Lecture Notes on

Commutative Algebra

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Notes on commutative algebra. Resources used:

- Atiyah and MacDonald's *Introduction to Commutative Algebra*.
- Altman and Kleiman's *A Term of Commutative Algebra*.
- Aluffi's *Algebra: Chapter 0*

Contents

1 Commutative rings

Just how groups are "sets decorated with structure", rings are "abelian groups decorated with multiplication". Formally, this means that

Definition 1. A **commutative ring** is an abelian group equipped with a multiplication operation that is distributive, associative, and commutative.

We take every ring to have 1_R, the multiplicative identity.

Example 1.1 (Examples of rings)
Here are some examples of rings:

- Fields are rings with division, i.e. $(\mathbb{Z}, +, \cdot)$, $(\mathbb{R}, +, \cdot)$, $(\mathbb{Q}, +, \cdot)$, $(\mathbb{C}, +, \cdot)$, etc. So every example of a field is an example of a ring.
- Polynomial rings over a field k in n variables: $k[x_1, \ldots, x_n]$. In fact, by the UMP for polynomials, every ring can be uniquely "embedded" into a polynomial ring in the obvious way. Some examples here are $\mathbb{R}[x]$, $\mathbb{C}[x]$, and $\mathbb{Z}[x]$.
- $\mathbb{Z}/p\mathbb{Z}$ is a ring if p is prime.
- Matrix rings, $M_n(R)$, where R is our ring. Concretely, these are matrices with entries taken from R. One can also take these over a finite field, which is cool.
- Rings of functions, like $C([0,1], \mathbb{R})$ (the set of real-valued continuous functions on $[0,1]$)
- Some more exotic examples: Boolean rings and the quaternions $\mathbb{H}$.

Rings form a category, Ring; its morphisms are constructed as to have the following diagram commute (if A and B are rings) for both addition and multiplication:

$$\begin{array}{ccc} A \times A & \xrightarrow{\phi \times \phi} & B \times B \\ {\scriptstyle m_A}\downarrow & & \downarrow{\scriptstyle m_B} \\ A & \xrightarrow{\phi} & B \end{array}$$

This guarantees **ring homomorphisms** preserve the ring structure, i.e. $\phi(a+b) = \phi(a) + \phi(b)$ and $\phi(ab) = \phi(a)\phi(b)$. A psychological remark here is that ring morphisms are more beefy than group morphisms, but that's expected, as rings are more beefy than groups.

We want to have some sort of natural quotient structure on a ring. The way we do this is through an **ideal**

Definition 2. An **ideal** $\mathfrak{a}$ is a subset of A that satisfies $A\mathfrak{a} \subseteq \mathfrak{a}$.

Intuitively, you "can't move out of $\mathfrak{a}$ with A", or (calling reference to representation theory) an ideal is an "invariant subspace" of a ring.

We now state a useful and obvious theorem: that some notion of size is preserved when you take the quotient (**finish**)