Lecture Notes on

QFT

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Resources used:

- Xi Yin's 253ab courses. These lecture notes are the best QFT resource on the planet. (**add more; this is main resource for these notes**)
- Steven Weinberg's two volumes.
- Daniel Harlow's notes.
- Sidney Coleman's QFT. A volume written by an absolute master of the subject. Indeed, Weinberg said that he learned more QFT from Sidney than from anyone else. (**add more when you finish reading**)
- *FIELDS* by Warren Siegel. (**i am so conflicted about this book bro**)

We mostly follow Weinberg's notation, except where his notation sucks.

Contents

1 Overview

(overview things here; talk about how single-particle rel qm is inconsistent)

2 Relativistic Quantum Mechanics

We start by recalling the axioms of quantum mechanics[1]:

Idea 2.1 (Axioms of quantum mechanics)
The axioms of quantum mechanics are:

- Quantum mechanical **states** are represented by **rays** in a **Hilbert space**. We will denote Hilbert spaces by $\mathcal{H}$.
- **Observables** are represented by self-adjoint operators on our Hilbert space. Observed quantities are the eigenvalues of vectors in $\mathcal{H}$.
- Let $\mathcal{R}$ be a ray. If $\mathcal{R}_1, \ldots, \mathcal{R}_n$ are a set of mutually orthogonal rays, the probabilty of observing $\mathcal{R}$ in $\mathcal{R}_i$ is

$$\mathbb{P}(\mathcal{R} \to \mathcal{R}_i) = |\langle \psi | \phi_i \rangle|^2,$$

  where $\psi \in \mathcal{R}$ and $\phi_i \in \mathcal{R}_i$.

Remarks:

- Recall that rays are equivalence classes of kets up to phase and normalization. Since this is an equivalence class, WLOG we can choose our representative to be normalized.
- Self-adjoint or Hermitian? **(comment on this)**
- In our third axiom, we make reference to experiment. How do we physically represent an experiment on our Hilbert space? **(good question)**

The most important kinds of operators are operators that realize *symmetry transformations* on our Hilbert space.

(finish)

2.1 Symmetries

(add the normal ones, and then ps in the next section)

(for whatever reason) , this continuity may also be found in the *representation* of the symmetry.

[1] Of course, there are different axioms than this that give identical results. "Axioms" is really a misnomer; they're just assumptions that let us build the rest of the theory. There is nothing intrinsically fundamental about this set of axioms as opposed to other ones.

2.2 Poincaré symmetry and the little group

Since we are doing relativistic quantum mechanics, let's see how the relativistic symmetry group acts on our Hilbert space. The Poincaré group in mostly-plus is given by

$$\mathcal{P} = \mathbb{R}^D \rtimes SO(D-1,1).$$

Consider some Poincaré transformation $(\Lambda^\mu{}_\nu, a^\mu)$. By Wigner's theorem, this is represented as a linear unitary operator on our Hilbert space; the group property is

$$U(\Lambda, a)U(\overline{\Lambda}, \overline{a}) = U(\Lambda\overline{\Lambda}, \Lambda\overline{a} + a).$$

Since the Poincaré group is a Lie group, things are continuous. Taylor-expanding $U(\Lambda, a)$ gives

$$U(\Lambda, a) \simeq \delta^\mu_\nu - i a_\mu \widehat{\mathcal{P}}^\mu + \frac{i}{2}\omega_{\mu\nu}\widehat{J}^{\mu\nu}.$$

The generators of $U(\Lambda, a)$ are very special: they are the four-momentum operator and the angular momentum/boost operator. Looking at $\widehat{J}^{\mu\nu}$ gives

$$\widehat{J}^{\mu\nu} = \Big\{ \textbf{(put the boost angular momentum ops here)}$$

(continue talking about the general properties of the operators here) We would now like to see how $U(\Lambda, a)$ acts on our states. Choose our states to be labeled as $|\mathbf{k}, \sigma\rangle$, where $\mathbf{k}$ is our spatial momentum and σ is some set of internal degrees of freedom (which have not yet been specified). Note that we may well use $|k, \sigma\rangle$ instead of $|\mathbf{k}, \sigma\rangle$, but I like $|\mathbf{k}, \sigma\rangle$ more because it shows that these particles are all on-shell (as $k^0 = \sqrt{|\mathbf{k}|^2 + m^2}$ here). First note that this basis diagonalizes translations, i.e.

$$U(0, a)|\mathbf{k}, \sigma\rangle = e^{-i\widehat{\mathcal{P}}a}|\mathbf{k}, \sigma\rangle = e^{-ika}|\mathbf{k}, \sigma\rangle.$$

Indeed, this is one of the reasons we chose our states to be labeled by momentum eigenvalues; they are conserved under translations. Let us now study how Lorentz transformations act on these states. Define an arbitrary ket state as

$$|\mathbf{k}, \sigma\rangle := \mathcal{N}(\mathbf{k})U(L(\mathbf{k}))|\mathbf{k}_R, \sigma\rangle,$$

where $\mathbf{k}_R$ is a given **reference momentum**[2], $L(\mathbf{k})$ is just some Lorentz transformation that takes $\mathbf{k}_R \xrightarrow{L(\mathbf{k})} \mathbf{k}$, and $\mathcal{N}(\mathbf{k})$ a normalization factor to be determined. Consider some $U(\Lambda)$ for arbitrary Λ. We have

$$\begin{aligned} U(\Lambda)|\mathbf{k}, \sigma\rangle &= \mathcal{N}(\mathbf{k})U(\Lambda)U(L(\mathbf{k}))|\mathbf{k}_R, \sigma\rangle \\ &= \mathcal{N}(\mathbf{k})\mathcal{N}(\boldsymbol{\Lambda k})^{-1}U(L(\boldsymbol{\Lambda k}))U(\underbrace{L(\boldsymbol{\Lambda k})^{-1}\Lambda L(\mathbf{k})}_{W})|\mathbf{k}_R, \sigma\rangle. \end{aligned}$$

Look at W. The following diagram commutes:

$$\begin{tikzcd}
\mathbf{k}_R \arrow[r, "L(\mathbf{k})"] \arrow[drr, "W"'] & \mathbf{k} \arrow[r, "\Lambda"] & \boldsymbol{\Lambda k} \arrow[d, "L(\boldsymbol{\Lambda k})^{-1}"] \\
& & \mathbf{k}_R
\end{tikzcd}$$

[2] This will be specific to the kinds of particles you are talking about, e.g. $k_R = (E, 0, 0, E)$ is a reference momentum for the massless little group in $D = 4$.

Thus W fixes $\mathbf{k}_R$. The set of W's forms a group called the **little group** for our specific case (e.g. massless $D = 4$, massive $D = 3$, etc.). Thus $U(W)$ must be a linear combination of $|\mathbf{k}_R, \sigma'\rangle$ vectors, which we can write as

$$U(W)|\mathbf{k}_R, \sigma\rangle = \sum_{\sigma'} D_{\sigma\sigma'}(W)|\mathbf{k}_R, \sigma'\rangle.$$

Notice that $D(W_1W_2) = D(W_2)D(W_1)$, as we can see from the definition. We now determine the normalization from orthogonality. Consider (**finish**) Putting this all together, we have that

$$\boxed{U(\Lambda)|\mathbf{k}, \sigma\rangle = \sqrt{\frac{\omega_{\mathbf{\Lambda} \boldsymbol{k}}}{\omega_{\mathbf{k}}}} \sum_{\sigma\sigma'} D_{\sigma\sigma'}(W(\mathbf{k}))|\mathbf{\Lambda k}, \sigma'\rangle.}$$

(**why does this hold for multiparticle states? I think because they are diff vectors, so doesn't matter; can use raising and lowering ops i think**)

$$\boxed{U(\Lambda, a)|\mathbf{k}_1, \sigma_1; \mathbf{k}_2, \sigma_2; \ldots; \mathbf{k}_n, \sigma_n\rangle = \prod_{i=1}^{n} e^{-ik_i a} \sqrt{\frac{\omega_{\mathbf{\Lambda k}_i}}{\omega_{\mathbf{k}_i}}} \sum_{\sigma_i\sigma_i'} D_{\sigma_i\sigma_i'}(W(\mathbf{k}_i))|\mathbf{\Lambda k}_1, \sigma_1'; \mathbf{\Lambda k}_2, \sigma_2'; \ldots, \mathbf{\Lambda k}_n, \sigma_n'\rangle}$$

This will be useful when we talk about scattering later.

We now take a closer look at the D matrices. (**finish**)

2.3 Lagrangian QM

Now we'll get into Lagrangian QM, which naturally introduces the path integral and accompanying ideas (regularization, renormalization, etc.).

Idea 2.2 (The path integral)

Let $\langle q_f|$ and $|q_i\rangle$ respectively be final and initial position eigenstates. Call the time-translation operator for a time T, $U(T)$. (**finish**)

(**add lagrangian qm; reg, renorm, path integral derivation; add exercises from yin and weinberg here too**)

3 Classical Field Theory

4 Spin-0 QFTs

5 Scattering

We now get into scattering theory. This is probably the most important part of QFT, so we'll go relatively deep into it.

5.1 The basics

Composite particles decay, and fundamental particles (**finish**)

5.2 Symmetries of the S-matrix

5.3 The LSZ reduction

Remark 5.1 (On our derivation). For the LSZ reduction, we follow Haag and Ruelle by way of Yin. You may see his lecture notes on this from 253a linked on the first page of these notes.

We will purposefully state many approximate results. One can get true equalities here, but the derivation is much more obtuse, so we will stick to our wave packets. Also, everything is basically the same anyways, so I don't care.

The goal of the LSZ reduction is to write our in and out states in terms of our field operators, thus giving us a connection between the S-matrix and perturbation theory. A priori, there is no connection here to speak about—indeed, the LSZ reduction is **the most important result in all of quantum field theory**.

We begin by defining a "smeared field operator" via

$$\widehat{\phi}_f := \int d^D x\, f(x)\widehat{\phi}(x).$$

We assume our smearing function's Fourier transform is supported near the mass shell:

$$\tilde{f}(k) \neq 0 \xleftrightarrow{\approx} k^2 + m^2 = 0.$$

How do we talk about "moving fields"? This is something we would like, as our in and out states are assumed to be asymptotically free; you can't get to an asympototic if you can't move anywhere. Consider the following transformation on f:

$$f(k) \mapsto f^{(T)}(k) \text{ by } f^{(T)}(k) = e^{i(k^0-\omega_{\mathbf{k}})T} f(k).$$

How does this transformation act on our field operators? **(add wave equation part here)** Furthermore, the support of $\tilde{f}$ kills the multi-particle contribution of $\widehat{\phi}_f|\Omega\rangle$. Recall

$$\widehat{\phi}(x)|\Omega\rangle = \int đ^{D-1}\mathbf{k}\, e^{-ikx}\, \mathcal{Z}^{\text{eff}}_{\mathbf{k}}|\mathbf{k}\rangle + \int_{\text{m.p.}} d\alpha\, e^{-ip_\alpha x}\, \mathcal{Z}^{\text{eff}}_{\alpha}|\alpha\rangle$$

Applying the definition of $\widehat{\phi}_f$ and taking the inverse Fourier transform of f gives

$$\begin{aligned}\widehat{\phi}_f|\Omega\rangle &= \int d^D x \int đ^{D-1}\mathbf{k}\, e^{-ikx}\, \mathcal{Z}^{\text{eff}}_{\mathbf{k}}|\mathbf{k}\rangle + \int d^D x \int_{\text{m.p.}} d\alpha\, e^{-ip_\alpha x}\, \mathcal{Z}^{\text{eff}}_{\alpha}|\alpha\rangle \\ &\approx \int đ^{D-1}\mathbf{k}\, \tilde{f}(\mathbf{k},\omega_{\mathbf{k}})\, \mathcal{Z}^{\text{eff}}_{\mathbf{k}}|\mathbf{k}\rangle + 0\end{aligned}$$

(explain why T transformation doesn't do anything) Thus,

$$\boxed{\widehat{\phi}_f|\Omega\rangle \approx \widehat{\phi}_{f^{(T)}}|\Omega\rangle.}$$

The intuition behind this is that "there are a lot more operators than states". Thus we may define our states as

$$|\phi_{f_1},\ldots,\phi_{f_n}\rangle := \widehat{\phi}_{f_n}\cdots\widehat{\phi}_{f_1}|\Omega\rangle.$$

Thus, by taking inner products of our states, we may get the S-matrix in terms of field operators acting on the vaccuum. We have **(add T variables on all of these guys below)**

5.4 Analyticity properties of the S-matrix

(my goat xi yin)

(unitarity bounds too?)

6 Spin-1/2 QFTs

(talk about fermions here)

7 Spin-1 QFTs

(photons!)

8 Spin-3/2 QFTs

(susy. . .)

9 Spin-2 QFTs

(gravity!)

(maybe I should find a better organization scheme for this)

10 Useful formulae

10.1 Integrals and special functions

(schwinger parameters, feynman parameters, equivalence of the two)

Schwinger Parameters

We have that

$$\int_0^\infty ds\, s^{-a-1} e^{-bs-c/s} = 2\left(\frac{b}{c}\right)^{1/2} K_a(2\sqrt{bc}).$$

10.2 Dirac matrices