Lecture Notes on

Groups and Representations

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References:

- Peter van Nieuwenhuizen's PHY 680 course at Stony Brook University, and their associated lecture notes. The most detailed group theory for physics resource in existence—no other book even comes close.
- Andre Lukas' groups and representations lecture notes. Good notes covering the essential topics from a more precise perspective (relative to physics standards; he is an M-theorist, after all). One downside of these notes is that they are telegraphic in places.
- Fuchs and Schweigert's *Symmetries, Lie Algebras and Representations: A Graduate Course for Physicists*. A solid resource that covers all of the standard material rigorously, and contains lots of important hep-th coded topics (e.g. Kac-Moody, univeral enveloping algebras, etc.).

Contents

1 Finite Groups and Representations

We begin by collecting the most important facts about the representation theory of finite groups. At the end, we touch on Majorana spinors and the normal modes of atomic molecules. For general background, see the notes on algebra here.

1.1 Finite groups

Example 1.1 (The most important finite groups for physics)

Of course, the most important groups for physics are simple Lie groups, but there are a few important finite ones as well:

- S_n. This group is good for talking about permutations of objects, and is necessary for talking about group actions.
- A_n. See above, when we only want even permutations.
- D_n. Because D_n is, by definition, the group of isometries of a planar n-point object, this is useful to look at when studing planar objects. We can also think about certain continuous groups as limits of D_n, e.g. emergent $O(N)$ symmetry in spin systems.
- $\mathbb{Z}/n\mathbb{Z}$. These groups are useful for a number of things. $\mathbb{Z}/2\mathbb{Z}$ is parity, and modding out by it also produces quite a few useful Lie groups (e.g. $SU(2)/(\mathbb{Z}/2\mathbb{Z}) \cong SO(3)$). It also catalogues cyclic groups, so it's useful when we want to think about those ideas. Additionally, these groups are closely connected to number theory when $n = p$ is prime, and there are more and more connections between physics and number theory as time goes on...

We now get into the most important definitions behind finite group theory for physics.

Definition 1. Consider $\gamma_g \colon G \to G$ by $\gamma_g(a) = gag^{-1}$. We say γ_g is an **inner automorphism** of G, and we denote the set of γ_g's by $\mathrm{Inn}_{\mathsf{Grp}}(G)$.

We may talk about the orbit of some a under the elements of $\mathrm{Inn}_{\mathsf{Grp}}(G)$; this is called the **class** of a. The practical algorithm for producing classes is as follows:

1. Take some $a \in G$. Consider gag^{-1} for all g in g.
2. If gag^{-1} is not already in your set, add it to the set.
3. Once you are done, take some $b \in G$ not in the set and repeat these steps.

This algorithm terminates, as conjugation is an equivalence relation, so the classes of G partition it.

Definition 2. The **center** of a group is defined as $\mathcal{Z}(G) := \{z \in G \mid zg = gz \ \forall g \in G\}$.

Intuitively, the center of a group tells you how abelian the group is. Indeed, we have that $\mathcal{Z}(G) = G \iff G$ is abelian. We may ask the question of how to take some non-abelian group and "turn it into" an abelian group. The only natural mechanism for this would be modding out by a subgroup. As it turns out, the smallest subgroup that makes this possible is the commutator subgroup.

Definition 3. The **commutator subgroup** $[G, G]$ of G is defined as the group containing the elements $aba^{-1}b^{-1}$ for all $a, b \in G$.

Definition 4. The **abelianization** of G is defined as $G^{\mathrm{ab}} = G/[G, G]$.

Note that every map $\varphi\colon G \to A$, where A is abelian, factors through the abelianization. This is true, as $\varphi([a, b]) = e$, so $[G, G] \subset \ker\varphi$. Thus the following diagram commutes.

$$\begin{tikzcd} G \arrow[rr, "\varphi"] \arrow[dr, two heads, "\pi"'] & & A \\ & G^{\mathrm{ab}} \arrow[ur, "\widetilde{\varphi}"'] & \end{tikzcd}$$

where $\pi\colon G \twoheadrightarrow G^{\mathrm{ab}}$ is the canonical projection and $\widetilde{\varphi}$ is the unique map $\widetilde{\varphi}\colon G^{\mathrm{ab}} \to A$. We now prove an obvious but important theorem on the abelianization of G.

Theorem 1. $[G, G]$ is the smallest normal subgroup of G such that G/N is abelian.

Proof. We first show that G^{ab} is abelian. Note that

$$[G, G] = b^{-1}a^{-1}ba[G, G] \implies ba[G, G] = ab[G, G]$$

so G^{ab} is abelian. Consider $\phi\colon G \to G/N$, where G/N is an abelian quotient of G. Then this map factorizes through G, and there is a surjection $G^{\mathrm{ab}} \twoheadrightarrow G/N$, so $|G^{\mathrm{ab}}| \geq |G/N|$.

Kind of how like the center measured how abelian a group is, the commutator subgroup measures how "non-abelian" it is. We can see that $[G, G] = 0$ for abelian groups, and if $[G, G] = G$ then we say G is **perfect**. Equivalently, a perfect group has no non-trivial abelian quotients. An interesting fact is that all non-abelian simple groups are perfect (e.g. A_5).

Another way to look at this is to notice that abelianization is a functor $F\colon \mathsf{Grp} \to \mathsf{Ab}$ by $G \to G^{\mathrm{ab}}$, and that it's the left adjoint to inclusion $\iota\colon \mathsf{Ab} \hookrightarrow \mathsf{Grp}$, so

$$\mathrm{Hom}_{\mathsf{Ab}}(F(G), A) \cong \mathrm{Hom}_{\mathsf{Grp}}(G, \iota(A)).$$

This encapsulates the universal property nicely: every map from $G \to A$ corresponds to a unique map $G^{\mathrm{ab}} \to A$.

Idea 1.1 (Most important facts about finite groups)

Whenever you see a finite group, you want to answer these questions:

- **What is the order of the group?** I find this question to be psychologically comforting, and it also tells me what the possible orders of the subgroups of G are by Lagrange's theorem.
- **What are its classes?** The reason this is useful is because characters are class functions, and so there are a number of useful facts we can draw from knowing the classes of a group (e.g. number of irreps, classes for characters to act on, possible normal subgroups, etc.). You should also check the order of the classes.
- **What is its commutator subgroup?** As we saw above, the commutator subgroup is the smallest subgroup of G such that G/H is abelian. This tells us that the 1D irreps of

G act through the abelianization of G, so $|G^{\mathrm{ab}}| = n_{1\,\mathrm{diml}}$

Some general algebraic constructions are also useful here for physics, namely products, group actions, and semi-direct products. For posterity, we recall the writings on these subjects from the algebra notes here.

The direct product group is the product in Grp. This means that by the universal property, for all $\varphi_G\colon A \to G$, $\varphi_H\colon A \to H$, there exists a unique map $\widetilde{\varphi}\colon A \to G \times H$ making the diagram

$$\begin{tikzcd} & & G \\ A \arrow[r, dashed, "\widetilde{\varphi}"] \arrow[urr, bend left, "\varphi_G"] \arrow[drr, bend right, "\varphi_H"'] & G \times H \arrow[ur, "\pi_G"] \arrow[dr, "\pi_H"'] & \\ & & H \end{tikzcd}$$

commute. Concretely, taking the binary operations on G and H and defining

$$m_G \times m_H\colon (G \times H) \times (G \times H) \to G \times H$$
$$(m_G \times m_H)((g_1, h_1),\, (g_2, h_2)) \mapsto (m_G((g_1, g_2)),\, m_H((h_1, h_2))).$$

gives the set $G \times H$ a group structure

$$(g_1, h_1) * (g_2, h_2) = (g_1 g_2, h_1 h_2).$$

Now onto group actions. Recall that an **action** of a group on the object A of a category C is a homomorphism $\sigma\colon G \to \mathrm{Aut}_{\mathsf{C}}(A)$. Specializing $\mathsf{C} = \mathsf{Set}$, we have the standard definition of a group action

$$\rho\colon G \times A \to A$$
$$\rho(e_G, a) = a \qquad \rho(gh, a) = \rho(g, \rho(h, a)).$$

This is reminiscent of the multiplication structure of Poincaré transformations, $U(\Lambda, a)U(\overline{\Lambda}, \overline{a}) = U(\Lambda\overline{\Lambda}, \Lambda\overline{a} + a)$. Writing ρ is annoying, so we abbreviate the above to simply $(gh)a = g(ha)$, so there is some sense of psuedo-associativity here. The semi-direct product follow naturally from this. **(finish)**

1.2 Representations of finite groups

Remark 1.1 (Why are groups so ubiquitous in physics?)**.** After seeing so much group theory in physics, one may begin to wonder if there is any motivation for *why* this is the case. Martin Roček gives the following nice explanation:

If you have some system, you can transform it in the following ways:

- You can do nothing to it.
- You can transform it.
- You can "un-transform" it by just doing whatever you did in reverse.
- You can transform it and then transform it again, and of course this is another transfor-

mation.

- If you do three transformations to something, it doesn't matter whether you do (12)3 or 1(23).

These are the axioms of a group. A nice example that I like to keep in mind is rotating a globe.

Some motivation for representation theory is that we know symmetries in nature can generally be given a group structure, but we tend to work with vector spaces in physics (e.g. Hilbert spaces, phase spaces, or configuration spaces[1]). How can we get a group to "act" on a vector space?

A natural choice may be a group action on Vect_k, but this cannot work; we need to do more than just permute operators and vector spaces: we need linear combinations of elements.

This naturally leads us to mapping into a set of operators on V. So, we intuitively need some sort of map into $\operatorname{End} V$ that respects the group structure of G, so it should be a homomorphism $G \to \operatorname{End} V$. This is what the definition of a representation encodes.

Definition 5. A **representation** of G on V is the pair (ρ, V), where $\rho\colon G \to GL(V)$ is a homomorphism and V a k-vector space.

Note 1.1 (Abuse of notation)**.** I will oftentimes refer to a representation by either ρ or V instead of (ρ, V).

Remark 1.2 (The importance of V in (ρ, V))**.** It is vitally important to note that a representation includes both a map *and* a vector space in its construction. Physicists usually think of a representation as just a set of matrices, one for each group element—neither the general map nor the vector space is emphasized; this is morally incorrect, and leads to issues later down the line. We do not take this unnatural view in these notes.

Note 1.2 (The categorical viewpoint)**.** Regard G as a one-object groupoid BG whose morphisms are the elements of G. Then we may define a finite representation as the *functor*

$$\rho\colon BG \to \mathsf{Vect}_k^{\mathrm{fd}},$$

sending the single object $G \to V$, and sending morphisms in g to operators on V, $g \to \varphi$.

As we can see from the categorical construction, this is a natural definition encoding everything that we want from V. We say that the **dimension** of a representation is the dimension of its underlying vector space, $\dim \rho = \dim V$. If we have structure on V, we can build up new representations from old ones by using this structure and translating it into the maps ρ:

- **Tensor representation**. We can tensor up two representations $V_1 \otimes V_2$ and their maps $\rho_1 \otimes \rho_2$ to get a new representation, $(\rho_1 \otimes \rho_2, V_1 \otimes V_2)$, where $\rho_1 \otimes \rho_2$ is naturally given by

$$(\rho_1 \otimes \rho_2)(v_1 \otimes v_2) = \rho_1(v_1) \otimes \rho_2(v_2).$$

  This is the **tensor representation** of $V_1 \otimes V_2$.

[1] It is not necessary that phase and configuration spaces are endowed with a linear structure, but it is often true that this is the case. See the notes on classical mechanics for more about this.

- **Dual representation**. If we have a metric on V, we can consider its dual space $V^\vee$. We'd like the duality to be G-invariant, so $\langle \varphi, v\rangle = \langle g^\vee \varphi, gv\rangle$. Renaming $gv = w$, we have

$$(g^\vee \varphi)(w) = \varphi(g^{-1}w) \implies (g^\vee \varphi)(v) = \varphi(g^{-1}v)$$

  So, if $\varphi \in V^\vee$, we define the **dual representation** to be

$$(\rho^\vee(g)\varphi)(v) = \varphi(\rho(g^{-1})v) \iff (g^\vee\varphi)(v) = \varphi(g^{-1}v).$$

- **Direct sum representation**. If $V = U \oplus W$, with U and W representations, we can define $\rho_V(u \oplus W) = \rho_U(u) + \rho_W(w)$ to be the **direct sum representation** of U and W. This will be useful later when we talk about reducibility.

Note 1.3 (A refresher on the tensor product)**.** We will have to actually compute with the tensor product in these notes, so we recall the basic construction here:

If we have two matrices A and B, $A \otimes B$ is defined as

$$A \otimes B \coloneqq \begin{pmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{n1}B & \cdots & A_{nn}B \end{pmatrix}.$$

That is, multiplying every entry of A by B. Note that if A is an $n \times n$ matrix and B is an $m \times m$ matrix, $A \otimes B$ is an $nm \times nm$ matrix. An example is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}.$$

This construction will be necessary later when we talk about γ-matrices.

We can map representations to one another by **similarity transformations**, or **intertwiners**. This is equivalent to saying that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\rho"] & W \arrow[d, "\widetilde{\rho}"'] \\ V & W \arrow[l, "\varphi^{-1}"'] \end{tikzcd}$$

If there is an intertwiner between two reps, we say they're **equivalent**. This is written as $V \cong W$, or $\rho = \varphi^{-1} \circ \widetilde{\rho} \circ \varphi$; in matrix notation, this reads $M = SAS^{-1}$, where M and A are reps. If $V = W$ this is just a change of basis. We may further classify representations according to their *reality* properties; we distinguish between three kinds of reprsentations:

- **Real representations**: these are representations such that $\rho^* = S\rho S^{-1}$ for some S, and there exists a similarity transformation such that $\widetilde{\rho} = A\rho A^{-1}$ is completely real.

- **Pseudoreal representations**: these are representations such that $\rho^* = S\rho S^{-1}$ for some S, but there *does not* exist a similarity transformation such that $\widetilde{\rho} = A\rho A^{-1}$ is completely real. **(mention here the fact that if its symmetric then real if asym then pseudoreal)**

- **Complex representations**: these are representations such that $\rho^* \neq S\rho S^{-1}$ for any S.

Similar to how we find bases in vector spaces, we may ask if there is a "basis" for representations. The answer is yes, and these are called *irreducible representations*.

Definition 6. A **subrepresentation** $U \subset V$ is a subspace of V that satisfies $\rho|_U U \subset U$ for all $g \in G$. An **invariant subspace** is a subrepresentation that satisfies $\rho|_U = \mathrm{id}_U$.

Definition 7. A representation is **reducible** if it has an invariant subspace.

Definition 8. An **irreducible representation** is a representation that has no invariant subspaces.

Intuitively, a representation being reducible means it has some number of "off-diagonal" elements. Irreducible representations are nice, as we can build up every representation as a direct sum of them, which is the content of the next few propositions.

Definition 9. A representation is **completely reducible** if it may be written as the direct sum of irreps.

> **Remark 1.3** (Unitarity)**.** The next theorem is almost always presented under the assumption that V has a metric. Indeed, it is a psychological problem amongst physicists to assume that every vector space have a metric, while in fact this is not always true[a]. Giving a vector space an unnatural metric is always almost a mistake, and is morally incorrect, so we present a proof that doesn't employ an inner product in its construction.
>
> [a]It is a reasonable assumption to make, though, as all Hilbert spaces necessarily have metrics, and physicists almost exclusively work in Hilbert spaces.

Theorem 2. Maschke's Theorem
All finite representations are completely reducible.

Proof. Let $U \subset V$ be an invariant subspace of V, and consider the projector $\pi_U \colon V \to U$. Average π_u over G by defining

$$\pi(v) := \frac{1}{|G|}\sum_g g\pi_u(g^{-1}v).$$

Recall that $V = \operatorname{Im}\pi \oplus \ker\pi$, where π is a projector. We claim $V = U \oplus \ker\pi$. Consider $\pi(u)$

$$\pi(u) = \frac{1}{|G|}\sum_g g\pi_u(g^{-1}u) = u.$$

So $\operatorname{Im}\pi = U$, and thus $W := \ker\pi$ has the right dimension. It is clear that $\pi^2 = \pi$. Note that W is G-invariant, as

$$\pi(hv) = \frac{1}{|G|}\sum_g g(\pi_u(g^{-1}hv)) = \frac{1}{|G|}\sum_{g'} hg'\pi_u((g')^{-1}v) = h\pi(v).$$

So, $\pi(gw) = g\pi(w) = 0$, and thus W is G-invariant. $\square$

> **Note 1.4** (The "group-averaging" trick)**.** The trick used in the previous proof is a common one called "group averaging". It basically means that we can make any operator A G-invariant by defining its averaged version,
>
> $$\langle A\rangle \coloneqq \frac{1}{|G|}\sum_g gA.$$
>
> Then clearly $h\langle A\rangle = \langle A\rangle$, which we can see by re-indexing.

Theorem 3. All finite representations can be made unitary by a similarity transformation.

Proof. Let $(\bullet,\bullet)_0$ be an inner product on V. Construct a new inner product

$$(v, w) \coloneqq \sum_g (\rho(g)v, \rho(g)w)_0.$$

Then V is unitary with respect to $(\bullet,\bullet)$. Explicitly,

$$(\rho(g)v, \rho(g)w) = \sum_{g'}(\rho(g')\rho(g)v, \rho(g')\rho(g)w)_0 = \sum_{g'}(\rho(g'g)v, \rho(g'g)w)_0 = \sum_{\tilde{g}}(\rho(\tilde{g})v, \rho(\tilde{g})w)_0 = (v, w).$$

□

> **Note 1.5** (The "re-indexing" trick)**.** The trick used in the previous proof is a common strategy. It's called "re-indexing", and it consists of noticing that if you sum over a group then it doesn't matter what element you use. This is effectively just a group "u-sub", e.g. like in QFT when we switch the measure $\ell - kx \to \ell$ because of translational invariance. In the immortal words of Georgi,
>
> > "... where the last line follows because hg runs over all elements of G when h does. QED." – Howard Georgi

Theorem 4. Schur's Lemma
Let V and W be irreps of G, and $\varphi\colon V \to W$ a linear map such that $\varphi\circ\rho_V = \rho_W\circ\varphi$, then

(a) Either φ is an isomorphism or zero.

(b) If $V = W$, then $\varphi = \lambda\,\mathrm{id}$ for some $\lambda\in\mathbb{C}$.

Proof.

(a) The first claim follows from the fact that $\ker\varphi$ and $\operatorname{Im}\varphi$ are invariant subspaces; we can see that for $v\in\ker\varphi$, $\varphi(\rho_V(g)v) = \rho_W(g)\varphi(v) = 0$, and similarly for $\operatorname{Im}\varphi$. Since these invariant subspaces must be trivial by assumption, the claim follows.

(b) If our field is algebraically closed, then ϕ has an eigenvalue; call it $\lambda\in k$. Then $\phi - \lambda\,\mathrm{id}$ must have non-zero kernel, and by (a), $\phi - \lambda\,\mathrm{id} \equiv 0$, so $\phi = \lambda\,\mathrm{id}$.

□

Corollary 1. We may uniquely decompose any representation as a direct sum of irreducible representations

$$V = \bigoplus_i V_i^{\oplus a_i}.$$

This is both psychologically and computationally useful, as it tells us that to understand every finite representation, we only have to understand the irreducible ones.

> **Remark 1.4** (Nascent connection to the little group). The previous corollary is kind of like the little group in QFT, where to understand all n-particle states, $|\mathbf{k}, \sigma, n\rangle$, we only have to understand the reference state, $|\mathbf{k}_R, \sigma, n\rangle$. This gives the following idea to play around with.

Idea 1.2 (References)

If a group of objects can be considered as being built up from or a transformed version of some reference object, it's usually easier to just work with the reference object and then get the other objects from there.

Example 1.2 (Examples of references)

Some examples of references:

- Bases in vector spaces, generators in algebra.
- Direct sums, tensor products.
- Abelian Lie algebras being the direct sum of $\mathfrak{u}(1)$'s.
- Irreducible representations in representation theory.
- States and polarization vectors from the little group.
- Prime numbers.

1.3 Characters

Recall that there are two algebraic invariants for any matrix: the trace and the determinant. Thus, it would make sense that these help us classify representations. Indeed, as we will see in this subsection, the trace of a representation is the most useful bit of information we can reasonably find out about it. We call this number the **character** of our rep.

Definition 10. Let ρ be a representation. The **character** is the trace of ρ, $\chi(g) = \operatorname{tr} \rho(g)$.

Just how we can build up representations from irreducible representations, we can do the same thing for characters

(read more about characters first; organize it in your own head)

1.4 Applications

1.4.1 Dirac, Majorana, and Weyl spinors

Remark 1.5 (Why do we care about spinors?). It turns out that spin-1/2 particles transform in the spinor representation of the Poincaré group; we call the particles that transform like this *spinors*. We can impose two broad classes of constraints on spinors: *reality* conditions and *chirality* conditions. The latter is imposed through the so-called *chirality matrix*, which we can construct from the Dirac matrices we will study below.

Example 1.3 (A motivating example)

When we're working with spin-1/2 massive particles, we naturally work in the $j = 1/2$ representation of the little group. So we want to find some $S^{\mu\nu}$ such that

$$D^{(1/2)}(W) = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}},$$

and $[S^{\mu\nu}, S^{\rho\sigma}]$ satisfies the same commutation relation as the $J^{\mu\nu}$'s. As it turns out, if you have matrices γ^μ such that $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, then we *can* construct such an S: it's given by

$$\boxed{S^{\mu\nu} := -\frac{i}{4}[\gamma^\mu, \gamma^\nu].}$$

We now study the properties of these γ-matrices below.

To start our analysis of the γ-matrices, we first need to talk about the Clifford algebra. Recall the definition of an R-algebra.

Definition 11. An **R-algebra** is an R-module with a bilinear multiplication defined on it.

Definition 12. **(definition of clifford algebra here)**

Definition 13. The **Dirac group** is a multiplicative subgroup of the Clifford algebra given by

$$\mathrm{Dir}(n) = \{\pm\gamma^{\mu_1}\cdots\gamma^{\mu_n} \mid \mu_i < \mu_{i+1},\ \ \gamma^{\mu_i} \in \mathrm{Cl}_n(\mathbb{C})\}.$$

(probably change the γ's to e's)

Remark 1.6 (What are the γ's?). **(probably get rid of this)** It is often stated that the Dirac group is spanned by the product of "γ-matrices". This is *false*; the elements of $\mathrm{Dir}(n)$ are just symbols. There is no inherent meaning to them. Indeed, that is what representation theory is for; giving them practical meaning.

Some points of contention with this:

- "But there is a μ index, so it looks like a vector": This is just a label for the group elements. μ ranges from $\mu = 0, \ldots, n-1$ or $\mu = 1, \ldots, n$ for Minkowski and Euclidean signature respectively. This is just to concisely label the abstract elements of the group with one index instead of explicitly writing out commutators for each element.

- "But $\eta^{\mu\nu}$ is a *matrix*, so $\gamma^\mu\gamma^\nu$ must *also* be a matrix": $\eta^{\mu\nu}$ is NOT a matrix here. It is just useful shorthand for expressing $\{\gamma^0, \gamma^0\} = -2I$ and $\{\gamma^a, \gamma^a\} = 2I$. You are free to arrange each component of $\eta^{\mu\nu}$ into a matrix, but a priori this is *not true*, and it leads to the wrong idea behind what the γ's are.

The takeaway from this is that the Clifford algebra and the Dirac group are spanned by abstract symbols which we happen to call γ^μ; these are *not* matrices yet.

However, one may think about the γ's as being $n \times n$ matrices even in the group, with composition given by matrix multiplication. This is simply because we may identify $\eta^{\mu\nu}$ on the RHS with the metric, which is indeed an $n \times n$ matrix. We would then naturally define the representation of the γ-matrices with the standard embedding representation, $\mathrm{Dir}(n) \hookrightarrow \mathrm{End}(V)$. An analogy to this is $SO(3)$, where its elements are naturally seen as 3×3 matrices which have to be embedded into $\mathbb{R}^3$, but we can also represent them as, say, spherical harmonics.

(**I think it may be morally correct to view them as matrices tbh**)

Some facts about the Dirac group:

- It has order 2^{n+1}. This is because of the skew-symmetry of the group, so for some product of k γ's, there are $\binom{n}{k}$ ways to do it, so summing k from $1 \to n$ and multiplying by two gives

$$2\left(\binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}\right) = 2(1+1)^n = 2^{n+1}.$$

- Its classes depend on the order of the group. This is just because of the last element; with even n, $\gamma^{\mu_1}\cdots\gamma^{\mu_n} \sim -\gamma^{\mu_1}\cdots\gamma^{\mu_n}$, but this is false with odd n. You can see this by counting swaps of γ's. So, there are $2^n + 1$ classes with n even and $2^n + 2$ classes with n odd. Note that $-e$ is in its own class.

- Its commutator subgroup is obviously $[\mathrm{Dir}(n), \mathrm{Dir}(n)] = \{\pm e\}$. This is because all γ's square to either 1 or -1, and you can move them around by picking up minus signs.

- The difference between Minkowski and Euclidean signature is in the sign of γ^2, as you can see from the definition. To do this, we simply add an i to as many matrices as we have time coordinates; note that this will generically add factors of i to γ_c.

We now get into representations of the Dirac group. This is where the fun[2] begins. As a psychological remark, this is going to be a relatively straightforward but tedious process of bootstrapping our way to whatever dimension representation we desire. The boostrap goes as follows:

- **Even $\to$ odd**: we first construct the **chirality matrix** as follows: $\gamma_c = \alpha\gamma^1\cdots\gamma^n$, with $(\gamma_c)^2 = 1$. Note that $\{\gamma_c, \gamma^\mu\} = 0$ for all $\gamma^\mu \in \mathrm{Dir}(n)$. We then add either $+\gamma_c$ or $-\gamma_c$ to our set of matrices $\gamma^{(2n)}$, and this is our new representation, $\{\gamma^{(n)}, \pm\gamma_c\}$. These are $2n \times 2n$ matrices.

- **Odd $\to$ even**: we tensor up all of the γ's with some sigma matrices. Concretely, given some representation $\gamma^{(n)}$, our new representation in even dimensions is $\{\gamma^{(n)} \otimes \sigma^1, I_{n-1} \otimes \sigma^2\}$, or the same with $(\sigma^1, \sigma^2 \leftrightarrow \sigma^1, \sigma^3)$, or $(\sigma^1, \sigma^2 \leftrightarrow \sigma^2, \sigma^3)$. It does not matter which sigmas you use to do this with. Note that these matrices are now $4k \times 4k$ in size.

[2]This is one way to describe it...

Example 1.4 (Bootstrapping our way to Heaven (***D* = 11** supergravity))

We start in $D = 2+0$. This is the easiest dimension to start in, as we instantly have the representation of

$$\gamma^1 = \sigma^1 \qquad\qquad \gamma^2 = \sigma^2.$$

Constructing the chirality matrix, we have $\gamma^3 = \alpha\gamma^1\gamma^2$, where $(\gamma^3)^2 = -\alpha^2(\gamma^1)^2(\gamma^2)^2 = -\alpha^2$, so WLOG $\alpha = -i$. This gives $\gamma^3 = \sigma^3$. So our rep in $D = 3+0$ is $\{I_2, \sigma^1, \sigma^2, \sigma^3\}$. For posterity, recall the explicit forms of the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Tensoring the identity with σ^1 and the γ's with $-\sigma^2$ gives

$$\gamma^a = -i\begin{pmatrix} 0 & \sigma^a \\ -\sigma^a & 0 \end{pmatrix} \qquad\qquad I_2 \otimes \sigma^1 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}.$$

Note that the identity here is $I_2 \otimes I_2$. If we want to switch from $D = 4+0 \to 3+1$, we simply add a factor of $\pm i$ to one of the matrices; it is traditional (at least in $D = 4$) to do it to $I_2 \otimes \sigma^1$, so in $D = 3+1$,

$$\gamma^a = -i\begin{pmatrix} 0 & \sigma^a \\ -\sigma^a & 0 \end{pmatrix} \qquad\qquad \gamma^4 = -i(I_2 \otimes \sigma^1) = -i\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}.$$

We now consider the chirality matrix in $D = 4+0$. Taking the product gives

$$\gamma_c = \alpha\gamma^1\gamma^2\gamma^3\gamma^4 = \alpha(\sigma^1 \otimes \sigma^2)(\sigma^2 \otimes \sigma^2)(i\sigma^3 \otimes \sigma^2)(I^2 \otimes \sigma^1) = \alpha(i\sigma^1\sigma^2\sigma^3 I_2 \otimes \sigma^2\sigma^2\sigma^2\sigma^1) = i\alpha(I_2 \otimes \sigma^3).$$

Squaring this and demanding $(\gamma_c)^2 = 1$ gives

$$(\gamma_c)^2 = -\alpha^2(I_2 \otimes I_2) \implies \alpha = -i.$$

So $\gamma_c = \gamma^5 = (I_2 \otimes \sigma^3)$, and our irreps for $D = 5+0$ are $\{\gamma^{(4)}, \pm\gamma^5\}$. For $D = 6+0$, we now tensor up with σ^1 and σ^2 to get $\{\gamma^{(5)} \otimes \sigma^1, I_4 \otimes \sigma^2\}$. These are 8×8 matrices. Explicitly, these are given by

$$\gamma^a \otimes \sigma^1 = -i\begin{pmatrix} 0_4 & \sigma^a \otimes \sigma^1 \\ -\sigma^a \otimes \sigma^1 & 0 \end{pmatrix} \qquad\qquad \gamma^4 \otimes \sigma^1 = -i\begin{pmatrix} 0_4 & I_2 \otimes \sigma^1 \\ -I_2 \otimes \sigma^1 & 0_4 \end{pmatrix}$$

$$\gamma^5 \otimes \sigma^1 = \begin{pmatrix} \sigma^3 \otimes \sigma^1 & 0_4 \\ 0_4 & \sigma^3 \otimes \sigma^1 \end{pmatrix} \qquad\qquad \gamma^6 = I_4 \otimes \sigma^2 = \begin{pmatrix} \sigma^2 & & & \\ & \sigma^2 & & \\ & & \sigma^2 & \\ & & & \sigma^2 \end{pmatrix}.$$

For $D = 6+0$, the chirality matrix is

$$\gamma^7 = \alpha\gamma^1 \cdots \gamma^6 = \alpha(\gamma^1 \cdots \gamma^5 \otimes (\sigma^1)^5)(I_4 \otimes \sigma^2) = \alpha i((\gamma^5)^2 \otimes \sigma^3) = \boxed{\alpha i(I_4 \otimes \sigma^3),}$$

so $\alpha = -i$ and $\gamma^7 = I_4 \otimes \sigma^3$. So our irrep in $D = 7+0$ is $\{\gamma^{(6)}, \pm\gamma^7\}$. Tensoring $\gamma^{(7)}$ with σ^2 and I_8 with σ^1 gives

$$\gamma^{(8)} = \left\{\gamma^{(7)} \otimes \sigma^2, I_8 \otimes \sigma^1\right\}.$$

Note that these are 16×16 matrices. Constructing $\gamma_c = \gamma_9$ gives

$$\begin{aligned}\gamma_c = \gamma_9 &= \alpha(\gamma^1 \otimes \sigma^2)\cdots(\gamma^7 \otimes \sigma^2)(I_8 \otimes \sigma^1)\\ &= \alpha((\gamma^1\cdots\gamma^6)\gamma^7) \otimes ((\sigma^2)^7\sigma^1)\\ &= -\alpha i(I_8 \otimes \sigma^3) \implies \alpha = i,\\ &= \boxed{I_8 \otimes \sigma^3.}\end{aligned}$$

So our irreps for $D = 9+0$ are $\{\gamma^{(8)}, \pm\gamma^9\}$. Tensoring $\gamma^{(9)}$ with σ^1 and I_{16} with σ^3 gives

$$\gamma^{(10)} = \left\{\gamma^{(9)} \otimes \sigma^1, I_{16} \otimes \sigma^3\right\}.$$

These are 32×32 matrices. Constructing $\gamma_c = \gamma_{11}$ gives

$$\begin{aligned}\gamma_c = \gamma_{11} &= \alpha(\gamma^1 \otimes \sigma^1)\cdots(\gamma^9 \otimes \sigma^1)(I_{16} \otimes \sigma^3)\\ &= \alpha((\gamma^9)^2) \otimes ((\sigma^1)^9\sigma^3)\\ &= \alpha i(I_{16} \otimes \sigma^2) \implies \alpha = -i,\\ &= \boxed{I_{16} \otimes \sigma^2.}\end{aligned}$$

So our irreps for $D = 11+0$ are $\{\gamma^{(10)}, \pm\gamma_{11}\}$, and we are done. Whew.

Idea 1.3 (Dirac vs. Majorana vs. Weyl vs. Majorana-Weyl spinors)

The representation of the Dirac group will depend on what kinds of spinors we would like to use the γ-matrices with. There are three (actually five) major distinctions here.

- **Dirac spinors**: these are just elements of the vector space $S \cong \mathbb{C}^{2^{\lfloor n/2 \rfloor}}$ of our representation.
- **Majorana spinors**: these are Dirac spinors with reality conditions imposed on them, namely that the Dirac conjugate of the spinor equals its Majorana conjugate.
- **Weyl spinors**: these are Dirac spinors with chirality conditions imposed on them; concretely, we construct the chirality matrix and impose eigenvalues under it. These are necessarily massless.
- **Majorana-Weyl spinors**: these are spinors that satisfy both the Majorana and Weyl conditions for spinors.
- **Sympletic Dirac spinors**: these are kind of the odd one out. In some sense these are "pseudo-Majorana spinors", in that you look for them if you can't have Majorana spinors in your space. Basically, if you have more than one spinor in some dimension, you can contract it with Ω_{ab} to get a "symplectic Majorana condition".

The taglines to remember are: "Dirac = normal", "Majorana = reality", and "Weyl = chiral".

We now get into the machinery behind these spinors.

Definition 14. A **Majorana spinor** is a spinor whose Dirac conjugate equals its Majorana conjugate. Recall ($\sigma = (1/2)n_t(n_t - 1) + 1$, where n_t is the number of time coordinates)

$$\overline{\lambda}_D := \lambda^\dagger i^\sigma \gamma^1 \cdots \gamma^{n_t} \qquad \overline{\lambda}_M := \lambda^T C.$$

So a Majorana spinor satisfies

$$\lambda^\dagger i^\sigma \gamma^1 \cdots \gamma^{n_t} = \lambda^T C,$$

where C is the **charge conjugation matrix**. We will denote Majorana spinors by ζ. If a representation is real, then Majorana spinors exist. A consistency condition for Majorana spinors is

$$C_+^T = (-1)^{\lfloor t/2 \rfloor} C_+ \qquad C_-^T = (-1)^{\lfloor (t+1)/2 \rfloor} C_-.$$

(**justify**)

Definition 15. The **projection operator** is defined to be

$$\mathcal{P}_\pm := \frac{1 \pm \gamma_c}{2}.$$

This is indeed a projector, as it's idempotent and its projected spaces are orthogonal.

Definition 16. A left/right-handed **Weyl spinor** is defined to be

$$\chi^\pm := \mathcal{P}_\pm \lambda,$$

where $\chi^\pm = \chi^{L/R}$. Another way of saying this is that Weyl spinors are eigenvectors of the chirality matrix,

$$\gamma_c \chi^\pm = \pm \chi^\pm.$$

Weyl spinors only exist in even dimensions. (**put consistency condition here**)

Definition 17. A **Majorana-Weyl** spinor is a spinor that is both a Majorana and a Weyl spinor. Concretely, for a MW spinor ξ,

$$\overline{\xi}^\pm_M = \overline{\xi}^\pm_D \qquad \gamma_c \xi^\pm = \pm \xi^\pm.$$

MW spinors only exist in Euclidean and Minkowski spacetimes $D_{E/M}$ such that

$$D_E \equiv 0 \bmod 8 \qquad D_M \equiv 2 \bmod 8.$$

Definition 18. A **symplectic Majorana spinor** is defined as a spinor whose *symplectic* Majorana conjugate equals its Dirac conjugate. Let the spinor exist in $D = 2n$ spacetime diensions. Recall that the definition of the symplectic Majorana conjugate is

$$\overline{\lambda}^a_{\text{SpM}} = (\lambda^b)^T \Omega^{ab} C,$$

where

$$\Omega^{ab} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Note that one requires at least two spinors for this construction to hold. If an irrep is pseudoreal, then symplectic Majorana spinors exist. (**justify this**)

The definition of the Majorana spinor introduces a new object, the charge conjugation matrix. This matrix may or may not exist in a given $D = s+t$, leaving us to derive the following consistency conditions on C. We gave the most useful one in the definition, but we list the rest here:

- **(finish)**

Reality condition for the Dirac matrices in $D = s + 1$ dimensions

$$\# = \sum_{k=0}^{D} (-1)^{k(k-1)/2} \left(\binom{n}{k} - \binom{n}{k-1} \right),$$

where $\binom{n}{-1} := 0$. Also you can use Bott periodicity, $s - t \equiv n \bmod 8$, where

- $n \in \{0,1,2\} \implies$ real.
- $n \in \{4,5,6\} \implies$ pseudoreal.
- $n \in \{3,7\} \implies$ complex.

Example 1.5 (Majorana and Weyl spinors in $\boldsymbol{D = 3+1}$)
Recall our irrep from the previous example,

$$\gamma^a = -i \begin{pmatrix} 0 & \sigma^a \\ -\sigma^a & 0 \end{pmatrix}, \qquad \gamma^4 = -i \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \qquad \gamma^5 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},$$

Constructing the Majorana and Weyl spinors gives us **(finish)**

Note that for constructing the similarity transformation, $S^T = S$ if the rep is real, and $S^T = -S$ if the rep is pseudoreal. The similarity transformation that works is $(C_+^{-1})^T$ for Euclidean dimensions, and $(C_+^{-1})^T \gamma^0$ for Minkowski dimensions. That is what ChatGPT says, at least.

$D = s + t$	Faithful	Hermitian	Reality	Similarity
$D = 2 + 0$	Yes	Yes	Real	γ^1
$D = 1 + 1$	Yes	No	Real	$\gamma^1\gamma^0$
$D = 3 + 0$	Yes	Yes	Complex	DNE
$D = 2 + 1$	Yes	No	Real $\gamma^3\gamma^1\gamma^0$	
$D = 4 + 0$	Yes	Yes	Pseudoreal	$\gamma^3\gamma^1$
$D = 3 + 1$	Yes	No	Real $\gamma^3\gamma^1\gamma^0$	
$D = 5 + 0$	No	Yes	Pseudoreal	$\gamma^3\gamma^1$
$D = 4 + 1$	Yes	No	Complex	DNE
$D = 6 + 0$	Yes	Yes	Pseudoreal	$\gamma^5\gamma^4\gamma^2$
$D = 5 + 1$	Yes	No	Pseudoreal	$\gamma^5\gamma^4\gamma^2\gamma^0$
$D = 7 + 0$	Yes	Yes	Complex	DNE
$D = 6 + 1$	No	No	Pseduoreal	**($\boldsymbol{C_-\gamma^0}$)**
$D = 8 + 0$	Yes	Yes	Real	$\gamma^7\gamma^5\gamma^4\gamma^2$
$D = 7 + 1$	Yes	No	Pseudoreal	$\gamma^7\gamma^5\gamma^4\gamma^2\gamma^0$
$D = 9 + 0$	No	Yes	Real	$\gamma^7\gamma^5\gamma^4\gamma^2$
$D = 8 + 1$	Yes	No	Complex	DNE
$D = 10 + 0$	Yes	Yes	Real	$\gamma^7\gamma^5\gamma^3\gamma^2$
$D = 9 + 1$	Yes	No	Real	$\gamma^7\gamma^5\gamma^3\gamma^2\gamma^0$
$D = 11 + 0$	Yes	Yes	Complex	DNE
$D = 10 + 1$	Yes	No	Real	$\gamma^{11}\gamma^7\gamma^5\gamma^4\gamma^2\gamma^0$

Symmetry of charge conjugation matrices:

- $C_+^T = (-1)^{\lfloor t/2 \rfloor} C_+$
- $C_-^T = (-1)^{\lfloor (t+1)/2 \rfloor} C_-$

Note that in odd dimensions, *either* C_+ or C_- exist. You inherit the two charge conjugation matrices from even dimensions, and then you test them on some test γ-matrix to see which one stays around. Recall that the charge conjugation matrices satisfy

$$\boxed{C_\pm \gamma C_\pm^{-1} = \pm \gamma^T.}$$

(the symmetry properties are wrong for the minkowskian entries in this table; redo later)

$D = s + t$	C_+/C_-	Sym.	Diag	Sp. Ind.	M	W	MW	SpM
$D = 2 + 0$	Both	Sym	Both (Weyl)	Weyl	Y	Y	Y	N
$D = 1 + 1$	Both	50/50	Both	Weyl	Y	Y	Y	Y
$D = 3 + 0$	$C_- = \gamma^2$	Sym	Off	None	N	N	N	N
$D = 2 + 1$	$C_- = \gamma^2$	50/50	Off	Norm	Y	N	N	Y
$D = 4 + 0$	Both	Sym	Both	SpWeyl	Y	Y	Y	Y
$D = 3 + 1$	Both	50/50	Both	Weyl	Y	Y	Y	Y
$D = 5 + 0$	$C_+ = \gamma^1\gamma^3$	Sym	Block	Sp	N	N	N	Y
$D = 4 + 1$	$C_+ = \gamma^1\gamma^3$	50/50	Block	None	N	N	N	N
$D = 6 + 0$	Both	Sym	Both	SpWeyl	N	Y	N	Y
$D = 5 + 1$	Both	50/50	Both	SpWeyl	N	Y	N	Y
$D = 7 + 0$	$C_- = \gamma^1\gamma^3\gamma^6$	Sym	Off	None	N	N	N	N
$D = 6 + 1$	$C_- = \gamma^1\gamma^3\gamma^6$	50/50	Off	Sp	N	N	N	Y
$D = 8 + 0$	Both	Sym	Both	Weyl	Y	Y	Y	N
$D = 7 + 1$	Both	50/50	Both	SpWeyl	N	Y	N	Y
$D = 9 + 0$	$C_+ = \gamma^2\gamma^4\gamma^5\gamma^7$	Sym	Block	Norm	Y	N	N	N
$D = 8 + 1$	$C_+ = \gamma^2\gamma^4\gamma^5\gamma^7$	50/50	Block	Norm	Y	N	N	Y
$D = 10 + 0$	Both	Sym	Both	Weyl	Y	Y	Y	N
$D = 9 + 1$	Both	50/50	Both	Weyl	Y	Y	Y	N
$D = 11 + 0$	$C_+ = \gamma^2\gamma^4\gamma^5\gamma^7\gamma^{11}$	Sym	Off	None	N	N	N	N
$D = 10 + 1$	$C_+ = \gamma^2\gamma^4\gamma^5\gamma^7\gamma^{11}$	50/50	Off	Norm	Y	N	N	Y

Table 1: Table for properties of the charge conjugation matrices and what kinds of spinors are allowed. See 1.4.1 for explicit constructions of C_+ and C_-. Sym goes to normal sym (see above) under time-like additions to the metric.

	γ^1	γ^2	γ^3	γ^4	γ^5	γ^6	γ^7	γ^8	γ^9	γ^{10}	γ^{11}
$D = 2$	+	−									
$D = 3$	+	−	+								
$D = 4$	−	+	−	+							
$D = 5$	−	+	−	+	+						
$D = 6$	−	+	−	+	+	−					
$D = 7$	−	+	−	+	+	−	+				
$D = 8$	+	−	+	−	−	+	−	+			
$D = 9$	+	−	+	−	−	+	−	+	+		
$D = 10$	+	−	+	−	−	+	−	+	+	+	
$D = 11$	+	−	+	−	−	+	−	+	+	+	−

Table 2: Symmetry properties of the γ's.

We now determine the charge conjugation matrices in every dimension. We can generally construct the matrix as follows for dimension D.

- Write out the D equations of commutation/anti-commutation for the charge conjugation matrix. You will get some number of pluses and minuses for each, e.g. $C\gamma^2 = -\gamma^2 C$, which implies that $\{C, \gamma^2\} = 0$.

- Take the product of all of the pluses and minuses. This will give you C_+ and C_-. Now take $C\gamma^1 C^{-1}$ and see if it's $\pm\gamma^1$. From this you can determine which is which.
- In odd dimensions, you just need to check what the value of the matrices are on the chirality matrix; one of them will fail. Then you discard that one and choose the other one as your matrix.

	C_+	C_-
$D = 2$	γ^1	γ^2
$D = 3$	DNE	$\gamma^1\gamma^3$
$D = 4$	$\gamma^1\gamma^3$	$\gamma^2\gamma^4$
$D = 5$	$\gamma^1\gamma^3$	DNE
$D = 6$	$\gamma^2\gamma^4\gamma^5$	$\gamma^1\gamma^3\gamma^6$
$D = 7$	DNE	$\gamma^1\gamma^3\gamma^6$
$D = 8$	$\gamma^2\gamma^4\gamma^5\gamma^7$	$\gamma^1\gamma^3\gamma^6\gamma^8$
$D = 9$	$\gamma^2\gamma^4\gamma^5\gamma^7$	DNE
$D = 10$	$\gamma^2\gamma^4\gamma^5\gamma^7$	$\gamma^1\gamma^3\gamma^6\gamma^8\gamma^9\gamma^{10}$
$D = 11$	DNE	$\gamma^2\gamma^4\gamma^5\gamma^7\gamma^{11}$

Table 3: Charge conjugation matrices up to $D = 11 + 0$ for the γ-matrix construction performed earlier. Note multiplying by $\pm i$ doesn't change anything, (transposes do nothing to those) so the Minkowski and Euclidean signatures are the same.

0	real
1	real
2	complex
3	pseudoreal
4	pseudoreal
5	complex
6	real
7	real

And for **chiral** Weyl spinors (when they exist, i.e. even d), the types differ:

- In $d = 2 \mod 8$: Weyl spinors are **complex** (this is SO(10)!)
- In $d = 6 \mod 8$: Weyl spinors are **pseudorea** .g. SO(6) ≅ SU(4)).
- In $d = 0 \mod 8$: Weyl spinors may be **real** (Majorana-Weyl), e.g. SO(8) or SO(16).

Figure 1: Reality properties of spinor irreps in various dimensions. Note that the dimension here is equivalent to the number of single γ-matrices, or the dimension of the underlying space, e.g. when we say the "10-dimensional" irrep we mean $\mathbb{R}^{10}$, etc.

Fact: $C_+ = C_-\gamma_c$.

Example 1.6 (A deep dive into six dimensions)

This is the second problem from the 2021 PvN midterm. We start in $D = 4+0$, with the standard representation (see previous example). We construct the chirality matrix $\gamma^5 = I_2 \otimes \sigma^1$, and get $\gamma^{(5)} = \{\gamma^{(4)}, \gamma^5\}$. This representation is not unique, as we could have also used $-\gamma^5$. Tensoring up by σ^1 and σ^2 gives the standard answer. Every matrix here is Hermitian. The chirality matrix is $\gamma_7 = I_4 \otimes \sigma^3$. This representation is unique. Going to $D = 5+1$ by instead tensoring up with $-i\sigma^2$ gives another unqiue representation, but with γ^6 anti-hermitian instead of Hermitian. We now explicitly construct the charge conjugation matrices in $D = 4$, 5, and 6, giving:

- $D = 4$. γ^1 and γ^3 are symmetric, and $2, 4$ are antisymmetric. So, $C_+ = \gamma^1\gamma^3$, and $C_- = \gamma^2\gamma^4$.
- $D = 5$. Only C_+ satisfies the condition that $C_\pm \gamma^5 = (\gamma^5)^T C_\pm$.
- $D = 6$. Symmetric ones: $2, 4, 5$. Asymmetric ones: $1, 3, 6$. So $C_+ = \gamma^2\gamma^4\gamma^5$, and $C_- = \gamma^1\gamma^3\gamma^6$. Explicitly, $C_+^6 = C_+^5 \otimes C_+^2$, and $C_-^6 = C_+^5 \otimes C_-^2$. $C_+^6 = C_-^6 \gamma_7$, as

$$(\gamma^1\gamma^3 \otimes \sigma^1) = (\gamma^1\gamma^3 \otimes (-i\sigma^2))(I_4 \otimes \sigma^3).$$

  Since adding an i to the $C_\pm$ doesn't do anything for transposition, it can't affect symmetry properties.

Note that the irrep in $D = 6+0$ is pseudoreal by Bott periodicity. The matrix S such that $S\gamma^M S^{-1} = (\gamma^M)^*$ is given by $S = C_+$. The $D = 5+1$ irrep is the same, but $S = \gamma^1\gamma^3 \otimes I_4$, which is clearly symmetric. Weyl spinors exist (6 is even), and since C_- is symmetric, Majorana spinors exist in $D = 6+0$. There are no Majorana spinors in $D = 5+1$, as neither C_+ is symmetric or C_- antisymmetric. Majorana-Weyl spinors do not exist in either dimension. To show that C_+ stays symmetric under a similarity transformation, see below.

Solution: If $C_+\gamma^\mu C_+^{-1} = \gamma^{\mu,T}$ for a particular matrix representation, and $\gamma^\mu = S\widehat{\gamma}^\mu S^{-1}$ is the similarity transformation, then $C_+ S\widehat{\gamma}^\mu S^{-1} C_+^{-1} = S^{-1,T}\widehat{\gamma}^{\mu,T} S^T$. Thus $(S^T C_+ S)\widehat{\gamma}^\mu (S^{-1} C_+ S^{T,-1}) = \gamma^{\mu,T}$. The new C_+ is $\widehat{C}_+ = S^T C_+ S$, which is again a symmetric matrix.

Process of finding reality properties of spinor irreps for $SO(N)$:

- Construct N Γ-matrices.
- Construct the spinor irreps: $M_{ij} = \frac{1}{2}\Gamma_i\Gamma_j$ for $i < j$.
- Now to find some S such that $M_{ij}^* = SM_{ij}S^{-1}$. Take the c.c. of all generators. Find the ones that turn negative. From this, obtain a set of constraints on the possible S's.
- If you find S, the symmetry of it obeys: real if symmetric, and psuedoreal if anti-symmetric. If you could not find such an S, then the irrep is *complex*.

Deligne's Bott periodicity for spinor irreps

signature $p-q \bmod 8$	real, complex or quaternionic
0	$\mathbb{R}, \mathbb{R}$
1 or 7	$\mathbb{R}$
2 or 6	$\mathbb{C}$
3 or 5	$\mathbb{H}$
4	$\mathbb{H}, \mathbb{H}$

1.4.2 Normal modes of atoms

Molecules with multiple atoms generally obey a periodic structure, and thus they are symmetric under certain operations (translations, generalized rotations, and generalized reflections). Since symmetries constrain dynamics, it's natural to find that we can use this information to determine the normal modes of the molecule. We do this through analyzing the characters of the representation of the symmetry group acting on our vector space, which we will take to be $\mathbb{R}^n$ for $n = 2$ and 3 (planar and $3D$ objects, respectively).

A molecule with N atoms can be described by the $3N$-dimensional vector $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_N)$, where $\mathbf{x}_i$ are the positions of the atoms. Considering deviations from equilibrium for each atom and defining $\boldsymbol{\xi}_a = \mathbf{x}_a - \mathbf{x}_a^0$ gives $\widetilde{\mathbf{X}} = (\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_N)$. The energy associated with the small deviations of the molecule is then

$$H = \frac{1}{2} m_\alpha (\partial_t \widetilde{X}_\alpha)^2 + \frac{1}{2} K_{\alpha\beta} \widetilde{X}_\alpha \widetilde{X}_\beta.$$

where $\alpha, \beta = 1, \ldots, 3N$. We may rescale $\boldsymbol{\xi}_a \to \boldsymbol{\xi}_a/\sqrt{m_a} = \boldsymbol{\lambda}_a$, and then orthogonalize $K'_{\alpha\beta}$ as it's symmetric; we then have the equation **(PvN's equation is wrong with index contractions, fix later)**

The physical interpretation of this is then clear: normal modes are the eigenvalues of **(finish after fixing his equation)**

There are three ingredients we need to consider to represent our atom using representation theory:

- We need **two** representations. One of them is a permutation representation, $A \colon G \to \mathrm{Aut}(V)$, and the other is a rotation representation, $R \colon G \to D_n \subset SO(3)$.
- We need **one** group action, $\pi \colon G \times G \to G$, which acts to permute the labels of our atoms, s.

We then need to consider the characters of these representations, and we need to find out how many irreps are contained within them. The use of the group action is that it just lets our representations talk to each other via projections through the following equation:

$$\boxed{\mathcal{P}_{\pi_s} A_g \widetilde{\mathbf{X}} = D(g) \mathcal{P}_s \widetilde{\mathbf{X}}.}$$

In words, this tells you that permuting atoms and rotating them is the same thing[3].

[3] PvN remarked upon this multiple times while showing us the demonstration that "To half of you this will be incredibly obvious, and half of you will not understand this no matter how many times I say it. Now you know if you're a physicist or a mathematician, no one told you before but now you know." **(fix quote)**

Consider the character for rotations, $\chi_{\mathbb{R}^3}$. This is related to the genuine normal modes by

$$\boxed{\chi_{\text{gen}} = \chi_S - \chi_{\text{tr}} - \chi_{\text{rot}} = \chi_{\mathbb{R}^3}(c_g - 1 - \det D),}$$

so $\chi_S = c_g \chi_{\mathbb{R}^3}$, $\chi_{\text{tr}} = \chi_{\mathbb{R}^3}$, taking all of the possible modes and subtracting out the trivial ones. It is only slightly reductionistic to say that this is the most important equation in this entire analysis. Our entire goal will be to find χ_{gen} and then take inner products of it with our irrep characters to find how many irreps are contained in the genuine motion.

(explain why the relations between the rotations and the other trivial modes are true)

The general process for determining the normal modes of a molecule goes as follows:

1. Analyze the number of normal modes using the formulae $n = 2N - 3$, $n = 3N - 6$.
2. Determine the symmetry group of the molecule.
3. Find the classes of the group and their orders.
4. Find the commutator subgroup of the group.
5. Find the dimensions of the irreps.
6. Find the character table for the irreps
7. Find χ_{tr}, χ_{rot}, and χ_S through $\chi_{\mathbb{R}^3}$.
8. Find χ_{gen}.
9. Take inner products of the normal modes to determine the decomposition of χ_{gen}.
10. Write the motion as a direct sum of irreps.
11. Using symmetry considerations, determine the physical realization of the normal modes.

We now show a number of canonical examples, answering the standard questions for each molecule.

Example 1.7 (Triangle)

We have:

1. The symmetry group is D_3.
2. The classes are $\{e\}$, $\{(12)\}$, $\{(123)\}$. Their orders are $1 + 3 + 2 = 6$.
3. The commutator subgroup is A_3.
4. There are 2 $1D$-irreps and 1 $2D$-irrep.
5. The character table is

	e	(12)	(123)
	1	3	2
id	1	3	2
sgn	1	-1	1
χ^2	2	0	-1
χ_{gen}	2	0	2

6. We can use the formula $\chi_{\text{gen}} = \chi^2(c_g - 1 - \det D)$, to get the last row of the table above.

7. Taking inner products gives

$$\begin{aligned}(\chi_{\text{gen}}, \text{id}) &= \frac{1}{6}(2+4) = 1,\\ (\chi_{\text{gen}}, \text{sgn}) &= \frac{1}{6}(2+4) = 1,\\ (\chi_{\text{gen}}, \chi^2) &= \frac{1}{6}(4-4) = 0.\end{aligned}$$

8. So our motion is

$$\boxed{\Omega = A_1 \oplus A_2.}$$

9. This corresponds to the breather and the stretcher.

Example 1.8 (Square)

We have:

1. There are $2N - 3 = 5$ non-trivial planar normal modes, and $3N - 6 = 6$ non-trivial 3D normal modes.

2. The symmetry group is D_4.

3. The classes are $\{e\}$, $\{r, r^3\}$, $\{r^2\}$, $\{\sigma r, \sigma r^3\}$, $\{\sigma, \sigma r^2\}$. Their orders are obvious.

4. The commutator subgroup is $\{e, r^2\}$.

5. There are 4 $1D$-irreps and there is 1 $2D$-irrep.

6. The character table is

	e	r^2	r^{2k+1}	σr^2	σr^{2k+1}
	1	1	2	2	2
id	1	1	1	1	1
sgn	1	1	-1	1	-1
χ^1	1	1	-1	1	-1
χ^2	1	1	-1	-1	1
χ_{2D}	2	-2	0	0	0
χ_{gen}	4	4	0	0	0

7. See above, use $\chi_{\text{gen}} = \chi_{2D}(c_g - 1 - \det D)$.

8. Taking inner products gives

$$\begin{aligned}(\chi_{\text{gen}}, \text{id}) &= \frac{1}{8}(4+4) = 1,\\ (\chi_{\text{gen}}, \text{sgn}) &= \frac{1}{8}(4+4) = 1,\\ (\chi_{\text{gen}}, \chi^1) &= \frac{1}{8}(4+4) = 1,\\ (\chi_{\text{gen}}, \chi^2) &= \frac{1}{8}(4+4) = 1,\\ (\chi_{\text{gen}}, \chi_{2D}) &= \frac{1}{8}(8-8) = 0.\end{aligned}$$

9. So our motion is

$$\boxed{\Omega = A_1 \oplus A_2 \oplus A_3 \oplus A_4.}$$

10. This corresponds to the breather, the (out-of-plane) twister, the (x-y) stretcher, and the (diagonal) shearer.

Example 1.9 (Tetrahedron)

We have:

1. The symmetry group is S_4.

2. The classes are $\{e\}$, $\{(12)\}$, $\{(123)\}$, $\{(12)(34)\}$, and $\{(1234)\}$. Their orders are $1+6+6+3+8 = 24$.

3. The commutator subgroup is A_4.

4. There are 2 $1D$-irreps, 1 $2D$-irrep, and 2 $3D$-irreps.

5. The character table is

	e 1	(12) 6	(123) 6	$(12)(34)$ 3	(1234) 8
id	1	1	1	1	1
sgn	1	-1	1	1	-1
χ^2	2	0	-1	2	0
χ^3	3	1	0	-1	-1
χ^3_{sgn}	3	-1	0	-1	1
χ_{gen}	6	2	0	2	0

Some explanation for the table:

- Identity and sign are obvious.
- χ^2 is difficult to find. You get it through orthogonality with the other rows, as there are four equations and four unknowns.
- χ^3 is given through finding $1 + 2\cos\theta$ for each entry, with 1 for reflections.
- χ^3_{sgn} is $\chi^3\,\text{sgn}$.

6. We can use the formula $\chi_{\text{gen}} = \chi^3(c_g - 1 - \det D)$ to fill in the last row of the table, as given above.

7. Taking inner products, we have

$$\begin{aligned}
(\chi_{\text{gen}}, \chi^2) &= \frac{1}{24}(6 + 12 + 0 + 6 + 0) = 1, \\
(\chi_{\text{gen}}, \chi^2) &= \frac{1}{24}(12 + 0 + 0 + 12 + 0) = 1, \\
(\chi_{\text{gen}}, \chi^3) &= \frac{1}{24}(18 + 12 + 0 - 6 + 0) = 1, \\
(\chi_{\text{gen}}, \chi^3_{\text{sgn}}) &= \frac{1}{24}(18 - 12 + 0 - 6 + 0) = 0.
\end{aligned}$$

8. Thus, our motion is

$$\boxed{\Omega = A_1 \oplus A_2 \oplus A_3.}$$

9. Physically, A_1 is the breather, A_2 is the four pumpers (along and against a line), and A_3 is the three twisters (twisting any two edges in the opposite way).

Example 1.10 (Triangle with dot)

We have

- There are $2N - 3 = 8 - 3 = 5$ non-trivial planar normal modes.
- The symmetry group is D_3.
- The classes are $\{e\}$, $\{r, r^2\}$, $\{\sigma, r\sigma, r^2\sigma\}$.

- The commutator subgroup is A_3.
- There are 2 $1D$-irreps and 1 $2D$-irrep.
- The character table is

	e 1	r 2	σr 3
id	1	1	1
sgn	1	1	-1
χ^2	2	-1	0
χ_{gen}	8	-1	0

- One A_1, two A_3's.
- Breather and two doublets.

2 Lie groups and algebras

We now turn our attention to continuous groups and their algebras. Let us first recall some definitions.

Definition 19. An ***R*-module** is an abelian group with a scalar multiplication on it that respects addition.

Definition 20. An ***R*-algebra** is an R-module with a bilinear map on it.

Specializing $R = k$ (a field), an R-module turns into a k-vector space, and an R-algebra a k-algebra. We will almost exclusively work over k-algebras in these notes. To turn a k-algebra into a Lie algebra, we require that the bilinear map on it satisfies two properties.

Definition 21. A **Lie algebra**, $\mathfrak{g}$, is a k-algebra whose multiplication satisfies $[x, x] = 0$ for all $x \in \mathfrak{g}$, and that satisfies the Jacobi identity, $\sum_{\text{cyc}}[x, [y, z]] = 0$.

The basis of a Lie algebra are called its **generators**, T_a. So, by assumption, there exists some coefficients in the base field such that

$$[T_a, T_b] = \sum_{c=1}^{\dim \mathfrak{g}} f_{ab}{}^{c} T_c.$$

These are called the **structure constants** of the Lie algebra. A nice example of a Lie algebra is $\mathbb{R}^3$ with $[\mathbf{x}, \mathbf{y}] = \mathbf{x} \times \mathbf{y}$. If the generators are $T_a = \mathbf{e}_a$, then the structure constants are $f_{abc} = \epsilon_{abc}$.

If some set of elements of $\mathfrak{g}$ forms a Lie algebra, we say it's a **Lie subalgebra**. This is equivalent to saying that $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$, for $\mathfrak{h} \subset \mathfrak{g}$. Furthermore, if the algebra is invariant under multiplication by $\mathfrak{g}$, then we say it's an **ideal**; concretely, an ideal is a Lie subalgebra that satisfies $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$. This should all be very reminiscent of normal subgroups for groups, and ideals for rings.

There are only two kinds of Lie algebras that we really care about in physics: abelian and simple Lie algebras. A Lie algebra is **abelian** if $[\mathfrak{g}, \mathfrak{g}] = 0$, and **simple** if it's not abelian and has no non-trivial ideals.

3 Highest weight decomposition

How to find Dynkin labels of irreps:

- $SU(N)$: this is the easiest case. The Dynkin labels are just the differences in the length of the rows of the Young Tableaux corresponding to the irrep. For example, the adjoint in $SU(3)$ is two boxes, then one box, then no boxes. The differences are $2-1=1$, and $1-0=1$, so the Dynkin label for the adjoint of $SU(3)$ is $(1,1)$.
- For general Lie Algebras: let the weight be $\boldsymbol{\zeta} = c_i \boldsymbol{\mu}^i_{\text{FW}}$. Then the Dynkin labels are

$$\boxed{a_i = 2\frac{(\boldsymbol{\zeta}, \boldsymbol{\alpha}^I)}{(\boldsymbol{\alpha}^I, \boldsymbol{\alpha}^I)}.}$$

  Note that this instantly tells you the Dynkin labels of the fundamental weights; just δ^K_I

Fact: the highest weight of an irrep is

$$\boxed{\boldsymbol{\zeta}_{\text{HW}} = a_i \boldsymbol{\mu}^i_{\text{FW}}.}$$

Examples:

- The highest weight of the fundamental and anti-fundamental reps of $SU(3)$ are $\boldsymbol{\mu}^1_{\text{FW}}$ and $\boldsymbol{\mu}^2_{\text{FW}}$, as we'd expect from the formula.
- The highest weight of the adjoint of $SU(3)$ is indeed $\boldsymbol{\mu}^1_{\text{FW}} + \boldsymbol{\mu}^2_{\text{FW}}$.

Spinor decompositions. Spinors decompose as

$$\psi^\alpha \chi^\beta = \sum_k a_k (\gamma^{\mu_1 \cdots \mu_k} C^{-1})^{\alpha\beta} (\psi^T C \gamma_{\mu_1 \cdots \mu_k} \chi).$$

The process for finding the Clebsch-Gordon decomp of these spinors is

- Find the structure of the charge conjugation matrix. If it's diagonal, then only even terms contribute. If it's off-diagonal, then only odd terms contribute. The terms go until N for $SO(N)$. You *flip* this for chiral spinors: for a BOD c.c. matrix, only *even* terms contribute, and for a block diagonal c.c. matrix only *odd* terms contribute.
- The dimensions of each term is $\binom{N}{k}$. If $k = N/2$, then you need another factor of $1/2$ because things split into self-dual and anti-self dual parts.
- Really important fact: in general, the *Dirac* spinor irrep has dimension 2^N for $SO(N)$, but the chiral spinor irreps have dimension 2^{N-1} for $SO(2N)$.

The Cartan-Weyl basis in terms of the Gell-Mann matrices:

$$\lambda_{1,2,3} = \begin{pmatrix} \sigma_{1,2,3} & 0 \\ 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$
$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

H_i and E_{jk}. In part (b) we already used that

$$T_3 = H_1, \quad T_8 = H_2. \tag{2.19}$$

For the other ones we have

$$T_1 = \frac{1}{2}\left(E_{12} + E_{21}\right), \quad T_2 = \frac{\imath}{2}\left(E_{21} - E_{12}\right), \quad T_4 = \frac{1}{2}\left(E_{13} + E_{31}\right), \tag{2.20}$$

$$T_5 = \frac{\imath}{2}\left(E_{31} - E_{13}\right), \quad T_6 = \frac{1}{2}\left(E_{23} + E_{32}\right), \quad T_7 = \frac{\imath}{2}\left(E_{32} - E_{23}\right). \tag{2.21}$$

The two diagonal ones,

$$\lambda_3 = \mathrm{diag}(1, -1, 0), \qquad \lambda_8 = \frac{1}{\sqrt{3}}\mathrm{diag}(1, 1, -2),$$

are the standard Cartan generators.

3. Raising/lowering operators

You can form combinations that correspond to the six roots:

$$E_{\alpha_1} = \frac{1}{2}(\lambda_1 + i\lambda_2), \quad E_{\alpha_2} = \frac{1}{2}(\lambda_6 + i\lambda_7), \quad E_{\alpha_1+\alpha_2} = \frac{1}{2}(\lambda_4 + i\lambda_5),$$

and their conjugates.

$d_{abc} = \mathrm{tr}(\{\lambda_a, \lambda_b\}\lambda_c)$ for $SU(3)$.

The raising operators are E_{12}, E_{13} and E_{23}. When acting with $C_2(R)$ on the highest weight state the only non-zero contributions are coming from the squares of the Cartan generators H_i^2 and the combinations of E_{jk} that have raising operators on the left and lowering on the right:

$$E_{12}E_{21}, \quad E_{13}E_{31}, \quad E_{23}E_{32}. \tag{2.22}$$

In other words we have

$$E_{12}^{(R)}|\vec{\mu}_{HW}, R_{\boldsymbol{q}}\rangle = E_{13}^{(R)}|\vec{\mu}_{HW}, R_{\boldsymbol{q}}\rangle = E_{23}^{(R)}|\vec{\mu}_{HW}, R_{\boldsymbol{q}}\rangle = 0. \tag{2.23}$$

Using the three positive roots of $SU(3)$

$$\vec{\alpha}^{12} = \vec{\alpha}_1 = (1,0)\,, \quad \vec{\alpha}^{23} = \vec{\alpha}_2 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \quad \vec{\alpha}^{13} = \vec{\alpha}_3 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \tag{2.24}$$

we compute

$$T_1^2 + T_2^2 = \frac{1}{2}\left(E_{12}E_{21} + E_{21}E_{12}\right) = E_{21}E_{12} + \vec{\alpha}_1 \cdot \vec{H}, \tag{2.25}$$

$$T_4^2 + T_5^2 = E_{31}E_{13} + \vec{\alpha}_3 \cdot \vec{H}, \tag{2.26}$$

$$T_6^2 + T_7^2 = E_{32}E_{23} + \vec{\alpha}_2 \cdot \vec{H}. \tag{2.27}$$

Now we are ready to act on the highest weight state:

$$C_2(R)\,|\vec{\mu}_{HW}, R_{\boldsymbol{q}}\rangle = \left(\vec{\mu}_{HW} \cdot \vec{\mu}_{HW} + \sum_{i=1}^{3} \alpha_i \cdot \vec{\mu}_{HW}\right)|\vec{\mu}_{HW}, R_{\boldsymbol{q}}\rangle. \tag{2.28}$$

This leads to the following answer:

$$C_2(R) = \frac{1}{3}\left(q_1^2 + q_2^2 + q_1 q_2 + 3\,q_1 + 3\,q_2\right) \mathrm{I}_{\dim R \times \dim R}. \tag{2.29}$$

As can be seen from the above expression, the quadratic Casimir operator does not distin-

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guish between any given representation and its complex conjugate (it is symmetric under the permutation of q_1 and q_2).

Solution: First, we need to figure out how the elements of $\mathbb{Z}_3$ are generated from the Lie algebra. Since the group elements z_k are proportional to the identity matrix, they can only be generated by the Cartan subalgebra. In the defining representation the two Cartan generators are proportional to the Gell-Mann matrices λ_3 and λ_8. Thus, we consider the following group elements:

$$U_3(\phi) = \exp(\imath\phi\lambda_3) = \begin{pmatrix} e^{\imath\phi} & 0 & 0 \\ 0 & e^{-\imath\phi} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.10}$$

$$U_8(\phi) = \exp(\imath\phi\lambda_8) = \begin{pmatrix} e^{\imath\phi/\sqrt{3}} & 0 & 0 \\ 0 & e^{\imath\phi/\sqrt{3}} & 0 \\ 0 & 0 & e^{-2\imath\phi/\sqrt{3}} \end{pmatrix}. \tag{2.11}$$

The group element $U_3(\phi)$ is proportional to the identity matrix only when $\phi = 0 + 2\pi n$, $n \in \mathbb{Z}$. For the other group element we have

$$U_8(0) = \mathrm{I}_{3\times 3}, \quad U_8\left(\frac{2\pi}{\sqrt{3}}\right) = e^{2\pi\imath/3}\,\mathrm{I}_{3\times 3}, \quad U_8\left(\frac{4\pi}{\sqrt{3}}\right) = e^{4\pi\imath/3}\,\mathrm{I}_{3\times 3}. \tag{2.12}$$

Thus, the group elements of $\mathbb{Z}_3$ are given by

$$k = 0, 1, 2: \quad z_k = \exp(\imath\phi_k\lambda_8) = \exp(2\,\imath\,\phi_k T_8), \tag{2.13}$$

where $\phi_k = \frac{2\pi k}{\sqrt{3}}$. In any given representation R the generator T_8 is replaced by $T_8^{(R)}$ and

$$z_k^{(R)} \propto \mathrm{I}_{\dim R\times\dim R}. \tag{2.14}$$

The coefficient in front of the identity matrix can be determined by acting with $z_k^{(R)}$ on the highest weight state $|\vec{\mu}_{HW}, R_{\boldsymbol{q}}\rangle$. We use (2.1) to get

$$T_8^{(R)}|\vec{\mu}_{HW}, R_{\boldsymbol{q}}\rangle = \frac{1}{2\sqrt{3}}(q_1 + 2\,q_2)\,|\vec{\mu}_{HW}, R_{\boldsymbol{q}}\rangle, \tag{2.15}$$

which gives

$$z_k^{(R)}|\vec{\mu}_{HW}, R_{\boldsymbol{q}}\rangle = \exp\left(\imath\,\frac{2\,\pi\,k}{3}(q_1 + 2\,q_2)\right)|\vec{\mu}_{HW}, R_{\boldsymbol{q}}\rangle. \tag{2.16}$$

Taking the trace of the identity matrix in a given representation $R_{\boldsymbol{q}}$ leads to

$$\chi_{\boldsymbol{q}}(z_k) = \dim R_{\boldsymbol{q}}\, e^{2\pi\imath k(q_1+2\,q_2)/3}, \quad k = 0, 1, 2. \tag{2.17}$$

A useful isomorphism is that

$$\boxed{U(N) \cong \frac{SU(N)\times U(1)}{\mathbb{Z}_N}.}$$

One can raise and lower indices using the epsilon symbol in $SU(N)$.
Young tableaux facts:

- The dimension of a totally antisymmetric tensor is $\binom{N}{k}$, a totally symmetric tensor is $\binom{N+k-1}{k}$, and a totally symmetric and traceless tensor is $\binom{N+k-1}{k} - \binom{N+k-3}{k-2}$.