Lecture Notes on

# Algebra

Will Lancer

will.m.lancer@gmail.com

Notes on algebra. Resources used:

- Aluffi's *Algebra: Chapter 0*

# Contents

# 1 Categorical preliminaries

**(fill-in)**

# 2 Groups

We review the properties of groups here. First, two high-brow definitions of a group:

- A **group** is a group object in Set.
- A **group** is a one-object groupoid.

These aren't bad definitions, but they're more explicitly useful when they're unpacked a bit.

**Definition 1.** A **group** is a set with an associative binary operation $m_G\colon G \to G$ that has identites and inverses.

We can phrase this by saying that a group has two additional unique maps, $\iota\colon G \to G$ by $g \mapsto g^{-1}$ and $\mathrm{id}\colon G \to G$ by $g \mapsto g$. This is the more categorical and "morally correct" way of putting this.

**Example 2.1** (Canonical group examples)

The most canonical group examples are

- Any field with addition, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, etc.. Some of them also work with multiplication.
- $\mathbb{Z}/n\mathbb{Z}$: the integers mod $n$ under addition. $(\mathbb{Z}/n\mathbb{Z})^\times$ are the units mod $n$.
- $D_n$. The isometries of a planar solid.
- $S_n$ and $A_n$. The permutations and even permutations of $n$ objects.
- Matrices over some field under addition or multiplication (if the operation is multiplication then needs invertibility).
- Types of functions under addition, e.g. smooth, periodic, etc..

Groups form a category, Grp. Its maps are **group homomorphisms**, which are maps $\varphi\colon G \to G'$ that respect the group structure, $\varphi(ab) = \varphi(a)\varphi(b)$. One way of saying this is that the following diagram commutes:

**(put diagram)**

We now investigate what other universal constructions exist in Grp.

- **Products**: the product in Grp is the **direct product group**. This means that for all $\varphi_G\colon A \to G$, $\varphi_H\colon A \to H$, there exists a unique map $\widetilde{\varphi}\colon A \to G \times H$ making the diagram

$$
\begin{tikzcd}
 & & G \\
A \arrow[urr, bend left, "\varphi_G"] \arrow[drr, bend right, "\varphi_H"'] \arrow[r, dashed, "\widetilde{\varphi}"] & G \times H \arrow[ur, "\pi_G"] \arrow[dr, "\pi_H"'] & \\
 & & H
\end{tikzcd}
$$

commute. Concretely, taking the binary operations on $G$ and $H$ and defining

$$m_G \times m_H \colon (G \times H) \times (G \times H) \to G \times H$$
$$(m_G \times m_H)((g_1, h_1), (g_2, h_2)) \mapsto (m_G((g_1, g_2)), m_H((h_1, h_2))).$$

gives the set $G \times H$ a group structure

$$(g_1, h_1) * (g_2, h_2) = (g_1 g_2, h_1 h_2).$$

- **Coproducts**: there is not a generic coproduct in Grp. **(talk about free product and Ab)**
- **Quotients** **(add)**