Lecture Notes on

Groups and Representations

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Notes on group and representations, as used in physics. Resources used:

- Peter van Nieuwenhuizen's PHY 680 course at SBU, and the lecture notes from the course.
- Andre Lukas' lecture notes on groups and representations.
- Fulton and Harris' *Representation Theory: A First Course*.
- Parts of Xi Yin's 253ab courses.

Contents

1 Finite Groups and Representations

We begin by collecting the most important facts about the representation theory of finite groups. At the end, we touch on Majorana spinors and the normal modes of atomic molecules. For general background, see the notes on algebra here.

Example 1.1 (The most important finite groups for physics)

Of course, the most important groups for physics are (semi-simple) Lie groups, but there are a few important finite ones as well:

- S_n. This group is good for talking about permutations of objects, and is necessary for talking about group actions.
- A_n. See above, but sometimes we only want even permutations.
- D_n. Because D_n is, by definition, the group of isometries of a planar n-point object, this is useful to look at when studing planar objects. We can also think about certain continuous groups as limits of D_n, e.g. emergent $O(N)$ symmetry in spin systems.
- $\mathbb{Z}/n\mathbb{Z}$. These groups are useful for a number of things. $\mathbb{Z}/2\mathbb{Z}$ is parity, and modding out by it also produces quite a few useful Lie groups (e.g. **(give examples here)**). It also catalogues cyclic groups, so it's useful when we want to think about those ideas. Additionally, these groups are closely connected to number theory when $n = p$ is prime, and there are more and more connections between physics and number theory as time goes on...

We now get into the most important definitions behind finite group theory for physics.

Definition 1. Consider $\gamma_g \colon G \to G$ by $\gamma_g(a) = gag^{-1}$. We say γ_g is an **inner automorphism** of G, and we denote the set of γ_g's by $\mathrm{Inn}_{\mathsf{Grp}}(G)$.

We may talk about the orbit of some a under the elements of $\mathrm{Inn}_{\mathsf{Grp}}(G)$; this is called the **class** of a. The practical algorithm for producing classes is as follows:

1. Take some $a \in G$. Consider gag^{-1} for all g in g.
2. If gag^{-1} is not already in your set, add it to the set.
3. Once you are done, take some $b \in G$ not in the set and repeat these steps.

This algorithm terminates, as conjugation is an equivalence relation, so the classes of G partition it.

Definition 2. The **center** of a group is defined as $\mathcal{Z}(G) \coloneqq \{z \in G \mid zg = gz \ \forall g \in G\}$.

Intuitively, the center of a group tells you how abelian the group is. Indeed, we have that $\mathcal{Z}(G) = G \iff G$ is abelian. We may ask the question of how to take some non-abelian group and "turn it into" an abelian group. The only natural mechanism for this would be modding out by a subgroup. As it turns out, the smallest subgroup that makes this possible is

Definition 3. The **commutator subgroup** $[G, G]$ of G is defined as the group containing the elements $aba^{-1}b^{-1}$ for all $a, b \in G$.

Definition 4. The **abelianization** of G is defined as $G^{\text{ab}} = G/[G,G]$.

Idea 1.1 (Most important facts about finite groups)

Whenever you see a finite group, you want to answer these questions:

- What is the order of the group? I find this question to be psychologically comforting, and it also tells me what the possible orders of the subgroups of G are.
- **What are its classes?** The reason this is useful is because characters are class functions, and so there are a number of useful facts we can draw from knowing the classes of a group.
- What are the orders of its classes? This is less important, but useful if one wants to check whether a map V is a representation or not.
- **What is its center?** This is useful, because if we mod out by $\mathcal{Z}(G)$, the order of this group is the number of one-dimensional representations of G (so it tells us how many representations we can promptly throw away).
- **What is its commutator subgroup?** As we saw above, the commutator subgroup is the smallest subgroup of G such that G/H is abelian. (**add more once you recall this**)