Lecture Notes on

QFT

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Resources used:

- Xi Yin's 253ab courses.
- Steven Weinberg's two volumes.
- Daniel Harlow's notes.
- Sidney Coleman's QFT.
- Peskin and Schroeder.
- Schwartz.
- Srednicki.
- *FIELDS* by Warren Siegel.

We mostly follow Weinberg's notation, except where his notation sucks.

Contents

1 Overview **3**

2 Relativistic Quantum Mechanics **3**
2.1 Symmetries . . . 4
2.2 Poincaré symmetry and the little group . . . 4
2.3 Lagrangian QM . . . 6

3 Classical Field Theory **6**

4 Spin-0 QFTs **6**

5 Scattering **6**
5.1 The basics . . . 6
5.2 Symmetries of the S-matrix . . . 6
5.3 The LSZ reduction . . . 6
5.4 Analyticity properties of the S-matrix . . . 7

6 Spin-1/2 QFTs **7**

7 Spin-1 QFTs **7**
7.1 Basic constructions . . . 8

8 Spin-3/2 QFTs **8**

9 Spin-2 QFTs **8**

10 Useful formulae **8**
10.1 Integrals and special functions . . . 8
10.2 Dirac matrices . . . 8

1 Overview

(overview things here; talk about how single-particle rel qm is inconsistent; talk about how particles are fundamental, and not fields (not so sure about this anymore))

Which is more fundamental, particles or fields? Most physicists are likely to say fields—they are *wrong*: particles are fundamental. To see this, just ask an experimentalist what they measure. They measure *particles*! In the calorimeter at the LHC, *particles* hit the walls of the collider, blazing paths through the detector that we can reconstruct as paths these particles take. The energy and momentum of these particles, combined with the ingenuity of experimentalists, tells you exactly what kind of particle this was. There is no mention of fields anywhere in this picture. **(can steelman this, but should also fight back)**

(think about this more)

An "axiom" of QFT is that it has local, microcausal field operators. These represent local "measurement" on our system. One can think of this as taking inspiration from electromagnetism, since that's our only example of a truly local observable (e.g. the electric field; you measure it at a *point*. This is obvious actually, as this is what *must* occur mathematically, i.e. $\phi\colon M \to \mathbb{R}$, so there is no sense of asking its value at anywhere but a point you "can reach"). Since these are, by definition, observables, they also must be Hermitian. Since things are local, they must obey Poincaré symmetry. Enumerating these conditions for the scalar field gives

1. **Poincaré symmetry**: under a Poincaré transformation $U(\Lambda, a)$, our fields must transform as
$$U(\Lambda, a)\hat{\phi}(x)U(\Lambda, a)^{-1} = \hat{\phi}(\Lambda x + a).$$
2. **Microcausality**: our fields must be local and causal, e.g. spacelike separated measurements can't effect each other. This corresponds to
$$[\phi(x), \phi(y)] = 0 \qquad\qquad (x - y)^2 > 0.$$
3. **Unitarity**: as stated, the fields must be Hermitian, $\hat{\phi} = \hat{\phi}^\dagger$.

2 Relativistic Quantum Mechanics

We start by recalling the axioms of quantum mechanics[1]:

Idea 2.1 (Axioms of quantum mechanics)

The axioms of quantum mechanics are:

- Quantum mechanical **states** are represented by **rays** in a **Hilbert space**. We will denote Hilbert spaces by $\mathcal{H}$.
- **Observables** are represented by self-adjoint operators on our Hilbert space. Observed quantities are the eigenvalues of vectors in $\mathcal{H}$.

[1] Of course, there are different axioms than this that give identical results. "Axioms" is really a misnomer; they're just assumptions that let us build the rest of the theory. There is nothing intrinsically fundamental about this set of axioms as opposed to other ones.

- Let $\mathcal{R}$ be a ray. If $\mathcal{R}_1, \ldots, \mathcal{R}_n$ are a set of mutually orthogonal rays, the probabilty of observing $\mathcal{R}$ in $\mathcal{R}_i$ is

$$\mathbb{P}(\mathcal{R} \to \mathcal{R}_i) = |\langle \psi | \phi_i \rangle|^2,$$

where $\psi \in \mathcal{R}$ and $\phi_i \in \mathcal{R}_i$.

Remarks:

- Recall that rays are equivalence classes of kets up to phase and normalization. Since this is an equivalence class, WLOG we can choose our representative to be normalized.
- Self-adjoint or Hermitian? (**comment on this**)
- In our third axiom, we make reference to experiment. How do we physically represent an experiment on our Hilbert space? (**good question**)

The most important kinds of operators are operators that realize *symmetry transformations* on our Hilbert space.

(**finish**)

2.1 Symmetries

(**add the normal ones, and then ps in the next section**)

(**for whatever reason**) , this continuity may also be found in the *representation* of the symmetry.

2.2 Poincaré symmetry and the little group

Since we are doing relativistic quantum mechanics, let's see how the relativistic symmetry group acts on our Hilbert space. The Poincaré group in mostly-plus is given by

$$\mathcal{P} = \mathbb{R}^D \rtimes SO(D-1,1).$$

Consider some Poincaré transformation $(\Lambda^\mu{}_\nu, a^\mu)$. By Wigner's theorem, this is represented as a linear unitary operator on our Hilbert space; the group property is

$$U(\Lambda, a)U(\overline{\Lambda}, \overline{a}) = U(\Lambda\overline{\Lambda}, \Lambda\overline{a} + a).$$

Since the Poincaré group is a Lie group, things are continuous. Taylor-expanding $U(\Lambda, a)$ gives

$$U(\Lambda, a) \simeq \delta^\mu_\nu - ia_\mu \widehat{\mathcal{P}}^\mu + \frac{i}{2}\omega_{\mu\nu}\widehat{\mathcal{J}}^{\mu\nu}.$$

The generators of $U(\Lambda, a)$ are very special: they are the four-momentum operator and the angular momentum/boost operator. Looking at $\widehat{\mathcal{J}}^{\mu\nu}$ gives

$$\widehat{\mathcal{J}}^{\mu\nu} = \Big\{ \textbf{(put the boost angular momentum ops here)}$$

(**continue talking about the general properties of the operators here**) We would now like to see how $U(\Lambda, a)$ acts on our states. Choose our states to be labeled as $|\mathbf{k}, \sigma\rangle$, where $\mathbf{k}$ is our spatial momentum and σ is some set of internal degrees of freedom (which have not yet been specified).

Note that we may well use $|k,\sigma\rangle$ instead of $|\mathbf{k},\sigma\rangle$, but I like $|\mathbf{k},\sigma\rangle$ more because it shows that these particles are all on-shell (as $k^0 = \sqrt{|\mathbf{k}|^2+m^2}$ here). First note that this basis diagonalizes translations, i.e.

$$U(0,a)|\mathbf{k},\sigma\rangle = e^{-i\widehat{\mathcal{P}}a}|\mathbf{k},\sigma\rangle = e^{-ika}|\mathbf{k},\sigma\rangle.$$

Indeed, this is one of the reasons we chose our states to be labeled by momentum eigenvalues; they are conserved under translations. Let us now study how Lorentz transformations act on these states. Define an arbitrary ket state as

$$|\mathbf{k},\sigma\rangle := \mathcal{N}(\mathbf{k})U(L(\mathbf{k}))|\mathbf{k}_R,\sigma\rangle,$$

where $\mathbf{k}_R$ is a given **reference momentum**[2], $L(\mathbf{k})$ is just some Lorentz transformation that takes $\mathbf{k}_R \xrightarrow{L(\mathbf{k})} \mathbf{k}$, and $\mathcal{N}(\mathbf{k})$ a normalization factor to be determined. Consider some $U(\Lambda)$ for arbitrary Λ. We have

$$\begin{aligned} U(\Lambda)|\mathbf{k},\sigma\rangle &= \mathcal{N}(\mathbf{k})U(\Lambda)U(L(\mathbf{k}))|\mathbf{k}_R,\sigma\rangle \\ &= \mathcal{N}(\mathbf{k})\mathcal{N}(\boldsymbol{\Lambda k})^{-1}U(L(\boldsymbol{\Lambda k}))U(\underbrace{L(\boldsymbol{\Lambda k})^{-1}\Lambda L(\mathbf{k})}_{W})|\mathbf{k}_R,\sigma\rangle. \end{aligned}$$

Look at W. The following diagram commutes:

$$\begin{tikzcd} \mathbf{k}_R \arrow[r, "L(\mathbf{k})"] \arrow[drr, "W"'] & \mathbf{k} \arrow[r, "\Lambda"] & \boldsymbol{\Lambda k} \arrow[d, "L(\boldsymbol{\Lambda k})^{-1}"] \\ & & \mathbf{k}_R \end{tikzcd}$$

Thus, W fixes $\mathbf{k}_R$. The set of W's forms a group called the **little group** for our specific case (e.g. massless $D=4$, massive $D=3$, etc.). So, $U(W)$ can't change $\mathbf{k}_R$, but it can change our σ labels; thus, the σ labels tell you how the particle transforms under the little group. The operator $U(W)$ generically gives a combination of particles with the same $\mathbf{k}_R$ but different σ labels, which we may write as

$$U(W)|\mathbf{k}_R,\sigma\rangle = \sum_{\sigma'} D_{\sigma\sigma'}(W)|\mathbf{k}_R,\sigma'\rangle.$$

Notice that $D(W_1W_2) = D(W_2)D(W_1)$, as we can see from the definition. We now determine the normalization from orthogonality. Consider **(finish)** Putting this all together, we have that

$$\boxed{U(\Lambda)|\mathbf{k},\sigma\rangle = \sqrt{\frac{\omega_{\boldsymbol{\Lambda}\mathbf{k}}}{\omega_{\mathbf{k}}}}\sum_{\sigma\sigma'} D_{\sigma\sigma'}(W(\mathbf{k}))|\boldsymbol{\Lambda}\mathbf{k},\sigma'\rangle.}$$

(why does this hold for multiparticle states? I think because they are diff vectors, so doesn't matter; can use raising and lowering ops i think)

$$\boxed{U(\Lambda,a)|\mathbf{k}_1,\sigma_1;\mathbf{k}_2,\sigma_2;\dots;\mathbf{k}_n,\sigma_n\rangle = \prod_{i=1}^{n} e^{-ik_ia}\sqrt{\frac{\omega_{\boldsymbol{\Lambda}\mathbf{k}_i}}{\omega_{\mathbf{k}_i}}}\sum_{\sigma_i\sigma_i'} D_{\sigma_i\sigma_i'}(W(\mathbf{k}_i))|\boldsymbol{\Lambda}\mathbf{k}_1,\sigma_1';\boldsymbol{\Lambda}\mathbf{k}_2,\sigma_2';\dots,\boldsymbol{\Lambda}\mathbf{k}_n,\sigma_n'\rangle}$$

This will be useful when we talk about scattering later.

We now take a closer look at the D matrices. **(finish)**

[2] This will be specific to the kinds of particles you are talking about, e.g. $k_R = (E,0,0,E)$ is a reference momentum for the massless little group in $D=4$.

2.3 Lagrangian QM

Now we'll get into Lagrangian QM, which naturally introduces the path integral and accompanying ideas (regularization, renormalization, etc.).

Idea 2.2 (The path integral)

Let $\langle q_f|$ and $|q_i\rangle$ respectively be final and initial position eigenstates. Call the time-translation operator for a time T, $U(T)$. (**finish**)

(**add lagrangian qm; reg, renorm, path integral derivation; add exercises from yin and weinberg here too**)

3 Classical Field Theory

4 Spin-0 QFTs

5 Scattering

We now get into scattering theory. This is probably the most important part of QFT, so we'll go relatively deep into it.

5.1 The basics

Composite particles decay, and fundamental particles (**finish**)

5.2 Symmetries of the S-matrix

5.3 The LSZ reduction

Remark 5.1 (On our derivation). For the LSZ reduction, we follow Haag and Ruelle by way of Yin. You may see his lecture notes on this from 253a linked on the first page of these notes.

We will purposefully state many approximate results. One can get true equalities here, but the derivation is much more obtuse, so we will stick to our wave packets. Also, everything is basically the same anyways, so I don't care.

The goal of the LSZ reduction is to write our in and out states in terms of our field operators, thus giving us a connection between the S-matrix and perturbation theory. A priori, there is no connection here to speak about—indeed, the LSZ reduction is **the most important result in all of quantum field theory**.

We begin by defining a "smeared field operator" via

$$\widehat{\phi}_f := \int d^D x\, f(x)\widehat{\phi}(x).$$

We assume our smearing function's Fourier transform is supported near the mass shell:

$$\tilde{f}(k) \neq 0 \overset{\approx}{\longleftrightarrow} k^2 + m^2 = 0.$$

How do we talk about "moving fields"? This is something we would like, as our in and out states are assumed to be asymptotically free; you can't get to an asympototic if you can't move anywhere. Consider the following transformation on f:

$$f(k) \mapsto f^{(T)}(k) \text{ by } f^{(T)}(k) = e^{i(k^0 - \omega_{\mathbf{k}})T} f(k).$$

How does this transformation act on our field operators? **(add wave equation part here)** Furthermore, the support of $\tilde{f}$ kills the multi-particle contribution of $\widehat{\phi}_f|\Omega\rangle$. Recall

$$\widehat{\phi}(x)|\Omega\rangle = \int đ^{D-1}\mathbf{k}\, e^{-ikx}\, \mathcal{Z}^{\text{eff}}_{\mathbf{k}}|\mathbf{k}\rangle + \int_{\text{m.p.}} d\alpha\, e^{-ip_\alpha x}\, \mathcal{Z}^{\text{eff}}_{\alpha}|\alpha\rangle$$

Applying the definition of $\widehat{\phi}_f$ and taking the inverse Fourier transform of f gives

$$\begin{aligned}\widehat{\phi}_f|\Omega\rangle &= \int d^D x \int đ^{D-1}\mathbf{k}\, e^{-ikx}\, \mathcal{Z}^{\text{eff}}_{\mathbf{k}}|\mathbf{k}\rangle + \int d^D x \int_{\text{m.p.}} d\alpha\, e^{-ip_\alpha x}\, \mathcal{Z}^{\text{eff}}_{\alpha}|\alpha\rangle \\ &\approx \int đ^{D-1}\mathbf{k}\, \tilde{f}(\mathbf{k}, \omega_{\mathbf{k}})\, \mathcal{Z}^{\text{eff}}_{\mathbf{k}}|\mathbf{k}\rangle + 0\end{aligned}$$

(explain why T transformation doesn't do anything) Thus,

$$\boxed{\widehat{\phi}_f|\Omega\rangle \approx \widehat{\phi}_{f^{(T)}}|\Omega\rangle.}$$

The intuition behind this is that "there are a lot more operators than states". Thus we may define our states as

$$|\phi_{f_1}, \ldots, \phi_{f_n}\rangle \coloneqq \widehat{\phi}_{f_n} \cdots \widehat{\phi}_{f_1}|\Omega\rangle.$$

Thus, by taking inner products of our states, we may get the S-matrix in terms of field operators acting on the vaccuum. We have **(add T variables on all of these guys below)**

5.4 Analyticity properties of the S-matrix

(my goat xi yin)

(unitarity bounds too?)

6 Spin-1/2 QFTs

(talk about fermions here)

7 Spin-1 QFTs

(motivation)

7.1 Basic constructions

There are two cases for spin-1 particles: massless or massive. They correspond to different representations of the little group, respectively being $SO(D-1)$ and $SO(D-2)$. In $D=4$, this means that a massive spin-1 particle has *three* spin DOF, while a massless spin-1 particle has *two* spin DOF. This discontinuous jump in the number of DOF between massive and massless is the root cause behind the difficulties of dealing with massless spin-1 particles.

(**general Lorentz transformation properties of spin-1 particles go here;**)

(

- **particles; massive vs massless**
- **little group representations**
- **particles -¿ fields; massless vs massive**
- **Path integral construction; gauge fixing**

)

8 Spin-3/2 QFTs

(**susy. . .**)

9 Spin-2 QFTs

(**gravity!**)

(**maybe I should find a better organization scheme for this**)

10 Useful formulae

10.1 Integrals and special functions

(**schwinger parameters, feynman parameters, equivalence of the two**)

Schwinger Parameters

We have that

$$\int_0^\infty ds\, s^{-a-1} e^{-bs-c/s} = 2\left(\frac{b}{c}\right)^{1/2} K_a(2\sqrt{bc}).$$

10.2 Dirac matrices

We reproduce some of the results covered in the notes on (**group theory**) here. We may form the Dirac matrices in any dimension. The most useful is $D=4$. A choice of Dirac matrices in $D=3+1$ is

$$\gamma^0 = -i\begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \qquad \gamma^a = -i\begin{pmatrix} 0 & \sigma^a \\ -\sigma^a & 0 \end{pmatrix}.$$

You can construct the Dirac matrices in any dimension by simply finding matrices that obey the Clifford algebra, $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. Practically, this is done by bootstrapping your way up from a given dimension via the following iterative process:

- **Even $\rightarrow$ odd**: to go from even to odd dimensions, add the chirality matrix to your basis, $\gamma_c = \alpha\gamma^1 \cdots \gamma^n$. Fix α by demanding $(\gamma_c)^2 = 1$. This gives two inequivalent irreps, $\{\gamma^{(n)}, \gamma_c\}$ and $\{\gamma^{(n)}, -\gamma_c\}$.

- **Odd $\rightarrow$ even**: to go from odd to even dimensions, tensor the existing representation by one Pauli matrix, and add the identity times another Pauli matrix to get your representation. Concretely, if your odd rep is $\gamma^{(2k+1)}$, then your even rep is $\{\gamma^{(2k+1)} \otimes \sigma^a, I_{2k} \otimes \sigma^b\}$.