Institute of Systems Science and Intelligent Control Technology 系统科学与智能控制技术研究所

鲁棒控制：建模、跟踪、抗扰、容错

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（第十讲）

提纲

1. 古典控制基础
2. 鲁棒控制理论基础
3. 鲁棒控制在迟滞系统中应用
4. 高精度跟踪与抗扰控制
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6. 教材2-16章

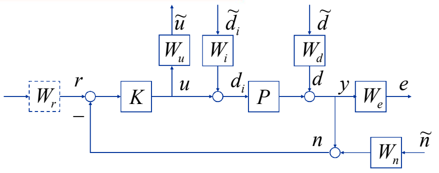
State Space Formula

- Let $P=\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a stabilizable and detectable realization, and let F and L be such that $A+BF$ and $A+LC$ are both stable.
- Define

$$\begin{bmatrix} M \\ N \end{bmatrix}=\left[\begin{array}{c|c} A+BF & B \\ \hline F & I \\ C+DF & D \end{array}\right] \quad \begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix}=\left[\begin{array}{c|cc} A+LC & B+LD & L \\ \hline C & D & I \end{array}\right]$$

Then $P=NM^{-1}=\tilde{M}^{-1}\tilde{N}$ are rcf and lcf, respectively.

Weighted H_2 and H_∞



***H_2* Performance:** Assume $d_1(t)=\eta\delta(t)$ and $E(\eta\eta^*) = I$

Minimize the expected energy of the error e:

$$E\left\{\|e\|_2^2\right\}=E\left\{\int_0^\infty \|e\|^2\,dt\right\}=\|W_eS_oW_d\|_2^2$$

Include the control signal u in the cost function:

$$E\left\{\|e\|_2^2+\rho^2\|\tilde{u}\|_2^2\right\}=\left\|\begin{matrix} W_eS_oW_d \\ \rho W_uKS_oW_d \end{matrix}\right\|_2^2$$

Robustness problem????

***H_∞* Performance:** under worst possible case

$$\sup_{\|\tilde{d}\|_2\le 1}\|e\|_2=\|W_eS_oW_d\|_\infty$$

restrictions on the control energy or control bandwidth:

$$\sup_{\|\tilde{d}\|_2\le 1}\|\tilde{u}\|_2=\|W_uKS_oW_d\|_\infty$$

Combined cost:

$$\sup_{\|\tilde{d}\|_2\le 1}\left\{\|e\|_2^2+\rho^2\|\tilde{u}\|_2^2\right\}=\left\|\begin{bmatrix} W_eS_oW_d \\ \rho W_uKS_oW_d \end{bmatrix}\right\|_\infty^2$$

Chapter 7: Balanced Model Reduction

- Lyapunov Equations
- Gramians
- Controllable Decomposition
- Observable Decomposition
- Balanced Realization
- Computing Balanced Realization
- Balanced Truncation and Matlab
- Bounds of Functions Norms
- Balanced Model Reduction
- Frequency Weighted Balanced Model Reduction
- Relative Reduction

Lyapunov Equations

Consider the following Lyapunov equation

$$A^*X+XA+Q=0$$

- **Assume that A is stable, then the following statements hold:**
  - $X=\int_0^\infty e^{A^*t}Qe^{At}\,dt.$
  - $X>0$ if $Q>0$ and $X\ge 0$ if $Q\ge 0$.
  - if $Q\ge 0$, then (Q,A) is observable iff $X>0$.
- **Proof:**

$$A^*X+XA=A^*\int_0^\infty e^{A^*t}Qe^{At}dt+\int_0^\infty e^{A^*t}Qe^{At}dt\;A$$

$$=\int_0^\infty (A^*e^{A^*t}Qe^{At}+e^{A^*t}Qe^{At}A)\,dt=\int_0^\infty \frac{d}{dt}(e^{A^*t}Qe^{At})\,dt=-Q$$

Let $Q = C^*C$ for some C matrix and let $v \in \mathbf{C}^n$.
Then $X \geq 0$ and

$$v^* X v = \int_0^\infty v^* e^{A^* t} Q e^{At} v \, dt = \int_0^\infty \| C e^{At} v \|^2 \, dt$$

Thus X is singular

$\Leftrightarrow$ there is a $v \in \mathbf{C}^n$ such that $Ce^{At}v = 0, \ \forall t$

$\Leftrightarrow$ there is a $v \in \mathbf{C}^n$ such that $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} v = 0$

$\Leftrightarrow$ (C, A) is not observable

$\Leftrightarrow$ (Q, A) is not obervable.

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$$A^*X+XA+Q = 0$$

- Suppose X is the solution of the Lyapunov equation, then
  - $\mathrm{Re}\lambda_i(A) \leq 0$ if $X > 0$ and $Q \geq 0$.
  - A is stable if $X > 0$ and $Q > 0$.
  - A is stable if $X \geq 0$, $Q \geq 0$ and (Q, A) is detectable.
- Proof: Let λ be an eigenvalue and v be a corresponding eigenvector of A, I.e., $Av=\lambda v$. Then

  $0=v^*(A^*X+XA)v+v^*Qv=(\lambda^*+\lambda)v^*Xv+v^*Qv=2\,\mathrm{Re}\,\lambda\, v^*Xv+v^*Qv$

  $\geq 2\,\mathrm{Re}\,\lambda\, v^*Xv$ since $Q \geq 0$. Thus $\mathrm{Re}\,\lambda \leq 0$.

  If $\mathrm{Re}\,\lambda = 0$, then $v^*Qv=0$. Thus $Qv=0$ and $Av=\lambda v$. By PBH test, (Q, A) is not detectable.

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Gramians

- Let A be stable. Then a pair(C,A) is observable iff the *observability Gramian* $Q > 0$

  $$A^*Q+QA+C^*C=0.$$

  Similarly, (A,B) is controllable iff the *controllability Gramian* $P > 0$

  $$AP+PA^*+BB^*=0$$

- When A is not necessarily stable, the above Lyapunov equations may still have solutions. But they are not gramians.

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Controllable Decomposition

Let $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a state space realization of a (not necessarily stable) transfer matrix $G(s)$. Suppose that there exists a symmetric matrix

$$P = P^* = \begin{bmatrix} P_1 & 0 \\ 0 & 0 \end{bmatrix}$$

(P is not necessarily the Gramian) with P_1 nonsingular such that

$$AP+PA^*+BB^*=0$$

Now partition the realization (A,B,C,D) compatibly with P as

$$\left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right].$$

Then $\left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D_1 \end{array}\right]$ is also a realization of G.
Moreover, (A_{11},B_1) is controllable if A_{11} is stable.

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Controllable Decomposition

<u>Proof</u> Using the partitioned equation $0=AP+PA^*+BB^*$ to get

$A_{11}P_1+P_1A_{11}^*+B_1B_1^*=0$, $P_1A_{21}^*+B_1B_2^*=0$, and $B_2B_2^*=0$

then we conclude that $B_2 = 0$ and $A_{21} = 0$. Hence, part of the realization is not controllable:

$$\left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ 0 & A_{22} & 0 \\ \hline C_1 & C_2 & D \end{array}\right] = \left[\begin{array}{c|c} A_{11} & B_{11} \\ \hline C_1 & D \end{array}\right].$$

$P_1 > 0$ if A_{11} is stable. So (A_{11}, B_1) is controllable.

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Observable Decomposition

Let $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a state space realization of a (not necessarily stable) transfer matrix $G(s)$. Suppose that there exists a symmetric matrix

$$Q = Q^* = \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix}$$

(Q is not necessarily the Gramian) with Q_1 nonsingular such that

$$A^*Q+QA+C^*C=0$$

Now partition the realization (A,B,C,D) compatibly with Q as

$$\left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right].$$

Then $\left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D_1 \end{array}\right]$ is also a realization of G.
Moreover, (C_1, A_{11}) is observable if A_{11} is stable.

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Balanced Realization

- Let P and Q be the controllability and observability Gramians,

$$AP+PA^*+BB^*=0, \qquad A^*Q+QA+C^*C=0$$

Suppose $P=Q=\Sigma=diag(\sigma_1, \sigma_2,\ldots, \sigma_n)$. Then the state space realization is called *internally balanced realization* and $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$, are called the *Hankel singular values* of the system.

- Two other closely related realizations are called

input normal realization with $P = I$ and $Q = \Sigma^2$

and

output normal realization with $P = \Sigma^2$ and $Q = I$.

Both realizations can be obtained easily from the balanced realization by a suitable scaling on the states.

Computing Balanced Realization

- In the special case where $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is a minimal realization, a balanced realization can be obtained through the following simplified procedure:
  1. Compute $P>0$ and $Q>0$.
  2. Find a matrix R such that $P=R^*R$.
  3. Diagonalize RQR^* to get $RQR^*=U\Sigma^2U^*$.
  4. Let $T^{-1}=R^*U\Sigma^{-1/2}$. Then $TPT^*=(T^*)^{-1}QT^{-1}=\Sigma$ and $\left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & D \end{array}\right]$ is balanced.
- In general, let P and Q be two positive semidefinite matrices. Then there exists a nonsingular matrix T such that

$$TPT^* = \begin{bmatrix} \Sigma_1 & & & \\ & \Sigma_2 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} \quad (T^{-1})^* QT^{-1} = \begin{bmatrix} \Sigma_1 & & & \\ & 0 & & \\ & & \Sigma_3 & \\ & & & 0 \end{bmatrix}$$

respectively with Σ_1, Σ_2, Σ_3 diagonal and positive definite.

Balanced Truncation and Matlab

- Suppose $\sigma_r \gg \sigma_{r+1}$ for some r then the balanced realization implies that those states corresponding to the singular values of $\sigma_{r+1},\ldots,\sigma_n$ are less controllable and observable than those states corresponding to $\sigma_1,\ldots,\sigma_r$. Therefore, truncating those less controllable and observable states will not lose much information about the system.
- MATLAB:

```
>>G=ss(A,B,C,D);
>>[Gb,sig,T,Tinv]=balreal(G); %sig is a vector of Hankel
   singular values and Tinv=T-1;
>>[Gred,redinfo] = balancmr(G,r); %r=reduced order
>>[Gred,redinfo] =
   reduce(G,'ErrorType','add','MaxError',[0.01,0.05]);
>>[Gred, redinfo] = reduce(G,[5:10],'ErrorType','mult')
```


Bounds of Function Norms

- Suppose $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \in RH_\infty$ is a balanced realization: that is, there exists

$$\Sigma=\text{diag}(\sigma_1 I_{s_1}, \sigma_2 I_{s_2},\ldots, \sigma_N I_{s_N}) \quad with \quad \sigma_1 > \sigma_2 > \ldots > \sigma_N \geq 0,$$

such that

$$A\Sigma+\Sigma A^*+BB^*=0, \qquad A^*\Sigma+\Sigma A+C^*C=0$$

Then

$$\sigma_1 \leq \|G\|_\infty \leq \int_0^\infty \|g(t)\| dt \leq 2\sum_{i=1}^{N} s_i\sigma_i$$

where $g(t)=Ce^{At}B$.

Bounds (proof)

- Proof: Consider $\begin{aligned} \dot{x} &= Ax + Bw \\ z &= Cx \end{aligned}$

where (A,B) is controllable and (C,A) is observable.

$$\frac{d}{dt}(x^*\Sigma^{-1}x) = \dot{x}^*\Sigma^{-1}x + x^*\Sigma^{-1}\dot{x} = x^*(A^*\Sigma^{-1}+\Sigma^{-1}A)x + 2\langle w, B^*\Sigma^{-1}x\rangle$$

$$\frac{d}{dt}(x^*\Sigma^{-1}x) = \|w\|^2 - \|w-B^*\Sigma^{-1}x\|^2$$

Integration from $t=-\infty$ to $t=0$ with $x(-\infty)=0$ and $x(0)=x_0$ gives

$$x_0^*\Sigma^{-1}x_0 = \|w\|_2^2 - \|w-B^*\Sigma^{-1}x\|_2^2 \leq \|w\|_2^2$$

$$\inf_{w\in L_2(-\infty,0]} \left\{ \|w\|_2^2 \ : \ x(0)=x_0 \right\} = x_0^*\Sigma^{-1}x_0.$$

Given $x(0)=x_0$ and $w=0$ for $t\geq 0$, the norm of $z(t)=Ce^{At}x_0$ can be found from

$$\int_0^\infty \|z(t)\|^2 dt = \int_0^\infty x_0^* e^{A^*t}C^*Ce^{At}x_0 dt = x_0^*\Sigma x_0$$

To show $\sigma_1 \leq \|G\|_\infty$, note that

$$\|G\|_\infty = \sup_{w\in L_2(-\infty,\infty)} \frac{\|g*w\|_2}{\|w\|_2} = \sup_{w\in L_2(-\infty,\infty)} \frac{\sqrt{\int_{-\infty}^{\infty}\|z(t)\|^2 dt}}{\sqrt{\int_{-\infty}^{\infty}\|w(t)\|^2 dt}} \geq \sup_{w\in L_2(-\infty,0]} \frac{\sqrt{\int_{-\infty}^{\infty}\|z(t)\|^2 dt}}{\sqrt{\int_{-\infty}^{\infty}\|w(t)\|^2 dt}}$$

$$\geq \sup_{w\in L_2(-\infty,0]} \frac{\sqrt{\int_0^{\infty}\|z(t)\|^2 dt}}{\sqrt{\int_{-\infty}^{0}\|w(t)\|^2 dt}} = \sup_{x_0\neq 0}\sqrt{\frac{x_0^*\Sigma x_0}{x_0^*\Sigma^{-1}x_0}} = \sigma_1$$

We shall now show the other inequalities. Since

$$G(s) := \int_0^\infty g(t)e^{-st}dt, \text{Re}(s)>0,$$

by the definition of H_∞ norm, we have

$$\|G\|_\infty = \sup_{\text{Re}(s)>0}\left\|\int_0^\infty g(t)e^{-st}dt\right\| \leq \sup_{\text{Re}(s)>0}\int_0^\infty \left\|g(t)e^{-st}\right\| dt \leq \int_0^\infty \|g(t)\| dt.$$

See next page for the proof of the last inequality.

To prove the last inequality, let e_i be the ith unit vector and note that $\sum_{i=1}^{n} e_i e_i^* = I$ and consider the case $s_i = 1$ (general case with a suitable change)

$$\int_0^\infty \|g(t)\| dt = \int_0^\infty \left\| Ce^{At/2} \sum_{i=1}^{n} e_i e_i^* e^{At/2} B \right\| dt \le \sum_{i=1}^{n} \int_0^\infty \left\| Ce^{At/2} e_i e_i^* e^{At/2} B \right\| dt$$
$$\le \sum_{i=1}^{n} \int_0^\infty \left\| Ce^{At/2} e_i \right\| \left\| e_i^* e^{At/2} B \right\| dt \le \sum_{i=1}^{n} \sqrt{\int_0^\infty \left\| Ce^{At/2} e_i \right\|^2 dt} \sqrt{\int_0^\infty \left\| e_i^* e^{At/2} B \right\|^2 dt}$$
$$\le 2 \sum_{i=1}^{n} \sigma_i$$

where we have used Cauchy-Schwarz inequality and the following relations

$$\int_0^\infty \left\| Ce^{At/2} e_i \right\|^2 dt = \int_0^\infty e_i^* e^{At/2} C^* C e^{At/2} e_i dt = 2 e_i^* \Sigma e_i = 2\sigma_i$$
$$= \int_0^\infty \left\| e_i^* e^{At/2} B \right\|^2 dt = \int_0^\infty e_i^* e^{At/2} B B^* e^{A^* t/2} e_i dt$$

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Balanced Model Reduction

Additive Model Reduction

$$G = G_r + \Delta_a, \Rightarrow \inf_{\deg(G_r) \le r} \| G - G_r \|_\infty$$

- Suppose

$$G(s) = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right]$$

is a balanced realization with Gramian Σ=diag (Σ_1, Σ_2)

$$A\Sigma + \Sigma A^* + BB^* = 0, \quad A^*\Sigma + \Sigma A + C^*C = 0$$

where

Σ_1 =diag ($\sigma_1 I_{s_1}$, $\sigma_2 I_{s_2}$,…, $\sigma_r I_{s_r}$) , Σ_2=diag ($\sigma_{r+1} I_{s_{r+1}}$, $\sigma_2 I_{s_{r+2}}$,…, $\sigma_N I_{s_N}$)

and $\sigma_1 > \sigma_2 > \ldots > \sigma_r > \sigma_{r+1} > \sigma_{r+2} > \ldots > \sigma_N \ge 0$,

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where σ_i has multiplicity s_i, $i = 1,2,\ldots,N$ and $s_1+s_2+\ldots+s_N = n$.

Then the truncated system

$$G_r(s) = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$$

is balanced and asymptotically stable. Furthermore

$\|G(s)-G_r(s)\|_\infty \le 2(\sigma_{r+1}+\sigma_{r+2}+\ldots+\sigma_N)$.

$\|G(s)-G(\infty)\|_\infty \le 2(\sigma_1+\sigma_2+\ldots+\sigma_N)$.

$\|G(s)-G_{N-1}(s)\|_\infty = 2\sigma_N$.

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Additive Reduction (proof)

- Proof. We shall first show the one step model reduction. Hence we shall assume $\Sigma_2 = \sigma_N I_{s_N}$. Define the approximation error

$$E_{11} := \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right] - \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right] = \left[\begin{array}{ccc|c} A_{11} & 0 & 0 & B_1 \\ 0 & A_{11} & A_{12} & B_1 \\ 0 & A_{21} & A_{22} & B_2 \\ \hline -C_1 & C_1 & C_2 & 0 \end{array}\right]$$

Apply a similarity transformation T to the preceding state-space realization with

$$T = \begin{bmatrix} I/2 & I/2 & 0 \\ I/2 & -I/2 & 0 \\ 0 & 0 & I \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} I & I & 0 \\ I & -I & 0 \\ 0 & 0 & I \end{bmatrix}$$

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to get

$$E_{11} = \left[\begin{array}{ccc|c} A_{11} & 0 & A_{12}/2 & B_1 \\ 0 & A_{11} & -A_{12}/2 & 0 \\ A_{21} & -A_{21} & A_{22} & B_2 \\ \hline 0 & -2C_1 & C_2 & 0 \end{array}\right]$$

Consider a dilation of $E_{11}(s)$:

$$E(s) = \begin{bmatrix} E_{11}(s) & E_{12}(s) \\ E_{21}(s) & E_{22}(s) \end{bmatrix} = \left[\begin{array}{ccc|cc} A_{11} & 0 & A_{12}/2 & B_1 & 0 \\ 0 & A_{11} & -A_{12}/2 & 0 & \sigma_N \Sigma_1^{-1} C_1^* \\ A_{21} & -A_{21} & A_{22} & B_2 & -C_2^* \\ \hline 0 & -2C_1 & C_2 & 0 & 2\sigma_N I \\ -2\sigma_N B_1^* \Sigma_1^{-1} & 0 & -B_2^* & -2\sigma_N I & 0 \end{array}\right]$$
$$=: \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \tilde{D} \end{array}\right]$$

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Then it is easy to verify that

$$\tilde{P} = \begin{bmatrix} \Sigma_1 & 0 & 0 \\ 0 & \sigma_N^2 \Sigma_1^{-1} & 0 \\ 0 & 0 & 2\sigma_N I_{s_N} \end{bmatrix}, \quad \tilde{A}\tilde{P} + \tilde{P}\tilde{A}^* + \tilde{B}\tilde{B}^* = 0, \ \tilde{P}\tilde{C}^* + \tilde{B}\tilde{D}^* = 0$$

Using these equations, we have

$$E(s)E^{\sim}(s) = \left[\begin{array}{cc|c} \tilde{A} & -\tilde{B}\tilde{B}^* & \tilde{B}\tilde{D}^* \\ 0 & -\tilde{A}^* & \tilde{C}^* \\ \hline C & -\tilde{D}\tilde{B}^* & \tilde{D}\tilde{D}^* \end{array}\right] = \left[\begin{array}{cc|c} \tilde{A} & -\tilde{A}\tilde{P} - \tilde{P}\tilde{A}^* - \tilde{B}\tilde{B}^* & \tilde{P}\tilde{C}^* + \tilde{B}\tilde{D}^* \\ 0 & -\tilde{A}^* & \tilde{C}^* \\ \hline \tilde{C} & -\tilde{C}\tilde{P} - \tilde{D}\tilde{B}^* & \tilde{D}\tilde{D}^* \end{array}\right]$$
$$= \left[\begin{array}{cc|c} \tilde{A} & 0 & 0 \\ 0 & -\tilde{A}^* & \tilde{C}^* \\ \hline \tilde{C} & 0 & \tilde{D}\tilde{D}^* \end{array}\right] = \tilde{D}\tilde{D}^* = 4\sigma_N^2 I$$

where a second quality is obtained by applying a similarity transformation

$$T = \begin{bmatrix} I & \tilde{P} \\ 0 & I \end{bmatrix}$$

Hence $\|E_{11}\|_\infty \le \|E\|_\infty = 2\sigma_N$, which is the desired result.

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The remainder of the proof is achieved by using the order reduction by one-step results and by noting that

$$G_k(s)=\left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$$

obtained by the "kth" order partitioning is internally balanced with balanced Gramian given by $\Sigma_1=\text{diag}\,(\sigma_1 I_{s_1}, \sigma_2 I_{s_2},\ldots, \sigma_r I_{s_r})$

Let $E_k(s)=G_{k+1}(s)-G_k(s)$ for $k=1,2,\ldots,N-1$ and let $G_N(s)=G(s)$. Then

$$\bar{\sigma}[E_k(j\omega)] \le 2\sigma_{k+1}$$

Since $G_k(s)$ is a reduced-order model obtained from the internally balanced realization of $G_{k+1}(s)$ and the bound for one-step order reduction holds.

Noting that $G(s)-G_r(s)=\sum_{k=r}^{N-1} E_k(s)$

by the definition of $E_k(s)$, we have

$$\bar{\sigma}[G(j\omega)-G_r(j\omega)] \le \sum_{k=r}^{N-1}\bar{\sigma}[E_k(j\omega)] \le 2\sum_{k=r}^{N-1}\sigma_{k+1}$$

This is the desired upper bound.

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Tightness of the Bounds

- Bound can be tight. For example,

$$G(s)=\sum_{j=1}^{N}\frac{b_i}{s+a_i}=\left[\begin{array}{cccc|c} -a_1 & & & & \sqrt{b_1} \\ & -a_2 & & & \sqrt{b_2} \\ & & \ddots & & \vdots \\ & & & -a_n & \sqrt{b_n} \\ \hline \sqrt{b_1} & \sqrt{b_2} & \cdots & \sqrt{b_n} & 0 \end{array}\right]$$

with $a_i>0$ and $b_i>0$. Then

$$\mathrm{P}=\mathrm{Q}=\left[\frac{\sqrt{b_i b_j}}{a_i+a_j}\right] \text{ and } \|G(s)\|_\infty = G(0)=\sum_{i=1}^{n}\frac{b_i}{a_i}=2\,\text{trace}(P)=2\sum_{i=1}^{n}\sigma_i$$

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- Bound can also be loose for systems with Hankel singular values close to each other. For example,

$$G(s)=\left[\begin{array}{cccc|c} -19.9579 & -5.4682 & 9.6954 & 0.9160 & -6.3180 \\ 5.4682 & 0 & 0 & 0.2378 & 0.0020 \\ -9.6954 & 0 & 0 & -4.0051 & -0.0067 \\ 0.9160 & -0.2378 & 4.0051 & -0.0420 & 0.2893 \\ \hline -6.3180 & -0.0020 & 0.0067 & 0.2893 & 0 \end{array}\right]$$

with Hankel singular values given by

$\sigma_1=1,\ \sigma_2=0.9977,\ \sigma_3=0.99570,\ \sigma_4=0.9952$

r	0	1	2	3
$\|G-G_r\|_\infty$	2	1.996	1.991	1.9904
Bounds: $2\sum_{i=r+1}^{4}\sigma_i$	7.9772	5.9772	3.9818	1.9904
$2\sigma_{r+1}$	2	1.9954	1.9914	1.9904

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Frequency-Weighted Reduction

- General Case: given weights W_o and W_i, find G_r such that

$$\inf_{\deg(G_r)\ r}\|W_0(G-G_r)W_i\|$$

Let

$$G=\left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right], W_i=\left[\begin{array}{c|c} A_i & B_i \\ \hline C_i & D_i \end{array}\right], W_0=\left[\begin{array}{c|c} A_0 & B_0 \\ \hline C_0 & D_0 \end{array}\right],$$

$$W_0 G W_i=\left[\begin{array}{ccc|c} A & 0 & BC_i & BD_i \\ B_0C & A_0 & 0 & 0 \\ 0 & 0 & A_i & B_i \\ \hline D_0C & C_0 & 0 & 0 \end{array}\right]=:\left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & 0 \end{array}\right]$$

and solving the following Lyapunov equations

$$\tilde{A}\tilde{P}+\tilde{P}\tilde{A}^*+\tilde{B}\tilde{B}^*=0$$
$$\tilde{Q}\tilde{A}+\tilde{A}^*\tilde{Q}+\tilde{C}^*\tilde{C}=0.$$

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Frequency-Weighted Reduction

The input/output weighted Gramians P and Q are defined by

$$P:=\begin{bmatrix} I_n & 0 \end{bmatrix}\tilde{P}\begin{bmatrix} I_n \\ 0 \end{bmatrix},\quad Q:=\begin{bmatrix} I_n & 0 \end{bmatrix}\tilde{Q}\begin{bmatrix} I_n \\ 0 \end{bmatrix}.$$

P and Q satisfy the following lower order equations

$$\begin{bmatrix} A & BC_i \\ 0 & A_i \end{bmatrix}\begin{bmatrix} P & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix}+\begin{bmatrix} P & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix}\begin{bmatrix} A & BC_i \\ 0 & A_i \end{bmatrix}^*+\begin{bmatrix} BD_i \\ B_i \end{bmatrix}\begin{bmatrix} BD_i \\ B_i \end{bmatrix}^*=0$$

$$\begin{bmatrix} Q & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix}\begin{bmatrix} A & 0 \\ B_0C & A_0 \end{bmatrix}+\begin{bmatrix} A & 0 \\ B_0C & A_0 \end{bmatrix}\begin{bmatrix} Q & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix}^*+\begin{bmatrix} C^*D_0^* \\ C_0^* \end{bmatrix}\begin{bmatrix} C^*D_0^* \\ C_0^* \end{bmatrix}^*=0.$$

If $W_i=I$, then P can be obtained from

$$PA^*+AP+BB^*=0$$

If $W_o=I$, then Q can be obtained from

$$QA+A^*Q+C^*C=0$$

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Frequency-Weighted Reduction

Now let T be a nonsingular matrix such that

$$TPT^*=(T^{-1})^*QT^{-1}=\begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix}$$

(i.e., balanced) and partition the system accordingly as

$$\left[\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & 0 \end{array}\right]=\left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & 0 \end{array}\right].$$

Then a reduced order model G_r is obtained as

$$G_r=\left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & 0 \end{array}\right].$$

Works well but without guarantee in general.

```
>>[Gred, redinfo] =
    reduce(G,'weight',{wo,wi},'order',[4:2:10])
```

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Relative Reduction

- Consider a special frequency-weighted model reduction problem:

$$G_r = G(I+\Delta_{rel}), \Rightarrow \inf_{\deg(G_r)\le r} \left\| G^{-1}(G-G_r) \right\|_\infty$$

This is a relative error approximation problem.

Consider also a related multiplicative error reduction problem

$$G = G_r(I+\Delta_{mul})$$

Let $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in RH_\infty$ be minimum phase and D be nonsingular.

Then

$$W_0 = G^{-1}(s) = \left[\begin{array}{c|c} A-BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array}\right].$$

Now apply the frequency-weighted model reduction to get

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(a) Then the input/output weighted Gramians P and Q are given by

$$PA^*+AP+BB^*=0$$

$$Q(A-BD^{-1}C)+(A-BD^{-1}C)^*Q+C^*(D^{-1})^*D^{-1}C=0$$

(b) Suppose P and Q are balanced:

$P=Q=\Sigma=\text{diag}\,(\Sigma_1, \Sigma_2)=\text{diag}\,(\sigma_1 I_{s_1}, \ldots, \sigma_r I_{s_r}, \sigma_{r+1} I_{s_{r+1}}, \sigma_N I_{s_N})$

and let G be partitioned compatibly with Σ_1, Σ_2 as

$$G(s) = \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right].$$

```
>> [Gr,redinfo]=bstmr(G,r)
```

Then $G_r(s) = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & D \end{array}\right]$ is stable and minimum phase.

Furthermore

$$\|\Delta_{rel}\|_\infty \le \prod_{i=r+1}^{N} \left(1+2\sigma_i\left(\sqrt{1+\sigma_i^2}+\sigma_i\right)\right)-1$$

$$\|\Delta_{mul}\|_\infty \le \prod_{i=r+1}^{N} \left(1+2\sigma_i\left(\sqrt{1+\sigma_i^2}+\sigma_i\right)\right)-1$$

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Chapter 8 Uncertainty and Robustness

- model uncertainty
- small gain theorem
- additive uncertainty
- multiplicative uncertainty
- coprime factor uncertainty
- other tests
- robust performance
- skewed specifications
- example: siso vs mimo

Model Uncertainty

Suppose P is the nominal model and K is a controller.

Nominal Stability (NS): if K stabilizes the nominal P.

Robust Stability (RS): if K stabilizes every plant in Π.

Nominal Performance (NP): if the performance objectives are satisfied for the nominal plant P.

Robust Performance(RP): if the performance objectives are satisfied for every plant in Π.

$$P_\Delta(s) = P(s) + w(s)\Delta(s), \quad \bar{\sigma}[\Delta(j\omega)] < 1, \forall\omega$$

$$P_\Delta(s) = (I + w(s)\Delta(s))P(s).$$

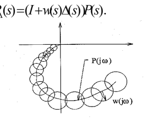
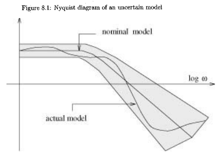


Figure 8.1: Nyquist diagram of an uncertain model



Example 1: Consider a uncertain transfer function

$$P(s,\alpha,\beta) = \frac{10((2+0.2\alpha)s^2+(2+0.3\alpha+0.4\beta)s+(1+0.2\beta))}{(s^2+0.5s+1)(s^2+2s+3)(s^2+3s+6)}, \quad \alpha,\beta \in [-1,1]$$

$$P(s,\alpha,\beta) \in \{P_0 + W\Delta \mid \|\Delta\| \le 1\}$$

with $P_0 := P(s,0,0)$ and

$$W(s) = P(s,1,1) - P(s,0,0) = \frac{10(0.2s^2+0.7s+0.2)}{(s^2+0.5s+1)(s^2+2s+3)(s^2+3s+6)}$$

The frequency response $P_0+W\Delta$ is shown in Figure 8.14 as circles

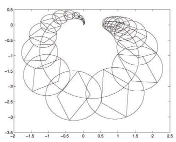


Figure 8.14: Nyquist diagram of uncertain system and disk covering

Another way to bound the frequency response is to treat and as norm bounded uncertainities; that is,

$P(s,\alpha,\beta) \in \{P_0+W_1\Delta_1+W_2\Delta_2 \mid \|\Delta_i\|_\infty \le 1\}$

with $P_0=P(s,0,0)$ and

$$W_1 = \frac{10(0.2s^2+0.3s)}{(s^2+0.5s+1)(s^2+2s+3)(s^2+3s+6)}, \quad W_2 = \frac{10(0.4s+0.2)}{(s^2+0.5s+1)(s^2+2s+3)(s^2+3s+6)}$$

It is in fact easy to show that

$\{P_0+W_1\Delta_1+W_2\Delta_2 \mid \|\Delta_i\|_\infty \le 1\} = \{P_0+W\Delta \mid \|\Delta\|_\infty \le 1\}$

with $|W|=|W_1|+|W_2|$.

The frequency response $P_0+W\Delta$ is shown in Figure 8.15. This bounding is clearly more conservative.

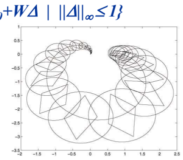


Figure 8.15: A conservative covering

Example 2: Consider a process control model

$$G(s)=\frac{ke^{-\tau s}}{Ts+1}, 4\le k\le 9, 2\le T\le 3, 1\le \tau\le 2.$$

Take the nominal model as

$$G_0(s)=\frac{6.5}{(2.5s+1)(1.5s+1)}$$

Then for each frequency, all possible frequency responses are in a box, as shown in Figure 8.16.

$\Delta_a(j\omega)=G(j\omega)-G_0(j\omega)$

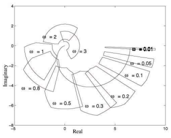


Figure 8.16: uncertain delay system and G_0

To get an additive weighting W_a, use the following Matlab procedure:

```
>> mf=ginput(50)   % pick 50 points: the first column of mf is the frequency points and the second column of mf is the corresponding magnitude responses.
>>magg=vpck(mf(:2),mf(:,1)); % pack them as a varying matrix.
>> Wa=fitmag(magg); % choose the order of Wa online. A third-order Wa is sufficient for this example.
>> [A,B,C,D]=unpck(Wa )   %converting into state-space.
>> [Z,P,K]=ss2zp(A,B,C,D) % converting into zero/pole/gain form.
```

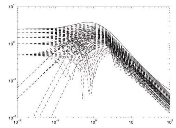


Figure 8.17: Δ_a (dashed line) and a bound W_a (solid line)

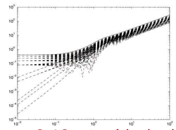


Figure 8.18: Δ_m (dashed line) and a bound W_m (solid line)

We get

$$W_a(s)=\frac{0.0376(s+116.4808)(s+7.4514)(s+0.2674)}{(s+1.2436)(s+0.5575)(s+4.95081)}$$

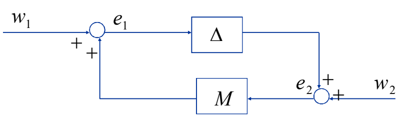
And the frequency response of W_a is also plotted in Figure 8.17. Similarly, define the multiplicative uncertainty

$$\Delta_m(s):=\frac{G(s)-G_0(s)}{G_0(s)}$$

and a W_m can be found such that $|\Delta_m(j\omega)|\le|W_m(j\omega)|$ as shown in Figure 8.18. A W_m is given by

$$W_m=\frac{2.8169(s+0.212)(s^2+2.6128s+1.732)}{s^2+2.2425s+2.6319}$$

Small Gain Theorem



- Small Gain Theorem: Suppose $M\in(RH_\infty)^{p\times q}$. Then the system is well-posed and internally stable for all $\Delta\in RH_\infty$ with

  (a) $\|\Delta\|_\infty\le 1/\gamma$ if and only if $\|M\|_\infty<\gamma$;

  (b) $\|\Delta\|_\infty< 1/\gamma$ if and only if $\|M\|_\infty\le\gamma$.

- Proof: Assume $\gamma = 1$. System is stable iff $\det(I-M\Delta)$ has no zero in the closed right-half plane for all $\Delta\in RH_\infty$ and $\|\Delta\|_\infty\le 1$.

  ($\Leftarrow$) $\det(I-M\Delta)\ne 0$ for all $\Delta\in RH_\infty$ and $\|\Delta\|_\infty\le 1$ since

  $$|\lambda(I-M\Delta)|\ge 1-\max|\lambda(M\Delta)|\ge 1-\|M\|_\infty>0$$

  ($\Rightarrow$) Suppose $\|M\|_\infty\ge 1$. There exists a $\Delta\in RH_\infty$ with $\|\Delta\|_\infty\le 1$ such that $\det(I-M(s)\Delta(s))$ has a zero on the imaginary axis, so the system is unstable. Suppose $\omega\in R_+\cup\{\infty\}$ is such that $\sigma_1(M(j\omega_0))\ge 1$. Let $M(j\omega_0)=U(j\omega_0)\Sigma(j\omega_0)V^*(j\omega_0)$ be a singular value decomposition with

  $$U(j\omega_0)=[u_1,u_2,\dots,u_p],\ V(j\omega_0)=[v_1,v_2,\dots,v_p],$$

  $$\Sigma(j\omega_0)=\mathrm{diag}[\sigma_1,\sigma_2,\dots]$$

We shall construct a $\Delta\in RH_\infty$ such that $\Delta(j\omega_0)=(1/\sigma_1)v_1u_1^*$ and $\|\Delta\|_\infty\le 1$. Indeed, for such $\Delta(s)$,

$$\det(I-M(j\omega_0)\Delta(j\omega_0))=\det(I-U(j\omega_0)\Sigma(j\omega_0)V^*(j\omega_0)(1/\sigma_1)v_1u_1^*)$$

$$=1-u_1^*U(j\omega_0)\Sigma(j\omega_0)V^*(j\omega_0)v_1(1/\sigma_1)=0$$

and thus the closed-loop system is either not well-posed (if $\omega_0=\infty$) or unstable (if $\omega_0\in R_+$). There are two different cases:

(1) $\omega_0=0$ or ∞: Then U and V are real matrices. Choose

$$\Delta=(1/\sigma_1)v_1u_1^*\in R^{q\times p}$$

(2) $0<\omega_0<\infty$: write u_1 and v_1 in the following form

$$u_1^*=\begin{bmatrix}u_{11}e^{j\theta_1} & u_{12}e^{j\theta_2} & \cdots & u_{1p}e^{j\theta_p}\end{bmatrix},\ v_1=\begin{bmatrix}v_{11}e^{j\phi_1}\\ v_{12}e^{j\phi_2}\\ \vdots\\ v_{1q}e^{j\phi_q}\end{bmatrix}$$

where $u_{1i}, v_{1j}\in R$ are chosen so that $\theta_i, \phi_j\in[-\pi,0)$.

Choose $\beta_i \geq 0$ and $\alpha_j \geq 0$ so that

$$\angle\left(\frac{\beta_i - j\omega_0}{\beta_i + j\omega_0}\right) = \theta_i, \quad \angle\left(\frac{\alpha_j - j\omega_0}{\alpha_j + j\omega_0}\right) = \phi_j$$

Let

$$\Delta(s) = \frac{1}{\sigma_1}\begin{bmatrix} v_{11}\frac{\alpha_1 - s}{\alpha_1 + s} \\ \vdots \\ v_{1q}\frac{\alpha_q - s}{\alpha_q + s} \end{bmatrix}\begin{bmatrix} u_{11}\frac{\beta_1 - s}{\beta_1 + s} & \cdots & u_{1p}\frac{\beta_p - s}{\beta_p + s} \end{bmatrix} \in RH_\infty.$$

Then $\|\Delta\|_\infty = 1/\sigma_1 \leq 1$ and $\Delta(j\omega_0) = (1/\sigma_1)\, v_1 u_1^*$.

The small gain theorem still holds even if Δ and M are infinite dimensional. This is summarized as the following corollary.

- Corollary: The following statements are equivalent:

(i) The system is well-posed and internally stable for all $\Delta \in H_\infty$ with $\|\Delta\|_\infty \leq 1/\gamma$.

(ii) The system is well-posed and internally stable for all $\Delta \in RH_\infty$ with $\|\Delta\|_\infty \leq 1/\gamma$.

(iii) The system is well-posed and internally stable for all $\Delta \in C^{q \times p}$ with $\|\Delta\| \leq 1/\gamma$.

(iv) $\|M\|_\infty \leq \gamma$

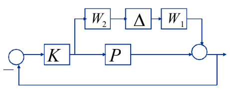
It can be shown that the small gain condition is sufficient to guarantee internal stability even if Δ is a nonlinear and time varying "stable" operator with an appropriately defined stability notion, see Desoer and Vidyasagar [1975].

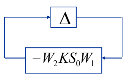
Additive Uncertainty

Define

$S_o = (I+PK)^{-1}$, $T_o = PK(I+PK)^{-1}$
$S_i = (I+KP)^{-1}$, $T_i = KP(I+KP)^{-1}$

- Let $\Pi = \{P + W_1 \Delta W_2 : \Delta \in RH_\infty\}$ and let K stabilize P. Then the closed-loop system is well-posed and internally stable for all $\|\Delta\|_\infty < 1$ if and only if $\|W_2 K S_o W_1\|_\infty \leq 1$.



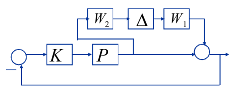


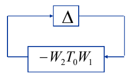
Multiplicative Uncertainty

Define

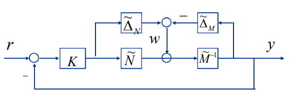
$S_o = (I+PK)^{-1}$, $T_o = PK(I+PK)^{-1}$
$S_i = (I+KP)^{-1}$, $T_i = KP(I+KP)^{-1}$

- Let $\Pi = \{(I + W_1 \Delta W_2)P : \Delta \in RH_\infty\}$ and let K stabilize P. Then the closed-loop system is well-posed and internally stable for all $\|\Delta\|_\infty < 1$ if and only if $\|W_2 T_o W_1\|_\infty \leq 1$.





Coprime Factor Uncertainty



- Let $P = \tilde{M}^{-1}\tilde{N}$ be stable left coprime factorization and let K stabilize P. Suppose

$$\Pi = (\tilde{M} + \tilde{\Delta}_M)^{-1}(\tilde{N} + \tilde{\Delta}_N), \quad \Delta := \begin{bmatrix} \tilde{\Delta}_N & \tilde{\Delta}_M \end{bmatrix}$$

with $\tilde{\Delta}_M, \tilde{\Delta}_N \in RH_\infty$. Then the closed-loop system is well-posed and internally stable for all $\|\Delta\|_\infty < 1$ if and only if

$$\left\| \begin{bmatrix} K \\ I \end{bmatrix} (I + PK)^{-1} \tilde{M}^{-1} \right\|_\infty \leq 1.$$

Other Tests

$W_1 \in RH_\infty$ $W_2 \in RH_\infty$ $\Delta \in RH_\infty$, $\|\Delta\|_\infty < 1$		
Perturbed Model Sets Π	Representative Types of Uncertainty Characterized	Robust Stability Tests
$(I + W_1 \Delta W_2)P$	output (sensor) errors neglected HF dynamics uncertain rhp zeros	$\|W_2 T_0 W_1\|_\infty \leq 1$
$P(I + W_1 \Delta W_2)$	Input (actuators) errors neglected HF dynamics uncertain rhp zeros	$\|W_2 T_i W_1\|_\infty \leq 1$
$(I + W_1 \Delta W_2)^{-1} P$	LF parameter errors uncertain rhp poles	$\|W_2 S_0 W_1\|_\infty \leq 1$

$P(I+W_1\Delta W_2)^{-1}$	LF parameter errors uncertain rhp poles	$\lVert W_2 S_i W_1\rVert_\infty \le 1$
$P+W_1\Delta W_2$	Additive plant errors neglected HF dynamics uncertain rhp zeros	$\lVert W_2 K S_0 W_1\rVert_\infty \le 1$
$P(I+W_1\Delta W_2 P)^{-1}$	LF parameter errors uncertain rhp poles	$\lVert W_2 S_0 P W_1\rVert_\infty \le 1$
$(\tilde{M}+\tilde{\Delta}_M)^{-1}(\tilde{N}+\tilde{\Delta}_N)$ $P=\tilde{M}^{-1}\tilde{N}$ $\Delta=\begin{bmatrix}\tilde{\Delta}_N & \tilde{\Delta}_M\end{bmatrix}$	LF parameter errors neglected HF dynamics uncertain rhp poles & zeros	$\left\lVert \begin{bmatrix}K\\ I\end{bmatrix} S_0 \tilde{M}^{-1}\right\rVert_\infty \le 1$
$(N+\Delta_N)(M+\Delta_M)^{-1}$ $P=NM^{-1}$ $\Delta=\begin{bmatrix}\Delta_N\\ \Delta_M\end{bmatrix}$	LF parameter errors neglected HF dynamics uncertain rhp poles & zeros	$\lVert M^{-1} S_i \begin{bmatrix}K & I\end{bmatrix}\rVert_\infty \le 1$

Robust Performance

- **Consider a feedback system with uncertain plant P_Δ, the weighted sensitivity function is defined as the transfer function from d to e:**

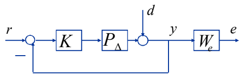
  $T_{ed} = W_e (I + P_\Delta K)^{-1}$

  Suppose our performance objective is

  $\|T_{ed}\|_\infty = \sup_{\{\|d\|_2 \le 1\}} \|e\|_2 \le 1$

- **Suppose $P_\Delta \in \{(I+\Delta W_2)P: \Delta \in RH_\infty, \|\Delta\|_\infty < 1\}$ and K stabilizes P. Then the**

  $\bar{\sigma}(W_e S_0) + \bar{\sigma}(W_2 T_0) \le 1.$



Proof:

$$\bar{\sigma}(T_{ed}) \le \bar{\sigma}(W_e S_0)\bar{\sigma}[(I+\Delta W_2 T_0)^{-1}] = \frac{\bar{\sigma}(W_e S_0)}{\underline{\sigma}(I+\Delta W_2 T_0)} \le \frac{\bar{\sigma}(W_e S_0)}{1-\bar{\sigma}(\Delta W_2 T_0)} \le \frac{\bar{\sigma}(W_e S_0)}{1-\bar{\sigma}(W_2 T_0)}.$$

Skewed Specifications

- **Suppose $P_\Delta \in \{P(I+w\Delta): \Delta \in RH_\infty, \|\Delta\|_\infty < 1\}$ and K stabilizes P.**

  robust stability:

  $\|wT_i\|_\infty \le 1$

  nominal performance:

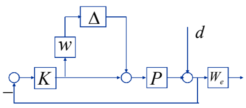
  $\|W_e S_o\|_\infty \le 1.$

  robust performance is guaranteed if

  $\bar{\sigma}(W_e S_0) + \kappa(P)\bar{\sigma}(wT_i) \le 1$

  $\bar{\sigma}(W_e S_0) + \kappa(P)\bar{\sigma}(wT_0) \le 1.$

  Condition number!!!!



Proof: Note that

$T_{ed} = W_e S_o (I + P\Delta w K S_o)^{-1}$

$= W_e S_o (I + P\Delta P^{-1} w T_o)^{-1}$

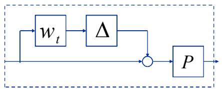
The Role of Condition Number

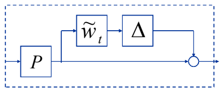
- **Covering Uncertainty:**

$$\Pi_1 := \{P(I+w_t\Delta): \Delta \in RH_\infty, \|\Delta\|_\infty < 1\}, \quad \Pi_2 := \{(I+\tilde{w}_t\Delta)P: \Delta \in RH_\infty, \|\Delta\|_\infty < 1\}.$$

$$\Pi_2 \supseteq \Pi_1 \ \text{ if } \ |\tilde{w}_t| \ge |w_t|\kappa(P) \ \ \forall\omega$$

since $P(I+w_t\Delta) = (I+\tilde{w}_t P\Delta P^{-1})P.$





$$P(s) = \left[\begin{array}{ccc|cc} -0.2 & 0.1 & 1 & 0 & 1 \\ -0.05 & 0 & 0 & 0 & 0.7 \\ 0 & 0 & -1 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right] = \frac{1}{a(s)}\begin{bmatrix} s & (s+1)(s+0.07) \\ -0.05 & 0.7(s+1)(s+0.13) \end{bmatrix}$$

$$P(s) = \left[\begin{array}{ccc|cc} -0.2 & 0.1 & 1 & 0 & 1 \\ -0.05 & 0 & 0 & 0 & 0.7 \\ 0 & 0 & -1 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right] = \frac{1}{a(s)}\begin{bmatrix} s & (s+1)(s+0.07) \\ -0.05 & 0.7(s+1)(s+0.13) \end{bmatrix}$$

where $a(s) = (s+1)(s+0.1707)(s+0.02929)$.

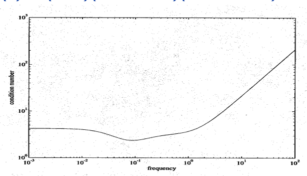


Figure 8.16: Condition number $\kappa(\omega) = \bar{\sigma}(P(j\omega))/\underline{\sigma}(P(j\omega))$

Example: SISO vs MIMO

- **Spinning body with constant velocity in the z direction.**

$$\begin{bmatrix}\dot{\omega}_1\\ \dot{\omega}_2\end{bmatrix} = \begin{bmatrix}0 & a\\ -a & 0\end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix} + \begin{bmatrix}u_1\\ u_2\end{bmatrix},$$

$$y = \begin{bmatrix}1 & a\\ -a & 1\end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}$$

$$P(s) = \frac{1}{s^2+a^2}\begin{bmatrix}s-a^2 & a(s+1)\\ -a(s+1) & s-a^2\end{bmatrix}.$$

$$S = (I+P)^{-1} = \frac{1}{s+1}\begin{bmatrix}s & -a\\ a & s\end{bmatrix},$$

$$T = P(I+P)^{-1} = \frac{1}{s+1}\begin{bmatrix}1 & a\\ -a & 1\end{bmatrix}.$$

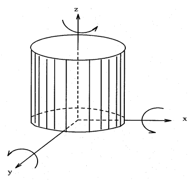
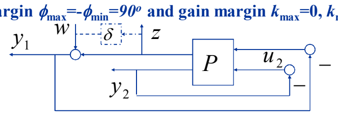


Figure 8.17: Spinning body

- Each loop has the open-loop transfer function as *1/s* so each loop has phase margin $\phi_{max}=-\phi_{min}=90^o$ and gain margin $k_{max}=0$, $k_{min}=\infty$



- Suppose one loop transfer function is perturbed

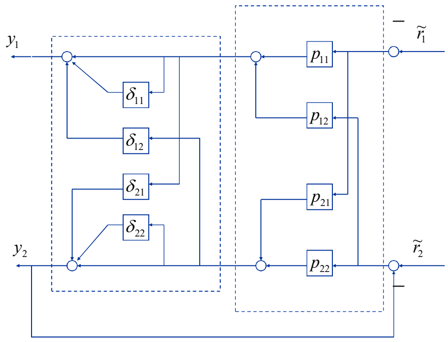
  Denote $z(s)/w(s)=-T_{11}(s) = -1/(s+1)$. Then the maximum allowable perturbation is $\|\delta\|_\infty < 1/\|T_{11}(s)\|_\infty = 1$, which is independent of *a*.

- However, if both loops are perturbed at the same time, then the maximum allowable perturbation is much smaller, as shown below

$$P_\Delta = (I+\Delta)P, \quad \Delta = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix} \in RH_\infty, \ \|\Delta\|_\infty < \gamma$$

The system is robustly stable for every such Δ iff

$$\gamma \le \frac{1}{\|T\|_\infty} = \frac{1}{\sqrt{1+a^2}} \quad (\ll 1 \text{ if } a \gg 1).$$



- In particular, consider

$$\Delta = \Delta_d = \begin{bmatrix} \delta_{11} & \\ & \delta_{22} \end{bmatrix} \in \mathrm{R}^{2\times 2}.$$

Then the closed-loop system is stable for every such Δ iff

$$\det(I+T\Delta_d) = \frac{(s^2+(2+\delta_{11}+\delta_{22})s+1+\delta_{11}+\delta_{22}+(1+a^2)\delta_{11}\delta_{22})}{(s+1)^2}$$

has no zero in the closed right-half plane.

Hence the stability region is given by

$2+\delta_{11}+\delta_{22}>0$,
$1+\delta_{11}+\delta_{22}+(1+a^2)\delta_{11}\delta_{22}>0$

The system is unstable with

$\delta_{11}=-\delta_{22}=(1+a^2)^{-1/2}$