Preface

Partial typed notes from Stanford course Math 115: Functions of Real Variable, taught in the Spring 2020-21 quarter by Prof. Zhenkun Li.

March 29, 2021 (Real Analysis)

Basic Stuff

A **set** is a collection of elements.

- We write $a \in A$ to signify that a is an element of set A.
- We consider A to be a subset of B, denoted $A \subseteq B$, if every element of A is also in B.
- Two sets are considered equal if they have the same elements in any order.

We define the intersection and union sets for A and B below in that order.

$$A \cap B = \{a \mid a \in A \ \text{ and } \ a \in B\}$$
$$A \cup B = \{a \mid a \in A \ \text{ or } \ a \in B\}$$

The **Cartesian product** of two sets is $AxB = \{(a, b) \mid a \in A, b \in C\}$. For example, the Cartesian product of R (set of real numbers) and R is the 2D Cartesian coordinate plane R^2.

Definition: Map

Suppose that A, B are two sets. A map $f : A \to B$ is an assignment that delegates an element in B to every element in A.

For example, a function on R is a map $R \to R$ such as $f(x) = x^2$.

Integer Axioms

The fundamental properties of $\mathbf{N}$, the set of natural numbers, are:

1. $1 \in \mathbf{N}$
2. Any element $n \in \mathbf{N}$ has a successor $n + 1$.
3. 1 is not the successor of any $n \in \mathbf{N}$.
4. If $m, n \in \mathbf{N}$ have the same successor, then $m = n$.
5. If $S \subseteq N$ such that $1 \in S$ and each element in S has a successor in S, then $S = N$.

Remark: These are called the Peano axiom for $\mathbf{N}$.

Other properties of $\mathbf{N}$ are *addition* (if $a, b \in \mathbf{N}$, then $a + b \in \mathbf{N}$), *multiplication* (if $a, b \in \mathbf{N}$, then $ab \in \mathbf{N}$), *order* (either $m < n$, $m = n$, or $m > n$ for any $m, n \in \mathbf{N}$), and *transitivity* of the order (if $a < b$ and $b < c$, then $a < c$).

Mathematical Induction

Suppose we have a list of statements $S_1, S_2, S_3, \cdots,$. If S_1 is true and S_{n+1} is true whenever S_n is true, then S_n is true for all $n \in \mathbf{N}$.

Proof: In homework

The following paradox is a famous problem in set theory.

Russell's Paradox

Consider S to be the set of all sets A so that A does not contain itself as an element. Is it true that $S \in S$ (that S contains itself as an element)?

Explication: Suppose that $S \in S$. This is impossible since S contains itself as an element, which violates the definition of S. Suppose $S \notin S$. Since S does not contain itself, then S belongs in the set by definition, so a contradiction. Hence, both $S \in S$ and $S \notin S$ lead to contradictions, creating a paradox.

March 31, 2021 (Real Analysis)

Theorem (Minimal Principle)

Any non-empty subset $S \subseteq N$ has a minimal element s_0 where $s_0 \in S$ and $s_0 < s_i$ for any other element $s_i \in S$.

Recall the theorem of **induction** from the previous lecture. If we have a list of statements $S_1, S_2, \cdots$ with S_1 true and S_{k+1} true whenever S_k is true, then S_n is true for all $n \in \mathbf{N}$. We will use induction to solve a medley of problems.

Example

Prove that for any $n \in N$, we have

$$1 + 2 + 3 \cdots + n = \frac{n(n+1)}{2}.$$

Proof: To address the base case, $n = 1$, we have $1 = \frac{1*2}{2}$. Suppose that $1 + 2 + 3 + \cdots + k = \frac{k(k+1}{2}$ for some $k \in N$ (*induction hypothesis*).

$$\begin{aligned} 1 + 2 + \cdots + (k+1) &= (1 + 2 + \cdots + k) + (k+1) \\ &= \frac{(k)(k+1)}{2} + (k+1) \\ &= \frac{(k+1)(k+2)}{2}. \end{aligned}$$

By induction, we now that $1 + 2 + \cdots + n = \frac{n(n+1}{2}$ for all $n \in \mathbf{N}$.

Definition: Operation

Suppose that A is a set. Then, an operation on A is the map $AxA \to A$, where AxA represents the Cartesian product of A with itself.

There are two operations in $\mathbf{Z}$: addition and multiplication. We can view subtraction as the inverse to addition: $a - b = c$ if $a = b + c$. To accommodate the negative values that can result from subtraction, we extend $\mathbf{N}$ to the set of all integers $\mathbf{Z}$.

Definition: We define the set of integers $\mathbf{Z} = \mathbf{N} \cup \{0\} \cup -\mathbf{N}$.

Note that the minimal principle does not apply for $\mathbf{Z}$ since for $n - 1 < n$ for any $n \in \mathbf{Z}$ and $n - 1 \in \mathbf{Z}$. We alter the minimal principle to incorporate this.

Modified Minimal Principle

If $S \subseteq Z$ is non-empty and S has a lower bound, which is an element $b \in Z$ so that $b < s$ for all $s \in S$, then S contains a minimal element. Note that it is not necessarily true that $b \in S$.

Moving onward, we can view *division* to be the inverse of multiplication. To define division, we must extend $\mathbf{Z}$ to $\mathbf{Q}$, the set of all rational numbers, to house fractions. Formally, $\frac{p}{q}$ is the solution to $p = qx$.

$$\mathbf{Q} = \{\frac{p}{q} \ \text{ for } \ p, q \in \mathbf{Z}, q \neq 0\}$$

We require $\frac{p}{q} = \frac{p'}{q'}$ if $pq' = p'q$. For instance, $\frac{2}{4} = \frac{1}{2}$. Note that the modified minimal principle does not apply for $\mathbf{Q}$, as the set of positive rational numbers has lower bound 0 but no minimal element.

A few definitions involving a subset $S \subseteq \mathbf{Q}$ for clarity.

- **Lower Bound:** An element $b \in \mathbf{Q}$ such that $b \leq s$ for all $s \in S$.
- **Minimal Element:** An element $b_0 \in S$ so that $s_0 \leq s$ for all $S \in S$.
- **Infimum:** An element $i \in \mathbf{Q}$ so that i is a lower bound for S and any rational number $r > i$ is not a lower bound on S. A synonym is the *greatest lower bound*.

Disprove

If $S \subseteq \mathbf{Q}$ is non-empty and S has a lower bound, then the infimum of S must exist in $\mathbf{Q}$.

Explication: Let S be the set of all rational numbers that are larger than 2 when squared. The infimum of this set is $\sqrt{2}$, but $\sqrt{2} \notin Q$.

April 2, 2021 (Real Analysis)

We introduce the concept of a field, noting that $\mathbf{Q}$ is a field.

Definition: Field

A field $\overrightarrow{F}$ is a set equipped with addition and subtraction which satisfy the following axioms.

- **Commutativity** of both operations, meaning that for every $a, b \in \overrightarrow{F}$, we have $ab = ba$ and $a + b = b + a$.
- **Associativity** of both operations, meaning that for every $a, b, c \in \overrightarrow{F}$, we have $(a+b)+c = a + (b + c)$, and similarly for multiplication.
- **Existence of identity elements for operations**, meaning that there exists elements $O \neq 1 \in F$ such that
$$a + 0 = a \text{ and } a * 1 = a.$$
- **Existence of additive inverses**. For every $a \in F$, there exists $a' \in F$ such that $a+a' = 0$.
- **Existence of multiplicative inverses** *for non-zero elements*. For every $a \in \overrightarrow{F}$, which is not an additive identity, there exists $a' \in F$ such that $a(a' = 1$.
- **Distributivity of multiplication over addition**. This means that $a(b + c) = ab + ac$ for every $a, b, c \in F$.

Remark: If we have associativity for addition, identity elements for addition, and additive inverses, we have a **group**. If commutativity is also true for addition, we have a **abelian group**. If we have the aforementioned properties along with associativity and commutativity of multiplication, and distributivity, we have a **commutative ring**.

Example

Prove that $x = \sqrt{2}$ is not rational.

Solution: Assume that $\sqrt{2}$ is rational, which means that there exists $p, q \in \mathbf{Q}$ with $p, q > 0$ such that $\sqrt{2} = \frac{p}{q}$. Let S be the set of all positive integers $q \in \mathbf{N}$ such that there exists a p such that $\frac{p}{q} = \sqrt{2}$. By the minimal principle for $S \subseteq N$, then there exists a minimal element $q_0 \in S$. Hence, there exists a $q_0 \in S$ and $p_0 \in N$ so that $\frac{p_0}{q_0} = x$.

Since $x^2 = 2$, then $\frac{p_0^2}{q_0^2} = 2$ so $p_0^2 = 2q_0^2$. This implies that $p_0 = 2p_1$ as p_0 is even for some integer p_1. Thus, $4p_1^2 = 2q_0^2$ so $q_0^2 = 2p_1^2$, which means that q_0 is also even so $q_0 = 2q_1$. Hence, we have

$$x = \frac{p_0}{q_0} = \frac{2p_1}{2q_1} = \frac{p_1}{q_1}$$

This implies that $q_1 \in S$, but $q_1 < q_0$ contradicts $q_0 = \min(S)$. There is a contradiction, so $\sqrt{2}$ is irrational.

Definition: We define $Q[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbf{Q}\}$. Note that $Q[\sqrt{2}]$ is a field and $\sqrt{2} \in Q[\sqrt{2}]$.

Definition: Dedekin Cut

A Dedekin cut is a pair (S, L) of a sets so that

1. $S \subseteq Q$ and $L \subseteq Q$
2. $S \cup L = \mathbf{Q}$
3. $S \neq \emptyset$ and $L \neq \emptyset$
4. For any $x \in S$ and $y \in L$, we have $x < y$.
5. S does not contain a maximum element.

We use this to define the set of **real numbers R** to be the set of all $r = (S, L)$, where (S, L) is a Dedekin cut. To embed $\mathbf{Q}$ into $\mathbf{R}$, we define the map $i : \mathbf{Q} \to \mathbf{R}$. For any rational $q \in \mathbf{Q}$, we designate

$$S_q = \{x \in \mathbf{Q} \mid x < q\}$$
$$L_q = \{y \in \mathbf{Q} \mid y \geq q\}$$

The pair (S_q, L_q) is a Dedekin cut. Define $i(q) = (S_q, L_q) \in \mathbf{R}$. We know that i is injective since for any $q_1 \neq q_2 \in \mathbf{Q}$, then $i(q_1) \neq i(q_2)$.

Definition (Order): Suppose $r_1 = (S_1, L_1) \in \mathbf{R}$ and $r_2 = (S_2, L_2) \in \mathbf{R}$. We say that $r_1 \leq r_2$ if $S_1 \subseteq S_2$. If $r = (S, L) \in R$ and $q \in \mathbf{Q}$ is a rational number, then $q < r$ if and only if $q \in S$.

Proposition

Suppose $r_1 = (S_1, L_1) \in \mathbf{R}$ and $r_2 = (S_2, L_2) \in \mathbf{R}$ so that $r_1 < r_2$. There exists a rational number $q \in \mathbf{Q}$ such that

$$r_2 < q < r_1$$

Proof: We have $r_1 < r_2$, which means that $S_1 \subset S_2$ and $S_1 \neq S_2$. Hence, there exists a q' so that $q' \in S_2$, which means that $q' < r_2$, but $q' \notin S_1$, which means that $r_1 \leq q'$. Hence, $r_1 \leq q' < r_2$. As an extensional exercise, we can find a q' such that the inequalities are strict.

Addition in Real Numbers

Suppose $r_1 = (S_1, L_1) \in R$ and $r_2 = (S_2, L_2) \in R$. We define:

$$S_1 + S_2 = \{a + b \mid a \in S_1, b \in S_2\} \in \mathbf{R}$$
$$r_1 + r_2 = (S_1 + S_2, \mathbf{Q} - (S_1 + S_2))$$

We can verify that $(S_1 + S_2, \mathbf{Q} - (S_1 + S_2) = L)$ is indeed a Dedekin cut. Referring to the definition, (1) and (2) are straightforward. We will show (4).

Proof of (4): Suppose, for the sake of contradiction, that there exists $x_0 \in S_1 + S_2$ and $y_0 \in L$ such that $x_0 \geq y_0$. As $x_0 \in S_1 + S_2$, there exists $a_1 \in S_1$, $a_2 \in S_2$ such that $a_1 + a_2 = x_0$. Hence, $a_1 + a_2 \geq y_0$ so $a_2 \geq y_0 - a_1$. We also know that $a_2 \in S_2$, which means $y_0 - a_1 \leq a_2$ implies $y_0 - a_1 \in S_2$.

Since $a_1 \in S_1$, then $y_0 = a_1 + (y_0 - a_1) \in S_1 + S_2$, This is a contradiction since $y_0 \in L$, which represents all elements in $\mathbf{Q}$ *excluding* $S_1 + S_2$. Hence, $x_0 < y_0$ for any $x_0 \in S_1 + S_2, y_0 \in L$.

April 5, 2021 (Real Analysis)

Recall that we defined addition in $\mathbf{R}$ to be the Dedekin cut $r_1 + r_2 = (S_1 + S_2, \mathbf{Q} - (S_1 + S_2))$ for $r_1 = (S_1, L_1) \in \mathbf{R}$ and $r_2 = (S_2, L_2) \in \mathbf{R}$. We can also define *multiplication* in $\mathbf{R}$ to be

$$r_1 * r_2 = (S_1 * S_2, \mathbf{Q} - S_1 * S_2)$$

Here, $S_1 * S_2 = \{a * b \mid a \in S_1, b \in S_2\}$.

Density of Q inside R

If $r_1, r_2 \in \mathbf{R}$ are two real numbers so that $r_1 < r_2$. There is a rational number $q \in \mathbf{Q}$ so that $r_1 < q < r_2$.

Proof: Take $r_3 = \frac{1}{2}(r_1 + r_2) \in \mathbf{R}$. This means that $r_1 < r_3 < r_2$. Suppose that $r_3 = (S_3, L_3)$ and $r_2 = (S_2, L_2)$, as all real numbers are Dedekin cuts. By definition, $r_3 < r_2$ so $S_3 \subseteq S_2$, and since $r_3 \neq r_2$, then $S_3 \neq S_2$. This means that there exists a $q \in S_2 \in \mathbf{Q}$ such that $q \notin S_3$.

Since $q \in S_2$, then $q < r_2$. Since $q \notin S_3$, then $r_1 < r_3 \leq q < r_2$, and we are done.

Fact: If $S \subseteq \mathbf{N}$ and $S \neq \emptyset$, then S has a minimum. If $S \subseteq \mathbf{Z}$ and $S \neq \emptyset$, and S has a lower bound, then S has a minimal element. We establish the following for $\mathbf{R}$.

Completeness Theorem

If $T \subseteq \mathbf{R}$, $T \neq \emptyset$, and T has a lower bound, then T has an *infimum*, which is a lower bound $t \in \mathbf{R}$ such that any $r > t$ is not a lower bound.

Proof: We construct a Dedekin cut for the infimum. We define S to be the set of all $q \in \mathbf{Q}$ such that q is a lower bound for T,. Let $L = \mathbf{Q} - S$ be the complemehnt of S. We can check that indeed (S, L) form a Dedekin cut as $S, L \subseteq \mathbf{Q}$, $S \cup L = \mathbf{Q}$, and $S \neq \emptyset$. Assume that T has a lower bound $b \in \mathbf{R}$, which means that $b = (S_b, L_b)$ where $S_b \subseteq S$, $S_b \neq \emptyset$. Note that $L \neq \emptyset$ (follows from $T \neq \emptyset$), and that if $x \in S, y \in L$, then $x < y$. We will prove this below.

We assume, for the sake of contradiction, that there exists a $x\ inS$, $y \in L$ such that $y \leq x$. Since $x \in S$, then x is a lower bound for L and as $y \leq x$, then y is also a lower bound. This means that $y \in S$, which is a contradiction since L and S are complementary sets.

Claim: If S has a maximal element, $\max(S) = S_0$ is the infimum of T

Proof: Since $S_0 \in S$, then S_0 is a lower bound for T. To show that any $y > s_0$ is not a lower bound, suppose there exists $y > s_0$ and y is a lower bound of T. This means that there exists a $q \in \mathbf{Q}$ such that $s_0 < q < y$ by the density of $\mathbf{Q}$ in $\mathbf{R}$. Hence, $q < y$, which means that q is a lower bound so that $q \in S$, which is a contradiction since q is larger than S_0, which is defined as $\max(S)$.

Claim: If S does not have a maximal element, $r = (S, L) \in \mathbf{R}$ is the infimum of T

Proof: Similar to the case where $\max(S)$ exists.

Definition: Supremum

Similar to infimum. The supremum of S is an upper bound s_0 of S so that any $y < s_0$ is not an upper bound of S. It is synonymous with the least upper bound.

The following is a famous application of the completeness theorem.

Theorem: Archimedean Property

If $a, b \in \mathbf{R}$ and $a, b > 0$, there exists a positive integer $n \in \mathbf{N}$ so that $na > b$.

Proof: Suppose that the theorem is false for some positive $a, b \in R$. We define the set $T = \{na \mid n \in \mathbf{N}\}$. This means that T has an upper bound, which is b. According to the **Completeness Theorem**, then T has a supremum t_0. Since $t_0 - a < t_0$, then $t_0 - a$ is no longer an upper bound. This implies that there exists an integer $n_0 \in \mathbf{N}$ so that $t_0 - a < na$. We get $t_0 < (n+1)a$, which is a contradiction as t_0 is an upper bound for T and $(n+1)a \in T$.

1 April 7, 2021 (Real Analysis)

We define a sequence in a numbering sequence, such as R, Q, Z as a map $N \to R$. Occasionally, we map from whole numbers and include 0 alongside N. We write $a(n)$ as a_n in this sequence. For instance, the sequence $a_n = \frac{1}{n}$ consists of $1, \frac{1}{2}, \frac{1}{3}, \cdots$.

One task using sequences include the approximation of $\sqrt{2}$, taking advantage of the fact that $(\sqrt{2}-1) = (\sqrt{2}+1)^{-1}$:

$$\begin{aligned}
\sqrt{2} &= 1 + (\sqrt{2}-1) \\
&= 1 + \frac{1}{\sqrt{2}+1} \\
&= 1 + \frac{1}{2 + (\sqrt{2}-1)} \\
&= 1 + \frac{1}{2 + \frac{1}{\sqrt{2}+1}} \\
&= 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{\sqrt{2}-1}}} \\
&= 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \cdots}}}
\end{aligned}$$

We describe the sequence $a_n = 1 + \underbrace{\frac{1}{2 + \frac{1}{2 + \cdots}}}_{(n-1)\text{ 2's}}$. We note that a_n will approach $\sqrt{2}$ rapidly. We observe the following theorem (we can take the proof for granted, albeit it relies on observing how much fraction is missing from each a_n).

Theorem

Suppose that $a_n = \frac{p_n}{q_n}$ for co-prime p_n, q_n and $q_n \geq n$. Then,

$$|a_n - \sqrt{2}| \leq \frac{1}{2q_n^2} \leq \frac{1}{2n^2}$$

We introduce the **definition of limit**. We say that $\{a_n\}$ converges to real number $r \in R$ if $\lim_{n \to \infty} a_n = r$, meaning that a_n approaches r as n grows without bound.

Formal Definition of Limit

We say that $\lim_{n\to\infty} a_n = r$ if for any $\epsilon > 0$, there exists an integer N such that for any number $n > N$, we have:

$$|a_n - r| < \epsilon.$$

Note that this means that a_n is a convergent sequence.

Example

Verify that $\lim_{n\to\infty} a_n = \sqrt{2}$, where a_n was the sequence defined above to approximate $\sqrt{2}$.

Solution: We know from the theorem that $|a_n - \sqrt{2}| < \frac{1}{2n^2}$. For any $\epsilon > 0$, we can take $N_e = \left\lfloor \sqrt{\frac{2}{\epsilon}} \right\rfloor$. This means that if $n > N_\epsilon$, we have

$$n > \frac{1}{2\epsilon} \to n^2 > \frac{1}{2\epsilon} \to \epsilon * n^2 > \frac{1}{2}.$$

Hence, we have $\frac{1}{2n^2} < \epsilon$. Combining this result with the theorem, we have $|a_n - \sqrt{2}| < \frac{1}{2n^2} < \epsilon$. Using the formal definition of a limit, we can confirm that a_n approaches $\sqrt{2}$ as $n \to \infty$.

We introduce a principle relating to divergence.

Divergence Definition

A sequence $\{a_n\}$ is said to be divergent if $\{a_n\}$ does not converge. More tangibly, to show divergence, we must show that for any $r \in R$, there exists an $\epsilon > 0$ such that for any $N \in \mathbf{N}$, there exists $n_\epsilon > N$ such that:

$$|a_n - r| \geq \epsilon.$$

Note that this is the *negation* of the convergence definition.

Example

Prove that $a_n = (-1)^n$ diverges.

Solution: We observe that either for any $r \in \mathbf{R}$, we have $|r-1| + |r-(-1)| \leq 2$, meaning that either $|r-1| \leq 1$ or $|r+1| \leq 1$. If we assume $|r-1| > 1$ (the other case proceeds similarly), we pick $\epsilon = 1$. Thus, for any $N \in \mathbf{N}$, we take an even $n > N$, and we find

$$|a_n - r| = |(-1)^n - r| = |1 - r| \leq 1 \leq \epsilon.$$

Example

Prove that $a_n = n^2$ is divergent.

Solution: Note that if $\{a_n\}$ is a sequence of real numbers, if there exists $N \in \mathbf{N}$ such that for any $n > N$, we have $a_n > M$ for any $M > 0$, then we say that $\lim_{n\to} a_n = +\infty$. Semantically, this definition means that no upper, finite boundary can possibly exist for a_n. Similarly, if for any $n > N$, we had $a_n < -M$ instead for any $M > 0$, we would say $\lim_{n\to} a_n = -\infty$, which means a_n diverges towards negative infinity.

To show that $\lim_{n\to\infty} a_n = +\infty$, we construct $N = \left\lfloor \sqrt{M} \right\rfloor + 1 \in \mathbf{N}$ for any $M > 0$. Then, for any $n > N$, we have

$$n^2 > N^2 = (\left\lfloor \sqrt{M} \right\rfloor + 1)^2 \geq (\sqrt{M} - 1 + 1)^2 = M.$$

Hence, $a_n > M$, meaning that $\lim_{n\to\infty} a_n = +\infty$ by the provided definition.

Example

Show that the sequence $a_n = \frac{2n+1}{3n+2}$ converges.

Solution: We hypothesize that the sequence converges toward $\frac{2}{3}$ as $\frac{2n+1}{3n+2}$ is very close to the fraction $\frac{2n}{3n} = \frac{2}{3}$, and especially when n gets large and the constant terms become less critical.

$$|a_n - \frac{2}{3}| = |\frac{2n+1}{3n+2} - \frac{2}{3}| = |\frac{(6n+3)-(6n+4)}{3(3n+2)}| = |\frac{1}{9n+6}| < \frac{1}{n}$$

Hence, for any $\epsilon > 0$, we take $N = \lfloor \frac{1}{\epsilon} \rfloor + 1 \in \mathbf{N}$, which means that for any $n > N$, then $\frac{1}{n} < \epsilon$. Hence,

$$|a_n - \frac{2}{3}| < \frac{1}{n} < \epsilon$$

This confirms that $\lim_{n\to\infty} a_n = \boxed{\frac{2}{3}}$.

Example

Show that the sequence $a_n = \sqrt{n^2+2n} - n$ converges.

Solution: We can make a guess for the convergence value by noting that n^2+2n is very close to $(n+1)^2$, which would lead to approximately $\sqrt{(n+1)^2} - n = n + 1 - n = 1$.

$$|a_n - 1| = |\sqrt{n^2+2n} - (n+1)| = |\frac{(\sqrt{n^2+2n})^2 - (n+1)^2}{\sqrt{n^2+2n} + (n+1)}| = |-\frac{1}{\sqrt{n^2+2n}+(n+1)}| < \frac{1}{n}$$

The fractionalization results from the fact that $(x-y) = \frac{x^2-y^2}{x+y}$. Using the same technique from the prior example, we can finish off the problem, eventually finding $\lim_{n\to\infty} a_n = \boxed{1}$.

2 April 9, 2021 (Real Analysis)

Last time, we defined a sequence to be a map from natural numbers to real numbers. We use the notation $\{a_n\}$ to represent a set, with $a(n) = a_n$ denoting the nth element of the sequence.

We define $\sup\{a_n\}$ to be the **supremum**, or least upper bound, of the set $\{a_n \mid n \in N\}$, which is the set that packages the elements of a sequence $\{a_n\}$. We define the **infinitum** $\inf\{a_n\}$, which is the greatest lower bound, of a sequence similarly.

Property 1

If $\lim_{n\to\infty} a_n$ exists, it is unique.

Proof: Suppose $r_1, r_2 \in R$ and $r_1 \neq r_2$ for $\lim_{n\to\infty} a_n = r_1$ and $\lim_{n\to\infty} a_n = r_2$. Recall the formal definition of a limit, which is that $\lim_{n\to\infty} a_n = r$ if for any $\epsilon > 0$, there exists a $N \in \mathbf{N}$ such that for any $n > N$, we have $|a_n - r| < \epsilon$. Here, we take $\epsilon_0 = \frac{1}{2}|r_1 - r_2| > 0$.

Since $\lim_{n\to\infty} a_n = r_1$, there exists $N_1 \in N$ so that $|a_n - r_1| < \epsilon_0$ for all $n > N_1$. Since $\lim_{n\to\infty} a_n = r_2$, then there exists $N_2 \in N$ such that $|a_n - r_2| < \epsilon_0$ for all $n > N_2$. To proceed we will use the triangle inequality.

Triangle Inequality

If $a, b \in R$, then $|a| + |b| \leq |a + b|$.

We take $n_0 > \max(N_1, N_2)$, as well as $a = a_{n_0} - r_1$ and $b = r_2 - a_{n_0}$. By the Triangle Inequality, we get

$$|r_1 - r_2| = |a + b| \leq |a| + |b| = |a_{n_0} - r_1| + |a_{n_0} - r_2| < \epsilon_0 + \epsilon_0 = 2\epsilon_0 = |r_1 - r_2|.$$

This is a contradiction, as we claimed that $|r_1 - r_2| < |r_1 - r_2|$. The last part works as we cleverly selected $\epsilon_0 = \frac{1}{2}|r_1 - r_2|$ so $2\epsilon_0 = |r_1 - r_2|$.

Property 1

Suppose that $\{a_n\}$ and $\{b_n\}$ are two convergent sequences. Suppose $a_n \leq b_n$ for any $n \in N$. Then,

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$$

Proof: Suppose that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Suppose, for the sake of contradiction, that $a > b$. We take $\epsilon_0 = \frac{1}{2}(a-b) > 0$. As $\lim_{n\to 0} a_n = a$, then there exists a $N_a \in \mathbf{N}$ such that $|a_n - a| < \epsilon_0$ for any $n > N_a$. Analagously, as $\lim_{n\to 0} b_n = b$, then there exists a $N_b \in \mathbf{N}$ such that $|b_n - b| < \epsilon_0$ for any $n > N_b$.

We can take $n_0 > \max N_a, N_b$, which means $|a_{n_0} - a|, |b_{n_0} - b| < \epsilon_0$. From $|b_{n_0} - b| < \epsilon_0$, we get $b_{n_0} - b < \epsilon_0$, so $b_{n_0} < b + \epsilon_0 = b + \frac{1}{2}(a - b) = \frac{1}{2}(a + b)$. From $|a_{n_0} - a| < \epsilon_0$, we get $a_{n_0} - a > -\epsilon_0$, so $a - \epsilon_0 < a_{n_0}$, eventually leading to $\frac{1}{2}(a + b) < a_{n_0}$ by plugging in our expression for ϵ_0.

This means that $b_{n_0} < \frac{1}{2}(a + b) < a_{n_0}$, which is a contradiction since $a_n \leq b_n$ for any $n \in N$. Thus, we confirm $a \leq b$.

Remark: Note that $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$ is also true for the stronger assumption that $a_n < b_n$ for any $n \in N$. Note that it is not necessarily true that $\lim_{n\to\infty} a_n < \lim_{n\to\infty} b_n$ even if we have $a_n < b_n$ for any $n \in N$. Consider $a_n = -\frac{1}{n}$ and $b_n = \frac{1}{n}$ for example, which both gravitate to 0 as $n \to \infty$.

Remark: Note that if $a_n < b$ for all $n \in N$ in sequence $\{a_n\}$, then $\lim_{n\to\infty} a_n \leq b$.

Sandwich Theorem

Suppose $\{a_n\}, \{b_n\}, \{c_n\}$ care three sequences, and that $a_n \leq b_n \leq c_n$ for all $n \in N$. Suppose that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = r \in R$. Then, b_n also converges, and in particular,

$$\lim_{n\to\infty} b_n = r$$

We define a **subsequence** to consist partially of the elements in a sequence. For instance, if we have a sequence $\{a_1, a_2, a_3, a_4, a_5, a_6, \cdots\}$, an example of a subsequence would be $\{a_1, a_4, a_5, a_7, \cdots\}$. More formally, we write $\{a_{n_k}\}_{k\in N}$ as a subsequence of $\{a_n\}_{n\in N}$, where $\{n_k\}$ is an integer sequence of positive indices indicating which elements to select from $\{a_n\}$. For example, if $\{a_n\}$ is the sequence $a_1, a_2, a_3, \cdots$, then $\{a_{2n}\}_{n\in N}$ is the sub-sequence consisting of $a_2, a_4, a_6, \cdots$.

Problem

Show that if $\{a_n\}$ converges, then so does $\{a_{2n}\}$.

Solution: Suppose that $\{a_n\}$ converges to r. This means that for all $\epsilon > 0$, there exists a $N_\epsilon \in N$ so that for all $n > N_\epsilon$, $|a_n - r| < \epsilon$. Now, for all $n > N_\epsilon$, then $2n > N_\epsilon$, meaning that

$$|a_{2n} - r| < \epsilon$$

By definition, $\{a_{2n}\}$ also converges to r.

Remark

If the sequence $\{a_n\}$ converges. then $\{a_n\}$ is bounded, meaning that it has an upper bound and a lower bound.

Proof: Suppose that $\{a_n\}$ converges to a, meaning that

$$\lim_{n\to\infty} a_n = a.$$

Take $\epsilon = 1$. Then, there exists an integer N_ϵ such that $|a_n - a| < 1$ for any $n > N_e$. This means that $|a_n| \leq a + 1$ for any $n > N_e$, which means that a_n is bounded.

We introduce the following toolbox of properties.

Sequence Limit Properties

Suppose that $\lim_{n\to\infty} a_n = a \in R$ and $\lim_{n\to\infty} b_n = b \in R$ for sequences $\{a_n\}, \{b_n\}$ for convergent sequences $\{a_n\}, \{b_n\}$. We have

1. $\lim_{n\to\infty} a_n + b_n = a + b$
2. $\lim_{n\to\infty} a_n * b_n = a * b$
3. $\lim_{n\to\infty} \frac{a_n}{b_n}$ is equal to $\frac{a}{b}$ if $b \neq 0$, $\pm\infty$ if $a \neq 0$ and $b = 0$, 0 if $a = 0$ and $b \neq 0$, and inconclusive if $a = b = 0$.

Proof of (2) Suppose that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. This means that there exists an M such that $|a_n| \leq M$ and $|b_n| \leq M$ for any $n \in N$, as both $\{a_n\}$ and $\{b_n\}$ are bounded by an earlier remark. For any $\epsilon > 0$, there exists N_a and N_b such that for $n > \max(N_a, N_b)$,

$$|a_n - a| \leq \frac{1}{2M} * \epsilon$$
$$|b_n - b| \leq \frac{1}{2M} * \epsilon$$

We chose $\frac{1}{2M} * \epsilon$ strategically and it works because $|a_n - a|$ and $|b_n - b|$ do not have a non-zero lower bounds. Using re-arranging and the triangle inequality,

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - a_n b + a_n b - ab| \\ &= |a_n(b_n - b) + (a_n - a)b| \\ &\leq |a_n| * |b_n - b| + |a_n - a| * |b| \\ &< M * \frac{1}{2M} * \epsilon + M * \frac{1}{2M} * \epsilon \\ &= \epsilon. \end{aligned}$$

By the formal definition of the limit, the sequence $\{a_n b_n\}$ converges toward the value ab and hence $\lim_{n\to\infty} a_n b_n = ab$.

April 12, 2021 (Real Analysis)

Monotonicity

A sequence $\{a_n\}$ is called monotonically

- increasing if $a_{n+1} \geq a_n$ for each $n \in \mathbf{N}$
- decreasing if $a_{n+1} \leq a_n$ for each $n \in \mathbf{N}$

We introduce the following theorem.

Theorem

Suppose $\{a_n\}$ is a monotonically increasing sequence and it has an upper bound M with $a_n \leq M$ for any $n \in \mathbf{N}$. Then. $\{a_n\}$ converges.

Similarly, if $\{a_n\}$ is monotonically decreasing and has a lower bound, then $\{a_n\}$ also converges.

Proof: We conjecture that the supremum $r = \sup_{n \in \mathbf{N}} \{a_n\}$ will be equal to $\lim_{n\to\infty} a_n$. We know that r exists using the Completeness Theorem as $\{a_n\}$ has an upper bound.

For any $\epsilon > 0$, we know that $r - \epsilon$ is no longer an upper bound for $\{a_n\}$. Then, there exists an $N \in \mathbf{N}$ such that $r - \epsilon < a_N \leq r$. Since $\{a_n\}$ is increasing, for any $n > N$, we have $r - \epsilon < a_N \leq a_n \leq r < r + \epsilon$. This means that $r - \epsilon < a_n < r + \epsilon$, so $|a_n - r| < \epsilon$ for any $n > N$, which completes the proof for $\lim_{n\to\infty} a_n = r = \sup_{n\in\mathbf{N}} \{a_n\}$.

Nested Interval Theorem || Cauchy-Cantor Theorem

Suppose that $[a_n, b_n]$ is a seqwuence of *closed* intervals so that $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$. Then, there exists a real number in the intersection of all intervals.

$$x \in \cap_{n=1}^{+\infty} [a_n, b_n]$$

Proof: We have a monotonically increasing sequence $\{a_n\}$, which has an upper bound b_1. This means that $\lim_{n\to\infty} a_n = a \in R$. Similarly, we have $\lim_{n\to\infty} b_n = b \in R$. Since we have $a_n \leq b_n$ for all $n \in \mathbf{N}$, then

$$a = \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n = b$$

Then, x can be chosen as any real number in $[a, b]$.

Supremum and Infimum Sequences

Suppose that $\{a_n\}$ is a sequence and is bounded, which means that there exists $M \in R$ such that $|a_n| < M$ for all $n \in \mathbf{N}$. Define sequences $S_n = \sup(a_k \mid k \geq n)$ and $I_n = \inf\{a_k \mid k \geq n\}$. Here, $S_{n+1} \leq S_n$, $I_{n+1} \geq I_n$, and $S_n \geq I_n$. We conclude that $\{S_n\}$ is monotonically decreasing and $\{I_n\}$ is monotonically increasing, and both are bounded. We define

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} S_n$$
$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} I_n$$

Properties for $\limsup a_n$ and $\liminf a_n$ for sequences $\{a_n\}$ are provided below.

- $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$
- $\{a_n\}$ converges if and only if $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$. If $\{a_n\}$ converges , then

$$\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \lim_{n\to\infty} a_n$$

- If $\{a_{n_k}\}$ is a sub-sequence of $\{a_n\}$ so that $\{a_{k_n}\}$ converges, then

$$\liminf_{n\to\infty} a_n \leq \lim_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$$

April 16, 2021

We define S_n and I_n as the monotonically increasing sequences

$$S_n = \sup(\{a_k \mid k \geq n\})$$
$$I_n = \inf(\{a_k \mid k \geq n\})$$

We have the following theorem.

Theorem

The sequence $\{a_n\}$ converges if and only if

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} S_n = \lim_{n\to\infty} I_n = \liminf_{n\to\infty} a_n.$$

Proof: Since we have an if and only if statement, we will need to prove both directions. Suppose that $\{a_n\}$ converges, which means that there is a real number $r = \lim_{n\to\infty} a_n$ such that for any $\epsilon > 0$, there exists a $N \in \mathbf{N}$ such that $|a_n - r| < \epsilon$ for all $n > N$, or $r - \epsilon < a_n < r + \epsilon$.

Recall that $S_n = \sup\{a_k \mid k \geq n\}$, which means that if $n > N$, then $S_n \leq r+\epsilon$ and similarly, $r - \epsilon \leq I_n$. Hence, we have

$$r - \epsilon \leq I_n \leq S_n \leq r + \epsilon.$$

Next, we need to prove the other direction. Suppose that $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$. Then,

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = r.$$

This means that for any $\epsilon > 0$, there exists a $N_S \in N$ so that 2
eating sushi rn, will fill in the missing proof later

Here is another property involving $\limsup$ and $\liminf$:

Example 1

Suppose that $\{a_n\}$ is a bounded sequence that may have some sub-sequences that converge. If $\{a_{n_k}\}$ is a convergent sub-sequence, then

$$\liminf_{n\to\infty} a_n \leq \lim_{k\to\infty} a_{n_k} \leq \limsup_{n\to\infty} a_n$$

We introduce the following **crucial** topic of Cauchy sequences.

Definition: Cauchy Sequence

A sequence $\{a_n\}$ is called Cauchy if for any $\epsilon > 0$, there exists a $N \in \mathbf{N}$ such that for any $m, n > N$, we have

$$|a_m - a_n| < \epsilon.$$

Recall that a sequence $\{a_n\}$ converges to $r \in \mathbf{R}$ if for any $\epsilon > 0$, there exists a $N \in \mathbf{N}$ such that for any $n > N$, we have

$$|a_n - r| < \epsilon.$$

Theorem

A sequence $\{a_n\}$ converges *if and only if* it is Cauchy!

Proof: We will need to prove both directions. Suppose that $\{a_n\}$ converges to r; we aim to prove that it is Cauchy. Hence, for any $\epsilon > 0$, we have a $N \in \mathbf{N}$ such that for any $n > N$, we have $|a_n - r| < \frac{\epsilon}{2}$ and, for any $m > N$, we have $|a_m - r| < \frac{\epsilon}{2}$. If $m, n > N$, then we have

$$|a_m - a_n| = |a_m - r + r - a_n| \leq |a_m - r| + |r - a_n| \leq |a_m - r| + |r - a_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence, we have $|a_m - a_n| < \epsilon$ for any $\epsilon > 0$ and some $m, n > N$, so $\{a_n\}$ is Cauchy.

Next, we prove the other direction. Suppose that $\{a_n\}$ is Cauchy. Our aim is to show that $\{a_n\}$ is bounded and that $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$. Since $\{a_n\}$ is Cauchy, we take $\epsilon = 1$. There exists a N such that, for all $m, n > N$,

$$|a_m - a_n| < \epsilon = 1$$

Take $m = (N + 1)$. Then, for all $n > N$, we have

$$a_{N+1} - 1 < a_n < a_{N+1} + 1$$

This confirms that $\{a_n\}$ is bounded. Next, we will show that $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$. Recall that $S_n = \sup a_k \mid k \leq n$, which implies that $a_n \leq S_n \leq a_{N+1} + \epsilon$. Similar for infimum, we get

$$a_{N+1} - \epsilon \leq I_n \leq a_n \leq S_n \leq a_{N+1} + \epsilon$$

This means that $|S_n - I_n| = 2\epsilon$ as 2ϵ is the length of the entire interval above. Hence, $\lim_{S_n - I_n} = 0$, which implies $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$. In conjunction with the fact that $\{a_n\}$ is bounded, this means that $\{a_n\}$ converges.