Introduction to Fourier transform and signal analysis

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June 15, 2019

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Outline

1. Continuous Fourier transform
2. Discrete Fourier transform
3. References

Orthogonal condition

- Any two vectors **a**, **b** satisfying following condition are mutually orthogonal.

$$\mathbf{a}^* \cdot \mathbf{b} = 0 \tag{1}$$

- Any two functions $a(x)$, $b(x)$ satisfying the following condition are mutually orthogonal.

$$\int a^*(x) \cdot b(x) dx = 0 \tag{2}$$

- * means complex conjugate.

Complete and orthogonal basis

- $\cos nx$ and $\sin mx$ are mutually orthogonal in which n and m are integers.

$$
\begin{aligned}
\int_{-\pi}^{\pi} \cos nx \cdot \sin mx dx &= 0 \\
\int_{-\pi}^{\pi} \cos nx \cdot \cos mx dx &= \pi\delta_{nm} \\
\int_{-\pi}^{\pi} \sin nx \cdot \sin mx dx &= \pi\delta_{nm}
\end{aligned}
\tag{3}
$$

- δ_{nm} is Dirac-delta symbol. It means $\delta_{nn} = 1$ and $\delta_{nm} = 0$ when $n \neq m$.

Fourier series

Since $\cos nx$ and $\sin mx$ are mutually orthogonal, we can expand an arbitrary periodic function $f(x)$ by them. we shall have a series expansion of $f(x)$ which has 2π period.

$$
\begin{aligned}
f(x) &= a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \\
a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \\
a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx \\
b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx
\end{aligned} \tag{4}
$$

Fourier series

If $f(x)$ has L period instead of 2π, x is replaced with $\pi x/L$.

$$\begin{aligned} f(x) &= a_0 + \sum_{k=1}^{\infty}\left(a_k \cos\frac{2k\pi x}{L} + b_k \sin\frac{2k\pi x}{L}\right) \\ a_0 &= \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)dx \\ a_k &= \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\cos\frac{2k\pi x}{L}dx, k = 1, 2, ... \\ b_k &= \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\sin\frac{2k\pi x}{L}dx, k = 1, 2, ... \end{aligned} \tag{5}$$

Fourier series of step function

$f(x)$ is a periodic function with 2π period and it's defined as follows.

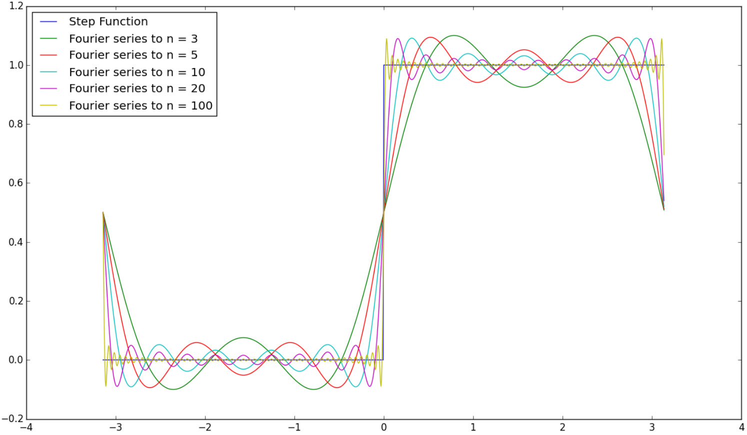
$$
\begin{aligned}
f(x) &= 0, -\pi < x < 0 \\
f(x) &= h, 0 < x < \pi
\end{aligned} \tag{6}
$$

Fourier series expansion of $f(x)$ is

$$
f(x) = \frac{h}{2} + \frac{2h}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + ...\right) \tag{7}
$$

$f(x)$ is piecewise continuous within the periodic region. Fourier series of $f(x)$ converges at speed of $1/n$.

Fourier series of step function



Fourier series of triangular function

$f(x)$ is a periodic function with 2π period and it's defined as follows.

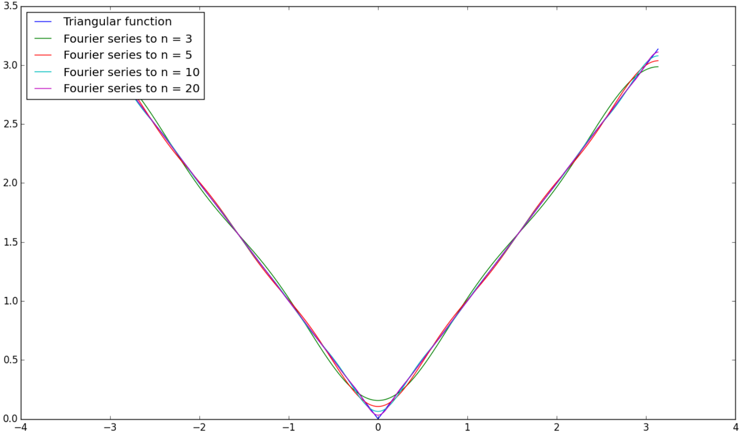
$$\begin{aligned} f(x) &= -x, -\pi < x < 0 \\ f(x) &= x, 0 < x < \pi \end{aligned} \tag{8}$$

Fourier series expansion of $f(x)$ is

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,5\ldots} \left(\frac{\cos nx}{n^2} \right) \tag{9}$$

$f(x)$ is continuous and its derivative is piecewise continuous within the periodic region. Fourier series of $f(x)$ converges at speed of $1/n^2$.

Fourier series of triangular function



Fourier series of full wave rectifier

$f(t)$ is a periodic function with 2π period and it's defined as follows.

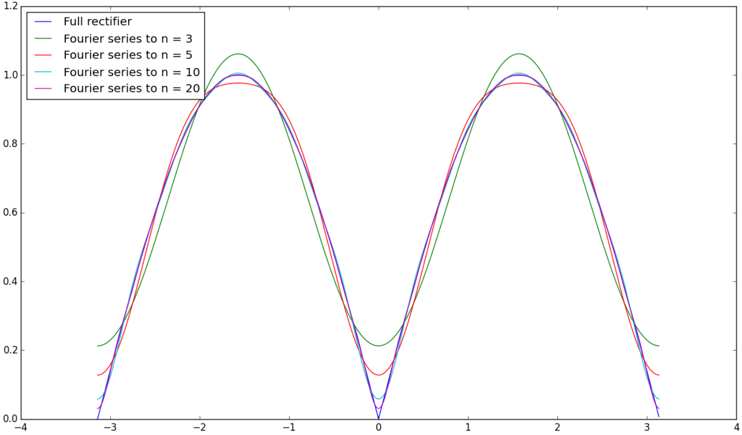
$$
\begin{aligned}
f(t) &= -\sin \omega t, -\pi < t < 0 \\
f(t) &= \sin \omega t, 0 < t < \pi
\end{aligned}
\tag{10}
$$

Fourier series expansion of $f(x)$ is

$$f(t) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=2,4,6...} \left(\frac{\cos n\omega t}{n^2 - 1} \right) \tag{11}$$

$f(x)$ is continuous and its derivative is piecewise continuous within the periodic region. Fourier series of $f(x)$ converges at speed of $1/n^2$.

Fourier series of full wave rectifier



Complex Fourier series

Using Euler's formula, Eq. 4 becomes

$$f(x) = a_0 + \sum_{k=1}^{\infty}\left(\frac{a_k - ib_k}{2}e^{ikx} + \frac{a_k + ib_k}{2}e^{-ikx}\right)$$

Let $c_0 \equiv a_0$, $c_k \equiv \frac{a_k - ib_k}{2}$ and $c_{-k} \equiv \frac{a_k + ib_k}{2}$, we have

$$\begin{aligned} f(x) &= \sum_{m=-\infty}^{\infty} c_m e^{imx} \\ c_m &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-imx}dx \end{aligned} \qquad (12)$$

e^{imx} and e^{inx} are also mutually orthogonal provided $n \neq m$ and it forms a complete set. Therfore, it can be used as orthogonal basis.

Complex Fourier series

If $f(x)$ has T period instead of 2π, x is replaced with $2\pi x/T$.

$$\begin{aligned} f(x) &= \sum_{m=-\infty}^{\infty} c_m e^{i\frac{2\pi mx}{T}} \\ c_m &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(x)e^{-i\frac{2\pi mx}{T}}dx, m = 0, 1, 2... \end{aligned} \tag{13}$$

Fourier transform

from Eq. 13, we define variables $k \equiv \frac{2\pi m}{T}$, $\hat{f}(k) \equiv \frac{c_m T}{\sqrt{2\pi}}$ and $\triangle k \equiv \frac{2\pi(m+1)}{T} - \frac{2\pi m}{T} = \frac{2\pi}{T}$.
We can have

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \hat{f}(k) e^{ikx} \triangle k \\ \hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) e^{-ikx} dx \end{aligned}$$

Fourier transform

Let $T \longrightarrow \infty$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk \tag{14}$$

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \tag{15}$$

Eq.15 is the *Fourier transform* of $f(x)$ and Eq.14 is the *inverse Fourier transform* of $\hat{f}(k)$.

Properties of Fourier transform

$f(x)$, $g(x)$ and $h(x)$ are functions and their Fourier transforms are $\hat{f}(k)$, $\hat{g}(k)$ and $\hat{h}(k)$. a, b x_0 and k_0 are real numbers.

- Linearity: If $h(x) = af(x) + bg(x)$, then Fourier transform of $h(x)$ equals to $\hat{h}(k) = a\hat{f}(k) + b\hat{g}(k)$.
- Translation: If $h(x) = f(x - x_0)$, then $\hat{h}(k) = \hat{f}(k)e^{-ikx_0}$
- Modulation: If $h(x) = e^{ik_0x}f(x)$, then $\hat{h}(k) = \hat{f}(k - k_0)$
- Scaling: If $h(x) = f(ax)$, then $\hat{h}(k) = \frac{1}{a}\hat{f}(\frac{k}{a})$
- Conjugation: If $h(x) = f^*(x)$, then $\hat{h}(k) = \hat{f}^*(-k)$. With this property, one can know that if $f(x)$ is real and then $\hat{f}^*(-k) = \hat{f}(k)$. One can also find that if $f(x)$ is real and then $|\hat{f}(k)| = |\hat{f}(-k)|$.

Properties of Fourier transform

- If $f(x)$ is even, then $\hat{f}(-k) = \hat{f}(k)$.
- If $f(x)$ is odd, then $\hat{f}(-k) = -\hat{f}(k)$.
- If $f(x)$ is real and even, then $\hat{f}(k)$ is real and even.
- If $f(x)$ is real and odd, then $\hat{f}(k)$ is imaginary and odd.
- If $f(x)$ is imaginary and even, then $\hat{f}(k)$ is imaginary and even.
- If $f(x)$ is imaginary and odd, then $\hat{f}(k)$ is real and odd.

Dirac delta function

Dirac delta function is a generalized function defined as the following equation.

$$
\begin{aligned}
f(0) &= \int_{-\infty}^{\infty} f(x)\delta(x)dx \\
\int_{-\infty}^{\infty} \delta(x)dx &= 1
\end{aligned} \tag{16}
$$

The Dirac delta function can be loosely thought as a function which equals to infinite at $x = 0$ and to zero else where.

$$
\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}
$$

Dirac delta function

From Eq.15 and Eq.14

$$\begin{aligned}\hat{f}(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{f}(k')e^{ik'x}dk'e^{-ikx}dx \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{f}(k')e^{i(k'-k)x}dxdk'\end{aligned}$$

Comparing to "Dirac delta function", we have

$$\begin{aligned}\hat{f}(k) &= \int_{-\infty}^{\infty}\hat{f}(k')\delta(k'-k)dk' \\ \delta(k'-k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i(k'-k)x}dx \qquad (17)\end{aligned}$$

Eq.17 doesn't converge by itself, it is only well defined as part of an integrand.

Convolution theory

Considering two functions $f(x)$ and $g(x)$ with their Fourier transform $F(k)$ and $G(k)$. We define an operation

$$f * g = \int_{-\infty}^{\infty} g(y) f(x-y) dy \tag{18}$$

as the convolution of the two functions $f(x)$ and $g(x)$ over the interval $\{-\infty \sim \infty\}$. It satisfies the following relation:

$$f * g = \int_{-\infty}^{\infty} F(k) G(k) e^{ikx} dt \tag{19}$$

Let $h(x)$ be $f * g$ and $\hat{h}(k)$ be the Fourier transform of $h(x)$, we have

$$\hat{h}(k) = \sqrt{2\pi} F(k) G(k) \tag{20}$$

Parseval relation

$$\int_{-\infty}^{\infty} f(x)g(x)^* dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)e^{ikx} dk \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G^*(k')e^{-ik'x} dk' dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k)G^*(k')e^{i(k-k')x} dk dk'$$

By using Eq. 17, we have the Parseval's relation.

$$\int_{-\infty}^{\infty} f(x)g^*(x)dx = \int_{-\infty}^{\infty} F(k)G^*(k)dk \qquad (21)$$

Calculating inner product of two fuctions gets same result as the inner product of their Fourier transform.

Cross-correlation

Considering two functions $f(x)$ and $g(x)$ with their Fourier transform $F(k)$ and $G(k)$. We define cross-correlation as

$$(f \star g)(x) = \int_{-\infty}^{\infty} f^*(x+y)g(x)dy \tag{22}$$

as the cross-correlation of the two functions $f(x)$ and $g(x)$ over the interval $\{-\infty \sim \infty\}$. It satisfies the following relation: Let $h(x)$ be $f \star g$ and $\hat{h}(k)$ be the Fourier transform of $h(x)$, we have

$$\hat{h}(k) = \sqrt{2\pi} F^*(k)G(k) \tag{23}$$

Autocorrelation is the cross-correlation of the signal with itself.

$$(f \star f)(x) = \int_{-\infty}^{\infty} f^*(x+y)f(x)dy \tag{24}$$

Uncertainty principle

One important properties of Fourier transform is the uncertainty principle. It states that the more concentrated $f(x)$ is, the more spread its Fourier transform $\hat{f}(k)$ is.
Without loss of generality, we consider $f(x)$ as a normalized function which means $\int_{-\infty}^{\infty} |f(x)|^2 dx = 1$, we have uncertainty relation:

$$\left(\int_{-\infty}^{\infty} (x - x_0)^2 |f(x)|^2 dx\right) \left(\int_{-\infty}^{\infty} (k - k_0)^2 |\hat{f}(k)|^2 dk\right) \geqq \frac{1}{16\pi^2} \quad (25)$$

for any x_0 and $k_0 \in \mathbf{R}$. [3]

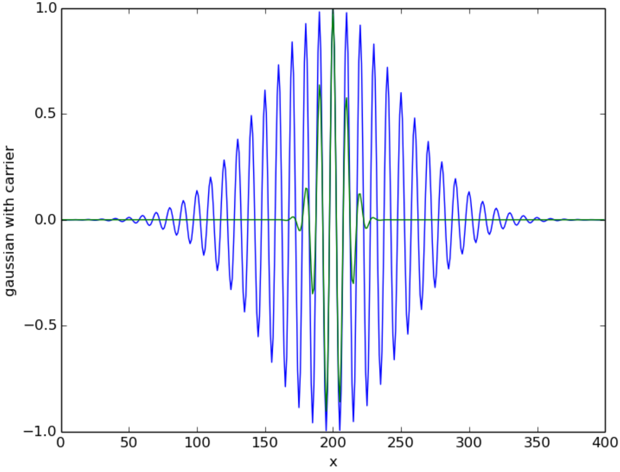
Fourier transform of a Gaussian pulse

$$
\begin{aligned}
f(x) &= f_0 e^{\frac{-x^2}{2\sigma^2}} e^{ik_0 x} \\
\hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \\
&= \frac{f_0}{1/\sigma^2} e^{\frac{-(k_0-k)^2}{2/\sigma^2}} \\
|\hat{f}(k)|^2 &\propto e^{\frac{-(k_0-k)^2}{1/\sigma^2}}
\end{aligned}
$$

Wider the $f(x)$ spread, the more concentrated $\hat{f}(k)$ is.

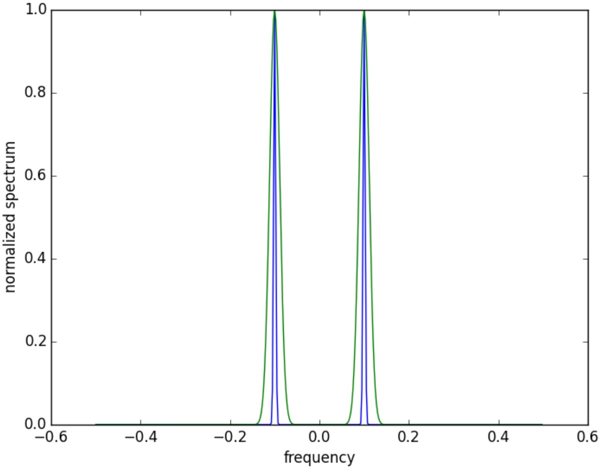
Fourier transform of a Gaussian pulse

Signals with different width.



Fourier transform of a Gaussian pulse

The bandwidth of the signals are different as well.



Outline

Nyquist critical frequency

Critical sampling of a sine wave is two sample points per cycle. This leads to *Nyquist critical frequency* f_c.

$$f_c = \frac{1}{2\Delta} \tag{26}$$

In above equation, Δ is the sampling interval.
Sampling theorem: If a continuos signal $h(t)$ sampled with interval Δ happens to be bandwidth limited to frequencies smaller than f_c. $h(t)$ is completely determined by its samples h_n. In fact, $h(t)$ is given by

$$h(t) = \Delta \sum_{n=-\infty}^{\infty} h_n \frac{\sin[2\pi f_c(t - n\Delta)]}{\pi(t - n\Delta)} \tag{27}$$

It's known as Whittaker - Shannon interpolation formula.

Discrete Fourier transform

Signal $h(t)$ is sampled with N consecutive values and sampling interval Δ. We have $h_k \equiv h(t_k)$ and $t_k \equiv k * \Delta$, $k = 0, 1, 2, ..., N-1$.
With N discrete input, we evidently can only output independent values no more than N. Therefore, we seek for frequencies with values

$$f_n \equiv \frac{n}{N\Delta}, n = -\frac{N}{2}, ..., \frac{N}{2} \tag{28}$$

Discrete Fourier transform

Fourier transform of Signal $h(t)$ is $H(f)$. We have discrete Fourier transform H_n.

$$
\begin{aligned}
H(f_n) &= \int_{-\infty}^{\infty} h(t) e^{-i2\pi f_n t} dt \approx \Delta \sum_{k=0}^{N-1} h_k e^{-i2\pi f_n t_k} \\
&= \Delta \sum_{k=0}^{N-1} h_k e^{-i2\pi kn/N} \\
H_n &\equiv \sum_{k=0}^{N-1} h_k e^{-i2\pi kn/N} \qquad (29)
\end{aligned}
$$

Inverse Fourier transform is

$$
h_k \equiv \frac{1}{N} \sum_{n=0}^{N-1} H_n e^{i2\pi kn/N} \qquad (30)
$$

Periodicity of discrete Fourier transform

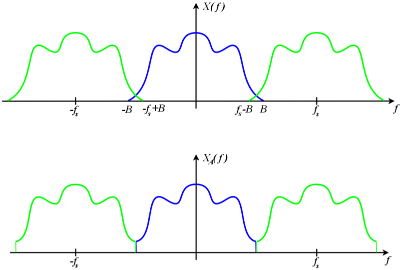
From Eq.29, if we substitute n with $n+N$, we have $H_n = H_{n+N}$.
Therefore, discrete Fourier transform has periodicity of N.

$$\begin{aligned} H_{n+N} &= \sum_{k=0}^{N-1} h_k e^{-i2\pi k(n+N)/N} \\ &= \sum_{k=0}^{N-1} h_k e^{-i2\pi k(n)/N} e^{-i2\pi kN/N} \\ &= H_n \end{aligned} \tag{31}$$

Critical frequency f_c corresponds to $\frac{1}{2\Delta}$.
We can see that discrete Fourier transform has f_s period where $f_s = 1/\Delta = 2 * f_c$ is the sampling frequency.

Aliasing

If we have a signal with its bandlimit larger than f_c, we have following spectrum due to periodicity of DFT.

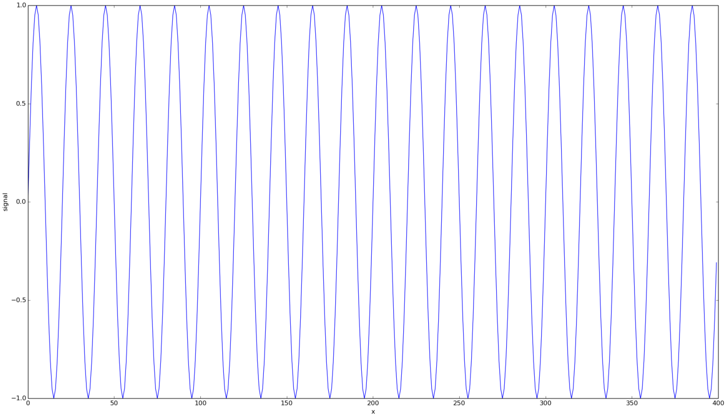


[5]

Aliased frequency is $f - N * f_s$ where N is an integer.

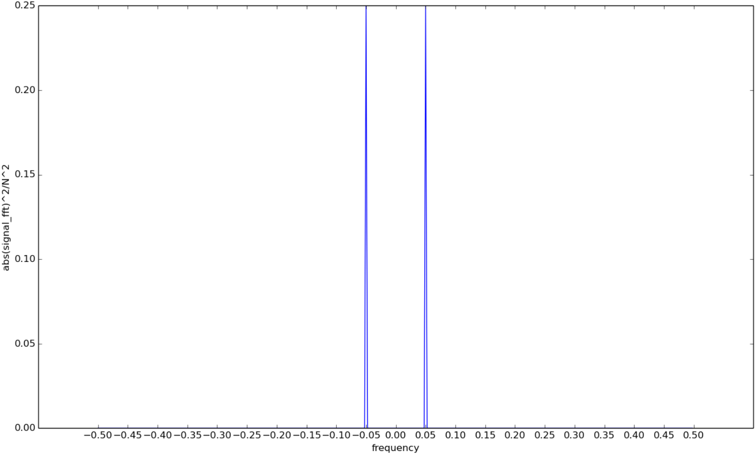
Aliasing example: original signal

Let's say we have a sinusoidal signal of frequency 0.05. The sampling interval is 1. We have the signal



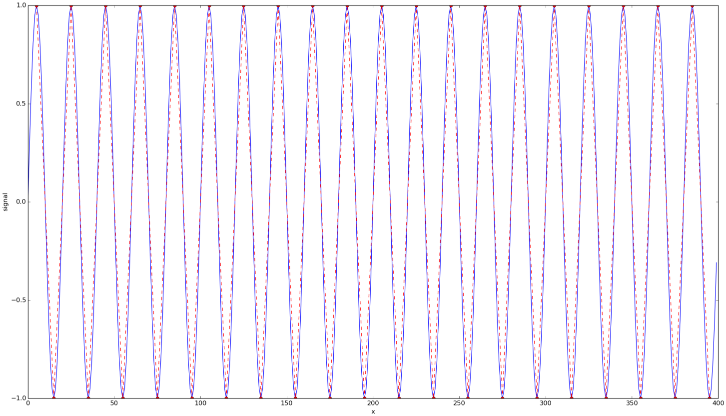
Aliasing example: spectrum of original signal

and we have its spectrum



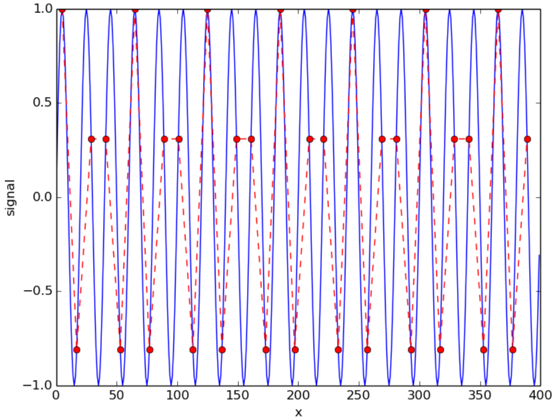
Aliasing example: critical sampling of original signal

The critical sampling interval of the original signal is 10 which is half of the signal period.



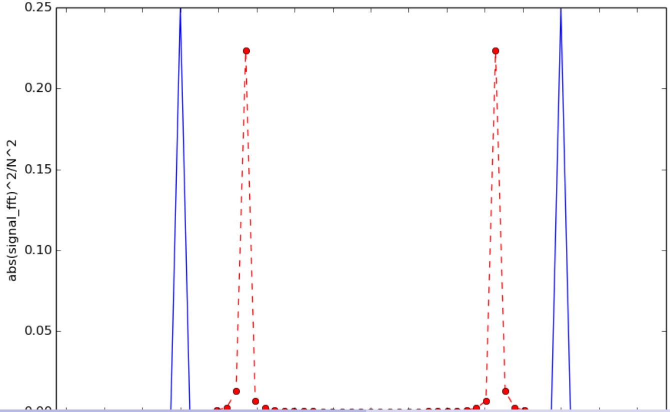
Aliasing example: under sampling of original signal

If we sampled the original sinusiodal signal with period 12, aliasing happens.



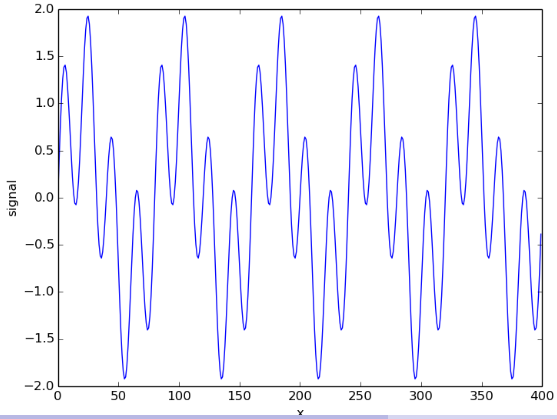
Aliasing example: DFT of under sampled signal

f_c of downsampled signal is $\frac{1}{2*12}$, aliased frequency is $f - 2 * f_c = -0.03333$ and it has symmetric spectrum due to real signal.



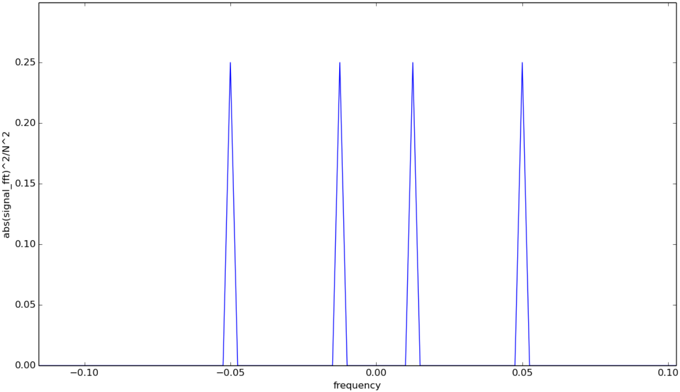
Aliasing example: two frequency signal

Let's say we have a signal containing two sinusoidal signal of frequency 0.05 and 0.0125. The sampling interval is 1. We have the signal



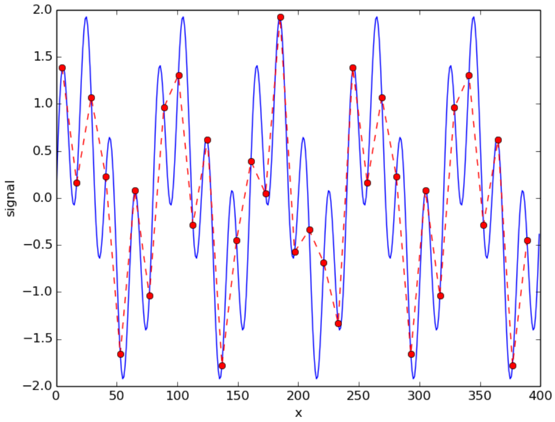
Aliasing example: spectrum of two frequency signal

and we have its spectrum



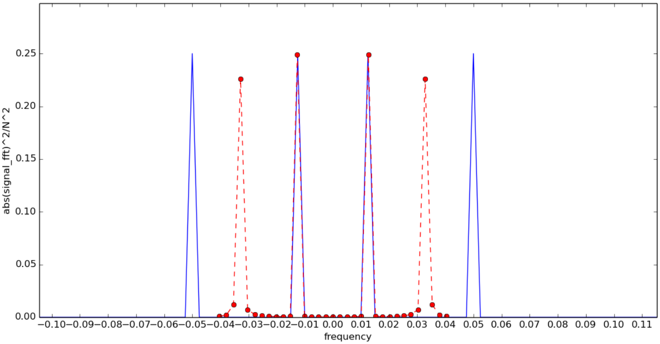
Aliasing example: downsampled two frequency signal

Doing same undersampling with interval 12.



Aliasing example: DFT of downsampled signal

We have the spectrum of downsampled signal.

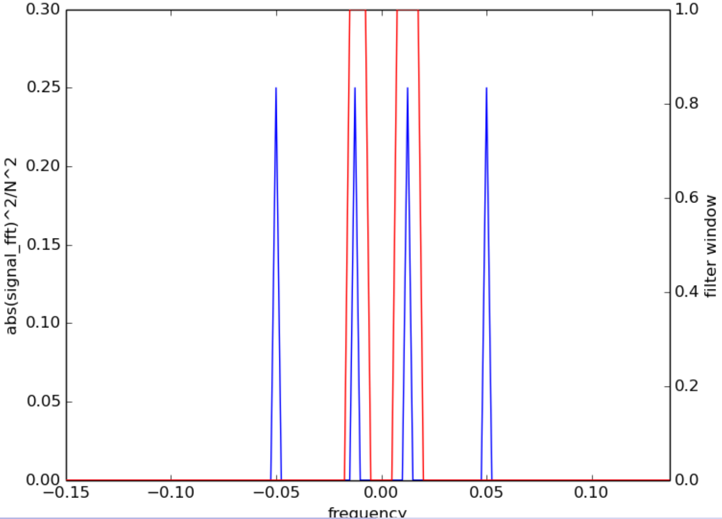


Filtering

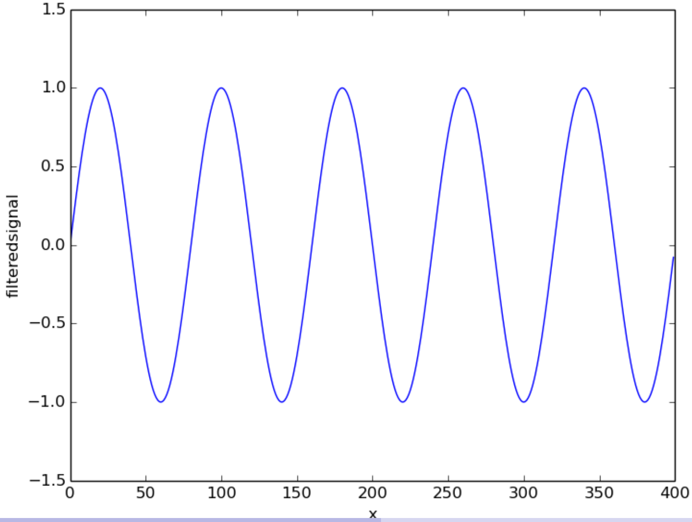
We now want to get one of the two frequency out of the signal.We will adapt a proper rectangular window to the spectrum.
Assuming we have a filter function $w(f)$ and a multi-frequency signal $f(t)$, we simply do following steps to get the frequency band we want.

$$\mathcal{F}^{-1}\{w(f)\mathcal{F}\{f(t)\}\} \tag{32}$$

Filtering example: filtering window and signal spectrum



Filtering example: filtered signal



Outline

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