\}$;

  5. $\|A\|_F^2=\sigma_1^2+\sigma_2^2+\dots+\sigma_r^2$;

  6. $\|A\|=\sigma_1$;

  7. $\sigma_i(U_0AV_0)=\sigma_i(A)$, $i=1,2,\dots,p$. for any appropriately dimensioned unitary matrices U_0 and V_0.

Generalized Inverses

- Let $A\in F^{m\times n}$. $X\in F^{n\times m}$ is a ***right inverse*** if $AX=I$. One of the right inverses is given by $X=A^*(AA^*)^{-1}$. $YA=I$ then Y is a ***left inverse*** of A.

- ***Pseudo-inverse*** or ***Moore-Penrose inverse*** A^+:

  (i) $AA^+A=A$; (ii) $A^+AA^+=A^+$; (iii) $(AA^+)^*=AA^+$; (iv) $(A^+A)^*=A^+A$.

  Pseudo-inverse is unique.

  Let $A=BC$ where B has a full column rank and C has full row rank. Then $A^+=C^*(CC^*)^{-1}(B^*B)^{-1}B^*$.

  or let $A=U\Sigma V^*$ with $\Sigma=\begin{bmatrix}\Sigma_r & 0\\ 0 & 0\end{bmatrix}$ and $\Sigma_r=\mathrm{diag}\{\sigma_1, \sigma_2, \dots ,\sigma_r\}$.

  Then $A^+=V\Sigma^+U^*$ with $\Sigma^+=\begin{bmatrix}\Sigma_r^{-1} & 0\\ 0 & 0\end{bmatrix}$.

Semidefinite Matrices

- $A=A^*$ is ***positive definite*** (semi-definite) denoted by $A>0(\geq 0)$, if $x^*Ax>0(\geq 0)$ for all $x\neq 0$.

- $A\in F^{n\times n}$ and $A=A^*\geq 0$, $\exists B\in F^{n\times r}$ with $r\geq \mathrm{rank}(A)$ such that $A=BB^*$.

- Let $B\in F^{m\times n}$ and $C\in F^{k\times n}$. Suppose $m\geq k$ and $B^*B=C^*C$. $\exists U\in F^{m\times k}$ such that $U^*U=I$ and $B=UC$.

- ***Square root*** for a positive semidefinite matrix A, $A^{1/2}=(A^{1/2})^*\geq 0$, such that $A=A^{1/2}A^{1/2}$.

  Clearly, $A^{1/2}$ can be computed by using spectral decomposition or SVD: let $A=U\Lambda U^*$, then $A^{1/2}=U\Lambda^{1/2}U^*$, where $\Lambda=\mathrm{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}$, $\Lambda^{1/2}=\mathrm{diag}\{(\lambda_1)^{1/2}, (\lambda_2)^{1/2}, \dots, (\lambda_n)^{1/2}\}$

- $A=A^*>0$ and $B=B^*\geq 0$. Then $A>B$ iff $\rho(BA^{-1})<1$.
- Let $X=X^*\geq 0$ be partitioned as

$$X=\begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix}.$$

Then $\operatorname{Ker} X_{22}\subset \operatorname{Ker} X_{12}$. Consequently, if X_{22}^+ is the pseudo-inverse of X_{22}, then $Y=X_{12}X_{22}^+$ solves

$$YX_{22}=X_{12}$$

and

$$\begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix}=\begin{bmatrix} I & X_{12}X_{22}^+ \\ 0 & I \end{bmatrix}\begin{bmatrix} X_{11}-X_{12}X_{22}^+X_{22}^* & 0 \\ 0 & X_{22} \end{bmatrix}\begin{bmatrix} I & 0 \\ X_{22}^+X_{12}^* & I \end{bmatrix}.$$

Chapter 3: Linear Systems

- Dynamical systems
- Controllability and stabilizability
- Observability and detectability
- Observer theory
- System interconnections
- Realizations
- Poles and zeros

Dynamical Systems

- Linear equations: $\dot{x}=Ax+Bu,\ x(t_0)=x_0$
  $y=Cx+Du$
- transfer matrix: $Y(s)=G(s)\,U(s)$, $G(s)=C(sI-A)^{-1}B+D$
- notation $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]:=C(sI-A)^{-1}B+D.$
- solution:

$$x(t)=e^{A(t-t_0)}x(t_0)+\int_{t_0}^{t}e^{A(t-\tau)}Bu(\tau)d\tau$$

$$y(t)=Cx(t)+Du(t).$$

- impulse matrix: $g(t)=L^{-1}\{G(s)\}=Ce^{At}B1_+(t)+D\delta(t).$
- input/output relationship:

$$y(t)=(g*u)(t)=\int_{-\infty}^{t}g(t-\tau)u(\tau)d\tau.$$

Matlab

```
>> G=ss(A,B,C,D)    % state space realization
>> [A,B,C,D] = ssdata(G)  % unpack the system matrix
>> [y,t,x] = step(g)
>> [y,t,x] = impulse(g) % impulse response
>> [y,x]=lsim(A,B,C,D,U,T) % U is a length(T)   column(B)
matrix input; T is the sampling points.
```

Controllability

- Controllability: (A,B) is controllable if, for any initial state $x(0)=x_0$, $t_1>0$ and final state x_1, there exists a (piecewise continuous) input $u(.)$ such that $x(t_1)=x_1$.
- The matrix $W_c(t):=\int_0^t e^{A\tau}BB^*e^{A^*\tau}d\tau$ is positive definite for any $t>0$.
- The controllability matrix $C=[B\ AB\ A^2B\ \ldots\ A^{n-1}B]$ has full row rank, i.e., $\langle A\,|\,\operatorname{Im}B\rangle:=\sum_{i=1}^{n}\operatorname{Im}(A^{i-1}B)=\mathbb{R}^n.$
- The eigenvalues of $A+BF$ can be freely assigned by a suitable F.

Controllability: PBH test

PBH (Popov-Belevitch-Hautus) test:

- The matrix $[A-\lambda I,\ B]$ has full row rank for all λ in C.
- Let λ and x be any eigenvalue and any corresponding left eigenvector of A, i.e., $x^*A=\lambda x^*$, then $x^*B\neq 0$.

Stability and Stabilizability

A is *stable* if *Re*$\lambda(A)<0$.

- (A,B) is stabilizable.
- $A+BF$ is stable for some F.

PBH test:

- The matrix $[A-\lambda I,\ B]$ has full row rank for all $Re\lambda \geq 0$.
- For all λ and x such that $x^*A=x^*\lambda$ and $Re\lambda \geq 0$, $x^*B\neq 0$.

Observability

- (C,A) is observable if, for any $t_1 > 0$, the initial state $x(0) = x_0$ can be determined from the time history of the input $u(t)$ and the output $y(t)$ in the interval of $[0,t_1]$.
- The matrix $W_0(t) := \int_0^t e^{A^*\tau}C^*Ce^{A\tau}d\tau$ is positive definite for any $t > 0$.
- The observability matrix O has full column rank, i.e., $\bigcap_{i=1}^{n} \mathrm{Ker}(CA^{i-1}) = 0$.

$$O = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

- The eigenvalues of $A+LC$ can be freely assigned by a suitable L.

Observability: PBH test

PBH test:

- The matrix $\begin{bmatrix} A-\lambda I \\ C \end{bmatrix}$ has full column rank for all λ in C.
- Let λ and y be any eigenvalue and any corresponding right eigenvector of A, i.e., $Ay=\lambda y$, then $Cy\neq 0$.

Duality:

- (C, A) is observable if and only if (A^*,C^*) is controllable.

Detectability

The following are equivalent:

- (C,A) is detectable.
- $A+LC$ is stable for a suitable L.
- (A^*,C^*) is stabilizable.

PBH test:

- The matrix $\begin{bmatrix} A-\lambda I \\ C \end{bmatrix}$ has full column rank for all $Re\lambda\geq 0$.
- For all λ and y such that $Ay=\lambda y$, $Re\lambda \geq 0$, and $Cy\neq 0$.

An example:

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cccc|c} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 1 \\ 0 & 0 & \lambda_1 & 0 & \alpha \\ 0 & 0 & 0 & \lambda_2 & 1 \\ \hline 1 & 0 & 0 & \beta & 0 \end{array}\right]$$

Not controllable if $\lambda_1 = \lambda_2$ or $\alpha=0$;

Not observable if $\lambda_1 = \lambda_2$ or $\beta=0$.

```
>> C =ctrb(A,B) ; O = obsv(A,C);
>> Wc(∞ )= gram(g,'o'); % if A is stable.
>> F = -place (A,B,P)  % P is a vector of desired
     eigenvalues.
```

Observer-Based Controllers

An observer is a dynamical system with input (u,y) and output, say $\hat{x}$ which asymptotically estimates the state x, i.e., $\hat{x}(t)-x(t)\to 0$ as $t\to\infty$ for all initial states and for every input.

An observer exists iff (C,A) is detectable. Further, if (C,A) is detectable, then a full order Luenberger observer is given by

$$\dot{q} = Aq+Bu+L(Cq+Du-y)$$
$$\hat{x} = q$$

where L is a matrix such that $A+LC$ is stable.

Observer-based controller:

$$\dot{\hat{x}} = (A+LC)\hat{x} + Bu + LDu - Ly$$
$$u = F\hat{x}$$
$$u = K(s)y$$

and

$$K(s) = \left[\begin{array}{c|c} A+BF+LC+LDF & -L \\ \hline F & 0 \end{array}\right].$$

Example

Let $A=\begin{bmatrix}1 & 2\\ 1 & 0\end{bmatrix}, B=\begin{bmatrix}1\\ 0\end{bmatrix}, \text{and } C=\begin{bmatrix}1 & 0\end{bmatrix}$

Design $u=Fx$ so that the closed-loop poles are at {-2, -3}.

F = [-6 -8] >> F = -place(A,B,[-2,-3])

Suppose observer poles are at {-10,-10}

Then $L=\begin{bmatrix}-21\\ -51\end{bmatrix}$ can be obtained by using

>> L = -acker(A',C',[-10,-10])'

and the observer-based controller is given by

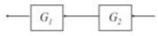
$$K(s)=\frac{-534\,(s+0.6966)}{(s+34.6564)(s-8.6564)}$$

which is unstable: this may not be desirable in practice.

Operations on Systems

$$G_1=\left[\begin{array}{c|c}A_1 & B_1\\ \hline C_1 & D_1\end{array}\right] \quad G_2=\left[\begin{array}{c|c}A_2 & B_2\\ \hline C_2 & D_2\end{array}\right].$$

- cascade:

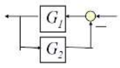


$$G_1G_2=\left[\begin{array}{cc|c}A_1 & B_1C_2 & B_1D_2\\ 0 & A_2 & B_2\\ \hline C_1 & D_1C_2 & D_1D_2\end{array}\right]=\left[\begin{array}{cc|c}A_2 & 0 & B_2\\ B_1C_2 & A_1 & B_1D_2\\ \hline D_1C_2 & C_1 & D_1D_2\end{array}\right].$$

- addition:

$$G_1+G_2=\left[\begin{array}{cc|c}A_1 & 0 & B_1\\ 0 & A_2 & B_2\\ \hline C_1 & C_2 & D_1+D_2\end{array}\right].$$

- feedback:



$$T=\left[\begin{array}{cc|c}A_1-B_1D_2R_{12}^{-1}C_1 & -B_1R_{21}^{-1}C_2 & B_1R_{21}^{-1}\\ B_2R_{12}^{-1}C_1 & A-B_2D_1R_{21}^{-1}C_2 & B_2D_2R_{21}^{-1}\\ \hline R_{12}^{-1}C_1 & -R_{12}^{-1}D_1C_2 & D_1R_{21}^{-1}\end{array}\right]$$

where $R_{21}=I+D_1D_2$ and $R_{21}=I+D_2D_1$.

- transpose or dual system $G\mapsto G^T(s)=B^*(sI-A^*)^{-1}C^*+D^*$

  or equivalently $\left[\begin{array}{c|c}A & B\\ \hline C & D\end{array}\right]\mapsto\left[\begin{array}{c|c}A^* & C^*\\ \hline B^* & D^*\end{array}\right].$

- conjugate system $G\mapsto G^{\sim}(s):=G^T(-s)=B^*(-sI-A^*)^{-1}C^*+D^*$

  or equivalently $\left[\begin{array}{c|c}A & B\\ \hline C & D\end{array}\right]\mapsto\left[\begin{array}{c|c}-A^* & -C^*\\ \hline B^* & D^*\end{array}\right].$

  In particular, we have $G^*(j\omega):=[G(j\omega)]^*=G^{\sim}(j\omega)$.

- inverse system: Let D^+ denote a right (left) inverse of D if D has full row (column) rank. Then

$$G^+=\left[\begin{array}{c|c}A-BD^+C & -BD^+\\ \hline D^+C & D^+\end{array}\right]$$

is a right (left) inverse of G, I.e, $GG^+=I$ ($G^+G=I$).

State Space Realizations

Given $G(s)$, find (A,B,C,D) such that $G(s)=\left[\begin{array}{c|c}A & B\\ \hline C & D\end{array}\right]$ which is a state space realization of $G(s)$.

A state space realization (A,B,C,D) of $G(s)$ is *minimal* if and only if (A,B) is controllable and (C,A) is observable.

Let (A_1,B_1,C_1,D) and (A_2,B_2,C_2,D) be two minimal realizations of $G(s)$. Then there exists a unique nonsingular T such that

$$A_2=TA_1T^{-1},\quad B_2=TB_1,\quad C_2=C_1T^{-1}.$$

Furthermore, T can be specified as

$$T=(O_2{}^*O_2)^{-1}O_2{}^*O_1 \quad \text{or } T^{-1}=C_1C_2{}^*(C_2C_2{}^*)^{-1}.$$

where C_1, C_2, O_1, and O_2 are the corresponding controllability and observability matrices respectively.

SIMO and MISO

SIMO Case: Let $G(s)=\begin{bmatrix}g_1(s)\\ g_2(s)\\ \vdots\\ g_m(s)\end{bmatrix}=\frac{\beta_1s^{n-1}+\beta_2s^{n-2}+\cdots+\beta_{n-1}s+\beta_n}{s^n+a_1s^{n-1}+\cdots+a_{n-1}s+a_n}+d,$

where $\beta_i\in R^m$ and $d\in R^m$. Then

$$G(s)=\left[\begin{array}{c|c}A & b\\ \hline C & d\end{array}\right],\ b\in R^n, C\in R^{m\times n}, d\in R^m$$

$$A:=\begin{bmatrix}-a_1 & -a_2 & \cdots & -a_{n-1} & -a_n\\ 1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\end{bmatrix} \qquad b:=\begin{bmatrix}1\\ 0\\ 0\\ \vdots\\ 0\end{bmatrix}$$

$$C=\begin{bmatrix}\beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n\end{bmatrix}$$

MISO Case: Let $G(s)=\begin{pmatrix}g_1(s) & g_2(s) & \cdots & g_m(s)\end{pmatrix}$

$$=\frac{\eta_1s^{n-1}+\eta_2s^{n-2}+\cdots+\eta_{n-1}s+\eta_n}{s^n+a_1s^{n-1}+\cdots+a_{n-1}s+a_n}+d$$

with $\eta_i{}^*$, $d^*\in R^p$. Then

$$G(s)=\left[\begin{array}{ccccc|c}-a_1 & 1 & 0 & \cdots & 0 & \eta_1\\ -a_2 & 0 & 1 & \cdots & 0 & \eta_2\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ -a_{n-1} & 0 & 0 & \cdots & 1 & \eta_{n-1}\\ -a_n & 0 & 0 & \cdots & 0 & \eta_n\\ \hline 1 & 0 & 0 & \cdots & 0 & d\end{array}\right]$$

Realizing Each Element

To illustrate, consider a 2×2 (block) matrix

$$G(s)=\begin{bmatrix} G_1(s) & G_2(s) \\ G_3(s) & G_4(s) \end{bmatrix}$$

and assume that $G_i(s)$ has a state space realization of

$$G_i(s)=\left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right], \quad i=1,\dots,4.$$

Note that $G_i(s)$ may itself be a MIMO transfer matrix.

Then a realization for $G(s)$ can be given by

$$G(s)=\left[\begin{array}{cccc|cc} A_1 & 0 & 0 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & 0 & 0 & B_2 \\ 0 & 0 & A_3 & 0 & B_3 & 0 \\ 0 & 0 & 0 & A_4 & 0 & B_4 \\ \hline C_1 & C_2 & 0 & 0 & D_1 & D_2 \\ 0 & 0 & C_3 & C_4 & D_3 & D_4 \end{array}\right].$$

Limitation: may not be minimal.

>>G=tf(num,den); G=zpk(zeros,poles,gain);

Gilbert's Realization

Let G(s) be a p×m transfer matrix $G(s) = N(s)/d(s)$

with $d(s)$ a scalar polynomial. For simplicity, we shall assume that $d(s)$ has only real and distinct roots $\lambda_i \neq \lambda_j$ if $i \neq j$ and

$$d(s)=(s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_r).$$

Then $G(s)$ has the following partial fractional expansion:

$$G(s)=D+\sum_{i=1}^{r}\frac{W_i}{s-\lambda_i}.$$

Suppose rank $W_i = k_i$ and let $B_i \in R^{k_i \times m}$ and $C_i \in R^{p \times k_i}$ be two constant matrices such that $W_i = C_i B_i$.

Then a realization is given by

$$G(s)=\left[\begin{array}{ccc|c} \lambda_1 I_{k_1} & & & B_1 \\ & \ddots & & \vdots \\ & & \lambda_r I_{k_r} & B_r \\ \hline C_1 & \cdots & C_r & D \end{array}\right].$$

This realization is controllable and observable (minimal) by PBH tests.

System Poles and Zeros

An example:

$$G(s)=\begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ \frac{2}{s+2} & \frac{1}{s+1} \end{bmatrix}$$

which is stable and each element of $G(s)$ has no finite zeros. Let

$$K=\begin{bmatrix} \frac{s+2}{s-\sqrt{2}} & -\frac{s+1}{s-\sqrt{2}} \\ 0 & 1 \end{bmatrix}$$

which is unstable.

However,

$$KG=\begin{bmatrix} -\frac{s+2}{(s+1)(s+2)} & 0 \\ \frac{2}{s+2} & \frac{1}{s+1} \end{bmatrix}$$

is stable. This implies that $G(s)$ must have an unstable zero at $\sqrt{2}$ that cancels the unstable pole of K.

Smith Form

- A square polynomial matrix $Q(s)$ is *unimodular* iff det $Q(s)$ is constant.
- Let $Q(s)$ be a ($p \times m$) polynomial matrix. Then the *normal rank* of $Q(s)$, denoted *normalrank* $(Q(s))$, is the maximally possible rank of $Q(s)$ for at least one $s \in$ C.

Example:

$$Q(s)=\begin{bmatrix} s & 1 \\ s^2 & 1 \\ s & 1 \end{bmatrix}.$$

$Q(s)$ has normal rank 2 since rank $Q(2) = 2$. However, $Q(0)$ has rank 1.

- *Smith form*: Let $P(s)$ be any polynomial matrix, then there exist unimodular matrices $U(s)$, $V(s) \in$ R[s] such that

$$U(s)P(s)V(s)=S(s):=\begin{bmatrix} \gamma_1(s) & 0 & \cdots & 0 & 0 \\ 0 & \gamma_2(s) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \gamma_r(s) & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

where $\gamma_i(s)$ divides $\gamma_{i+1}(s)$ and r is the normal rank of $P(s)$.

$S(s)$ is called the *Smith form* of $P(s)$.

Example:

$$P(s)=\begin{bmatrix} s+1 & (s+1)(2s+1) & s(s+1) \\ s+2 & (s+2)(s^2+5s+3) & s(s+2) \\ 1 & 2s+1 & s \end{bmatrix}.$$

$P(s)$ has normal rank 2 since $\det(P(s)) \equiv 0$ and

$$\det\begin{bmatrix} s+1 & (s+1)(2s+1) \\ s+2 & (s+2)(s^2+5s+3) \end{bmatrix}=(s+1)^2(s+2)^2 \neq 0.$$

Let

$$U=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -(s+2) \\ 1 & 0 & -(s+1) \end{bmatrix} \qquad V(s)=\begin{bmatrix} 1 & -(2s+1) & -s \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S(s)=U(s)P(s)V(s)=\begin{bmatrix} 1 & 0 & 0 \\ 0 & (s+1)(s+2)^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Smith-McMillan Form

- Let $G(s)$ be any proper real rational transfer matrix, then there exist unimodular matrices $U(s)$, $V(s) \in$ R[s] such that

$$U(s)G(s)V(s)=M(s):=\begin{bmatrix} \frac{\alpha_1(s)}{\beta_1(s)} & 0 & \cdots & 0 & 0 \\ 0 & \frac{\alpha_2(s)}{\beta_2(s)} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{\alpha_r(s)}{\beta_r(s)} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

and $\alpha_i(s)$ divides $\alpha_{i+1}(s)$ and $\beta_{i+1}(s)$ divides $\beta_i(s)$.

- Write $G(s)$ as $G(s) = N(s)/d(s)$ such that $d(s)$ is a scalar polynomial and $N(s)$ is a $p \times m$ polynomial matrix.

Let the Smith form of $N(s)$ be $S(s) = U(s)N(s)V(s)$.

Then $M(s) = S(s)/d(s)$.

Poles and Transmission Zeros

- *McMillan* degree of $G(s) = \sum \deg(\beta_i(s))$ where $\deg(\beta_i(s))$ denotes the degree of the polynomial $\beta_i(s)$.
- McMillan degree of $G(s)$ = the dimension of a minimal realization of $G(s)$.
- Poles of $G(s)$ = roots of $\beta_i(s)$
- transmission zeros of $G(s)$ = the roots of $\alpha_i(s)$

$z_0 \in \mathbb{C}$ is a blocking zero of $G(s)$ if $G(z_0) = 0$.

An example:

$$G(s)=\begin{bmatrix} \frac{1}{(s+1)(s+2)} & \frac{2s+1}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \\ \frac{1}{(s+1)^2} & \frac{s^2+5s+3}{(s+1)^2} & \frac{s}{(s+1)^2} \\ \frac{1}{(s+1)^2(s+2)} & \frac{2s+1}{(s+1)^2(s+2)} & \frac{s}{(s+1)^2(s+2)} \end{bmatrix}.$$

Then $G(s)$ can be written as

$$G(s)=\frac{1}{(s+1)^2(s+2)}\begin{bmatrix} s+1 & (s+1)(2s+1) & s(s+1) \\ s+2 & (s^2+5s+3)(s+2) & s(s+2) \\ 1 & 2s+1 & s \end{bmatrix}.$$

$G(s)$ has the McMillan form

$$M(s)=\begin{bmatrix} \frac{1}{(s+1)^2(s+2)} & 0 & 0 \\ 0 & \frac{s+2}{s+1} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

McMillan degree of $G(s) = 4$.

Poles: {-1,-1,-1,-2}.

Transmission zero: {-2}.

The transfer matrix has pole and zero at the same location {-2}; this is the unique feature of multivariable systems.

Alternative Characterizations

- Let $G(s)$ have full column normal rank. Then $z_0 \in \mathbb{C}$ is a transmission zero if and only if there exists a vector $0 \neq u_0$ such that $G(z_0)u_0 = 0$.

not true if $G(s)$ does not have full column normal rank:

$$G(s)=\frac{1}{s+1}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},\ u_0=\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

G has no transmission zero but $G(s)u_0 = 0$ for all s.

z_0 can be a pole of $G(s)$ even though $G(z_0)$ is not defined. (however $G(z_0)\,u_0$ may be well defined.)

For example,

$$G(s)=\begin{bmatrix} \frac{s-1}{s+1} & 0 \\ 0 & \frac{s+2}{s-1} \end{bmatrix},\ u_0=\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Then $G(1)\,u_0 = 0$. Therefore, 1 is a transmission zero.

- Let $G(s)$ have full row normal rank. Then $z_0 \in \mathbb{C}$ is a transmission zero if and only if there exists a vector $\eta_0 \neq 0$ such that $\eta_0^* G(z_0) = 0$.
- Suppose $z_0 \in \mathbb{C}$ is not a pole of $G(s)$. Then z_0 is a transmission zero if and only if

$$\operatorname{rank}(G(z_0)) < \operatorname{normalrank}(G(s)).$$

- Let $G(s)$ be a square $m \times m$ matrix and $\det G(s) \neq 0$. Suppose $z_0 \in \mathbb{C}$ is not a pole of $G(s)$. Then z_0 is a transmission zero if and only if $\det G(z_0) = 0$.

$$\det\begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ \frac{2}{s+2} & \frac{1}{s+1} \end{bmatrix} = \frac{2-s^2}{(s+1)^2(s+2)^2}.$$

Invariant Zeros (state space)

- The poles and zeros of a transfer matrix can also be characterized in terms of its state space realizations:

$$G(s)=\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Consider the following system matrix

$$Q(s)=\begin{bmatrix} A-sI & B \\ C & D \end{bmatrix}.$$

z_0 is an invariant zero of the realization if it satisfies

$$\operatorname{rank}\begin{bmatrix} A-z_0 I & B \\ C & D \end{bmatrix} < \operatorname{normalrank}\begin{bmatrix} A-sI & B \\ C & D \end{bmatrix}.$$

- Suppose $\begin{bmatrix} A-sI & B \\ C & D \end{bmatrix}$ has full column normal rank. Then z_0 is an invariant zero iff there exist $0 \neq x \in \mathbb{C}^n$ and $u \in \mathbb{C}^m$ such that

$$\begin{bmatrix} A-z_0 I & B \\ C & D \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix}=0.$$

Moreover, if $u = 0$, then z_0 is also a non-observable mode.

- Suppose $\begin{bmatrix} A-sI & B \\ C & D \end{bmatrix}$ has full row normal rank. Then z_0 is an invariant zero iff there exist $0 \neq y \in \mathbb{C}^n$ and $v \in \mathbb{C}^p$ such that

$$\begin{bmatrix} y^* & v^* \end{bmatrix}\begin{bmatrix} A-z_0 I & B \\ C & D \end{bmatrix}=0.$$

Moreover, if $v = 0$, then z_0 is also a non-controllable mode.

- G(s) has full column(row) normal rank if and only if $\begin{bmatrix} A-sI & B \\ C & D \end{bmatrix}$ has full column (row) normal rank.

This follows by noting that

$$\begin{bmatrix} A-sI & B \\ C & D \end{bmatrix}=\begin{bmatrix} I & 0 \\ C(A-sI)^{-1} & I \end{bmatrix}\begin{bmatrix} A-sI & B \\ 0 & G(s) \end{bmatrix}$$

and

$$\operatorname{normalrank}\begin{bmatrix} A-sI & B \\ C & D \end{bmatrix}=n+\operatorname{normalrank}(G(s)).$$

- Let $G(s)=\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a minimal realization. Then z_0 is a transmission of $G(s)$ iff it is an invariant zero of the minimal realization.

- Let $G(s)$ be a $p\times m$ transfer matrix and let (A,B,C,D) be a minimal realization. Let the input be $u(t)=u_0e^{\lambda t}$, where λ is not a pole of $G(s)$ and $u_0\in C^m$ is an arbitrary constant vector, then the output with the initial state $x(0)=(\lambda I-A)^{-1}Bu_0$ is $y(t)=G(\lambda)u_0e^{\lambda t}$.
- Let $G(s)$ be a $p\times m$ transfer matrix and let (A,B,C,D) be a minimal realization. Suppose that z_0 is a transmission zero of $G(s)$ and is not a pole of $G(s)$. Then for any nonzero vector $u_0\in C^m$ such that $G(z_0)u_0=0$, the output of the system due to the initial state $x(0)=(z_0I-A)^{-1}Bu_0$ and the input $u(t)=u_0e^{z_0 t}$ is identically zero: $y(t)=G(z_0)\, u_0e^{z_0 t}=0$.
- Computing Invariant Zeros: generalized eigenvalue problem

$$\underbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}_{M}\begin{bmatrix} x \\ u \end{bmatrix} = z_0\underbrace{\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}}_{N}\begin{bmatrix} x \\ u \end{bmatrix}$$

- MATLAB command: eig(M,N).

Example

Let

$$G(s)=\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]=\left[\begin{array}{ccc|ccc} -1 & -2 & 1 & 1 & 2 & 3 \\ 0 & 2 & -1 & 3 & 2 & 1 \\ -4 & -3 & -2 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 4 & 0 & 0 & 0 \end{array}\right].$$

invariant zeros:

\>> G=ss(A,B,C,D), z_0 = tzero(G), % or

\>> z_0 = tzero(A,B,C,D)

which gives $z_0 = 0.2$. Since $G(s)$ is full-row rank, we can find y and v:

$$\begin{bmatrix} y^* & v^* \end{bmatrix}\begin{bmatrix} A-z_0I & B \\ C & D \end{bmatrix}=0,$$

\>> null([A-z_0*eye(3),B;C,D]')

$$\Rightarrow \begin{bmatrix} y \\ v \end{bmatrix}=\left[\begin{array}{c} 0.0466 \\ 0.0466 \\ -0.1866 \\ \hline -0.9702 \\ 0.1399 \end{array}\right].$$

Chapter 4: H_2 and H_∞ Spaces

- **Hilbert Space**
- **H_2 and H_∞ Functions**
- **State Space Computation of H_2 and H_∞ norms**

Inner Product

- **Inner Product: Let V be a vector space over C. An inner product on V is a complex valued function, <•, •>: $V\times V\to$ C Such that for any $x, y, z\in V$ and $a, b\in$C**
  (i) $\langle x, \alpha y+\beta z\rangle=\alpha\langle x, y\rangle+\beta\langle x, z\rangle$
  (ii) $\langle x, y\rangle=\langle y, z\rangle^*$ (complex conjugate)
  (iii) $\langle x, x\rangle>0$ if $x\neq 0$.
- **Inner product on C^n:**

$$\langle x,y\rangle := x^*y=\sum_{i=1}^{n}\bar{x}_iy_i \quad \forall x=\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},\ y=\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}\in \mathbf{C}^n,$$

$$\|x\|:=\sqrt{\langle x,x\rangle},\quad \cos\angle(x,y)=\frac{\langle x,y\rangle}{\|x\|\|y\|},\ \angle(x,y)\in[0,\pi].$$

x and y are orthogonal if $\angle(x,y)=\frac{1}{2}\pi$

Properties of Inner Product

A vector space V with an inner product is called an *inner product space*.

Inner product induced norm $\|x\| := \sqrt{\langle x,x\rangle}$
Distance between vectors x and y : $d(x,y) = \|x - y\|$.
Two vectors x and y are *orthogonal* if $\langle x,y\rangle = 0$, denoted $x \perp y$.

- **Properties of Inner Product:**
  - $|\langle x, y\rangle|\le\|x\|\,\|y\|$ (Cauchy-Schwarz inequality). Equality holds iff $x=\alpha y$ for some constant α or $y=0$.
  - $\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2$ (Parallelogram law)
  - $\|x+y\|^2=\|x\|^2+\|y\|^2$ if $x\perp y$.

Hilbert Spaces

- **Hilbert Space: a complete inner product space. (We shall not discuss the completeness here.)**

Examples:

- C^n with the usual inner product.
- $C^{n\times m}$ with the following inner product
  $\langle A, B\rangle$:= Trace A^*B $\quad\forall A, B\in C^{n\times m}$
- $L_2[a,b]$: all square integrable and Lebesgue measurable functions defined on an interval $[a,b]$ with the inner product

  $\langle f, g\rangle := \int_a^b f(t)^*g(t)dt$, Matrix form: $\langle f, g\rangle := \int_a^b \mathrm{Trace}[f(t)^*g(t)]dt$.
- $L_2 = L_2(-\infty, \infty)$: $\langle f, g\rangle := \int_{-\infty}^{\infty}$ Trace$[f(t)^*g(t)]dt$.
- $L_{2+} = L_2[0, \infty)$: subspace of $L_2(-\infty, \infty)$.
- $L_{2-} = L_2(-\infty, 0]$: subspace of $L_2(-\infty, \infty)$.

Analytic Functions

- Let $S \subset C$ be an open set, and let $f(s)$ be a complex valued function defined on S, $f(s) : S \rightarrow C$. Then $f(s)$ is *analytic at a point z_0* in S if it differentiable at z_0 and also at each point in some neighborhood of z_0.

  It is a fact that if $f(s)$ is analytic at z_0 then f has continuous derivatives of all orders at z_0. Hence, it has a power series representation at z_0.

  A function $f(s)$ is said to be *analytic in S* if it has a derivative or is analytic at each point of S.

- *Maximum Modulus Theorem*: If $f(s)$ is defined and continuous on a closed-bounded set S and analytic on the interior of S, then

$$\max_{s \in S} |f(s)| = \max_{s \in \partial S} |f(s)|$$

  where ∂S denotes the boundary of S.

L_2 and H_2 Spaces

- $L_2(jR)$ Space: all complex matrix functions F such that the integral below is bounded:

$$\int_{-\infty}^{\infty} \text{Trace}[F^*(j\omega)F(j\omega)]d\omega < \infty$$

  with the inner product

$$\langle F, G \rangle := \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}\,[F^*(j\omega)G(j\omega)]d\omega.$$

  and the inner product induced norm is given by $\|F\|_2 := \sqrt{\langle F, F \rangle}$.

  $RL_2(jR)$ or simply RL_2: all real rational strictly proper transfer matrices with no poles on the imaginary axis.

- H_2 Space: a (closed) subspace of $L_2(jR)$ with functions $F(s)$ analytic in $Re(s) > 0$.

$$\|F\|_2^2 := \sup_{\sigma > 0}\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}[F^*(\sigma + j\omega)F(\sigma + j\omega)]d\omega\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}[F^*(j\omega)F(j\omega)]d\omega.$$

  RH_2 (real rational subspace of H_2): all strictly proper and real rational stable transfer matrices.

- $H_2^{\perp}$ Space: the orthogonal complement of H_2 in L_2, i.e., the (closed) subspace of functions in L_2 that are analytic in $Re(s)<0$.

  $RH_2^{\perp}$ (the real rational subspace of $H_2^{\perp}$): all strictly proper rational antistable transfer matrices.

- *Parseval's relations*: (between time domain and frequency domain)

$$L_2(-\infty, \infty) \cong L_2(jR) \qquad L_2[0, \infty) \cong H_2 \qquad L_2(-\infty, 0] \cong H_2^{\perp}$$

$$\|G\|_2 = \|g\|_2 \qquad \text{where} \quad G(s) = L[g(t)] \in L_2(jR)$$

L_∞ and H_∞ Spaces

- $L_\infty(jR)$ Space: $L_\infty(jR)$ or simply L_∞ is a Banach space of matrix-valued (or scalar-valued) functions that are (essentially) bounded on jR, with norm

$$\|F\|_\infty := ess \sup_{\omega \in R} \bar{\sigma}[F(j\omega)].$$

  $RL_\infty(jR)$ or simply RL_∞: all proper and real rational transfer matrices with no poles on the imaginary axis.

- H_∞ Space: H_∞ is a (closed) subspace of L_∞ with functions that are analytic and bounded in the open right-half plane. The H_∞ norm is defined as

$$\|F\|_\infty := \sup_{Re(s)>0} \bar{\sigma}[F(s)] = \sup_{\omega \in R} \bar{\sigma}[F(j\omega)].$$

  The second equality can be regarded as a generalization of the maximum modulus theorem for matrix functions. See Boyd and Desoer [1985] for a proof.

  RH_∞: all proper and real rational stable transfer matrices.

L_∞ and H_∞ Spaces

- H_∞^- Space: H_∞^- is a (closed) subspace of L_∞ with functions that are analytic and bounded in the open left-half plane. The H_∞^- norm is defined as

$$\|F\|_\infty := \sup_{Re(s)<0} \bar{\sigma}[F(s)] = \sup_{\omega \in R} \bar{\sigma}[F(j\omega)].$$

  RH_∞^- : all proper real rational antistable transfer matrices.

  Examples: H_2 functions: 1/s+1, e^{-hs}/s+2, …

  H_∞ functions: 5, 1/s+1, 5s+1/s+2, e^{-hs}/s+2, 1/s+1+0.1e^{-hs}, …

  L_∞ functions: 5, 1/s+1,1/(s+1)(s-2), 1/s-1+0.1e^{-hs}

H_∞ Norm: Induced H_2 Norm

- Let $G(s)$ be a $p \times q$ transfer matrix. Then a *multiplication operator* is defined as $M_G: L_2 \rightarrow L_2$, $M_G f = Gf$

  Then

$$\|M_G\| = \sup_{f \in L_2} \frac{\|Gf\|_2}{\|f\|_2} = \|G\|_\infty.$$

  Proof: It is clear that $||G||_\infty$ is the upper bound:

$$\|Gf\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} f^*(j\omega)G^*(j\omega)G(j\omega)f(j\omega)d\omega$$

$$\leq \|G\|_\infty^2 \frac{1}{2\pi}\int_{-\infty}^{\infty} \|f(j\omega)\|^2 d\omega = \|G\|_\infty^2 \|f\|_2^2.$$

  To show that $||G||_\infty$ is the least upper bound, first choose a frequency ω_0 where $\bar{\sigma}[G(j\omega)]$ is maximum, i.e.,

$$\bar{\sigma}[G(j\omega_0)] = \|G\|_\infty$$

and denote the singular value decomposition of $G(j\omega_0)$ by

$$G(j\omega_0) = \overline{\sigma} u_1(j\omega_0) v_1^*(j\omega_0) + \sum_{i=2}^{r} \sigma_i u_i(j\omega_0) v_i^*(j\omega_0)$$

where r is the rank of $G(j\omega_0)$ and u_i, v_i have unit length.

If $\omega_0 < \infty$, write $v_1(j\omega_0)$ as

$$v_1(j\omega_0) = \begin{bmatrix} \alpha_1 e^{j\theta_1} \\ \alpha_2 e^{j\theta_2} \\ \vdots \\ \alpha_q e^{j\theta q} \end{bmatrix}$$

where $\alpha_i \in R$ is such that $\theta_i \in (-\pi, 0]$. Now let $0 \le \beta_i \le \infty$ be such that

$$\theta_i = \angle\left(\frac{\beta_i - j\omega_0}{\beta_i + j\omega_0}\right)$$

(with $\beta_i = \infty$ if $\theta_i = 0$) and let f be given by

$$f(s) = \begin{bmatrix} \alpha_1 \frac{\beta_1 - s}{\beta_1 + s} \\ \alpha_2 \frac{\beta_2 - s}{\beta_2 + s} \\ \vdots \\ \alpha_q \frac{\beta_q - s}{\beta_q + s} \end{bmatrix} \hat{f}(s)$$

(with 1 replacing $\frac{\beta_i - s}{\beta_i + s}$ if $\theta_i = 0$) where a scalar function is chosen so that

$$\left|\hat{f}(j\omega)\right| = \begin{cases} c, & \text{if } |\omega - \omega_0| < \varepsilon \text{ or } |\omega + \omega_0| < \varepsilon \\ 0, & \text{otherwise} \end{cases}$$

where ε is a small positive number and c is chosen so that f(s) has unit 2-norm, i.e., $c = \sqrt{\pi / 2\varepsilon}$. This in turn implies that f has unit 2-norm.

$$\|Gf\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} f^*(j\omega) G^*(j\omega) G(j\omega) f(j\omega) d\omega$$

$$\approx \overline{\sigma}\left[G(-j\omega_0)\right]^2 \frac{1}{2\pi} \int_{-\omega_0-\varepsilon}^{-\omega_0+\varepsilon} \hat{f}^*(j\omega) \hat{f}(j\omega) d\omega$$

$$+ \overline{\sigma}\left[G(j\omega_0)\right]^2 \frac{1}{2\pi} \int_{\omega_0-\varepsilon}^{\omega_0+\varepsilon} \hat{f}^*(j\omega) \hat{f}(j\omega) d\omega$$

$$\approx \frac{1}{2\pi}\left[\overline{\sigma}\left[G(-j\omega_0)\right]^2 \pi + \overline{\sigma}\left[G(j\omega_0)\right]^2 \pi\right] = \overline{\sigma}\left[G(j\omega_0)\right]^2 = \|G\|_\infty^2 .$$

Similarly, if $\omega_0 = \infty$, the conclusion follows by letting $\omega_0 \to \infty$ in the above.

Computing L_2 and H_2 Norms

- Let $G(s) \in L_2$ and $g(t) = L^{-1}[G(s)]$. Then

$$\|G\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}\{G^*(j\omega) G(j\omega)\} d\omega = \frac{1}{2\pi j} \oint \text{trace}\{G^T(-s) G(s)\} ds.$$

$$= \sum \begin{matrix}\text{the residues of trace}\{G^T(-s)G(s)\} \\ \text{at its polesin theleft half plane.}\end{matrix}$$

$$= \int_{-\infty}^{\infty} \text{trace}\{g^*(t) g(t)\} dt = \|g(t)\|_2^2$$

Consider $G(s) = C(sI-A)^{-1}B \in RH_2$. Then we have

$$\|G(s)\|_2^2 = \text{trace}(B^* L_0 B) = \text{trace}(C L_c C^*)$$

where L_0 and L_c are observability and controllability Gramians:

$$AL_c + L_c A^* + BB^* = 0 \qquad A^* L_0 + L_0 A + C^* C$$

- Proof: Note that $g(t) = L^{-1}[G(s)] = Ce^{At}B,\ t \ge 0$, and

$$L_o = \int_0^\infty e^{A^* t} C^* C e^{At} \, dt, \qquad L_c = \int_0^\infty e^{At} BB^* e^{A^* t} \, dt$$

Then

$$\|G\|_2^2 = \int_0^\infty \text{trace}\{g^*(t) g(t)\} dt = \int_0^\infty \text{trace}\{B^* e^{A^* t} C^* C e^{At} B\} dt$$

$$= \text{trace}\left\{B^* \int_0^\infty e^{A^* t} C^* C e^{At} \, dt \, B\right\} = \text{trace}\{B^* L_o B\}$$

$$= \int_0^\infty \text{trace}\{g(t) g^*(t)\} dt = \text{trace}\left\{C \int_0^\infty e^{At} BB^* e^{A^* t} \, dt \, C^*\right\} = \text{trace}\{C L_c C^*\}$$

Computing L_2 and H_2 Norms

- Hypothetical input-output experiments:

Apply the impulsive input $\delta(t)e_i$ ($\delta(t)$ is the unit impulse and e_i is the i^{th} standard basis vector) and denote the output by $z_i(t) (= g(t)e_i)$. Then $z_i \in L_{2+}$ (assuming $D = 0$) and

$$\|G\|_2^2 = \sum_{i=1}^{m} \|z_i\|_2^2$$

Can be used for nonlinear time varying systems.

- Example: Consider a transfer matrix

$$G = \begin{bmatrix} \frac{3(s+3)}{(s-1)(s+2)} & \frac{2}{s-1} \\ \frac{s+1}{(s+2)(s+3)} & \frac{1}{s-4} \end{bmatrix} = G_s + G_u$$

with

$$G_s = \left[\begin{array}{cc|cc} -2 & 0 & -1 & 0 \\ 0 & -3 & 2 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right], \quad G_u = \left[\begin{array}{cc|cc} 1 & 0 & 4 & 2 \\ 0 & 4 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right].$$

Then the command norm(G_s) gives $\|G_s\|_2 = 0.6055$ and norm(G_u(-s)) gives $\|G_u\|_2 = 3.182$. Hence

$$\|G\|_2 = \sqrt{\|G_s\|_2^2 + \|G_u\|_2^2} = 3.2393$$

>> P = gram(A,B); Q = gram(A′,C′); or P = lyap(A,B*B′);

>> [Gs,Gu] = stabsep(G); % decompose into stable and antistable parts.

Computing L_∞ and H_∞ Norms

- **Rational Functions: Let $G(s) \in RL_\infty$:**
  - the farthest distance the Nyquist plot of G from the origin $\|G\|_\infty := \sup_{\omega \in \mathbb{R}} \bar{\sigma}[G(j\omega)]$.
  - the peak on the Bode magnitude plot
  - estimation: set up a fine grid of frequency points, $\{\omega_1, \cdots, \omega_N\}$.

$$\|G\|_\infty \approx \max_{1 \le k \le N} \bar{\sigma}\{G(j\omega_k)\}.$$

- **Characterization: Let $\gamma > 0$ and $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in RL_\infty$. Then**

$$\|G\|_\infty < \gamma \Leftrightarrow \bar{\sigma}(D) < \gamma \ \& \ H \text{ has no } j\omega \text{ eigenvalues}$$

where

$$H := \begin{bmatrix} A + BR^{-1}D^*C & BR^{-1}B^* \\ -C^*(I + DR^{-1}D^*)C & -(A + BR^{-1}D^*C)^* \end{bmatrix}$$

and $R = \gamma^2 I - D^*D$.

- **Proof: Let $\Phi(s) = \gamma^2 I - G^\sim(s)G(s)$.**

Then $\|G\|_\infty < \gamma \Leftrightarrow \Phi(j\omega) > 0, \forall\, \omega \in \mathbb{R} \cup \{\infty\} \Leftrightarrow \det \Phi(j\omega) \neq 0, \forall\, \omega \in \mathbb{R}$ since $\Phi(\infty) = R > 0$ and $\Phi(j\omega)$ is continuous. $\Leftrightarrow$ $\Phi(s)$ has no imaginary axis zero. $\Leftrightarrow$ $\Phi^{-1}(s)$ has no imaginary axis pole.

$$\Phi^{-1}(s) = \left[\begin{array}{c|c} H & \begin{bmatrix} BR^{-1} \\ -C^*DR^{-1} \end{bmatrix} \\ \hline \begin{bmatrix} R^{-1}D^*C & R^{-1}B^* \end{bmatrix} & R^{-1} \end{array}\right].$$

$\Leftrightarrow$ H has no $j\omega$ axis eigenvalues if the above realization has neither uncontrollable modes nor unobservable modes on the imaginary axis.

- **We now show that the above realization for $\Phi^{-1}(s)$ indeed has neither uncontrollable modes nor unobservable modes on the imaginary axis.**

Assume that $j\omega_0$ is an eigenvalue of H but not a pole of $\Phi^{-1}(s)$. Then $j\omega_0$ must be either an unobservable mode of $([R^{-1}D^*C \quad R^{-1}B^*], H)$ or an uncontrollable mode of $\left(H, \begin{bmatrix} BR^{-1} \\ -C^*DR^{-1} \end{bmatrix}\right)$. Suppose $j\omega_0$ is an unobservable mode of

$([R^{-1}D^*C \quad R^{-1}B^*], H)$. Then there exists an $x_0 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq 0$ such that

$$Hx_0 = j\omega_0 x_0, \ [R^{-1}D^*C \quad R^{-1}B^*]x_0 = 0. \ \Leftrightarrow$$

$$(j\omega_0 I - A)x_1 = 0, \ (j\omega_0 I + A^*)x_2 = -C^*Cx_1, \ D^*Cx_1 + B^*x_2 = 0.$$

Since A has no imaginary axis eigenvalues, we have $x_1 = 0$ and $x_2 = 0$. Contradiction!!!

Similarly, a contradiction will also be arrived if $j\omega_0$ is assumed to be an uncontrollable mode of $\left(H, \begin{bmatrix} BR^{-1} \\ -C^*DR^{-1} \end{bmatrix}\right)$.

Bisection Algorithm

- **(a) select an upper bound γ_u and a lower bound γ_l such that**

$$\gamma_l \le \|G\|_\infty \le \gamma_u$$

(b) if $(\gamma_u - \gamma_l)/\gamma_l \le$ specified level, stop; $\|G\|_\infty \approx (\gamma_u + \gamma_l)/2$. Otherwise go to next step.

(c) set $\gamma = (\gamma_l + \gamma_u)/2$;

(d) test if $\|G\|_\infty < \gamma$ by calculating the eigenvalues of H with this γ;

(e) if H has an eigenvalue on $j\mathbb{R}$ set $\gamma_l = \gamma$; otherwise set $\gamma_u = \gamma$; go back to step (b).

- **In all the subsequent discussions, WLOG we can assume $\gamma = 1$ by a suitable scaling since $\|G\|_\infty < \gamma \Leftrightarrow \|\gamma^{-1}G\|_\infty < 1$.**

Estimating the H_∞ Norm

- **Estimating the H_∞ norm experimentally: the maximum magnitude of the steady-state response to all possible unit amplitude sinusoidal input signals.**

$z(t) = |G(j\omega)| \sin(\omega t + \angle G(j\omega))$ $\qquad$ $u(t) = \sin \omega t$

$$z \longleftarrow \boxed{G(s)} \longleftarrow u$$

Let the sinusoidal input u(t) as shown below. Then the steady-state response of the system can be written as

$$u(t) = \begin{bmatrix} u_1 \sin(\omega_0 t + \phi_1) \\ u_2 \sin(\omega_0 t + \phi_2) \\ \vdots \\ u_q \sin(\omega_0 t + \phi_q) \end{bmatrix}, \quad y(t) = \begin{bmatrix} y_1 \sin(\omega_0 t + \theta_1) \\ y_2 \sin(\omega_0 t + \theta_2) \\ \vdots \\ y_p \sin(\omega_0 t + \theta_p) \end{bmatrix}, \quad \hat{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_q \end{bmatrix}, \quad \hat{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix}$$

for some y_i, $i = 1,2,\ldots,p$, and furthermore,

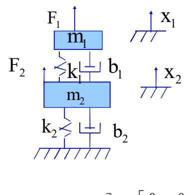
where $\|\bullet\|$ is the Euclidean norm.

$$\|G\|_\infty = \sup_{\phi_i, \omega_0, \hat{u}} \frac{\|\hat{y}\|}{\|\hat{u}\|}$$

Examples

- **Consider a mass/spring/damper system as shown in Figure 4.2.**

The dynamical system can be described by the following differential equations:



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + B \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{m_1} & \frac{k_1}{m_1} & -\frac{b_1}{m_1} & \frac{b_1}{m_1} \\ \frac{k_1}{m_2} & -\frac{k_1 + k_2}{m_2} & \frac{k_1}{m_2} & -\frac{b_1 + b_2}{m_2} \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}$$

Suppose that $G(s)$ is the transfer matrix from (F_1, F_2) to (x_1, x_2); that is,

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad D = 0$$

and suppose k_1=1, k_2 =4, b_1 =0.2, b_2 = 0.1, m_1=1, and m_2=2 with appropriate units.

>>G=ss(A,B,C,D);

>>norm(G,inf,0.0001) % relative error ≤0.0001

>>w=logspace(-1,1,200); %200 points between 0.1=10^{-1} and 10=10^1;

>>Gf=freqresp(G,w); %computing frequency response;

>>sigma(G,w), grid %plot both singular values and grid.

$\|G\|_\infty$=11.47=the peak of the largest singular value Bode plot in Figure 4.3.

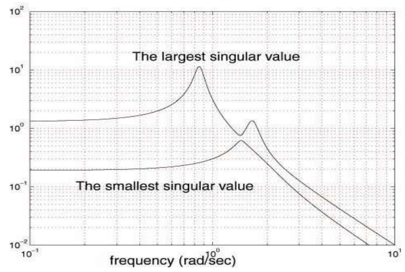


Figure 4.3: $\|G\|_\infty$ is the peak of the largest singular value of $G(j\,\omega)$

$\|G\|_\infty$ =11.47, ω_{max} = 0.8483

Since the peak is achieved at ω_{max} = 0.8483, exciting the system using the following sinusoidal input

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} 0.9614\ \sin(0.8483\,t) \\ 0.2753\ \sin(0.8483\,t - 0.12) \end{bmatrix}$$

gives the steady-state response of the system as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 11.47 \times 0.9614 \sin(0.8483t - 1.5483) \\ 11.47 \times 0.2753 \sin(0.8483t - 1.4283) \end{bmatrix}$$

This shows that the system response will be amplified 11.47 times for an input signal at the frequency ω_{max}, which could be undesirable if F_1 and F_2 are disturbance force and x_1 and x_2 are the positions to be kept steady.

- Example 2: Consider a two-by-two transfer matrix

$$G(s) = \begin{bmatrix} \dfrac{10(s+1)}{s^2+0.2s+100} & \dfrac{1}{s+1} \\ \dfrac{s+2}{s^2+0.1s+10} & \dfrac{5(s+1)}{(s+2)(s+3)} \end{bmatrix}$$

A state-space realization of G can be obtained by using the following MATLAB commands:

>>G11=tf([10,10],[1,0.2,100]);

>>G12=tf(1,[1,1]);

>>G21=tf([1,2],[1,0.1,10]);

>>G22=tf([5,5],[1,5,6]);

>>G=[G11,G21;G12,G22];

Next, we setup a frequency grid to compute the frequency response of G and the singular values of $G(j\omega)$ over a suitable range of frequency.

>>w = logspace(0,2,200); % 200 points between 1=10^0 and 100=10^2;

>>sigma(G,w), grid %plot both singular values and grid;

The singular values of G(j ω) are plotted in Figure 4.4, which gives an estimate of $\|G\|_\infty \approx$32.861. The state-space bisection algorithm described previously leads to $\|G\|_\infty$ = 50.25±0.01 and the corresponding MATLAB command is

>>norm(G,inf,0.0001) % relative error ≤0.0001.

The preceding computational results show clearly that the graphical method can lead to a wrong answer for a lightly damped system if the frequency grid is not sufficiently dense. Indeed, we would get $\|G\|_\infty \approx$43.525, 48.286 and 49.737 from the graphical method if 400,800, and 1600 frequency points are used respectively.

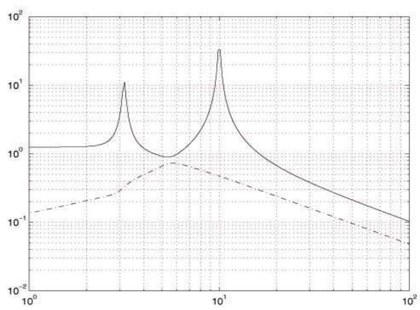


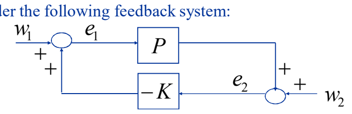
Figure 4.4 The largest and the smallest singular values of $G(j\,\omega)$

Chapter 5: Internal Stability

- Internal stability
- Coprime factorization over RH_∞

Internal Stability

Consider the following feedback system:



- well-posed if $I+K(\infty)P(\infty)$ is invertible.
- **Internal Stability:** if

$$\begin{bmatrix} I & K \\ -P & I \end{bmatrix}^{-1} = \begin{bmatrix} (I+KP)^{-1} & -K(I+PK)^{-1} \\ P(I+KP)^{-1} & (I+PK)^{-1} \end{bmatrix} \in RH_\infty$$

- Need to check all Four transfer matrices. For example,

$$P = \frac{s-1}{s+1}, \quad K = \frac{1}{s-1}. \qquad \begin{bmatrix} I & K \\ -P & I \end{bmatrix}^{-1} = \begin{bmatrix} \frac{s+1}{s+2} & -\frac{s+1}{(s-1)(s+2)} \\ \frac{s-1}{s+2} & \frac{s+1}{s+2} \end{bmatrix}$$

Special Cases

- ❑ Suppose $K \in RH_\infty$. Internal stability ⇔ $P(I+KP)^{-1} \in RH_\infty$

  This is because $K \in RH_\infty$ and $P(I+KP)^{-1} \in RH_\infty$ ⇒ $KP(I+KP)^{-1} \in RH_\infty$ ⇒ $I-KP(I+KP)^{-1}=(I+KP)^{-1} \in RH_\infty$ ⇒ $K(I+PK)^{-1} \in RH_\infty$

  and $P(I+KP)^{-1} \in RH_\infty$ ⇒ $(I+PK)^{-1} = I-P(I+KP)^{-1}K \in RH_\infty$

- ❑ Suppose $P \in RH_\infty$. Internal stability ⇔ $K(I+PK)^{-1} \in RH_\infty$
- ❑ Suppose $P, K \in RH_\infty$. Internal stability ⇔ $(I+PK)^{-1} \in RH_\infty$ ⇔ $P(I+KP)^{-1} \in RH_\infty$ ⇔ $K(I+PK)^{-1} \in RH_\infty$
- ❑ Suppose no unstable pole-zero cancellation in PK. (#PK=#P+#K)

  Internal stability ⇔ $(I+PK)^{-1} \in RH_\infty$.

(note [1,0] [1;1/s-1]=1 but no unstable pole-zero cancellation)

Example

Let P and K be two-by-two transfer matrices

$$P = \begin{bmatrix} \frac{1}{s-1} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix}, \quad K = \begin{bmatrix} \frac{s-1}{s+1} & 1 \\ 0 & 1 \end{bmatrix}.$$

Then

$$PK = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s-1} \\ 0 & \frac{1}{s+1} \end{bmatrix}, \quad (I+PK)^{-1} = \begin{bmatrix} \frac{s+1}{s+2} & -\frac{(s+1)^2}{(s-1)(s+2)^2} \\ 0 & \frac{s+1}{s+2} \end{bmatrix}$$

So the closed-loop system is not stable even though

$$\det(I+PK) = \frac{(s+2)^2}{(s+1)^2}$$

has no zero in the closed right-half plane and the number of unstable poles of $PK=n_k+n_p=1$. Hence, in general, $det(I+PK)$ having no zeros in the closed right-half plane does not necessarily imply $(I+PK)^{-1} \in RH_\infty$.

Coprime Factorization

- **Two polynomials $m(s)$ and $n(s)$ are coprime if the only common factors are constants.**
- **Two transfer functions $m(s)$ and $n(s)$ in RH_∞ are coprime if the only common factors are stable and invertible transfer functions (units): I.e. $h, mh^{-1}, nh^{-1} \in RH_\infty$ ⇒ $h^{-1} \in RH_\infty$**

  Equivalent, there exists $x,y \in RH_\infty$ such that

  $$xm+yn=1.$$

- **Matrices M and N in RH_∞ are right coprime if there exist matrices X_r and Y_r in RH_∞ such that**

  $$\begin{bmatrix} X_r & Y_r \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} = X_r M + Y_r N = I.$$

- **Matrices $\tilde{M}$ and $\tilde{N}$ in RH_∞ are left coprime if there exist matrices X_l and Y_l in RH_∞ such that**

  $$\begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix} \begin{bmatrix} X_l \\ Y_l \end{bmatrix} = \tilde{M} X_l + \tilde{N} Y_l = I.$$

State Space Formula

- Let $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a stabilizable and detectable realization, and let F and L be such that $A+BF$ and $A+LC$ are both stable.

  Define

  $$\begin{bmatrix} M & -Y_l \\ N & X_l \end{bmatrix} = \left[\begin{array}{c|cc} A+BF & B & -L \\ \hline F & I & 0 \\ C+DF & D & I \end{array}\right] \qquad \begin{bmatrix} X_r & Y_r \\ -\tilde{N} & \tilde{M} \end{bmatrix} = \left[\begin{array}{c|cc} A+LC & -(B+LD) & L \\ \hline F & I & 0 \\ C & -D & I \end{array}\right].$$

  Then

  $$\begin{bmatrix} X_r & Y_r \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & -Y_l \\ N & X_l \end{bmatrix} = I$$

- Hence $P = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ are rcf and lcf, respectively.

Closed-loop Stability

- Let $P = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ and $-K = UV^{-1} = \tilde{V}^{-1}\tilde{U}$ be rcf and lcf, respectively. Then the following conditions are equivalent:

1. The feedback system is internally stable.
2. $\begin{bmatrix} M & U \\ N & V \end{bmatrix}$ is invertible in RH_∞ 3. $\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix}$ is invertible in RH_∞
4. $\tilde{M}V - \tilde{N}U$ is invertible in RH_∞ 5. $\tilde{V}M - \tilde{U}N$ is invertible in RH_∞

$$\begin{bmatrix} I & K \\ -P & I \end{bmatrix}^{-1} = \begin{bmatrix} I & -UV^{-1} \\ -NM^{-1} & I \end{bmatrix}^{-1} = \begin{bmatrix} -M & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} M & U \\ N & V \end{bmatrix}^{-1} \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}$$

$$\begin{bmatrix} I & K \\ -P & I \end{bmatrix}^{-1} = \begin{bmatrix} I & -\tilde{V}^{-1}\tilde{U} \\ -\tilde{M}^{-1}\tilde{N} & I \end{bmatrix}^{-1} = \begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{V} & 0 \\ 0 & \tilde{M} \end{bmatrix}$$

$$\begin{bmatrix} \tilde{V} & -\tilde{U} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & U \\ N & V \end{bmatrix} = \begin{bmatrix} \tilde{V}M - \tilde{U}N & \tilde{V}U - \tilde{U}V \\ -\tilde{N}M + \tilde{M}N & -\tilde{N}U + \tilde{M}V \end{bmatrix} = \begin{bmatrix} \tilde{V}M - \tilde{U}N & 0 \\ 0 & -\tilde{N}U + \tilde{M}V \end{bmatrix}$$

Example

Let $P(s) = \frac{s-2}{s(s+3)}$ and $\alpha = (s+1)(s+3)$. Then $P(s) = n(s)/m(s)$ with

$$n(s) = \frac{s-2}{(s+1)(s+3)} \text{ and } m(s) = \frac{s}{(s+1)}$$

forms a coprime factorization. To find an $x(s)$ and a $y(s)$ such that $x(s)n(s) + y(s)m(s) = 1$, consider a stabilizing controller for P:

$$K = \frac{s-1}{s+10}.$$

Then $-K = u/v$ with $u = -K$ and $v = 1$ is a coprime factorization and

$$m(s)v(s) - n(s)u(s) = \frac{(s+11.7085)(s+2.214)(s+0.077)}{(s+1)(s+3)(s+10)} =: \beta(s)$$

Then we can take

$$x(s) = -u(s)/\beta(s) = \frac{(s-1)(s+1)(s+3)}{(s+11.7085)(s+2.214)(s+0.077)}$$

$$y(s) = v(s)/\beta(s) = \frac{(s+1)(s+3)(s+10)}{(s+11.7085)(s+2.214)(s+0.077)}$$

MATLAB programs can be used to find the appropriate F and L matrices in state-space so that the desired coprime factorization can be obtained. Let $A \in \mathbf{R}^{n \times n}$, $B \in \mathbf{R}^{n \times m}$ and $C \in \mathbf{R}^{p \times n}$. Then an F and an L can be obtained from

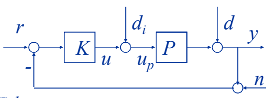
- **F = -lqr(A,B, eye(n), eye(m));** % or
- **F = -place(A, B, Pf);** % Pf= poles of A+BF
- **L = -lqr(A,C,eye(n), eye(p));** % or
- **L = -place(A,C,Pl);** % Pl=poles of A+LC

Chapter 6: Perf Specs & Lim

- **Feedback Properties**
- **Weighted and Performance**
- **Selection of Weighting Performance**
- **Bode's Gain and Phase Relation**
- **Bode's Sensitivity Integral**
- **Analyticity Constraints**

Feedback Properties

Consider a feedback system



and define

$S_i=(I+KP)^{-1}$, $S_o=(I+PK)^{-1}$

$T_i=I-S_i=KP(I+KP)^{-1}$, $T_o=I-S_o=PK(I+PK)^{-1}$

$Y=T_o(r-n)+S_oPd_i+S_od$

$U=KS_o(r-n)-KS_od-T_id_i$ $U_p=KS_o(r-n)-KS_od+S_id_i$

- **Disturbance rejection at the plant output (low**

$$\bar{\sigma}(S_0) = \bar{\sigma}((I+PK)^{-1}) = \frac{1}{\underline{\sigma}(I+PK)} \; (<<1)$$

$$\bar{\sigma}(S_0P) = \bar{\sigma}((I+PK)^{-1}P) = \bar{\sigma}(PS_i) \; (<<1)$$

- **Disturbance rejection at the plant input (low frequency):**

$$\bar{\sigma}(S_i) = \bar{\sigma}((I+KP)^{-1}) = \frac{1}{\underline{\sigma}(I+KP)} \; (<<1)$$

$$\bar{\sigma}(S_iK) = \bar{\sigma}(K(I+PK)^{-1}) = \bar{\sigma}(KS_0) \; (<<1)$$

- **Sensor noise rejection and robust stability (high frequency):**

$$\bar{\sigma}(T_0) = \bar{\sigma}(PK(I+PK)^{-1}) \; (<<1)$$

Note that

$$\bar{\sigma}(S_0) << 1 \Leftrightarrow \underline{\sigma}(PK) >> 1$$
$$\bar{\sigma}(S_i) << 1 \Leftrightarrow \underline{\sigma}(KP) >> 1$$
$$\bar{\sigma}(T_0) << 1 \Leftrightarrow \bar{\sigma}(PK) << 1$$

Now suppose *P* and *K* are invertible, then

$$\underline{\sigma}(PK) >> 1 \text{ or } \underline{\sigma}(KP) >> 1$$

$$\Leftrightarrow \begin{cases} \bar{\sigma}(S_0P) = \bar{\sigma}((I+PK)^{-1}P) \approx \bar{\sigma}(K^{-1}) = \dfrac{1}{\underline{\sigma}(K)} \\ \bar{\sigma}(KS_0) = \bar{\sigma}(K(I+PK)^{-1}) \approx \bar{\sigma}(P^{-1}) = \dfrac{1}{\underline{\sigma}(P)} \end{cases}$$

Desired Loop Shape

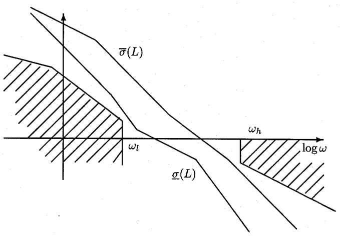
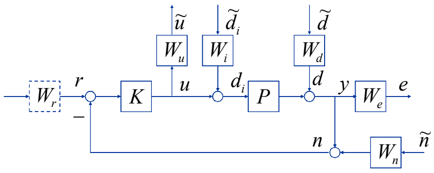


Figure 6.2: Desired loop gain

Weighted H_2 and H_∞



H_2 Performance: Assume $d_1(t)=\eta\delta(t)$ and $E(\eta\eta^*) = I$

Minimize the expected energy of the error e:

[redacted]

Include the control signal u in the cost function:

$$E\left\{\|e\|_2^2 + \rho^2\|\tilde{u}\|_2^2\right\} = \left\| \begin{matrix} W_e S_o W_d \\ \rho W_u K S_o W_d \end{matrix} \right\|_2^2$$

Robustness problem????

H_∞ Performance: under worst possible case

$$\sup_{\|\tilde{d}\|_2 \le 1} \|e\|_2 = \|W_e S_o W_d\|_\infty$$

restrictions on the control energy or control bandwidth:

$$\sup_{\|\tilde{d}\|_2 \le 1} \|\tilde{u}\|_2 = \|W_u K S_o W_d\|_\infty$$

Combined cost:

$$\sup_{\|\tilde{d}\|_2 \le 1} \left\{\|e\|_2^2 + \rho^2\|\tilde{u}\|_2^2\right\} = \left\| \begin{bmatrix} W_e S_o W_d \\ \rho W_u K S_o W_d \end{bmatrix} \right\|_\infty^2$$

Bode's Gain and Phase

L stable and minimum phase:

$$\angle L(j\omega_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d\ln|L|}{dv} \ln\coth\frac{|v|}{2} dv \quad v := \ln(\omega/\omega_0)$$

$$|1+L(j\omega_c)| = |1+L^{-1}(j\omega_c)| = 2\left|\sin\frac{\pi + \angle L(j\omega_c)}{2}\right|,$$

which must not be too small for good stability robustness. It is important to keep the slope of L near the crossover frequency not much smaller than -1 for a reasonably wide range of frequencies in order to guarantee some reasonable performance.

L stable and nonminimum phase with RHP zeros: $z_1, \ldots, z_k$:

$$L(s) = \frac{-s+z_1}{s+z_1}\frac{-s+z_2}{s+z_2}\cdots\frac{-s+z_k}{s+z_k} L_{mp}(s)$$

$$\angle L(j\omega_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d\ln|L|}{dv} \ln\coth\frac{|v|}{2} dv + \sum_{i=1}^{k} \angle \frac{-j\omega_0 + z_i}{j\omega_0 + z_i},$$

In this case we conclude that the crossover frequency must satisfy the following in order to guarantee the closed-loop stability and some reasonable closed-loop performance.

$$\omega_c < \begin{cases} |z|/4, \Re(z) >> \Im(z) \\ |z|/3, \Re(z) \approx \Im(z) \\ |z|, \Re(z) << \Im(z) \end{cases}$$

Bode's Sensitivity Integral

Let $p_1, p_2, \ldots, p_m$ be the open right-half plane poles of L

$$\int_0^\infty \ln|S(j\omega)|\, d\omega = \pi\sum_{i=1}^{m}\Re(p_i)$$

In the case where L is stable, the integral simplifies to

$$\int_0^\infty \ln|S(j\omega)|\, d\omega = 0$$

Analyticity Constraints

Let $p_1, p_2, \ldots, p_m$ and $z_1, z_2, \ldots, z_k$ be the open right-half plane poles and zeros of L, respectively.

$$S(p_i) = 0,\ T(p_i) = 1,\ i = 1, 2, \ldots, m$$

and

$$S(z_j) = 1,\ T(z_j) = 0,\ j = 1, 2, \ldots, k$$

Suppose $S=(I+L)^{-1}$ and $T=L(I+L)^{-1}$ are stable. Then $p_1, p_2, \ldots, p_m$ are the right-half plane zeros of S and $z_1, z_2, \ldots, z_k$ are the right-half plane zeros of T. Let

$$B_p(s) = \prod_{i=1}^{m} \frac{s - p_i}{s + p_i}, \quad B_z(s) = \prod_{i=1}^{k} \frac{s - z_j}{s + z_j}$$

Then $|B_p(j\omega)| = 1$ and $|B_z(j\omega)| = 1$ for all frequencies and, moreover,

$$B_p^{-1}(s)S(s) \in RH_\infty,\ B_z^{-1}(s)T(s) \in RH_\infty$$

Hence, by the maximum modulus theorem, we have

$$\|S(s)\|_\infty = \|B_p^{-1}(s)S(s)\|_\infty \geq |B_p^{-1}(z)S(z)| = |B_p^{-1}(z)|$$

for any z with z>0. Let z be a right-half plane zero of L; then

$$\|S(s)\|_\infty \geq |B_p^{-1}(z)| = \prod_{i=1}^{m} \left| \frac{z + p_i}{z - p_i} \right|$$