Institute of Systems Science and Intelligent Control Technology
系统科学与智能控制技术研究所

鲁棒控制：建模、跟踪、抗扰、容错

周克敏
山东科技大学
电气与自动化工程学院
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（第十二讲）

提纲

1. 古典控制基础
2. 鲁棒控制理论基础
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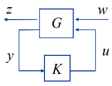
Chapter 11: Controller Parameterization

- Consider again the general framework with

$$G(s)=\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]=\begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix}.$$

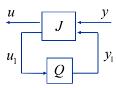


- Youla parameterization: *all controllers K that internally stabilize G.*
- Suppose $G\in RH_\infty$. Then $K=Q(I+G_{22}Q)^{-1}$, $Q\in RH_\infty$ and $I+G_{22}Q(\infty)$ nonsingular.
- Proof: K stabilizes a stable plant G_{22} iff $K(I-G_{22}K)^{-1}$ is stable. So the conclusion follows by letting $Q=K(I-G_{22}K)^{-1}$.

$$\begin{bmatrix} (I-G_{22}K)^{-1} & G_{22}(I-G_{22}K)^{-1} \\ K(I-G_{22}K)^{-1} & (I-KG_{22})^{-1} \end{bmatrix}$$

- General Case: Let F and L be such that $A+LC_2$ and $A+B_2F$ are stable. Then $K=F_l(J,Q)$:

$$J=\left[\begin{array}{c|cc} A+B_2F+LC_2+LD_{22}F & -L & B_2+LD_{22} \\ \hline F & 0 & I \\ -(C_2+D_{22}F) & I & -D_{22} \end{array}\right]$$



with any $Q\in RH_\infty$ and $I+D_{22}Q(\infty)$ nonsingular.

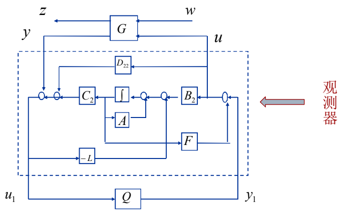


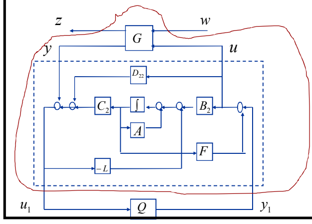
Figure 11.3 Structure of Stabilizing Controllers

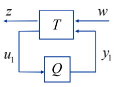
- Closed-loop Matrix:

$$F_l(G,K)=F_l(G,F_l(J,Q))=F_l(T,Q)$$
$$=\{T_{11}+T_{12}QT_{21}:\ Q\in RH_\infty \text{ and } I+G_{22}Q(\infty) \text{ nonsingular}\}.$$

where T ($T_{22}=0$) is given by

$$T=\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}=\left[\begin{array}{cc|cc} A+B_2F & -B_2F & B_1 & B_2 \\ 0 & A+LC_2 & B_1+LD_{21} & 0 \\ \hline C_1+D_{12}F & -D_{12}F & D_{11} & D_{12} \\ 0 & C_2 & D_{21} & 0 \end{array}\right].$$





$T_{22}=0$

- Coprime Factorization Approach: Let $G_{22}=NM^{-1}=\tilde{M}^{-1}\tilde{N}$ be rcf and lcf of G_{22} over RH_∞ respectively. And let them satisfy the Bezout identity:

$$\begin{bmatrix} \tilde{V}_0 & -\tilde{U}_0 \\ -\tilde{N} & \tilde{M} \end{bmatrix}\begin{bmatrix} M & U_0 \\ N & V_0 \end{bmatrix}=\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

$$\begin{bmatrix} M & -Y_l \\ N & X_l \end{bmatrix}=\left[\begin{array}{c|cc} A+BF & B & -L \\ \hline F & I & 0 \\ C+DF & D & I \end{array}\right]$$

$$\begin{bmatrix} X_r & Y_r \\ -\tilde{N} & \tilde{M} \end{bmatrix}=\left[\begin{array}{c|cc} A+LC & -(B+LD) & L \\ \hline F & I & 0 \\ C & -D & I \end{array}\right].$$

Then

$$\begin{aligned} K &= (U_0+MQ_y)(V_0+NQ_y)^{-1} \\ &= (\tilde{V}_0+Q_y\tilde{N})^{-1}(\tilde{U}_0+Q_y\tilde{M}) \\ &= F_\ell(J_y,Q_y),\ Q_y\in RH_\infty \end{aligned}$$

where

$$J_y:=\begin{bmatrix} U_0V_0^{-1} & \tilde{V}_0^{-1} \\ V_0^{-1} & -V_0^{-1}N \end{bmatrix}==\left[\begin{array}{c|cc} A+BF+LC+LDF & -L & B+LD \\ \hline F & 0 & I \\ -(C+DF) & I & -D \end{array}\right]$$

and $(I+V_0^{-1}NQ_y)(\infty)$ is invertible

Chapter 12: Algebraic Riccati Equations

- Algebraic Riccati Equation
- Solving ARE
- Bounded Real Lemma
- Standard ARE

Algebraic Riccati Equations

- Algebraic Riccati Equation:

  $A^*X+XA+XRX+Q=0, \quad R=R^*,\ Q=Q^*$

  An ARE may have many solutions. We are only interested in symmetric solutions. In particular, we are interested in the symmetric solution such that $A+RX$ is stable. This solution is called stabilizing solution.

  Consider the associated Hamiltonian matrix:

  $$H:=\begin{bmatrix} A & R \\ -Q & -A^* \end{bmatrix}$$

  Then

  $$J^{-1}HJ=-JHJ=-H^*, J:=\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$$

  so H and $-H^*$ are similar. Thus λ is an eigenvalue iff $-\lambda^*$ is.
- Thus we conclude: eig(H) ≠ jω ⇔ H has n eigenvalues in Re s<0 and n eigenvalues in Re s>0.

Solving ARE

Let $X\_(H)$ be the n-dimensional spectral subspace corresponding to eigenvalues in Re s <0:

$$X\_(H)=\operatorname{Im}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

where $X_1, X_2 \in C^{n\times n}$ (X_1 and X_2 can be chosen to be real matrices.)
If X_1 is nonsingular, define

$X:=Ric(H)=X_2X_1^{-1}$: $dom(Ric) \subset R^{2n\times 2n} \rightarrow R^{n\times n}$

where $dom(Ric)$ consists of all H matrices such that

- H has no eigenvalues on the imaginary axis
- $X\_(H), \operatorname{Im}\begin{bmatrix} 0 \\ I \end{bmatrix}$ are complementary (or X_1 is nonsingular.)

Then X is a solution of the ARE. (see next theorem)

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>>[X1,X2] = ric_schr(H), X = X2/X1
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- Theorem 12.1: Suppose $H \in dom(Ric)$ and $X=Ric(H)$. Then

  (i) X is real symmetric

  (ii) X satisfies the algebraic Riccati equation

  $A^*X+XA+XRX+Q=0$

  (iii) $A+RX$ is stable.
- Proof: (i) Let $X\_(H)=\operatorname{Im}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$. We show $X_1^*X_2$ is symmetric.

  Note that there exists a stable matrix $H\_$ in $R^{n\times n}$ such that

  $$H\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}=\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}H\_.$$

  Pre-multiply this equation by

  $$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}^* J$$

  to get

  $$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}^* JH\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}=\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}^* J\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}H\_$$

Since JH is symmetric ⇒ :

$(-X_1^*X_2+X_2^*X_1)\,H\_=H\_^*(-X_1^*X_2+X_2^*X_1)^* = -H\_^*(-X_1^*X_2+X_2^*X_1)$

This is Lyapunov equation. Since $H\_$ is stable, the unique solution is

$-X_1^*X_2+X_2^*X_1=0.$

i.e., $X_1^*X_2$ is symmetric. ⇒ $X=(X_1^{-1})^*(X_1^*X_2)X_1^{-1}$ is symmetric.

(ii) Start with the equation

$$H\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}=\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}H\_.$$

and post-multiply by X_1^{-1} to get

$$H\begin{bmatrix} I \\ X \end{bmatrix}=\begin{bmatrix} I \\ X \end{bmatrix}X_1H\_X_1^{-1}.$$

now pre-multiply by $[X \quad -I]$:

$$\begin{bmatrix} X & -I \end{bmatrix}H\begin{bmatrix} I \\ X \end{bmatrix}=0.$$

This is precisely the Riccati equation.

(iii) $$\begin{bmatrix} I & 0 \end{bmatrix}\left(H\begin{bmatrix} I \\ X \end{bmatrix}=\begin{bmatrix} I \\ X \end{bmatrix}X_1H\_X_1^{-1}\right) \Rightarrow A+RX=X_1H\_X_1^{-1}.$$

Thus $A + RX$ is stable because $H\_$ is.

- Theorem 12.2: Suppose eig(H) ≠ jω and R is semi-definite (≥ 0 or ≤0). Then $H \in dom(Ric)$ ⇔ (A, R) is stabilizable.
- Proof. (⇐) Note that

  $$X\_(H)=\operatorname{Im}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \quad H\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}=\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}H\_$$

  We need to show that X_1 is nonsingular, i.e., Ker X_1=0.

  Claim: Ker X_1 is $H\_$-invariant.

  Let $x \in$ Ker X_1 and note that $X_2^*X_1$ is symmetric and

  $AX_1+RX_2=X_1H\_$.

  Pre-multiply by $x^* X_2^*$, post multiply by x to get

  $x^* X_2^* RX_2 x=0 \Rightarrow RX_2 x=0 \Rightarrow X_1H\_ x=0$

  i.e., $H\_ x \in$ Ker X_1.

  Suppose Ker $X_1 \neq 0$. Then $H\_|_{\text{Ker } X_1}$ has an eigenvalue, λ, and a corresponding eigenvector, x:

  $H\_ x=\lambda x$, Re$\lambda<0$, $0 \neq x \in$ Ker X_1

Note that

$$-QX_1-A^*X_2=X_2H\_$$

Post-multiply the above equation by x:

$$(A^*+\lambda I)X_2x=0$$

Recall that $RX_2x = 0$, we have

$$x^*X_2^*[A+\lambda^*I\ \ R]=0.$$

(A,R) stabilizable $\Rightarrow$ $X_2x=0 \Rightarrow \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} x=0 \Rightarrow x=0$ since $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ has full column rank, which is a contradiction.

($\Rightarrow$) $H \in dom(Ric) \Rightarrow$ $A+RX$ is stable $\Rightarrow$ (A, R) is stabilizable.

Computing L_∞ and H_∞ Norms

- Rational Functions: Let $G(s) \in RL_\infty$:
  - the farthest distance the Nyquist plot of G from the origin $\|G\|_\infty := \sup_{\omega\in R} \bar{\sigma}[G(j\omega)]$.
  - the peak on the Bode magnitude plot
  - estimation: set up a fine grid of frequency points, $\{\omega_1, \cdots, \omega_N\}$.

$$\|G\|_\infty \approx \max_{1\le k\le N} \bar{\sigma}\{G(j\omega_k)\}.$$

- Characterization: Let $\gamma > 0$ and $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in RL_\infty$. Then

$$\|G\|_\infty < \gamma \Leftrightarrow \bar{\sigma}(D) < \gamma \ \&\ H \text{ has no } j\omega \text{ eigenvalues}$$

where
$$H := \begin{bmatrix} A+BR^{-1}D^*C & BR^{-1}B^* \\ -C^*(I+DR^{-1}D^*)C & -(A+BR^{-1}D^*C)^* \end{bmatrix}$$

and $R = \gamma^2 I - D^*D$.

- Proof: Let $\Phi(s) = \gamma^2 I - G^\sim(s)G(s)$.

$\|G\|_\infty < \gamma \Leftrightarrow \Phi(j\omega)>0, \forall\ \omega \in R \cup \{\infty\}$

$\Leftrightarrow$ $\det\Phi(j\omega) \neq 0, \forall\ \omega \in R$ since $\Phi(\infty)=R>0$ and $\Phi(j\omega)$ is continuous.

$\Leftrightarrow$ $\det\Phi(j\omega) \neq 0, \forall\ \omega \in R$ since $\Phi(\infty)=R>0$ and $\Phi(j\omega)$ is continuous.

$\Leftrightarrow$ $\Phi(s)$ has no imaginary axis zero.

$\Leftrightarrow$ $\Phi^{-1}(s)$ has no imaginary axis pole.

$$\Phi^{-1}(s) = \left[\begin{array}{c|c} H & \begin{bmatrix} BR^{-1} \\ -C^*DR^{-1} \end{bmatrix} \\ \hline \begin{bmatrix} R^{-1}D^*C & R^{-1}B^* \end{bmatrix} & R^{-1} \end{array}\right].$$

$\Leftrightarrow$ H has no $j\omega$ axis eigenvalue.

Bounded Real Lemma

- Corollary 12.3: Let $\gamma>0$. $G(s)=C(sI-A)^{-1}B+D \in RH_\infty$ and

$$H := \begin{bmatrix} A+BR^{-1}D^*C & BR^{-1}B^* \\ -C^*(I+DR^{-1}D^*)C & -(A+BR^{-1}D^*C)^* \end{bmatrix}$$

where $R=\gamma^2 I-D^*D$. Then the following conditions are equivalent:

(i) $\|G\|_\infty < \gamma$.

(ii) $\|D\| < \gamma$ and H has no eigenvalues on the imaginary axis.

(iii) $\|D\| < \gamma$ and $H \in dom(Ric)$, i.e., there exists an $X \ge 0$ such that

$$X(A+BR^{-1}D^*C)+(A+BR^{-1}D^*C)^*X+XBR^{-1}B^*X+C^*(I+DR^{-1}D^*)C=0$$

and $A+BR^{-1}D^*C+BR^{-1}B^*X$ is stable.

(iv) $\|D\| < \gamma$, $H \in dom(Ric)$ and $Ric(H) \ge 0$ ($Ric(H)>0$ if (C,A) is observable).

(v) $\|D\| < \gamma$ and there exists an $X \ge 0$ such that

$$X(A+BR^{-1}D^*C)+(A+BR^{-1}D^*C)^*X+XBR^{-1}B^*X+C^*(I+DR^{-1}D^*)C=0$$

and $A+BR^{-1}D^*C+BR^{-1}B^*X$ has no eigenvalues on the imaginary axis.

(vi) $\|D\| < \gamma$ and there exists an $X > 0$ such that

$$X(A+BR^{-1}D^*C)+(A+BR^{-1}D^*C)^*X+XBR^{-1}B^*X+C^*(I+DR^{-1}D^*)C<0$$

(vii) there exists and $X > 0$ such that

$$\begin{bmatrix} XA+A^*X & XB & C^* \\ B^*X & -\gamma I & D^* \\ C & D & -\gamma I \end{bmatrix} < 0.$$

- Proof: We have already known: (i) $\Leftrightarrow$ (ii). (iii) $\Rightarrow$ (ii) is obvious. To show that (ii) $\Rightarrow$ (iii), we need to show that $(A+BR^{-1}D^*C, BR^{-1}B^*)$ is stabilizable (Theorem 12.2). In fact, we will show that $A+BR^{-1}D^*C$ is stable for all those γ such that $\|G\|_\infty < \gamma$.

Note that we can write

$$A+BR^{-1}D^*C = A+B(\gamma^2 I - D^*D)^{-1}D^*C = A+B(I-\Delta D_1)^{-1}\Delta C_1$$

with $\Delta = D^*/\gamma$, $D_1 = D/\gamma$, and $C_1 = C/\gamma$. Then $\|\Delta\| < 1$ and

$$\|C_1(sI-A)^{-1}B+D_1\|_\infty = \gamma^{-1}\|G\|_\infty < 1.$$

Hence by small gain theorem, $A+B(I-\Delta D_1)^{-1}\Delta C_1$ is stable for all Δ with $\|\Delta\|<1$. Thus $A+BR^{-1}D^*C$ is stable for all γ such that $\|G\|_\infty < \gamma$.

(iii) $\Rightarrow$ (iv) follows from the fact that the ARE

$$X(A+BR^{-1}D^*C)+(A+BR^{-1}D^*C)^*X+XBR^{-1}B^*X+C^*(I+DR^{-1}D^*)C=0$$

can be regarded as a Lyapunov equation with

$$A_1 := A+BR^{-1}D^*C, \qquad Q := XBR^{-1}B^*X+C^*(I+DR^{-1}D^*)C$$

Hence $X \geq 0$ since A_1 is stable and $Q \geq 0$.

(v) $\Rightarrow$ (i): Assume $D = 0$ for simplicity. Then there is an $X \geq 0$

$$XA+A^*X+XBB^*X/\gamma^2+C^*C=0$$

and $A+BB^*X/\gamma^2$ has no $j\omega$-axis eigenvalue.

Hence

$$W(s) := \left[\begin{array}{c|c} A & -B \\ \hline B^*X/\gamma & \gamma I \end{array}\right]$$

has no zeros on the imaginary axis since

$$W^{-1}(s) = \left[\begin{array}{c|c} A+BB^*X/\gamma^2 & B/\gamma \\ \hline B^*X/\gamma^2 & I/\gamma \end{array}\right]$$

has no poles on the imaginary axis. Next, note that

$$-X(j\omega I-A)-(j\omega I-A)^*X+XBB^*X/\gamma^2 I+C^*C=0$$

Multiply $B^*\{(j\omega I-A)^*\}^{-1}$ on the left and $(j\omega I-A)^{-1}B$ on the right of the above equation to get

$$-B^*\{(j\omega I-A)^*\}^{-1}XB-B^*X(j\omega I-A)^{-1}B+B^*\{(j\omega I-A)^*\}^{-1}C^*C(j\omega I-A)^{-1}B$$

$$+B^*\{(j\omega I-A)^*\}^{-1}XBB^*X(j\omega I-A)^{-1}B/\gamma^2=0$$

Completing square, we have

$$G^*(j\omega)G(j\omega) = \gamma^2 I - W^*(j\omega)W(j\omega)$$

Since $W(s)$ has no $j\omega$-axis zeros, we conclude that $\|G\|_\infty < \gamma$.

(vi) $\Leftrightarrow$ (vii): follows from Schur complement.

(vi) $\Rightarrow$ (i): by following the similar procedure as above.

(i) $\Rightarrow$ (vi): let

$$G_\varepsilon = \left[\begin{array}{c|c} A & B \\ \hline C & D \\ \varepsilon I & 0 \end{array}\right].$$

Then there exists an $\varepsilon>0$ such that $\|G_\varepsilon\|_\infty < \gamma$. Now (vi) follows by applying part (v) to G_ε.

Standard ARE

- Theorem 12.4: Suppose H has the form

$$H = \begin{bmatrix} A & -BB^* \\ -C^*C & -A^* \end{bmatrix}$$

Then $H \in dom(Ric)$ iff (A,B) is stabilizable and (C,A) has no unobservable modes on the imaginary axis. Furthermore, $X=Ric(H)\geq 0$. And $X>0$ if and only if (C,A) has no stable unobservable modes.

- Proof: Only need to show that, assuming (A,B) is stabilizable, H has no $j\omega$ eigenvalues iff (C,A) has no unobservable modes on the imaginary axis.

Suppose that $j\omega$ is an eigenvalue and $0 \neq \begin{bmatrix} x \\ z \end{bmatrix}$ is a corresponding eigenvector. Then

$$Ax-BB^*z=j\omega x, \quad -C^*Cx-A^*z=j\omega z$$

Re-arrange: $(A-j\omega I)x=BB^*z, \quad -(A-j\omega I)^*z=C^*Cx$

Thus $\langle z,(A-j\omega I)x\rangle = \langle z,BB^*z\rangle = \|B^*z\|^2$

$-\langle x,(A-j\omega I)^*z\rangle = \langle x,C^*Cx\rangle = \|Cx\|^2$

so $\langle x,(A-j\omega I)^*z\rangle$ is real and

$$-\|Cx\|^2 = \langle (A-j\omega I)x,z\rangle = \langle z,(A-j\omega I)x\rangle^* = \|B^*z\|^2$$

Therefore $B^*z = 0$ and $Cx = 0$. So

$$(A-j\omega I)x=0, \quad (A-j\omega I)^*z=0$$

Combine the last four equations to get

$$z^*\begin{bmatrix} A-j\omega I & B \end{bmatrix}=0, \qquad \begin{bmatrix} A-j\omega I \\ C \end{bmatrix}x=0.$$

The stabilizability of (A,B) gives $z=0$. Now it is clear that $j\omega$ is an eigenvalue of H if $j\omega$ is an unobservable mode of (C,A).

$$(A-BB^*X)^*X+X(A-BB^*X)+XBB^*X+C^*C=0$$

$X\geq 0$ since $A-BB^*X$ is stable.

- Corollary 12.5: Suppose (A,B) is stabilizable and (C,A) is detectable. Then

$$A^*X+XA-XBB^*X+C^*C=0$$

has a unique positive semidefinite solution. Moreover, it is stabilizing.

- Corollary 12.7: Suppose D has full column rank and denote $R = D^*D>0$. Let H have the form

$$H = \begin{bmatrix} A & 0 \\ -C^*C & -A^* \end{bmatrix} - \begin{bmatrix} B \\ -C^*D \end{bmatrix} R^{-1}\begin{bmatrix} D^*C & B^* \end{bmatrix} = \begin{bmatrix} A-BR^{-1}D^*C & -BR^{-1}B^* \\ -C^*(I-DR^{-1}D^*)C & -(A-BR^{-1}D^*C)^* \end{bmatrix}.$$

Then $H \in dom(Ric)$ iff (A,B) is stabilizable and $\begin{bmatrix} A-j\omega I & B \\ C & D \end{bmatrix}$ has full column rank for all ω. Furthermore, $X=Ric(H)\geq 0$ if $H \in dom(Ric)$ and $\mathrm{Ker}(X)=0$ if and only if $(D_\perp^*C, A-BR^{-1}D^*C)$ has no stable unobservable modes.

- Proof: This is because $\begin{bmatrix} A-j\omega I & B \\ C & D \end{bmatrix}$ has full column rank for all ω $\Leftrightarrow$ $((I-DR^{-1}D^*)C, A-BR^{-1}D^*C)$ has no observable modes on $j\omega$-axis.

Chapter 13: H_2 Optimal Control

- ❑ *H_2* optimal control
- ❑ stability margins of *H_2* controllers

Computing L_2 and H_2 Norms

- Let $G(s) \in L_2$ and $g(t) = L^{-1}[G(s)]$. Then

$$\|G\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{trace}\{G^*(j\omega)G(j\omega)\}d\omega = \frac{1}{2\pi j}\oint \text{trace}\{G^T(-s)G(s)\}ds.$$

$$= \sum \begin{matrix}\text{the residues of trace}\{G^T(-s)G(s)\} \\ \text{at its poles in the left half plane.}\end{matrix}$$

$$= \int_{-\infty}^{\infty} \text{trace}\{g^*(t)g(t)\}dt = \|g(t)\|_2^2$$

Consider $G(s)=C(sI-A)^{-1}B \in RH_2$. Then we have

$$\|G(s)\|_2^2 = \text{trace}(B^*L_0B) = \text{trace}(CL_cC^*)$$

where L_0 and L_c are observability and controllability Gramians:

$$AL_c+L_cA^*+BB^* = 0 \qquad A^*L_0+L_0A+C^*C$$

- Proof: Note that $g(t) = L^{-1}[G(s)]=Ce^{At}B,\ t \geq 0$, and

$$L_o = \int_0^{\infty} e^{A^*t}C^*Ce^{At}\,dt\,, \qquad L_c = \int_0^{\infty} e^{At}BB^*e^{A^*t}\,dt$$

Then

$$\|G\|_2^2 = \int_0^{\infty}\text{trace}\{g^*(t)g(t)\}dt = \int_0^{\infty}\text{trace}\{B^*e^{A^*t}C^*Ce^{At}B\}dt$$

$$= \text{trace}\left\{B^*\int_0^{\infty} e^{A^*t}C^*Ce^{At}\,dt\,B\right\} = \text{trace}\{B^*L_oB\}$$

$$= \int_0^{\infty}\text{trace}\{g(t)g^*(t)\}dt = \text{trace}\left\{C\int_0^{\infty} e^{At}BB^*e^{A^*t}\,dt\,C^*\right\} = \text{trace}\{CL_cC^*\}$$

- Example: Consider a transfer matrix

$$G = \begin{bmatrix} \frac{3(s+3)}{(s-1)(s+2)} & \frac{2}{s-1} \\ \frac{s+1}{(s+2)(s+3)} & \frac{1}{s-4} \end{bmatrix} = G_s + G_u$$

with

$$G_s = \left[\begin{array}{cc|cc} -2 & 0 & -1 & 0 \\ 0 & -3 & 2 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right], \quad G_u = \left[\begin{array}{cc|cc} 1 & 0 & 4 & 2 \\ 0 & 4 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right].$$

Then the command norm(G_s) gives $\|G_s\|_2 = 0.6055$ and norm(G_u(-s)) gives $\|G_u\|_2 = 3.182$. Hence

$$\|G\|_2 = \sqrt{\|G_s\|_2^2 + \|G_u\|_2^2} = 3.2393$$

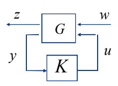
>> P = gram(A,B); Q = gram(A´,C´); or P = lyap(A,B*B´);

>> [Gs,Gu] = stabsep(G); % decompose into stable and antistable parts.

H_2 Optimal Control

- ❑ Consider a general LFT system

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right].$$



- ❑ Assumptions:

(i) (A,B_2) is stabilizable and (C_2,A) is detectable;

(ii) D_{12} has full column rank with $[D_{12}\ D_\perp]$ unitary, and D_{21} has full row rank with $\begin{bmatrix} D_{12} \\ \tilde{D}_\perp \end{bmatrix}$ unitary;

(iii) $\begin{bmatrix} A-j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω;

(iv) $\begin{bmatrix} A-j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all ω.

- ❑ *H_2* Problem: *find a stabilizing controller K that minimizes* $\|T_{zw}\|_2$

Let X_2 and Y_2 be stabilizing solutions to

$$X_2(A-B_2D_{12}^*C_1)+(A-B_2D_{12}^*C_1)^*X_2 - X_2B_2B_2^*X_2 + C_1^*D_\perp D_\perp^*C_1 = 0$$

$$Y_2(A-B_1D_{12}^*C_2)^*+(A-B_1D_{21}^*C_2)Y_2 - Y_2C_2^*C_2Y_2 + B_1\tilde{D}_\perp^*\tilde{D}_\perp B_1^* = 0$$

Define

$$F_2 := -(B_2^*X_2 + D_{12}^*C_1), \quad L_2 := -(Y_2C_2^* + B_1D_{21}^*)$$

$$G_c(s) := \left[\begin{array}{c|c} A+B_2F_2 & I \\ \hline C_1+D_{12}F_2 & 0 \end{array}\right], \quad G_f(s) := \left[\begin{array}{c|c} A+L_2C_2 & B_1+L_2D_{21} \\ \hline I & 0 \end{array}\right].$$

There exists a unique optimal controller

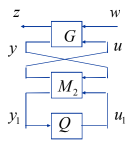
$$K_{opt}(s) := \left[\begin{array}{c|c} A+B_2F_2+L_2C_2 & -L_2 \\ \hline F_2 & 0 \end{array}\right].$$

Moreover,

$$\min \|T_{zw}\|_2^2 = \|G_cB_1\|_2^2 + \|F_2G_f\|_2^2 = \|G_cL_2\|_2^2 + \|C_1G_f\|_2^2$$

- **Proof: Note that**
  - $U := \left[\begin{array}{c|c} A+B_2F_2 & B_2 \\ \hline C_1+D_{12}F_2 & D_{12} \end{array}\right] \in RH_\infty$ is inner and $U^{\sim}G_c \in RH_2^{\perp}$.
  - $V := \left[\begin{array}{c|c} A+L_2C_2 & B_1+L_2D_{21} \\ \hline C_2 & D_{21} \end{array}\right] \in RH_\infty$ is co-inner and $G_fV^{\sim} \in RH_2^{\perp}$
  - all stabilizing controllers $K(s) = F_\ell(M_2, Q), Q \in RH_\infty$

$$M_2(s) = \left[\begin{array}{c|cc} A+B_2F_2+L_2C_2 & -L_2 & B_2 \\ \hline F_2 & 0 & I \\ -C_2 & I & 0 \end{array}\right].$$



  - Closed−loop with K

$$T_{zw} = G_cB_1 - UF_2G_f + UQV.$$

$\|T_{zw}\|_2^2 = \|G_cB_1\|_2^2 + \|F_2G_f - QV\|_2^2 = \|G_cB_1\|_2^2 + \|F_2G_f\|_2^2 + \|Q\|_2^2$

And Q = 0 gives the unique optimal control: $K=F_l(M_2,0)$.

Stability Margins of H_2 Controllers

- **LQR margin: ≥ 60º phase margin and ≥ 6dB gain margin.**
- **LQG or H_2 Controller: No guaranteed margin.**
- **Example:**

$$G(s) = \left[\begin{array}{c|cc} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} \sqrt{\sigma} & 0 \\ \sqrt{\sigma} & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \hline \begin{bmatrix} \sqrt{q} & \sqrt{q} \\ 0 & 0 \end{bmatrix} & 0 & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \end{bmatrix} & 0 \end{array}\right].$$

Then

$$X_2 = \begin{bmatrix} 2\alpha & \alpha \\ \alpha & \alpha \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 2\beta & \beta \\ \beta & \beta \end{bmatrix}$$

and

$$F_2 = -\alpha \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad L_2 = -\beta \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where

$$\alpha = 2 + \sqrt{4+q}, \quad \beta = 2 + \sqrt{4+\sigma}.$$

$$K_{opt} = \left[\begin{array}{cc|c} 1-\beta & 1 & \beta \\ -(\alpha+\beta) & 1-\alpha & \beta \\ \hline -\alpha & -\alpha & 0 \end{array}\right].$$

- **Suppose the controller implemented in the system (or plant G_{22}) is actually**

$K = kK_{opt}$,

with a nominal value k=1. Then the closed-loop system A-matrix becomes

$$\tilde{A} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -k\alpha & -k\alpha \\ \beta & 0 & 1-\beta & 1 \\ \beta & 0 & -\alpha-\beta & 1-\alpha \end{bmatrix}.$$

The characteristic polynomial has the form

$$\det(sI - \tilde{A}) = a_4s^4 + a_3s^3 + a_2s^2 + a_1s + a_0$$

With $a_1 = \alpha+\beta-4+2(k-1)\alpha\beta$, $a_0 = 1+(1-k)\alpha\beta$

- necessary for stability: $a_0>0$ and $a_1>0$.
- α>>1 and β>>1 and k≠1 ⇒ $a_0 \approx (1-k)\alpha\beta$, and $a_1 \approx 2(k-1)\alpha\beta$
- α>>1 and β>>1 (q and σ), the system is unstable for arbitrarily small perturbations in k in either direction. Thus, by choice of q and σ the gain margins may be made arbitrarily small.

It is interesting to note that the margins deteriorate as control weight (1/q) gets small (large q) and/or system driving noise gets large (large σ). In modern control folklore, these have often been considered ad hoc means of improving sensitivity.

- **H_2 (LQG) controllers have no global system-independent guaranteed robustness properties.**
- **Improve the robustness of a given design by relaxing the optimality of the filter with respect to error properties. LQG loop transfer recovery (LQG/LTR) design technique. The idea is to design a filtering gain in such way so that the LQG (or H_2) control law will approximate the loop properties of the regular LQR control.**