Introduction to Fourier Transform and Signal Analysis

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Outline

Orthogonal Condition

- Any two vectors **a**, **b** satisfied the following condition are mutually orthogonal.

$$\mathbf{a}^* \cdot \mathbf{b} = 0 \tag{1}$$

- Any two functions $a(x)$, $b(x)$ satisfied the following condition are mutually orthogonal.

$$\int a^*(x) \cdot b(x) dx = 0 \tag{2}$$

- * means complex conjugate.

Complete and Orthogonal Basis

- $\cos nx$ and $\sin mx$ are mutually orthogonal in which n and m are integers.

$$\begin{aligned} \int_{-\pi}^{\pi} \cos nx \cdot \sin mx dx &= 0 \\ \int_{-\pi}^{\pi} \cos nx \cdot \cos mx dx &= \pi\delta_{nm} \\ \int_{-\pi}^{\pi} \sin nx \cdot \sin mx dx &= \pi\delta_{nm} \end{aligned} \tag{3}$$

- δ_{nm} is Dirac-delta symbol. It means $\delta_{nn} = 1$ and $\delta_{nm} = 0$ when $n \neq m$.

Fourier Series

Since $\cos nx$ and $\sin mx$ are mutually orthogonal, we can expand an arbitrary periodic function $f(x)$ by them. we shall have a series expansion of $f(x)$ which has 2π period.

$$\begin{aligned}
f(x) &= a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) \\
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \\
a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kxdx \\
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kxdx
\end{aligned} \tag{4}$$

Fourier Series

If $f(x)$ has $2L$ period instead of 2π, x is replaced with $\pi x/L$.

$$
\begin{aligned}
f(x) &= a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \\
a_0 &= \frac{1}{2L} \int_{-L}^{L} f(x) dx \\
a_k &= \frac{1}{L} \int_{-L}^{L} f(x) \cos kx dx \\
b_k &= \frac{1}{L} \int_{-L}^{L} f(x) \sin kx dx
\end{aligned} \tag{5}
$$

Complex Fourier Series

Using Euler's formula, equation (4) becomes

$$f(x) = a_0 + \sum_{k=1}^{\infty} \left(\frac{a_k - ib_k}{2} e^{ikx} + \frac{a_k + ib_k}{2} e^{-ikx} \right)$$

Let $c_0 \equiv a_0$, $c_k \equiv \frac{a_k - ib_k}{2}$ and $c_{-k} \equiv \frac{a_k + ib_k}{2}$, we have

$$f(x) = \sum_{m=-\infty}^{\infty} c_m e^{imx} \tag{6}$$

e^{imx} and e^{inx} are also mutually orthogonal provided $n \neq m$ and it forms a complete set. Therfore, it can be used as orthogonal basis.

Fourier Series of Step Function

$f(x)$ is a periodic function with 2π period and it's defined as follows.

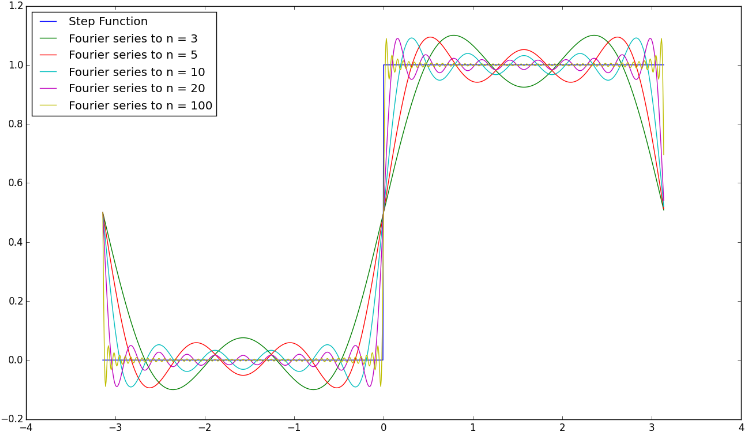
$$\begin{aligned} f(x) &= 0, -\pi < x < 0 \\ f(x) &= h, 0 < x < \pi \end{aligned} \tag{7}$$

Fourier series of $f(x)$ is

$$f(x) = \frac{h}{2} + \frac{2h}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\right) \tag{8}$$

$f(x)$ is piecewise continuous within the periodic region. Fourier series of $f(x)$ converges at speed of $1/n$.

Fourier Series of Step Function



Fourier Series of Saw Tooth Function

$f(x)$ is a periodic function with 2π period and it's defined as follows.

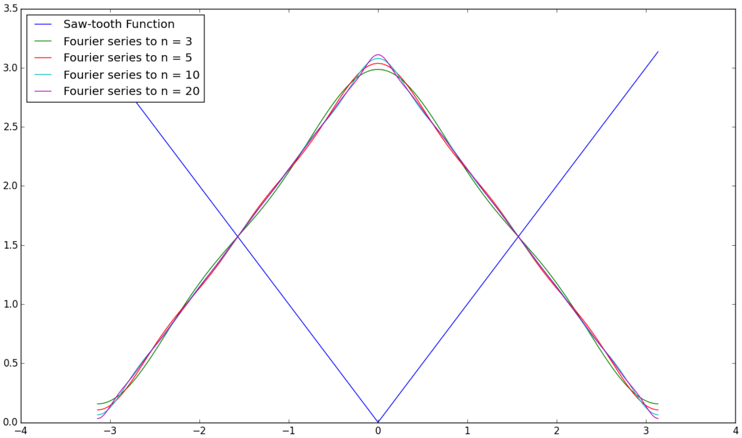
$$\begin{aligned} f(x) &= -x, -\pi < x < 0 \\ f(x) &= x, 0 < x < \pi \end{aligned} \tag{9}$$

Fourier series of $f(x)$ is

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,5\ldots} \left(\frac{\cos nx}{n^2} \right) \tag{10}$$

$f(x)$ is continuous and its derivative is piecewise continuous within the periodic region. Fourier series of $f(x)$ converges at speed of $1/n^2$.

Fourier Series of Saw Tooth Function



Fourier Series of Full Wave Rectifier

$f(t)$ is a periodic function with 2π period and it's defined as follows.

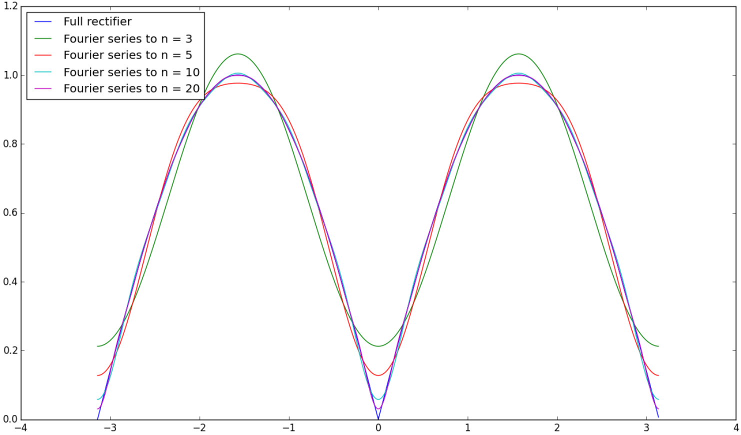
$$\begin{aligned} f(t) &= -\sin\omega t, -\pi < t < 0 \\ f(t) &= \sin\omega t, 0 < t < \pi \end{aligned} \tag{11}$$

Fourier series of $f(x)$ is

$$f(t) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=2,4,6\ldots} \left(\frac{\cos n\omega t}{n^2 - 1} \right) \tag{12}$$

$f(x)$ is continuous and its derivative is piecewise continuous within the periodic region. Fourier series of $f(x)$ converges at speed of $1/n^2$.

Fourier Series of Full Wave Rectifier



Fourier Transform

Convolution Theory and Parseval Relation

Fourier Transform of a Gaussian Function

Fourier Transform of a Gaussian Function with Carrier

Transfer Function

Outline

From Continuous to Discrete Fourier Transform

Frequency Bins and Nyquist Frequency

Concept of Nyquist-Shannon Frequency

Aliasing

Window Functions

Filter

Fast Fourier Transform

Outline

1. Continuous Fourier Transform
2. Discrete Fourier Transform
3. Calculate DFT with Python Numpy Package
4. References

Discrete Fourier Transform in Numpy or Scipy

Transform One Dimensional Data

Transform More than One Dimensional Data

Outline

References

- http://idv.sinica.edu.tw/jwang/SNGP/SNGP20090621.pdf
- MATHEMATICAL METHODS FOR PHYSICISTS by George B. Arfken and Hans J. Weber. ISBN-13: 978-0120598762
- Numerical Recipes 3rd Edition: The Art of Scientific Computing by William H. Press (Author), Saul A. Teukolsky. ISBN-13: 978-0521880688
- Chapter 12 and 13 in http://www.nrbook.com/a/bookcpdf.php

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