Lecture Notes on

The Four Mechanics

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These are notes on basic physics, covering classical mechanics, classical field theory, statistical mechanics, and quantum mechanics. Resources used:

Remark 0.1 (Notes on notation)**.** A list of notation I will use:

- $\not{d} = d/2\pi$. $\not{\delta}^D(x) = (2\pi)^D \delta^D(x)$.
- My standard "volume" will be Ω. I'll denote its boundary using the boundary operator ∂, so my standard surface element is $\partial\Omega$.
- I use the mostly-plus metric, so $\eta_{\mu\nu} = \text{diag}(-1, 1, \cdots, 1)$. I'll also work in D space-time dimensions frequently, so $(D-1)$ spatial dimensions plus one time dimension.
- We work in natural units almost everywhere, with the notable exception of non-relativistic electromagnetism.

Contents

1 Classical Mechanics

1.1 Newtonian mechanics

All of Newtonian mechanics is predicated upon the fact that we live in Galilean space. Gaililean space is the triple $G = (\mathbb{A}^4, t, g)$, where $\mathbb{A}$ is an affine space, t is a time function, and g is the Euclidean metric. t and g do exactly what you think they do. Note that g is restricted to act on the space components of our vectors, which we take to be four-vectors at the outset.

It's an experimental observation that symmetry exists. A symmetry is defined as an action on a mechanical system that doesn't change any observables of that system. The set of transformations that we can perform on Galilean space to give us another Galilean space forms a group, which we call the Galilean group,

$$\mathcal{G} = \text{span}\{\mathbf{a}, \mathbf{V}, R_{ab}, \tau\}.$$

It has ten generators, as you can see. Experimentally, we can observe that a system being symmetric under certain generators of the Galilean group implies that that system has a conserved quantity associated with that generator. While we will have a systematic way to define these later (Noether's theorem), right now we can just look at experiments and come up with the following definitions (particle definitions, not extended bodies).

Symmetry	Conserved quantity
Translational symmetry	Linear momentum: $\mathbf{p} \coloneqq m\mathbf{v}$
Rotational symmetry	Angular momentum: $\boldsymbol{\ell} \coloneqq \mathbf{r} \times \mathbf{p}$
Time-translation symmetry	Energy: $E \coloneqq T + V$

Here, $T \coloneqq (1/2)m|\dot{\mathbf{r}}|^2$, and V is the potential energy, which is calculated on a system by system basis. Note that it makes intuitive sense that the only scalar conserved quantity is associated with time, which is a scalar, as opposed to translations or rotations, which are inherently associated with vectors.

We may experimentally observe that symmetries are broken. To do one of these very difficult experiments in the comfort of your own home, take a pen and drop it onto the floor. It does not have translational invariance; this symmetry was *broken* by *something*. We now define those somethings:

Symmetry	Conserved quantity	Symmetry breaker
Translational symmetry	Linear momentum: $\mathbf{p} \coloneqq m\mathbf{v}$	Force: $\mathbf{F} \coloneqq d\mathbf{p}/dt$
Rotational symmetry	Angular momentum: $\boldsymbol{\ell} \coloneqq \mathbf{r} \times \mathbf{p}$	Torque: $\boldsymbol{\tau} = d\boldsymbol{\ell}/dt$
Time-translation symmetry	Energy: $E \coloneqq T + V$	Power: $P = dE/dt$

Again, we see energy is the odd one out here; this is because the practical way of changing the energy of a system is by doing *work* on it, which is defined as the one-form $đW = -F\,dx$. Yet we define the symmetry breaker as power. This suggest a fundamental fault in the definitions we use, which we get into now.

Remark 1.1 (Why is energy so weird?)**.** Why does energy not follow the nice schema we have laid out for our conserved quantities? After all, the defintions of force and torque are quite

intuitive, and then out flops power. What? I forgot about power after high school; why is that there? (**maybe something to do with entropy**)

1.2 Configuration space

We would like to catalog where things are; a **configuration space** allows us to do this.

Definition 1.1 (Configuration space)

A **configuration space** is the space of all possible positions of our system.

Example 1.1 (Examples of configuration spaces)

Some examples:

- The standard configuration space: $\mathbb{R}^3$. This corresponds to one particle in three dimensions, or three particles in one dimension, or what have you.
- An important example: $\mathbb{R}^6$. This is the configuration space corresponding to two particles moving in three dimensions. Note that spaces like these show that the wave function isn't a field like most people think it is.
- The circle: S^1. This corresponds to a particle that is constrained to move on, you guessed it, a circle!
- The circle^2: $S^1 \times S^1$. Physically, this corresponds to *two* things constrained to move on circles. An example of this is the double pendulum.

Remark 1.2 (The intuition behind this)**.** Intuitively, the configuration space of your system partitions a dimension for each degree of freedom. Look at the previous examples for confirmation of this heuristic ((**fact?**)).

Ok, so configuration space seems like a natural place to try and define things on (it tells us where our particles are, and predicting the positions of paricles is the goal of classical mechanics). We can define some sort of tensor function on this space to tell us the dynamics of our system.

The most natural tensors are going to be scalars and vectors; this is just because we have those nice conserved quantities from before, and they are all scalars and vectors. Let's study the simplest case first to get some constraints on this function.

Remark 1.3. The simplest case is the most symmetric one with the fewest components: so a scalar function with Galilean symmetry.

Idea 1.1 (Scalar function on configuration space)

Call this scalar function L. Denote a point in configuration space by $\mathbf{q}$, and its time derivative by $\dot{\mathbf{q}}$. Then the most general L is some function of powers of these quantities, $L = L(\mathbf{q}, \dot{\mathbf{q}}, |\mathbf{q}|^2, |\dot{\mathbf{q}}|^2, \ldots, t)$. Note that we implemented time-dependence somewhat ad hoc; this is

because we want to talk about dynamics, which implicitly refers to time.

We now implement our symmetry considerations onto L. Note that by translational and rotational invariance, L can only be a function of $|\mathbf{q}-\mathbf{q}'|$, and furthermore, only a function of $|\mathbf{q}|^{2n}$ and $|\dot{\mathbf{q}}|^{2n}$. By time-translation invariance, $L \neq L(t)$. The most natural rotationally invariant objects we can make are $\mathbf{q}\cdot\mathbf{q} = |\mathbf{q}|^2$ and $\dot{\mathbf{q}}\cdot\dot{\mathbf{q}} = |\dot{\mathbf{q}}|^2$.

So the form of L is

$$L(|\mathbf{q}|^2, |\dot{\mathbf{q}}|^2, \ldots, |\mathbf{q}|^{2n}, |\dot{\mathbf{q}}|^{2n}, |\mathbf{q}-\mathbf{q}'|^n) = \frac{1}{2}A_{mn}\dot{q}_m\dot{q}_n - \frac{1}{2}B_{mn}q_mq_n + (\text{other stuff}),$$

where A_{mn} and B_{mn} are matrices.

So we've constrained L as much as we can **(I think)** . We are now going to invoke the action principle to constrain L more.

Idea 1.2 (The action principle)

It's an experimental observation that the following quantity, when minimized, produces the correct equations of motion for dynamical systems:

$$S[\mathbf{q}] = \int dt\, L.$$

This quantity is called the **action**.

Let's continue to constrain L. Add a constant shift in position or velocity to L. We want this to contribute a total derivative to S, as this should be a symmetry of L. Doing this, we can see that the only way for this to be a symmetry is if **(BLAH)**

1.3 Examples of Lagrangians

Most important example: oscillations

The ideas behind oscillations are *linearity* and *time-translation invariance*. As we'll see in this section, these considerations give rise to the idea of **normal modes**, which are just the choice of coordinates that makes these ideas most manifest.

Integrable systems

Let's define the term I used in the title of this section.

Definition 1.2 (Integrability)

A system is said to be **integrable** if it has as many first integrals as it has degrees of freedom.

Example 1.2 (Examples of integrable systems)

Here are some common examples of integrable systems.

- **(orbits)**

- (**any system with one symmetry and one DOF**)
- (**something else idk**)

1.4 Phase space

(**chungus?**)

2 Classical Electrodynamics

2.1 Mathematical identities

(basically input the front cover of Jackson here)

2.2 Generalities

The Lagrangian for electromagnetism is

$$\mathcal{L} = \frac{1}{2}(|\mathbf{E}|^2 + |\mathbf{B}|^2) + \rho\Phi - \mathbf{j}\cdot\mathbf{A}.$$

This may be recast into relativistic notation in the form of

$$\mathcal{L} = \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\text{fields}} - \underbrace{A_\mu J^\mu}_{\text{interaction}}.$$

Note that you can also get the Lorentz force law by adding the kinetic term $(1/2)m|\dot{\mathbf{r}}|^2$ to $\mathcal{L}$. (For mostly minus, the interaction term is $-A_\mu J^\mu$.) This tells us how our fields propagate freely, and how they interact with charged matter (charge and current densities). Note the definition of the Faraday tensor, $F^{\mu\nu} := \partial^\mu A^\nu - \partial^\nu A^\mu$. Using the Euler-Lagrange equations on $\mathcal{L}$ give Maxwell's equations:

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) - \frac{\partial\mathcal{L}}{\partial A_\nu} = 0$$

Note that the gauge-redundancy of S ($A_\mu \to A_\mu + \partial_\mu\chi$) gives $\partial_\mu J^\mu = 0$, which is continuity.

Derivation 2.1 (Maxwell's equations)

Writing out $\mathcal{L}$ more explicitly gives

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - A_\mu J^\mu.$$

Considering each term in the ELE gives

$$\frac{\partial\mathcal{L}}{\partial(\partial_\alpha A_\beta)} = -\frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\delta^\alpha_\mu\delta^\beta_\nu - \delta^\alpha_\nu\delta^\beta_\mu) = -(\partial^\alpha A^\beta - \partial^\beta A^\alpha) = -F^{\alpha\beta},$$
$$\frac{\partial\mathcal{L}}{\partial A_\beta} = -\delta^\beta_\mu J^\mu = -J^\beta.$$

So the ELE are

$$\boxed{\partial_\mu F^{\mu\nu} = J^\nu.}$$

Writing this equation out in components gives the inhomogeneous ME,

$$\nabla\cdot\mathbf{E} = \rho \qquad\qquad \nabla\times\mathbf{B} - \partial_t\mathbf{E} = \mathbf{J}.$$

Inherent in the construction of the four-potential is the constraint (due to skew-symmetry and Clairaut's theorem; as we'll see in a second, really just because $d^2 = 0$),

$$\partial_{[\mu} F_{\nu\rho]} = 0.$$

Writing out these components gives the homogeneous ME,

$$\nabla \times \mathbf{E} + \partial_t \mathbf{B} = \mathbf{0}. \qquad \nabla \cdot \mathbf{B} = 0.$$

Writing out all of the ME at once shows you the lack of symmetry between $\mathbf{E}$ and $\mathbf{B}$ (namely, the lack of magnetic monopoles and currents).

	Inhomogeneous	Homogeneous
Constraint	$\nabla \cdot \mathbf{E} = \rho$	$\nabla \cdot \mathbf{B} = 0$
Dynamics	$\nabla \times \mathbf{B} - \partial_t \mathbf{E} = \mathbf{J}$	$\nabla \times \mathbf{E} + \partial_t \mathbf{B} = \mathbf{0}$.

A short remark is that equations of dynamics are *also* constraints: by definition, they constrain how a system can evolve in time.

Let's cover this again from the language of forms. Define the one-form

$$A \coloneqq -\Phi\, dt + A_i\, dx^i.$$

Then $F = dA$ gives

$$F = \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu.$$

We also define the three-form J by

$$J = J^\mu \epsilon_{\mu\nu\rho\sigma}\, dx^\nu \wedge dx^\rho \wedge dx^\sigma.$$

The Lagrangian for this system is then

$$\mathcal{L} = -\frac{1}{2} F \wedge \star F - A \wedge \star J.$$

Demanding $\delta S = 0$ to first-order and using the fact that d and δ commute gives the inhomogeneous ME,

$$\boxed{d \star F = \star J} \iff \partial_\mu F^{\mu\nu} = J^\nu.$$

Using the fact that $d^2 = 0$ gives the homogeneous ME,

$$\boxed{dF = 0} \iff \partial_{[\mu} F_{\nu\rho]} = 0.$$

Note that gauge redundancy ($A \to A + d\chi$) again demands that $d \star J = 0$ (continuity).

2.3 Electrostatics

The equations of motion in electrostatics are

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \mathbf{E} = \mathbf{0}$$
$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0} \qquad \mathbf{F} = q\mathbf{E}.$$

Since we're dealing with statics, we should think of these as equations of constraint, not dynamics. The first EOM tells you exactly how $\mathbf{E}$ must be sourced, the second is a constraint on $\mathbf{E}$'s components, and the last tells you how we relate the field picture to the measurable quantity of force. Poisson's equation just combines the Maxwell's equations into a more useful form.

Remark 2.1 (On $\nabla \times \mathbf{E} = 0 \implies \mathbf{E} = -\nabla\Phi$)**.** We often conclude that the curl-free condition gives a *global* scalar potential Φ such that $\mathbf{E} = -\nabla\Phi$. This isn't quite right. While we may always define a local Φ that satisfies such a condition, a global potential function requires the domain we're working on to be simply connected, so $H^1(M) = 0$, where M is our domain, and H^1 is the first de Rham cohomology (lest things become multivalued). This is intuitively obvious, as the statement $\nabla \times \mathbf{E} = \mathbf{0}$ is really just the statement that $dE = 0$, so if $H^1(M) \neq 0$, then Poincaré's lemma doesn't hold, so E isn't necessarily $-d\Phi$ for some Φ.

Vacuum and conductors

There are two broad divisions to vacuum and conductor electrostatics:

- Potential problems.
- Field properties, conductor properties, and field-conductor interactions.

We first treat potential problems. The only way we solve Poisson's equation in practice is through Green's functions[1].

Idea 2.1 (Green's functions)

Let $\mathcal{L}$ be a linear differential operator, $\psi(x)$ a field (x being a parameter), and $S(x)$ a source term. If we would like to solve

$$(\mathcal{L}\psi)(x) = S(x),$$

then we may consider the following alternative problem:

$$(\mathcal{L}G)(x, x') = \delta(x - x'). \tag{1}$$

We call G a **Green's function**. The reason it's useful is because we may recover ψ by integrating G over our source; instead of solving the (probably intractably difficult) PDE each time for a new source term $S(x)$, we may just input it into our general integral solution.

Claim. $\psi(x) = \int_\Omega dx'\, G(x, x')S(x')$.

[1] See Littlejohn's notes on Green's functions here.

Proof. Apply $\mathcal{L}$ to the RHS.

$$\begin{aligned}(\mathcal{L}\psi)(x) &= \mathcal{L}\int_\Omega dx'\, G(x,x')S(x') \\ &= \int_\Omega dx'\, (\mathcal{L}G)(x,x')S(x') \\ &= \int_\Omega dx'\, \delta(x-x')S(x') \\ &= S(x).\end{aligned}$$

□

In words, a Green's function tells you about the response of a linear system to a delta-function source; heuristically, it tells you how much to weight each delta function source term in considering your solution field ψ.

Some comments on Green's functions, both in generality and in the context of electrostatics:

- Green's functions are a priori not unique. This is because we may add homogeneous solutions to G and still satisfy (1). This is solved by implementing boundary conditions on G. This is to be expected, of course, as we're still solving a PDE, so boundary conditions are necessary to specify unique solutions. Before we specify boundary conditions, solving (1) actually solves for an entire equivalence class of Green's functions for the operator $\mathcal{L}$. More concretely, if Λ obeys $(\mathcal{L}\Lambda)(\mathbf{x},\mathbf{x}') = 0$, then

  $$G(x,x') \sim G(x,x') + \Lambda(x,x'),$$

  where we define equivalence by $(\mathcal{L}G) = (\mathcal{L}G')$, where $G' = G + \Lambda$.

- The way to think about specifying boundary conditions on G is that you're adding "homogeneous solution" to G until the boundary conditions fit. In the bulk, this homogeneous solution does nothing; only when you get to the boundary does it start helping G become unique. The physical interpretation of Λ in electrostatics is some system of charge extraneous to the domain in question; it then follows that $\nabla^2\Lambda = 0$, but clearly Λ would affect the boundary conditions on our domain.

- Note that we are free to choose the boundary conditions on G as long as the boundary conditions on Φ are satisfied. Physically, this is just because we only care about what's happening inside of our domain, so adding homogeneous solution (extraneous charges) doesn't change anything that we care about. We will use this freedom in our choice to make things as easy as possible for ourselves.

- The physical interpretation of boundary conditions on Φ and G in electrostatics is some potential and/or charge distribution on the boundary of our domain.

Applying this to electrostatics gives some nice results. We first quickly mention Green's theorem, which you can get from the divergence theorem by considering $\mathbf{A} = \psi\nabla\Phi$.

$$\int_\Omega d^3\mathbf{x}'\,(\psi\nabla^2\Phi - \Phi\nabla^2\psi) = \int_{\partial\Omega} dS'\,\left(\psi\frac{\partial\Phi}{\partial n'} - \Phi\frac{\partial\psi}{\partial n'}\right).$$

For electrostatics, the Green's function problem is:

$$(\nabla^2 G)(\mathbf{x}, \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}'),$$

Implementing scattering boundary conditions on G (i.e. $G \to 0$ as $|\mathbf{x}| \to \infty$) gives the unique solution of

$$\boxed{G_{FF}(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|}.}$$

This is called the **free-field Green's function**. You can find this via the Fourier transform of G. Integrating this gives the potential for a localized charge distribution:

$$\boxed{\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_\Omega d^3\mathbf{x}' \, \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}.} \tag{2}$$

If we have prescribed boundary conditions on our domain (physical interpretation: some potential or charge configuration on conducting surfaces), we may use Green's theorem to get

$$\Phi_D(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_\Omega d^3\mathbf{x}' \, \rho(\mathbf{x}') G_D(\mathbf{x}, \mathbf{x}') - \frac{1}{4\pi} \oint_{\partial\Omega} dS' \, \Phi(\mathbf{x}') \frac{\partial G_D}{\partial n'},$$
$$\Phi_N(\mathbf{x}) = \langle\Phi\rangle_{\partial\Omega} + \frac{1}{4\pi\epsilon_0} \int_\Omega d^3\mathbf{x}' \, \rho(\mathbf{x}') G_N(\mathbf{x}, \mathbf{x}') + \frac{1}{4\pi} \oint_{\partial\Omega} dS' \, G_N \frac{\partial \Phi}{\partial n'}.$$

Note the inevitable fact that both of these solutions have the form of "free-field" + "boundary terms".

Remark 2.2 (More general boundary conditions). **(check this)** You may wonder how the Green's function would be specified in the case of Robin boundary conditions (linear combinations of Dirichlet and Neumann BC), or other more exotic boundary conditions. Here is the general procedure:

Let $\mathcal{L} = -\nabla^2$, and $\alpha\Phi + \beta\partial_n\Phi = 0$ on $\partial\Omega$. This implies that $\mathcal{L}$ is self-adjoint. We aim to solve $\mathcal{L}\Phi_n = \lambda_n\Phi_n$ with $\alpha\Phi + \beta\partial_n\Phi = 0$ on $\partial\Omega$. Note that we have $\int_\Omega d^3\mathbf{x}' \, \Phi_m\Phi_n = \delta_{nm}$, and $\lambda_n \in \mathbb{R}$. Since Ω is bounded by assumption, and $\mathcal{L}$ is an elliptic differential operator, then $(\mathcal{L} + \mu I)^{-1}$ is compact[a]. The Spectral theorem then tells us that the Φ_n are a complete orthonormal basis of $L^2(\Omega)$. Completeness means, by definition, that

$$\sum_{n=1}^{\infty} \Phi_n(x)\Phi_n(x') = \delta(x - x'),$$

and by definition $(\mathcal{L}G)(x, x') = \delta(x - x')$, so

$$G(x, x') = \sum_{n=1}^{\infty} \frac{\Phi_n(x)\Phi_n(x')}{\lambda_n}.$$

This series is guaranteed to converge (in the Sobolev sense) to a function that's smooth outside of the diagonal $x = x'$ due to the fact that $\mathcal{L}$ is elliptic. Then since G is *literally* the kernel of

$-\nabla^2$, we then get the solution

$$\Phi(\mathbf{x}) = \int_\Omega d^3\mathbf{x}'\, \rho(\mathbf{x}')G(\mathbf{x},\mathbf{x}') + (\text{boundary terms}).$$

[a] See more here. More accurately, the inclusion $H^2(\Omega) \hookrightarrow L^2(\Omega)$ from the Sobolev space H^2 to L^2 is compact.

The above remark tells us about an interesting and useful idea: expanding G in the eigenfunctions of ∇^2[2]. This is actually quite useful, as it allows us to construct an *approximate* solution to the oftentimes intractable expression (2). The eigenfunctions of ∇^2_{sph} and ∇^2_{cyl} are respectively spherical harmonics and Bessel functions. Expanding G_{FF} in each coordinate system gives[3]

$$\begin{aligned}
\frac{1}{|\mathbf{x}-\mathbf{x}'|} &\simeq \frac{1}{r} + \mathbf{x}\cdot\frac{\mathbf{x}'}{r^3} + \frac{1}{2}x_a x_b \frac{3x'_a x'_b - (r')^2\delta_{ab}}{r^5} + \mathcal{O}((r')^3/r^4) \\
\frac{1}{|\mathbf{x}-\mathbf{x}'|} &= \sum_{\ell=0}^{\infty} \frac{4\pi}{2\ell+1}\left(\frac{r_<^\ell}{r_>^{\ell+1}}\right) \sum_{m=-\ell}^{\ell} Y_{\ell m}(\Omega)Y^*_{\ell m}(\Omega') = \sum_{\ell=0}^{\infty}\left(\frac{r_<^\ell}{r_>^{\ell+1}}\right) P_\ell(\cos\gamma) \\
\frac{1}{|\mathbf{x}-\mathbf{x}'|} &= \frac{2}{\pi}\sum_{m=-\infty}^{\infty}\int_0^\infty dq\, e^{im(\phi-\phi')}\cos(q(z-z'))I_m(q\rho_<)K_m(q\rho_>)
\end{aligned}$$

Note that the expansion in spherical harmonics uses the addition theorem in its second equality.

Idea 2.2 (Addition theorem of spherical harmonics)

Place $\mathbf{x}'$ on the z-axis, so $\theta' = 0$. Then we get that

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{\ell=0}^{\infty}\left(\frac{r_<^\ell}{r_>^{\ell+1}}\right) P_\ell(\cos\theta).$$

Note that the LHS is rotationally invariant, so the RHS must be as well. For this to occur, interpret θ as the angle between $\mathbf{x}$ and $\mathbf{x}'$; call it γ. We have that $\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}' = \cos\gamma$, where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$. So we have

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \sum_{\ell=0}^{\infty}\left(\frac{r_<^\ell}{r_>^{\ell+1}}\right) P_\ell(\cos\gamma).$$

This result always holds, not just for the times when $\mathbf{x}'$ is along the z-axis. Comparing this to the normal eigenfunction expansion in spherical harmonics gives the **addition theorem for spherical harmonics**:

$$\boxed{P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell} Y_{\ell m}(\Omega)Y^*_{\ell m}(\Omega').}$$

Now that we've covered Poisson's equation in painful detail, we move to its homogeneous counterpart: **Laplace's equation**.

[2] Or $-\nabla^2$, it doesn't really matter.

[3] For the procedure of deriving the second two, see the first LJ lecture here, or page 257 of Zangwill. This is called the **method of direct integration**.

Idea 2.3 (Laplace's equation)

Laplace's equation is the homogeneous version of Poisson's equation, i.e.

$$\nabla^2 \Phi = 0.$$

There are two main ways to solve this:

- Separation of variables. The standard way of finding eigenfunctions.
- Fourier transform. More standard in field theory at large (QFT), but still quite useful (more useful for finding Green's functions imo).

It should be noted that both of these methods are just ways of finding eigenfunction expansions of Φ. We may separate variables in any coordinate system we feel like. The most common are Cartesian, spherical, and cylindrical. The solutions for each are listed below:

- **Cartesian:** the solutions all go like products of exponentials (complex and real). **(finish)**

Idea 2.4 (Eigenfunction expansion)

(talk about more general eigenfunction expansion here; delete the previous section in the laplace's equation idea)

Dielectric media

Remark 2.3 (The Wilsonian view of dielectrics)**.** This section uses a lot of ideas common in the Wilsonian view of QFT, like effective field theories, coarse-graining, and organizing by scale. It's quite beautiful.

We start by defining what a dielectric is physically[4]. To do this, we first have to consider what a conductor is.

Idea 2.5 (Dielectrics and conductors)

Let a charge distribution have length scale L. Consider the length scale at which the *intrinsic* charges of a body can move. "Intrinsic" just means the charge the body has by virtue of existing, so its electrons and protons.

For **conductors**, the length scale at which charges can move is L. This aligns with our intuition for conductors, as we know that charges can rearrange on them however they like.

For **dielectrics**, the length scale at which charges can move is less than L. That's the distinction between conductors and dielectrics: the length scale of the movement of their

[4]The deep considerations of this section are due to Paul Goldbart's incredible lecturing on the subject of dielectrics. I have expanded on what he said, but the core ideas of an "effective field theory" approach to dielectrics is all him.

charges.

2.4 Magnetostatics

Magnetostatics is basically just electrostatics upon the following identifications:

$$\begin{aligned} \text{Dot products} &\longrightarrow \text{Cross products} \\ \text{Charge densities} &\longrightarrow \text{Current densities} \\ \text{Scalar potential} &\longrightarrow \text{Vector potential} \end{aligned}$$

Great. The core equations of magnetostatics are

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$
$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \qquad \mathbf{F} = q\mathbf{v} \times \mathbf{B}.$$

You can see that the source shows up in the "curl sector" of the Maxwell's equation for the mangetic field instead of the "divergence sector" like it did for the electric field. This fundamental asymmetry produces everything different in magnetostatics, more or less.

The first ME gives a constraint on $\mathbf{B}$, and tells us that we may introduce $\mathbf{A}$, the vector potential. The second ME tells us how the $\mathbf{B}$ field is sourced. Combining these two equations and choosing the Coulomb gauge gives our Poisson equation, and the Lorentz force law tells us how the magnetic field measurably couples to charge.

Vacuum

We are going to be quick here, because a lot of this is review from electrostatics. Solving Poisson's equation via Green's function methods gives

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|},$$

so our vector potential is

$$\boxed{\mathbf{A} = \frac{\mu_0}{4\pi} \int d^3\mathbf{x}' \, \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}.}$$

The approximation schemes for the vector potential are significantly harder, as we're now dealing with a vector instead of a scalar.

(talk about vector spherical harmonics here)

2.5 Electrodynamics

So far we've assumed that our fields are not time-dependent. This is not a realistic assumption for most purposes, so now we introduce time-dependence. The good thing is that our previous techinques mostly work with some modifications.

2.6 Retarded potentials

We now solve for the Green's function **(finish, ts is trivial)**

2.7 Waves

3 Statistical Mechanics

Idea 3.1 (Statistical mechanics)

Statistical mechanics is mechanics, treated statistically.

After that rousing battle cry, we are ready to begin the study of this subject.

Statistical mechanics aims to make predictions about the bulk properties of systems with a large number of degrees of freedom. The "thermodynamic limit" is about $\sim 10^{23}$ particles. (**explain**) To do this, we need statistics, because solving for the dynamics of $\sim 10^{23}$ molecules is intractable.

Core ideas; the microscopic picture

Since we're doing mechanics, we will use a Hamiltonian. Furthermore, because all of our particles are tiny, everything is elastic, so energy is conserved.

Definition 1. Let our system have N particles. We define the Hamiltonian of our system to be

$$H(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{N} \frac{|\mathbf{p}_i|^2}{2m_i^2} + V_i(\mathbf{r}_1, \ldots, \mathbf{r}_N).$$

Definition 2. Call the phase space of our system $\mathcal{H}$. Note that necessarily, $\dim \mathcal{H} = 6N$. Let the energy of the system be E.

$H(\mathbf{p}, \mathbf{q}) - E = 0$ defines a submanifold $\Omega \subset \mathcal{H}$ by the Regular Value Theorem.

4 Quantum Mechanics

4.1 Assumptions

(insert intermediate stuff)

4.2 Perturbation theory

Hamiltonian

We first cover bound-state perturbation theory. Only studying bound states simplifies things, as continuous spectra require more complicted techniques (i.e. TD-perturbation theory). We follow the Brillouin-Wigner (BW) derivation for bound-state perturbation theory, showing equivalence to Rayleigh-Schrodinger perturbation theory (RSPT) along the way (as the latter is generally more immediately useful).

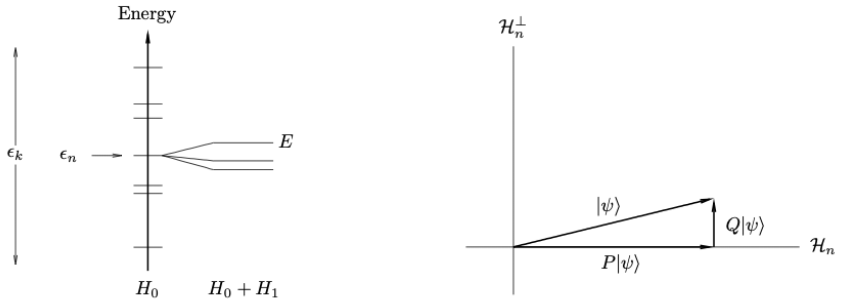

Figure 1: Algebraic and geometric pictures associated with perturbation theory.

It is often true that useful physical systems take the form of a known system plus a small perturbation; indeed, every prediction we see from colliders like the LHC tacitly assume this statement. So we should have a formalism to deal with these kinds of problems; this is the study of **perturbation theory**. We cover the quantum mechanical version here[5].

Consider a Hamiltonian H, and assume that it may be written as

$$H = H_0 + \lambda H_1 = H_0 + \widetilde{H}_1.$$

where $\lambda \in [0, 1]$ interpolates between the free system (H_0) and the full system (H). Another way of looking at this is brought to us by Wilsonian QFT; we may think of λH_1 has being an operator $\widetilde{H}_1 = g\mathcal{O}$, where $\mathcal{O}$ is composed of a's and $a^\dagger$'s. Then λ is defined through $g = \Lambda^{d-\Delta_\mathcal{O}}\lambda$, with $\Delta_\mathcal{O} \coloneqq [\mathcal{O}]$ being the mass dimension of $\mathcal{O}$, and we may proceed as usual. As a concrete example, consider the anharmonic oscillator—in this case, $H_0 = p^2/2m + (1/2)m\omega^2 q^2$ and $\widetilde{H}_1 = gq^4/4!$. We have that $\lambda = L^{d-\Delta_\mathcal{O}}g$, where L is the characteristic length scale of our system, and d the dimension.

We now study the geometry of this problem. Introduce

$$\mathcal{H}_n = \{\text{unperturbed eigenspace}\} \qquad \mathcal{H}_n^\perp = \{\text{perturbed eigenspace}\}.$$

These spaces are orthogonal, as they're eigenspaces of different eigenvalues. Label the kets in the spaces respectively by $\{|n\alpha\rangle\}$ and $\{|k\alpha\rangle\}$, where α is a label used to resolve degeneracies. These

[5]This idea is also found in classical mechanics and quantum field theory, and more generally in analysis.

states are orthonormal, $\langle n\alpha|k\beta\rangle = \delta_n^k\delta_\alpha^\beta$. Note that kets correspond to upper indices and bras to lower indices. Introduce commuting projectors onto the eigenspaces

$$P \coloneqq \sum_\alpha |n\alpha\rangle\langle n\alpha|, \qquad\qquad Q \coloneqq \sum_{\substack{k\neq n\\ \alpha}} |k\alpha\rangle\langle k\alpha|.$$

Then $|\psi\rangle = P|\psi\rangle + Q|\psi\rangle$. $P|\psi\rangle$ is known by assumption, and $Q|\psi\rangle$ is not. This is the goal of perturbation theory: solve for the small hard part $Q|\psi\rangle$ in terms of big easy part $P|\psi\rangle$. The full Hamiltonian is assumed to satisfy

$$H|\psi\rangle = (H_0 + \lambda H_1)|\psi\rangle = E|\psi\rangle, \tag{3}$$

where E and $|\psi\rangle$ are assumed to be λ-dependent. As $\lambda \to 0$, $E \to \epsilon_n$ and $|\psi\rangle \to |n\alpha\rangle \in \mathcal{H}_n$. This second statement shows the difficulty of degenerate perturbation theory; since $\mathcal{H}_n$ is degenerate, the "tracing back" of $|\psi\rangle$ as λ turns off is not unique, and this is also true in reverse—as we turn on λ, $|\psi\rangle$ generically splits into one of many energy eigenstates, the number of which determined by $\dim \mathcal{H}_n$.

As stated before, we may write $|\psi\rangle = P|\psi\rangle + Q|\psi\rangle$, where the perturbation is entirely found in $Q|\psi\rangle$. Rewriting eq. (3) gives

$$(E - H_0)|\psi\rangle = \lambda H_1|\psi\rangle. \tag{4}$$

So, we'd like to invert $E - H_0$. Write $E - H_0$ as a function of the energy eigenstates,

$$E - H_0 = \sum_{k,\,\alpha} \frac{|k\alpha\rangle\langle k\alpha|}{E - \epsilon_k}.$$

This expression has a zero eigenvalue at $E = \epsilon_n$, and so it is meaningless as it stands. To resolve this, introduce

$$R = \sum_{\substack{k\neq n\\ \alpha}} \frac{|k\alpha\rangle\langle k\alpha|}{E - \epsilon_k}.$$

R satisfies $RQ = QR = R$, and $[R, P] = 0$. We also note that $R(E - H_0) = (E - H_0)R = Q$, and so $R|_{\mathcal{H}_n^\perp} = \mathrm{id}_{\mathcal{H}_n^\perp}$, and $Q|\psi\rangle = \lambda R H_1|\psi\rangle$ (which we can see by multiplying (4) through by R). So,

$$|\psi\rangle = P|\psi\rangle + Q|\psi\rangle = P|\psi\rangle + \lambda R H_1|\psi\rangle.$$

Solving for $|\psi\rangle$ and expanding gives

$$(1 - \lambda R H_1)|\psi\rangle = P|\psi\rangle \implies \boxed{|\psi\rangle = \sum_{n=0}^{\infty} \lambda^n (RH_1)^n P|\psi\rangle.} \tag{5}$$

So we've accomplished our goal of writing $|\psi\rangle$ in terms of the easy part $P|\psi\rangle$. We now explore the consequences of this equation with a few general examples.

Example 4.1 (Non-degenerate perturbation theory)

Assume $\dim \mathcal{H}_n = 1$—call this state $|n\rangle$. Note that $\langle\psi|n\rangle = 1$ by assumption of $\langle n|n\rangle = 1$. P takes an especially simple form in this case, $P = |n\rangle\langle n|$. Referencing eq. (5), we have

$$\begin{aligned}|\psi\rangle &= |n\rangle + \lambda RH_1|n\rangle + \lambda^2(RH_1)^2|n\rangle + \cdots \\ &= |n\rangle + \sum_{\substack{k\neq n \\ \alpha}} |k\alpha\rangle \frac{\langle k\alpha|H_1|n\rangle}{E-\epsilon_k} + \cdots\end{aligned}$$

The energies are given by bra-ing through eq. (4) by $\langle n|$,

$$\langle n|E - H_0|\psi\rangle = E - \epsilon_n = \lambda\langle n|H_1|\psi\rangle,$$

so

$$E = \epsilon_n + \lambda\langle n|H_1|n\rangle + \lambda^2\langle n|H_1RH_1|n\rangle + \cdots$$

Substituting the zeroth-order approximation for E ($E = \epsilon_n$) gives

$$|\psi\rangle = |n\rangle + \lambda \sum_{\substack{k\neq n \\ \alpha}} |k\alpha\rangle \frac{\langle k\alpha|H_1|n\rangle}{\epsilon_n - \epsilon_k} + \cdots$$

which, along with the equation for $|\psi\rangle$, agrees with RSPT to first order. We remark upon a general fact here: the λ-expansion for E is always "one-behind" the expansion for $|\psi\rangle$. Concretely, if we know E to order λ^n, then we know $|\psi\rangle$ to order λ^{n+1}. This is because E only shows up in $|\psi\rangle$ *starting at* order λ, so $|\psi\rangle$ is always "one-ahead" of $E(\lambda)$.

In RSPT, we simply identify things slightly differently. $|n\rangle \to |n_0\rangle$, and $\lambda^k(RH_1)^k|n\rangle \to |n_k\rangle$. Each of the states $\{n_k\}$ is assumed to be orthogonal to $|n_0\rangle$, which follows from the fact that R projects out all $k = n$ states, or from using eq. (4) and noting that that $\langle P\psi|Q\psi\rangle = 0$. Everything else follows from this.

Example 4.2 (Degenerate perturbation theory)

We assume that the unperturbed eigenspace is degenerate, and thus $P = \sum_\alpha |n\alpha\rangle\langle n\alpha|$. P acting on $|\psi\rangle$ gives some combination of the $|n\alpha\rangle$'s, $P|\psi\rangle = \sum_\alpha c_\alpha|n\alpha\rangle$. The main difference between non-degenerate and degenerate perturbation theory is that we must solve for the state (c_α) and the energy E simultaneously. Subbing $|\psi\rangle$ into the energy equation we previously found gives

$$\underline{\text{State}}: \qquad |\psi\rangle = c_\alpha|n\alpha\rangle + \lambda RH_1(c_\alpha|n\alpha\rangle) + \cdots$$

$$\underline{\text{Energies}}: \qquad (E-\epsilon_n)c_\alpha = \left\{ \lambda\langle n\alpha|H_1|n\beta\rangle + \lambda^2 \sum_{\substack{k\neq n \\ \gamma}} \frac{\langle n\alpha|H_1|k\gamma\rangle\langle k\gamma|H_1|n\beta\rangle}{E-\epsilon_k} + \cdots \right\} c_\beta.$$

From the second equation we can see what the c_α's are: eigenvectors with respect to the $g \times g$

matrix in the braces (with $g = \dim \mathcal{H}_n$). Their eigenvalues are $(E - \epsilon_n)$. In practice, we truncate the energy series at $\mathcal{O}(\lambda)$.

(**show equivalence to RSPT**)

Example 4.3 (Nearly degenerate perturbation theory)

Lagrangian