Math 53 Section Notes

Spring 2021

These are supplementary notes from Math 53, a course mainly on differential equations. My section TA was Francois-Simon Fauteux-Chapleau.

1 March 30, 2021

Example 1 (i)

Solve the following differential equation.

$$y' = y$$
$$y(0) = 10$$

Solution: We know that the solution is based on the exponential function. We can guess $y(t) = 10e^t$ and verify it via $y'(t) = 10(e^t)' = 10e^t$ and $y(0) = 10e^0 = 10$. More generally, for any constant $c\ inR$, $y(t) = ce^t$ is a solution to $y'(t) = y(t)$.

Example 1 (ii)

Solve the following differential equation.

$$y' = 2y$$
$$y(0) = 1$$

Solution: We claim that $y(t) = e^{2t}$, and utilizing the Chain Rule means that

$$\frac{dy}{dt} = \frac{dy}{ds} * \frac{ds}{dt} = e^s * 2 = 2 * y(t).$$

We can easily validate that $y(0) = e^{2*0} = 1$.

Example 1 (iii)

Solve the following differential equation.

$$y' = -y \text{ and } y(0) = 100$$

Solution: We claim that the solution is $y(t) = 100e^{-t}$, which we can readily greenlight through differentiation and computing $y(0)$.

Example 1 (iv)

Solve the following differential equation.

$$y' = 1 - y \text{ and } y(0) = 1$$

Solution: We perform a change of variables via $z = 1 - y$, which means that $z' = 0 - y' = -y'$. Hence, $y' = 1 - y$, meaning that $z' = -z$, which was the equation in part (iii). We have the initial conditions $z(0) = 1 - y(0) = 0$. We can similarly solve it through $z(t) = 0 * e^{-t} = 0$. Tracing backward, we get $y(t) = 1 - z(t) = 1 - 0 = 1$.

Example 1 (v)

Solve the following differential equation.

$$y' = 1 - y \text{ and } y(0) = 2$$

Solution: We perform the same change of variable as in part (iv), which leads us to $z' = -z$ with $z = 1 - y$. However, we have a new initial condition $z(0) = 1 - y(0) = 1 - 2 = -1$. The revamped solution will be $z(t) = -e^{-t}$. From here, we can produce $y(t) = 1 - (-e^{-t}) = 1 + e^{-t}$.

Example 2

Draw the graphs of all five functions in a single ty-plane by hand. Make all qualitative features of your graphs visible.

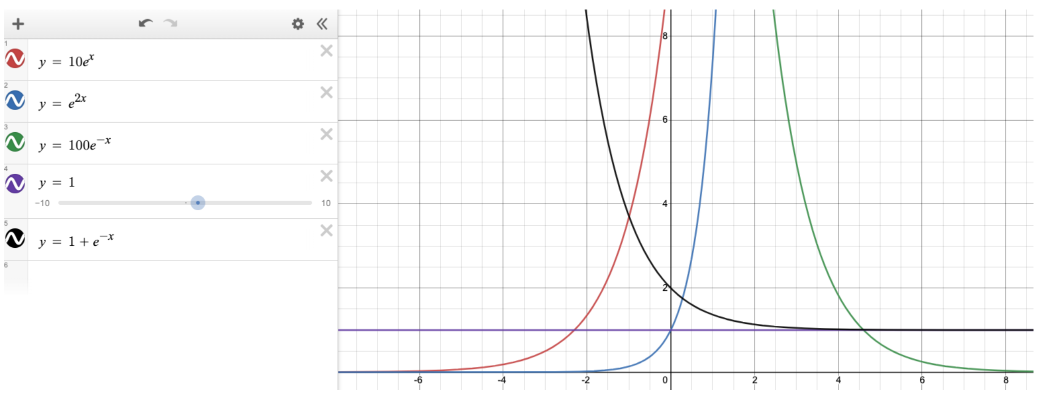


Figure 1: *Graphs of the five functions listed in order.*

The features we can consider are (1) positive or negative, (2) increasing or decreasing, and the (3) value at $t = 0$. It is important to consider the (4) asymptotic features as well, studying the behavior of y as $t \to \pm\infty$. We can also examine the (5) growth rate of each function.

Infection Model

We simplify our model for the spread of a pathogen from Lecture 1 even further. We will only consider the discrete model where people move once every day and explore the consequences of lots of travel quantitatively.

Let us continue thinking of the world as a very large square, but also draw a regular grid with a fine mesh in this square (think of the square as drawn in a graph paper). Now we assume that at any given day there is a single person at each crossing of the grid. Also assume that any person who is adjacent (horizontal, vertical or diagonal) to an infected person becomes infected the next day.

Example 3

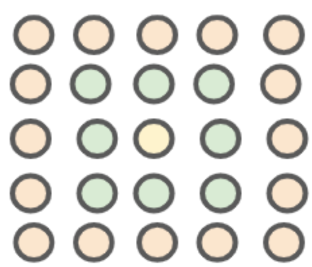
- Write down the corresponding difference equation assuming lots of travel, which we take to mean that people rearrange themselves ("travel") in this square world so heavily every day that the "spheres of influence" of two infected persons never intersect. Note that we can determine the proportionality constant here. Assume that on day 0 there is only one infected person and solve the difference equation.
- Now assume there is no travel. With only one person being infected on day 0, compute the number of people who are infected on day N. No need to write a difference equation.
- Compare your results. Which option is worse? Explain your answer without computation

The first assumption is that there is lots of travel. At day N, an infected person is always infecting 8 new people in the square grid (the people surrounding them). Let $I(n)$ be the number of infected people on day n. We can write

$$\begin{aligned} I(n) - I(n-1) &= 8 * I(n-1) \\ I(n) &= 9 * I(n-1) \\ I(n) &= 9^N * I(0) = 9^N. \end{aligned}$$

The second assumption is that no travel. We have a growing square of infected people where one person infects their neighbors, and those neighbors infect their neighbors in circularly outward fashion.

We can envision that on day N, there is a square of infected people. The side length of this square in *persons* grows by two each day: on Day 0, we have a side length of 1, on Day 2, we have a side length of 3, and On Day 3, we have a side length of 5. Thus, the side length on day n will be $2n+1$, which means that there will be $I(n) = (2n+1)^2$ people infected on day n.



With travel, we have exponential growth with $I(N) \sim O(9^N)$, and with no travel, we have polynomial growth with $I(N) \sim O(N^2)$.

April 1, 2021

Problem 1

Find the constant solutions to the following differential equation. Note that for each a, this equation has a unique solution. Examine the limits for the other solutions.

$$y' = y(1-y)(10-y)$$
$$y(0) = a$$

Solution: First, we aim to find a constant solution. If $y(t) = a$, then $y' = y(1-y)(10-y)$ devolves into $0 = a(1-a)(10-a)$. Hence, a is either 0, 1, or 10 in this scenario, so $y = 0, 1, 10$ are all compatible. Next, we consider the possible limit values for other solutions. We have an equation of the form $y' = f(y)$, which means that the other solutions $y(t)$ must approach either $\pm\infty$ or one of the constant solutions as $t \to \infty$.

If $1 < a < 10$, we see that $y'(t) < 0$, so we have a decreasing solution. If $0 < a < 1$, then $y'(t) > 0$, meaning that we have an increasing solution. If $a < 0$, the solution is decreasing, and if $a > 10$, the solution is increasing. This helps us examine the asymptotic behavior of the solutions $y(t)$ as $t \to \infty$.

Problem 2

Craft a differential equation of the form $y' = Cy(y+A)(y+B)$ so that all limit values are real (not equivalent to $\pm\infty$).

Solution: The claim is that if $C < 0$, the end behavior of $y(t)$ is invariant on the values of A and B. If y is very large, then $y + A > 0$, $y + B > 0$, $y > 0$, $C < 0$, which means that $y' < 0$. If y is very small, then $y + A$, $y + B$, y, and C will also be negative, meaning that $y' > 0$ via cancellation. Hence, $y'(t)$ will never "blow up" to infinity as $t \to \infty$ since $y(t)$ travels, in layman terms, downward at high values and upward at low values.

Problem 3

We can create a differential equation of the form $y' = Cy(y+A)(y+B)$ so that we only get one of $\pm\infty$ as a limit value?

Solution: If $c < 0$, as demonstrated in Problem 1, each solution will approach a real number. If $c > 0$, the signs invert, meaning that $y(t)$ will blow up toward $\pm\infty$ as $y \to \infty$. At large values of y, $y'(t) > 0$, and at very small (large negative) values of y, $y'(t) < 0$. Thus, the task is not possible, since either both ∞ and $-\infty$ will be approached or neither will be. Disproving the $c = 0$ case is relatively trivial.

Problem 4

A bank is advertising a simple interest loan of P_0 dollars with an annual interest rate r and interest period of $\frac{1}{n}$ years. Let $p(t)$ be the debt at time t. Simple interest means that every time the interest kicks in, your debt increases by $r * P_0$.

- Write down the difference equation for all n.
- Explicitly solve these difference equations.

The difference equation is

$$p(t + \frac{1}{n}) - p(t) = \frac{r}{n} * P_0.$$

Hence, with $P(0) = P_0$, we have

$$
\begin{aligned}
P(\frac{1}{n}) &= P(0) + \frac{r}{n}P_0 = P_0 + \frac{r}{n}P_0 \\
P(\frac{2}{n}) &= P(\frac{1}{n}) + \frac{r}{n}P_0 = P_0 + \frac{2r}{n} * P_0 \\
&\cdots \\
P(\frac{k}{n}) &= P_0 + \frac{kr}{n} * P_0 = P_0(1 + \frac{k}{n}r)
\end{aligned}
$$

If $t = \frac{k}{n}$, then $P(t) = P_0(1 + tr)$. Hence, we see that simple interest $p(t)$ increases linearly with time. We can build the *differential equation* from the difference equation as shown.

$$
\begin{aligned}
P(t + \frac{1}{n}) - P(t) &= \frac{r}{n} * P_0 \\
\frac{P(t + \frac{1}{n}) - P(t)}{\frac{1}{n}} &= rP_0 \\
P'(t) &= rP_0
\end{aligned}
$$

The third equation results from the definition of the derivative via applying a limit as $n \to \infty$. Solving this differential equation leads to

$$P(t) = \int rP_0 \; dt = trP_0 + C$$

Since $P(0) = P_0$, then $C = P_0$, so the final solution is $P(t) = \boxed{P_0(1 + tr)}$. We observe that two solutions are equal and behave in linear fashion.

2 April 6, 2021

Problem 1

Find the general solution to
$$P'(t) = rP(t)$$

Solution: We can readily observe that $P(t) = 0$ is a constant solution. Due to the first order nature of the ODE, different solutions cannot intersect and hence all non-constant solutions need to obey $P(t) \neq 0$. Using *separation of variables*,

$$\begin{aligned}\frac{P'(t)}{P(t)} &= r\\ \int \frac{1}{P}\, dP &= \int r\ dt\\ \log|P| &= rt + c\\ |P| &= e^{rt+c}\end{aligned}$$

This means that either $P(t) = e^{rt+c} = e^c * e^{rt}$ or $P(t) = -e^{rt+c} = -e^c * e^{rt}$. Recall that $P(t) = 0 = 0 * e^{rt}$ is also a solution. Hence, multiplied to e^{rt} is a constant, e^c, $-e^c$, or 0, that is either positive, negative, or zero. Our general solution is thus $\boxed{P(t) = Ce^{rt}}$ for any $C \in R$.

Problem 2

Find the general solution to the logistic ODE, where $k > 0$,
$$y'(t) = ry(t)(1 - \frac{y(t)}{k})$$

Solution: We consider the constant solutions $y(t) = 0$ and $y(t) = k$. Since we have a first order ODE, we cannot have intersections, meaning that any non-constant solutions will fall into one of the following ranges.

- $y(t) < 0$
- $0 < y(t) < k$
- $y(t) > k$

We will perform separation of variables, rearranging the equation and integrating both sides

$$\begin{aligned}\frac{Ky'(t)}{y(K-y)} &= r\\ \int \frac{K}{y(K-y)}y'(t)\ dt &= \int r\ dt\\ \underbrace{\int \frac{1}{y} + \frac{1}{K-y}\ dy}_{y'(t) \text{ vanishes due to Chain Rule of } dy = y'(t)\ dt} &= rt + c\end{aligned}$$

We proceed through integration and logarithm properties.

$$\begin{aligned}\log|y| - \log|K-y| &= rt + c\\ \log|\frac{y}{K-y}| &= rt + c\\ |\frac{y}{K-y}| &= e^{rt+c}\end{aligned}$$

We will consider each of the three cases enumerated above. In the $y < 0$ case, we have $\frac{y}{K-y} < 0$; in the $0 < y < K$ case, we have $\frac{y}{K-y} > 0$; in the $y > K$ case, we have $\frac{y}{K-y} < 0$. We proceed through casework anchored on the sign of the fraction $\frac{y}{K-y}$.

- *The fraction $\frac{y}{K-y}$ is negative:* Here, $y < 0$ or $y < k$. Through rearranging, we obtain

$$\begin{aligned} |\frac{y}{K-y}| = \frac{y}{y-k} &= e^{rt+c} \\ y &= (y-k)e^{rt+c} \\ \boxed{y(t) = \frac{-Ke^{rt+c}}{1-e^{rt+c}}} \end{aligned}$$

- *The fraction $\frac{y}{K-y}$ is positive:* Here, $0 < y < k$. Through rearranging, we obtain

$$\begin{aligned} \frac{y}{K-y} &= e^{rt+c} \\ (e^{rt+c}+1)y &= Ke^{rt+c} \\ \boxed{y(t) = \frac{Ke^{rt+c}}{e^{rt+c}+1}} \end{aligned}$$

Problem 3

Find the general solution to the following ODE for $r > 0$, $t \geq 0$, and $C(0) = C_0 \geq 0$.

$$\frac{dC}{dt} = -rC^2$$

Solution: We have the constant solution $C(t) = 0$ when $C_0 = 0$. The next step is to explore the case where $C_0 > 0$. Through separation of variables, we obtain

$$\begin{aligned} \frac{1}{C^2}\frac{dC}{dt} &= -r \\ \int \frac{1}{C^2}\frac{dC}{dt}\ dt &= \int -r\ dt \\ \int \frac{1}{C^2}\ dc &= \int -r\ dt \end{aligned}$$

Performing the integration and rearranging results in

$$\begin{aligned} \frac{-1}{C} &= -rt + k \\ C(t) = \frac{-1}{-rt+k} &= \frac{1}{rt-k}. \end{aligned}$$

Here, we get $C(0) = \frac{1}{0-k} = -\frac{1}{k}$, which is equal to C_0 due to the provided initial value. Hence, $k = -\frac{1}{C_0}$.
This means that our solutions are the following listed:

- $C_0 = 0 \to C(t) = \boxed{0}$
- $C_0 > 0 \to C(t) = \frac{1}{rt-k} = \boxed{\frac{1}{rt + \frac{1}{C_0}}}$.

3 April 8, 2021

Example 1 + 2

Consider and solve the following Bernoulli differential equation for $a \in R$:

$$y'(t) + q(t)y = r(t)y^a$$

Solution: We briefly discuss each case of a. If $a = 0$, we have a non-homogenuous linear ODE $y' + q(t)y = r(t)$. If we have $a = 1$, we have the ODE $y' + q(t)y = r(t)y$, which leads to a homogenuous ODE of form $y' + (q(t) - r(t))y = 0$. If a is another integer, we will have a non-linear ODE.

We consider the general $y' + q(t)y = r(t)y^a$, and particularly want to multiply the function $u = y^{1-a}$ to both sides, with the motivation of reducing the exponent of y on the right side. Through Chain Rule, we get

$$\frac{du}{dt} = \frac{du}{dy} * \frac{dy}{dt} = (1-a)y^{(1-a-1)} * \frac{dy}{dt} = (1-a)y^{-a} * \frac{dy}{dt}.$$

We knowan expression for $\frac{dy}{dt} = y'$ from the differentiation equation itself! Thus,

$$\begin{aligned}\frac{du}{dt} &= (1-a)y^{-a} * (-q(t)y^1 + r(t)y^a) \\ &= -(1-a)q(t)y^{1-a} + (1-a)r(t)y^0\end{aligned}$$

This results in $u' + (1-a)q(t)y^{1-a} = (1-a)r(t)$. Noting that $u = y^{1-a}$, we plug it to retreive $\boxed{u' + (1-a)q(t)u(t) = (1-a)r(t)}$. Through u substitution, we now have a truly linear ODE that we can solve more conventionally.

Example 3

Consider and solve the following differential equation for $a \in R$ using the method demonstrated in the prior example.

$$y'(t) = ry(1 - \frac{y}{K})$$

We note that we can re-arrange this logistic ODE into $y' - ry = \frac{-r}{K}y^2$, which is the form we examined in Example 1. Note that $a = 2$, $q(t) = -r$, and $r(t) = \frac{-r}{K}$ here for the Bernoulli

$$y'(t) + q(t)y = r(t)y^a.$$

Hence, we will consider $u = y^{1-a} = y^{1-2} = y^{-1}$. Using the general, simplified ODE we found in Example 1, which was $u' + (1-a)q(t)u = (1-a)r(t)$, we plug in the new numbers and variables, with $a = 2$, $q(t) = -r$, and $r(t) = \frac{-r}{K}$ leading to

$$\begin{aligned}u' + (1-a)q(t)u &= (1-a)r(t) \\ u' + (-1)(-r)u &= (-1)(-\frac{r}{K}) \\ u' + ru &= \frac{r}{k}\end{aligned}$$

We multiply the integrating factor e^{rt} to both sides, which leads to

$$u'e^{rt} + rue^{rt} = \frac{r}{k}e^{rt}$$

The left-hand side is actually equal to $(ue^{rt})'$, which can be confirmed via product rule and expansion. Hence, through integrating both sides with respect to t, we get

$$\int (ue^{rt})' \; dt = \int \frac{r}{k} e^{rt}$$
$$ue^{rt} = \frac{r}{k} * \frac{1}{r} e^{rt} + C$$
$$u = \frac{\frac{r}{k} * \frac{1}{r} e^{rt} + C}{e^{rt}}$$
$$u = \frac{\frac{1}{k} e^{rt} + C}{e^{rt}}$$

Since $u = \frac{1}{y}$ by how we defined u, then $y = \frac{1}{u} = \frac{e^{rt}}{\frac{1}{k}e^{rt}+C}$, which can be represented through $u = \boxed{\dfrac{k}{1 + Ce^{-rt}}}$. This occurs by dividing both the numerator and denominator by ke^{rt} and flexibly treating our integration constant C.

We note that we have the constant solution $y = K$ accommodated by using $C = 0$, but it turns out that we are currently missing the solution $y = 0$. We are missing the solution because we used the intermediate variable $u = \frac{1}{y}$, which artificially constrained $y \neq 0$.

4 April 13, 2021 (Differential Equations)

Section Notes // TA: Francois-Simon Fauteux-Chapleau

Example 1

Consider $x' = Ax$, where A is a 2x2 matrix with repeated eigenvalue λ. Find the general solution if v_1 and v_2 are linearly independent eigenvectors of A, and find matrix A.

Explication: The characteristic polynomial is $(z-\lambda)^2$. From lecture, we know that $x_1(t) = e^{\lambda}v_1$ and $x_2(t) = e^{\lambda t}v_2$ are solutions. The general solution is

$$\boxed{x(t) = c_1 e^{\lambda t} v_1 + c_2 e^{\lambda t} v_2}.$$

Since v_1, v_2 are linearly independent vectors in R^2, they form a basis of R^2, meaning for any $v \in R^2$, we can write $v = c_1 v_1 + c_2 v_2$ for some $c_1, c_2 \in R$. This means that $Av = \lambda v$ for any $v \in R^2$, which follows from $Av_1 = \lambda v_1$ and $Av_2 = \lambda v_2$. This implies that A is the identity matrix multiplied by λ, so

$$A = \lambda I_2 = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}}.$$

Example 2

Let $A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$. Find the general solution to the differential equation $x' = Ax$.

Explication: We note that $\mathrm{tr}(A) = 2\lambda$ and $\det(A) = \lambda^2$, so the characteristic polynomial is equal to $z^2 - \mathrm{tr}(A)z + \det(A) = z^2 - 2\lambda z + \lambda^2 = (z-\lambda)^2$. We solve for our eigenvectors directly.

$$\begin{gathered} Av = \lambda v \\ \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \end{bmatrix} \\ \left\{ \begin{matrix} \lambda a + b = \lambda a \\ \lambda b = \lambda b \end{matrix} \right\} \end{gathered}$$

This means that $b = 0$, so our eigenvector is of the form $\begin{bmatrix} a \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus, our solution here is $x_1 = e^{\lambda t} v_1 = e^{\lambda t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so $x_1(t) = \begin{bmatrix} e^{\lambda t} \\ 0 \end{bmatrix}$ (we incorporate the constant a into the general solution). We can acquire other solutions through directly solving $x_2'(t) = Ax_2(t)$ for $x_2(t) = \begin{bmatrix} a(t) \\ b(t) \end{bmatrix}$.

$$\begin{gathered} x_2'(t) = Ax_2(t) \\ \begin{bmatrix} a'(t) \\ b'(t) \end{bmatrix} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} a(t) \\ b(t) \end{bmatrix} \\ \left\{ \begin{matrix} a'(t) = \lambda a(t) + b(t) \\ b'(t) = \lambda b(t) \end{matrix} \right\} \end{gathered}$$

From here, we know that $b(t) = e^{\lambda t}$ from the second equation in the system. We will solve for $a(t)$ next. We have $a'(t) = \lambda a(t) + e^{\lambda t}$ from the first equation, which leads to $a'(t) - \lambda a(t) = e^{\lambda t}$. We use the integrating factor $e^{-\int \lambda \, dt} = e^{-\lambda t}$. This leads to

$$\begin{aligned} (a(t)e^{-\lambda t})' &= e^{\lambda t} e^{-\lambda t} = 1 \\ a(t)e^{-\lambda t} &= t \\ a(t) &= te^{\lambda t}. \end{aligned}$$

Thus, we have $x_2(t) = \begin{bmatrix} te^{\lambda t} \\ e^{\lambda t} \end{bmatrix}$. Note that we have largely ignored the integration constants. We can incorporate them in the general solution.

$$x(t) = c_1 x_1(t) + c_2 x_2(t) = c_1 \begin{bmatrix} e^{\lambda t} \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} te^{\lambda t} \\ e^{\lambda t} \end{bmatrix} = \boxed{c_1 \begin{bmatrix} e^{\lambda t} \\ 0 \end{bmatrix} + c_2 \left(te^{\lambda t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{\lambda t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)}.$$

Example 3

Assume that all eigenvalues of A are multiples of v. This means that there exists some vector w such that $(A - \lambda I)w = V$. Prove that the general solution is

$$c_1 e^{\lambda t} v + c_2 (te^{\lambda t} v + e^{\lambda t} w)$$

Explication: We remark that when $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we obtain the solution to Example 2.

We want to show that $x_1(t) = e^{\lambda t} v$ solves $x_1'(t) = Ax_1(t)$. Here, $x_1'(t) = \lambda e^{\lambda t} v$ by Chain Rule, and $Ax_1(t) = e^{\lambda t} Av = e^{\lambda t} \lambda v$. Thus, $x_1(t)$ is a solution.

We want to prove that $x_2(t) = te^{\lambda t} v + e^{\lambda t} w$ is also a solution. Here,

$$\begin{aligned} x_2'(t) &= (e^{\lambda t} + t\lambda e^{\lambda t})v + \lambda e^{\lambda t} Aw \\ Ax_2(t) &= te^{\lambda t} Av + e^{\lambda t} Aw \\ &= te^{\lambda t} \lambda v + e^{\lambda t} (\lambda w + v) \\ &= (t\lambda e^{\lambda t} + e^{\lambda t})v + \lambda e^{\lambda t} w. \end{aligned}$$

Thus, $x_2'(t) = Ax_2(t)$, which confirms $x_2(t)$ as a solution. We proceed to confirm that $x_1(t) = e^{\lambda t} v$ and $x_2(t) = te^{\lambda t} v + e^{\lambda t} w$ are linearly independent. We have $x_1(0) = v$ and $x_2(0) = w$. From $(A - \lambda I)w = v$, we have $Aw = \lambda w + v$. Suppose, for the sake of contradiction, that $w = cv$ for some $c\ inR$. This means

$$\lambda w + v = Aw = A(cv) = cAv = c\lambda v = \lambda w$$

This means that $v = 0$, which is a contradiction. Thus, v and w are linearly independent, which means that the general solution incarnates the form

$$x(t) = c_1 x_1(t) + c_2 x_2(t) = \boxed{c_1 e^{\lambda t} v + c_2 (te^{\lambda t} v + e^{\lambda t} w)}.$$

April 15, 2021

Section Notes // TA: Francois-Simon Fauteux-Chapleau

Example 1

Consider the mass-spring system with damping. Assume $m, \kappa > 0$ and $\gamma \geq 0$. Find a general solution.

$$mx'' + \gamma x' + \kappa x = 0$$

Explication for $\gamma = 0$: In the case where $\gamma = 0$, we have $mx'' + \kappa x = 0$, which leads to $x'' + \frac{\kappa}{m}x = 0$. Recall from **Lecture 8** that for an equation of the form $y'' + a_1 y' + a_0 y = 0$ with characteristic polynomial $z^2 + a_1 z + a_0 = 0$ and its complex roots $r + i\omega$ and $r - i\omega$, the general solution is

$$y(t) = c_1 e^{rt}\cos(\omega t) + c_2 e^{rt}\sin(\omega t).$$

In the specific case, we have characteristic polynomial $z^2 + \frac{\kappa}{m} = 0$ so $z^2 = -\frac{\kappa}{m} < 0$. Hence, $z = \pm i\sqrt{\frac{\kappa}{m}}$. This means that $r = 0$ and $\omega = \sqrt{\frac{\kappa}{m}}$, so the general solution is

$$x(t) = c_1 \cos(\sqrt{\frac{k}{m}}t) + c_2 \sin(\sqrt{\frac{k}{m}}t).$$

We can explicitly check that $x(t)$ works by plugging it into $mx'' + \gamma x' + \kappa x = 0$, or verify that certain values of c_1, c_2 work. **Spoiler:** It works.

Example 2

Suppose that $m = \kappa = 1$, which means that $x'' + \gamma x' + x = 0$. Find a general solution.

Explication: We set up a system of equations by using $y = x'$, which means that we have $y' + \gamma y + x = 0$ and hence $y' = -\gamma y - x$. We form the first order system (it aligns with our equation upon matrix multiplication)

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\gamma \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The characteristic polynomial is equal to $z^2 - \mathrm{tr}(A)z + \det(A)$, where A is the 2 x 2 matrix above, so we have $z^2 + \gamma z + 1 = 0$. The discriminant here is $\gamma^2 - 4$, which is positive when $\gamma > 2$, 0 when $\gamma = 0$, and negative when $0 \leq \gamma < 2$.

Example 3

Assume that $0 \leq \gamma < 2$ and we have complex roots. Find the general solution to the matrix equation above. This precipitates the solution for the original $x'' + \gamma x' + x = 0$.

Explication: Recall from **Lecture 8** that for an equation of the form $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, and root $r + i\omega$ of the characteristic polynomial, the *general solution* is equal to

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{rt}\left(\cos(\omega t)\begin{bmatrix} 1 \\ r \end{bmatrix} - \sin(\omega t)\begin{bmatrix} 0 \\ \omega \end{bmatrix}\right) + c_2 e^{rt}\left(\sin(\omega t)\begin{bmatrix} 1 \\ r \end{bmatrix} + \cos(\omega t)\begin{bmatrix} 0 \\ \omega \end{bmatrix}\right)$$

The characteristic polynomial is equal to $z^2 + \gamma y + 1$ with roots

$$z = -\frac{-\gamma \pm \sqrt{\gamma^2 - 4}}{2} = -\frac{\gamma \pm i\sqrt{4 - \gamma^2}}{2} = -\frac{\gamma}{2} \pm i\frac{\sqrt{4 - \gamma^2}}{2}.$$

We use $r = -\frac{\gamma}{2}$ and $\omega = \frac{\sqrt{4-\gamma^2}}{2}$, so the general solution is

$$\begin{aligned}\begin{bmatrix} x \\ y \end{bmatrix} &= c_1 e^{-\frac{\gamma}{2}t}\left(\cos(\frac{\sqrt{4-\gamma^2}}{2}t)\begin{bmatrix} 1 \\ -\frac{\gamma}{2} \end{bmatrix} - \sin(\frac{\sqrt{4-\gamma^2}}{2}t)\begin{bmatrix} 0 \\ \frac{\sqrt{4-\gamma^2}}{2} \end{bmatrix}\right) \\ &+ c_2 e^{-\frac{\gamma}{2}t}\left(\sin(\frac{\sqrt{4-\gamma^2}}{2}t)\begin{bmatrix} 1 \\ -\frac{\gamma}{2} \end{bmatrix} + \cos(\frac{\sqrt{4-\gamma^2}}{2}t)\begin{bmatrix} 0 \\ \frac{\sqrt{4-\gamma^2}}{2} \end{bmatrix}\right)\end{aligned}$$

The first component of the vector informs us that

$$\boxed{x(t) = c_1 e^{-\frac{\gamma t}{2}}\cos(\frac{\sqrt{4-\gamma^2}}{2}t) + c_2 e^{-\frac{\gamma t}{2}}\sin(\frac{\sqrt{4-\gamma^2}}{2}t)}.$$

Example 4

Suppose that we have initial conditions $x(0) = 1$ and $x'(0) = 0$, which equates to $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

We use the vector formula obtained from Example 3. Note that $\sin(0) = 0$ and $\cos(0) = 1$, which clears out most of the trigonometry fluff.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 e^0 \begin{bmatrix} 1 \\ -\frac{\gamma}{2} \end{bmatrix} + c_2 e^0 \begin{bmatrix} 0 \\ \frac{\sqrt{4-\gamma^2}}{2} \end{bmatrix} = \begin{bmatrix} c_1 \\ -\frac{\gamma}{2}c_1 + \frac{\sqrt{4-\gamma^2}}{2}c_2 \end{bmatrix}$$

This means that $c_1 = 1$ and plugging this into the second equation leads to $c_2 = \frac{\gamma}{\sqrt{4-\gamma^2}}$. Hence,

$$\boxed{x(t) = e^{-\frac{\gamma t}{2}}\cos(\frac{\sqrt{4-\gamma^2}}{2}t) + \frac{\gamma}{\sqrt{4-\gamma^2}}e^{-\frac{\gamma t}{2}}\sin(\frac{\sqrt{4-\gamma^2}}{2}t)}$$

Using trigonometry properties, recall from **Lecture 5** that $A\cos(\omega t) + B\sin(\omega t) = \sqrt{A^2+B^2}\sin(\omega t + \phi)$ for some ϕ. We simplify $x(t)$ to, for some ϕ,

$$x(t) = e^{-\frac{\gamma t}{2}}\sqrt{1^2 + (\frac{\gamma}{\sqrt{4-\gamma^2}})^2}\sin(\frac{\sqrt{4-\gamma^2}}{2}t + \phi)$$

This makes it more apparent that $x(t)$ describes a sine-based oscillation of the mass moving up and down the spring with a decaying amplitude over time with the $e^{-\frac{\gamma t}{2}}$ term.

April 20, 2021 (Differential Equations)

Problem (On Practice Midterm)

Solve the following ODE.

$$u'' = -u'(1-u')(2-u')$$

Explication: We perform the substitution $y = u'$, which leads to the ODE $y' = -y(1-y)(2-y)$. We will start searching for constant solutions. Noting that $-y(1-y)(2-y)$ appears like a polynomial, we find $y(t) = 0$, $y(t) = 1$, and $y(t) = 2$, so $u(t) = c, u(t) = t + c, u(t) = 2t + c$ are all solutions for $c \in \mathbb{R}$. Presume that $y(t) \neq 0, 1, 2$. Next, we perform separation of variables.

$$\frac{y'}{y(1-y)(2-y)} = -1$$
$$\int \frac{dy}{y(1-y)(2-y)} = \int -1 \; dt$$

We use partial fraction decomposition (essentially, solve the system of equations that results from $\frac{1}{y(1-y)(2-y)} = \frac{A}{y} + \frac{B}{1-y} + \frac{C}{2-y}$) to terraform the above.

$$\int \left(\frac{\frac{1}{2}}{y} + \frac{1}{1-y} + \frac{-\frac{1}{2}}{2-y} \right) \; dy = -t + C$$
$$\frac{1}{2}\log|y| - \log|1-y| + \frac{1}{2}\log|2y| = c - t$$
$$\log|y| - 2\log|1-y| + \log|2-y| = 2c - 2t$$
$$\log|\frac{y(2-y)}{(1-y)^2}| = 2c - 2t$$

Exponentiating both sides leads to

$$|\frac{y(2-y)}{(1-y)^2}| = e^{2c-2t} = e^{2c}e^{-2t}.$$

Due to the absolute value, we have either $\frac{y(2-y}{(1-y)^2} = \pm e^{2c}e^{-2t}$. Note that $(1-y)^2 > 0$. If $y(2-y) > 0$, then we have $0 < y < 2$. If $y(2-y) < 0$, then $y < 0$ or $y > 2$. To proceed, we let $C = \pm e^{2c}$ for simplicity. Hence, we have $\frac{y(2-y)}{(1-y)^2} = Ce^{-2t}$. With some re-arranging,

$$y(2-y) = Ce^{-2t}(1-y)^2$$
$$2y - y^2 = Ce^{-2t}(y^2 - 2y + 1)$$
$$(1 + Ce^{-2t})y^2 - 2(1 + Ce^{-2t})y + Ce^{-2t} = 0.$$

This is a quadratic equation, where the discriminant $b^2 - 4ac$ is equal to

$$\begin{aligned} b^2 - 4ac &= (2(1+Ce^{-2t})^2 - 4(1+Ce^{-2t})(Ce^{-2t}) \\ &= (1+Ce^{-2t})\left[4(1+Ce^{-2t}) - 4Ce^{-2t})\right] \\ &= 4(1+Ce^{-2t}). \end{aligned}$$

Hence, solving for y in the quadratic leads to

$$y = \frac{2(1+Ce^{-2t}) \pm \sqrt{4(1+Ce^{-2t})}}{2(1+Ce^{-2t}} = 1 \pm \frac{2\sqrt{1+Ce^{-2t}}}{2(1+Ce^{-2t})} = 1 \pm \frac{1}{\sqrt{1+Ce^{-2t}}}.$$

Note that we must we have $1 + Ce^{-2t} > 0$. Since $y(t) = u'(t)$, then $u(t) = \int y(t) \; dt$, so

$$u(t) = \int 1 \pm \frac{1}{\sqrt{1+Ce^{-2t}}} \; dt = t \pm \int \frac{dt}{\sqrt{1+Ce^{-2t}}}. \tag{1}$$

The final step is to compute the integral $\int \frac{dt}{\sqrt{1+Ce^{-2t}}}$, which requires a substition. We use $s = \sqrt{1+Ce^{-rt}}$, which means that $\frac{ds}{dt} = \frac{-Ce^{-rt}}{\sqrt{1+Ce^{-2t}}}$ after differentiating both sides and thus $ds = \frac{-Ce^{-rt}}{\sqrt{1+Ce^{-2t}}}\ dt$. This reduces into $ds = \frac{1-s^2}{1-s}\ dt$ when cleverly incorporating our formula for $s = \sqrt{1+Ce^{-rt}}$. Hence, $dt = \frac{s}{1-s^2}\ ds$. Using the substitution, we have

$$\begin{aligned}
\int \frac{1}{\sqrt{1+Ce^{-2t}}}\ dt &= \int \frac{1}{s} * \frac{s}{1-s^2}\ ds \\
&= \int \frac{1}{1-s^2}\ ds \\
&= \int \frac{1}{(1+s)(1-s)}\ ds \\
&= \int \frac{1}{2}\left(\frac{1}{1-s} + \frac{1}{1+s}\right)\ ds \\
&= \frac{1}{2}\left(-\log|1-s| + \log|1+s| + D\right)
\end{aligned}$$

This means that we have $\frac{1}{2}\log|\frac{1+s}{1-s}| + D$ on the right hand side for integration constant $D \in \mathbb{R}$. Plugging in the formula for s means that we have

$$\int \frac{1}{\sqrt{1+Ce^{-2t}}}\ dt = \frac{1}{2}\log|\frac{1+\sqrt{1+Ce^{-rt}}}{1-\sqrt{1+Ce^{-rt}}}| + D.$$

Finally, we embed this into our full representation of $u(t)$ (equation 1) to produce

$$\boxed{u(t) = t \pm \frac{1}{2}\log|\frac{1+\sqrt{1+Ce^{-rt}}}{1-\sqrt{1+Ce^{-rt}}}| + D}.$$

Recall that $u(t) = C$, $u(t) = t + C$, $u(t) = 2t + C$ are all solutions resulting from constant solutions for $y(t)$.