PHY680: Midterm Exam

Thursday, October 14, 2021.

This exam consists of 3 pages.

1 The spectrum of a tetrahedron

We are going to determine the spectrum of the normal modes of the small fluctuations of the atoms of a molecule in the shape of a tetrahedron. Below follows a large number of mostly easy questions; answer them briefly.

a) Check Euler's formula $V - E + F = c$ where c is an integer for a tetrahedron. To check your value of c, repeat this calculation for a cube.

b) How many genuine normal modes are there?

c) The symmetry group G of a tetrahedron is S_4. Which symmetries correspond to each of the classes. Write the classes of S_4 in the following order

$$e,\ (ab)(cd),\ (abc),\ (ab),\ (abcd)\,. \tag{1.1}$$

What is the order of each class?

d) How many nontrivial normal subgroups does S_4 have? Prove your answer, only using the orders of the classes.

e) What are the coset elements for the largest of these normal subgroups, and what are the coset elements for the smallest of these normal subgroups? What are the corresponding quotient groups?

f) What is the commutator subgroup of S_4? What are the one-dimensional irreps of S_4? How many irreps are there for S_4?

g) Find the character χ_3 of the 3-dimensional rep of S_4 corresponding to the isometries of the tetrahedron in physical 3-dimensional space. Prove that this rep is an irrep.

h) Construct the character $\chi_{3'}$ of another, inequivalent, 3-dimensional irrep. What are the dimensions of all irreps?

i) Construct the character table for S_4. To find the character of the last irrep, use the orthogonality relations for characters.

j) Rederive the result in i) from tensor methods by constructing the Clebsch-Gordan decomposition for the tensor product $\mathbf{3} \times \mathbf{3}$ where $\mathbf{3}$ denotes the irrep in g).

k) How many atoms n_S are held fixed by the symmetries of each of the classes?

l) Now construct the character for the molecule as a whole (the character we have called χ_S in class). Then subtract the characters for the zero modes. Call the resulting character for the genuine normal modes χ_{gen}.

m) Decompose $\chi_{\text{gen}}(g) = \sum_i n_i \chi_i(g)$ where $\chi_i(g)$ are the characters of the irreps of S_4. Obtain the result for $\sum_i (n_i)^2$. What is $\sum_i d_i n_i$ (where d_i is the dimension of the irrep whose character is χ_i)?

n) How often are the irreps of S_4 contained in χ_{gen}? How many multiplets (frequencies) are there in the spectrum of this molecule?

o) (Most interesting, but also most difficult.) What motions of the atoms correspond to each multiplet of normal modes?

2 Dirac matrices

We consider Dirac matrices in $d = 6$ dimensions, and construct them from the following representation of the Dirac matrices γ^μ with $\mu = 1, \cdots, 4$ in $d = 4+0$ Euclidean dimensions by tensoring (as explained in the notes for $d = 7, 9, 10, 11$)

$$\gamma^\mu(d = 4) = \begin{pmatrix} 0 & -i\sigma^\mu \\ i\bar{\sigma}^\mu & 0 \end{pmatrix} \tag{2.1}$$

with $\sigma^\mu = (\vec{\sigma}, iI)$ and $\bar{\sigma}^\mu = (\vec{\sigma}, -iI)$.

a) First construct the Dirac matrices $\widehat{\gamma}^m$ with $m = 1, \cdots, 5$ in $d = 5+0$ Euclidean space. Are all matrices in $d = 5+0$ hermitian and unitary? Is this representation in $d = 5+0$ unique?

b) Now construct a set of Dirac matrices Γ^M with $M = 1, \cdots, 6$ for $d = 6+0$ Euclidean space from those in $d = 5+0$. Is this representation unique? Are all Dirac matrices for $d = 6+0$ hermitian and do they square to $+I$? What is the chirality matrix $\Gamma_c^{(6)}$ in 6 dimensions? (Normalize such that $\left(\Gamma_c^{(6)}\right)^2 = I$.)

c) Next construct a representation of the Dirac matrices in $d = 5+1$ Minkowski space. Is this representation unique? Are all matrices hermitian, and do they all square to $+I$?

d) Construct for these explicit matrix representations the charge conjugation matrices in $d = 4$, $d = 5$ and $d = 6$. Explain why they are the same in Minkowski space and Euclidean space. Check that $C_+^{(6)} = C_-^{(6)} \Gamma_c^{(6)}$.

e) Is the irrep of Γ^M in $d = 6 + 0$ Euclidean space real, pseudoreal or complex? If it real or pseudoreal, find the matrix S in $S\Gamma^M S^{-1} = (\Gamma^M)^*$.

f) Is the irrep of Γ^M in $d = 5 + 1$ Minkowski space real, pseudoreal or complex? Find again S if it exists.

g) Do Majorana spinors exist in $d = 6 + 0$ and/or $d = 5 + 1$? Do Weyl spinors exist in $d = 6 + 0$ and/or $d = 5 + 1$? Do Majorana-Weyl spinors exist in $d = 6 + 0$ and/or $d = 5 + 1$?

h) The properties of Dirac matrices and charge conjugation matrices do not depend on the particular representation chosen. Show that if C_+ is symmetric for a particular representation of the Dirac matrices, then after a similarity transformation of the Dirac matrices the new C_+ is again symmetric.