Institute of Systems Science and Intelligent Control Technology 系统科学与智能控制技术研究所

鲁棒控制：建模、跟踪、抗扰、容错

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（第十一讲）

提纲

1. 古典控制基础
2. 鲁棒控制理论基础
3. 鲁棒控制在迟滞系统中应用
4. 高精度跟踪与抗扰控制
5. 故障诊断与容错控制
6. 教材2-16章

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Chapter 9: LFT

- LFT: Definition
- Properties
- Examples
- General Technique
- HIMAT Example
- Redheffer Star Product

LFT: Definition

- Let M be a partitioned matrix

$$M=\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

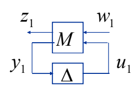
- A (lower) linear fractional transformation (LFT) of M over Δ is defined as

$$F_l(M,\Delta):=M_{11}+M_{12}\Delta(I-M_{22}\Delta)^{-1}M_{21}$$

where Δ has suitable dimensions and $I-M_{22}\Delta$ is invertible.

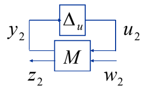
$$\begin{bmatrix} z_1 \\ y_1 \end{bmatrix}=M\begin{bmatrix} w_1 \\ u_1 \end{bmatrix}=\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\begin{bmatrix} w_1 \\ u_1 \end{bmatrix},$$

$$u_1=\Delta y_1$$



- Similarly, an (upper) LFT is defined as

$$F_u(M,\Delta_u):=M_{22}+M_{21}\Delta_u(I-M_{11}\Delta_u)^{-1}M_{12}$$



Properties

- $F_l(M,\Delta)$ is well-posed if $I-M_{22}\Delta$ is invertible.
- $(F_u(M,\Delta_u))^{-1}=F_u(N,\Delta_u)$ with N given by

$$N=\begin{bmatrix} M_{11}-M_{12}M_{22}^{-1}M_{21} & -M_{12}M_{22}^{-1} \\ M_{22}^{-1}M_{21} & M_{22}^{-1} \end{bmatrix}.$$

- Suppose C is invertible. Then
$(A+BQ)(C+DQ)^{-1}=F_l(M,Q)$, $\qquad(C+DQ)^{-1}(A+BQ)=F_l(N,Q)$

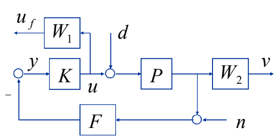
$$M=\begin{bmatrix} AC^{-1} & B-AC^{-1}D \\ C^{-1} & -C^{-1}D \end{bmatrix},\quad N=\begin{bmatrix} C^{-1}A & C^{-1} \\ B-DC^{-1}A & -DC^{-1} \end{bmatrix}.$$

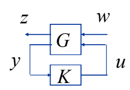
- If M_{12} is invertible, then $F_l(M,Q)=(C+DQ)^{-1}(A+BQ)$
with $A=M_{12}^{-1}M_{11}$, $B=M_{21}-M_{22}M_{12}^{-1}M_{11}$, $C=M_{12}^{-1}$, and $D=-M_{22}M_{12}^{-1}$
- If M_{21} is invertible, then $F_l(M,Q)=(A+BQ)(C+DQ)^{-1}$
with $A=M_{11}M_{21}^{-1}$, $B=M_{12}-M_{11}M_{21}^{-1}M_{22}$, $C=M_{21}^{-1}$, and $D=-M_{21}^{-1}M_{22}$

Examples

- Feedback System: consider a feedback system with disturbance d, sensor noise n, we can write this system in the LFT form with external inputs (d, n) and controlled outputs (v,u_f) such that $z=F_l(G,K)w$ with

$$w=\begin{pmatrix} d \\ n \end{pmatrix}\quad z=\begin{pmatrix} v \\ u_f \end{pmatrix}\quad G=\left[\begin{array}{cc|c} W_2P & 0 & W_2P \\ 0 & 0 & W_1 \\ \hline -FP & -F & -FP \end{array}\right].$$

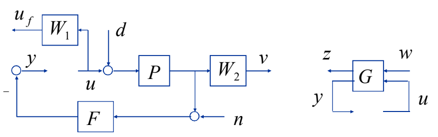




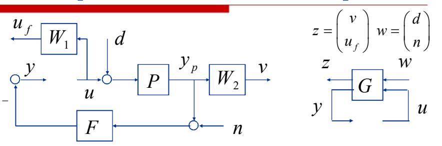
Examples

- Here we show more steps for putting in the LFT form. Note that G is the transfer matrix between (w,u) and (z,y). It does not include the controller K. It is better to ``pull out'' the K in the first place as shown below.
- Now find the G as the transfer matrix between (w,u) and (z,y):

$$w=\begin{pmatrix} d \\ n \end{pmatrix} \quad z=\begin{pmatrix} v \\ u_f \end{pmatrix} \quad \begin{bmatrix} v \\ u_f \\ y \end{bmatrix}=\left[\begin{array}{cc|c} W_2P & 0 & W_2P \\ 0 & 0 & W_1 \\ \hline -FP & -F & -FP \end{array}\right]\begin{bmatrix} d \\ n \\ u \end{bmatrix}=G\begin{bmatrix} d \\ n \\ u \end{bmatrix}$$



Examples: *in State Space*



$$\begin{aligned} \dot{x}_p &= A_p x_p + B_p(d+u), y_p = C_p x_p, \\ \dot{x}_f &= A_f x_f + B_f(y_p+n), -y = C_f x_f + D_f(y_p+n), \\ \dot{x}_u &= A_u x_u + B_u u, u_f = C_u x_u + D_u u, \\ \dot{x}_v &= A_v x_v + B_v y_p, v = C_v x_v + D_v y_p. \end{aligned}$$

$$P=\left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & 0 \end{array}\right], F=\left[\begin{array}{c|c} A_f & B_f \\ \hline C_f & D_f \end{array}\right], W_1=\left[\begin{array}{c|c} A_u & B_u \\ \hline C_u & D_u \end{array}\right], W_2=\left[\begin{array}{c|c} A_v & B_v \\ \hline C_v & D_v \end{array}\right]$$

Now define a new state vector *$x=(x_p, x_f, x_u, x_v)$* and eliminate the variable y_p to get a realization of G as

$$A=\begin{bmatrix} A_p & 0 & 0 & 0 \\ B_fC_p & A_f & 0 & 0 \\ 0 & 0 & A_u & 0 \\ B_vC_p & 0 & 0 & A_v \end{bmatrix}, B_1=\begin{bmatrix} B_p & 0 \\ 0 & B_f \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$B_2=\begin{bmatrix} B_p \\ 0 \\ B_u \\ 0 \end{bmatrix}, C_1=\begin{bmatrix} D_vC_p & 0 & 0 & C_v \\ 0 & 0 & C_u & 0 \end{bmatrix}, D_{11}=0,$$

$$D_{12}=\begin{bmatrix} 0 \\ D_u \end{bmatrix}, C_2=\begin{bmatrix} -D_fC_p & -C_f & 0 & 0 \end{bmatrix},$$

$$D_{21}=\begin{bmatrix} 0 & -D_f \end{bmatrix}, D_{22}=0$$

$$\begin{aligned} \dot{x} &= Ax + B_1w + B_2u \\ z &= C_1x + D_{11}w + D_{12}u \\ y &= C_2x + D_{21}w + D_{22}u \end{aligned}$$

General Technique

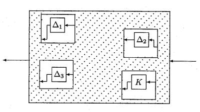


Figure 9.4: Multiple source of uncertain structure

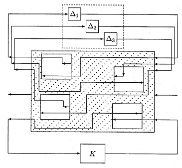
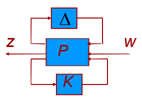
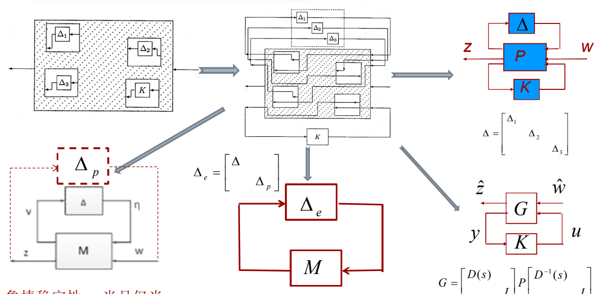


Figure 9.5: Pulling out the Δ's

After pulling out the Deltas and Ks, we have the right diagram. G is the TF from right to the left in the shadow.



一般综合框架



鲁棒稳定性： 当且仅当$(\mathrm{I}-M_{11}(\mathrm{s})\Delta)^{-1}$对所有$\Delta\in\mathbf{\Delta}$满足$\|\Delta\|_\infty\le 1$都稳定

鲁棒性能： 当且仅当$(\mathrm{I}-M(\mathrm{s})\Delta_e)^{-1}$对所有$\Delta_e\in\mathbf{\Delta}_e$满足$\|\Delta_e\|_\infty\le 1$都稳定

$$G=\begin{bmatrix} D(s) & \\ & I \end{bmatrix}P\begin{bmatrix} D^{-1}(s) & \\ & I \end{bmatrix}$$

$$D^{-1}(s)\Delta D(s)=\Delta$$

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$$P=\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}, M=\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

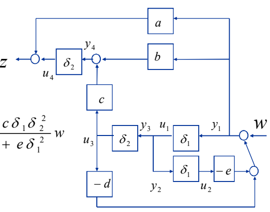
$$M(s)\triangleq\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}+\begin{bmatrix} P_{13} \\ P_{23} \end{bmatrix}K(I-P_{33}(s)K)^{-1}\begin{bmatrix} P_{31} & P_{32} \end{bmatrix}$$

$$G=\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}=\left[\begin{array}{c|c} D(s) & \\ & I \\ \hline & I \end{array}\right]\left[\begin{array}{cc|c} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ \hline P_{31} & P_{32} & P_{33} \end{array}\right]\left[\begin{array}{c|c} D(s) & \\ & I \\ \hline & I \end{array}\right]^{-1}$$

$$=\begin{bmatrix} D(s) & \\ & I \end{bmatrix}P\begin{bmatrix} D^{-1}(s) & \\ & I \end{bmatrix}$$

- **Uncertain Function: assume each coefficient has some perturbation.**

$$z=\frac{a+b\delta_2+c\delta_1\delta_2^2}{1+d\delta_1\delta_2+e\delta_1^2}w$$

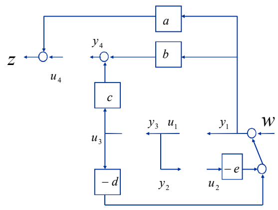


$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ z \end{bmatrix}=M\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ w \end{bmatrix}$$

where $M=\left[\begin{array}{cccc|c} 0 & -e & -d & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -be & -bd+c & 0 & b \\ \hline 0 & -ae & -ad & 1 & a \end{array}\right].$

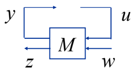
Then $z=F_u(M,\Delta)w, \Delta=\begin{bmatrix} \delta_1 I_2 & 0 \\ 0 & \delta_2 I_2 \end{bmatrix}.$

- **More steps: pulling out the deltas.**



$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ z \end{bmatrix} = M \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ w \end{bmatrix}$$

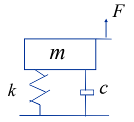
where $M = \left[\begin{array}{cccc:c} 0 & -e & -d & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -be & -bd+c & 0 & b \\ \hdashline 0 & -ae & -ad & 1 & a \end{array}\right]$.



Then $z = F_u(M,\Delta)w, \Delta = \begin{bmatrix} \delta_1 I_2 & 0 \\ 0 & \delta_2 I_2 \end{bmatrix}$.

- **Parametric Uncertainty--A Mass/Spring/Damper System: assume each coefficient has some perturbation.**

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{F}{m}.$$

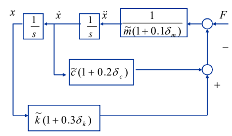


Then $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = F_l(M,\Delta)\begin{bmatrix} x_1 \\ x_2 \\ F \end{bmatrix}$

where

$$M = \left[\begin{array}{ccc:ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{\bar{k}}{\bar{m}} & -\frac{\bar{c}}{\bar{m}} & \frac{1}{\bar{m}} & -\frac{1}{\bar{m}} & -\frac{1}{\bar{m}} & -\frac{0.1}{\bar{m}} \\ \hdashline 0.3\bar{k} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.2\bar{c} & 0 & 0 & 0 & 0 \\ -\bar{k} & -\bar{c} & 1 & -1 & -1 & -0.1 \end{array}\right]$$

$$\Delta = \begin{bmatrix} \delta_k & 0 & 0 \\ 0 & \delta_c & 0 \\ 0 & 0 & \delta_m \end{bmatrix}.$$



- **HIMAT Example: Consider a HIMAT system with**

$$W_{del} = \begin{bmatrix} \frac{50(s+100)}{s+10000} & 0 \\ 0 & \frac{50(s+100)}{s+10000} \end{bmatrix},$$

$$W_p = \begin{bmatrix} \frac{0.5(s+3)}{s+0.03} & 0 \\ 0 & \frac{0.5(s+3)}{s+0.03} \end{bmatrix},$$

$$W_n = \begin{bmatrix} \frac{2(s+1.28)}{s+320} & 0 \\ 0 & \frac{2(s+1.28)}{s+320} \end{bmatrix},$$

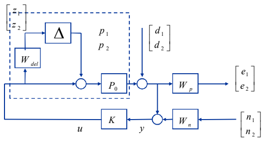


Figure 9.7: HIMAT closed-loop interconnection

$$P_0 = \left[\begin{array}{cccc|cc} -0.0226 & -36.6 & -18.9 & -32.1 & 0 & 0 \\ 0 & -1.9 & 0.983 & 0 & -0.414 & 0 \\ 0.0123 & -11.7 & -2.63 & 0 & -77.8 & 22.4 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 57.3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 57.3 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} z_1 \\ z_2 \\ e_1 \\ e_2 \\ y_1 \\ y_2 \end{bmatrix} = \hat{G}(s)\begin{bmatrix} p_1 \\ p_2 \\ d_1 \\ d_2 \\ n_1 \\ n_2 \\ u_1 \\ u_2 \end{bmatrix}.$$

The open-loop interconnection matrix is $\hat{G}(s)$

- **The interconnection matrix:**

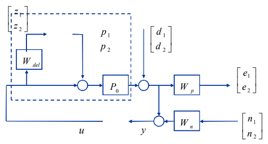


Figure 9.7: HIMAT closed-loop interconnection

$$\begin{bmatrix} z_1 \\ z_2 \\ e_1 \\ e_2 \\ y_1 \\ y_2 \end{bmatrix} = \hat{G}(s)\begin{bmatrix} p_1 \\ p_2 \\ d_1 \\ d_2 \\ n_1 \\ n_2 \\ u_1 \\ u_2 \end{bmatrix} = \left[\begin{array}{ccc:c} 0 & 0 & 0 & W_{del} \\ W_p P_0 & W_p & 0 & W_p P_0 \\ \hdashline P_0 & I & W_n & P_0 \end{array}\right]\begin{bmatrix} p_1 \\ p_2 \\ d_1 \\ d_2 \\ n_1 \\ n_2 \\ u_1 \\ u_2 \end{bmatrix}$$

The SIMULINK block diagram:

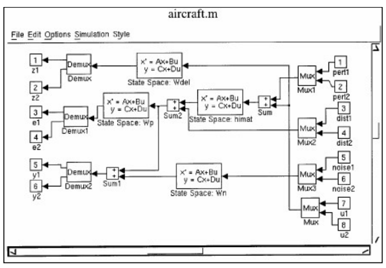


Figure 9.8: SIMULINK block diagram for HIMAT (aircraft.m)

The state space for $\hat{G}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ can be computed by

>> [A,B,C,D] = linmod('aircraft')

which gives

$$A = \begin{bmatrix} -10000\,I_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.0226 & -36.6 & -18.9 & -32.1 & 0 & 0 & 0 \\ 0 & 0 & -1.9 & 0.983 & 0 & 0 & 0 & 0 \\ 0 & 0.0123 & -11.7 & -2.63 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -54.087 & 0 & 0 & -0.018 & 0 & 0 \\ 0 & 0 & 0 & 0 & -54.087 & 0 & -0.018 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -320\,I_2 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & -703.5624 & 0 \\ 0 & 0 & 0 & 0 & 0 & -703.5624 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.4140 & 0 & 0 & 0 & -0.4140 & 0 \\ -77.8 & 22.4 & 0 & 0 & -77.8 & 22.4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.9439\,I_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -25.2476\,I_2 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 703.5624\,I_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 28.65 & 0 & 0 & -0.9439 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 28.65 & 0 & -0.9439 & 0 & 0 \\ 0 & 0 & 57.3 & 0 & 0 & 0 & 0 & 25.2476 & 0 \\ 0 & 0 & 0 & 0 & 57.3 & 0 & 0 & 0 & 25.2476 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 50 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 50 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 \end{bmatrix}$$

Redheffer Star Product

- Let P and K be two matrices with appropriate dimensions.

$$P=\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, K=\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}, \quad P*K:=\begin{bmatrix} F_l(P,K_{11}) & P_{12}(I-K_{11}P_{22})^{-1}K_{12} \\ K_{21}(I-P_{22}K_{11})^{-1}P_{21} & F_u(K,P_{22}) \end{bmatrix}.$$

Then the following interconnection, called a ***star product***, is also a matrix, denoted by $P*K$

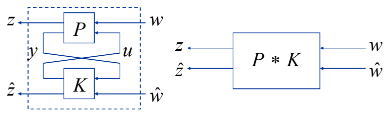


Figure 9.9: Interconnection of LFTs

- Let P and K be two transfer matrices with state space realizations. Then the transfer matrix $P*K$ is given by

$$P=\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right], \quad K=\left[\begin{array}{c|cc} A_K & B_{K1} & B_{K2} \\ \hline C_{K1} & D_{K11} & D_{K12} \\ C_{K2} & D_{K21} & D_{K22} \end{array}\right], \quad P*K=\left[\begin{array}{c|cc} \overline{A} & \overline{B}_1 & \overline{B}_2 \\ \hline \overline{C}_1 & \overline{D}_{11} & \overline{D}_{12} \\ \overline{C}_2 & \overline{D}_{21} & \overline{D}_{22} \end{array}\right]=\left[\begin{array}{c|c} \overline{A} & \overline{B} \\ \hline \overline{C} & \overline{D} \end{array}\right]$$

$$\overline{A}=\begin{bmatrix} A+B_2\tilde{R}^{-1}D_{K11}C_2 & B_2\tilde{R}^{-1}C_{K1} \\ B_{K1}R^{-1}C_2 & A_K+B_{K1}R^{-1}D_{22}C_{K1} \end{bmatrix}=\begin{bmatrix} A & B_2 \\ C_2 & D_{22} \end{bmatrix}*\begin{bmatrix} D_{K11} & C_{K1} \\ B_{K1} & A_K \end{bmatrix},$$

$$\overline{B}=\begin{bmatrix} B_1+B_2\tilde{R}^{-1}D_{K11}D_{21} & B_2\tilde{R}^{-1}D_{K12} \\ B_{K1}R^{-1}D_{21} & B_{K2}+B_{K1}R^{-1}D_{22}D_{K12} \end{bmatrix}=\begin{bmatrix} B_1 & B_2 \\ D_{21} & D_{22} \end{bmatrix}*\begin{bmatrix} D_{K11} & D_{K12} \\ B_{K1} & B_{K2} \end{bmatrix},$$

$$\overline{C}=\begin{bmatrix} C_1+D_{12}D_{K11}R^{-1}C_2 & D_{12}\tilde{R}^{-1}C_{K1} \\ D_{K21}R^{-1}C_2 & C_{K2}+D_{K21}R^{-1}D_{22}C_{K1} \end{bmatrix}=\begin{bmatrix} C_1 & D_{12} \\ C_2 & D_{22} \end{bmatrix}*\begin{bmatrix} D_{K11} & C_{K1} \\ D_{K21} & C_{K2} \end{bmatrix},$$

$$\overline{D}=\begin{bmatrix} D_{11}+D_{12}D_{K11}R^{-1}D_{21} & D_{12}\tilde{R}^{-1}D_{K12} \\ D_{K21}R^{-1}D_{21} & D_{K22}+D_{K21}R^{-1}D_{22}D_{K12} \end{bmatrix}=\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}*\begin{bmatrix} D_{K11} & D_{K12} \\ D_{K21} & D_{K22} \end{bmatrix}$$

$$R=I-D_{22}D_{K11}, \quad \tilde{R}=I-D_{K11}D_{22}.$$

- Matlab

```
>> P*K = lft(P, K, dimu, dimy)
>> F_l(P, K) = lft(P, K)
```

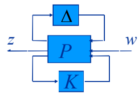
Chapter 10: μ and μ-Synthesis

- general framework
- analysis and synthesis methods for unstructured uncertainty
- stability with structured uncertainties
- unstructured perturbation
- structured perturbation
- definition of SSV
- examples
- tighter bounds
- tightness of bounds
- computing bounds using Matlab
- structured robust stability
- structured robust stability theorem
- robust performance
- extension to nonlinear time varying uncertainties
- HIMAT example
- skewed problem
- overview on μ-synthesis

General Framework

- Every problem can be put in the general framework with

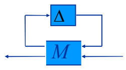
$$P(s)=\begin{bmatrix} P_{11}(s) & P_{12}(s) & P_{13}(s) \\ P_{21}(s) & P_{22}(s) & P_{23}(s) \\ P_{31}(s) & P_{32}(s) & P_{33}(s) \end{bmatrix}$$



$$z=F_u(F_l(P,K),\Delta)w$$
$$=F_l(F_u(P,\Delta),K)w$$

- For analysis, the controller can be absorbed into the system:

$$M(s)=F_l(P(s),K(s))=\begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{bmatrix},$$
$$z=F_u(M,\Delta)w=[M_{22}+M_{21}\Delta(I-M_{11}\Delta)^{-1}M_{12}]w.$$



Analysis and Synthesis Methods for Unstructured Uncertainty

Input Assumptions	Performance Specifications	Perturbation Assumptions	Analysis Tests	Synthesis Methods
$E(w(t)w(\tau)^*) = \delta(t-\tau)I$	$E(z(t)^*z(t))\le 1$			LQG Wiener - Hopf
$w=U_0\delta(t)$ $E(U_0U_0^*)=I$	$E(\lVert z\rVert_2^2)\le 1$	$\Delta=0$	$\lVert M_{22}\rVert_2\le 1$	H_2
$\lVert w\rVert_2\le 1$	$\lVert z\rVert_2\le 1$	$\Delta=0$	$\lVert M_{22}\rVert_\infty\le 1$	
$\lVert w\rVert_2\le 1$		$\lVert\Delta\rVert_\infty<1$	$\lVert M_{11}\rVert_\infty\le 1$	H_∞

Stability with Structured Uncertainties

- Assume $\Delta(s)=\mathrm{diag}[\delta_1 I_{r1}, \ldots, \delta_s I_{rs}, \Delta_1, \ldots, \Delta_F] \in RH_\infty$ with $\|\delta_i\|_\infty<1$ and $\|\Delta_j\|_\infty<1$.

  Robust Stability ⇔ The interconnection is stable.

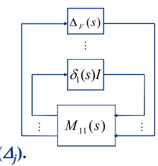
- Stability Conditions:

  (1) (sufficient condition) $\|M_{11}\|_\infty\le 1$

  Conservative: ignoring structure of Δ(s).

  (2) (necessary conditions) Test for each δ_i (Δ_j) individually (assuming no uncertainty in other channels): $\|(M_{11})_{ii}\|_\infty\le 1$

  Optimistic: ignoring interaction between the δ_i (Δ_j).



Unstructured Perturbation

- **Problem: Given $M \in C^{p\times q}$, find a smallest $\Delta \in C^{p\times q}$ (no structure imposed on Δ) in the sense of $\overline{\sigma}(\Delta)$ such that det($I-M\Delta$)=0.**

It is easy to see that

$$\alpha_{\min} := \inf\{\overline{\sigma}(\Delta) : \det(I - M\Delta) = 0, \Delta \in \mathbf{C}^{p\times q}\}$$
$$= \inf\{\alpha : \det(I - \alpha M\Delta) = 0, \overline{\sigma}(\Delta) \le 1, \Delta \in \mathbf{C}^{p\times q}\}$$

and $\max_{\overline{\sigma}(\Delta)\le 1} \rho(M\Delta) = \alpha_{\min}^{-1} = \overline{\sigma}(M)$

with a smallest "destabilizing" $\Delta_{des} = \frac{1}{\overline{\sigma}(M)} v_1 u_1^*, \det(I - M\Delta_{des}) = 0$

where $M = \overline{\sigma}(M) u_1 v_1^* + \sigma_2 u_2 v_2^* + \cdots$

So the largest singular value of M can be defined as

$$\overline{\sigma}(M) := \frac{1}{\inf\{\overline{\sigma}(\Delta) : \det(I - M\Delta) = 0, \Delta \in \mathbf{C}^{p\times q}\}}$$

Structured Perturbation

- **Problem: Given $M \in C^{p\times q}$, find a smallest $\Delta \in \mathbf{\Delta}$**

  $\mathbf{\Delta}$={diag[$\delta_1 I_{r1}$, …, $\delta_s I_{rs}$, Δ_1, …, Δ_F] $\in C^{p\times q}$: $\delta_i \in$ C, $\Delta_j \in C^{mj\times mj}$}

  such that det($I-M\Delta$)=0.

$$\alpha_{\min} := \inf\{\overline{\sigma}(\Delta) : \det(I - M\Delta) = 0, \Delta \in \mathbf{\Delta}\}$$
$$= \inf\{\alpha : \det(I - \alpha M\Delta) = 0, \overline{\sigma}(\Delta) \le 1, \Delta \in \mathbf{\Delta}\}$$

and

$$\max_{\Delta\in\mathbf{\Delta},\ \overline{\sigma}(\Delta)\le 1} \rho(M\Delta) = \alpha_{\min}^{-1} \le \overline{\sigma}(M)$$

- **We shall call $1/\alpha_{\min}$ as structured singular value (SSV).**

Definition of SSV

- **For $M \in C^{n\times n}$, $\mu_\Delta(M)$ is defined as**

$$\mu_\Delta(M) := \frac{1}{\min\{\overline{\sigma}(\Delta) : \Delta \in \mathbf{\Delta}, \det(I - M\Delta) = 0\}}$$

unless no $\Delta \in \mathbf{\Delta}$ makes $I-M\Delta$ singular, in which case $\mu_\Delta(M)$:=0.

  - If $\mathbf{\Delta}$={δI_n : $\delta \in$ C} (S=1, F=0, r_1=n), then $\mu_\Delta(M)=\rho(M)$, the spectral radius of M.
  - If $\mathbf{\Delta}$={$\Delta \in C^{n\times n}$} (S=0, F=1, m_1=n), then $\mu_\Delta(M) = \overline{\sigma}(M)$.

- **Thus we have the following bounds:**

$$\rho(M) \le \mu_\Delta(M) \le \overline{\sigma}(M)$$

Examples

- **Let $\mathbf{\Delta}$={diag[δ_1, δ_2] $\in C^{2\times 2}$: $\delta_i \in$ C}. Consider two matrices:**

(1) $M = \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix}$ for any $\beta > 0$. Then $\rho(M) = 0$ and $\overline{\sigma}(M) = \beta$.

But $\mu(M) = 0$ since $\det(I - M\Delta) = 1 \ne 0$ for all admissible Δ.

(2) $M = \begin{bmatrix} -1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$. Then $\rho(M) = 0$ and $\overline{\sigma}(M) = 1$. Since

$$\det(I - M\Delta) = 1 + \frac{\delta_1 - \delta_2}{2} = 0$$

if $\delta_1 = -\delta_2 = -1$. So $\mu(M) = 1$.

- **Thus neither $\rho(M)$ nor $\overline{\sigma}(M)$ provides useful bounds even in these simple cases.**

Tighter Bounds

- **To obtain tighter bounds, define $\mathbf{U}$={$U \in \mathbf{\Delta}$: $UU^*=I_n$}**

$$\mathbf{D} = \{diag[D_1, \ldots, D_S, d_1 I_{m_1}, \ldots, d_{F-1} I_{m_{F-1}}, I_{m_F}] :$$
$$D_i \in \mathbf{C}^{r_i\times r_i}, D_i = D_i^* > 0, d_j \in \mathbf{R}\, d_j > 0\}$$

Note that for any $\Delta \in \mathbf{\Delta}, U \in \mathbf{U}$, and $D \in \mathbf{D}$,

$$U^* \in \mathbf{U}, U\Delta \in \mathbf{\Delta}, \Delta U \in \mathbf{\Delta}, \overline{\sigma}(U\Delta) = \overline{\sigma}(\Delta U) = \overline{\sigma}(\Delta), \qquad D\Delta = \Delta D.$$

For all $U \in \mathbf{U}$ and $D \in \mathbf{D}$

$$\mu_\Delta(MU) = \mu_\Delta(UM) = \mu_\Delta(M) = \mu_\Delta(DMD^{-1}).$$

$$\max_{U\in\mathbf{u}} \rho(UM) \le \max_{\Delta\in\mathbf{B\Delta}} \rho(\Delta M) = \mu_\Delta(M) \le \inf_{D\in\mathbf{D}} \overline{\sigma}(DMD^{-1})$$
$$\max_{U\in\mathbf{u}} \rho(UM) \le \mu_\Delta(M) \le \inf_{D\in\mathbf{D}} \overline{\sigma}(DMD^{-1}).$$

Tightness of Bounds

[*Doyle,1982*] $\max_{U\in\mathbf{u}} \rho(UM) = \mu_\Delta(M)$.

Not Convex or Concave.

$$\mu_\Delta(M) = \inf_{D\in\mathbf{D}} \overline{\sigma}(DMD^{-1}) \quad \textit{if } 2S + F \le 3.$$

F= S=	0	1	2	3	4
0		Yes	Yes	Yes	No
1	Yes	Yes	No	No	No
2	No	No	No	No	No

Computing Bounds

- Example: Let M be a 13x13 matrix and suppose Δ is given by

$$\Delta=\begin{bmatrix}\delta_1 I_2 & 0 & 0 & 0 & 0 & 0\\ 0 & \delta_2 & 0 & 0 & 0 & 0\\ 0 & 0 & \Delta_3 & 0 & 0 & 0\\ 0 & 0 & 0 & \Delta_4 & 0 & 0\\ 0 & 0 & 0 & 0 & \delta_5 I_3 & 0\\ 0 & 0 & 0 & 0 & 0 & \Delta_6\end{bmatrix} \qquad \mathbf{blk}=\begin{bmatrix}2 & 0\\ 1 & 1\\ 2 & 3\\ 3 & 3\\ 3 & 0\\ 2 & 1\end{bmatrix}.$$

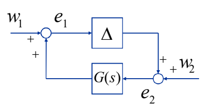
$$\delta_1,\delta_2,\delta_5,\in \mathbf{C}, \Delta_3\in \mathbf{C}^{2\times 3}, \Delta_4\in \mathbf{C}^{3\times 3}, \Delta_6\in \mathbf{C}^{2\times 1}$$

where the size of Δ is specified by the matrix blk.

```
>> [bounds,rowd] = mu(M,blk)
>> [Dl,Dr] = unwrapd(rowd,blk)
```

Structured Robust Stability

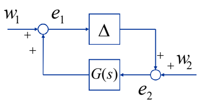
- Consider a general feedback interconnection where Δ is structured uncertainty block and $G(s)$ is the interconnection



- We are interested in the following question: How large Δ (in the sense of $\|\Delta\|_\infty$) can be without destabilizing the feedback system?

  Since the closed-loop poles are given by det(I-$G(s)\Delta$)=0 the feedback system becomes unstable if det(I-$G(s)\Delta(s)$)=0 for some s in the closed right half plane. Now let α>0 be a sufficiently small number such that the closed-loop system is stable for all stable $\|\Delta\|_\infty<\alpha$. Next increase α until α_{max} so that the closed-loop system becomes unstable. So α_{max} is the robust stability margin.

Structured Robust Stability Theorem



Define

$$\boldsymbol{\Delta} := \{\Delta(\cdot)\in RH_\infty : \Delta(s_0)\in \boldsymbol{\Delta} \text{ for all } s_0\in \overline{\mathbf{C}}_+\}$$

- Theorem: Let β>0. The interconnected system is well-posed and internally stable for all $\Delta(\cdot)\in\boldsymbol{\Delta}$ with $\|\Delta\|_\infty<1/\beta$ if and only if

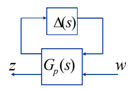
$$\sup_{\omega\in\mathbb{R}}\mu_\Delta(G(j\omega))\le\beta$$

Robust Performance

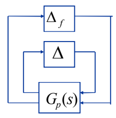
Let

$$G_p(s)=\begin{bmatrix}G_{11} & G_{12}\\ G_{21} & G_{22}\end{bmatrix}$$

$$\Delta_P := \left\{\begin{bmatrix}\Delta & 0\\ 0 & \Delta_f\end{bmatrix} : \Delta\in\boldsymbol{\Delta}, \Delta_f\in\mathbf{C}^{q_2\times p_2}\right\}.$$



- Theorem: Let β>0. For all $\Delta(s)\in\boldsymbol{\Delta}$ with $\|\Delta\|_\infty<1/\beta$, the system is well-posed, internally stable, and $\|F_u(G_p,\Delta)\|_\infty\le\beta$ if and only if $\sup_{\omega\in\mathbb{R}}\mu_{\Delta_P}(G_p(j\omega))\le\beta$.

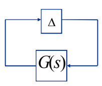


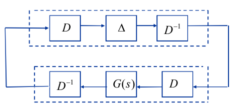
Extension to Nonlinear Time Varying Uncertainty

- Suppose $\Delta\in\Delta_N$ is a structured Nonlinear (Time-varying) Uncertainty and suppose D is constant scaling matrix such that $D\Delta D^{-1}\in\Delta_N$. (Note that we do not require $D\Delta=\Delta D$.)

  Then a sufficient stability condition for all $\Delta\in\Delta_N$ with $\|\Delta\|_\infty<1$, is (by small gain theorem)

  $\|D^{-1}G(s)D\|_\infty\le 1$

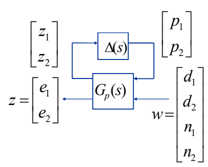




HIMAT Example

- We shall first find a H_∞ controller using Matlab (details will be discussed in later chapters. Just follow the steps here.), which gives γ=1.8612=$\|G_p\|_\infty$, a stabilizing controller K, and a closed loop transfer matrix G_p:

```
>> [A,B,C,D] = linmod('aircraft')
>> Ĝ = pck(A,B,C,D);
>> [K, Gp, γ] = hinfsyn(Ĝ,2,2,0,10, 0.001,2) ;
```



$$\begin{bmatrix}z_1\\ z_2\\ e_1\\ e_2\end{bmatrix}=G_p(s)\begin{bmatrix}p_1\\ p_2\\ d_1\\ d_2\\ n_1\\ n_2\end{bmatrix}, G_p(s)=\begin{bmatrix}G_{p11} & G_{p12}\\ G_{p21} & G_{p22}\end{bmatrix}.$$

- Now generate the singular value frequency responses of G_p:

```
>> w = logspace(-3,3,300);
>> Gpf = frsp(Gp,w);  % Gpf is the frequency response of Gp;
>> [u,s,v] = vsvd(Gpf);
>> vplot('liv, m', s)
```

The singular value frequency responses of G_p are shown in Figure 10.6

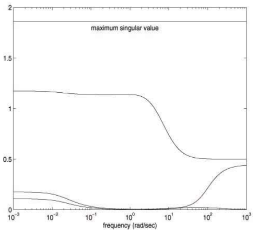


Figure 10.6: Singular values of $G_p(j\omega)$

- To test the robust stability, we need to compute $\|G_{p11}\|_\infty$,

```
>> Gp11 = sel(Gp, 1:2, 1:2);
>> norm_of_Gp11 = hinfnorm(Gp11,0.001);
```

which gives $\|G_{p11}\|_\infty$=0.933<1. So the system is robustly stable.

To check the robust performance, we shall compute the $\mu_{\Delta_P}(G_p(j\omega))$ for each frequency with

$$\Delta_P = \begin{bmatrix} \Delta & \\ & \Delta_f \end{bmatrix}, \Delta \in \mathbf{C}^{2\times 2},\ \Delta_f \in \mathbf{C}^{4\times 2}.$$

```
>> blk = [2,2;4,2];
>> [bnds,dvec,sens,pvec] = mu(Gpf,blk);
>> vplot('liv,m',vnorm(Gpf),bnds)
>> title('Maximum Singular Value and mu')
>> xlabel('frequency(rad/sec)')
>> text(0.01,1.7,'maximum singular value')
>> text(0.5,0.8,' mu bounds')
```

- The structured singular value $\mu_{\Delta_P}(G_p(j\omega))$ and $\overline{\sigma}(G_p(j\omega))$ are shown in Figure 10.7. It is clear that the robust performance is not satisfied. Note that

$$\max_{\|\Delta\|_\infty \le 1} \|F_u(G_p,\Delta)\|_\infty \le \gamma \Leftrightarrow \sup_\omega \mu_{\Delta_P}\left(\begin{bmatrix} G_{p11} & G_{p12} \\ G_{p21}/\gamma & G_{p22}/\gamma \end{bmatrix}\right) \le 1.$$

- Using a bisection algorithm, we can also find the worst performance

$$\max_{\|\Delta\|_\infty \le 1} \|F_u(G_p,\Delta)\|_\infty = 12.7824.$$

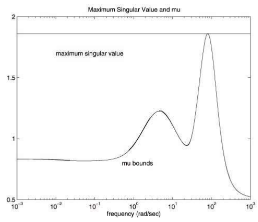


Figure 10.7: $\mu_{\Delta_P}(G_p(j\omega))$ and $\overline{\sigma}(G_p(j\omega))$

Skewed Problem

- Recall the skewed problem in Chapter 8. It can be shown that robust performance problem is equivalent to a 2 block structure singular value with the following interconnection matrix

$$G = \begin{bmatrix} -W_2 T_i W_1 & -W_2 K S_o W_d \\ W_e S_o P W_1 & W_e S_o W_d \end{bmatrix}.$$

So the robust performance condition is

$$\mu_\Delta(G(j\omega)) = \inf_{d_\omega \in \mathbb{R}_+} \overline{\sigma}\left(\begin{bmatrix} -W_2 T_i W_1 & -\frac{1}{d_\omega} W_2 K S_o W_d \\ d_\omega W_e S_o P W_1 & W_e S_o W_d \end{bmatrix}\right) \le 1$$

for all ω≥0.

Now assume $W_e = w_s I$, $W_d = I$, $W_1 = I$, $W_2 = w_t I$ and P is stable and has a stable inverse (I.e., minimum phase) and $K(s)=P^{-1}(s)l(s)$ such that $K(s)$ is proper and the closed-loop is stable. Then

$$S_o = S_i = 1/(1+l(s))I = \varepsilon(s)I,\ T_o = T_i = l(s)/(1+l(s))I = \tau(s)I$$

$$G = \begin{bmatrix} -w_t\tau I & -w_t\tau P^{-1} \\ w_s\varepsilon P & w_s\varepsilon I \end{bmatrix}$$

and

$$\mu_\Delta(G(j\omega)) = \inf_{d \in \mathbb{R}_+} \overline{\sigma}\left(\begin{bmatrix} -w_t\tau I & -w_t\tau(dP)^{-1} \\ w_s\varepsilon dP & w_s\varepsilon I \end{bmatrix}\right).$$

Let the SVD of $P(j\omega)$ be

$$P(j\omega) = U\Sigma V^*,\ \Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_m)$$

with $\sigma_1 = \overline{\sigma}$ and $\sigma_m = \underline{\sigma}$ where m is the dimension of P.

Then

$$\mu_\Delta(G(j\omega)) = \inf_{d \in \mathbb{R}_+} \overline{\sigma}\left(\begin{bmatrix} -w_t\tau I & -w_t\tau(d\Sigma)^{-1} \\ w_s\varepsilon d\Sigma & w_s\varepsilon I \end{bmatrix}\right)$$

Note that

$$\begin{bmatrix} -w_t\tau I & -w_t\tau(d\Sigma)^{-1} \\ w_s\varepsilon d\Sigma & w_s\varepsilon I \end{bmatrix} = P_1 \mathrm{diag}(M_1, M_2, \ldots, M_m) P_2$$

$$M_i = \begin{bmatrix} -w_t\tau & -w_t\tau(d\sigma_i)^{-1} \\ w_s\varepsilon d\sigma_i & w_s\varepsilon \end{bmatrix}.$$

where P_1 and P_2 are permutation matrices.

Hence

$$\begin{aligned} \mu_\Delta(G(j\omega)) &= \inf_{d \in \mathbb{R}_+} \max_i \overline{\sigma}\left(\begin{bmatrix} -w_t\tau & -w_t\tau(d\sigma_i)^{-1} \\ w_s\varepsilon d\sigma_i & w_s\varepsilon \end{bmatrix}\right) \\ &= \inf_{d \in \mathbb{R}_+} \max_i \overline{\sigma}\left(\begin{bmatrix} -w_t\tau \\ w_s\varepsilon d\sigma_i \end{bmatrix}\begin{bmatrix} 1 & (d\sigma_i)^{-1} \end{bmatrix}\right) \\ &= \inf_{d \in \mathbb{R}_+} \max_i \sqrt{\left(1 + |d\sigma_i|^{-2}\right)\left(|w_s\varepsilon d\sigma_i|^2 + |w_t\tau|^2\right)} \\ &= \inf_{d \in \mathbb{R}_+} \max_i \sqrt{|w_s\varepsilon|^2 + |w_s\varepsilon d\sigma_i|^2 + |w_t\tau|^2 + \left|\frac{w_t\tau}{d\sigma_i}\right|^2}. \end{aligned}$$

The maximum is achieved at

$$d^2 = \frac{|w_t \tau|}{|w_s \varepsilon| \underline{\sigma}\overline{\sigma}},$$

and

$$\mu_\Delta(G(j\omega)) = \sqrt{|w_s\varepsilon|^2 + |w_t\tau|^2 + |w_s\varepsilon||w_t\tau|[\kappa(P) + \frac{1}{\kappa(P)}]}.$$

$$\mu_\Delta(G(j\omega)) \approx \sqrt{|w_s\varepsilon||w_t\tau|\kappa(P)}.$$

- Conclusion: The structure singular value of the skewed problem is (approximately) proportional to the square root of the condition number of the plant.

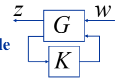
Overview on μ Synthesis

- Consider a general feedback system with interconnection G. Find a stabilizing controller K so that the following μ-norm is minimized:

$$\min_K \|F_\ell(G,K)\|_\mu = \min_K \sup_\omega \mu_\Delta(F_\ell(G(j\omega), K(j\omega)))$$

$$F_\ell(G,K) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}$$

This problem is called μ-synthesis.



- The μ-synthesis is not yet fully solved. But a reasonable approach is to "solve"

$$\min_K \inf_{D, D^{-1} \in H_\infty} \|DF_\ell(G,K)D^{-1}\|_\infty$$

by iteratively solving for K and D, i.e., first minimizing over K with D fixed, then minimizing point-wise over D with K fixed,

- Fix D $\min_K \|DF_\ell(G,K)D^{-1}\|_\infty$

  is a standard H_∞ optimization problem.

- Fix K $\inf_{D,D^{-1} \in H_\infty} \|DF_\ell(G,K)D^{-1}\|_\infty$

  is a standard convex optimization problem and it can be solved point-wise in the frequency domain:

$$\sup_\omega \inf_{D_\omega \in D} \overline{\sigma}\left[D_\omega F_\ell(G,K)(j\omega)D_\omega^{-1}\right]$$

Note that when $S = 0$, (no scalar blocks)

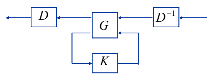
$$D_\omega = \mathrm{diag}(d_1^\omega I, \ldots, d_{F-1}^\omega I, I) \in D,$$

- Details of D-K Iterations:

  (i) Fix an initial estimate of the scaling matrix $D_\omega \in D$ point-wise across frequency.

  (ii) Find scalar transfer functions $d_i(s)$, $d_i^{-1}(s) \in RH_\infty$ for $i = 1, \ldots, (F-1)$ such that $|d_i(j\omega)| \approx d_i^\omega$.

  (iii) Let $D(s) = \mathrm{diag}(d_1(s)I, \ldots, d_{F-1}(s)I, I)$. Construct a state space model for system



$$\hat{G}(s) = \begin{bmatrix} D(s) & \\ & I \end{bmatrix} G(s) \begin{bmatrix} D^{-1}(s) & \\ & I \end{bmatrix}.$$

  (iv) Solve an H_∞ optimization problem to minimize

$$\|F_\ell(\hat{G}, K)\|_\infty$$

  over all stabilizing K's. Denote the minimizing controller by $\hat{K}$.

  (v) Minimize

$$\overline{\sigma}[D_\omega F_\ell(G,\hat{K})D_\omega^{-1}]$$

  over $D_\omega \in D$ point-wise across frequency. The minimization itself produces a new scaling function $\hat{D}_\omega$.

  (vi) Compare $\hat{D}_\omega$ with the previous estimate D_ω, Stop if they are close, otherwise, replace D_ω with $\hat{D}_\omega$ and return to step (ii).

- The joint optimization of D and K is not convex and the global convergence is not guaranteed, many designs have shown that this approach works well.