Lecture Notes on

Algebra

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Notes on algebra. Resources used:

 \bullet Aluffi's Algebra: Chapter 0

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1 Categorical preliminaries

(fill-in)

2 Groups

We review the properties of groups here. First, two high-brow definitions of a group:

- A group is a group object in Set.
- A group is a one-object groupoid.

These aren't bad definitions, but they're more explicitly useful when they're unpacked a bit.

Definition 1. A group is a set with an associative binary operation $m_G: G \to G$ that has identities and inverses.

We can phrase this by saying that a group has two additional unique maps, $\iota \colon G \to G$ by $g \mapsto g^{-1}$ and id: $G \to G$ by $g \mapsto g$. This is the more categorical and "morally correct" way of putting this.

Example 2.1 (Canonical group examples)

The most canonical group examples are

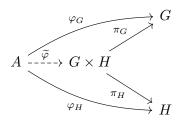
- Any field with addition, \mathbb{Z} , \mathbb{R} , \mathbb{C} , \mathbb{Q} , etc.. Some of them also work with multiplication.
- $\mathbb{Z}/n\mathbb{Z}$: the integers mod n under addition. $(\mathbb{Z}/n\mathbb{Z})^{\times}$ are the units mod n.
- D_n . The isometries of a planar solid.
- S_n and A_n . The permutations and even permutations of n objects.
- Matrices over some field under addition or multiplication (if the operation is multiplication then needs invertibility).
- Types of functions under addition, e.g. smooth, periodic, etc...

Groups form a category, Grp. Its maps are **group homomorphisms**, which are maps $\varphi \colon G \to G'$ that respect the group structure, $\varphi(ab) = \varphi(a)\varphi(b)$. One way of saying this is that the following diagram commutes:

(put diagram)

We now investigate what other universal constructions exist in Grp.

• **Products**: the product in Grp is the direct product group. This means that for all $\varphi_G \colon A \to G$, $\varphi_H \colon A \to H$, there exists a unique map $\widetilde{\varphi} \colon A \to G \times H$ making the diagram



commute. Concretely, taking the binary operations on G and H and defining

$$m_G \times m_H \colon (G \times H) \times (G \times H) \to G \times H$$

 $(m_G \times m_H)((g_1, h_1), (g_2, h_2)) \mapsto (m_G((g_1, g_2)), m_H((h_1, h_2))).$

gives the set $G \times H$ a group structure

$$(g_1, h_1) * (g_2, h_2) = (g_1g_2, h_1h_2).$$

- Coproducts: there is not a generic coproduct in Grp. (talk about free product and Ab)
- Quotients (add)