Institute of Systems Science and Intelligent Control Technology
系统科学与智能控制技术研究所

鲁棒控制：建模、跟踪、抗扰、容错

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2020年6月25日
（第十六讲）

提纲

2

Normalized Coprime H_∞ Control

Let $D=0$ and let $P=\tilde{M}^{-1}\tilde{N}$ be normalized coprime factorization. Then

$$\gamma_{\min} := \inf_{K\ stabilizing} \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I+PK)^{-1}\tilde{M}^{-1} \right\|_\infty = \frac{1}{\sqrt{1-\lambda_{\max}(YQ)}}$$

where Y and Q are the solutions to

$$AY + YA^* - YC^*CY + BB^* = 0 \qquad Q(A-YC^*C)+(A-YC^*C)^*Q + C^*C = 0.$$

Moreover, for any $\gamma > \gamma_{\min}$ a controller achieving

$$b_{P,K} := \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I+PK)^{-1}\tilde{M}^{-1} \right\|_\infty = \left\| \begin{bmatrix} K \\ I \end{bmatrix} (I+PK)^{-1}\begin{bmatrix} I & P \end{bmatrix} \right\|_\infty < \gamma$$

is given by $K(s) = \left[\begin{array}{c|c} A-BB^*X_\infty - YC^*C & -YC^* \\ \hline -B^*X_\infty & 0 \end{array}\right]$ where $X_\infty = \frac{\gamma^2}{\gamma^2-1}Q\left(I - \frac{\gamma^2}{\gamma^2-1}YQ\right)^{-1}$.

- $P_\Delta = \left(\tilde{M}+\tilde{\Delta}_M\right)^{-1}\left(\tilde{N}+\tilde{\Delta}_N\right)$ with $\left\|\begin{bmatrix}\tilde{\Delta}_N & \tilde{\Delta}_M\end{bmatrix}\right\|_\infty < \varepsilon.$

Then there is a robustly stabilizing controller for P_Δ if and only if

$$\varepsilon \le \sqrt{1-\lambda_{\max}(YQ)} = b_{\mathrm{opt}}(P) = \min_K b_{P,K}$$

为什么这个指标？

$b_{P,K} > 0$ 意味着 K 也鲁棒镇定:

- $P_a = P + \Delta_a$（加性不确定性）其中 P_a 与 P 具有相同的不稳定极点并且 $\|\Delta_a\|_\infty < b_{P,K}$.
- $P_m = (I+\Delta_m)P$（乘性不确定性）其中 P_m 与 P 具有相同的不稳定极点并且 $\|\Delta_m\|_\infty < b_{P,K}$.
- $P_f = (I+\Delta_f)^{-1}P$（反馈不确定性）其中 P_f 与 P 具有相同的不稳定极点并且 $\|\Delta_f\|_\infty < b_{P,K}$.
- $P_{fac} = (M+\Delta_m)^{-1}(N+\Delta_n)$, $P = M^{-1}N$ 并且 $\|[\Delta_m\ \Delta_n]\|_\infty < b_{P,K}$.

鲁棒性对P 与 K是一样的： $b_{P,K} = b_{K,P}$

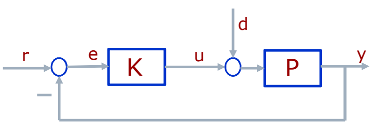
（没有控制器的脆弱性）

为什么这个指标？

SISO P：增益裕度 $\ge \frac{1+b_{P,K}}{1-b_{P,K}}$ 相位裕度 $\ge 2\arcsin(b_{P,K})$

$$\begin{bmatrix} I \\ K \end{bmatrix}(I+PK)^{-1}\begin{bmatrix} I & P \end{bmatrix} = \begin{bmatrix} (I+PK)^{-1} & (I+PK)^{-1}P \\ K(I+PK)^{-1} & K(I+PK)^{-1}P \end{bmatrix} = \begin{bmatrix} T_{er} & -T_{ed} \\ T_{ur} & -T_{ud} \end{bmatrix}$$

性能不要求每项都小，也不可能



Proof. Note that for SISO system

$$b_{P,K} \le \frac{|1+P(j\omega)K(j\omega)|}{\sqrt{1+|P(j\omega)|^2}\sqrt{1+|K(j\omega)|^2}}, \quad \forall\omega.$$

So, at frequencies where $k := -P(j\omega)K(j\omega) \in \mathrm{R}^+$,

$$b_{P,K} \le \frac{|1-k|}{\sqrt{\left(1+|P|^2\right)(1+\frac{k^2}{|P|^2})}} \le \frac{|1-k|}{\sqrt{\min_P\left\{\left(1+|P|^2\right)(1+\frac{k^2}{|P|^2})\right\}}} = \left|\frac{1-k}{1+k}\right|,$$

which implies that $k \le \frac{1-b_{P,K}}{1+b_{P,K}}$ or $k \ge \frac{1+b_{P,K}}{1-b_{P,K}}$

From which the gain margin result follows.

Similarly, at frequencies where $P(j\omega)K(j\omega) = -e^{j\theta}$

$$b_{P,K} \le \frac{\left|1-e^{j\theta}\right|}{\sqrt{\left(1+|P|^2\right)(1+\frac{1}{|P|^2})}} \le \frac{\left|2\sin\frac{\theta}{2}\right|}{\sqrt{\min_P\left\{\left(1+|P|^2\right)(1+\frac{1}{|P|^2})\right\}}} = \frac{\left|2\sin\frac{\theta}{2}\right|}{2},$$

which implies $\theta \ge 2\arcsin(b_{P,K})$.

For example, $b_{\mathrm{P,K}} = 1/2$ guarantees a gain margin of 3 and a phase margin of 60°.

>> $b_{p,k}$ = emargin (P,K); % given *P* and *K*, compute $b_{P,K}$

>> [K_{opt}, $b_{p,k}$]=ncfsyn(P,1); % find the optimal controller K_{opt}.

>> [K_{sub}, $b_{p,k}$] = ncfsyn(P,2); % find a suboptimal controller K_{sub}.

H_∞ Loop Shaping Design

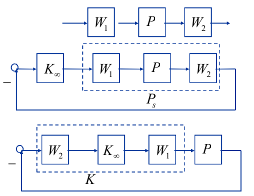
Given nominal model *P*(*s*).

- **(1) Loop Shaping: Obtain a desired open-loop shape (singular values) by using a precompensator W_1 and/or a postcompensator W_2,**

  $$P_s = W_2 P W_1$$

  Assume that W_1 and W_2 are such that P_s contains no hidden modes.
- **(2) (a) Calculate robust stability margin $b_{opt}(P_s)$. If $b_{opt}(P_s)$ <<1, return to (1) and adjust W_1 and W_2. (b) Select $\varepsilon \le b_{opt}(P_s)$, then synthesize a stabilizing controller K_∞ which satisfies**

  $$\left\| \begin{bmatrix} I \\ K_\infty \end{bmatrix} (I+P_s K_\infty)^{-1} \tilde{M}_s^{-1} \right\|_\infty \le \varepsilon^{-1}.$$
- **(3) The final controller $K = W_1 K_\infty W_2$**



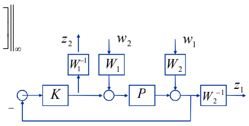
- **A typical design works as follows: the designer inspects the open-loop singular values of the nominal plant, and shapes these by pre- and/or postcompensation until nominal performance (and possibly robust stability) specifications are met. (Recall that the open-loop shape is related to closed-loop objectives.) A feedback controller K_∞ with associated stability margin (for the shaped plant) $\varepsilon \le b_{opt}(P_s)$ is then synthesized. If $b_{opt}(P_s)$ is small, then the specified loop shape is incompatible with robust stability requirements, and should be adjusted accordingly, then K_∞ is reevaluated.**
- ***Note that the final controller is $K = W_1 K_\infty W_2$, so it is necessary to check if the loop properties are significantly changed. It is helpful to choose W_1 and W_2 with small condition numbers.***
- ***Only W_1 or W_2 is needed if P is SISO.***

Weighted H_∞ Control Interpretation

$$\left\| \begin{bmatrix} I \\ K_\infty \end{bmatrix} (I+P_s K_\infty)^{-1} \tilde{M}_s^{-1} \right\|_\infty = \left\| \begin{bmatrix} I \\ K_\infty \end{bmatrix} (I+P_s K_\infty)^{-1} \begin{bmatrix} I & P_s \end{bmatrix} \right\|_\infty$$

$$= \left\| \begin{bmatrix} W_2 & \\ & W_1^{-1} \end{bmatrix} \begin{bmatrix} I \\ K \end{bmatrix} (I+PK)^{-1} \begin{bmatrix} I & P \end{bmatrix} \begin{bmatrix} W_2^{-1} & \\ & W_1 \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} I \\ P_s \end{bmatrix} (I+K_\infty P_s)^{-1} \begin{bmatrix} I & K_\infty \end{bmatrix} \right\|_\infty$$

$$= \left\| \begin{bmatrix} W_1^{-1} & \\ & W_2 \end{bmatrix} \begin{bmatrix} I \\ P \end{bmatrix} (I+KP)^{-1} \begin{bmatrix} I & P \end{bmatrix} \begin{bmatrix} W_1 & \\ & W_2^{-1} \end{bmatrix} \right\|_\infty$$



This shows how all the closed - loop objective are incorporat ed.

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \left\| \begin{bmatrix} W_2 & \\ & W_1^{-1} \end{bmatrix} \begin{bmatrix} I \\ K \end{bmatrix} (I+PK)^{-1} \begin{bmatrix} I & P \end{bmatrix} \begin{bmatrix} W_2^{-1} & \\ & W_1 \end{bmatrix} \right\|_\infty \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

Chapter 17: Gap Metric and v-Gap Metric

Measure of Distance:

$$P_1(s) = \frac{1}{s}, \quad P_2(s) = \frac{1}{s+0.1}.$$

Closed-loop:

$$\left\| P_1(I+P_1)^{-1} - P_2(I+P_2)^{-1} \right\|_\infty = 0.0909,$$

Open-loop:

$$\left\| P_1 - P_2 \right\|_\infty = \infty.$$

Need new measure.

Gap Metric

Normalized right and left stable coprime factorizations:

$$P = NM^{-1} = \tilde{M}^{-1}\tilde{N}.$$

$$M^{\sim}M + N^{\sim}N = I, \quad \tilde{M}\tilde{M}^{\sim} + \tilde{N}\tilde{N}^{\sim} = I$$

The graph of the operator P is the subspace of H_2 consisting of all pairs (u,y) such that $y=Pu$. This is given by

$$\begin{bmatrix} M \\ N \end{bmatrix} H_2$$

and is a closed subspace of H_2. The gap between two systems P_1 and P_2 is defined by

$$\delta_g(P_1, P_2) = \left\| \Pi_{\begin{bmatrix} M_1 \\ N_1 \end{bmatrix} H_2} - \Pi_{\begin{bmatrix} M_2 \\ N_2 \end{bmatrix} H_2} \right\|$$

where Π_K denotes the orthogonal projection onto K and $P_1 = N_1 M_1^{-1}$ and $P_2 = N_2 M_2^{-1}$ are normalized right coprime factorizations.

Computing Gap Metric

Theorem 0.1 Let $P_1 = N_1 M_1^{-1}$ and $P_2 = N_2 M_2^{-1}$ be normalized right coprime factorizations. Then

$$\delta_g(P_1,P_2) = \max\{\vec{\delta}(P_1,P_2), \vec{\delta}(P_2,P_1)\}$$

where $\vec{\delta}(P_1,P_2)$ is the directed and can be computed by

$$\vec{\delta}_g(P_1,P_2) = \inf_{Q\in H_\infty} \left\| \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} - \begin{bmatrix} M_2 \\ N_2 \end{bmatrix} Q \right\|_\infty .$$

- If $\delta_g(P_1,P_2) < 1$, then $\delta_g(P_1,P_2) = \vec{\delta}_g(P_1,P_2) = \vec{\delta}_g(P_2,P_1)$.

$>>\delta_g\ \mathbf{(P_1,P_2) = gap(P_1,P_2,tol)}$

$$G(s) = \begin{bmatrix} \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} & \begin{bmatrix} M_2 \\ N_2 \end{bmatrix} \\ -I & 0 \end{bmatrix}$$

$$\delta_g\left(\frac{1}{s}, \frac{1}{s+0.1}\right) = 0.0995$$

Gap 的下界

Let

$$\Phi = \begin{bmatrix} M_2^{\sim} & N_2^{\sim} \\ -\tilde{N}_2 & \tilde{M}_2 \end{bmatrix}.$$

Then $\Phi^{\sim}\Phi = \Phi\Phi^{\sim} = I$ and

$$\begin{aligned} \vec{\delta}_g(P_1,P_2) &= \inf_{Q\in H_\infty} \left\| \begin{bmatrix} M_2^{\sim} & N_2^{\sim} \\ -\tilde{N}_2 & \tilde{M}_2 \end{bmatrix} \left\{ \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} - \begin{bmatrix} M_2 \\ N_2 \end{bmatrix} Q \right\} \right\|_\infty \\ &= \inf_{Q\in H_\infty} \left\| \begin{bmatrix} M_2^{\sim} M_1 + \tilde{N}_2 N_1 - Q \\ -\tilde{N}_2 M_1 + \tilde{M}_2 N_1 \end{bmatrix} \right\|_\infty \\ &\boxed{\geq \|\Psi(P_1,P_2)\|_\infty} \end{aligned}$$

where

$$\Psi(P_1,P_2) := -\tilde{N}_2 M_1 + \tilde{M}_2 N_1 = \begin{bmatrix} \tilde{M}_2 & \tilde{N}_2 \end{bmatrix} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} M_1 \\ N_1 \end{bmatrix}.$$

$\|\Psi(P_1,P_2)\|_\infty$ is related to the v-gap metric.

$$\boxed{\Psi(P_1,P_2) = (I + P_2 P_2^{\sim})^{-1/2}(P_1 - P_2)(I + P_1^{\sim} P_1)^{-1/2}}$$

$$P_1 = \frac{k_1}{s+1}, \quad P_2 = \frac{k_2}{s+1}.$$

Then it is easy to verify that $P_i = N_i / M_i$, i=1,2,with

$$N_i = \frac{k_i}{s + \sqrt{1+k_i^2}}, \quad M_i = \frac{s+1}{s+\sqrt{1+k_i^2}},$$

Are normalized coprime factorizations and it can be further shown, as in Georgiou and Smith [1990], that

$$\delta_g(P_1,P_2) = \|\Psi(P_1,P_2)\|_\infty = \begin{cases} \dfrac{|k_1-k_2|}{|k_1|+|k_2|}, & \text{if } |k_1 k_2| > 1; \\ \dfrac{|k_1-k_2|}{\sqrt{(1+k_1^2)(1+k_2^2)}}, & \text{if } |k_1 - k_2| \leq 1. \end{cases}$$

举例：最优标称模型

Question: Given an uncertain plant

$$P(s) = \frac{k}{s-1}, \quad k \in [k_1, k_2],$$

(a) Find the best nominal design model $P_0 = \dfrac{k_0}{s-1}$ in the sense

$$\inf_{k_0\in[k_1,k_2]} \sup_{k\in[k_1,k_2]} \delta_g(P,P_0).$$

For simplicity, suppose $k_1 \geq 1$. It can be shown that

$$\delta_g(P,P_0) = \frac{|k_0 - k|}{k_0 + k}.$$

Then the optimal k_0 for question (a) satisfies

$$\frac{k_0-k_1}{k_0+k_1} = \frac{k_2-k_0}{k_2+k_0};$$

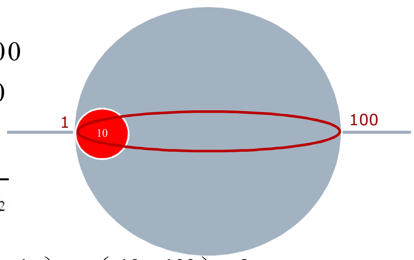
that is, $k_0 = \sqrt{k_1 k_2}$ and

$$\inf_{k_0\in[k_1,k_2]} \sup_{k\in[k_1,k_2]} \delta_g(P,P_0) = \frac{\sqrt{k_2}-\sqrt{k_1}}{\sqrt{k_2}+\sqrt{k_1}}.$$

$k_1 = 1$

$k_2 = 100$

$k_0 = 10$



$k_0 = \sqrt{k_1 k_2}$

$$\delta_v\left(\frac{10}{s-1}, \frac{1}{s-1}\right) = \delta_v\left(\frac{10}{s-1}, \frac{100}{s-1}\right) = \frac{9}{11}$$

$$\delta_v\left(\frac{50.5}{s-1}, \frac{1}{s-1}\right) = \delta_v\left(\frac{50.5}{s-1}, \frac{2525}{s-1}\right) = \frac{49.5}{51.5} > \frac{9}{11}$$

Example

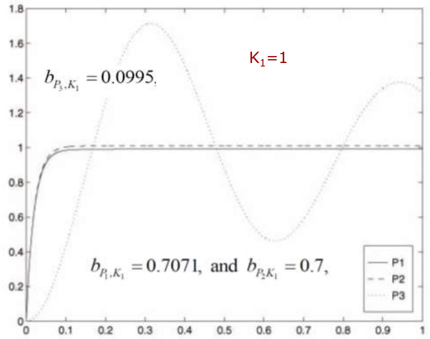
$$P_1 = \frac{100}{2s+1}, \quad P_2 = \frac{100}{2s-1}, \quad P_3 = \frac{100}{(s+1)^2}.$$

$$\delta_v(P_1,P_2) = \delta_g(P_1,P_2) = 0.02, \delta_v(P_1,P_3) = \delta_g(P_1,P_3) = 0.8988,$$
$$\delta_v(P_2,P_3) = \delta_g(P_2,P_3) = 0.8941,$$

Which show that P_1 and P_2 are very close while P_1 and P_2 (or P_2 and P_3) are quite far away. It is not surprising that any reasonable controller for P_1 will do well for P_2 but not necessarily for P_3.

$$b_{obt} = 0.7106, \text{ and } b_{obt}(P_2) = 0.7036$$

(in fact, the optimal controllers for P_1 and P_2 are K =0.99 and K = 1.01, respectively).



和互质因子不确定性关系

Corollary 0.2 Let P have a normalized coprime factorization $P = NM^{-1}$. Then for all $0 < b \le 1$,

$$\{P_1 : \vec{\delta}_g(P, P_1) < b\}$$

$$= \left\{P_1 : P_1 = (N + \Delta_N)(M + \Delta_M)^{-1}, \Delta_N, \Delta_M \in H_\infty, \left\| \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} \right\|_\infty < b \right\}.$$

互质因子不确定性=**gap**不确定性

Theorem 0.3 *Suppose the feedback system with the pair (P_0, K_0) is stable. Let*

$$\mathcal{P} := \{P : \delta_g(P, P_0) < r_1\} \text{ and } \mathcal{K} := \{K : \delta_g(K, K_0) < r_2\}$$

Then

(a) The feedback system with the pair (P,K) is also stable for all $P \in \mathcal{P}$ and $K \in \mathcal{K}$ if and only if

$$\arcsin b_{P_0, K_0} \ge \arcsin r_1 + \arcsin r_2.$$

(b) The worst possible performance resulting from these sets of plants and controllers is given by

$$\inf_{P \in \mathcal{P}, K \in \mathcal{K}} \arcsin b_{P,K} = \arcsin b_{P_0, K_0} - \arcsin r_1 - \arcsin r_2.$$

one can take either $r_1 = 0$ or $r_2 = 0$.

v-Gap Metric

Definition 0.2 *The winding number of g(s) with respect to this contour, denoted by* wno(*g*)*, is the number of counterclockwise encirclements around the origin by g(s) evaluated on the Nyquist contour Γ. (A clockwise encirclement counts as a negative encirclement.)*

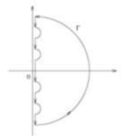


Figure 0.33: The Nyquist contour

Lemma 0.4 (The Argument Principle) *Let Γ be a closed contour in the complex plane. Let f(s) be a function analytic along the contour; that is, f(s) has no poles on Γ. Assume f(s) has Z zeros and P poles inside Γ. Then f(s) evaluated along the contour Γ once in an anti-clockwise direction will make Z - P anti-clockwise encirclements of the origin.*

Properties of wno

Denote $\eta(G)$ and $\eta_0(G)$, respectively, the number of open right-half plane and imaginary axis poles of G(s).

Lemma 0.5 *Let g and h be biproper rational scalar transfer functions and let F be a square transfer matrix. Then*

(a) wno(*gh*) = wno(*g*) + wno(*h*);

(b) $\text{wno}(g) = \eta(g^{-1}) - \eta(g);$

(c) $\text{wno}(g^{\sim}) = -\text{wno}(g) - \eta_0(g^{-1}) + \eta_0(g);$

(d) $\text{wno}(1 + g) = 0$ *if* $g \in RL_\infty$ *and* $\|g\|_\infty < 1;$

(e) $\text{wno}\det(I + F) = 0$ *if* $F \in RL_\infty$ *and* $\|F\|_\infty < 1;$

Example

$$g_1 = \frac{1.2(s+3)}{s-5}, g_2 = \frac{s-1}{s-2}, g_3 = \frac{2(s-1)(s-2)}{(s+3)(s+4)}, g_4 = \frac{(s-1)(s+3)}{(s-2)(s-4)}.$$

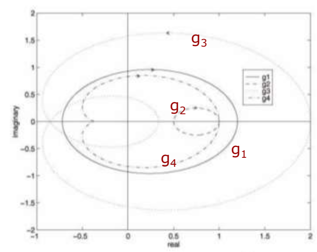


Figure 0.34: g_1, g_2, g_3 and g_4 evaluated on Γ

wno(g_1) = -1, wno(g_2) = 0, wno(g_3)= 2, wno(g_4) = -1

v-Gap Metric

Definition 0.3 *The v-gap metric is defined as*

$$\delta_\nu(P_1,P_2) == \begin{cases} \|\Psi(P_1,P_2)\|_\infty & \text{if } \det\Theta(j\omega)\neq 0\ \forall\omega \\ & \text{and wno}\det\Theta(s)=0. \\ 1, & \text{otherwise} \end{cases}$$

where $\Theta(s) := N_2^{\sim}N_1 + M_2^{\sim}M_1$ *and* $\Psi(P_1,P_2) := -\widetilde{N}_2 M_1 + \widetilde{M}_2 N_1$.

$$\delta_\nu(P_1,P_2)=\delta_\nu(P_2,P_1)=\delta_\nu(P_1^T,P_2^T)$$

$$>>\boldsymbol{\delta_\nu(P_1,P_2)=\mathbf{nugap}(P_1,P_2,\mathbf{tol})}$$

where tol is the computational tolerance.

Theorem 0.6 *The v-gap metric is defined as*

$$\delta_\nu(P_1,P_2) == \begin{cases} \|\Psi(P_1,P_2)\|_\infty & \text{if } \det(I+P_2^{\sim}P_1)\neq 0\ \forall\omega \text{ and} \\ & \text{wno}\det(I+P_2^{\sim}P_1)+\eta(P_1) \\ & -\eta(P_2)-\eta_0(P_2)=0, \\ 1, & \text{otherwise} \end{cases}$$

where $\Psi(P_1,P_2)$ *can be written as*

$$\Psi(P_1,P_2)=(I+P_2P_2^{\sim})^{-1/2}(P_1-P_2)(I+P_1^{\sim}P_1)^{-1/2}.$$

ν-间隙度量(读作nu)

- 在满足某个**winding number**条件下，两个系统$P_1(s)$和$P_2(s)$之间的ν-间隙度量(**Vinnicombe,1993**)：

$$\delta_\nu(P_1,P_2)=\sup_{\omega\in R}\frac{|P_1(j\omega)-P_2(j\omega)|}{\sqrt{1+|P_1(j\omega)|^2}\sqrt{1+|P_2(j\omega)|^2}} \quad (\leq 1)$$

- 如果$\tilde{P}(s)$和$\tilde{K}(s)$满足 $\delta_\nu(\tilde{P},P)\leq r_P$ $\delta_\nu(\tilde{K},K)\leq r_K$
则此系统也稳定，当且仅当

$$\arcsin b_{P,K} > \arcsin r_P + \arcsin r_K$$

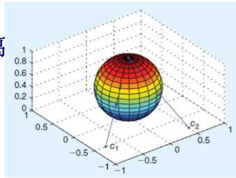
并且有

$$\arcsin b_{\tilde{P},\tilde{K}} \geq \arcsin b_{P,K} - \arcsin r_P - \arcsin r_K$$

立体投影

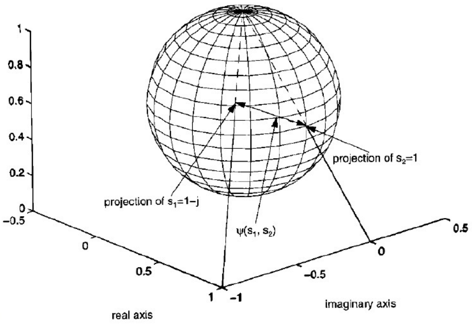
- 两个实(复)数距离：$d=|c_1-c_2|$。
- 不能描述距离的相对大小：$d=|1-2|=|100-101|=1$。但是由**1**到**2**的变化是**100%**，而由**100**到**101**的变化仅仅**1%**。
- 立体投影后球面上两点弦距离

$$\delta_\nu(c_1,c_2)=\frac{|c_1-c_2|}{\sqrt{1+|c_1|^2}\sqrt{1+|c_2|^2}}$$



- 弧距离是 **arcsin** δ_ν
- 那么，$\delta_\nu(1,2)=\frac{1}{\sqrt{10}}$，$\delta_\nu(100,101)\approx 10^{-4}$

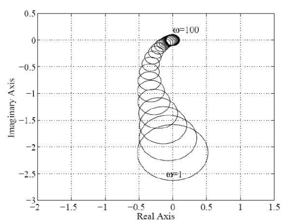
$$\psi(P_1(j\omega),\ P_2(j\omega))=\frac{|P_1(j\omega)-P_2(j\omega)|}{\sqrt{1+|P_1(j\omega)|^2}\ \sqrt{1+|P_2(j\omega)|^2}}$$



ν-间隙度量下不确定性

$$G(s)=\frac{10(s+1)}{s(s+2)(s+3)}$$

The corresponding uncertain Nyquist diagram for $\delta_\nu(G(s),\tilde{G}(s))\leq 0.1$ with $\omega\in[1,100]$ at each frequency.



低频不确定性危害性小

Computing v-Gap

Theorem 0.7. *Let $P_1 = N_1 M_1^{-1}$ and $P_2 = N_2 M_2^{-1}$ be normalized right coprime factorizations. Then*

$$\delta_v(P_1, P_2) = \inf_{Q, Q^{-1} \in L_\infty} \left\| \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} - \begin{bmatrix} M_2 \\ N_2 \end{bmatrix} Q \right\|_\infty .$$

$$\operatorname{wno}\det(Q) = 0$$

Moreover, $\delta_g(P_1, P_2) \le b_{obt}(P_1) \le \delta_v(P_1, P_2) \le \delta_g(P_1, P_2)$.

It is now easy to see that

$$\{P : \delta_v(P_0, P) < r\}$$

$$\supset \left\{ P = (N_0 + \Delta_N)(M_0 + \Delta_M)^{-1} : \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} \in H_\infty, \left\| \begin{bmatrix} \Delta_N \\ \Delta_M \end{bmatrix} \right\|_\infty < r \right\}.$$

互质因子不确定性 ⊂ **v-gap**不确定性

Theorem 0.9 *Let P_0 be a nominal plant and $\beta \le \alpha < b_{obt}(P_0)$.*

(I) For a given controller K,

$$\arcsin b_{P,K} > \arcsin \alpha - \arcsin \beta$$

for all P satisfying $\delta_v(P_0, P) \le \beta$ *if and only if* $b_{P_0,K} > \alpha$.

(ii) For a given plant P,

$$\arcsin b_{P,K} > \arcsin \alpha - \arcsin \beta$$

for all K satisfying $b_{P_0,K} > \alpha$ *if and only if* $\delta_v(P_0, P) \le \beta$

Theorem 0.10 Suppose the feedback system with the pair (P_0, K_0) is stable. Then

$$\boxed{\arcsin b_{P,K} \ge \arcsin b_{P_0,K_0} - \arcsin \delta_v(P_0, P) - \arcsin \delta_v(K_0, K)}$$

for any P and K.

Define

$$\frac{1}{b_{P,K}(\omega)} := \bar{\sigma}\left(\begin{bmatrix} I \\ K(j\omega) \end{bmatrix} (I + P(j\omega)K(j\omega))^{-1} \begin{bmatrix} I & P(j\omega) \end{bmatrix} \right)$$

and

$$\Psi(P_1(j\omega), P_2(j\omega)) = \bar{\sigma}\left(\Psi(P_1(j\omega), P_2(j\omega))\right).$$

The following theorem states that robust stability can be checked using the frequency-by-frequency test.

Theorem 0.8 *Suppose (P_0, K) is stable and Then (P_1, K) is stable if*

$$b_{P_0,K}(\omega) > \Psi(P_0(j\omega), P_1(j\omega)), \quad \forall \omega$$

Moreover,

$$\arcsin b_{P_1,K}(\omega) \ge \arcsin b_{P_0,K}(\omega) - \arcsin \Psi(P_0(j\omega), P_1(j\omega)), \quad \forall \omega$$

and

$$\arcsin b_{P_1,K} \ge \arcsin b_{P_0,K} - \arcsin \delta_v(P_0, P_1).$$

$$\arcsin b_{P_1,K}(\omega) \ge \arcsin b_{P_0,K}(\omega) - \arcsin \Psi(P_0(j\omega), P_1(j\omega)), \quad \forall \omega$$

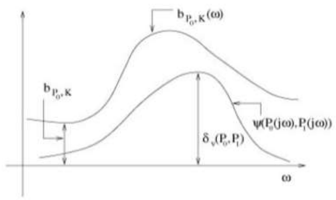


Figure 0.35: K stabilizes both P_0 and P_1 since $b_{P_0,K}(\omega) > \Psi(P_0, P_1)$ for all ω