Institute of Systems Science and Intelligent Control Technology
系统科学与智能控制技术研究所

鲁棒控制：建模、跟踪、抗扰、容错

周克敏
山东科技大学
电气与自动化工程学院
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（第十三讲）

提纲

1. 古典控制基础
2. 鲁棒控制理论基础
3. 鲁棒控制在迟滞系统中应用
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Chapter 14: H_∞ Control

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H_∞ Background

- Initial theory was SISO (Zames, Helton, Tannenbaum)
- Nevanlinna-Pick interpolation
- Operator-theoretic methods (Sarason, Adamjan *et al*, Ball-Helton)
- Initial work handled restricted problems (1-block and 2-block)
- Solution to 2x2-block problem

(1984 Honeywell-ONR Workshop)

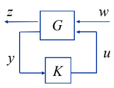
H_∞: 1984 H/ONR Workshop Approach

Solution approach:

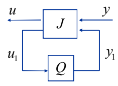
- Parametrize all stabilizing controllers via [Youla *et al*]
- Obtain realizations of the closed-loop transfer matrix
- Transform to 2x2 block general distance problem
- Reduce to the Nehari problem and solve via Glover

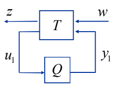
Properties of the solution:

- State-space using standard operations
- Computationally intensive (many Ric. eqns.)
- Potentially high-order controllers
- Find solution < γ, iterate for optimal



控制器参数化（Youla Parameterization）





$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

$T_{zw}=T_{11}+T_{12}QT_{21}$

$$\|T_{11}+T_{12}QT_{21}\|_\infty = \|T_{11}+UT_{12o}QT_{21o}V\|_\infty = \|T_{11}+UQ_1V\|_\infty$$

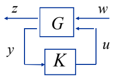
$$= \left\| T_{11} + \begin{bmatrix} U & U_\perp \end{bmatrix} \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V \\ V_\perp \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} U & U_\perp \end{bmatrix}^\sim T_{11} \begin{bmatrix} V \\ V_\perp \end{bmatrix}^\sim + \begin{bmatrix} Q_1 & 0 \\ 0 & 0 \end{bmatrix} \right\|_\infty$$

$$= \left\| \begin{bmatrix} U^\sim T_{11} V^\sim + Q_1 & U^\sim T_{11} V_\perp^\sim \\ U_\perp^\sim T_{11} V^\sim & U_\perp^\sim T_{11} V_\perp^\sim \end{bmatrix} \right\|_\infty \Leftrightarrow \|W+\tilde{Q}_1\|_\infty$$

Problem Formulation and Solutions

- Consider a general LFT system

$$G(s)=\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right].$$



- Assumptions:

(i) (A, B_1) is controllable and (C_1,A) is observable;

(ii) (A,B_2) is stabilizable and (C_2,A) is detectable;

(iii) $D_{12}^*[C_1, D_{12}]=[0 \;\; I\,]$

(iv) $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^* = \begin{bmatrix} 0 \\ I \end{bmatrix}$

$$\|z\|^2 = \|C_1x+D_{12}u\|^2 = (C_1x+D_{12}u)'(C_1x+D_{12}u)$$
$$= x'C_1'C_1x+u'u = \|C_1x\|^2+\|u\|^2$$

(i) Together with (ii) guarantees that the two H_2 AREs have nonnegative stabilizing solutions.

(ii) Necessary and sufficient for G to be internally stabilizable.

(iii) The penalty on $z = C_1x + D_{12}u$ includes a nonsingular, normalized penalty on the control u. In the conventional H_2 setting this means that there is no cross weighting between the state and control and that the control weight matrix is the identity.

(iv) w includes both plant disturbance and sensor noise, these are orthogonal, and the sensor noise weighting is normalized and nonsingular.

These assumptions simplify the theorem statements and proofs, and can be relaxed..

Output Feedback H_∞ Control

- **Solution: $\exists$ K such that $||T_{zw}||_\infty < \gamma$ if and only if**

(i) $X_\infty > 0$

$$X_\infty A+A^*X_\infty +X_\infty (B_1B_1^*/\gamma^2-B_2B_2^*)X_\infty+C_1^*C_1=0$$

(ii) $Y_\infty > 0$

$$AY_\infty +Y_\infty A^*+Y_\infty (C_1^*C_1/\gamma^2-C_2^*C_2)Y_\infty+B_1B_1^*=0$$

(iii) $\rho(X_\infty Y_\infty) < \gamma^2$

Furthermore,
$$K_{sub}(s):=\left[\begin{array}{c|c} \hat{A}_\infty & -Z_\infty L_\infty \\ \hline F_\infty & 0 \end{array}\right]$$

where
$$\hat{A}_\infty := A+B_1B_1^*X_\infty/\gamma^2+B_2F_\infty+ Z_\infty L_\infty C_2$$
$$F_\infty := -B_2^*X_\infty, \quad L_\infty := -Y_\infty C_2^*, \quad Z_\infty = (I-Y_\infty X_\infty/\gamma^2)^{-1}$$

A Matrix Fact

[Packard, 1994] Suppose X, $Y \in R^{n\times n}$, and $X=X^*>0$, $Y=Y^*>0$. Let r be a positive integer. Then there exist matrices $X_{12} \in R^{n\times r}$, $X_2 \in R^{r\times r}$ such that $X_2=X_2^*$, and

$$\begin{bmatrix} X & X_{12} \\ X_{12}^* & X_2 \end{bmatrix} > 0 \quad \& \quad \begin{bmatrix} X & X_{12} \\ X_{12}^* & X_2 \end{bmatrix}^{-1} = \begin{bmatrix} Y & * \\ * & * \end{bmatrix}$$

if and only if

$$\begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} \ge 0 \quad \& \quad \operatorname{rank}\begin{bmatrix} X & I_n \\ I_n & Y \end{bmatrix} \le n+r.$$

Proof.($\Leftarrow$) By assumption, there is a matrix $X_{12} \in R^{n\times r}$ such that $X-Y^{-1}=X_{12}X_{12}^*$. Defining $X_2:=I_r$ completes the construction.

($\Rightarrow$) Using Schur complements,

$$Y=X^{-1}+X^{-1}X_{12}(X_2-X_{12}^*X^{-1}X_{12})^{-1}X_{12}^*X^{-1}$$

Inverting, using the matrix inversion lemma, gives

$$Y^{-1}=X-X_{12}^*X^{-1}X_{12}^*$$

Hence, $X-Y^{-1}=X_{12}X_2^{-1}X_{12}^*\ge 0$, and indeed,

$\operatorname{rank}(X-Y^{-1})=\operatorname{rank}(X_{12}X_2^{-1}X_{12})\le$ r.

Inequality Characterization

Lemma IC: $\exists$ r-th order K such that $||T_{zw}||_\infty < \gamma$ only if

$(i)\ \exists Y_1 > 0$

$$AY_1+Y_1A^*+Y_1C_1^*C_1Y_1/\gamma^2+B_1B_1^*-\gamma^2B_2B_2^*<0$$

$(ii)\ \exists X_1 > 0$

$$X_1A+A^*X_1+X_1B_1B_1^*X_1/\gamma^2+C_1^*C_1-\gamma^2C_2^*C_2<0$$

$$(iii)\begin{bmatrix} X_1/\gamma & I_n \\ I_n & Y_1/\gamma \end{bmatrix} \ge 0 \text{ and rank}\begin{bmatrix} X_1/\gamma & I_n \\ I_n & Y_1/\gamma \end{bmatrix} \le n+r.$$

Proof. Suppose that there exists an r-th order controller $K(s)$ such that $\|T_{zw}\|_\infty < \gamma$. Let $K(s)$ have a state space realization

$$K(s) = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$$

Then

$$T_{zw} = \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right] := \left[\begin{array}{cc|c} A + B_2\hat{D}C_2 & B_2\hat{C} & B_1 + B_2\hat{D}D_{21} \\ \hat{B}C_2 & \hat{A} & \hat{B}D_{21} \\ \hline C_1 + D_{12}\hat{D}C_2 & D_{12}\hat{C} & D_{12}\hat{D}D_{21} \end{array}\right].$$

Denote $\quad R = \gamma^2 I - D_c^* D_c, \quad \widetilde{R} = \gamma^2 I - D_c D_c^*.$

By Bounded Real Lemma, $\exists \widetilde{X} = \begin{bmatrix} X_1 & X_{12} \\ X_{12}^* & X_2 \end{bmatrix} > 0$ such that

$$\tilde{X}\left(A_c + B_c R^{-1} D_c^* C_c\right) + \left(A_c + B_c R^{-1} D_c^* C_c\right)^* \tilde{X} + \tilde{X} B_c R^{-1} B_c^* \tilde{X} + \gamma^2 C_c^* \tilde{R}^{-1} C_c < 0$$

This gives after much algebraic manipulation

$$X_1 A + A^* X_1 + X_1 B_1 B_1^* X_1 / \gamma^2 + C_1^* C_1 - \gamma^2 C_2^* C_2$$
$$+ (X_1 B_1 \hat{D} + X_{12}\hat{B} + \gamma^2 C_2^*)(\gamma^2 I - \hat{D}^*\hat{D})^{-1}(X_1 B_1 \hat{D} + X_{12}\hat{B} + \gamma^2 C_2^*)^* < 0$$

which implies that

$$X_1 A + A^* X_1 + X_1 B_1 B_1^* X_1 / \gamma^2 + C_1^* C_1 - \gamma^2 C_2^* C_2 < 0.$$

Let $\widetilde{Y} = \gamma^2 \widetilde{X}^{-1}$ and partition $\widetilde{Y}$ as $\widetilde{Y} = \begin{bmatrix} Y_1 & Y_{12} \\ Y_{12}^* & Y_2 \end{bmatrix} > 0$

Then

$$\left(A_c + B_c R^{-1} D_c^* C_c\right)\widetilde{Y} + \widetilde{Y}\left(A_c + B_c R^{-1} D_c^* C_c\right)^* + \widetilde{Y} C_c^* \widetilde{R}^{-1} C_c \widetilde{Y} + B_c R^{-1} B_c^* < 0$$

This gives

$$AY_1 + Y_1 A^* + B_1 B_1^* - \gamma^2 B_2 B_2^* + Y_1 C_1^* C_1 Y_1 / \gamma^2$$
$$+ (Y_1 C_1^* \hat{D}^* + Y_{12}\hat{C}^* + \gamma^2 B_2)(\gamma^2 I - \hat{D}\hat{D}^*)^{-1}(Y_1 C_1^* \hat{D}^* + Y_{12}\hat{C}^* + \gamma^2 B_2)^* < 0$$

which implies that

$$AY_1 + Y_1 A^* + B_1 B_1^* - \gamma^2 B_2 B_2^* + Y_1 C_1^* C_1 Y_1 / \gamma^2 < 0.$$

By the matrix fact, given $X_1 > 0$ and $Y_1 > 0$, there exists X_{12} and X_2 such that $\widetilde{Y} = \gamma^2 \widetilde{X}^{-1}$ or $\widetilde{Y}/\gamma = \left(\widetilde{X}/\gamma\right)^{-1}$:

$$\begin{bmatrix} X_1/\gamma & X_{12}/\gamma \\ X_{12}^*/\gamma & X_2/\gamma \end{bmatrix}^{-1} = \begin{bmatrix} Y_1/\gamma & * \\ * & * \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} X_1/\gamma & I_n \\ I_n & Y_1/\gamma \end{bmatrix} \geq 0 \text{ and } \operatorname{rank}\begin{bmatrix} X_1/\gamma & I_n \\ I_n & Y_1/\gamma \end{bmatrix} \leq n + r.$$

Connection between ARE and ARI (LMI)

Lemma ARE: [Ran abd Vreugdenhil, 1988] Suppose (A,B) is controllable and there is an $X = X^*$ such that

$$Q(X) := XA + A^*X + XBB^*X + Q < 0.$$

Then there exists a solution $X_+ > X$ to the Riccati equation

$$X_+A + A^*X_+ + X_+BB^*X_+ + Q = 0 \qquad (0.7)$$

such that $A + BB^*X_+$ is antistable

Proof. Let X be such that $Q(X) < 0$. Choose F_0 such that $A_0 := A - BF_0$ is antistable. Let $X_0 = X_0^*$ solve

$$X_0A_0 + A_0^*X_0 - F_0^*F_0 + Q = 0.$$

Define $G_0 := F_0 + B^*X$. Then

$$(X_0 - X)A_0 + A_0^*(X_0 - X) = G_0^*G_0 - Q(X) > 0$$

and $X_0 > X$ (by anti-stability of A_0).

Define a non-increasing sequence of hermitian matrices $\{X_i\}$:

$$X_0 \geq X_1 \geq \ldots \geq X_{n-1} > X,$$

$$A_i = A - BF_i \text{, is anti-stable, } i = 0, \ldots, n-1;$$

$$F_i = -B^*X_{i-1},\ i = 1, \ldots, n-1;$$

$$X_iA_i + A_i^*X_i = F_i^*F_i - Q,\ i = 0, 1, \ldots, n-1. \qquad (0.8)$$

By Induction: We show this sequence can indeed be defined.

Introduce $F_n = -B^*X_{n-1}$, $A_n = A - BF_n$.

We show that A_n is antistable. Using (0.8), with $i = n-1$, we get

$$X_{n-1}A_n + A_n^*X_{n-1} + Q - F_n^*F_n - (F_n - F_{n-1})^*(F_n - F_{n-1}) = 0.$$

Let $G_n := F_n + B^*X$. Then

$$(X_{n-1} - X)A_n + A_n^*(X_{n-1} - X) = G_n^*G_n - Q(X) + (F_n - F_{n-1})^*(F_n - F_{n-1}) > 0$$

which implies that A_n is antistable by Lyapunov stability theorem since $X_{n-1} - X > 0$.

Let X_n be the unique solution of

$$X_n A_n + A_n^*X_n = F_n^*F_n - Q. \qquad (0.9)$$

Then X_n is hermitian. Next, we have

$$(X_n - X)A_n + A_n^*(X_n - X) = G_n^*G_n - Q(X) > 0$$

$$(X_{n-1} - X_n)A_n + A_n^*(X_{n-1} - X_n) = (F_n - F_{n-1})^*(F_n - F_{n-1}) \geq 0$$

Since A_n is antistable, we have $X_{n-1} \geq X_n > X$.

Therefore, we have a non-increasing sequence $\{X_i\}$.

Since the sequence is bounded below by $X_i > X$. Hence the limit

$$X_+ := \lim_{n\to\infty} X_n$$

exists and is hermitian, and we have $X_+ \geq X$. Passing the limit $n\to\infty$ in (0.9), we get $\boldsymbol{Q}(X_+)=0$. So X_+ is a solution of (0.7).

Note that $X_+ - X \geq 0$ and

$$(X_+-X)A_+ + A_+^*(X_+-X) = -\boldsymbol{Q}(X\ ) + (X_+-X)^*BB^*(X_+-X) > 0$$

Hence, $X_+ - X > 0$ and $A_+ = A + BB^*X_+$ is antistable.

Proof for Necessary

There exists a controller such that $\|T_{zw}\|_\infty < \gamma$ only if the following conditions hold:

(i) there exists a stabilizing solution $X_\infty > 0$ to

$$X_\infty A + A^* X_\infty + X_\infty (B_1 B_1^*/\gamma^2 - B_2 B_2^*) X_\infty + C_1^* C_1 = 0$$

(ii) there exists a stabilizing solution $Y_\infty > 0$ to

$$AY_\infty + Y_\infty A^* + Y_\infty (C_1^* C_1/\gamma^2 - C_2^* C_2) Y_\infty + B_1 B_1^* = 0$$

(iii) $\begin{bmatrix} \gamma Y_\infty^{-1} & I_n \\ I_n & \gamma X_\infty^{-1} \end{bmatrix} > 0$ or $\rho(X_\infty Y_\infty) < \gamma^2$.

Proof. Applying Lemma ARE to part (I) of Lemma IC, we conclude that there exists a $Y > Y_1 > 0$ such that

$$AY + YA^* + YC_1^*C_1Y/\gamma^2 + B_1B_1^* - \gamma^2 B_2 B_2^* = 0$$

and $A^* + C_1^*C_1Y/\gamma^2$ is antistable. Let $X_\infty := \gamma^2 Y^{-1}$, we have

$$X_\infty A + A^* X_\infty + X_\infty\left(B_1B_1^*/\gamma^2 - B_2B_2^*\right)X_\infty + C_1^*C_1 = 0$$

and

$$A + \left(B_1B_1^*/\gamma^2 - B_2B_2^*\right)X_\infty = -X_\infty^{-1}(A^* + C_1^*C_1X_\infty^{-1})X_\infty$$
$$= -X_\infty^{-1}(A^* + C_1^*C_1Y/\gamma^2)X_\infty$$

is stable.

Similarly, applying Lemma ARE to part (ii) of Lemma IC, we conclude that there exists an $X > X_1 > 0$ such that

$$XA + A^*X + XB_1B_1^*X/\gamma^2 + C_1^*C_1 - \gamma^2 C_2^*C_2 = 0$$

and $A + B_1B_1^*X/\gamma^2$ is antistable. Let $Y_\infty := \gamma^2 X^{-1}$, we have $AY_\infty + Y_\infty A^* + Y_\infty\left(C_1^*C_1/\gamma^2 - C_2^*C_2\right)Y_\infty + B_1B_1^* = 0$

and $A + \left(C_1^*C_1/\gamma^2 - C_2^*C_2\right)Y_\infty$ is stable.

Finally, note that the rank condition in part (iii) of Lemma IC is automatically satisfied by $r \geq n$, and

$$\begin{bmatrix} \gamma Y_\infty^{-1} & I_n \\ I_n & \gamma X_\infty^{-1} \end{bmatrix} = \begin{bmatrix} X/\gamma & I_n \\ I_n & Y/\gamma \end{bmatrix} > \begin{bmatrix} X_1/\gamma & I_n \\ I_n & Y_1/\gamma \end{bmatrix} \geq 0.$$

or $\rho(X_\infty Y_\infty) < \gamma^2$.

Proof for Sufficiency

Show K_{sub} renders $\|T_{zw}\|_\infty < \gamma$.

The closed-loop transfer function with K_{sub}:

$$T_{zw} = \left[\begin{array}{cc|c} A & B_2F_\infty & B_1 \\ -Z_\infty L_\infty C_2 & \hat{A}_\infty & -Z_\infty L_\infty D_{21} \\ \hline C_1 & D_{12}F_\infty & 0 \end{array}\right] =: \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right]$$

Define:

$$P = \begin{bmatrix} \gamma^2 Y_\infty^{-1} & -\gamma^2 Y_\infty^{-1} Z_\infty^{-1} \\ -\gamma^2 (Z_\infty^*)^{-1} Y_\infty^{-1} & \gamma^2 X_\infty Z_\infty^{-1} \end{bmatrix}$$

Then P > 0 and

$$PA_c + A_c^*P + PB_cB_c^*P/\gamma^2 + C_c^*C_c = 0.$$

Moreover

$$A_c + B_cB_c^*P/\gamma^2 = \begin{bmatrix} A + B_1B_1^*Y_\infty^{-1} & B_2F_\infty - B_1B_1^*Y_\infty^{-1}Z_\infty^{-1} \\ 0 & A + B_1B_1^*X_\infty/\gamma^2 + B_2F_\infty \end{bmatrix}$$

has no eigenvalues on the imaginary axis since

$A + B_1B_1^*Y_\infty^{-1}$ is antistable

and

$A + B_1B_1^*X_\infty/\gamma^2 + B_2F_\infty$ is stable

By Bounded Real Lemma, $\|T_{zw}\|_\infty < \gamma$.

Comments

The conditions in Lemma IC is in fact necessary and sufficient.

But the three conditions have to be checked simultaneously. This is because if one finds an $X_I > 0$ and a $Y_1 > 0$ satisfying conditions (i) and (ii) but not condition (iii), this does not imply that there is no admissible H_∞ controller since there might be other $X_I > 0$ and $Y_I > 0$ that satisfy all three conditions.

We will demonstrate this in the next page.

For example, consider γ=1 and

$$G(s) = \left[\begin{array}{c|c|c} -1 & \begin{bmatrix} 1 & 0 \end{bmatrix} & 1 \\ \hline \begin{bmatrix} 1 \\ 0 \end{bmatrix} & 0 & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \hline 1 & \begin{bmatrix} 0 & 1 \end{bmatrix} & 0 \end{array}\right].$$

It is easy to check that X_I=Y_I=0.5 satisfy (i) and (ii) but not (iii). Nevertheless, we can show that γ_{opt} = 0.7321 and thus a suboptimal controller exists for $\gamma = 1$. In fact, we can check that $1 < X_I < 2$, $1 < Y_I < 2$ also satisfy (i) , (ii) and (iii). For this reason, Riccati equation approach is usually preferred over the Riccati inequality and LMI approaches whenever possible.

Example

Consider the feedback system shown in Figure 0.4 with

$$P = \frac{50(s+1.4)}{(s+1)(s+2)},\; W_e = \frac{2}{s+0.2},\; W_u = \frac{s+1}{s+10}.$$

Design a K to minimize the H_∞ norm from $w = \begin{bmatrix} d \\ d_i \end{bmatrix}$, to $z = \begin{bmatrix} e \\ \tilde{u} \end{bmatrix}$:

$$\begin{bmatrix} e \\ \tilde{u} \end{bmatrix} = \begin{bmatrix} W_e(I+PK)^{-1} & W_e(I+PK)^{-1}P \\ -W_uK(I+PK)^{-1} & -W_uK(I+PK)^{-1}P \end{bmatrix} \begin{bmatrix} d \\ d_i \end{bmatrix} =: T_{zw} \begin{bmatrix} d \\ d_i \end{bmatrix}.$$

LFT framework:

$$G(s) = \left[\begin{array}{cc:c} W_e & W_eP & -W_eP \\ 0 & 0 & -W_u \\ \hdashline I & P & -P \end{array}\right] = \left[\begin{array}{cccc|cc:c} -0.2 & 2 & 2 & 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 20 & -20 \\ 0 & 0 & -2 & 0 & 0 & 30 & -30 \\ 0 & 0 & 0 & -10 & 0 & 0 & -3 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 & -1 \\ \hdashline 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{array}\right].$$

>> [K,T_{zw}, γ_{subopt}]=hinfsyn($G, n_y, n_u, \gamma_{min}, \gamma_{max}, tol$)

where n_y = dimensions of y, n_u = dimensions of u, $\gamma_{\min}$ = a lower bound, $\gamma_{\max}$= an upper bound for γ_{opt} and tol is a tolerance to the optimal value. Set $n_y = 1$, $n_u = 1$, $\gamma_{\min} = 0$, $\gamma_{\max}$= 10, $tol = 0.0001$; we get γ_{subopt}=0.7849 and a suboptimal controller

$$K = \frac{12.82(s/10+1)(s/7.27+1)(s/1.4+1)}{(s/32449447.67+1)(s/22.19+1)(s/1.4+1)(s/0.2+1)}.$$

If we set $tol = 0.01$, we would get γ_{subopt}=0.7875 and a suboptimal controller

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$$\tilde{K} = \frac{12.78(s/10+1)(s/7.27+1)(s/1.4+1)}{(s/2335.59+1)(s/21.97+1)(s/1.4+1)(s/0.2+1)}.$$

The only significant difference between K and $\tilde{K}$ is the exact location of the far-away stable controller pole.

Figure 0.25 shows the closed-loop frequency response of $\bar{\sigma}(T_{zw})$ and Figure 0.26 shows the frequency responses of S, T, KS and SP.

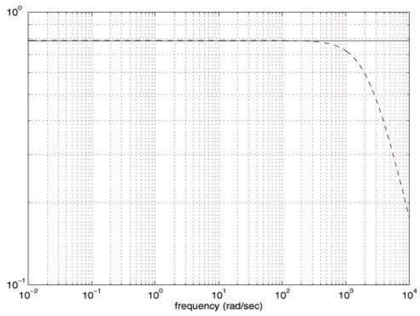


Figure 0.25: The closed-loop frequency responses of $\overline{\sigma}(T_{zw})$ with K (solid line) and $\tilde{K}$ (dashed line)

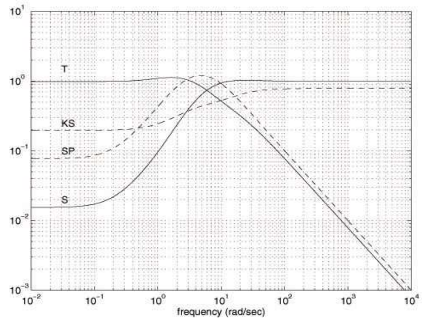


Figure 0.26: The frequency responses of S, T, KS and SP with K

Example

Consider again the two-mass/spring/damper system shown in Figure 0.1. Assume that F_1 is the control force, F_2 is the disturbance force, and the measurements of x_1 and x_2 are corrupted by measurement noise:

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + W_n \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}, \quad W_n = \begin{bmatrix} \frac{0.01(s+10)}{s+100} & 0 \\ 0 & \frac{0.01(s+10)}{s+100} \end{bmatrix}.$$

Our objective is to design a control law so that the effect of the disturbance force F_2 on the positions of the two masses x_1 and x_2, are reduced in a frequency range $0 \le \omega \le 2$.

The problem can be set up as shown in Figure 0.27, where $W_e = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}$ is the performance weight and W_u is the control weight. In order to limit the control force, we shall choose

$$W_u = \frac{s+5}{s+50}.$$

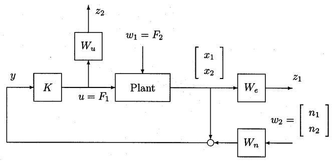


Figure 14.3: Rejecting the disturbance force F_2 by a feedback control

Let

$$u = F_1, w = \begin{bmatrix} F_2 \\ n_1 \\ n_2 \end{bmatrix}: \qquad G(s) = \begin{bmatrix} \begin{bmatrix} W_e P_1 & 0 \\ 0 & 0 \\ P_1 & Wn \end{bmatrix} & \begin{bmatrix} W_e P_2 \\ W_u \\ P_2 \end{bmatrix} \end{bmatrix}$$

where P_1 and P_2 denote the transfer matrices from F_1 and F_2 to $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, respectively.

- $W_1 = \dfrac{5}{s/2+1}, W_2 = 0$: only reject the effect of the disturbance force F_2 on the position x_1.

$$\|F_\ell(G, K_2)\|_2 = 2.6584,$$
$$\|F_\ell(G, K_2)\|_\infty = 2.6079,$$
$$\|F_\ell(G, K_\infty)\|_\infty = 1.6101.$$

This means that the effect of the disturbance force F_2 in the desired frequency range $0 \le \omega \le 2$ will be effectively reduced with the H_∞ controller K_∞ by 5/1.6101 = 3.1054 times at x_1.

- $W_1 = 0, W_2 = \dfrac{5}{s/2+1}$: only reject the effect of the disturbance force F_2 on the position x_2.

$$\|F_\ell(G,K_2)\|_2 = 0.1659$$
$$\|F_\ell(G,K_2)\|_\infty = 0.5202$$
$$\|F_\ell(G,K_\infty)\|_\infty = 0.5189.$$

This means that the effect of the disturbance force F_2 in the desired frequency range $0 \le \omega \le 2$ will be effectively reduced with the H_∞ controller K_∞ by 5/0.5189 = 9.6358 times at x_2.

- $W_1 = W_2 = \dfrac{5}{s/2+1}$: want to reject the effect of the disturbance force F_2 on both x_1 and x_2.

$$\|F_\ell(G,K_2)\|_2 = 4.087$$
$$\|F_\ell(G,K_2)\|_\infty = 6.0921$$
$$\|F_\ell(G,K_\infty)\|_\infty = 4.3611.$$

This means that the effect of the disturbance force F_2 in the desired frequency rang $0 \le \omega \le 2$ will only be effectively reduced with the H_∞ controller K_∞ by 5/4.3611 =1.1465 times at both x_1 and x_2.

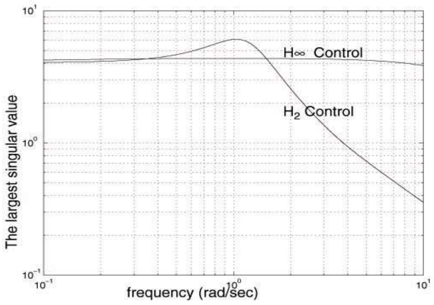


Figure 0.28: The largest singular value plot of the closed-loop system T_{zw} with an H_2 controller and an H_∞ controller.

This result shows clearly that it is very hard to reject the disturbance effect on both positions at the same time. The largest singular value Bode plots of the closed-loop system are shown in Figure 0.28. We note that the H_∞ controller typically gives a relatively flat frequency response since it tries to minimize the peak of the frequency response. On the other hand, the H_2 controller would typically produce a frequency response that rolls off fast in the high-frequency range but with a large peak in the low-frequency range.

Optimality and dependence on γ

There exists an admissible controller such that $||T_{zw}||_\infty < \gamma$ iff the following three conditions hold:

(i) $\exists$ a stabilizing $X_\infty > 0$

(ii) $\exists$ a stabilizing $Y_\infty > 0$

(iii) $\rho(X_\infty Y_\infty) < \gamma^2$

- Denote by γ_0 the infimum over all γ such that (i)-(iii) are satisfied. Descriptor formulae can be obtained for $\gamma = \gamma_0$
- As $\gamma \to \infty$, $H_\infty \to H_2$, $X_\infty \to X_2$, etc., and $K_{sub} \to K_2$

- At $\gamma = \gamma_0$, any one of the 3 conditions can fail. It is most likely that condition (iii) will fail first.
- If $\gamma_2 \ge \gamma_1 > \gamma_0$ then $X_\infty(\gamma_1) \ge X_\infty(\gamma_2)$ and $Y_\infty(\gamma_1) \ge Y_\infty(\gamma_2)$. Thus X_∞ and Y_∞ are decreasing fun. of γ, as is $\rho(X_\infty Y_\infty)$.
- To understand this, consider (i) and let γ_1 be the largest γ for which H_∞ fails to be in *dom(Ric),* because it fails to have either the stability property or the complementary property. The same remarks will apply to (ii) by duality.
- If the stability property fails at $\gamma = \gamma_1$, then $H_\infty \notin$ *dom(Ric),* but *Ric* can be extended to obtain X_∞ and the controller $u = -B_2^* X_\infty x$ is stabilizing and makes $||T_{zw}||_\infty = \gamma_1$.The stability property will also not hold for any $\gamma \le \gamma_1$, and no controller whatsoever exists which makes $||T_{zw}||_\infty < \gamma_1$.

In other words, if stability breaks down first then the infimum over stabilizing controllers equals the infimum over all controllers, stabilizing or otherwise.

- In view of this, we would expect that typically complementary would fail first.
- Complementary failing at $\gamma=\gamma_1$ means $\rho(X_\infty)\to\infty$ as $\gamma\to\gamma_1$ so condition (iii) would fail at even larger values of γ, unless the eigenvectors associated with $\rho(X_\infty)$ as $\gamma\to\gamma_1$ are in the null space of Y_∞
- Thus condition (iii) is the most likely of all to fail first.

H_∞ Controller Structure

$$K_{sub}(s) := \left[\begin{array}{c|c} \hat{A}_\infty & -Z_\infty L_\infty \\ \hline F_\infty & 0 \end{array}\right]$$

$$\hat{A}_\infty := A+\gamma^{-2}B_1B_1^*X_\infty + B_2F_\infty + Z_\infty L_\infty C_2$$

$$F_\infty := -B_2^*X_\infty, L_\infty := -Y_\infty C_2^*, Z_\infty := (I-\gamma^{-2}Y_\infty X_\infty)^{-1}$$

$$\dot{\hat{x}} = A\hat{x} + B_1\hat{w}_{worst} + B_2u + Z_\infty L_\infty (C_2\hat{x}-y)$$

$$u = F_\infty \hat{x}, \quad \hat{w}_{worst} = \gamma^{-2}B_1^*X_\infty \hat{x}$$

1) $\hat{w}_{worst}$ is the worst estimate of w

2) $Z_\infty L_\infty$ is the filter gain for the OE problem of estimating w_{worst}

$F_\infty x$ in the presence of the "worst-case" w,

3) The H_∞ controller has a separate interpretation.

Optimal Controller:

$$(I-\gamma_{opt}^{-2}Y_\infty X_\infty)\dot{\hat{x}} = A_s\hat{x} - L_\infty y \qquad (0.12)$$

$$u = F_\infty \hat{x} \qquad (0.13)$$

$$A_s := A + B_2F_\infty + L_\infty C_2 + \gamma_{opt}^{-2}Y_\infty A^* X_\infty + \gamma_{opt}^{-2}B_1B_1^*X_\infty + \gamma_{opt}^{-2}Y_\infty C_1^*C_1$$

See the example below.

Example

$$G(s) = \left[\begin{array}{c|cc} a & \begin{bmatrix}1 & 0\end{bmatrix} & 1 \\ \hline \begin{bmatrix}1\\0\end{bmatrix} & 0 & \begin{bmatrix}0\\1\end{bmatrix} \\ 1 & \begin{bmatrix}0 & 1\end{bmatrix} & 0 \end{array}\right].$$

Then all assumptions for output feedback problem are satisfied and

$$H_\infty = \begin{bmatrix} a & \frac{1-\gamma^2}{\gamma^2} \\ -1 & -a \end{bmatrix}, \quad J_\infty = \begin{bmatrix} a & \frac{1-\gamma^2}{\gamma^2} \\ -1 & -a \end{bmatrix}.$$

The eigenvalues of H_∞ and J_∞ are given, respectively, by

$$\sigma(H_\infty) = \left\{\pm\frac{\sqrt{(a^2+1)\gamma^2-1}}{\gamma}\right\}, \quad \sigma(J_\infty) = \left\{\pm\frac{\sqrt{(a^2+1)\gamma^2-1}}{\gamma}\right\}$$

If $\gamma > \frac{1}{\sqrt{a^2+1}}$, then $\chi\_(H_\infty)$ and $\chi\_(J_\infty)$ exist and

$$\chi\_(H_\infty) = \mathrm{Im}\begin{bmatrix} \frac{\sqrt{(a^2+1)\gamma^2-1}-a\gamma}{\gamma} \\ 1 \end{bmatrix}$$

$$\chi\_(J_\infty) = \mathrm{Im}\begin{bmatrix} \frac{\sqrt{(a^2+1)\gamma^2-1}-a\gamma}{\gamma} \\ 1 \end{bmatrix}.$$

Note that if $\gamma>1$, then $H_\infty \in dom(Ric)$, $J_\infty \in dom(Ric)$, and

$$X_\infty = \frac{\gamma}{\sqrt{(a^2+1)\gamma^2-1}-a\gamma} > 0$$

$$Y_\infty = \frac{\gamma}{\sqrt{(a^2+1)\gamma^2-1}-a\gamma} > 0.$$

It can be shown that

$$\rho(X_\infty Y_\infty) = \frac{\gamma^2}{\left(\sqrt{(a^2+1)\gamma^2-1}-a\gamma\right)^2} < \gamma^2$$

is satisfied if and only if

$$\gamma > \sqrt{a^2+2}+a.$$

So condition (iii) will fail before either (i) or (ii) fails.

The optimal γ for the output feedback is given by

$$\gamma_{opt} = \sqrt{a^2+2}+a$$

and the optimal controller given by the descriptor formula in equations (0.12) and (0.13) is a constant. In fact,

$$u_{opt} = \frac{\gamma_{opt}}{\sqrt{(a^2+1)\gamma_{opt}^2 - 1 - a\gamma_{opt}}}\, y.$$

For instance, let $a = -1$ then $\gamma_{opt} = \sqrt{3}-1 = 0.7321$ and Further, $u_{opt} = -0.7321y$.

$$T_{zw} = \left[\begin{array}{c|c|c} -1.7321 & 1 & -0.7321 \\ \hline 1 & 0 & 0 \\ \hline -0.7321 & 0 & -0.7321 \end{array}\right].$$

It is easy to check that $\|T_{zw}\|_\infty = 0.7321.$

An Optimal Controller

There exists an admissible controller such that $\|T_{zw}\|_\infty \le \gamma$ iff the following three conditions hold:

(i) there exists a full column rank matrix $\begin{bmatrix} X_{\infty 1} \\ X_{\infty 2} \end{bmatrix} \in \mathrm{R}^{2n\times n}$

such that

$$H_\infty \begin{bmatrix} X_{\infty 1} \\ X_{\infty 2} \end{bmatrix} = \begin{bmatrix} X_{\infty 1} \\ X_{\infty 2} \end{bmatrix} T_X, \quad \mathrm{Re}\lambda_i(T_X) \le 0\ \forall i$$

and $X_{\infty 1}^* X_{\infty 2} = X_{\infty 2}^* X_{\infty 1}$;

(ii) there exists a full column rank matrix $\begin{bmatrix} Y_{\infty 1} \\ Y_{\infty 2} \end{bmatrix} \in \mathrm{R}^{2n\times n}$

such that

$$J_\infty \begin{bmatrix} Y_{\infty 1} \\ Y_{\infty 2} \end{bmatrix} = \begin{bmatrix} Y_{\infty 1} \\ Y_{\infty 2} \end{bmatrix} T_Y, \quad \mathrm{Re}\lambda_i(T_Y) \le 0\ \forall i$$

and $Y_{\infty 1}^* Y_{\infty 2} = Y_{\infty 2}^* Y_{\infty 1}$;

(iii) $\begin{bmatrix} X_{\infty 2}^* X_{\infty 1} & \gamma^{-1} X_{\infty 2}^* Y_{\infty 2} \\ \gamma^{-1} Y_{\infty 2}^* X_{\infty 2} & Y_{\infty 2}^* Y_{\infty 1} \end{bmatrix} \ge 0.$

Moreover, when these conditions hold, one such controller is

$$K_{opt}(s) := C_K (sE_K - A_K)^+ B_K$$

$$E_K := Y_{\infty 1}^* X_{\infty 2} - \gamma^{-2} Y_{\infty 2}^* X_{\infty 2}, \quad B_K := Y_{\infty 2}^* C_2^*, \quad C_K := -B_2^* X_{\infty 2}$$

$$A_K := E_K T_X - B_K C_2 X_{\infty 1} = T_Y^* E_K + Y_{\infty 1}^* B_2 C_K.$$

H_∞ Control: General Case

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

Assumptions:

(A1) (A,B_2) is stabilizable and (C_2,A) is detectable;

(A2) $D_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix}$ and $D_{21} = \begin{bmatrix} 0 & I \end{bmatrix}$;

(A3) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω;

(A4) $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all ω.

$$R := D_{1\bullet}^* D_{1\bullet} - \begin{bmatrix} \gamma^2 I_{m_1} & 0 \\ 0 & 0 \end{bmatrix}, \text{ where } D_{1\bullet} := \begin{bmatrix} D_{11} & D_{12} \end{bmatrix}$$

$$\tilde{R} := D_{\bullet 1} D_{\bullet 1}^* - \begin{bmatrix} \gamma^2 I_{p_1} & 0 \\ 0 & 0 \end{bmatrix}, \text{ where } D_{\bullet 1} := \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix}$$

$$H_\infty := \begin{bmatrix} A & 0 \\ -C_1^* C_1 & -A^* \end{bmatrix} - \begin{bmatrix} B \\ -C_1^* D_{1\bullet} \end{bmatrix} R^{-1} \begin{bmatrix} D_{1\bullet}^* C_1 & B^* \end{bmatrix}$$

$$J_\infty := \begin{bmatrix} A^* & 0 \\ -B_1 B_1^* & -A \end{bmatrix} - \begin{bmatrix} B \\ -B_1 D_{\bullet 1}^* \end{bmatrix} \tilde{R}^{-1} \begin{bmatrix} D_{\bullet 1} B_1^* & C \end{bmatrix}$$

$$X_\infty := Ric(H_\infty) \qquad Y_\infty := Ric(J_\infty)$$

$$F := \begin{bmatrix} F_{1\infty} \\ F_{2\infty} \end{bmatrix} := -R^{-1}\left[D_{1\bullet}^* C_1 + B^* X_\infty\right]$$

$$L := \begin{bmatrix} L_{1\infty} & L_{2\infty} \end{bmatrix} := -\left[B_1 D_{\bullet 1}^* + Y_\infty C^*\right] \tilde{R}^{-1}$$

D, $F_{1\infty}$, and $L_{1\infty}$ are partitioned as follows:

$$\left[\begin{array}{c|c} & F' \\ \hline L' & D \end{array}\right] = \left[\begin{array}{c|ccc} & F_{11\infty}^* & F_{12\infty}^* & F_{2\infty}^* \\ \hline L_{11\infty}^* & D_{1111} & D_{1112} & 0 \\ L_{12\infty}^* & D_{1121} & D_{1122} & I \\ L_{2\infty}^* & 0 & I & 0 \end{array}\right].$$

There exists a stabilizing controller *K(s)* such that

$$\|F_\ell(G,K)\|_\infty < \gamma$$

if and only if

$$(i)\ \gamma > \max(\bar{\sigma}[D_{1111}, D_{1112}], \bar{\sigma}[D_{1111}^*, D_{1121}^*]);$$
$$(ii)\ H_\infty \in dom(Ric) \text{ with } X_\infty = Ric(H_\infty) \geq 0;$$
$$(iii)\ J_\infty \in dom(Ric) \text{ with } Y_\infty = Ric(J_\infty) \geq 0;$$
$$(iv)\ \rho(X_\infty Y_\infty) < \gamma^2.$$

$$K = F_\ell(M_\infty, Q),\ Q \in RH_\infty,\ \|Q\|_\infty < \gamma$$

$$M_\infty = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & 0 \end{array}\right]$$

where

$$\hat{D}_{11} = -D_{1121}D_{1111}^*(\gamma^2 I - D_{1111}D_{1111}^*)^{-1}D_{1112} - D_{1122},$$

$\hat{D}_{12} \in \mathbb{R}^{m_2 \times m_2}$ and $\hat{D}_{21} \in \mathbb{R}^{p_2 \times p_2}$ are any matrices satisfying

$$\hat{D}_{12}\hat{D}_{12}^* = I - D_{1121}(\gamma^2 I - D_{1111}^* D_{1111})^{-1}D_{1121}^*,$$
$$\hat{D}_{21}^*\hat{D}_{21} = I - D_{1112}^*(\gamma^2 I - D_{1111}D_{1111}^*)^{-1}D_{1112},$$

and

$$\hat{B}_2 = Z_\infty(B_2 + L_{12\infty})\hat{D}_{12},\quad \hat{C}_2 = -\hat{D}_{21}(C_2 + F_{12\infty}),$$
$$\hat{B}_1 = -Z_\infty L_{2\infty} + \hat{B}_2\hat{D}_{21}^{-1}\hat{D}_{11},\ \hat{C}_1 = F_{2\infty} + \hat{D}_{11}D_{21}^{-1}\hat{C}_2,$$
$$\hat{A} = A + BF + \hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2$$
$$Z_\infty = (I - \gamma^{-2}Y_\infty X_\infty)^{-1}.$$

Some Special Cases:

- $D_{12} = I$. Then (i) becomes $\gamma > \bar{\sigma}(D_{1121})$ and

$$\hat{D}_{11} = -D_{1122},\quad \hat{D}_{12}\hat{D}_{12}^* = I - \gamma^2 D_{1121}D_{1121}^*,\ \hat{D}_{21}^*\hat{D}_{21} = I.$$

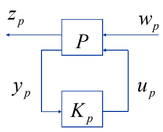
- $D_{21} = I$. Then (i) becomes $\gamma > \bar{\sigma}(D_{1112})$ and

$$\hat{D}_{11} = -D_{1122},\quad \hat{D}_{12}\hat{D}_{12}^* = I,\ \hat{D}_{21}^*\hat{D}_{21} = I - \gamma^2 D_{1121}D_{1121}^*.$$

- $D_{12} = I\ \&\ D_{21} = I$. Then (i) drops out and

$$\hat{D}_{11} = -D_{1122},\quad \hat{D}_{12}\hat{D}_{12}^* = I,\ \hat{D}_{21}^*\hat{D}_{21} = I.$$

Relaxing Assumptions



$$P(s) = \left[\begin{array}{c|cc} A_p & B_{p1} & B_{p2} \\ \hline C_{p1} & D_{p11} & D_{p12} \\ C_{p2} & D_{p21} & D_{p22} \end{array}\right]$$

Assume D_{p12} has full column rank and D_{p21} has full row rank.

Normalize D_{12} and D_{21}

Perform SVD

$$D_{p12} = U_p \begin{bmatrix} 0 \\ I \end{bmatrix} R_p,\quad D_{p21} = \tilde{R}_p \begin{bmatrix} 0 & I \end{bmatrix} \tilde{U}_p$$

such that U_p and $\tilde{U}_p$ are square and unitary. Now let

$$z_p = U_p z,\quad w_p = \tilde{U}_p^* w,\quad y_p = \tilde{R}_p y,\quad u_p = R_p^{-1} u$$
$$K(s) = R_p K_p(s) \tilde{R}_p$$
$$G(s) = \begin{bmatrix} U_p^* & 0 \\ 0 & \tilde{R}_p^{-1} \end{bmatrix} P(s) \begin{bmatrix} \tilde{U}_p^* & 0 \\ 0 & R_p^{-1} \end{bmatrix}$$

$$= \left[\begin{array}{c|cc} A_p & B_{p1}\tilde{U}_p^* & B_{p2}R_p^{-1} \\ \hline U_p^* C_{p1} & U_p^* D_{p11}\tilde{U}_p^* & U_p^* D_{p12}R_p^{-1} \\ \tilde{R}_p^{-1}C_{p2} & \tilde{R}_p^{-1}D_{p21}\tilde{U}_p^* & \tilde{R}_p^{-1}D_{p22}R_p^{-1} \end{array}\right]$$

$$=: \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Then

$$D_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix},\quad D_{21} = \begin{bmatrix} 0 & I \end{bmatrix},$$
$$\|F_\ell(P, K_p)\|_\infty = \|F_\ell(G, K)\|_\infty$$

Remove the Assumption $D_{22} = 0$

Suppose $K(s)$ is a controller for G with D_{22} set to zero. Then the controller for $D_{22} \neq 0$ is $K(I + D_{22}K)^{-1}$.

Relaxing A3 and A4

Complicated. Suppose that $G = \left[\begin{array}{c|cc} 0 & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}\right]$

which violates both A3 and A4 and corresponds to the robust stabilization of an integrator. If the controller $u=-\varepsilon x$ where $\varepsilon>0$ is used, then

$$T_{zw} = \frac{-\varepsilon s}{s+\varepsilon}, \text{ with } \|T_{zw}\|_\infty = \varepsilon.$$

Hence the norm can be made arbitrarily small as $\varepsilon \to 0$, but $\varepsilon=0$ is not stabilizing.

Relaxing A1

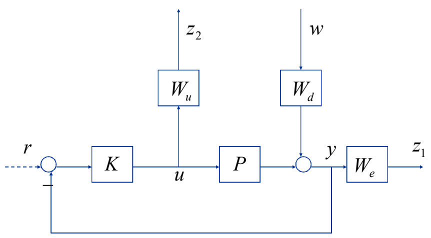
Complicated.

Relaxing A2

Singular Problem: reduced ARE or LMI,…

H_2 and H_∞ Integral Control

H_2 and H_∞ design frameworks do not in general produce integral control.



Ways to achieve the integral control:

- 1. Introduce an integral in the performance weight W_e:

$$z_1 = W_e(I+PK)^{-1}W_d\, w.$$

Now if the norm (2-norm or ∞-norm) between w and z_1 is finite, then K must have a pole at s = 0 which is the zero of the sensitivity function.

The standard H_2 (or H_∞) control theory can not be applied to this problem formulation directly because the pole of s =0 of W_e becomes an uncontrollable pole of the feedback system (A1 is violated).

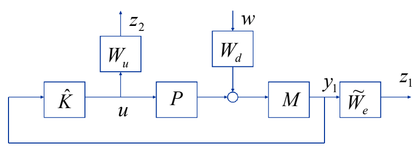
Suppose W_e can be factorized as follows

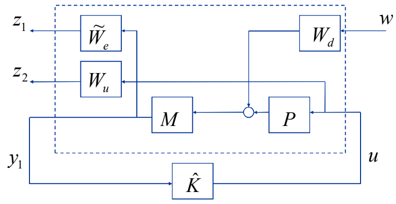
$$W_e = \tilde{W}_e(s)M(s)$$

Where $M(s)$ is proper, containing all the imaginary axis poles of W_e, and $M^{-1}(s) \in RH_\infty, \tilde{W}_e(s)$ is stable and minimum phase. Now suppose there exists a controller $K(s)$ which contains the same imaginary axis poles that achieves the performance. Then without loss of generality, K can be factorized as

$$K(s) = -\hat{K}(s)M(s)$$

Now the problem can be reformulated as





A simple numerical example:

$$P=\frac{s-2}{(s+1)(s-3)}=\left[\begin{array}{cc|c}0&1&0\\3&2&1\\\hline -2&1&0\end{array}\right],\ W_d=1,$$

$$W_u=\frac{s+10}{s+100}=\left[\begin{array}{c|c}-100&-90\\\hline 1&1\end{array}\right],\ W_e=\frac{1}{s}.$$

Then we can choose without loss of generality that

$$M=\frac{s+\alpha}{s},\ \widetilde{W}_e=\frac{1}{s+\alpha},\ \alpha>0.$$

This gives the following generalized system

$$G(s)=\left[\begin{array}{ccccc|cc}-\alpha&0&1&-2&1&1&0\\0&-100&0&0&0&0&-90\\0&0&0&-2\alpha&\alpha&\alpha&0\\0&0&0&0&1&0&0\\0&0&0&3&2&0&1\\\hline 1&0&0&0&0&0&0\\0&1&0&0&0&0&1\\0&0&1&-2&1&1&0\end{array}\right]$$

Suboptimal H_∞ controller $\hat{K}_\infty$:

$$\hat{K}_\infty=\frac{-2060381.4(s+1)(s+\alpha)(s+100)(s-0.1557)}{(s+\alpha)^2(s+32.17)(s+262343)(s-19.89)}$$

which gives the closed-loop ∞ norm 7.854.

$$K_\infty=-\hat{K}_\infty(s)M(s)=\frac{2060381.4(s+1)(s+100)(s-0.1557)}{s(s+32.17)(s+262343)(s-19.89)}$$
$$\approx\frac{7.85(s+1)(s+100)(s-0.1557)}{s(s+32.17)(s-19.89)}$$

An optimal H_2 controller

$$\hat{K}_2=\frac{-43.487(s+1)(s+\alpha)(s+100)(s-0.069)}{(s+\alpha)^2(s^2+30.94s+411.81)(s-7.964)}$$

and

$$K_2(s)=-\hat{K}_2M(s)=\frac{43.487(s+1)(s+100)(s-0.069)}{s(s^2+30.94s+411.81)(s-7.964)}.$$

❑ 2. An approximate integral control:

$$W_e=\widetilde{W}_e=\frac{1}{s+\varepsilon},\quad M(s)=1$$

for a sufficiently small ε>0. For example, a controller for ε=0.001 is given by

$$K_\infty=\frac{316880(s+1)(s+100)(s-0.1545)}{(s+0.001)(s+32)(s+40370)(s-20)}$$
$$\approx\frac{7.85(s+1)(s+100)(s-0.1545)}{s(s+32)(s-20)}$$

which gives the closed-loop H_∞ norm of 7.85.

$$K_2=\frac{43.47(s+1)(s+100)(s-0.0679)}{(s+0.001)(s^2+30.93s+411.7)(s-7.9718)}.$$

H_∞ Filtering

System Description:

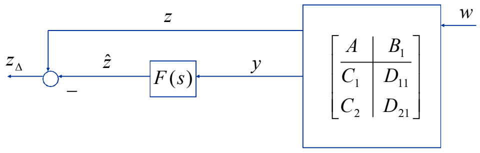
$$\dot{x}=Ax+B_1w(t),\quad x(0)=0$$
$$y=C_2x+D_{21}w(t)$$
$$z=C_1x,\quad B_1D_{21}^*=0,\ D_{21}D_{21}^*=I$$

H_∞ Filtering: Given a γ>0, find a causal filter $F(s)\in RH_\infty$

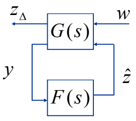
if it exists such that

$$J:=\sup_{w\in L_2[0,\infty)}\frac{\|z-\hat{z}\|_2^2}{\|w\|_2^2}<\gamma^2$$

with $\hat{z}=F(s)y$.



This filtering problem can be regarded as a H_∞ control problem without internal stability requirement.



$$G(s) = \left[\begin{array}{c|cc} A & B_1 & 0 \\ \hline C_1 & D_{11} & -I \\ C_2 & D_{21} & 0 \end{array}\right]$$

H_∞ Filtering Solution

There exists a causal filter $F(s) \in RH_\infty$ such that $J < \gamma^2$ if and only if $J_\infty \in dom(Ric)$ and $Y_\infty = Ric(J_\infty) \geq 0$

$$\hat{z} = F(s)y = \left[\begin{array}{c|c} A - Y_\infty C_2^* C_2 & Y_\infty C_2^* \\ \hline C_1 & 0 \end{array}\right] y$$

where Y_∞ is the stabilizing solution to

$$Y_\infty A^* + AY_\infty + Y_\infty(\gamma^{-2} C_1^* C_1 - C_2^* C_2) Y_\infty + B_1 B_1^* = 0.$$