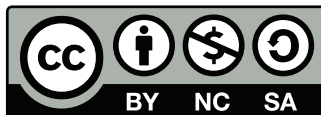# Introduction to Fourier transform and signal analysis

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Introduction to Fourier Transform and Signal Analysis by
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Outline

1 Continuous Fourier transform

2 Discrete Fourier transform

3 Calculate DFT with Python Numpy Package

4 References

Orthogonal condition

- Any two vectors **a**, **b** satisfied the following condition are mutually orthogonal.

$$\mathbf{a}^* \cdot \mathbf{b} = 0 \tag{1}$$

- Any two functions $a(x)$, $b(x)$ satisfied the following condition are mutually orthogonal.

$$\int a^*(x) \cdot b(x) dx = 0 \tag{2}$$

- * means complex conjugate.

Complete and orthogonal basis

- cos nx and sin mx are mutually orthogonal in which n and m are integers.

$$\int_{-\pi}^{\pi} \cos nx \cdot \sin mx dx = 0$$

$$\int_{-\pi}^{\pi} \cos nx \cdot \cos mx dx = \pi \delta_{nm}$$

$$\int_{-\pi}^{\pi} \sin nx \cdot \sin mx dx = \pi \delta_{nm} \qquad (3)$$

- δ_{nm} is Dirac-delta symbol. It means $\delta_{nn} = 1$ and $\delta_{nm} = 0$ when $n \neq m$.

Fourier series

Since $\cos nx$ and $\sin mx$ are mutually orthogonal, we can expand an arbitrary periodic function $f(x)$ by them. we shall have a series expansion of $f(x)$ which has 2π period.

$$
\begin{aligned}
f(x) &= a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \\
a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \\
a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx \\
b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx
\end{aligned}
\tag{4}
$$

Fourier series

If $f(x)$ has L period instead of 2π, x is replaced with $\pi x/L$.

$$
\begin{aligned}
f(x) &= a_0 + \sum_{k=1}^{\infty} \left(a_k \cos \frac{2k\pi x}{L} + b_k \sin \frac{2k\pi x}{L} \right) \\
a_0 &= \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) dx \\
a_k &= \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) \cos \frac{2k\pi x}{L} dx, k = 1, 2, ... \\
b_k &= \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) \sin \frac{2k\pi x}{L} dx, k = 1, 2, ...
\end{aligned}
\tag{5}
$$

Complex Fourier series

Using Euler's formula, equation (4) becomes

$$f(x) = a_0 + \sum_{k=1}^{\infty} \left(\frac{a_k - ib_k}{2} e^{ikx} + \frac{a_k + ib_k}{2} e^{-ikx} \right)$$

Let $c_0 \equiv a_0$, $c_k \equiv \frac{a_k - ib_k}{2}$ and $c_{-k} \equiv \frac{a_k + ib_k}{2}$, we have

$$
\begin{aligned}
f(x) &= \sum_{m=-\infty}^{\infty} c_m e^{imx} \\
c_m &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-imx} dx
\end{aligned}
\tag{6}
$$

e^{imx} and e^{inx} are also mutually orthogonal provided $n \neq m$ and it forms a complete set. Therfore, it can be used as orthogonal basis.

Complex Fourier series

If $f(x)$ has T period instead of 2π, x is replaced with $2\pi x/T$.

$$
\begin{aligned}
f(x) &= \sum_{m=-\infty}^{\infty} c_m e^{i\frac{2\pi mx}{T}} \\
c_m &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) e^{-i\frac{2\pi mx}{T}}\,dx, m = 0, 1, 2... \quad (7)
\end{aligned}
$$

Fourier Series of step function

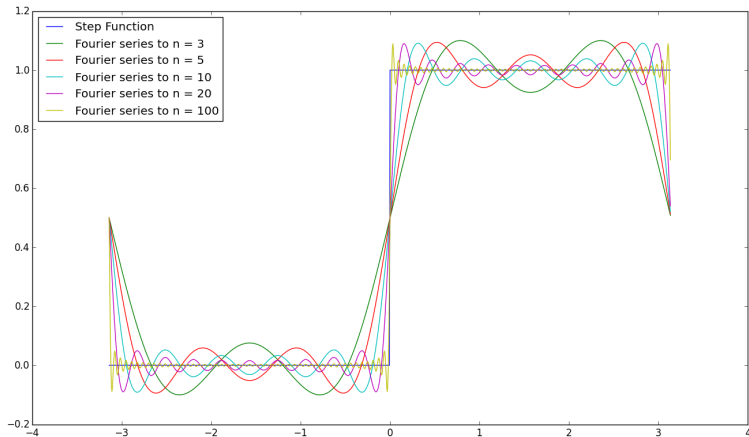$f(x)$ is a periodic function with 2π period and it's defined as follows.

$$
\begin{aligned}
f(x) &= 0, -\pi < x < 0 \\
f(x) &= h, 0 < x < \pi
\end{aligned}
\tag{8}
$$

Fourier series of $f(x)$ is

$$
f(x) = \frac{h}{2} + \frac{2h}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + ...\right)
\tag{9}
$$

$f(x)$ is piecewise continuous within the periodic region. Fourier series of $f(x)$ converges at speed of $1/n$.

Fourier series of step function

Fourier series of saw tooth function

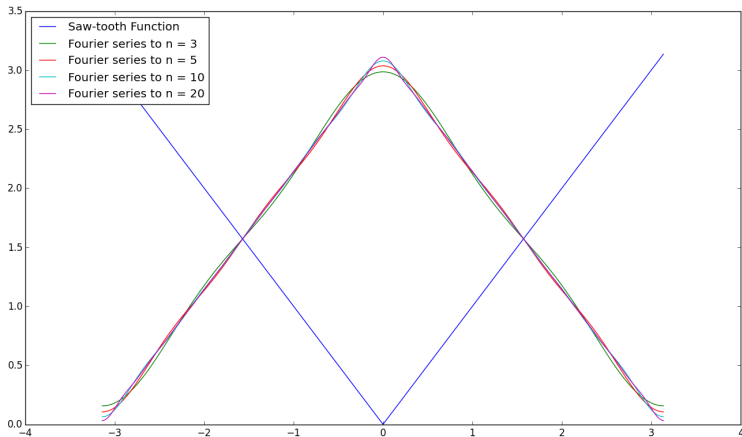$f(x)$ is a periodic function with 2π period and it's defined as follows.

$$
\begin{aligned}
f(x) &= -x, -\pi < x < 0 \\
f(x) &= x, 0 < x < \pi
\end{aligned}
\tag{10}
$$

Fourier series of $f(x)$ is

$$
f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,5...} \left(\frac{\cos nx}{n^2} \right)
\tag{11}
$$

$f(x)$ is continuous and its derivative is piecewise continuous within the periodic region. Fourier series of $f(x)$ converges at speed of $1/n^2$.

Fourier series of saw tooth function

Fourier series of full wave rectifier

$f(t)$ is a periodic function with 2π period and it's defined as follows.
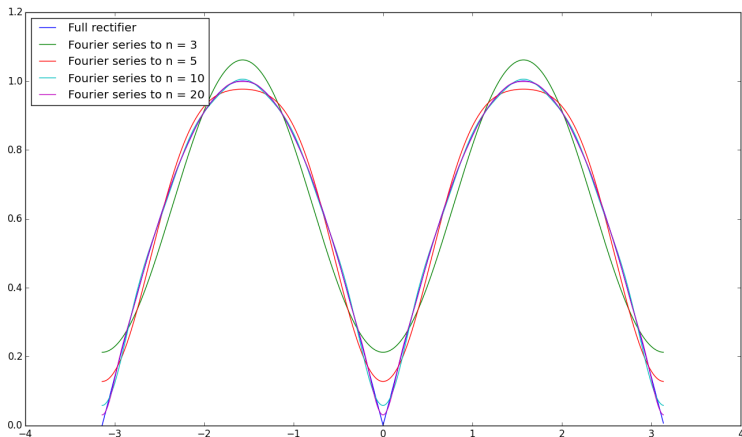
$$
\begin{aligned}
f(t) &= -\sin \omega t, -\pi < t < 0 \\
f(t) &= \sin \omega t, 0 < t < \pi
\end{aligned}
\tag{12}
$$

Fourier series of $f(x)$ is

$$
f(t) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=2,4,6...} \left(\frac{\cos n\omega t}{n^2 - 1} \right)
\tag{13}
$$

$f(x)$ is continuous and its derivative is piecewise continuous within the periodic region. Fourier series of $f(x)$ converges at speed of $1/n^2$.

Fourier series of full wave rectifier

Fourier transform

from Eq. 7, we define variables $k \equiv \frac{2\pi m}{T}$, $\hat{f}(k) \equiv \frac{c_m T}{\sqrt{2\pi}}$ and
$\triangle k \equiv \frac{2\pi(m+1)}{T} - \frac{2\pi m}{T} = \frac{2\pi}{T}$.
We can have

$$
\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \hat{f}(k) e^{ikx} \triangle k \\
\hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) e^{-ikx} dx
\end{aligned}
$$

Fourier transform

Let $T \longrightarrow \infty$

$$
\begin{align}
f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk \tag{14} \\
\hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \tag{15}
\end{align}
$$

Eq.15 is the *Fourier transform* of $f(x)$ and Eq.14 is the *inverse Fourier transform* of $\hat{f}(k)$.

Properties of Fourier transform

$f(x)$, $g(x)$ and $h(x)$ are functions and their Fourier transforms are $\hat{f}(k)$, $\hat{g}(k)$ and $\hat{h}(k)$. a, b x_0 and k_0 are real numbers.

- Linearity: If $h(x) = af(x) + bg(x)$, then Fourier transform of $h(x)$ equals to $\hat{h}(k) = a\hat{f}(k) + b\hat{g}(k)$.

- Translation: If $h(x) = f(x - x_0)$, then $\hat{h}(k) = \hat{f}(k)e^{-ikx_0}$

- Modulation: If $h(x) = e^{ik_0x}f(x)$, then $\hat{h}(k) = \hat{f}(k - k_0)$

- Scaling: If $h(x) = f(ax)$, then $\hat{h}(k) = \frac{1}{a}\hat{f}(\frac{k}{a})$

- Conjugation: If $h(x) = f^*(x)$, then $\hat{h}(k) = \hat{f}^*(-k)$. With this property, one can know that if $f(x)$ is real and then $\hat{f}^*(-k) = \hat{f}(k)$. One can also find that if $f(x)$ is real and then $|\hat{f}(k)| = |\hat{f}(-k)|$.

Properties of Fourier transform

- If $f(x)$ is even, then $\hat{f}(-k) = \hat{f}(k)$.
- If $f(x)$ is odd, then $\hat{f}(-k) = -\hat{f}(k)$.
- If $f(x)$ is real and even, then $\hat{f}(k)$ is real and even.
- If $f(x)$ is real and odd, then $\hat{f}(k)$ is imaginary and odd.
- If $f(x)$ is imaginary and even, then $\hat{f}(k)$ is imaginary and even.
- If $f(x)$ is imaginary and odd, then $\hat{f}(k)$ is real and odd.

Dirac delta function

Dirac delta function is a generalized function defined as the following equation.

$$f(0) = \int_{-\infty}^{\infty} f(x)\delta(x)dx$$

$$\int_{-\infty}^{\infty} \delta(x)dx = 1 \qquad (16)$$

The Dirac delta function can be loosely thought as a function which equals to infinite at $x = 0$ and to zero else where.

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

Dirac delta function

From Eq.15 and Eq.14

$$
\begin{aligned}
\hat{f}(k) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(k') e^{ik'x} dk' e^{-ikx} dx \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(k') e^{i(k'-k)x} dx dk'
\end{aligned}
$$

Comparing to "Dirac delta function", we have

$$
\begin{aligned}
\hat{f}(k) &= \int_{-\infty}^{\infty} \hat{f}(k') \delta(k'-k) dk' \\
\delta(k'-k) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k'-k)x} dx
\end{aligned} \tag{17}
$$

Eq.17 doesn't converge by itself, it is only well defined as part of an integrand.

Convolution theory

Considering two functions $f(x)$ and $g(x)$ with their Fourier transform $F(k)$ and $G(k)$. We define an operation

$$f * g = \int_{-\infty}^{\infty} g(y)f(x-y)dy \tag{18}$$

as the convolution of the two functions $f(x)$ and $g(x)$ over the interval $\{-\infty \sim \infty\}$. It satisfies the following relation:

$$f * g = \int_{-\infty}^{\infty} F(k)G(k)e^{ikx}dt \tag{19}$$

Let $h(x)$ be $f * g$ and $\hat{h}(k)$ be the Fourier transform of $h(x)$, we have

$$\hat{h}(k) = \sqrt{2\pi}F(k)G(k) \tag{20}$$

Parseval relation

$$
\begin{aligned}
\int_{-\infty}^{\infty} f(x)g(x)^* dx &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)e^{ikx} dk \frac{1}{\sqrt{2\pi}} \\
&\qquad \int_{-\infty}^{\infty} G^*(k')e^{-ik'x} dk' dx \\
&= \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k)G^*(k')e^{i(k-k')x} dk dk'
\end{aligned}
$$

By using 17, we have the Parseval's relation.

$$
\int_{-\infty}^{\infty} f(x)g^*(x)dx = \int_{-\infty}^{\infty} F(k)G^*(k)dk \qquad (21)
$$

Calculating inner product of two fuctions gets same result as the inner product of their Fourier transform.

Cross-correlation

Considering two functions $f(x)$ and $g(x)$ with their Fourier transform $F(k)$ and $G(k)$. We define cross-correlation as

$$(f \star g)(x) = \int_{-\infty}^{\infty} f^*(x+y)g(x)dy \tag{22}$$

as the cross-correlation of the two functions $f(x)$ and $g(x)$ over the interval $\{-\infty \sim \infty\}$. It satisfies the following relation: Let $h(x)$ be $f \star g$ and $\hat{h}(k)$ be the Fourier transform of $h(x)$, we have

$$\hat{h}(k) = \sqrt{2\pi}F^*(k)G(k) \tag{23}$$

Autocorrelation is the cross-correlation of the signal with itself.

$$(f \star f)(x) = \int_{-\infty}^{\infty} f^*(x+y)f(x)dy \tag{24}$$

Uncertainty principle

One important properties of Fourier transform is uncertainty
principle. It states that the more concentrated $f(x)$ is, the more
spread its Fourier transform $\hat{f}(k)$ is. Without loss of generality, we
consider $f(x)$ as a normalized function which means
$\int_{-\infty}^{\infty} |f(x)|^2 dx = 1$, we have uncertainty relation:

$$\left(\int_{-\infty}^{\infty} (x - x_0)^2 |f(x)|^2 dx \right) \left(\int_{-\infty}^{\infty} (k - k_0)^2 |\hat{f}(k)|^2 dk \right) \geqq \frac{1}{16\pi^2} (25)$$

for any x_0 and $k_0 \in \mathbf{R}$. [3]

Fourier transform of a Gaussian function

Fourier transform of a Gaussian function with carrier

Transfer function

Outline

1. Continuous Fourier transform

2. Discrete Fourier transform

3. Calculate DFT with Python Numpy Package

4. References

Nyquist critical frequency

Critical sampling of a sine wave is two sample points per cycle. This leads to *Nyquist critical frequency* f_c.

$$f_c = \frac{1}{2\Delta} \tag{26}$$

Where Δ is the sampling interval. Sampling theorem: If a continuos signal $h(t)$ sampled with interval Δ happens to be bandwidth limited to frequencies smaller than f_c. $h(t)$ is completely determined by its samples h_n

$$h(t) = \Delta \sum_{n=-\infty}^{\infty} h_n \frac{\sin[2\pi f_c(t - n\Delta)]}{\pi(t - n\Delta)} \tag{27}$$

Discrete Fourier transform

Signal $h(t)$ is sampled with N consecutive values and sampling interval Δ. We have $h_k \equiv h(t_k)$ and $t_k \equiv k * \Delta$,
$k = 0, 1, 2, ..., N - 1$.
With N discrete input, we evidently can only output independent values no more than N. Therefore, we seek for frequencies with values

$$f_n \equiv \frac{n}{N\Delta}, n = -\frac{N}{2}, ..., \frac{N}{2} \tag{28}$$

Discrete Fourier transform

Fourier transform of Signal $h(t)$ is $H(f)$. We have discrete Fourier transform H_n.

$$
\begin{aligned}
H(f_n) &= \int_{-\infty}^{\infty} h(t)e^{-i2\pi f_n t}dt \approx \Delta \sum_{k=0}^{N-1} h_k e^{-i2\pi f_n t_k} \\
&= \Delta \sum_{k=0}^{N-1} h_k e^{-i2\pi kn/N} \\
H_n &\equiv \sum_{k=0}^{N-1} h_k e^{-i2\pi kn/N}
\end{aligned}
\tag{29}
$$

Inverse Fourier transform is

$$
h_k \equiv \frac{1}{N} \sum_{n=0}^{N-1} H_n e^{i2\pi kn/N}
\tag{30}
$$

Periodicity of discrete Fourier transform

From Eq.29, if we substitute n with $n + N$, we have $H_n = H_{n+N}$.
Therefore, discrete Fourier transform has periodicity of N.

$$
\begin{aligned}
H_{n+N} &= \sum_{k=0}^{N-1} h_k e^{-i2\pi k(n+N)/N} \\
&= \sum_{k=0}^{N-1} h_k e^{-i2\pi k(n)/N} e^{-i2\pi kN/N} \\
&= H_n
\end{aligned}
\tag{32}
$$

Critical frequency f_c corresponds to $\frac{1}{2\Delta}$.

Aliasing

Filter

FIR

IIR

Fast Fourier transform

Outline

1. Continuous Fourier transform

2. Discrete Fourier transform

3. Calculate DFT with Python Numpy Package

4. References

Discrete Fourier transform in Numpy or Scipy

Transform one dimensional data

Transform more than one dimensional data

Outline

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