Lecture Notes on

Groups and Representations

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Notes on group and representations, as used in physics. Resources used:

- Peter van Nieuwenhuizen's PHY 680 course at SBU, and the lecture notes from the course. This is the primary source of content; I just explain things in my own way, and make things more formal and precise.
- Andre Lukas' lecture notes on groups and representations.
- Fulton and Harris' *Representation Theory: A First Course*.
- Georgi's *Lie Algebras in Particle Physics*.

Contents

1 Finite Groups and Representations

We begin by collecting the most important facts about the representation theory of finite groups. At the end, we touch on Majorana spinors and the normal modes of atomic molecules. For general background, see the notes on algebra here.

1.1 Finite groups

Example 1.1 (The most important finite groups for physics)

Of course, the most important groups for physics are (semi-simple) Lie groups, but there are a few important finite ones as well:

- S_n. This group is good for talking about permutations of objects, and is necessary for talking about group actions.
- A_n. See above, but sometimes we only want even permutations.
- D_n. Because D_n is, by definition, the group of isometries of a planar n-point object, this is useful to look at when studing planar objects. We can also think about certain continuous groups as limits of D_n, e.g. emergent $O(N)$ symmetry in spin systems.
- $\mathbb{Z}/n\mathbb{Z}$. These groups are useful for a number of things. $\mathbb{Z}/2\mathbb{Z}$ is parity, and modding out by it also produces quite a few useful Lie groups (e.g. $SU(2)/\mathbb{Z}/2\mathbb{Z} \cong SO(3)$). It also catalogues cyclic groups, so it's useful when we want to think about those ideas. Additionally, these groups are closely connected to number theory when $n = p$ is prime, and there are more and more connections between physics and number theory as time goes on...

We now get into the most important definitions behind finite group theory for physics.

Definition 1. Consider $\gamma_g : G \to G$ by $\gamma_g(a) = gag^{-1}$. We say γ_g is an **inner automorphism** of G, and we denote the set of γ_g's by $\mathrm{Inn}_{\mathsf{Grp}}(G)$.

We may talk about the orbit of some a under the elements of $\mathrm{Inn}_{\mathsf{Grp}}(G)$; this is called the **class** of a. The practical algorithm for producing classes is as follows:

1. Take some $a \in G$. Consider gag^{-1} for all g in g.
2. If gag^{-1} is not already in your set, add it to the set.
3. Once you are done, take some $b \in G$ not in the set and repeat these steps.

This algorithm terminates, as conjugation is an equivalence relation, so the classes of G partition it.

Definition 2. The **center** of a group is defined as $\mathcal{Z}(G) := \{z \in G \mid zg = gz \ \forall g \in G\}$.

Intuitively, the center of a group tells you how abelian the group is. Indeed, we have that $\mathcal{Z}(G) = G \iff G$ is abelian. We may ask the question of how to take some non-abelian group and "turn it into" an abelian group. The only natural mechanism for this would be modding out by a subgroup. As it turns out, the smallest subgroup that makes this possible is the commutator subgroup.

Definition 3. The **commutator subgroup** $[G,G]$ of G is defined as the group containing the elements $aba^{-1}b^{-1}$ for all $a,b \in G$.

Definition 4. The **abelianization** of G is defined as $G^{\text{ab}} = G/[G,G]$.

Note that every map $\varphi\colon G \to A$, where A is abelian, factors through the abelianization. This is true, as $\varphi([a,b]) = e$, so $[G,G] \subset \ker\varphi$. Thus the following diagram commutes.

$$\begin{tikzcd} G \arrow[rr, "\varphi"] \arrow[dr, two heads, "\pi"'] & & A \\ & G^{\text{ab}} \arrow[ur, "\widetilde{\varphi}"'] & \end{tikzcd}$$

where $\pi\colon G \twoheadrightarrow G^{\text{ab}}$ is the canonical projection and $\widetilde{\varphi}$ is the unique map $\widetilde{\varphi}\colon G^{\text{ab}} \to A$. We now prove an obvious but important theorem on the abelianization of G.

Theorem 1. $[G,G]$ is the smallest normal subgroup of G such that G/N is abelian.

Proof. We first show that G^{ab} is abelian. Note that

$$[G,G] = b^{-1}a^{-1}ba[G,G] \implies ba[G,G] = ab[G,G]$$

so G^{ab} is abelian. Consider $\phi\colon G \to G/N$, where G/N is an abelian quotient of G. Then this map factorizes through G, and there is a surjection $G^{\text{ab}} \twoheadrightarrow G/N$, so $|G^{\text{ab}}| \geq |G/N|$.

Kind of how like the center measured how abelian a group is, the commutator subgroup measures how "non-abelian" it is. We can see that $[G,G] = 0$ for abelian groups, and if $[G,G] = G$ then we say G is **perfect**. Equivalently, a perfect group has no non-trivial abelian quotients. An interesting fact is that all non-abelian simple groups are perfect (e.g. A_5).

Another way to look at this is to notice that abelianization is a functor $F\colon \mathsf{Grp} \to \mathsf{Ab}$ by $G \to G^{\text{ab}}$, and that it's the left adjoint to inclusion $\iota\colon \mathsf{Ab} \hookrightarrow \mathsf{Grp}$, so

$$\mathrm{Hom}_{\mathsf{Ab}}(F(G), A) \cong \mathrm{Hom}_{\mathsf{Grp}}(G, \iota(A)).$$

This encapsulates the universal property nicely: every map from $G \to A$ corresponds to a unique map $G^{\text{ab}} \to A$.

Idea 1.1 (Most important facts about finite groups)

Whenever you see a finite group, you want to answer these questions:

- **What is the order of the group?** I find this question to be psychologically comforting, and it also tells me what the possible orders of the subgroups of G are.
- **What are its classes?** The reason this is useful is because characters are class functions, and so there are a number of useful facts we can draw from knowing the classes of a group.
- What are the orders of its classes? This is less important, but useful if one wants to check whether a map V is a representation or not.
- **What is its commutator subgroup?** As we saw above, the commutator subgroup is the smallest subgroup of G such that G/H is abelian. This tells us that 1D reps of G acts

on the abelianization of G, so $|G^{\mathrm{ab}}| = n_{1\,\mathrm{diml}}$

Some general algebraic constructions are also useful here for physics, namely products, group actions, and semi-direct products. For posterity, we recall the writings on these subjects from the algebra notes here.

The direct product group is the product in Grp. This means that by the universal property, for all $\varphi_G\colon A \to G$, $\varphi_H\colon A \to H$, there exists a unique map $\widetilde{\varphi}\colon A \to G \times H$ making the diagram

$$\begin{tikzcd} & & G \\ A \arrow[r, dashed, "\widetilde{\varphi}"] \arrow[urr, bend left, "\varphi_G"] \arrow[drr, bend right, "\varphi_H"'] & G \times H \arrow[ur, "\pi_G"'] \arrow[dr, "\pi_H"] & \\ & & H \end{tikzcd}$$

commute. Concretely, taking the binary operations on G and H and defining

$$\begin{gathered} m_G \times m_H\colon (G \times H) \times (G \times H) \to G \times H \\ (m_G \times m_H)((g_1, h_1),\, (g_2, h_2)) \mapsto (m_G((g_1, g_2)),\, m_H((h_1, h_2))). \end{gathered}$$

gives the set $G \times H$ a group structure

$$(g_1, h_1) * (g_2, h_2) = (g_1 g_2, h_1 h_2).$$

Now onto group actions. Recall that an **action** of a group on the object A of a category C is a homomorphism $\sigma\colon G \to \mathrm{Aut}_{\mathsf{C}}(A)$. Specializing $\mathsf{C} = \mathsf{Set}$, we have the standard definition of a group action

$$\begin{gathered} \rho\colon G \times A \to A \\ \rho(e_G, a) = a \qquad\qquad \rho(gh, a) = \rho(g, \rho(h, a)). \end{gathered}$$

This is reminiscent of the multiplication structure of Poincaré transformations, $U(\Lambda, a)U(\overline{\Lambda}, \overline{a}) = U(\Lambda\overline{\Lambda}, \Lambda\overline{a} + a)$. Writing ρ is annoying, so we abbreviate the above to simply $(gh)a = g(ha)$, so there is some sense of psuedo-associativity here. The semi-direct product follow naturally from this. (**finish**)

1.2 Representations of finite groups

Remark 1.1 (Why are groups so ubiquitous in physics?)**.** After seeing so much group theory in physics, one may begin to wonder if there is any motivation for *why* this is the case. Martin Roček gave the following nice explanation for why:

If you have some system, you can transform it in the following ways:

- You can do nothing to it.
- You can transform it.
- You can "un-transform" it by just doing whatever you did in reverse.
- You can transform it and then transform it again, and of course this is another transfor-

mation.

- If you do three transformations to something, it doesn't matter whether you do (12)3 or 1(23).

These are the axioms of a group. A nice example that I like to keep in mind is rotating a globe.

Some motivation for representation theory is that we know symmetries in nature can generally be given a group structure, but we tend to work with vector spaces in physics (e.g. Hilbert spaces, phase spaces, or configuration spaces[1]). How can we get a group to "act" on a vector space?

A natural choice may be a group action on Vect_k, but this cannot work; we need to do more than just permute the elements of our vector space: we need linear combinations of elements.

(is it just a group action on vect k?)

This naturally leads us to operators on V. So, we intuitively need some sort of map into $\operatorname{End} V$ that respects the group structure of G. This is what the definition of a representation encodes.

Definition 5. A **representation** of G on V is the pair (ρ, V), where $\rho\colon G \to GL(V)$ is a homomorphism and V a k-vector space.

Note 1.1 (Abuse of notation)**.** I will oftentimes refer to a representation by either ρ or V instead of (ρ, V).

Remark 1.2 (The importance of V in (ρ, V))**.** It is *vitally* important to note that a representation includes the vector space you're acting on in it. This point is not emphasized by physicists, but not realizing this causes a number of conceptual difficulties later. Just realize that when you say "representation", you are referring *both* to a homomorphism and to a vector space on which your operators act.

Note 1.2 (The categorical viewpoint)**.** Regard G as a one-object groupoid BG whose morphisms are the elements of G. Then we may define a finite representation as the *functor*

$$\rho\colon BG \to \mathsf{Vect}_k^{\mathrm{fd}},$$

sending the single object $G \to V$, and sending morphisms in g to operators on V, $g \to \varphi$.

As we can see from the categorical construction, this is a natural definition encoding everything that we want from V. We say that the **dimension** of a representation is the dimension of its underlying vector space, $\dim \rho = \dim V$. If we have structure on V, we can build up new representations from old ones by using this structure and translating it into the maps ρ:

- **Tensor representation**. We can tensor up two representations $V_1 \otimes V_2$ and their maps $\rho_1 \otimes \rho_2$ to get a new representation, $(\rho_1 \otimes \rho_2, V_1 \otimes V_2)$, where $\rho_1 \otimes \rho_2$ is naturally given by

$$(\rho_1 \otimes \rho_2)(v_1 \otimes v_2) = \rho_1(v_1) \otimes \rho_2(v_2).$$

  This is the **tensor representation** of $V_1 \otimes V_2$.

[1] It is not necessary that phase and configuration spaces are endowed with a linear structure, but it is often true that this is the case. See the notes on classical mechanics for more about this.

- **Dual representation**. If we have a metric on V, we can consider its dual space $V^\vee$. We'd like the duality to be G-invariant, so $\langle \varphi, v\rangle = \langle g^\vee \varphi, gv\rangle$. Renaming $gv = w$, we have

$$(g^\vee \varphi)(w) = \varphi(g^{-1}w) \implies (g^\vee \varphi)(v) = \varphi(g^{-1}v)$$

So, if $\varphi \in V^\vee$, we define the **dual representation** to be

$$(\rho^\vee(g)\varphi)(v) = \varphi(\rho(g^{-1})v) \iff (g^\vee\varphi)(v) = \varphi(g^{-1}v).$$

- **Direct sum representation**. If $V = U \oplus W$, with U and W representations, we can define $\rho_V(u \oplus W) = \rho_U(u) + \rho_W(w)$ to be the **direct sum representation** of U and W.

Note 1.3 (A refresher on the tensor product)**.** We will have to actually compute with the tensor product in these notes, so we recall the basic computational construction here:

If we have two matrices A and B, $A \otimes B$ is defined as

$$A \otimes B := \begin{pmatrix} A_{11}B & \cdots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{n1}B & \cdots & A_{nn}B \end{pmatrix}.$$

That is, multiplying every entry of A by B. Note that if A is an $n \times n$ matrix and B is an $m \times m$ matrix, $A \otimes B$ is an $nm \times nm$ matrix. An example is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}.$$

This construction will be necessary later when we talk about γ-matrices.

We can map representations to one another by **similarity transformations**, or **intertwiners**. This is equivalent to saying that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\rho"] & W \arrow[d, "\widetilde{\rho}"'] \\ V & W \arrow[l, "\varphi^{-1}"'] \end{tikzcd}$$

If there is an intertwiner between two reps, we say they're **equivalent**. This is written as $V \cong W$, or $\rho = \varphi^{-1} \circ \widetilde{\rho} \circ \varphi$[2]. If $V = W$ this is just a change of basis. Similar to how we find bases in vector spaces, we may ask if there is a "basis" for representations. The answer is yes, and these are called *irreducible representations*.

Definition 6. A **subrepresentation** $U \subset V$ is a subspace of V that satisfies $\rho|_U U \subset U$ for all $g \in G$. An **invariant subspace** is a subrepresentation that satisfies $\rho|_U = \mathrm{id}_U$.

Definition 7. A representation is **reducible** if it has an invariant subspace.

[2]Physicists usually write this as $M = SAS^{-1}$, where M and A are representations of G.

Definition 8. An **irreducible representation** is a representation that has no invariant subspaces.

Intuitively, a representation being reducible means it has some number of "off-diagonal" elements. Irreducible representations are nice, as we can build up every representation as a direct sum of them, which is the content of the next few propositions.

Definition 9. A representation is **completely reducible** if it may be written as the direct sum of irreps.

> **Remark 1.3** (Unitarity)**.** The next theorem is oftentimes presented under the assumption that V has a metric. It is a psychological problem amongst physicists to assume that every vector space have a metric, while in fact this is not always true[a]. Giving a vector space an unnatural metric is always almost a mistake, and is morally incorrect.
>
> [a] It is a reasonable assumption to make, though, as all Hilbert spaces necessarily have metrics, and physicists almost exclusively work in Hilbert spaces.

Theorem 2. Maschke's Theorem
All finite representations are completely reducible.

Proof. Let $U \subset V$ be an invariant subspace of V, and consider the projector $\pi_U \colon V \to U$. Average π_u over G by defining

$$\pi(v) := \frac{1}{|G|} \sum_g g\pi_u(g^{-1}v).$$

Recall that $V = \operatorname{Im}\pi \oplus \ker\pi$, where π is a projector. We claim $V = U \oplus \ker\pi$. Consider $\pi(u)$

$$\pi(u) = \frac{1}{|G|} \sum_g g\pi_u(g^{-1}u) = u.$$

So $\operatorname{Im}\pi = U$, and thus $W := \ker\pi$ has the right dimension. It is clear that $\pi^2 = \pi$. Note that W is G-invariant, as

$$\pi(hv) = \frac{1}{|G|} \sum_g g(\pi_u(g^{-1}hv)) = \frac{1}{|G|} \sum_{g'} hg'\pi_u((g')^{-1}v) = h\pi(v).$$

So, $\pi(gw) = g\pi(w) = 0$, and thus W is G-invariant. □

> **Note 1.4** (The "group-averaging" trick)**.** The trick used in the previous proof is a common one called "group averaging". It basically means that we can make any operator A G-invariant by defining its averaged version,
>
> $$\overline{A} := \frac{1}{|G|} \sum_g gA.$$
>
> Then clearly $h\overline{A} = \overline{A}$, which we can see by re-indexing.

Theorem 3. All finite representations can be made unitary by a similarity transformation.

Proof. Let $(\bullet,\bullet)_0$ be an inner product on V. Construct a new inner product

$$(v,w) := \sum_g (\rho(g)v, \rho(g)w)_0.$$

Then V is unitary with respect to $(\bullet,\bullet)$. Explicitly,

$$(\rho(g)v,\rho(g)w) = \sum_{g'}(\rho(g')\rho(g)v,\rho(g')\rho(g)w)_0 = \sum_{g'}(\rho(g'g)v,\rho(g'g)w)_0 = \sum_{\tilde{g}}(\rho(\tilde{g})v,\rho(\tilde{g})w)_0 = (v,w).$$

□

Note 1.5 (The "re-indexing" trick)**.** The trick used in the previous proof is a common strategy. It's called "re-indexing", and it consists of noticing that if you sum over a group then it doesn't matter what element you use. This is effectively just a group "u-sub", e.g. like in QFT when we switch the measure $\ell - kx \to \ell$ because we're integrating over all space. In the immortal words of Georgi,

> "... where the last line follows because hg runs over all elements of G when h does. QED."

Theorem 4. Schur's Lemma
Let V and W be irreps of G, and $\varphi\colon V \to W$ a linear map such that $\varphi \circ \rho_V = \rho_W \circ \varphi$, then

(a) Either φ is an isomorphism or zero.

(b) If $V = W$, then $\varphi = \lambda\,\mathrm{id}$ for some $\lambda \in \mathbb{C}$.

Proof.

(a) The first claim follows from the fact that $\ker\varphi$ and $\operatorname{Im}\varphi$ are invariant subspaces; we can see that for $v \in \ker\varphi$, $\varphi(\rho_V(g)v) = \rho_W(g)\varphi(v) = 0$, and similarly for $\operatorname{Im}\varphi$. Since these invariant subspaces must be trivial by assumption, the claim follows.

(b) If our field is algebraically closed, then ϕ has an eigenvalue; call it $\lambda \in k$. Then $\phi - \lambda\,\mathrm{id}$ must have non-zero kernel, and by (a), $\phi - \lambda\,\mathrm{id} \equiv 0$, so $\phi = \lambda\,\mathrm{id}$.

□

Corollary 1. We may uniquely decompose any representation as a direct sum of irreducible representations

$$V = \bigoplus_i V_i^{\oplus a_i}$$

1.3 Applications

Dirac, Majorana, and Weyl spinors

Remark 1.4 (Why do we care about spinors?). It turns out that spin-1/2 particles transform in the spinor representation of the Poincaré group; we call the particles that transform like this *spinors*. We can impose two broad classes of constraints on spinors: *reality* conditions and *chirality* conditions. The latter is imposed through the so-called *chirality matrix*, which we can construct from the Dirac matrices we will study below.

Example 1.2 (A motivating example)

When we're working with spin-1/2 massive particles, we naturally work in the $j = 1/2$ representation of the little group. So we want to find some $S^{\mu\nu}$ such that

$$D^{(1/2)}(W) = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}},$$

and $[S^{\mu\nu}, S^{\rho\sigma}]$ satisfies the same commutation relation as the $J^{\mu\nu}$'s. As it turns out, if you have matrices γ^μ such that $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, then we *can* construct such an S: it's given by

$$\boxed{S^{\mu\nu} := -\frac{i}{4}[\gamma^\mu, \gamma^\nu].}$$

We now study the properties of these γ-matrices below.

To start our analysis of the γ-matrices, we first need to talk about the Clifford algebra. Recall the definition of an R-algebra.

Definition 10. An **R-algebra** is (finish)

Definition 11. The complex **Clifford algebra** is a $\mathbb{C}$-algebra spanned by vectors e_i such that $\{e^i, e^j\} = 2\eta^{ij}$. **(or something like this)**

(add general definition, and universal property) (then add physicist definition, which is what is useful for our purposes)

Definition 12. The **Dirac group** is a multiplicative subgroup of the Clifford algebra given by

$$\mathrm{Dir}(n) = \{\pm\gamma^{\mu_1}\cdots\gamma^{\mu_n} \mid \mu_i < \mu_{i+1},\ \ \gamma^{\mu_i} \in \mathrm{Cl}_n(\mathbb{C})\}.$$

Important note: a priori, the γ-elements in $\mathrm{Dir}(n)$ are *not* matrices. This is a point that is touched on in the next remark. We may think of them as matrices, though.

Remark 1.5 (What are the γ's?). It is often stated that the Dirac group is spanned by the product of "γ-matrices". This is *false*; the elements of $\mathrm{Dir}(n)$ are just symbols. There is no inherent meaning to them. Indeed, that is what representation theory is for; giving them practical meaning.

Some points of contention with this:

- "But there is a μ index, so it looks like a vector": This is just a label for the group elements. μ ranges from $\mu = 0, \ldots, n-1$ or $\mu = 1, \ldots, n$ for Minkowski and Euclidean signature respectively. This is just to concisely label the abstract elements of the group with one index instead of explicitly writing out commutators for each element.

- "But $\eta^{\mu\nu}$ is a *matrix*, so $\gamma^\mu\gamma^\nu$ must *also* be a matrix": $\eta^{\mu\nu}$ is *NOT* a matrix here. It is just useful shorthand for expressing $\{\gamma^0, \gamma^0\} = -2I$ and $\{\gamma^a, \gamma^a\} = 2I$. You are free to arrange each component of $\eta^{\mu\nu}$ into a matrix, but a priori this is *not true*, and it leads to the wrong idea behind what the γ's are.

The takeaway from this is that the Clifford algebra and the Dirac group are spanned by abstract symbols which we happen to call γ^μ; these are *not* matrices yet.

However, one may think about the γ's as being $n \times n$ matrices even in the group, with composition given by matrix multiplication. This is simply because we may identify $\eta^{\mu\nu}$ on the RHS with the metric, which is indeed an $n \times n$ matrix. We would then naturally define the representation of the γ-matrices with the standard embedding representation, $\mathrm{Dir}(n) \hookrightarrow \mathrm{End}(V)$. An analogy to this is $SO(3)$, where its elements are naturally seen as 3×3 matrices which have to be embedded into $\mathbb{R}^3$, but we can also represent them as, say, spherical harmonics.

(I think it may be morally correct to view them as matrices tbh)

Some facts about the Dirac group:

- It has order 2^{n+1}. This is because of the skew-symmetry of the group, so for some product of k γ's, there are $\binom{n}{k}$ ways to do it, so summing k from $1 \to n$ and multiplying by two gives
$$2\left(\binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}\right) = 2(1+1)^n = 2^{n+1}.$$

- Its classes depend on the order of the group. This is just because of the last element; with even n, $\gamma^{\mu_1}\cdots\gamma^{\mu_n} \sim -\gamma^{\mu_1}\cdots\gamma^{\mu_n}$, but this is false with odd n. You can see this by counting swaps of γ's. So, there are $2^n + 1$ classes with n even and $2^n + 2$ classes with n odd. Note that $-e$ is in its own class.

- Its commutator subgroup is obviously $[\mathrm{Dir}(n), \mathrm{Dir}(n)] = \{\pm e\}$. This is because all γ's square to either 1 or -1, and you can move them around by picking up minus signs.

- The difference between Minkowski and Euclidean signature is in the sign of γ^2, as you can see from the definition. To do this, we simply add an i to as many matrices as we have time coordinates; note that this will generically add factors of i to γ_c.

We now get into representations of the Dirac group. This is where the fun[3] begins. As a psychological remark, this is a relatively straightforward but tedious process of bootstrapping our way to whatever dimension representation we desire. The boostrap goes as follows:

- **Even $\to$ odd**: we first construct the **chirality matrix** as follows: $\gamma_c = \alpha\gamma^1\cdots\gamma^n$, with $(\gamma_c)^2 = 1$. Note that $\{\gamma_c, \gamma^\mu\} = 0$ for all $\gamma^\mu \in \mathrm{Dir}(n)$. We then add either $+\gamma_c$ or $-\gamma_c$ to our set of matrices $\gamma^{(2n)}$, and this is our new representation, $\{\gamma^{(2n)}, \pm\gamma_c\}$. These are $2n \times 2n$ matrices.

- **Odd $\to$ even**: we tensor up all of the γ's with some sigma matrices. Concretely, given some representation $\gamma^{(2k+1)}$, our new representation in even dimensions is $\{\gamma^{(2k+1)} \otimes \sigma^1, I_{2k} \otimes \sigma^2\}$, or the same with $(\sigma^1, \sigma^2 \leftrightarrow \sigma^1, \sigma^3)$, or $(\sigma^1, \sigma^2 \leftrightarrow \sigma^2, \sigma^3)$. You can choose this flippantly, **(as it does not change anything)** , but if you want to retain reality properties you should not choose to tensor with σ^2. Note that these matrices are now $4k \times 4k$ in size.

[3] This is one way to describe it…

Example 1.3 (Bootstrapping our way to Heaven (***D* = 11** supergravity))

We start in $D = 2+0$. This is the easiest dimension to start in, as we instantly have the representation of

$$\gamma^1 = \sigma^1 \qquad\qquad \gamma^2 = \sigma^2.$$

Constructing the chirality matrix, we have $\gamma^3 = \alpha\gamma^1\gamma^2$, where $(\gamma^3)^2 = -\alpha^2(\gamma^1)^2(\gamma^2)^2 = -\alpha^2$, so WLOG $\alpha = -i$. This gives $\gamma^3 = \sigma^3$. So our rep in $D = 3+0$ is $\{I_2, \sigma^1, \sigma^2, \sigma^3\}$. Tensoring the identity with σ^1 and the γ's with $-\sigma^2$ gives

$$\gamma^a = -i\begin{pmatrix} 0 & \sigma^a \\ -\sigma^a & 0 \end{pmatrix} \qquad\qquad I_2 \otimes \sigma^1 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}.$$

Note that the identity here is $I_2 \otimes I_2$. If we want to switch from $D = 4+0 \to 3+1$, we simply add a factor of $\pm i$ to one of the matrices; it is traditional (at least in $D = 4$) to do it to $I_2 \otimes \sigma^1$, so in $D = 3+1$,

$$\gamma^a = -i\begin{pmatrix} 0 & \sigma^a \\ -\sigma^a & 0 \end{pmatrix} \qquad\qquad \gamma^4 = I_2 \otimes \sigma^1 = -i\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}.$$

We now consider the chirality matrix in $D = 4+0$. Taking the product gives

$$\gamma_c = \alpha\gamma^1\gamma^2\gamma^3\gamma^4 = \alpha(\sigma^1 \otimes \sigma^2)(\sigma^2 \otimes \sigma^2)(i\sigma^3 \otimes \sigma^2)(I^2 \otimes \sigma^1) = \alpha(i\sigma^1\sigma^2\sigma^3 I_2 \otimes \sigma^2\sigma^2\sigma^2\sigma^1) = i\alpha(I_2 \otimes \sigma^3).$$

Squaring this and demanding $(\gamma_c)^2 = 1$ gives

$$(\gamma_c)^2 = -\alpha^2(I_2 \otimes I_2) \implies \alpha = -i.$$

So $\gamma_c = \gamma^5 = (I_2 \otimes \sigma^3)$, and our irreps for $D = 5+0$ are $\{\gamma^{(4)}, \pm\gamma^5\}$. For $D = 6+0$, we now tensor up with σ^1 and σ^2 to get $\{\gamma^{(5)} \otimes \sigma^1, I_4 \otimes \sigma^2\}$. These are 8×8 matrices. Explicitly, these are given by

$$\gamma^a \otimes \sigma^1 = -i\begin{pmatrix} 0_4 & \sigma^a \otimes \sigma^1 \\ -\sigma^a \otimes \sigma^1 & 0 \end{pmatrix} \qquad\qquad \gamma^4 \otimes \sigma^1 = -i\begin{pmatrix} 0_4 & I_2 \otimes \sigma^1 \\ -I_2 \otimes \sigma^1 & 0_4 \end{pmatrix}$$

$$\gamma^5 \otimes \sigma^1 = \begin{pmatrix} \sigma^3 \otimes \sigma^1 & 0_4 \\ 0_4 & \sigma^3 \otimes \sigma^1 \end{pmatrix} \qquad\qquad \gamma^6 = I_4 \otimes \sigma^2 = \begin{pmatrix} \sigma^2 & & & \\ & \sigma^2 & & \\ & & \sigma^2 & \\ & & & \sigma^2 \end{pmatrix}.$$

For $D = 7+0$, the chirality matrix is

$$\gamma^7 = \alpha\gamma^1 \cdots \gamma^6 = \alpha(\gamma^1 \cdots \gamma^5 \otimes (\sigma^1)^5)(I_4 \otimes \sigma^2) = \alpha i((\gamma^5)^2 \otimes \sigma^3) = \boxed{\alpha i(I_4 \otimes \sigma^3),}$$

so $\alpha = -i$ and $\gamma^7 = I_4 \otimes \sigma^3$. So our irrep in $D = 7+0$ is $\{\gamma^{(6)}, \pm\gamma^7\}$. Tensoring $\gamma^{(7)}$ with σ^2 and I_8 with σ^1 gives

$$\gamma^{(8)} = \left\{\gamma^{(7)} \otimes \sigma^2, I_8 \otimes \sigma^1\right\}.$$

Note that these are 16×16 matrices. Constructing $\gamma_c = \gamma_9$ gives

$$\begin{aligned}\gamma_c = \gamma_9 &= \alpha(\gamma^1 \otimes \sigma^2) \cdots (\gamma^7 \otimes \sigma^2)(I_8 \otimes \sigma^1)\\ &= \alpha((\gamma^1 \cdots \gamma^6)\gamma^7) \otimes ((\sigma^2)^7 \sigma^1)\\ &= -\alpha i (I_8 \otimes \sigma^3) \implies \alpha = i,\\ &= \boxed{I_8 \otimes \sigma^3.}\end{aligned}$$

So our irreps for $D = 9 + 0$ are $\{\gamma^{(8)}, \pm\gamma^9\}$. Tensoring $\gamma^{(9)}$ with σ^1 and I_{16} with σ^3 gives

$$\gamma^{(10)} = \left\{\gamma^{(9)} \otimes \sigma^1, I_{16} \otimes \sigma^3\right\}.$$

These are 32×32 matrices. Constructing $\gamma_c = \gamma_{11}$ gives

$$\begin{aligned}\gamma_c = \gamma_{11} &= \alpha(\gamma^1 \otimes \sigma^1) \cdots (\gamma^9 \otimes \sigma^1)(I_{16} \otimes \sigma^3)\\ &= \alpha((\gamma^9)^2) \otimes ((\sigma^1)^9 \sigma^3)\\ &= \alpha i (I_{16} \otimes \sigma^2) \implies \alpha = -i,\\ &= \boxed{I_{16} \otimes \sigma^2.}\end{aligned}$$

So our irreps for $D = 11 + 0$ are $\{\gamma^{(10)}, \pm\gamma_{11}\}$, and we are done. Whew.

Idea 1.2 (Dirac vs. Majorana vs. Weyl vs. Majorana-Weyl spinors)

The representation of the Dirac group will depend on what kinds of spinors we would like to use the γ-matrices with. There are three (actually five) major distinctions here.

- **Dirac spinors**: these are just elements of the vector space $S \cong \mathbb{C}^{2^{\lfloor n/2 \rfloor}}$ of our representation.
- **Majorana spinors**: these are Dirac spinors with reality conditions imposed on them, namely that the Dirac conjugate of the spinor equals its Majorana conjugate.
- **Weyl spinors**: these are Dirac spinors with chirality conditions imposed on them; concretely, we construct the chirality matrix and impose eigenvalues under it. These are necessarily massless.
- **Majorana-Weyl spinors**: these are spinors that satisfy both the Majorana and Weyl conditions for spinors.
- **Sympletic Dirac spinors**: these are kind of the odd one out. In some sense these are "pseudo-Majorana spinors", in that you look for them if you can't have Majorana spinors in your space. Basically, if you have more than one spinor in some dimension, you can contract it with Ω_{ab} to get a "symplectic Majorana condition".

The taglines to remember are: "Dirac = normal", "Majorana = reality", and "Weyl = chiral".

We now get into the machinery behind these spinors.

Definition 13. A **Majorana spinor** is a spinor whose Dirac conjugate equals its Majorana conjugate. Recall ($\sigma = (1/2)n_t(n_t - 1) + 1$, where n_t is the number of time coordinates)

$$\overline{\lambda}_D := \lambda^\dagger i^\sigma \gamma^1 \cdots \gamma^{n_t} \qquad\qquad \overline{\lambda}_M := \lambda^T C.$$

So a Majorana spinor satisfies

$$\lambda^\dagger i^\sigma \gamma^1 \cdots \gamma^{n_t} = \lambda^T C,$$

where C is the **charge conjugation matrix**. We will denote Majorana spinors by ζ. (**put consistency condition here**)

Definition 14. The **projection operator** is defined to be

$$\mathcal{P}_\pm := \frac{1 \pm \gamma_c}{2}.$$

This is indeed a projector, as it's idempotent and its projected spaces are orthogonal.

Definition 15. A left/right-handed **Weyl spinor** is defined to be

$$\chi^\pm := \mathcal{P}_\pm \lambda,$$

where $\chi^\pm = \chi^{L/R}$. Another way of saying this is that Weyl spinors are eigenvectors of the chirality matrix,

$$\gamma_c \chi^\pm = \pm \chi^\pm.$$

(**put consistency condition here**)

Definition 16. A **Majorana-Weyl** spinor is a spinor that is both a Majorana and a Weyl spinor. Concretely, for a MW spinor ξ,

$$\overline{\xi}^\pm_M = \overline{\xi}^\pm_D \qquad\qquad \gamma_c \xi^\pm = \pm \xi^\pm.$$

MW spinors only exist in Euclidean and Minkowski spacetimes $D_{E/M}$ such that

$$D_E \equiv 0 \bmod 8 \qquad\qquad D_M \equiv 2 \bmod 8.$$

Definition 17. A **symplectic Majorana spinor** is defined as a spinor whose *symplectic* Majorana conjugate equals its Dirac conjugate. Let the spinor exist in $D = 2n$ spacetime diensions. Recall that the definition of the symplectic Majorana conjugate is

$$\overline{\lambda}^a_{\text{SpM}} = (\lambda^b)^T \Omega^{ab} C,$$

where

$$\Omega^{ab} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

The symplectic Majorana conjugate is only defined if you have pairs of spinors in even dimensions. (**is this true**) ? If you *do not have* symplectic/normal Majorana spinors in some given $D = s + t$, then you *do have* normal/symplectic Majorana spinors in $D = s + t$. You may also have both in a given dimension.

The definition of the Majorana spinor introduces a new object, the charge conjugation matrix. This matrix may or may not exist in a given $D = s + t$, leaving us to derive the following consistency conditions on C. We gave the most useful one in the definition, but we list the rest here:

- **(finish)**

Example 1.4 (Majorana and Weyl spinors in $\boldsymbol{D = 3 + 1}$)
Recall our irrep from the previous example,

$$\gamma^a = -i\begin{pmatrix} 0 & \sigma^a \\ -\sigma^a & 0 \end{pmatrix}, \qquad \gamma^4 = -i\begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \qquad \gamma^5 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},$$

Constructing the Majorana and Weyl spinors gives us **(finish)**

(make table with the following columns: which charge conj exists, what it is symmetric? block diagonal? spinor indices? M? W? MW? SpM? reality properties)

(add reality properties, spinor indices, and everything like that for each entry later; basically do every single case that PvN could possibly ask about in your notes)

Normal modes of atoms

Molecules with multiple atoms generally obey a periodic structure, and thus they are symmetric under certain operations (translations and rotations). Generally, symmetry constrains dynamics, and it is no different here: we can decompose a molecule's normal modes through group theory, specifically through decomposing the characters of reducible representations into irreps, thus giving us **(multiplets? or smth like that. idk.)**

Let us define a few things first. A molecule with N atoms can be described by the $3N$-dimensional vector $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_N)$. Considering deviations from equilibrium for each atom and defining $\boldsymbol{\xi}_a = \mathbf{x}_a - \mathbf{x}_a^0$ gives $\widetilde{\mathbf{X}} = (\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_N)$. The energy associated with the small deviations of the molecule is then

$$H = \frac{1}{2}m_\alpha(\partial_t \widetilde{X}_\alpha)^2 + \frac{1}{2}K_{\alpha\beta}\widetilde{X}_\alpha\widetilde{X}_\beta.$$

where $\alpha, \beta = 1, \ldots, 3N$. We may rescale $\boldsymbol{\xi}_a \to \boldsymbol{\xi}_a/\sqrt{m_a} = \boldsymbol{\lambda}_a$, and then orthogonalize $K'_{\alpha\beta}$ as it's symmetric; we then have the equation **(PvN's equation is wrong with index contractions, fix later)**

There are three ingredients we need to consider to represent our atom using representation theory:

- We need **two** representations. One of them is a permutation representation, $A\colon G \to \mathrm{Aut}(V)$, and the other is a rotation representation, $R\colon G \to D_n \subset SO(3)$.
- We need **one** group action, $\pi\colon G \times G \to G$, which acts to permute the labels of our atoms, s.

We then need to consider the characters of these representations, and we need to find out how many irreps are contained within them. The use of the group action is that it just lets our representations talk to each other via projections through the following equation:

$$\boxed{\mathcal{P}_{\pi_s} A_g \widetilde{\mathbf{X}} = D(g)\mathcal{P}_s \widetilde{\mathbf{X}}.}$$

In words, this tells you that permuting atoms and rotating them is the same thing[4].

Consider the character for rotations, $\chi^{\mathbb{R}^3}$. This is related to the genuine normal modes by

$$\boxed{\chi_{\text{gen}} = \chi_S - \chi_{\text{tr}} - \chi_{\text{rot}} = \chi^{\mathbb{R}^3}(c_g - 1 - \det D),}$$

so $\chi_S = c_g \chi^{\mathbb{R}^3}$, $\chi_{\text{tr}} = \chi^{\mathbb{R}^3}$, taking all of the possible modes and subtracting out the trivial ones. It is only slightly reductionistic to say that this is the most important equation in this entire analysis. Our entire goal will be to find χ_{gen} and then take inner products of it with our irrechs to find how many irreps are contained in the genuine motion.

(**explain why the relations between the rotations and the other trivial modes are true**)

[4] PvN remarked upon this multiple times while showing us the demonstration that "To half of you this will be incredibly obvious, and half of you will not understand this no matter how many times I say it. Now you know if you're a physicist or a mathematician, no one told you before but now you know." (**fix quote**)