

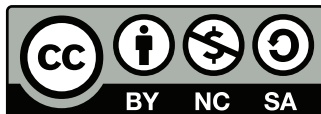
# Introduction to Fourier transform and signal analysis

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# Outline

- 1 Continuous Fourier transform
- 2 Discrete Fourier transform
- 3 References

# Orthogonal condition

- Any two vectors **a**, **b** satisfied the following condition are mutually orthogonal.

$$\mathbf{a}^* \cdot \mathbf{b} = 0 \quad (1)$$

- Any two functions  $a(x)$ ,  $b(x)$  satisfied the following condition are mutually orthogonal.

$$\int a^*(x) \cdot b(x) dx = 0 \quad (2)$$

- \* means complex conjugate.

# Complete and orthogonal basis

- $\cos nx$  and  $\sin mx$  are mutually orthogonal in which  $n$  and  $m$  are integers.

$$\begin{aligned}\int_{-\pi}^{\pi} \cos nx \cdot \sin mx dx &= 0 \\ \int_{-\pi}^{\pi} \cos nx \cdot \cos mx dx &= \pi \delta_{nm} \\ \int_{-\pi}^{\pi} \sin nx \cdot \sin mx dx &= \pi \delta_{nm}\end{aligned}\tag{3}$$

- $\delta_{nm}$  is Dirac-delta symbol. It means  $\delta_{nn} = 1$  and  $\delta_{nm} = 0$  when  $n \neq m$ .

# Fourier series

Since  $\cos nx$  and  $\sin mx$  are mutually orthogonal, we can expand an arbitrary periodic function  $f(x)$  by them. we shall have a series expansion of  $f(x)$  which has  $2\pi$  period.

$$\begin{aligned}f(x) &= a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \\a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \\a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx \\b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx\end{aligned}\tag{4}$$

# Fourier series

If  $f(x)$  has  $L$  period instead of  $2\pi$ ,  $x$  is replaced with  $\pi x/L$ .

$$\begin{aligned}f(x) &= a_0 + \sum_{k=1}^{\infty} \left( a_k \cos \frac{2k\pi x}{L} + b_k \sin \frac{2k\pi x}{L} \right) \\a_0 &= \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) dx \\a_k &= \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) \cos \frac{2k\pi x}{L} dx, k = 1, 2, \dots \\b_k &= \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) \sin \frac{2k\pi x}{L} dx, k = 1, 2, \dots\end{aligned} \tag{5}$$

## Fourier series of step function

$f(x)$  is a periodic function with  $2\pi$  period and it's defined as follows.

$$\begin{aligned}f(x) &= 0, -\pi < x < 0 \\f(x) &= h, 0 < x < \pi\end{aligned}\tag{6}$$

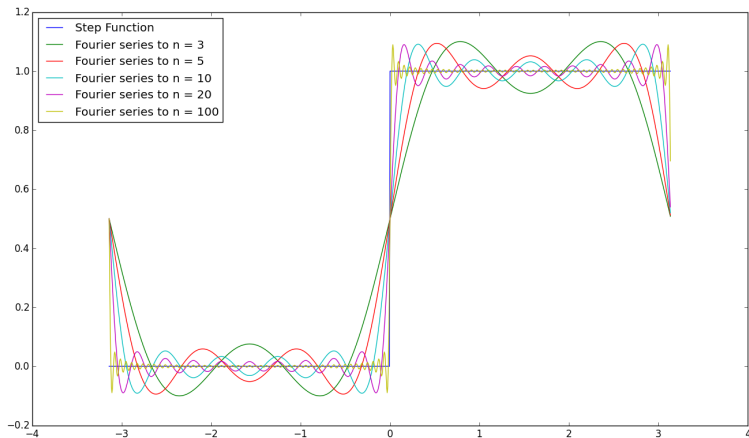
Fourier series of  $f(x)$  is

$$f(x) = \frac{h}{2} + \frac{2h}{\pi} \left( \frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots \right)\tag{7}$$

$f(x)$  is piecewise continuous within the periodic region. Fourier series of  $f(x)$  converges at speed of  $1/n$ .



# Fourier series of step function



## Fourier series of saw tooth function

$f(x)$  is a periodic function with  $2\pi$  period and it's defined as follows.

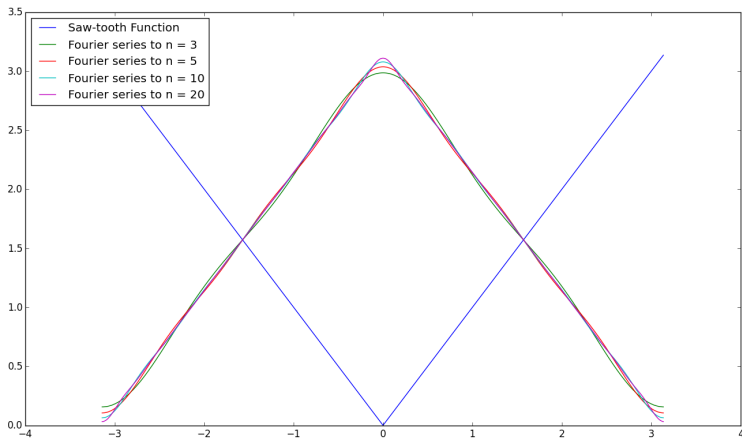
$$\begin{aligned}f(x) &= -x, -\pi < x < 0 \\f(x) &= x, 0 < x < \pi\end{aligned}\tag{8}$$

Fourier series of  $f(x)$  is

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,5,\dots} \left( \frac{\cos nx}{n^2} \right)\tag{9}$$

$f(x)$  is continuous and its derivative is piecewise continuous within the periodic region. Fourier series of  $f(x)$  converges at speed of  $1/n^2$ .

# Fourier series of saw tooth function



# Fourier series of full wave rectifier

$f(t)$  is a periodic function with  $2\pi$  period and it's defined as follows.

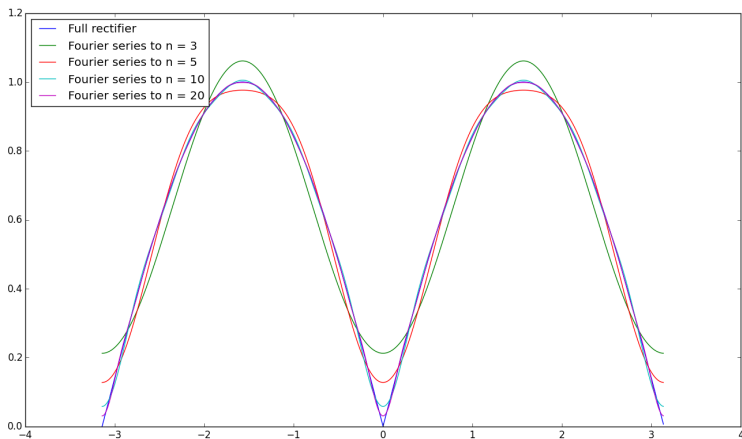
$$\begin{aligned}f(t) &= -\sin \omega t, -\pi < t < 0 \\f(t) &= \sin \omega t, 0 < t < \pi\end{aligned}\tag{10}$$

Fourier series of  $f(x)$  is

$$f(t) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=2,4,6\dots} \left( \frac{\cos n\omega t}{n^2 - 1} \right)\tag{11}$$

$f(x)$  is continuous and its derivative is piecewise continuous within the periodic region. Fourier series of  $f(x)$  converges at speed of  $1/n^2$ .

# Fourier series of full wave rectifier



# Complex Fourier series

Using Euler's formula, equation (4) becomes

$$f(x) = a_0 + \sum_{k=1}^{\infty} \left( \frac{a_k - ib_k}{2} e^{ikx} + \frac{a_k + ib_k}{2} e^{-ikx} \right)$$

Let  $c_0 \equiv a_0$ ,  $c_k \equiv \frac{a_k - ib_k}{2}$  and  $c_{-k} \equiv \frac{a_k + ib_k}{2}$ , we have

$$\begin{aligned} f(x) &= \sum_{m=-\infty}^{\infty} c_m e^{imx} \\ c_m &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-imx} dx \end{aligned} \quad (12)$$

$e^{imx}$  and  $e^{inx}$  are also mutually orthogonal provided  $n \neq m$  and it forms a complete set. Therefore, it can be used as orthogonal basis.

# Complex Fourier series

If  $f(x)$  has  $T$  period instead of  $2\pi$ ,  $x$  is replaced with  $2\pi x/T$ .

$$\begin{aligned}f(x) &= \sum_{m=-\infty}^{\infty} c_m e^{i \frac{2\pi m x}{T}} \\c_m &= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) e^{-i \frac{2\pi m x}{T}} dx, m = 0, 1, 2, \dots\end{aligned}\quad (13)$$

# Fourier transform

from Eq. 13, we define variables  $k \equiv \frac{2\pi m}{T}$ ,  $\hat{f}(k) \equiv \frac{c_m T}{\sqrt{2\pi}}$  and

$$\Delta k \equiv \frac{2\pi(m+1)}{T} - \frac{2\pi m}{T} = \frac{2\pi}{T}.$$

We can have

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \hat{f}(k) e^{ikx} \Delta k$$

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(x) e^{-ikx} dx$$



# Fourier transform

Let  $T \rightarrow \infty$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk \quad (14)$$

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \quad (15)$$

Eq.15 is the *Fourier transform* of  $f(x)$  and Eq.14 is the *inverse Fourier transform* of  $\hat{f}(k)$ .

# Properties of Fourier transform

$f(x)$ ,  $g(x)$  and  $h(x)$  are functions and their Fourier transforms are  $\hat{f}(k)$ ,  $\hat{g}(k)$  and  $\hat{h}(k)$ .  $a$ ,  $b$ ,  $x_0$  and  $k_0$  are real numbers.

- Linearity: If  $h(x) = af(x) + bg(x)$ , then Fourier transform of  $h(x)$  equals to  $\hat{h}(k) = a\hat{f}(k) + b\hat{g}(k)$ .
- Translation: If  $h(x) = f(x - x_0)$ , then  $\hat{h}(k) = \hat{f}(k)e^{-ikx_0}$
- Modulation: If  $h(x) = e^{ik_0x}f(x)$ , then  $\hat{h}(k) = \hat{f}(k - k_0)$
- Scaling: If  $h(x) = f(ax)$ , then  $\hat{h}(k) = \frac{1}{a}\hat{f}\left(\frac{k}{a}\right)$
- Conjugation: If  $h(x) = f^*(x)$ , then  $\hat{h}(k) = \hat{f}^*(-k)$ . With this property, one can know that if  $f(x)$  is real and then  $\hat{f}^*(-k) = \hat{f}(k)$ . One can also find that if  $f(x)$  is real and then  $|\hat{f}(k)| = |\hat{f}(-k)|$ .

# Properties of Fourier transform

- If  $f(x)$  is even, then  $\hat{f}(-k) = \hat{f}(k)$ .
- If  $f(x)$  is odd, then  $\hat{f}(-k) = -\hat{f}(k)$ .
- If  $f(x)$  is real and even, then  $\hat{f}(k)$  is real and even.
- If  $f(x)$  is real and odd, then  $\hat{f}(k)$  is imaginary and odd.
- If  $f(x)$  is imaginary and even, then  $\hat{f}(k)$  is imaginary and even.
- If  $f(x)$  is imaginary and odd, then  $\hat{f}(k)$  is real and odd.

# Dirac delta function

Dirac delta function is a generalized function defined as the following equation.

$$\begin{aligned} f(0) &= \int_{-\infty}^{\infty} f(x)\delta(x)dx \\ \int_{-\infty}^{\infty} \delta(x)dx &= 1 \end{aligned} \tag{16}$$

The Dirac delta function can be loosely thought as a function which equals to infinite at  $x = 0$  and to zero else where.

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

# Dirac delta function

From Eq.15 and Eq.14

$$\begin{aligned}\hat{f}(k) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(k') e^{ik'x} dk' e^{-ikx} dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(k') e^{i(k'-k)x} dx dk'\end{aligned}$$

Comparing to "Dirac delta function", we have

$$\begin{aligned}\hat{f}(k) &= \int_{-\infty}^{\infty} \hat{f}(k') \delta(k' - k) dk' \\ \delta(k' - k) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k'-k)x} dx\end{aligned}\tag{17}$$

Eq.17 doesn't converge by itself, it is only well defined as part of an integrand.

# Convolution theory

Considering two functions  $f(x)$  and  $g(x)$  with their Fourier transform  $F(k)$  and  $G(k)$ . We define an operation

$$f * g = \int_{-\infty}^{\infty} g(y)f(x - y)dy \quad (18)$$

as the convolution of the two functions  $f(x)$  and  $g(x)$  over the interval  $\{-\infty \sim \infty\}$ . It satisfies the following relation:

$$f * g = \int_{-\infty}^{\infty} F(k)G(k)e^{ikx}dt \quad (19)$$

Let  $h(x)$  be  $f * g$  and  $\hat{h}(k)$  be the Fourier transform of  $h(x)$ , we have

$$\hat{h}(k) = \sqrt{2\pi}F(k)G(k) \quad (20)$$

# Parseval relation

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)g(x)^* dx &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k)e^{ikx} dk \frac{1}{\sqrt{2\pi}} \\ &\quad \int_{-\infty}^{\infty} G^*(k')e^{-ik'x} dk' dx \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k)G^*(k')e^{i(k-k')x} dk dk'\end{aligned}$$

By using 17, we have the Parseval's relation.

$$\int_{-\infty}^{\infty} f(x)g^*(x)dx = \int_{-\infty}^{\infty} F(k)G^*(k)dk \quad (21)$$

Calculating inner product of two functions gets same result as the inner product of their Fourier transform.

# Cross-correlation

Considering two functions  $f(x)$  and  $g(x)$  with their Fourier transform  $F(k)$  and  $G(k)$ . We define cross-correlation as

$$(f \star g)(x) = \int_{-\infty}^{\infty} f^*(x+y)g(x)dy \quad (22)$$

as the cross-correlation of the two functions  $f(x)$  and  $g(x)$  over the interval  $\{-\infty \sim \infty\}$ . It satisfies the following relation: Let  $h(x)$  be  $f \star g$  and  $\hat{h}(k)$  be the Fourier transform of  $h(x)$ , we have

$$\hat{h}(k) = \sqrt{2\pi}F^*(k)G(k) \quad (23)$$

Autocorrelation is the cross-correlation of the signal with itself.

$$(f \star f)(x) = \int_{-\infty}^{\infty} f^*(x+y)f(x)dy \quad (24)$$



# Uncertainty principle

One important properties of Fourier transform is the uncertainty principle. It states that the more concentrated  $f(x)$  is, the more spread its Fourier transform  $\hat{f}(k)$  is.

Without loss of generality, we consider  $f(x)$  as a normalized function which means  $\int_{-\infty}^{\infty} |f(x)|^2 dx = 1$ , we have uncertainty relation:

$$\left( \int_{-\infty}^{\infty} (x - x_0)^2 |f(x)|^2 dx \right) \left( \int_{-\infty}^{\infty} (k - k_0)^2 |\hat{f}(k)|^2 dk \right) \geq \frac{1}{16\pi^2} \quad (25)$$

for any  $x_0$  and  $k_0 \in \mathbf{R}$ . [3]

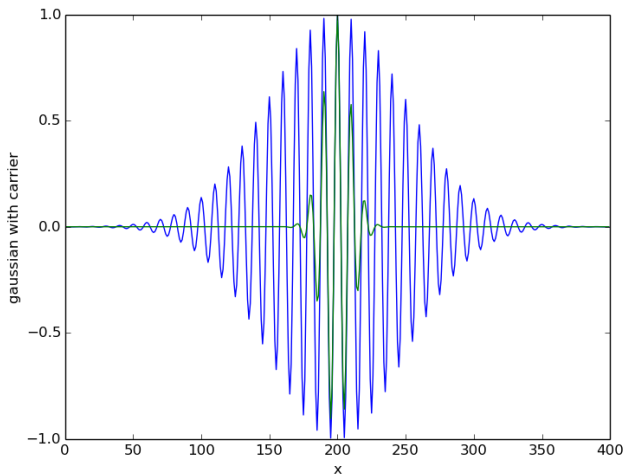
# Fourier transform of a Gaussian pulse

$$\begin{aligned}f(x) &= f_0 e^{\frac{-x^2}{2\sigma^2}} e^{ik_0 x} \\ \hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \\ &= \frac{f_0}{1/\sigma^2} e^{\frac{-(k_0 - k)^2}{2/\sigma^2}} \\ |\hat{f}(k)|^2 &\propto e^{\frac{-(k_0 - k)^2}{1/\sigma^2}}\end{aligned}$$

Wider the  $f(x)$  spread, the more concentrated  $\hat{f}(k)$  is.

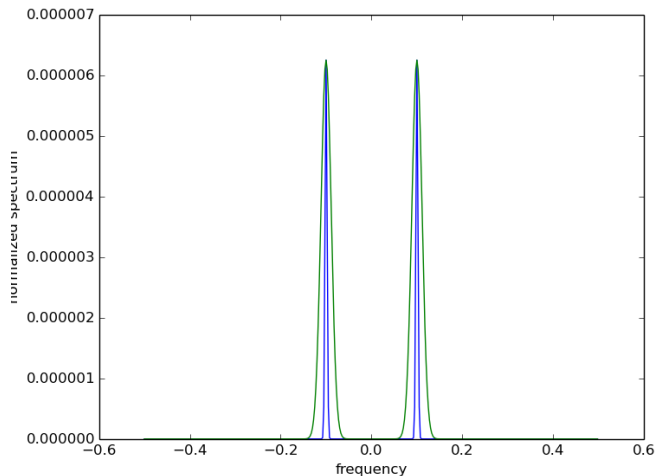
# Fourier transform of a Gaussian pulse

Signals with different width.



# Fourier transform of a Gaussian pulse

The bandwidth of the signals are different as well.



# Outline

- 1 Continuous Fourier transform
- 2 Discrete Fourier transform
- 3 References

## Nyquist critical frequency

Critical sampling of a sine wave is two sample points per cycle.  
This leads to *Nyquist critical frequency*  $f_c$ .

$$f_c = \frac{1}{2\Delta} \quad (26)$$

Where  $\Delta$  is the sampling interval.

Sampling theorem: If a continuous signal  $h(t)$  sampled with interval  $\Delta$  happens to be bandwidth limited to frequencies smaller than  $f_c$ .  $h(t)$  is completely determined by its samples  $h_n$ . In fact,  $h(t)$  is given by

$$h(t) = \Delta \sum_{n=-\infty}^{\infty} h_n \frac{\sin[2\pi f_c(t - n\Delta)]}{\pi(t - n\Delta)} \quad (27)$$

It's known as Whittaker - Shannon interpolation formula.

# Discrete Fourier transform

Signal  $h(t)$  is sampled with  $N$  consecutive values and sampling interval  $\Delta$ . We have  $h_k \equiv h(t_k)$  and  $t_k \equiv k * \Delta$ ,  $k = 0, 1, 2, \dots, N - 1$ .

With  $N$  discrete input, we evidently can only output independent values no more than  $N$ . Therefore, we seek for frequencies with values

$$f_n \equiv \frac{n}{N\Delta}, n = -\frac{N}{2}, \dots, \frac{N}{2} \quad (28)$$

# Discrete Fourier transform

Fourier transform of Signal  $h(t)$  is  $H(f)$ . We have discrete Fourier transform  $H_n$ .

$$\begin{aligned} H(f_n) &= \int_{-\infty}^{\infty} h(t) e^{-i2\pi f_n t} dt \approx \Delta \sum_{k=0}^{N-1} h_k e^{-i2\pi f_n t_k} \\ &= \Delta \sum_{k=0}^{N-1} h_k e^{-i2\pi kn/N} \\ H_n &\equiv \sum_{k=0}^{N-1} h_k e^{-i2\pi kn/N} \end{aligned} \quad (29)$$

Inverse Fourier transform is

$$h_k \equiv \frac{1}{N} \sum_{n=0}^{N-1} H_n e^{i2\pi kn/N} \quad (30)$$



# Periodicity of discrete Fourier transform

From Eq.29, if we substitute  $n$  with  $n + N$ , we have  $H_n = H_{n+N}$ . Therefore, discrete Fourier transform has periodicity of  $N$ .

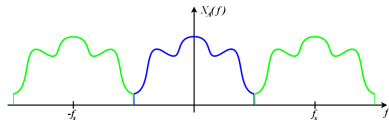
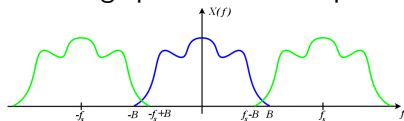
$$\begin{aligned} H_{n+N} &= \sum_{k=0}^{N-1} h_k e^{-i2\pi k(n+N)/N} \\ &= \sum_{k=0}^{N-1} h_k e^{-i2\pi k(n)/N} e^{-i2\pi kN/N} \\ &= H_n \end{aligned} \tag{31}$$

Critical frequency  $f_c$  corresponds to  $\frac{1}{2\Delta}$ .

We can see that discrete Fourier transform has  $f_s$  period where  $f_s = 1/\Delta = 2 * f_c$  is the sampling frequency.

# Aliasing

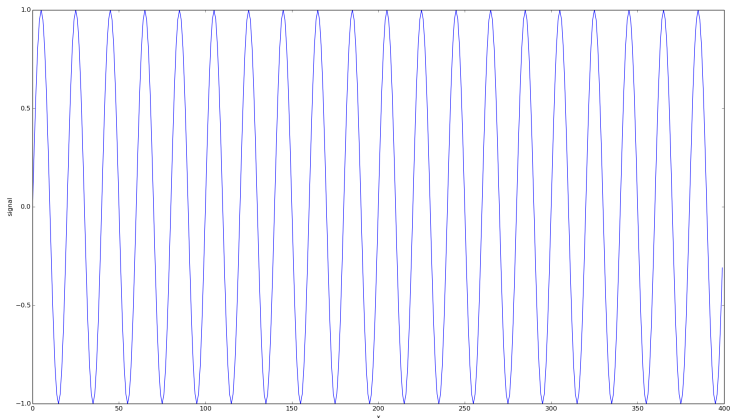
If we have a signal with its bandlimit larger than  $f_c$ , we have following spectrum due to periodicity of DFT.



Aliased frequency is  $f - N * f_s$  where  $N$  is an integer.

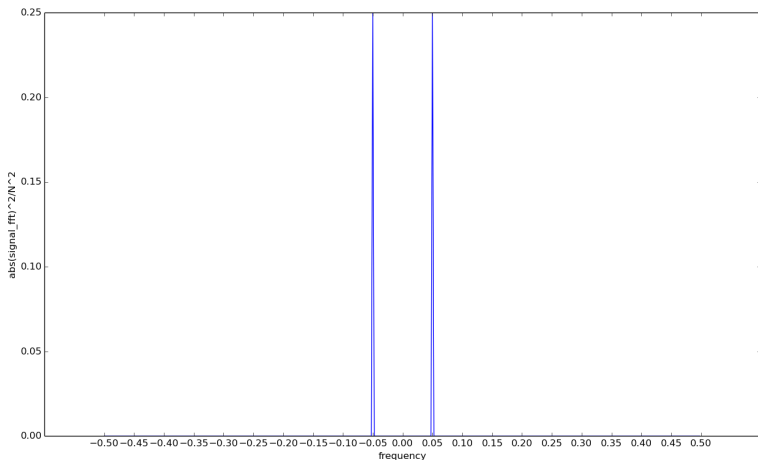
# Aliasing example: original signal

Let's say we have a sinusoidal signal of frequency 0.05. The sampling interval is 1. We have the signal



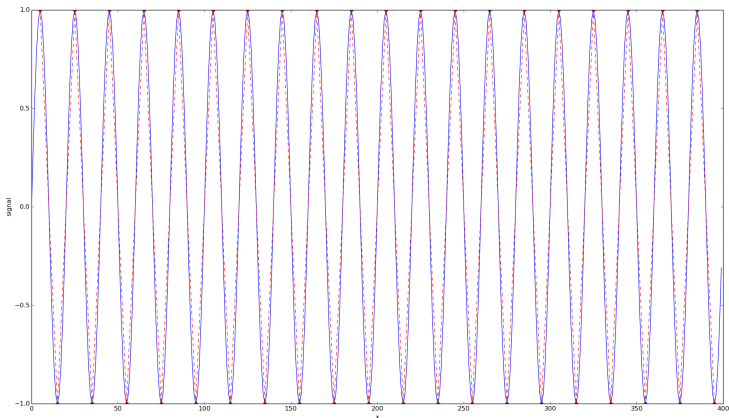
# Aliasing example: spectrum of original signal

and we have its spectrum



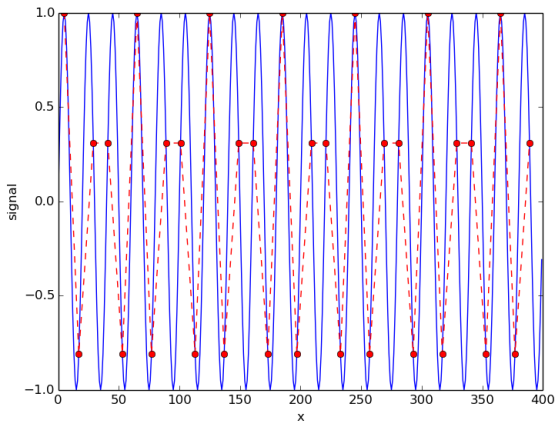
# Aliasing example: critical sampling of original signal

The critical sampling interval of the original signal is 10 which is half of the signal period.



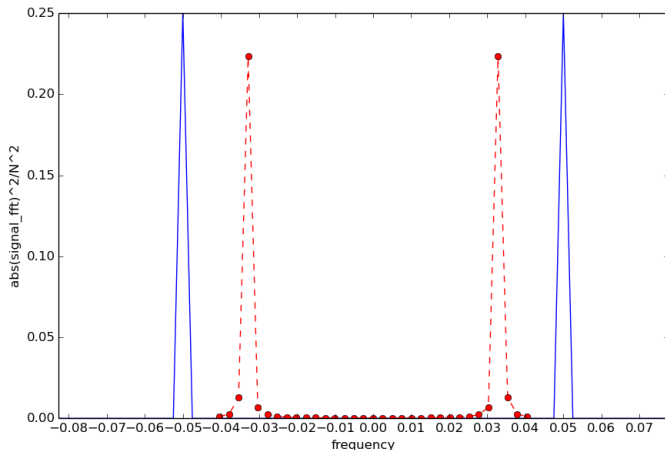
# Aliasing example: under sampling of original signal

If we sampled the original sinusoidal signal with period 12, aliasing happens.



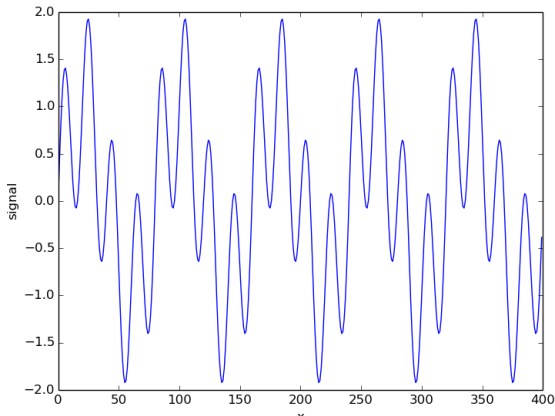
## Aliasing example: DFT of under sampled signal

$f_c$  of undersampling is  $\frac{1}{2 \times 12}$ , aliased frequency is  $f - 2 * f_c =$  and it has symmetric spectrum due to real signal.



## Aliasing example: two frequency signal

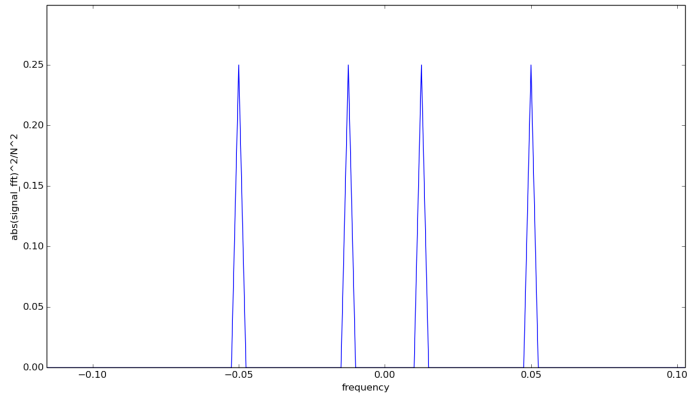
Let's say we have a signal containing two sinusoidal signal of frequency 0.05 and 0.0125. The sampling interval is 1. We have the signal





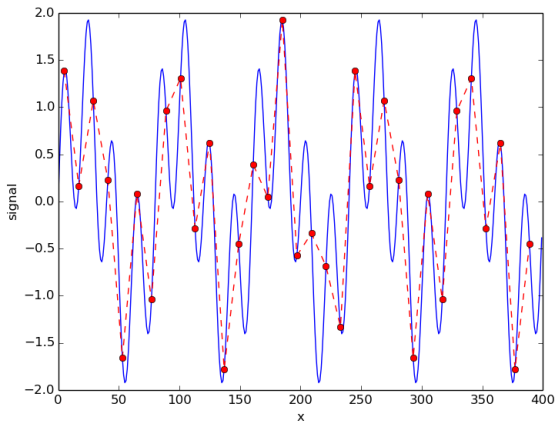
# Aliasing example: spectrum of two frequency signal

and we have its spectrum



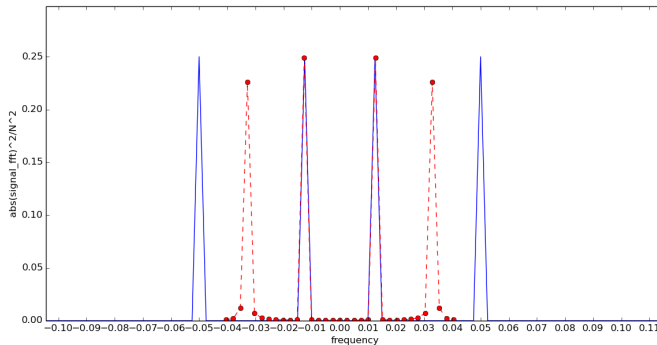
# Aliasing example: downsampled two frequency signal

Doing same undersampling with interval 12.



# Aliasing example: DFT of downsampled signal

We have the spectrum of downsampled signal.



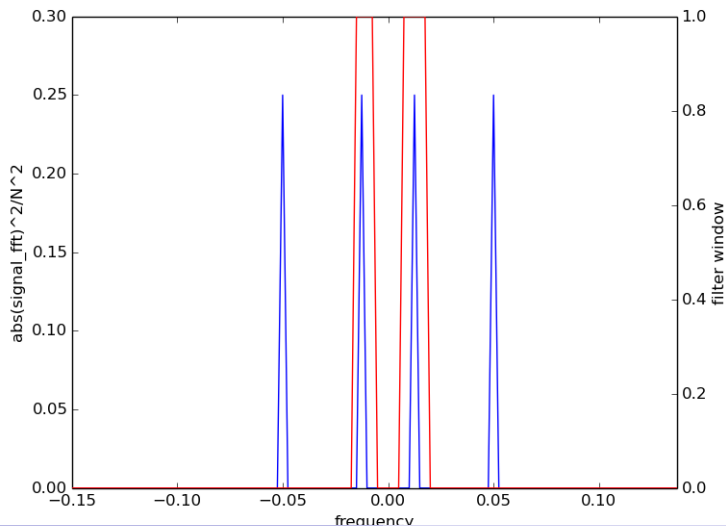
# Filtering

We now want to get one of the two frequency out of the signal. We will adapt a proper rectangular window to the spectrum.

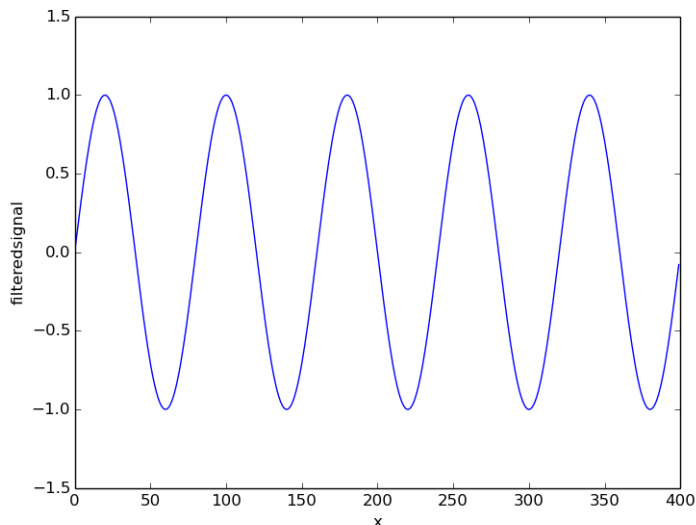
Assuming we have a filter function  $w(f)$  and a multi-frequency signal  $f(t)$ , we simply do following steps to get the frequency band we want.

$$\mathcal{F}^{-1}\{w(f)\mathcal{F}\{f(t)\}\} \quad (32)$$

# Filtering example: filtering window and signal spectrum










# Filtering example: filtered signal



# Outline

- 1 Continuous Fourier transform
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