Institute of Systems Science and Intelligent Control Technology 系统科学与智能控制技术研究所

鲁棒控制：建模、跟踪、抗扰、容错

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提纲

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Chapter 14a: Understanding H_∞ Control

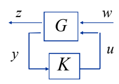
- *Objective: Derivation of H_∞ controller*
- *Methods: Intuition and hand-waving*
- *Background: State Feedback and Observer*

- **Problem Formulation and Solutions**
- **Bounded Real Lemma: another characterization**
- **An intuitive Proof**

Problem Formulation and Solutions

- **Consider a general LFT system**

$$G(s)=\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right].$$



- **Assumptions:**

(i) (A, B_1) is stabilizable and (C_1,A) is detectable;

(ii) (A,B_2) is stabilizable and (C_2,A) is detectable;

(iii) $D_{12}{}^*[C_1, D_{12}]=[0\ \ I\,]$

(iv) $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^* = \begin{bmatrix} 0 \\ I \end{bmatrix}$

$$\|z\|^2=\|C_1x+D_{12}u\|^2=(C_1x+D_{12}u)'(C_1x+D_{12}u)$$
$$=x'C_1'C_1x+u'u=\|C_1x\|^2+\|u\|^2$$

(i) Together with (ii) guarantees that the two H_2 AREs have nonnegative stabilizing solutions.

(ii) Necessary and sufficient for G to be internally stabilizable.

(iii) The penalty on $z = C_1x + D_{12}u$ includes a nonsingular, normalized penalty on the control u. In the conventional H_2 setting this means that there is no cross weighting between the state and control and that the control weight matrix is the identity.

(iv) w includes both plant disturbance and sensor noise, these are orthogonal, and the sensor noise weighting is normalized and nonsingular.

These assumptions can be relaxed.

- **Solution: $\exists$ K such that $\|T_{zw}\|_\infty < \gamma$ if and only if**

（i） $X_\infty \geq 0$

$$X_\infty A+A^*X_\infty+X_\infty(B_1B_1^*/\gamma^2-B_2B_2^*)X_\infty+C_1^*C_1=0$$

（ii） $Y_\infty \geq 0$

$$AY_\infty+Y_\infty A^*+Y_\infty(C_1^*C_1/\gamma^2-C_2^*C_2)Y_\infty+B_1B_1^*=0$$

（iii） $\rho(X_\infty Y_\infty)<\gamma^2$

Furthermore, $K_{sub}(s):=\left[\begin{array}{c|c} A_\infty & -Z_\infty L_\infty \\ \hline F_\infty & 0 \end{array}\right]$

where $A_\infty:=A+B_1B_1^*X_\infty/\gamma^2+B_2F_\infty+Z_\infty L_\infty C_2$

$F_\infty:=-B_2^*X_\infty$, $L_\infty:=-Y_\infty C_2^*$, $Z_\infty=(I-Y_\infty X_\infty/\gamma^2)^{-1}$

Bounded Real Lemma

- Corollary 12.3: Let $\gamma>0$. $G(s)=C(sI-A)^{-1}B+D \in RH_\infty$ and

$$H := \begin{bmatrix} A+BR^{-1}D^*C & BR^{-1}B^* \\ -C^*(I+DR^{-1}D^*)C & -(A+BR^{-1}D^*C)^* \end{bmatrix}$$

where $R=\gamma^2 I-D^*D$. Then the following conditions are equivalent:

(i) $\|G\|_\infty < \gamma$.

(ii) $\|D\| < \gamma$ and H has no eigenvalues on the imaginary axis.

(iii) $\|D\| < \gamma$ and $H \in dom(Ric)$, i.e., there exists an $X \geq 0$ such that

$X(A+BR^{-1}D^*C)+(A+BR^{-1}D^*C)^*X+XBR^{-1}B^*X+C^*(I+DR^{-1}D^*)C=0$

and $A+BR^{-1}D^*C+BR^{-1}B^*X$ is stable.

(iv) $\|D\| < \gamma$, $H \in dom(Ric)$ and $Ric(H) \geq 0$ ($Ric(H)>0$ if (C,A) is observable).

Bounded Real Lemma

- Let $z=G(s)w$, $G(s)=C(sI-A)^{-1}B \in RH_\infty$

$$\|G\|_\infty = \sup_w \frac{\|z\|_2}{\|w\|_2} := \sup_w \frac{\sqrt{\int_0^\infty \|z\|^2 dt}}{\sqrt{\int_0^\infty \|w\|^2 dt}},$$

$$\|G\|_\infty < \gamma \;\Leftrightarrow\; \int_0^\infty (\|z\|^2 - \gamma^2\|w\|^2)dt < 0,\ \forall w \neq 0$$

$$\Updownarrow$$

$\exists\, X=X^* \geq 0$ such that $XA+A^*X+XBB^*X/\gamma^2+C^*C=0$ and $A+BB^*X/\gamma^2$ is stable.

$$\Leftrightarrow$$

$Y=Y^* \geq 0$ such that $YA^*+AY+YC^*CY/\gamma^2+BB^*=0$ and $A+YC^*C/\gamma^2$ is stable.

$$\Updownarrow$$

$$\|G^T\|_\infty = \|B^T(sI-A^T)^{-1}C^T\|_\infty < \gamma$$

An Intuitive Proof

Note that the system equation can be written as

$$\dot{x} = Ax + B_1 w + B_2 u$$
$$z = C_1 x + D_{12} u$$
$$y = C_2 x + D_{21} w$$

- State Feedback: $u=Fx$

$$\dot{x} = (A+B_2F)x + B_1 w$$
$$z = (C_1 + D_{12}F)x$$

By Bounded Real Lemma: $\|T_{zw}\|_\infty < \gamma \;\Leftrightarrow\; \exists\, X=X^* \geq 0$ such that

$X(A+B_2F)+(A+B_2F)^*X+XB_1B_1^*X/\gamma^2+(C_1+D_{12}F)^*(C_1+D_{12}F)=0$

and $A+B_2F+B_1B_1^*X/\gamma^2$ is stable.

- (By completing square with respect to F to get) $\Leftrightarrow$
  $\exists\, X=X^* \geq 0$ such that
  $XA+A^*X+XB_1B_1^*X/\gamma^2-XB_2B_2^*X+C_1^*C+(F+B_2^*X)^*(F+B_2^*X)=0$
  and $A+B_2F+B_1B_1^*X/\gamma^2$ is stable.
- (Intuition suggests that $F=-B_2^*X$) $\Leftrightarrow$
  $\exists\, X=X^* \geq 0$ such that
  $XA+A^*X+XB_1B_1^*X/\gamma^2-XB_2B_2^*X+C_1^*C_1=0$
  and $A-B_2B_2^*X+B_1B_1^*X/\gamma^2$ is stable.
- $\Rightarrow$ $F=F_\infty$ and $X=X_\infty$.

- **Output Feedback: Converting to State Estimation**

  Suppose $\exists$ a K such that $\|T_{zw}\|_\infty < \gamma$

  Then $x(\infty)=0$ by stability (note also $x(0)=0$)

Consider the following integral

$$\int_0^\infty \left(\|z\|^2 - \gamma^2\|w\|^2\right)dt = \int_0^\infty \left(\|z\|^2 - \gamma^2\|w\|^2 + \frac{d}{dt}(x^*X_\infty x)\right)dt$$
$$= \int_0^\infty \left(\|z\|^2 - \gamma^2\|w\|^2 + \dot{x}^*X_\infty x + x^*X_\infty \dot{x}\right)dt$$

Substituting $\dot{x} = Ax + B_1w + B_2u$ and $z = C_1x + D_{12}u$ to get

$$= \int_0^\infty (\|C_1x\|^2 + \|u\|^2 - \gamma^2\|w\|^2 + 2x^*A^*X_\infty x + 2x^*X_\infty B_1 w + 2x^*X_\infty B_2 u)dt$$
$$= \int_0^\infty (x^*(C_1^*C_1 + X_\infty A + A^*X_\infty)x + \|u\|^2 - \gamma^2\|w\|^2 + 2x^*X_\infty B_1 w + 2x^*X_\infty B_2 u)dt$$

Using X_∞ equation to get

$$= \int_0^\infty (x^*(-X_\infty B_1B_1^*X_\infty/\gamma^2 + X_\infty B_2B_2^*X_\infty)x + \|u\|^2$$
$$-\gamma^2\|w\|^2 + 2x^*X_\infty B_1 w + 2x^*X_\infty B_2 u)dt$$
$$= \int_0^\infty (-\|B_1^*X_\infty/\gamma\|^2 - \gamma^2\|w\|^2 + 2x^*X_\infty B_1 w + \|B_2^*X_\infty x\|^2 + \|u\|^2 + 2x^*X_\infty B_2 u)dt$$

completing the squares with respect to u and w

$$= \int_0^\infty \left(\|u + B_2^*X_\infty x\|^2 - \gamma^2\|w - \gamma^{-2}B_1^*X_\infty x\|^2\right)dt$$

- **Summary:**

$$\int_0^\infty \left(\|z\|^2 - \gamma^2\|w\|^2\right)dt = \int_0^\infty \left(\|v\|^2 - \gamma^2\|r\|^2\right)dt$$

$$v = u + B_2^*X_\infty x = u - F_\infty x, \qquad r = w - \gamma^{-2}B_1^*X_\infty x$$

Rewrite the system equation with: $w = r + \gamma^{-2}B_1^*X_\infty x$

$$\dot{x} = (A + B_1B_1^*X_\infty/\gamma^2)x + B_1 r + B_2 u$$
$$v = -F_\infty x + u$$
$$y = C_2 x + D_{21} w = C_2 x + D_{21} r$$

$$\|T_{zw}\|_\infty < \gamma \Leftrightarrow \|T_{vr}\|_\infty < \gamma \Leftrightarrow \int_0^\infty \left(\|u - F_\infty x\|^2 - \gamma^2\|r\|^2\right)dt < 0$$

If state is availabe; $u=F_\infty x$

worst disturbance: $w_*=\gamma^{-2}B_1^*X_\infty x$

State is not available : using estimated state $u = F_\infty \hat{x}$

A standard observer :

$$\dot{\hat{x}} = (A + B_1 B_1^* X_\infty / \gamma^2)\hat{x} + B_2 u + L(C_2 \hat{x} - y)$$

where L is the observer gain is to be determined.

Let $e := x - \hat{x}$. Then

$$\dot{e} = (A + B_1 B_1^* X_\infty / \gamma^2 + LC_2)e + (B_1 + LD_{21})r$$

$$v = -F_\infty e$$

$\|T_{zw}\|_\infty < \gamma \Leftrightarrow \|T_{vr}\|_\infty < \gamma$ by bounded real lemma

$$Y(A + B_1 B_1^* X_\infty / \gamma^2 + LC_2)^* + (A + B_1 B_1^* X_\infty / \gamma^2 + LC_2)Y + YF_\infty^* F_\infty Y / \gamma^2 + (B_1 + LD_{21})(B_1 + LD_{21})^* = 0$$

Complete square w.r.t. L

$$(A + B_1 B_1^* X_\infty / \gamma^2)^* + (A + B_1 B_1^* X_\infty / \gamma^2)Y + YF_\infty^* F_\infty Y / \gamma^2 + B_1 B_1^* - YC_2^* C_2 Y + (L + YC_2^*)(L + YC_2^*)^* = 0$$

Again, intuition suggests that we can take $L = -YC_2^*$

which gives

$$Y(A + B_1 B_1^* X_\infty / \gamma^2)^* + (A + B_1 B_1^* X_\infty / \gamma^2)Y + YF_\infty^* F_\infty Y / \gamma^2 - YC_2^* C_2 Y + B_1 B_1^* = 0$$

It is easy to verify that

$$Y = Y_\infty (I - \gamma^{-2} X_\infty Y_\infty)^{-1}$$

Since $Y \ge 0$, we must have $\rho(X_\infty Y_\infty) < \gamma^2$

Hence $L = Z_\infty L_\infty$ and the controller is given by

$$\dot{\hat{x}} = (A + B_1 B_1^* X_\infty / \gamma^2)\hat{x} + B_2 u + Z_\infty L_\infty (C_2 \hat{x} - y)$$

$$u = F_\infty \hat{x}$$