Introduction to Fourier transform and signal analysis

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Outline

1. Continuous Fourier transform
2. Discrete Fourier transform
3. Calculate DFT with Python Numpy Package
4. References

Orthogonal condition

- Any two vectors **a**, **b** satisfied the following condition are mutually orthogonal.

$$\mathbf{a}^* \cdot \mathbf{b} = 0 \tag{1}$$

- Any two functions $a(x)$, $b(x)$ satisfied the following condition are mutually orthogonal.

$$\int a^*(x) \cdot b(x) dx = 0 \tag{2}$$

- * means complex conjugate.

Complete and orthogonal basis

- $\cos nx$ and $\sin mx$ are mutually orthogonal in which n and m are integers.

$$\begin{aligned}\int_{-\pi}^{\pi} \cos nx \cdot \sin mx dx &= 0 \\ \int_{-\pi}^{\pi} \cos nx \cdot \cos mx dx &= \pi\delta_{nm} \\ \int_{-\pi}^{\pi} \sin nx \cdot \sin mx dx &= \pi\delta_{nm}\end{aligned} \tag{3}$$

- δ_{nm} is Dirac-delta symbol. It means $\delta_{nn} = 1$ and $\delta_{nm} = 0$ when $n \neq m$.

Fourier series

Since $\cos nx$ and $\sin mx$ are mutually orthogonal, we can expand an arbitrary periodic function $f(x)$ by them. we shall have a series expansion of $f(x)$ which has 2π period.

$$\begin{aligned} f(x) &= a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \\ a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \\ a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx \\ b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx \end{aligned} \tag{4}$$

Fourier series

If $f(x)$ has L period instead of 2π, x is replaced with $\pi x/L$.

$$\begin{aligned}
f(x) &= a_0 + \sum_{k=1}^{\infty}\left(a_k \cos\frac{2k\pi x}{L} + b_k \sin\frac{2k\pi x}{L}\right) \\
a_0 &= \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)dx \\
a_k &= \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\cos\frac{2k\pi x}{L}dx, k = 1, 2, ... \\
b_k &= \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\sin\frac{2k\pi x}{L}dx, k = 1, 2, ... \qquad (5)
\end{aligned}$$

Complex Fourier series

Using Euler's formula, equation (4) becomes

$$f(x) = a_0 + \sum_{k=1}^{\infty} \left(\frac{a_k - ib_k}{2} e^{ikx} + \frac{a_k + ib_k}{2} e^{-ikx} \right)$$

Let $c_0 \equiv a_0$, $c_k \equiv \frac{a_k - ib_k}{2}$ and $c_{-k} \equiv \frac{a_k + ib_k}{2}$, we have

$$\begin{aligned} f(x) &= \sum_{m=-\infty}^{\infty} c_m e^{imx} \\ c_m &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-imx} dx \end{aligned} \tag{6}$$

e^{imx} and e^{inx} are also mutually orthogonal provided $n \neq m$ and it forms a complete set. Therfore, it can be used as orthogonal basis.

Complex Fourier series

If $f(x)$ has T period instead of 2π, x is replaced with $2\pi x/T$.

$$
\begin{aligned}
f(x) &= \sum_{m=-\infty}^{\infty} c_m e^{i\frac{2\pi mx}{T}} \\
c_m &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)e^{-i\frac{2\pi mx}{T}}dx, m = 0, 1, 2...
\end{aligned} \tag{7}
$$

Fourier Series of step function

$f(x)$ is a periodic function with 2π period and it's defined as follows.

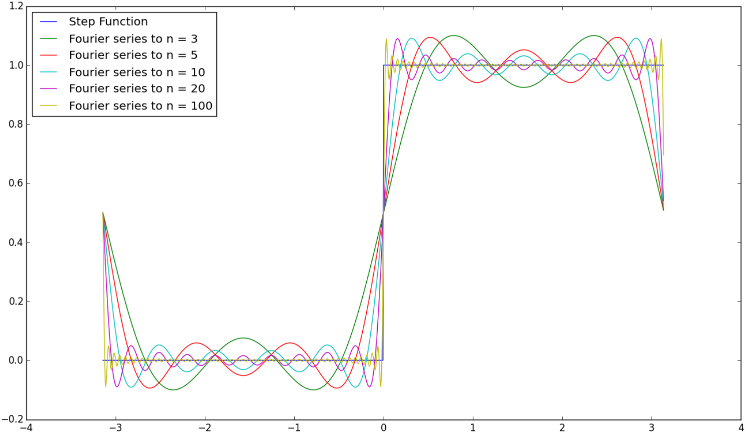
$$\begin{aligned} f(x) &= 0, -\pi < x < 0 \\ f(x) &= h, 0 < x < \pi \end{aligned} \tag{8}$$

Fourier series of $f(x)$ is

$$f(x) = \frac{h}{2} + \frac{2h}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\right) \tag{9}$$

$f(x)$ is piecewise continuous within the periodic region. Fourier series of $f(x)$ converges at speed of $1/n$.

Fourier series of step function



Fourier series of saw tooth function

$f(x)$ is a periodic function with 2π period and it's defined as follows.

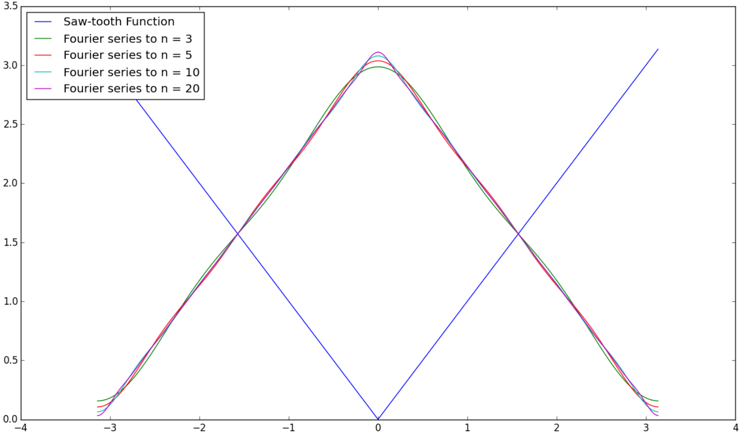
$$\begin{aligned} f(x) &= -x, -\pi < x < 0 \\ f(x) &= x, 0 < x < \pi \end{aligned} \tag{10}$$

Fourier series of $f(x)$ is

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,5\ldots} \left(\frac{\cos nx}{n^2} \right) \tag{11}$$

$f(x)$ is continuous and its derivative is piecewise continuous within the periodic region. Fourier series of $f(x)$ converges at speed of $1/n^2$.

Fourier series of saw tooth function



Fourier series of full wave rectifier

$f(t)$ is a periodic function with 2π period and it's defined as follows.

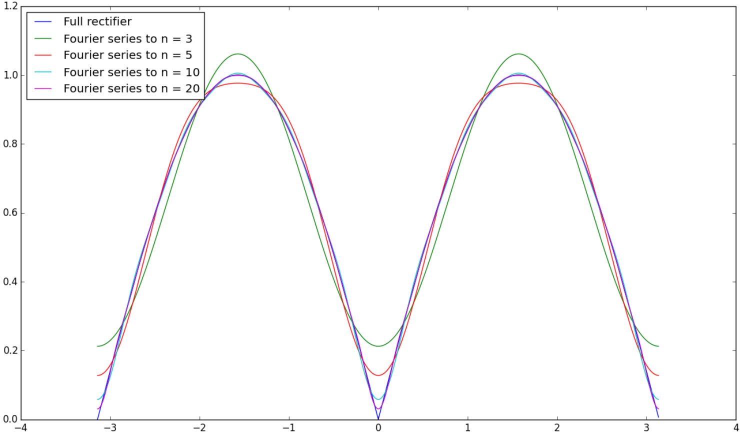
$$\begin{aligned} f(t) &= -\sin \omega t, -\pi < t < 0 \\ f(t) &= \sin \omega t, 0 < t < \pi \end{aligned} \tag{12}$$

Fourier series of $f(x)$ is

$$f(t) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=2,4,6\ldots} \left(\frac{\cos n\omega t}{n^2 - 1} \right) \tag{13}$$

$f(x)$ is continuous and its derivative is piecewise continuous within the periodic region. Fourier series of $f(x)$ converges at speed of $1/n^2$.

Fourier series of full wave rectifier



Fourier transform

from Eq. 7, we define variables $k \equiv \frac{2\pi m}{T}$, $\hat{f}(k) \equiv \frac{c_m T}{\sqrt{2\pi}}$ and $\triangle k \equiv \frac{2\pi(m+1)}{T} - \frac{2\pi m}{T} = \frac{2\pi}{T}$.
We can have

$$
\begin{aligned}
f(x) &= \frac{1}{\sqrt{(2\pi)}} \sum_{m=-\infty}^{\infty} \hat{f}(k) e^{ikx} \triangle k \\
\hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) e^{-ikx} dx
\end{aligned}
$$

Fourier transform

Let $T \longrightarrow \infty$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk \tag{14}$$

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \tag{15}$$

Eq.15 is the *Fourier transform* of $f(x)$ and Eq.14 is the *inverse Fourier transform* of $\hat{f}(k)$.

Convolution theory and Parseval relation

Considering two functions $f(x)$ and $g(x)$ with their Fourier transform $F(t)$ and $G(t)$. We define an operation

$$f * g = \int_{-\infty}^{\infty} g(y) f(x-y) dy \tag{16}$$

as the convolution of the two functions $f(x)$ and $g(x)$ over the interval $\{-\infty \sim \infty\}$. It satisfies the following relation:

$$f * g = \int_{-\infty}^{\infty} F(t) G(t) e^{itx} dt \tag{17}$$

Correlation

Uncertainty principle

Fourier transform of a Gaussian function

Fourier transform of a Gaussian function with carrier

Transfer function

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From continuous to discrete Fourier transform

Frequency bins and nyquist frequency

Concept of Nyquist-Shannon frequency

Aliasing

Window functions

Filter

Fast Fourier transform

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Discrete Fourier transform in Numpy or Scipy

Transform one dimensional data

Transform more than one dimensional data

Outline

References

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