PHY680: Midterm Exam Solutions

1 The spectrum of a tetrahedron

We are going to determine the spectrum of the normal modes of the small fluctuations of the atoms of a molecule in the shape of a tetrahedron. Below follows a large number of mostly easy questions; answer them briefly.

a) Check Euler's formula $V - E + F = c$ where c is an integer for a tetrahedron. To check your value of c, repeat this calculation for a cube.

Solution: For a tetrahedron $V - E + F = 4 - 6 + 4 = 2$. So $c = 2$. For a cube $V - E + F = 8 - 12 + 6 = 2$.

b) How many genuine normal modes are there?

Solution: There are $3N - 6 = 3 \times 4 - 6 = 6$ genuine normal modes.

c) The symmetry group G of a tetrahedron is S_4. Which symmetries correspond to each of the classes. Write the classes of S_4 in the following order

$$e,\ (ab)(cd),\ (abc),\ (ab),\ (abcd)\,. \tag{1.1}$$

What is the order of each class?

Solution: The classes and their order are

$$e\,[1],\ (ab)(cd)\,[3],\ (abc)\,[8],\ (ab)\,[6],\ (abcd)\,[6]\,. \tag{1.2}$$

Further, $(ab)(cd)$ is a rotation over π, (abc) a rotation over $\frac{2\pi}{3}$, (ab) a reflection wrt a plane, and $(abcd)$ a rotation followed by a reflection.

d) How many nontrivial normal subgroups does S_4 have? Prove your answer, only using the orders of the classes.

Solution: There are two nontrivial normal subgroups of S_4, namely A_4 and V. Any normal subgroup is composed of whole classes, and its order must be a divisor of 24. Any group element can be written as a product of transpositions, so the class (ab) is out. The class $(abcd)$ is also out since $1 + 6+$(any combination of 3 and 8) is not a divisor of 24. That leaves $1 + 3$, $1 + 8$, $1 + 3 + 8$. The first is V, the latter is A_4, and $1 + 8$ is out since it is not a divisor of 24.

e) What are the coset elements for the largest of these normal subgroups, and what are the coset elements for the smallest of these normal subgroups? What are the corresponding quotient groups?

Solution: For $N = V$, the elements of S_4/V are

$$\{V,\, (12)V,\, (13)V,\, (14)V,\, (123)V,\, (132)V\}\,. \tag{1.3}$$

The coset elements contain the following group elements

$$\begin{aligned} (12)V &= \{(12),(34),(1324),(1423)\} & (123)V &= \{(123),(134),(243),(142)\} \\ (13)V &= \{(13),(24),(1234),(1432)\} & (132)V &= \{(132),(124),(234),(143)\} \\ (14)V &= \{(14),(23),(1243),(1342)\} \end{aligned} \tag{1.4}$$

For $N = A_4$, the elements of S_4/A_4 are

$$\{A_4,\, (12)A_4\}\,. \tag{1.5}$$

The quotient groups are $S_4/V = S_3$ and $S_4/A_4 = Z_2$.

f) What is the commutator subgroup of S_4? What are the one-dimensional irreps of S_4? How many irreps are there for S_4?

Solution: $C(S_4) = A_4$, see homeworks. Since $\frac{|S_4|}{|A_4|} = 2$, there are two one-dimensional irreps. There are

$$\begin{aligned} \chi_1 &= (1,1,1,1,1) \\ \chi_{1'} &= (1,1,1,-1,-1) \end{aligned} \tag{1.6}$$

There are 5 irreps.

g) Find the character χ_3 of the 3-dimensional rep of S_4 corresponding to the isometries of the tetrahedron in physical 3-dimensional space. Prove that this rep is an irrep.

Solution: Using $\chi(g) = 1 + 2\cos\phi$ for rotations, and $\chi(g) = 1$ for reflections, we get

$$\chi_3 = (3,-1,0,1,-1) \tag{1.7}$$

The last entry -1 follows from the orthogonality relations for characters, or from constructing a particular 3×3 matrix for this isometry and then taking the trace. Since $\sum_g \chi_3(g)\chi_3^*(g) = 9\times 1 + 1\times 3 + 1\times 6 + 1\times 6 = 24$, this rep is an irrep.

h) Construct the character $\chi_{3'}$ of another, inequivalent, 3-dimensional irrep. What are the dimensions of all irreps?

Solution: Since the product of characters is again a character, $\chi_3\chi_{1'} = (3,-1,0,-1,1)$ is also a rep, and also an irrep, and linearly independent of χ_3, it is a second 3-dimensional irrep $\chi_{3'}$. From $24 = \sum_i (d_i)^2 = 1^2 + 1^2 + 3^2 + 3^2 + x^2$ we see that the fifth irrep of S_4 has dimension 2. So the dimensions of all irreps are $1, 1, 3, 3, 2$.

i) Construct the character table for S_4. To find the character of the last irrep, use the orthogonality relations for characters.

Solution:

S_4	$e[1]$	$(ab)(cd)[3]$	$(abc)[8]$	$(ab)[6]$	$(abcd)[6]$
χ_1	1	1	1	1	1
$\chi_{1'}$	1	1	1	-1	-1
χ_3	3	-1	0	1	-1
$\chi_{3'}$	3	-1	0	-1	1
χ_2	2	2	-1	0	0

j) Rederive the result in i) from tensor methods by constructing the Clebsch-Gordan decomposition for the tensor product $\mathbf{3} \times \mathbf{3}$ where $\mathbf{3}$ denotes the irrep in g).

Solution: The character of $\mathbf{3} \times \mathbf{3}$ is $\chi_{3\times 3} = \chi_3\chi_3 = (9,1,0,1,1)$. Since $\chi_{3\times 3} = \sum_i n_i\chi_i$, we find n_i from $n_i = \frac{1}{|G|}(\chi_i, \chi_{3\times 3})$. The result is

$$\chi_{3\times 3} = \chi_1 + \chi_2 + \chi_3 + \chi_{3'} \,. \tag{1.8}$$

Subtracting χ_1, χ_3, $\chi_{3'}$ from $\chi_{3\times 3}$ yields χ_2.

k) How many atoms n_S are held fixed by the symmetries of each of the classes?

Solution: $n_S = (4,0,1,2,0)$.

l) Now construct the character for the molecule as a whole (the character we have called χ_S in class). Then subtract the characters for the zero modes. Call the resulting character for the genuine normal modes χ_{gen}.

Solution:

$$\begin{aligned} \chi_3 &= (3,-1,0,1,-1) \\ \chi_S = n_S\chi_3 &= (12,0,0,2,0) \\ \chi_{\text{trans}} &= (3,-1,0,1,-1) \\ \chi_{\text{rot}} &= (3,-1,0,-1,1) \\ \chi_{\text{gen}} &= (6,2,0,2,0)\,. \end{aligned} \tag{1.9}$$

m) Decompose $\chi_{\text{gen}}(g) = \sum_i n_i\chi_i(g)$ where $\chi_i(g)$ are the characters of the irreps of S_4. Obtain the result for $\sum_i (n_i)^2$. What is $\sum_i d_i n_i$ (where d_i is the dimension of the irrep whose character is

χ_i)?

Solution: Since $(\chi_{\text{gen}}, \chi_{\text{gen}}) = 36 \times 1 + 4 \times 3 + 4 \times 6 = 72 = 3 \times 24$ we have $\sum_i n_i^2 = 3$. Further, $\chi_{\text{gen}}(e) = 6 = \sum_i d_i n_i = n_1 + n_{1'} + 2n_2 + 3n_3 + 3n_{3'}$.

n) How often are the irreps of S_4 contained in χ_{gen}? How many multiplets (frequencies) are there in the spectrum of this molecule?

Solution: Direct evaluation yields $(\chi_1, \chi_{gen}) = 24$ so $n_1 = 1$. Similarly $n_{1'} = 0$, $n_2 = 1$ and $n_3 = 1$. These numbers agree with the results in m)

$$\chi_{\text{gen}} = \chi_1 + \chi_2 + \chi_3\,. \tag{1.10}$$

o) (Most interesting, but also most difficult.) What motions of the atoms correspond to each multiplet of normal modes?

Solution: χ_1 corresponds to the breather, χ_3 corresponds to the 4 "pumpers", three of which are linearly independent. Finally, χ_2 corresponds to the 3 "twistors" (twisting any two opposite edges in opposite ways) of which 2 are linearly independent.

2 Dirac matrices

We consider Dirac matrices in $d = 6$ dimensions, and construct them from the following representation of the Dirac matrices γ^μ with $\mu = 1, \cdots, 4$ in $d = 4{+}0$ Euclidean dimensions by tensoring (as explained in the notes for $d = 7, 9, 10, 11$)

$$\gamma^\mu(d=4) = \begin{pmatrix} 0 & -i\sigma^\mu \\ i\bar{\sigma}^\mu & 0 \end{pmatrix} \tag{2.1}$$

with $\sigma^\mu = (\vec{\sigma}, iI)$ and $\bar{\sigma}^\mu = (\vec{\sigma}, -iI)$.

a) First construct the Dirac matrices $\widehat{\gamma}^m$ with $m = 1, \cdots, 5$ in $d = 5 + 0$ Euclidean space. Are all matrices in $d = 5 + 0$ hermitian and unitary? Is this representation in $d = 5 + 0$ unique?

Solution: $\widehat{\gamma}^m = \{\gamma^\mu, \gamma^5\}$ where $\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4$. All γ^μ and γ^5 are hermitian, and $\gamma^5 = \left(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right)$. This representation is not unique (except for similarity transformations; a second irrep is obtained by replacing γ^5 by $-\gamma^5$).

b) Now construct a set of Dirac matrices Γ^M with $M = 1, \cdots, 6$ for $d = 6{+}0$ Euclidean space from those in $d = 5{+}0$. Is this representation unique? Are all Dirac matrices for $d = 6{+}0$ hermitian and do they square to $+I$? What is the chirality matrix $\Gamma_c^{(6)}$ in 6 dimensions? (Normalize such that $\left(\Gamma_c^{(6)}\right)^2 = I$.)

Solution: $\Gamma^M = \{\widehat{\gamma}^m \otimes \sigma^1, \mathbb{1} \otimes \sigma^2\}$. All $\widehat{\gamma}^m \otimes \sigma^1$ and $\mathbb{1} \otimes \sigma^2$ are hermitian and their square is I. The chirality matrix $\Gamma_c^{(6)}$ is $\Gamma_c^{(6)} = \mathbb{1} \otimes \sigma^3$. This representation is unique.

c) Next construct a representation of the Dirac matrices in $d = 5+1$ Minkowski space. Is this representation unique? Are all matrices hermitian, and do they all square to $+I$?

Solution: $\Gamma^M = \{\widehat{\gamma}^m \otimes \sigma^1, \mathbb{1} \otimes (-i\sigma^2)\}$. Also this representation is unique, but Γ^6 is antihermitian and squares to $-I$.

d) Construct for these explicit matrix representations the charge conjugation matrices in $d = 4$, $d = 5$ and $d = 6$. Explain why they are the same in Minkowski space and Euclidean space. Check that $C_+^{(6)} = C_-^{(6)}\Gamma_c^{(6)}$.

Solution: For $d = 4$

$$\begin{aligned} C_+^{(4)} &= \gamma^1\gamma^3 \\ C_-^{(4)} &= \gamma^2\gamma^4 \end{aligned} \qquad \text{since} \qquad \begin{aligned} \gamma^{1T} &= -\gamma^1 \quad & \gamma^{3T} &= -\gamma^3 \\ \gamma^{2T} &= \gamma^2 \quad & \gamma^{4T} &= \gamma^4 \end{aligned} \tag{2.2}$$

$d = 5$: $C_\pm^{(5)}$ can only be equal to $C_+^{(4)}$ or $C_-^{(4)}$ but we must also satisfy

$$C_\pm^{(5)}\gamma^5 = \pm\gamma^{5,T}C_\pm^{(5)} = \pm\gamma^5 C_\pm^{(5)} \tag{2.3}$$

Only $C_+^{(4)}$ satisfies $C_+^{(4)}\gamma_5 = \gamma_5 C_+^{(4)}$. So $C_+^{(5)} = C_+^{(4)}$.

$d = 6$: We must satisfy $C_\pm^{(6)}\Gamma^M = \pm\Gamma^{M,T}C_\pm^{(6)}$ where

$$\begin{aligned} \Gamma^{1,T} &= -\Gamma^1\,; \qquad & \Gamma^{4,T} &= \Gamma^4 \\ \Gamma^{2,T} &= \Gamma^2\,; \qquad & \Gamma^{5,T} &= \Gamma^5 \\ \Gamma^{3,T} &= -\Gamma^3\,; \qquad & \Gamma^{6,T} &= -\Gamma^6 \end{aligned} \tag{2.4}$$

So

$$\begin{aligned} C_\pm^{(6)}\left(\Gamma^1 \text{ or } \Gamma^3 \text{ or } \Gamma^6\right) &= \mp\left(\Gamma^1 \text{ or } \Gamma^3 \text{ or } \Gamma^6\right)C_\pm^{(6)} \\ C_\pm^{(6)}\left(\Gamma^2 \text{ or } \Gamma^4 \text{ or } \Gamma^5\right) &= \pm\left(\Gamma^2 \text{ or } \Gamma^4 \text{ or } \Gamma^5\right)C_\pm^{(6)} \end{aligned} \tag{2.5}$$

The solutions are

$$\begin{aligned} C_+^{(6)} &= \Gamma^2\Gamma^4\Gamma^5 = \gamma^2\gamma^4\gamma^5 \otimes \sigma^1 = \gamma^1\gamma^3 \otimes \sigma^1 = C_+^{(5)} \otimes C_+^{(2)} \\ C_-^{(6)} &= \Gamma^1\Gamma^3\Gamma^6 = \gamma^1\gamma^3 \otimes (-i\sigma^2) = C_+^{(5)} \otimes C_-^{(2)} \end{aligned} \tag{2.6}$$

Clearly $C_+^{(6)}$ is antisymmetric and $C_-^{(6)}$ is symmetric. They satisfy $C_+^{(6)} = C_-^{(6)}\Gamma_c^{(6)}$ since

$$\left(\gamma^1\gamma^3 \otimes \sigma^1\right) = \left(\gamma^1\gamma^3 \otimes (-i\sigma^2)\right)\left(\mathbb{1} \otimes \sigma^3\right). \tag{2.7}$$

Since adding a factor i to a Dirac matrix does not change its properties under transposition, the matrices C_+ and C_- are the same in Euclidean and Minkowski space.

e) Is the irrep of Γ^M in $d = 6 + 0$ Euclidean space real, pseudoreal or complex? If it real or pseudoreal, find the matrix S in $S\Gamma^M S^{-1} = (\Gamma^M)^*$.

Solution: In $d = 6 + 0$, the Dirac matrices form a pseudoreal representation as follows from the Frobenius-Schur criterion

$$\sum_g \chi(g^2) = 8 \times 2 \times [1 + 6 - 15 - 20 + 15 + 6 - 1] = -128 \tag{2.8}$$

The matrix S is $S = C_+$ which is indeed antisymmetric.

f) Is the irrep of Γ^M in $d = 5 + 1$ Minkowski space real, pseudoreal or complex? Find again S if it exists.

Solution: In $d = 5 + 1$ the Dirac matrices are pseudoreal as follows again from the Frobenius-Schur criterion

$$\sum_g \chi(g^2) = 8 \times 2 \times [1 + (5 - 1) - (10 - 5) - (10 - 10) + (5 - 10) + (1 - 5) + 1] = -128 \tag{2.9}$$

The matrix S is now $\gamma^1\gamma^3 \otimes \mathbb{1}$ which is indeed antisymmetric.

g) Do Majorana spinors exist in $d = 6 + 0$ and/or $d = 5 + 1$? Do Weyl spinors exist in $d = 6 + 0$ and/or $d = 5 + 1$? Do Majorana-Weyl spinors exist in $d = 6 + 0$ and/or $d = 5 + 1$?

Solution: Weyl spinors always exist in even dimensions, whether Euclidean or Minkowskian. Majorana spinors exist in Euclidean space if C_+ and/or C_- are symmertric. So Majorana spinors exist in $d = 6 + 0$ (using C_-). In $d = 5 + 1$ Majorana spinors exist if C_+ is symmetric or C_- is antisymmetric. There are no Majorana spinors in $d = 5 + 1$. Finally, Majorana-Weyl spinors only exist in $d = 9 + 1$ or $d = 8 + 0$, so not in $d = 6 + 0$ or $d = 5 + 1$.

h) The properties of Dirac matrices and charge conjugation matrices do not depend on the particular representation chosen. Show that if C_+ is symmetric for a particular representation of the Dirac matrices, then after a similarity transformation of the Dirac matrices the new C_+ is again symmetric.

Solution: If $C_+\gamma^\mu C_+^{-1} = \gamma^{\mu,T}$ for a particular matrix representation, and $\gamma^\mu = S\widehat{\gamma}^\mu S^{-1}$ is the similarity transformation, then $C_+S\widehat{\gamma}^\mu S^{-1}C_+^{-1} = S^{-1,T}\widehat{\gamma}^{\mu,T}S^T$. Thus $(S^TC_+S)\widehat{\gamma}^\mu(S^{-1}C_+S^{T,-1}) = \gamma^{\mu,T}$. The new C_+ is $\widehat{C}_+ = S^TC_+S$, which is again a symmetric matrix.