Math 53: Ordinary Differential Equations with Linear Algebra

Notes by Ben Yan for the Stanford Math 53 course, taught by Prof. Umut Varolgunes in the Spring 2021 quarter. As a disclaimer, notes for the last few weeks of lecture content are absent.

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1 March 29, 2021 (Difference Equations and Pathogen Modeling)

Example: Spread of Disease

Think of the world as a very large square Sq with the people symbolized as dots.

- Every day $N = 0, 1, 2,$ every person is in a certain location in Sq, but they move a lot.
- On day 0, a small number of people are infected. Every day after, the people who are sufficiently close to an infected person are also infected.

Remark: It is reasonable to assume that the number of new people infected is *proportional*, for some proportionality constant K, to the number of people infected that day. Let $I(N)$ be the number of infected people on day N.

We can write the **difference equation**, with $I(N+1) - I(N)$ equal to the new infections on day $N+1$.

$$\boxed{I(N+1) - I(N) = K * I(N)}$$

Note the initial condition $I(0) = A$. We can write $I(N+1) = (K+1)*I(N)$, which means that $I(1) = (K+1)A$, $I(2) = (K+1)^2 A$, and so forth. More concretely, $I(N) = (K+1)^N A$.

Difference Equations → Differential Equations

To translate from *difference* equations to *differential* equations, we assume that the number of people infected does not alter substantially within the same day. Suppose that each person moves every hour. Adding up the difference equations on hours h through $h+23$ leads to

$$I(h+24) - I(h) = K_{\text{hour}} \left(\sum_{i=0}^{23} I(h+i) \right)$$
$$I(h+24) - I(h) \approx 24 K_{\text{hour}} I(h)$$

Hence, we have $K = K_{\text{day}} \approx 24 K_{\text{hour}}$ via another assumption that our day and hour difference equations are compatible. A corollary is that $K_{\text{hour}} \approx 60 * K_{\text{min}}$.

Remark: We can think of time t as a continuous variable with a unit of days, with $f(t)$ representing the number of infected people. We can write

$$f(t+1) - f(t) = Kf(t).$$

Let $\mathbf{\Delta}$ be any small time interval. This implies that $f(t+\mathbf{\Delta}) - f(t) = \mathbf{\Delta} * K * f(t)$. Dividing both sides by $\mathbf{\Delta}$ results in

$$\frac{f(t+\mathbf{\Delta}) - f(t)}{\mathbf{\Delta}} \approx K * f(t).$$

The final stage presumes that $f(t)$ varies smoothly with t and that $f(t)$ can span across real values. We take the limit $\mathbf{\Delta} \to 0$ on the left, resulting in

$$\lim_{\mathbf{\Delta} \to 0} \frac{f(t+\mathbf{\Delta}) - f(t)}{\mathbf{\Delta}} = Kf(t)$$
$$\boxed{f'(t) = Kf(t)}.$$

The creation of the **differential equation** works since the left side is the textbook definition of a derivative. With the condition $f(0) = A$, meaning that $f(t) = Ae^{kt}$. As a caution, it is not true that $f(N) = I(N)$ due to differences in annually/monthly/continouously compounded values.

2 March 31, 2021 (Substitution and Newton's Cooling)

We created a crude, differential equation for the spread of a pathogen in the prior lecture.

Example: Pathogen IVP

If $f(t)$ is the "number" people infected at time t, then for some $K > 0$,

$$f'(t) = Kf(t)$$

We have an initial value problem (IVP) establishing $f(0) = A$, with the solution $f(t) = Ae^{kt}$.

Compound Interest

A bank may advertise a compound interest loan of P_0 dollars with an annual interest rate r and compounding period of a year / quarter-year / month / day. If the period is $\frac{1}{n}$ years, they will apply $\frac{1}{n} * r$ interest rate every $\frac{1}{n}$ years. Hence, for $A = 0, \frac{1}{n}, \frac{2}{n}, \cdots$, via the *difference equation*,

$$P(A + \frac{1}{n}) - P(A) = \frac{1}{n} * r * P(A)$$
$$\boxed{P(A + \frac{1}{n}) = \left(\frac{r}{n} + 1\right) P(A)}$$

For example, with $P_0 = 100$, $r = 0.10$ and $n = 2$, we would have $P(0.5) = 100 * (1 + 0.1/2) = 105$ and $P(1) = P(0.5) * (1 + 0.1/2) = 110.5$. With $n = 1$, we would have simply $P(1) = 100 * (1 + 0.1) = 110$. As a note, different periods (selection of n) lead to different amounts of debt over time.

Remark

The smaller the period, the larger the debt for the individual. Since the debt gets slightly larger every time interest is applied, the more times the interest kicks in, the debt rises.

Continuous Compounding

If our debt at time t is $P(t)$, our debt satisfies the differential equation $P'(t) = rP(t)$ with $P(0) = P_0$. Hence, $P(t) = 100e^{0.1t}$, so $P(1) = 110.5170$ for the same banking example above. The solution $P(t)$ can be proven to be unique, but the proof is highly intricate and complex.

Oh no Physics

Example: Newton's Cooling

Imagine a small metal ball hanging in the air with a lot of wind. The initial ball temperature is u_0, and the environmental temperature is T_0. We denote the temperature at time to be $u(t)$. The evolution in time of the temperature of the ball is approximately governed by Newtons' law for some $k > 0$,

$$u'(t) = k(T_0 - u(t)).$$

If $u_0 = T_0$, then $u(t) = T_0$ is a solution of the IVP. If $u_0 > T_0$, the object will cool initially as $T_0 - u(t) < 0$ as $t \sim 0$. Also, note that temperature will never actually reach T_0 unless it started that way, but it will approach asymptotically.

One technique for solving differential equations is via **substitution**. Define $v(t) = T_0 - u(t)$, which means that

$$v' = (T_0 - u)' = (T_0)' - u' = -u'$$

Now, we have $v'(t) = -kv(t)$ with $v(0) = T_0 - u_0$. Using the covid DE, we solved this to be

$$v(t) = (T_0 - u_0)e^{-kt}$$

Performing substitution results in

$$u(t) = T_0 - v(t) = \boxed{T_0 - (T_0 - u_0)e^{-kt}}.$$

Note that e^{-kt} is a positive and decreasing function (exponential decay) and that is asymptotic to 0 as $t \to \infty$.

3 April 2, 2021 (ODE Terminologies)

Scalar and Linearity

Definition: Scalar ODE

A *scalar* ODE (ordinary differential equation) is an equation involving the derivatives of a single function depending on a single variable. If more than one variable is present, we would refer to them as **partial diff equations (PDE)**.

$$F(t, y(t), y'(t), \cdots, y^{(n)}(t)) = 0$$

The COVID and cooling differential equations are all first-order scalar ODEs. The **order** of a scalar ODE is the maximum order of the derivative that appears in the equation.

Definition: Linear ODE

A linear ODE is one which can be rearranged in the form

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_0(t)y = g(t)$$

Here, $a_n(t)$ through $a_0(t)$ and $g(t)$ are known functions of t.

An example of a linear ODE would be $\sin(t)y^{(6)} + (t^3 - 5)y^{(4)} + t^2y = e^{-t}$; the coefficients can be "nonlinear" in t as long as we linearly combine y and its derivatives.

Homogeneity

Definition: Homogeneous Linear ODE

A linear ODE for $y(t)$ is called **homogeneous** if $g(t) = 0$ in

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_0(t)y = g(t)$$

If $g(t) \neq 0$, we call the ODE non-homogeneous. Note that if $y(t)$ is the solution to a homogeneous linear scalar ODE, then $s * y(t)$ for $s \in R$ is also a solution since we can factor it out. If $y_1(t)$ and $y_2(t)$ solve a homogeneous linear ODE, then $y_1(t) + y_2(t)$ is also a solution by linearity of differentiation.

Initial Value Problem

If we have a first order ODE for $y(t)$, requiring $y'(t_0) = a$ for some time t_0 and $a \in R$ leads to a unique ODE solution. For higher order ODE, we will need more initial conditions. In particular, for an ODE with order n, we will need the n initial conditions $y(t_0) = y_0, y'(t_0) = y_1, \cdots, y^{(n-1)}(t_0) = y_{n-1}$.

Principle of Uniqueness and Existence

A linear scalar ODE of order n in standard form

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_0(t)y = g(t)$$

with initial conditions $y(t_0) = y_0, y'(t_0) = y_1, \cdots, y^{(n-1)}(t_0) = y_{n-1}$ has a unique solution*.

- Less initial conditions may cause non-uniqueness and more conditions may lead to non-existence.
- The principle, more precisely, states that a solution exists for some time interval $(t_0 - e_1, t_0 + e_2)$ with $e_1, e_2 > 0$.

- The uniqueness part argues that the solution is unique in any time interval with t_0.

For a linear scalar ODE of order n devoid of initial conditions, we expect a **general solution** (family of functions) to depend on n undetermined parameters. For instance, $f''(t) = 0$ has a general solution $f(t) = c_1 t + c_2$. Additionally, $P'(t) = rP(t)$ has a general solution $P(t) = ce^{rt}$.

Example: General Solutions

Search for a general solution to the scalar, order 2, linear, and homogeneous equation:

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 0$$

Noting the polynomial-esque appearance of the equation, we can trial solutions of the form $y(t) = e^{bt}$ for $b \in R$. Using the Chain Rule, we obtain

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = (b^2 - 3b + 2)e^{bt} = 0$$

Hence, $b^2 - 3b + 2 = 0$ so $b = 1$ or $b = 2$. We can confirm that e^t and e^{2t} are solutions. By linearity and homogeneity, we reason that $c_1e^t + c_2e^{2t}$ is a (general) solution for $c_1, c_2 \in R$. The general solution, with a number of free parameters equal to the order of the equation, acts as the mother of all solutions, or ”the one ring to rule them all”.

4 April 5, 2021 (Separation of Variables)

Method of Separation of Variables

Split Form

Consider a first order scalar ODE of the form $y' = f(y)g(t)$, for given functions f and g and unknown $y(t)$. We terraform such ODE's into the following split form:

$$\frac{1}{f(y)}y' = g(t)$$

For instance, $\frac{dy}{dt} = \cos(t)y^2$ can be transformed into $\frac{1}{y^2}\frac{dy}{dt} = \cos(t)$. Note that the solution of $y(t) = 0$ is the missing in the new equation, so we have to keep a memento of it. Next, recall that the solution to $y'(t) = u(t)$ is $U(t) = \int u(t)\ dt$ through antiderivatives. Hence, for function $y = y(t)$,

$$\int h(y)\ dy = \underbrace{\int h(y(t))\frac{dy}{dt}\ dt}_{\text{Chain Rule}}$$

We can solve the split form above through integrating both sides

$$\int \frac{1}{f(y)}y'\ dt = \int g(t)\ dt$$
$$\boxed{\int \frac{1}{f(y)}\ dy = \int g(t)\ dt}$$

Examining the COVID Differential

Consider $P'(t) = KP(t)$—the equation used to model the spread of a pathogen–which we shift into $\frac{1}{P(t)}P'(t) = K$. Integrating both sides leads to

$$\int \frac{1}{P(t)}P'(t)\ dt = \int K\ dt$$
$$\int \frac{1}{P}dP = \int K\ dt$$
$$\log(|P(t)|) = Kt + c$$

Exponentiating both sides leads to $|P(t)| = e^{Kt+c} = Ce^{kt}$ for $C = e^c$.

Logistic ODE

Definition: Logistic Growth Model

Consider a rabbit population in an infinite garden with evergreen grass. If $y(t)$ is the number of rabbits at time t, then we would expect $y'(t) = ry(t)$ for growth rate r. However, resources and grass are limited so for carrying capacity $K > 0$, we have

$$y'(t) = ry(t)(1 - \frac{y(t)}{K})$$

While growth will initially accelerate, intimately approaching the maximum population K will lead to $y'(t) \to 0$, inhibiting growth. We can rearrange this logistic ODE into

$$
\begin{aligned}
y'(t) &= \frac{r}{K}y(t)(K-y(t)) \\
\frac{K}{y(t)(K-y(t))}y'(t) &= r \\
\int \frac{K}{y(K-y)}\ dy &= \int r\ dt \\
\int \frac{1}{y} + \frac{1}{K-y}\ dy &= rt + c
\end{aligned}
$$

The last segment results from partial fraction decomposition.

$$
\int \frac{1}{y} + \frac{1}{K-y}\ dy = rt + c
$$

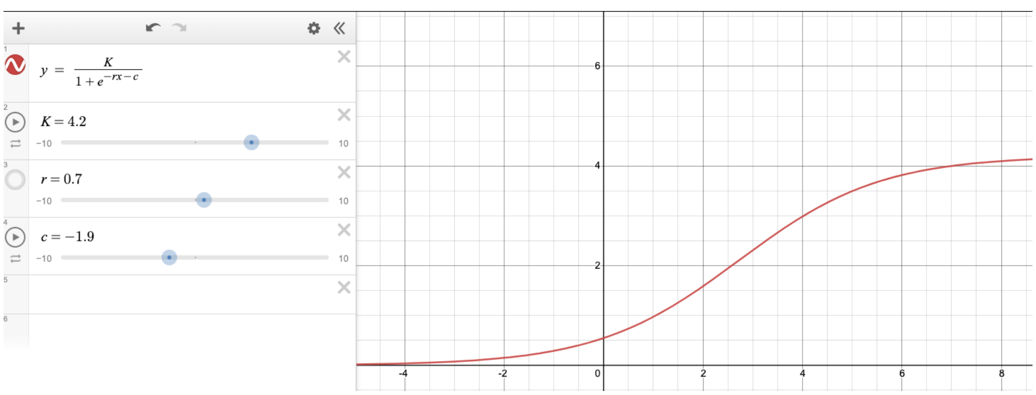
$$
\boxed{\log(|y|) - \log(|K-y|) = rt + c}
$$

We can synthesize understanding also by examining the differential equation $y'(t) = ry(t)(1-\frac{y(t)}{K})$. We note that the constant solutions are $y(t) = 0$ and $y(t) = K$. By the Principle of Uniqueness, all other solutions are located either in $y(t) < 0$, $0 < y(t) < K$ or $K < y(t)$. If we presume that $0 < y(t) < K$, we get

$$
\begin{aligned}
\log(y) - \log(K-y) = \log(\frac{y}{K-y}) &= rt + c \\
\frac{y}{K-y} &= e^{rt+c} \\
(e^{rt+c}+1)y &= Ke^{rt+c}
\end{aligned}
$$

$$
\boxed{y(t) = \frac{Ke^{rt+c}}{e^{rt+c}+1} = \frac{K}{1+e^{-rt-c}}}
$$

This represents the logistic growth model for our resource-limited rabbit population.



5 April 7, 2021 (Integrating Factors)

Integration Factor Trick

The solution to the differential equation of form $y' + a(t)y = g(t)$, where $a(t), g(t)$ are given functions and $y(t)$ is the unknown, is

$$y(t) = \frac{\int \mu(t)g(t)\ dt + C}{\mu(t)}.$$

Here, $\mu(t) = e^{\int a(t)\ dt}$ is the integration factor, and $C \in R$. We will show this works below.

Homogenuous Case

We start with the homogenuous case where $g(t) = 0$. This means that $y' + a(t)y = 0$ and thus $\frac{1}{y}y' = -a(t)$. This is just the template for *separation of variables*! We can solve it analagously (note that k and C are integration constants).

$$\begin{aligned} \int \frac{1}{y}y'\ dt &= \int -a(t)\ dt \\ ln(y) + k &= \int -a(t)\ dt \\ &\boxed{y(t) = Ce^{-\int a(t)\ dt}} \end{aligned}$$

Non-Homogenuous Case

The expression $y' + a(t)y$ makes us think of the product rule of differentiation. We multiply a function $u(t)$ to both sides to get $u(t)y' + u(t)a(t)y$. However, having a one-term expression like $(u(t)y(t))'$ would be **cleaner**, so we want to find a way to transform $u(t)y' + u(t)a(t)y$ into $(u(t)y(t))'$.

$$(u(t)y(t))' = \underbrace{u'(t)y(t)} + u(t)y'(t) = \underbrace{u(t)a(t)y} + u(t)y'$$

Hence, we want to chose $u(t)$ such that $u'(t) = u(t)a(t)$. We define $u(t) = e^{\int a(t)\ dt}$ (this works! it was the homogeneous case above). Tracing back to the original differential equation, this means that

$$\begin{aligned} y' + a(t)y &= g(t) \\ u(t)y' + u(t)a(t)y &= u(t)g(t) \\ \underbrace{(u(t)y(t))'}_{\text{Red to blue transform we showed above}} &= u(t)g(t) \end{aligned}$$

We integrate both sides, which leads to

$$\begin{aligned} \int (u(t)y(t))'\ dt &= \int u(t)g(t)\ dt \\ u(t)y(t) &= \int u(t)g(t)\ dt \\ y(t) &= \frac{\int u(t)g(t)dt + C}{u(t)} \end{aligned}$$

Plugging in our formula for $u(t) = e^{\int a(t)\ dt}$, this equation crystallizes into this monster:

$$\boxed{y(t) = \frac{\int e^{\int a(t)\ dt} g(t)dt + C}{e^{\int a(t)\ dt}}}.$$

Law of Cooling with Oscillating Air Temperature

Example: Cooling with Variegated Air Temperature

Consider a small metal ball in the air with plenty of wind and a fluctuating air temperature. Suppose that the air temperature at time t is given by $T_0 + A\sin(\omega t)$ for constants T_0, A, ω, which models the day/night cycle. We can express the cooling differential equation as

$$u'(t) = k(T_{\text{air temperature}} - u(t)) = k(T_0 + A\sin(\omega t) - u(t))$$

Using the Trick

Bringing the equation into standard form, we get (where $\phi(t)$ is the selected integration factor)

$$u'(t) + ku(t) = kA\sin(\omega t) + kT_0$$
$$\phi(t)u'(t) + \phi(t)ku(t) = \phi(t)[kA\sin(\omega t) + kT_0]$$

Seeing $ku(t)$ on the LHS, we select the integration factor $\phi(t) = e^{kt}$, observing that the derivative of $(\phi(t)u(t))' = \phi'(t)u(t) + u'(t)\phi(t) = ke^{kt} + u'(t)\phi(t) = k\phi(t) + u'(t)\phi(t)$, which is equal to the LHS.

Through red to blue substitution and integrating both sides, we obtain

$$(\phi(t)u(t))' = \phi(t)[kA\sin(\omega t) + kT_0]$$
$$\phi(t)u(t) = \int e^{kt}[kA\sin(\omega t) + kT_0]\ dt$$
$$\phi(t)u(t) = \int T_0 * ke^{kt}\ dt + A\int e^{kt}k\sin(\omega t)\ dt$$

This culminates in

$$\phi(t)u(t) = T_0e^{kt} + kA\int e^{kt}\sin(\omega t)\ dt + C = T_0e^{kt} + kAI + C \tag{1}$$

Finding I

Next, we need to compute $I = \int e^{kt}\sin(\omega t)\ dt$. We will use integration by parts.

$$I = \int e^{kt}\sin(\omega t)\ dt = -\omega^{-1}e^{kt}\cos(\omega t) + kw^{-1}\int e^{kt}\cos(\omega t)\ dt \tag{2}$$

We use integration by parts again (wtf) to the blue integral, leading to

$$\int e^{kt}\cos(\omega t)\ dt = w^{-1}e^{kt}\sin(\omega t) - k\omega^{-1}\int e^{kt}\sin(\omega t)\ dt \tag{3}$$

Hold up, $\int e^{kt}\sin(\omega t)\ dt = I$ is what we originally tried to solve. Thus, by substituting equation (3) into (2), we retrieve (with a lot of re-arranging!)

$$I = -\omega^{-1}e^{kt}\cos(\omega t) + kw^{-1}[w^{-1}e^{kt}\sin(\omega t) - k\omega^{-1}I]$$
$$I = \frac{e^{kt}}{w^2 + k^2}(k\sin(\omega t) - \omega\cos(\omega t))$$

Putting the Puzzle Together

Referencing equation (1) and bringing in our expression for I, we get

$$\phi(t)u(t) = T_0e^{kt} + kAI + C = T_0e^{kt} + kA\frac{e^{kt}}{w^2 + k^2}(k\sin(\omega t) - \omega\cos(\omega t)) + C$$
$$e^{kt}u(t) = T_0e^{kt} + kA\frac{e^{kt}}{w^2 + k^2}(k\sin(\omega t) - \omega\cos(\omega t)) + C$$

Dividing both sides e^{kt}, the general solution is equal to (yay!)

$$\boxed{u(t) = T_0 + \frac{kA}{\omega^2 + k^2}(k\sin(\omega t) - \omega\cos(\omega t)) + Ce^{-kt}}$$

Extra Cherry on Top: Simplification

The expression $k\sin(\omega t) - \omega\cos(\omega t)$ is fairly enigmatic and confusing. Using trig. identities, we can turn it into $R\sin(\omega t + \tau)$, where $\tau \in [0, 2\pi)$ is an angle measure. In particular, we select a τ such that

$$\cos(\tau) = \frac{k}{\omega^2 + k^2}$$
$$\sin(\tau) = \frac{-\omega}{\omega^2 + k^2}$$

This construction works since $\cos(\tau)^2 + \sin(\tau)^2 = 1$, meaning we can find τ in the unit circle. Hence, using $k = \cos(\tau)(\omega^2 + k^2)$ and $\omega = -\sin(\tau)(\omega^2 + k^2)$ and the sum of sine angles formula,

$$\begin{aligned} k\sin(\omega t) - \omega\cos(\omega t) &= (\omega^2 + k^2)(\cos(\tau)\sin(\omega t) + \sin(\tau)\cos(\omega t)) \\ &= (\omega^2 + k^2)\sin(\omega t + \tau) \end{aligned}$$

Our general solution morphs into

$$u(t) = T_0 + \frac{kA}{\omega^2 + k^2}((\omega^2 + k^2)\sin(\omega t + \tau)) + Ce^{-kt}$$
$$\boxed{u(t) = T_0 + kA\sin(\omega t + \tau) + Ce^{-kt}}$$

Note that for a large value of t, Ce^{-kt} experiences exponential decay to 0 and thus $u(t) \approx T_0 + kA\sin(\omega t + \tau)$. This makes sense, as the average temperature of the object (over the day) comes closer to the average day temperature but there is some sine-based oscillation.

6 April 9, 2021 (Systems of ODEs)

Predator-Prey Models

Rabbit-Wolf Dynamical System

Consider a population of rabbits (prey) and wolves (predator) in a grassy environment.

- When left to themselves, rabbit numbers grow and wolf numbers decay, both exponentially.
- The number of rabbits eaten at time t, as well as the increase in the number of wolves, is proportional to the number of rabbits times the number of wolves.

Let $r(t)$ and $w(t)$ be the number of rabbits and wolves at time t. We have the following Lotka-Volterra nonlinear system of ODE's for $A, B, C, D > 0$

$$\begin{aligned} r'(t) &= Ar(t) - Br(t)w(t) \\ w'(t) &= -Cw(t) + Dr(t)w(t) \end{aligned}$$

More generally, a system of n ODE's consists of n unknown functions and n equations each involving those unknown functions and their derivatives.

First-Order, Linear System of Two ODE's

Call our unknown functions are $x_1(t)$ and $x_2(t)$. In standard form, we have

$$\begin{aligned} x_1'(t) &= a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + b_1(t) \\ x_2'(t) &= a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + b_2(t) \end{aligned}$$

We form the vector $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, and wrap the known functions into the matrix $A = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}$. Finally, we wrap the the known $b_1(t), b_2(t)$ into the vector $b(t) = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

Derivative of Vector-Valued Function

The derivative of vector $x(t)$ is acquired by taking the derivative of each component.

$$x'(t) = \begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \cdots \\ x_n'(t) \end{bmatrix}$$

Thus, the original system of two ODEs can be written compactly as

$$\boxed{x'(t) = A(t)x(t) + b(t).}$$

Our system is homogeneous if and only if $b(t) = \overrightarrow{0}$.

Remark: Any Nth order linear, scalar ODE can be transformed into a first order linear system of N ODEs.

Proof for $n = 2$: Consider a second-order linear scalar ODE of the form

$$a_2(t)y''(t) + a_1(t)y'(t) + a_0(t)y(t) = g(t)$$

We introduce $z(t) = y'(t)$, which leads to $a_2(t)z'(t) + a_1(t)z(t) + a_0(t)y(t) = g(t)$. Re-arranging both equations leads to a linear system of 2 ODEs with 2 unknowns: $z(t)$ and $y(t)$.

$$\boxed{\left\{ \begin{matrix} y'(t) = z(t) \\ z'(t) = -\frac{a_0(t)}{a_2(t)}y(t) - \frac{a_1(t)}{a_2(t)}z(t) + \frac{g(t)}{a_2(t)} \end{matrix} \right\}}$$

In matrix form, we condense both equations into the singular:

$$\begin{bmatrix} y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{a_0(t)}{a_2(t)} & -\frac{a_1(t)}{a_2(t)} \end{bmatrix} \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{g(t)}{a_2(t)} \end{bmatrix}.$$

Newton's Law as a Differential Equation

Newton's Equation

Assume that at time t, if our object has position x and velocity $v = x'(t)$, a force $F(t, x, x'(t))$ acts on it. We introduce:

$$mx''(t) = F(t, x(t), x'(t))$$

Here, m is the object mass. Note that this equation resembles $F = ma$, where $a = x''(t)$.

Suppose that our object is a box on a friction-less surface attached to a wall with a spring. Hooke's Law states that $F(t, x, v) = -kx$, where $x(t)$ is the displacement of the object from rest and $k > 0$. Using Newton s Law, we have

$$\boxed{mx''(t) = -kx(t)}$$

If $k = 1$ and $m = 1$, we note that $x''(t) = -x(t)$ for $x = \cos(t)$ and $y(t) = \sin(t)$, which collate into the general solution $x(t) = a\sin(t) + b\cos(t)$. Finding a solution for all m, k is reserved for later units.

For $v(t) = x'(t)$, the corresponding first order system of ODEs is:

$$\begin{bmatrix} x'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}.$$

7 April 12, 2021 (Characteristic Polynomials and Solving Systems)

We consider ODEs of the form, $x'(t) = Ax(t)$, where $A(t)$ is the matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. The initial conditions are $x(t_0) = x_0 = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, and the principles of uniqueness and existence apply.

Remark

If $x_1(t)$ and $x_2(t)$ are two solutions, any linear combination $c_1x_1(t) + c_2x_2(t)$ is also a solution for $c_1, c_2 \in R$.

Proof: By the linearity of differentiation and matrix multiplication, we have

$$(c_1x_1 + c_2x_2)' = c_1x_1' + c_2x_2'$$
$$A(c_1x_1 + c_2x_2) = c_1Ax_1 + c_2Ax_2$$

Since $x_1' = Ax_1$ and $x_2' = Ax_2$, as they are solutions, then we have $c_1x_1' = c_1Ax_1$ and $c_2x_2' = c_2Ax_2$, and adding these equations together produces $c_1x_1' + c_2x_2' = c_1Ax_1 + c_2Ax_2$. Using the above, this implies $(c_1x_1 + c_2x_2)' = A(c_1x_1 + c_2x_2)$, so $c_1x_1(t) + c_2x_2(t)$ is a solution.

- To find a general solution, it suffices to find two solutions $x_1(t)$ and $x_2(t)$ that are linearly independent for some t. For 2D vectors, this means they are not multiples of one another.

Eigenvalue and Eigenvector Method

Eigenvector

A real eigenvector of matrix A is a vector v that satisfies, for some $\lambda \in R$,

$$Av = \lambda v.$$

We call λ an eigenvalue for non-zero v. It is possible to have multiple eigenvectors for some λ.

Suppose $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Recall that $\det(A) = a_{11}a_{22} - a_{12}a_{21}$ and $\operatorname{tr}(A) = a_{11} + a_{22}$. The characteristic polynomial of A is equal to $z^2 - \operatorname{tr}(A)z + \det(A)$. If this polynomial has distinct roots $\lambda_1, \lambda_2 \in R$, any of their eigenvectors v_1 and v_2 form a basis of R^2, and we have

$$z^2 - \operatorname{tr}(A)z + \det(A) = (z - \lambda_1)(z - \lambda_2)$$

The discriminant $\Delta(A) = (\operatorname{tr}(A))^2 - 4\det(A)$ informs us whether the polynomial has distinct real (positive Δ), repeated real (zero Δ), or complex roots (negative Δ).

Claim

If matrix A has a real eigenvector v with real eigenvalue λ, the equation $x'(t) = Ax(t)$ has a solution $x(t) = e^{\lambda t}v$.

Proof: On the left-hand side, we have $x'(t) = (e^{\lambda t}v)' = (e^{\lambda t})'v = \lambda e^{\lambda t}v$, as v is a constant vector that does not depend on t. On the right hand side, we have $Ax(t) = Ae^{\lambda t}v = e^{\lambda t}Av = e^{\lambda t}\lambda v$. Thus, the left hand and right hand side both culminate in $e^{\lambda t}\lambda v$, so we are done.

Remark

If v_1 and v_2 are not multiples of one another and real eigenvectors of matrix A, then $e^{\lambda_1 t}v_1$ and $e^{\lambda_2 t}v_2$ are linearly independent solutions. The general solution is

$$c_1 e^{\lambda_1 t}v_1 + c_2 e^{\lambda_2 t}v_2.$$

Now, suppose that we have a non-homogeneous term b with ODE $x'(t) = Ax(t) + b$. Assume that A is invertible and define $x_{eq} = -A^{-1}b$. The only constant solution is $x(t) = x_{eq}$, as $Ax(t) + B = A(-A^{-1}b) + b = -b + b = 0$. We define $\boxed{y(t) = x(t) - x_{eq}}$, which means $x(t) = y(t) + x_{eq}$, and substitute it into the ODE to produce

$$\begin{aligned} (y(t) + x_{eq})' &= A(y(t) + x_{eq}) + b \\ y'(t) + [x'_{eq} - Ax_{eq} - b] &= Ay(t) \\ &\boxed{y'(t) = Ay(t)} \end{aligned}$$

The last part works as x_{eq} is a solution so $x'_{eq} - Ax_{eq} - b = 0$ by definition. This is stellar since the *substitution reduces the problem to a homogeneous ODE.*

Example: Salt Mixing

There are fluctuating concentrations of salt, $Q_1(t)$ and $Q_2(t)$, in two tanks filled with water that interact and mix with one another. Initial state is $Q_1(0) = 3$ and $Q_2(0) = 5$.

$$\begin{aligned} Q_1'(t) &= -4Q_1(t) + 3Q_2(t) + 1 \\ Q_2'(t) &= 4Q_1(t) - 8Q_2(t) + 4 \end{aligned}$$

In matrix form, we have, for $Q(t) = \begin{bmatrix} Q_1(t) \\ Q_2(t) \end{bmatrix}$ and initial salt concentration values $Q(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$,

$$\begin{aligned} Q'(t) &= AQ(t) + b \\ Q'(t) &= \begin{bmatrix} -4 & 3 \\ 4 & -8 \end{bmatrix} Q(t) + \begin{bmatrix} 1 \\ 4 \end{bmatrix} \end{aligned}$$

For make the equation homogeneous, we introduce $x_{eq} = -A^{-1}b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We consider $y(t) = Q(t) - \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We must solve $y'(t) = Ay(t)$, where $\det(A) = 20$ and $\operatorname{tr}(A) = -12$, so the characteristic polynomial is equal to $z^2 + 12z + 20 = 0$. This factors into $(z+2)(z+10) = 0$, so the eigenvalues/vectors are $\lambda_1 = -2$ with $v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\lambda_2 = -10$ with $v_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. We obtain these vectors v by solving $Av = \lambda v$ for each $\lambda \in \{-2, -10\}$. The general solution is therefore

$$Q(t) = y(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 e^{\lambda_1 t}v_1 + c_2 e^{\lambda_2 t}v_2 + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\boxed{Q(t) = c_1 e^{-2t}\begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 e^{-10t}\begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}}.$$

8 April 14, 2021 (Complex Eigenvalues and Eigenvectors)

Complex Numbers

Definition: A complex number in standard form is an expression of the form $a + bi$,where a and b are real numbers. We can add and subtract complex numbers as shown.

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

We use the fact that $i^2 = -1$ to describe complex multiplication.

$$(a + bi)(c + di) = (ac - bd) + i(ad + bc)$$

Fact: Two complex numbers $a + bi$ and $c + di$ are equal if and only if $a = c$ and $b = d$.

Discriminant

For a quadratic equation $z^2 + bz + c = 0$, the solution is equal to

$$z = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

If the **discriminant** $b^2 - 4c < 0$, there are two complex solutions.

Using the quadratic formula and replacing $\sqrt{-1}$ with i, we obtain

$$z = \frac{-b}{2} \pm \frac{\sqrt{-1}\sqrt{4c - b^2}}{2} = -\frac{b}{2} \pm i\frac{\sqrt{4c - b^2}}{2}.$$

Complex Eigenvalues and Vectors

Consider $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ to be a real matrix. The characteristic polynomial is $z^2 - \operatorname{tr}(A) + \det(A)$, and the *determinant* is equal to $\boldsymbol{\Delta}(A) = (\operatorname{tr}(A))^2 - 4\det(A)$. If $\boldsymbol{\Delta}(A) < 0$, let λ be a complex root of the characteristic polynomial. We search for a non-zero, complex vector v with

$$Av = \lambda v.$$

Suppose $\lambda = r + i\omega$ and $v = \begin{bmatrix} p + iq \\ s + it \end{bmatrix}$. This means that

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} p + iq \\ s + it \end{bmatrix} = (r + i\omega) \begin{bmatrix} p + iq \\ s + it \end{bmatrix}$$
$$\begin{bmatrix} a_{11}(p + iq) + a_{12}(s + it) \\ a_{21}(p + iq) + a_{22}(s + it) \end{bmatrix} = \begin{bmatrix} (r + i\omega)(p + iq) \\ (r + i\omega)(s + it) \end{bmatrix}$$

Re-arranging leads eventually (more arithmetic in Umut's slides) to a complex eigenvector solution. We find that $p = a_{12}$, $q = 0$, $s = r - a_{11}$, and $t = \omega$.

$$\boxed{v = \begin{bmatrix} a_{12} \\ (r - a_{11}) + i\omega \end{bmatrix}} \tag{4}$$

Linearly Independent Solutions for $\Delta(A) < 0$

Assume that in the differential equation $x'(t) = Ax(t)$, matrix A has a complex eigenvector $v = \mathbf{u} + i\mathbf{w}$ with eigenvalue $\lambda = r + i\omega$. We have two linearly independent solutions.

$$x_1(t) = e^{rt}(\cos(\omega t)\mathbf{u} - \sin(\omega t)\mathbf{w})$$
$$x_2(t) = e^{rt}(\sin(\omega t)\mathbf{u} + \cos(\omega t)\mathbf{w})$$

Remark: This formula arises partially from the use of Euler's formula.

$$e^{\lambda t} = e^{(r+i\omega)t} = e^{rt}e^{i\omega t} = e^{rt}(\cos(\omega t) + i\sin(\omega t)).$$

Note that $x_1(t)$ and $x_2(t)$ are real and imaginary parts of $e^{\lambda t}v = e^{\lambda t}(\mathbf{u} + i\mathbf{w})$ when expanded out.

Problem

Find general solutions to the second order homogeneous linear scalar ODE's with constant coefficients of the form

$$y''(t) + a_1 y'(t) + a_0 y(t) = 0$$
$$\begin{bmatrix} y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}$$

Here, we set $z(t) = y'(t)$. For the characteristic polynomial $z^2 + a_1 z + a_0$, assume the discriminant $a_1^2 - 4a_0 < 0$ and $r + i\omega$ is a root. The eigenvector, using equation (1), is

$$v = \begin{bmatrix} a_{12} \\ (r - a_{11}) + i\omega \end{bmatrix} = \begin{bmatrix} 1 \\ r + i\omega \end{bmatrix}$$

Plugging $v = \begin{bmatrix} 1 \\ r + i\omega \end{bmatrix} = \begin{bmatrix} 1 \\ r \end{bmatrix} + i \begin{bmatrix} 0 \\ \omega \end{bmatrix} = \mathbf{u} + i\mathbf{w}$, we get $\mathbf{u} = \begin{bmatrix} 1 \\ r \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 0 \\ \omega \end{bmatrix}$ Hence, the general solution, using the formula in the red box above, is equal to

$$\begin{aligned}
\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} &= c_1 x_1(t) + c_2 x_2(t) \\
&= c_1 e^{rt}(\cos(\omega t)\mathbf{u} - \sin(\omega t)\mathbf{w}) + c_2 e^{rt}(\sin(\omega t)\mathbf{u} + \cos(i\omega t)\mathbf{w}) \\
&= c_1 e^{rt}\left(\cos(\omega t)\begin{bmatrix} 1 \\ r \end{bmatrix} - \sin(\omega t)\begin{bmatrix} 0 \\ \omega \end{bmatrix}\right) + c_2 e^{rt}\left(\sin(\omega t)\begin{bmatrix} 1 \\ r \end{bmatrix} + \cos(\omega t)\begin{bmatrix} 0 \\ \omega \end{bmatrix}\right) \\
&= c_1 e^{rt}\begin{bmatrix} \cos(\omega t) \\ r\cos(\omega t) - \omega\sin(\omega t) \end{bmatrix} + c_2 e^{rt}\begin{bmatrix} \sin(\omega t) \\ r\sin(\omega t) + \omega\cos(\omega t) \end{bmatrix} \\
&= \begin{bmatrix} c_1 e^{rt}\cos(\omega t) + c_2 e^{rt}\sin(\omega t) \\ c_1 e^{rt}(r\cos(\omega t) - \omega\sin(\omega t)) + c_2 e^{rt}(r\sin(\omega t) + \omega\cos(\omega t)) \end{bmatrix}
\end{aligned}$$

Exmaining the first component, we have the general solution $y(t)$.

$$\boxed{y(t) = c_1 e^{rt}\cos(\omega t) + c_2 e^{rt}\sin(\omega t)}$$

9 April 16, 2021 (N-Dimensional Systems and Spectral Theorem)

Second-Order Homogeneous ODEs

Summary

For ODEs of form $y''(t) + a_1 y'(t) + a_0 y(t) = 0$, we have the characteristic polynomial $z^2 + a_1 z + a_0$. There are 3 cases for discriminant $a_1^2 - 4a_0$, which is either positive, negative, or zero.

- If $a_1^2 - 4a_0 > 0$, then for two distinct real roots λ_1 and λ_2, the general solution is
$$c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}.$$
- If $a_1^2 - 4a_0 = 0$, we have a repeated root λ and the general solution is
$$c_1 e^{\lambda t} + c_2 t e^{\lambda t}.$$
- If $a_1^2 - 4a_0 < 0$, we have complex roots $r \pm i\omega$, and the general solution is
$$c_1 e^{rt}\cos(\omega t) + c_2 e^{rt}\sin(\omega t).$$

Remark: When serviced with initial conditions $y(0) = a$ and $y'(0) = b$, we have a system of two equations to solve for c_1 and c_2.

Dealing with N-Dimensional Matrices

We will explore solving $x'(t) = Ax(t)$ when A is a N by N real matrix and $x(t)$ is a vector-valued function with N components. To find the general solution $x'(t) = Ax(t)$, it suffices to find N solutions

$$x_1(t), x_2(t), \cdots, x_N(t)$$

such that for some t_0, $x_1(t_0), \cdots, x_N(t_0)$ are linearly independent vectors in $\mathbf{R}^n$. Here, the general solution is, for $c_1, \cdots, c_N \in \mathbf{R}$,

$$c_1 x_1(t) + c_2 x_2(t) + \cdots + c_N x_N(t).$$

Recall that $e^{\lambda_1 t} v_1$ is a solution to $x'(t) = Ax(t)$ for eigenvalue λ_1 and corresponding eigenvector v_1.

Matrix Solution

In the (happy) case where A has eigenvectors $v_1, \cdots, v_N$, which form a basis of $\mathbf{R}^n$, and corresponding eigenvalues $\lambda_1, \cdots, \lambda_N$, the general solution to $x'(t) = Ax(t)$ is equal to

$$x(t) = c_1 e^{\lambda_1 t} v_1 + \cdots + c_N e^{\lambda_N t} v_N.$$

To determine whether we are in the happy case, we use **spectral theorem**.

- If A is symmetric or if A has N distinct real eigenvalues, we are in the happy case.
- We can also compute the eigenvalues directly, such as finding the roots of the characteristic polynomial $\det(A - zI) = 0$ for identity matrix $I = I_N$.
- As a shortcut, if A happens to be a triangular matrix, the eigenvalues are its diagonal entries.

Example

Find the general solution to the following ODE.

$$x'(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 5 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} x(t) = Ax(t)$$

Solution: We compute the determinant of $A - zI$, which is equivalent to

$$\det(A - zI) = \det(\begin{bmatrix} 1-z & 0 & 0 & 0 \\ 0 & 2-z & 2 & 0 \\ 0 & 0 & 5-z & 1 \\ 1 & 0 & 0 & -z \end{bmatrix}).$$

With some arithmetic heavy-lifting, the determinant comes out to be $-z(5-z)(2-z)(1-z)$, which has 4 real roots. We have eigenvalues $\lambda = 0, 1, 2, 5$, and we can find eigenvectors v_1, v_2, v_3, v_4 (for the sake of time, this stone is left unturned). The general solution is

$$c_1 v_1 + c_2 e^t v_2 + c_3 e^{2t} v_3 + c_4 e^{5t} v_4.$$

10 April 19, 2021 (ODEs with Complex Functions)

Complex Number Properties

Recall that a complex number has standard form $a + ib$ for real $a, b \in \mathbb{R}$. Division in the complex numbers is defined as, for $a + ib \neq 0$,

$$\frac{c+id}{a+ib} = (c+id)(a+ib)^{-1}$$

Complex Inverse

The inverse of a complex number $a + bi$ is equal to

$$(a+ib)^{-1} = \frac{1}{a+ib} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} + i\frac{-b}{a^2+b^2}.$$

We can also represent e^{a+ib} as a complex number.

$$e^{a+ib} = e^a\left(\cos(b) + i\sin(b)\right) = e^a\cos(b) + ie^a\sin(b).$$

The case where $a = 0$ leads to Euler's formula, which is the groundwork for the formula $e^{i\pi} + 1 = 0$.

Euler's Formula

Suppose that b is a real number. We have

$$e^{ib} = \cos(b) + i\sin(b)$$

Remark: The exponentials of complex numbers behave similarly to real numbers.

$$e^{a+ib}e^{c+id} = e^{(a+c)+i(b+d)}.$$

Complex-Valued Functions

Complex functions take the form $x(t) = x_1(t) + ix_2(t)$, where $x_1(t), x_2(t)$ are real functions. Two complex functions are equal if and only if their real and imaginary parts are identical.

Calculus with Complex Functions

When taking derivatives or integrals of complex-valued functions, we perform the operation to the real and imaginary parts separately.

$$f(t)' = f_1'(t) + f_2'(t)$$
$$\int_a^b f(t)\ dt = \int_a^b f_1(t)\ dt + i\int_a^b f_2(t)\ dt.$$

The indefinite integral $\int f(t)\ dt$ is a complex function whose derivative is $f(t)$.

Examples of such functions are polynomials with complex coefficients $a_nt^n + \cdots + a_1t + a_0$ and the exponential function $e^{\lambda t}$ for $\lambda = a + bi$, where

$$e^{\lambda t} = e^{at}\cos(bt) + ie^{at}\sin(bt).$$

Linear Scalar ODEs for Complex Functions

Consider the standard form of a linear scalar ODE for complex unknown $y(t) = y_1(t) + iy_2(t)$.

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t) = g(t).$$

We will still use real-valued $a_0(t), a_1(t), \cdots, a_n(t)$ but also accommodate the case where $g(t) = g_1(t) + ig_2(t)$ is complex. Note that $a(t)y(t) = a(t)y_1(t) + ia(t)y_2(t)$ by distribution, which means that:

Solution to Linear Scalar ODE

The function $y(t) = y_1(t) + iy_2(t)$ is a solution if and only if $y_1(t)$ solves the real part and $y_2(t)$ solves the imaginary part.

$$a_n(t)y_1^{(n)} + a_{n-1}(t)y_1^{(n-1)} + \cdots + a_1(t)y_1' + a_0(t) = g_1(t)$$
$$a_n(t)y_2^{(n)} + a_{n-1}(t)y_2^{(n-1)} + \cdots + a_1(t)y_2' + a_0(t) = g_2(t)$$

System of ODEs

Problem

Let A be a N by N matrix. We consider complex and vector-valued solutions of

$$x'(t) = Ax(t).$$

Remark: A complex $x(t) = x_1(t) + ix_2(t)$ is a solution if and only if its real and imaginary parts $x_1(t)$ and $x_2(t)$ are also solutions.

Assume that A has a possibly complex eigenvector v with eigenvalue λ, which implies that $x(t) = e^{\lambda t}v$ is a solution. To decompose its real and imaginary parts, we let $v = \mathbf{u} + i\mathbf{w}$ and $\lambda = r + i\omega$. Using Euler's formula (above), we have

$$\begin{aligned} e^{\lambda t}v &= e^{rt}(\cos(\omega t) + \sin(\omega t))(\mathbf{u} + i\mathbf{w}) \\ &= e^{rt}(\cos(\omega t)\mathbf{u} - \sin(\omega t)\mathbf{w}) + ie^{rt}(\sin(\omega t)\mathbf{u} + \cos(\omega t)\mathbf{w}). \end{aligned}$$

The real and imaginary parts of $e^{\lambda t}$ are linearly independent solutions on their own of $x'(t) = Ax(t)$. Hence, we can produce a real solution by linearly combining them.

Solution for $N = 2$ Case

The general real-valued solution to $x'(t) = Ax(t)$ for 2 by 2 matrix A is, for $c_1, c_2 \in \mathbb{R}$,

$$x(t) = c_1e^{rt}(\cos(\omega t)\mathbf{u} - \sin(\omega t)\mathbf{w}) + c_2e^{rt}(\sin(\omega t)\mathbf{u} + \cos(\omega t)\mathbf{w}).$$

11 April 21, 2021 (Polar Form and Non-Homogeneous ODEs)

Polar Form

We define the **magnitude** $|a+bi|$ of a complex number $a+bi$ to be $\sqrt{a^2+b^2}$, its distance to the complex plane origin. We define its **argument** $\arg(a+bi)$ to be the angle θ that satisfies

$$\cos(\theta) = \frac{a}{\sqrt{a^2+b^2}}, \quad \sin(\theta) = \frac{b}{\sqrt{a^2+b^2}}$$

The **polar form** of $a+bi$ is $re^{i\theta}$, where r is the magnitude and θ is the argument.

Proof: This representations works since due to Euler's formula

$$\begin{aligned} re^{i\theta} &= r(\cos(\theta) + i\sin(\theta)) \\ &= \sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}} + i\frac{b}{\sqrt{a^2+b^2}}\right) \\ &= a+ib. \end{aligned}$$

Remark: The inverse of $a+bi = re^{i\theta}$ is $(a+ib)^{-1} = r^{-1}e^{-i\theta}$. This means that $|(a+bi)^{-1}| = |a+bi|^{-1}$ and $\arg((a+bi)^{-1})) = -\arg(a+bi)$.

Second Order Linear Scalar Non-Homogeneous ODE

We consider ODE's (the dampened mass-spring system) of the form $x''+\gamma x'+kx = g(t)$, where γ and k are real numbers. We call $g(t)$ the input, which occurs when there is an extra external force to the box in the system.

Property of Non-Homogeneous Case

Let $x_p(t)$ be a particular solution of $x''+\gamma x'+kx = g(t)$. The general solution is

$$x_p(t) + x_h(t),$$

where $x_h(t)$ is the general solution of homogeneous $x''+\gamma x'+kx = 0$.

Proof: Plug $x_p(t)+x_h(t)$ into the ODE and use linearity of differentiation.

Corollary: If $\tilde{x_p}(t)$ is another solution to $x''+\gamma x'+kx = g(t)$, then $x_p(t) - \tilde{x_p}(t)$ solves the homogeneous $x''+\gamma x'+kx = 0$.

The next procedure is to find a particular solution to the non-homogeneous ODE. Due to complexity, we will only deal with the case where $g(t) = \sin(\omega t)$ or $g(t) = \cos(\omega t)$, where $\omega \in \mathbb{R}$.

Problem

Find a complex valued solution to

$$x''+\gamma x'+kx = e^{i\omega t}$$

By Euler's, the real part solves $g(t) = \cos(\omega t)$ and the imaginary part solves $g(t) = \sin(\omega t)$.

Solution: We tinker with solutions of the form $x_p(t) = Ae^{i\omega t}$, where $A \in \mathbb{C}$ is a complex number. After plugging it in,

$$\begin{aligned} (x_p(t))'' + \gamma(x_p(t))' + kx_p(t) &= e^{i\omega t} \\ A(i\omega)^2 e^{i\omega t} + \gamma A(i\omega)e^{i\omega t} + ke^{i\omega t} &= e^{i\omega t} \\ e^{i\omega t}(A(i\omega t)^2 + \gamma A(i\omega) + k) &= e^{i\omega t} \end{aligned}$$

After canceling the $e^{i\omega t}$ terms and re-arranging, we get

$$A(-\omega^2 + i\gamma\omega + k) = 1$$

This means that we need inverse $A = (-\omega^2 + i\gamma\omega + k)^{-1}$. Hence, a particular solution is $x_p(t) = Ae^{i\omega t} = (-\omega^2 + k + i\gamma\omega)^{-1}e^{i\omega t}$. To simplify, we will write $(-\omega^2 + k + i\gamma\omega)^{-1}$ in polar form $Ge^{i\phi}$, where

$$G = \frac{1}{|(-\omega^2 + k) + i(\gamma\omega)|} = \frac{1}{\sqrt{(\omega^2 - k)^2 + \gamma^2\omega^2}}.$$

Noting that $e^{i\omega t}e^{\phi} = e^{i\omega t + \phi}$, our final form is

$$\boxed{x_p(t) = \frac{1}{\sqrt{(\omega^2 - k)^2 + \gamma^2\omega^2}} e^{i\omega t + \phi}}$$

Solutions to Problem

The real part (equation 1 below) of $x_p(t)$ solves $x'' + \gamma x' + kx = \cos(\omega t)$.

$$\frac{1}{\sqrt{(\omega^2 - k)^2 + \gamma^2\omega^2}} \cos(\omega t + \phi) \tag{5}$$

The imaginary part (equation 2 below) of $x_p(t)$ solves $x'' + \gamma x' + kx = \sin(\omega t)$.

$$\frac{1}{\sqrt{(\omega^2 - k)^2 + \gamma^2\omega^2}} \sin(\omega t + \phi) \tag{6}$$

Long-term Behavior of Homogeneous Equation

Consider $x'' + \gamma x' + kx = 0$ and assume that $\gamma, k > 0$ in the dampened mass-spring system. We expect that it converges to rest position $x = 0$ as time $t \to \infty$ due to damping, which is true due to exponential decaying in the solution equations.

Long-Term Behavior of the Non-Homogeneous Equation

For any general solution $x(t) = x_p(t) + x_h(t)$, we have $x_h(t) \to 0$ as $t \to \infty$ as $x_h(t)$ is the homogeneuous solution, whose behavior is described above. Thus, the solution of any IVP behaves like $x_p(t)$ as time progresses.

We consider the case where $x'' + \gamma x' + kx = \cos(\omega t)$ with $\gamma, k > 0$. Note that for $t >> 1$, we have

$$x(t) \approx x_p(t) = \frac{1}{\sqrt{(\omega^2 - k)^2 + \gamma^2\omega^2}} \cos(wt + \theta)$$

If we input a input with frequency ω, amplitude 1, the output is also periodic with the same frequency (with a phase lag) and amplitude (below) which is called the **gain** and depends on input.

$$G(\omega) = \frac{1}{\sqrt{(\omega^2 - k)^2 + \gamma^2\omega^2}}$$

The gain can be much larger than 1 is an event called the *resonance phenomenon*, which occurs at the resonant frequency $\omega \approx \sqrt{k}$. The actual maximum of $G(\omega)$ occurs at $\omega = \sqrt{k}\sqrt{1 - \frac{\gamma^2}{2k}}$, where the resonant gain (amplitude) is around $\frac{1}{\gamma\omega} \approx \frac{1}{\gamma\sqrt{k}}$.

12 April 26, 2021 (Autonomous ODEs and Dynamical Systems)

Definition: Autonomy

An **autonomous** ODE takes the form, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a vector-valued function with n components.

$$x'(t) = f(x(t))$$

The right hand side does not depend explicitly on time. In contrast, a **non-autonomous** ODE appears as $x'(t) = f(x(t), t)$.

Remark: In an autonomous ODE, if $x(t)$ is a solution, then $x(t + T)$ is also a solution for any real number T, which follows from Chain Rule.

- $n = 1$, where $f(y) = -y(y - 1)(y - 2)$

$$y'(t) = -y(t)(y(t) - 1)(y(t) - 2)$$

- $n = 2$, where we have the predator-prey model $f(r, w) = (Ar - Brw, -Cw + Drw)$

$$r'(t) = Ar(t) - Br(t)w(t)$$
$$w'(t) = -Cw(t) + Dr(t)w(t)$$

Dyanmical Systems

From a dynamical systems perspective, we can think of $x(t)$ as a trajectory moving in $\mathbb{R}^n$ with time. At the point $x(t)$ at time t, its velocity is $x'(t)$.

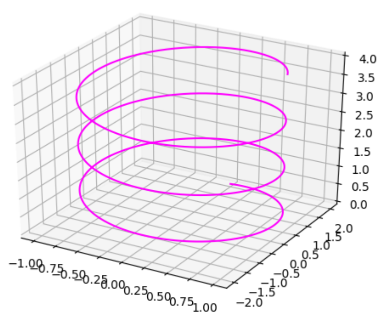


Figure 1: Trajectory of $x(t) = (\sin(t), 2\cos(t), \frac{t}{5})$, graphed in Python

We consider $f(x)$ to be *vector* that is assigned to every point x of $\mathbb{R}^n$, which is considered a vector field. If $x(t)$ is a solution of $x'(t) = f(x(t))$, then the velocity vector of its trajectory at time t is the vector delegated by f.

Consider an initial condition $x(t_0) = x_0$ at time t_0. We have the *principle of uniqueness and existence*, as a location at a specified time reveals how it got to that location and where it will continue moving along the vector field.

Remark

If x_0 lies on the trajectory of a solution $\tilde{x}(t)$, the solution $x(t)$ of the IVP with $x(t_0) = x_0$ is equal to $\tilde{x}(t + T)$ for some real number T.

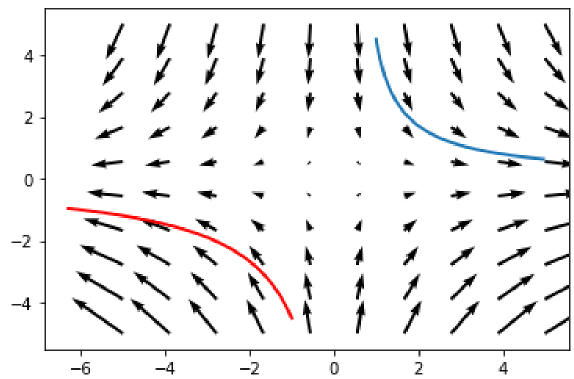


Figure 2: Vector field of a predator-prey model with $(A, B, C, D) = (1, 0.1, 1, 0.01)$. Each solution (the red and blue trajectories are examples) has its velocity tangent to some vectors

1D Dynamical Systems

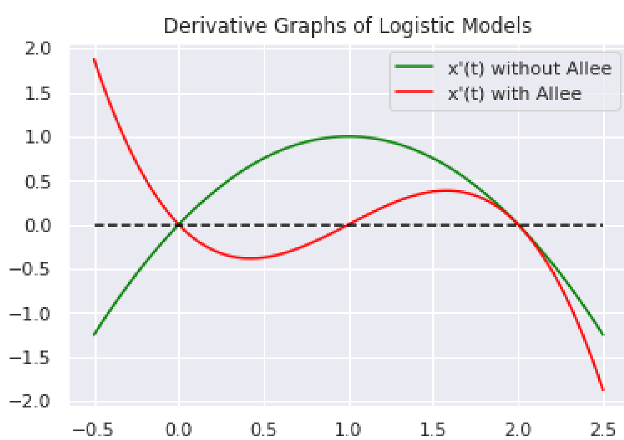
We consider ODEs of the form $x'(t) = f(x(t))$ where $x(t)$ is a scalar function and $f(x)$ is a real-valued function we interpret as a vector field. The magnitude of $f(x)$ is $|f(x)|$, the vector points left if $f(x) < 0$ and right if $f(x) > 0$, and $f(x) = 0$ designates a zero vector. There are three possible asymptotic outcomes for a solution $x(t)$.

- $x(t)$ converges to an equilibrium point (x_0 where $f(x_0) = 0$)
- $x(t)$ converges to ∞ or $-\infty$
- $x(t)$ blows up to ∞ or $-\infty$ in finite time

Logistic Equation

We consider the growth of a rabbit population $x(t)$ described through $x'(t) = x(t)(2 - x(t))$. It has a carrying capacity of 2 so any positive initial condition will converge there.

Next, we consider logistic growth with an Allee effect, where $x'(t) = x(t)(1 - x(t))(2 - x(t))$. This adjustment accounts for the rabbits dying off if there are <1 rabbits initially, but approaching carrying capacity 2 if the initial condition >1.



13 April 28, 2021 (Dynamical Systems in 2D)

We consider autonomous first order ODEs for $n = 2$, which inhabit the form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$$

The right hand side can be interpreted as a vector field in the $xy-$plane, where $(f(x,y), g(x,y))$ is a vector at point (x,y). We are interested in solution behavior, past and future, given initial conditions $(x(0), y(0)) = (x_0, y_0)$.

Definition: Equilibrium Points

Equilibrium points are points (x_0, y_0) where the constant map below solves the ODE.

$$(x(t), y(t)) = (x_0, y_0)$$

These are points where the vector field is the zero vector.

Remark: Here are a couple possibilities (there are more!) for long-term solution behavior in $n = 2$.

- Converging to an equilibrium point.
- Converging to ∞, which means leaving every bounded region in finite time and never going back in (note: this is the 2D generalization of the formal definition of a limit).
- Being a periodic solution that circulates around in a loop.
- Converging to a periodic solution.

Due to the *principle of uniqueness*, trajectories of two solutions either never intersect or completely align. The trajectory of a solution inside a periodic solution (which resembles a loop) cannot converge to ∞ as that forces it to cross the loop.

Lotka-Volterra Models

Predator-prey Model

We have a first order ODE of the form below, for positive A, B, C, D,

$$\begin{aligned} r'(t) &= Ar(t) - Br(t)w(t) \\ w'(t) &= -Cw(t) + Dr(t)w(t). \end{aligned}$$

Here, $r(t)$ and $w(t)$ quantify the rabbit and wolf populations at time t. Suppose that $A = 3, B = 1, C = 7, D = 1$, and we want to find its equilibrium points, where

$$\begin{bmatrix} 3r - rw \\ -7w + rw \end{bmatrix} = \begin{bmatrix} r' \\ t' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence, we have $r(3 - w) = 0$ and $w(r - 7) = 0$, so our equilibrium points are $(0, 0)$ and $(7, 3)$ in the $rw-$plane. We can study the solution behavior by tracing along the vectors in the field below.

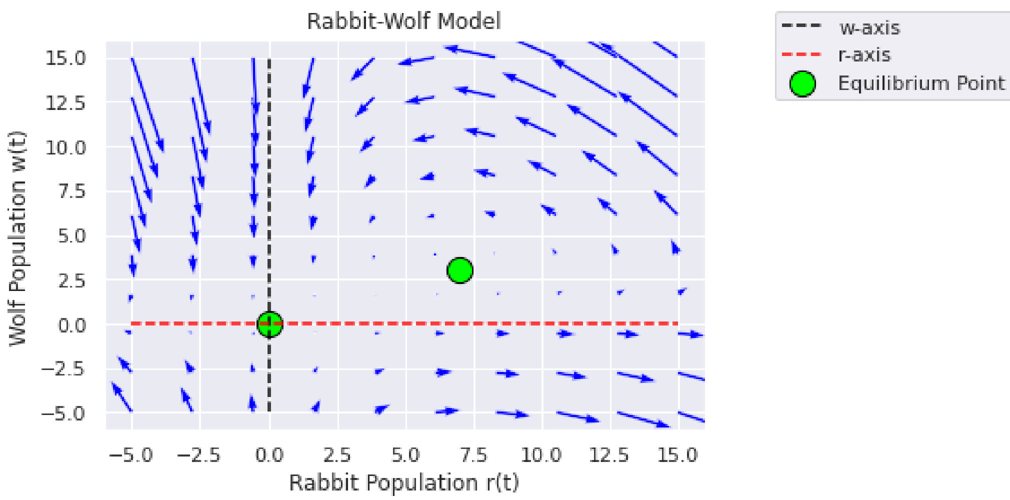


Figure 3: Link for open-source graph code.

We focus on $r, w > 0$. A solution that enters this region at any moment has to stay in the region eternally. All the solutions that reside in $r, w > 0$ are actually periodic (note the cyclical arrangement of the vectors) except for the equilibrium points.

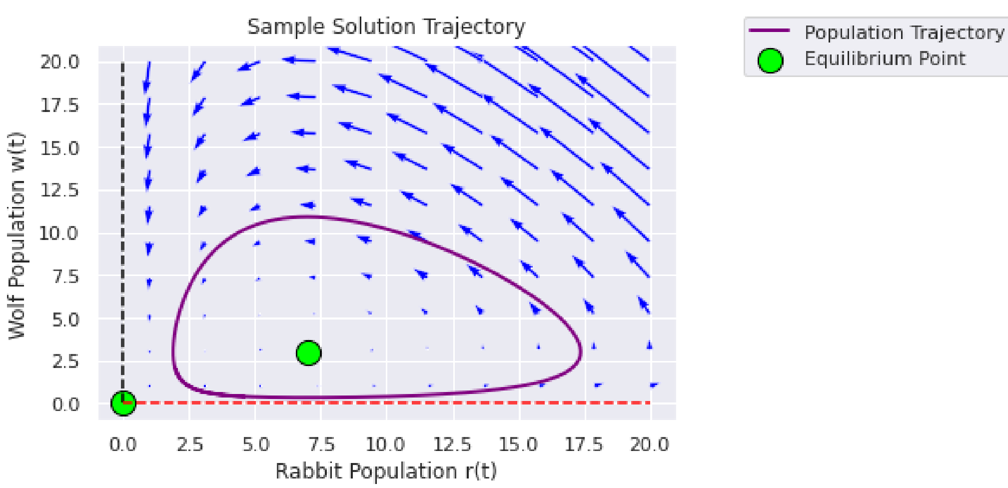


Figure 4: Link for open-source graph code.

Intuition for Periodic Solutions: Consider this cycle. (1)When we have a small number of wolves and rabbits, the rabbit population increases. (2) When there are lots of rabbits, the wolf population increases. (3) When there are lots of wolves, the rabbit population rapidly descends. (4) When there are little rabbits, the wolf lose their prey and their population descends, and the cycles begins anew.

14 April 30, 2021 (Equilibrium Classification and Trajectories)

We consider autonomous linear first ODEs of the form:

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A} \begin{bmatrix} x \\ y \end{bmatrix}$$

Three Main Trajectory Cases

1. $\mathbf{A}$ has real eigenvectors v_1 and v_2 which form a basis of $\mathbb{R}^2$ with eigenvalues λ_1 and λ_2.
2. $\mathbf{A}$ has a complex eigenvector $u + iw$ with eigenvalue $r + i\omega$.
3. $\mathbf{A}$ has a real (repeated) eigenvector v with eigenvalue λ, and there is a vector $\tilde{v}$ such that

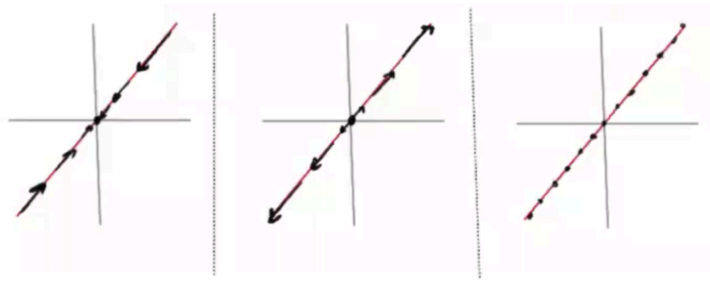
$$(\mathbf{A} - \lambda I)\tilde{v} = v.$$

Recall that an **equilibrium point** is a $(x, y) \in \mathbb{R}^2$ such that $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. These points compose the **null space** of the matrix $\mathbf{A}$. If A is invertible, the only equilibrium point is the origin.

Assume that v is an eigenvector of A with eigenvalue λ, where $Av = \lambda v$. Consider the line containing v in the xy-plane, which has points cv for $c \in \mathbb{R}$. The vector at point cv is

$$Acv = c(Av) = c\lambda v.$$

Remark: The vector gets linearly larger in magnitude as we go further from the origin. If $\lambda < 0$, the vectors point toward the origin (left); if $\lambda > 0$, the vectors point outward from the origin (middle); if $\lambda = 0$, the line consists of equilibrium points (right). The solutions whose trajectories are described by the vectors on the line are of the form $ce^{\lambda t}v$ for $c \in \mathbb{R}$.



Eigenbasis

If A has real eigenvectors v_1 and v_2 which form a (eigen)basis of $\mathbb{R}^2$ with eigenvalues λ_1, λ_2, the general solution is

$$c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2.$$

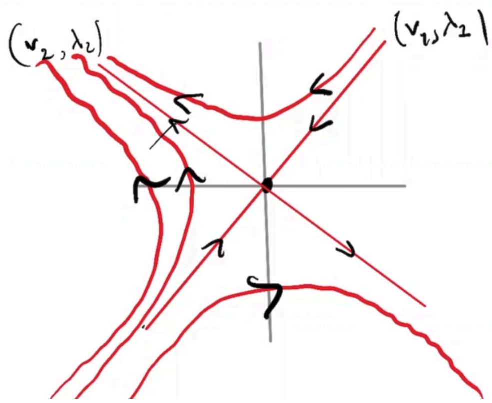
If $\lambda_1 < \lambda_2$ and $c_1, c_2 \neq 0$, then

1. $c_1 e^{\lambda_1 t} v_1$ dominates for $t << 0$
2. $c_2 e^{\lambda_2 t} v_2$ dominates for $t >> 0$

If $\lambda_1 = \lambda_2$ (equal contribution), we simply have $e^{\lambda_1 t}(c_1 v_1 + c_2 v_2)$.

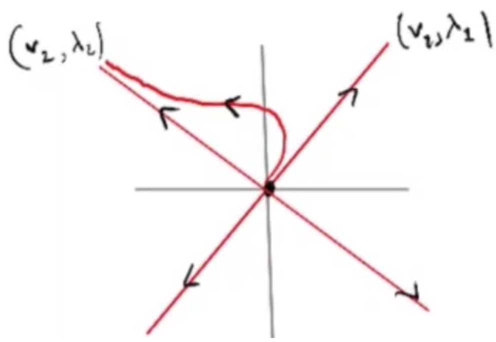
Saddle

In the case where $\lambda_1 < 0 < \lambda_2$, we have that our trajectories asymptotically approach the eigenline with v_2 in forward time ($t >> 0$) as it dominates. Thinking in reverse time, our trajectories are coming from the eigenline with v_1 as $t << 0$.

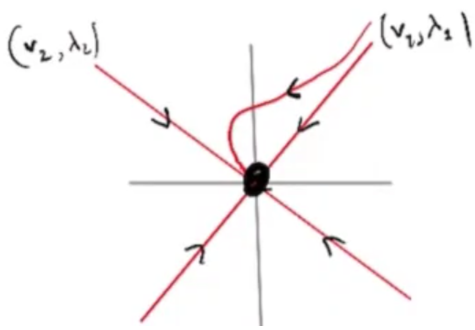


Stable/Unstable Node

Stable Node: In the case where $0 < \lambda_1 < \lambda_2$, we have that our solutions asymptotically approach the eigenline with v_2 and travel away from the origin.

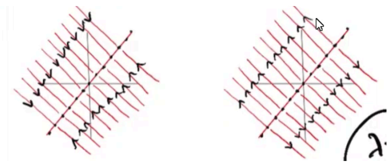


Unstable Node: In the case where $\lambda_1 < \lambda_2 < 0$, our solutions asymptotically converge on the origin in the diagram (right).



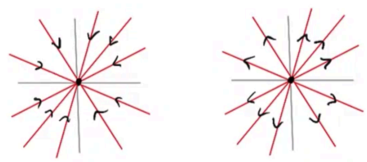
Comb

If one of the eigenvalues is 0, then we have a solution of form $c_1 + c_2 e^{\lambda_2}$, so we have several parallel solutions that are created by adjusting c_1. On the left, we have $\lambda_2 < 0$ (pointing toward origin), and on the right $\lambda_2 > 0$ (pointing away from origin).



Star

Here, we have equal non-zero eigenvalues, in which case if $\lambda < 0$ (left), our solutions point toward the origin (exponential decay), and if $\lambda > 0$ (right), our solutions grow away from the origin (exponential growth).

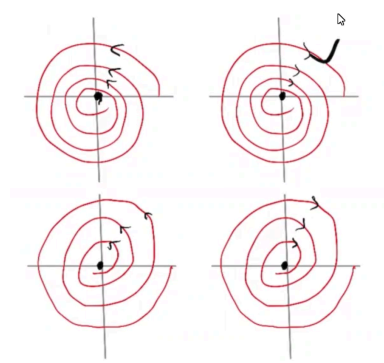


Complex Eigenvalues

Suppose matrix A has a complex eigenvector $u + iw$ with eigenvalue $r + i\omega$. The general solution is

$$c_1 e^{rt}(\cos(\omega t)u - \sin(\omega t)w) + c_2 e^{rt}(\sin(\omega t)u + \cos(\omega t)w)$$

This is one constant solution at the origin. If $r \neq 0$, the other solutions will resemble spirals around the origin, which converge to the origin as $t \to \infty$ (for $r < 0$) or as $t \to -\infty$ (for $r > 0$). Rotation is either clockwise or counter-clockwise. The 4 cases are shown below.



If $r = 0$, all solutions loop periodically around ellipses that can also gyrate in either direction.

Determining Rotation

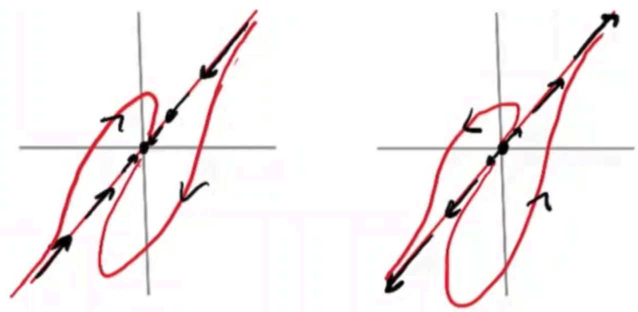
Compute the vector at point $(1, 0)$, which is equal to $A \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. If the resulting vector points upward, the solution rotates counter-clockwise; if it points downward, it rotates clockwise.

Real Eigenvalues (One Eigenvector Case)

Suppose that A has real eigenvector v with eigenvalue λ and vector $\tilde{v}$ with $(A - \lambda I)\tilde{v} = v$. As $v, \tilde{v}$ is a basis of $\mathbb{R}^2$, the genreal solution is

$$c_1 e^{\lambda t} + c_2(te^{\lambda t} v + e^{\lambda t}\tilde{v}) = (c_1 + tc_2)e^{\lambda t} v + c_2 e^{\lambda t}\tilde{v}.$$

For $\lambda < 0$ (left), our solution trajectories will essentially move toward the origin due to exponential decay. Note that we also have a solution curve consisting of the eigenline (straight red line in image) with v. With $\lambda > 0$ (right), the term with $tc_2 e^{\lambda t} v$ dominates as $t >> 0$, so the solution curves will spread away from the origin but asymptotically approach the eigenline with v.



15 May 3, 2021 (Linearization and Stability)

Linearization Near an Equilibrium Point

We will examine $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$ near the equilibrium points, which includes two main steps.

1. Linaerize to find the best approxmating linear vector field.
2. Implant the linearization for $\begin{bmatrix} x \\ y \end{bmatrix}' = A \begin{bmatrix} x \\ y \end{bmatrix}$ (A is a scalar matrix) near the origin to the vicinity of the equilibrium point.

We use the following linear approximation for multivariate functions. Let (x_0, y_0) be an equilibrium where $f(x_0, y_0) = g(x_0, y_0) = 0$, which means

$$\begin{bmatrix} f(x_0 + \mathbf{\Delta} x, y_0 + \mathbf{\Delta} y) \\ g(x_0 + \mathbf{\Delta} x, y_0 + \mathbf{\Delta} y) \end{bmatrix} \approx \begin{bmatrix} \frac{\partial f}{\partial x}(x_0, y_0)\mathbf{\Delta} x + \frac{\partial f}{\partial y}(x_0, y_0)\mathbf{\Delta} y \\ \frac{\partial g}{\partial x}(x_0, y_0)\mathbf{\Delta} x + \frac{\partial g}{\partial y}(x_0, y_0)\mathbf{\Delta} y \end{bmatrix}$$

These arise from $F(x + \mathbf{\Delta} x, y + \mathbf{\Delta} y) \approx F(x, y) + \frac{\partial F}{\partial x}(x, y)\mathbf{\Delta} x + \frac{\partial F}{\partial y}(x, y)\mathbf{\Delta} y = \frac{\partial F}{\partial x}(x, y)\mathbf{\Delta} x + \frac{\partial F}{\partial y}(x, y)\mathbf{\Delta} y$, as we are evaluating $F(x, y)$ for some function F at an equilibrium point.

Linearization at Equilibrium Point

Meshing the above multivariate estimation with $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$, we get the approximating linear system near (x_0, y_0).

$$\begin{bmatrix} \mathbf{\Delta} x \\ \mathbf{\Delta} y \end{bmatrix}' = \begin{bmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{bmatrix} \begin{bmatrix} \mathbf{\Delta} x \\ \mathbf{\Delta} y \end{bmatrix} = \mathbf{A} \begin{bmatrix} \mathbf{\Delta} x \\ \mathbf{\Delta} y \end{bmatrix}$$

We remark that $\mathbf{A}$ is a 2 by 2 matrix that represents the *Jacobian matrix* of $\begin{bmatrix} f \\ g \end{bmatrix}$.

Competing Species

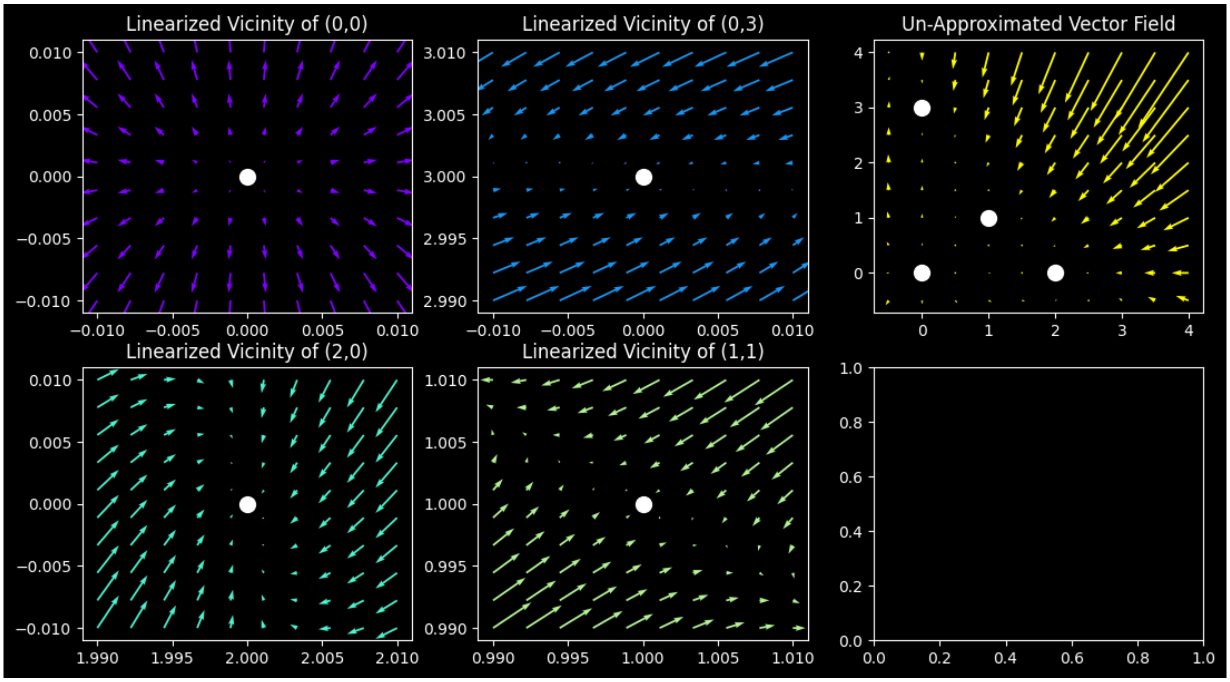
Consider the ODE system described by $x' = x(2 - x - y)$ and $y' = y(3 - 2x - y)$, where we have equilibrium points $(0, 0), (0, 3), (2, 0), (1, 1)$ from setting $x' = 0$ and $y' = 0$ and solving. To approximate our system linearly, we take the Jacobian matrix

$$\begin{bmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{bmatrix} = \begin{bmatrix} 2 - 2x - y & -x \\ -2y & 3 - 2x - 2y \end{bmatrix}$$

Let's find the linearization at $(0, 0)$, where the Jacobian matrix is $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, which has an eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ with $\lambda_1 = 2$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ with $\lambda_2 = 3$. We see that for trajectories near $(0, 0)$, the origin is an unstable node with two positive real eigenvalues.

Next, we consider the linearization at $(0, 3)$, which features the Jacobian matrix $\begin{bmatrix} -1 & 0 \\ -6 & -3 \end{bmatrix}$. This matrix has eigenvectors $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ with $\lambda_1 = -3$ and $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ with $\lambda_2 = -1$. We see that $(0, 3)$ is a stable node as $\lambda_1 < \lambda_2 < 0$.

We can find the linearizations at $(2, 0)$ and $(1, 1)$ using similar methods; we find that they are a stable node and saddle, respectively. An illustration of the general vector field and the local approximation field for each equilibrium point is provided below.



We remark that linearization at an equilibrium point (center) is not good if the matrix has a purely imaginary complex eigenvalue. For instance, the system $x' = y + ax(x^2 + y^2)$ and $y' = -x + ay(x^2 + y^2)$ has the same Jacobian $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ with equilibrium $(0, 0)$ regardless of $a \in \mathbb{R}$.

Equilibrium Stability

We call an equibrilum point *stable* if any trajectory that comes sufficiently close to it will stay sufficiently close for all future times.

Stable equilibrium points in linear systems include stable nodes, stable stars, inward spiraling spirals, centers, and stable degenerate nodes. Except for centers, we can generally check stability through the linearization.

16 May 5, 2021 (Nullclines and Competing Species)

Competing Species Population Dynamics

Let $x(t)$ and $y(t)$ be the number of rabbits and sheep, respectively, competing for meadow grass.

$$x' = r_x x(1 - \frac{x}{K_x} - \frac{y}{E_y})$$
$$y' = r_y y(1 - \frac{x}{E_x} - \frac{y}{K_y})$$

Here, r_x and r_y represent the growth rates if left alone with unlimited resources, K_x and K_y are the carrying capacities if left alone with limited resources, and E_x evaluates the *negative* effect of rabbits on sheep (E_y defined analagously), with a larger E_x, E_y implying a smaller effect.

Definition: Nullclines

Consider a system $x' = f(x, y), y' = g(x, y)$. A **horizontal nullcline** is the points in the plane where the vector field is fully horizontal, which are the $(x, y) \in \mathbb{R}^2$ such that $g(x, y) = 0$. A **vertical nullcline** is the points $(x, y) \in \mathbb{R}^2$ such that $f(x, y) = 0$ (fully vertical).

Remark: The intersecting points of the horizontal and vertical nullclines are the equilibrium points.

Recall from Monday, we had the system $x' = x(2 - x - y)$ and $y' = y(3 - 2x - y)$.

- To find the horizontal nullcline, we take $y' = 0$, so $y(3 - 2x - y) = 0$. These are the lines $y = 0$ and $2x + y = 3$ in the Cartesian plane.
- To find the vertical nullcline, we take $x' = 0$ so $x(2 - x - y) = 0$. These are just the lines $x = 0$ and $x + y = 2$ in the Cartesian plane.

We can analyze vectors along the nullclines to help determine trajectory behavior or identify bounded solutions.

General Observations on Competing Species

Revisiting the rabbit-sheep model, we make a few generalities. Along the axes, the population of a species just grows as if it was left alone. Also, converging to ∞ as $t \to \infty$ is impossible for $x, y \geq 0$, at (x, y), one coordinate is large enough to make the vector field's components negative. Finally, all solutions will converge to some equilibrium point as $t \to \infty$.

Equilibrium Analysis

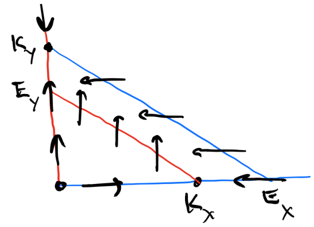
The points $(0, 0), (K_x, 0), (0, K_y)$ are always equilibrium points. There is also the Jacobian matrix

$$\begin{bmatrix} r_x - \frac{2r_x x}{K_x} - \frac{r_x y}{E_y} & -\frac{r_x x}{E_y} \\ -\frac{r_y y}{E_x} & r_y - \frac{2r_y y}{K_y} - \frac{r_y x}{E_x} \end{bmatrix}$$

.

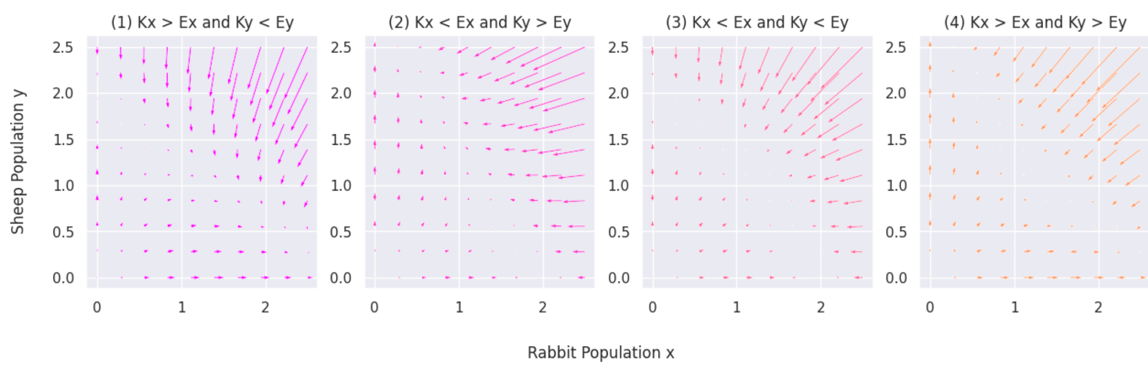
Nullcline Analysis

A horizontal nullcline is $y' = y(1 - \frac{x}{E_x} - \frac{y}{K_y}) = 0$, which is the union of the x-axis and the line segment connecting $(E_x, 0)$ and $(0, K_y)$. The vertical nullcline is $x(1 - \frac{x}{K_x} - \frac{y}{E_y}) = 0$, which is the union of the y-axis and the line segment connecting $(E_y, 0)$ and $(0, K_x)$.



Assume $E_x \neq K_x$ and $E_y \neq K_y$, so there are four cases.

1. $K_x > E_x$ and $K_y < E_y$: The rabbit population $x(t)$ will dominate. The nullclines do not intersect in quadrant I here, which means there is no equilibrium point with $x, y > 0$.
2. $K_x < E_x$ and $K_y > E_y$: The sheep population $y(t)$ will dominate.
3. $K_x < E_x$ and $K_y < E_y$: Co-existence between rabbits and sheep, implying the diagonal lines of the nullclines do intersect at an equilibrium point inside quadrant I.
4. $K_x > E_x$ and $K_y > E_y$: Initial condition determines the dominance.



We remark that for (1), the solution trajectories will head toward the bottom right as $t \to \infty$, which is where equilibrium point $(K_x, 0)$ is located, so rabbits dominate. For (2), the trajectories haed toward the top left as $t \to \infty$, which is where $(0, K_y)$ is located, so sheep dominate. For (3), the trajectories converge toward some population equilibrium (a, b) (co-existence) as $t \to \infty$. In (4), depending on initial conditions, nearly all trajectories converge to either $(K_x, 0)$ or $(0, K_y)$ as $t \to \infty$.

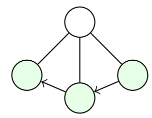
ODE	$(0,0)$	$(K_x, 0)$	$(0, K_y)$	$(a > 0, b > 0)$
$K_x > E_x$ and $K_y < E_y$	unstable node/star	stable node	saddle	NA
$K_x < E_x$ and $K_y > E_y$	unstable node/star	saddle	stable node	NA
$K_x < E_x$ and $K_y < E_y$	unstable node/star	saddle	saddle	stable node
$K_x > E_x$ and $K_y > E_y$	unstable node/star	stable node	stable node	saddle

Seperatrix

A solution trajectory such that the solution trajectories which are at the opposite ends of it behave very differently. Consider the trajectory between $(0, 0)$ and (a, b) in case (4), where solutions on top of it go to $(0, K_y)$ as $t \to \infty$ and solutions below it go to $(K_x, 0)$.

17 May 7, 2021 (Pendulum Modeling and Basins of Attraction)

Consider a pendulum, whose state at any given moment is its angular position (counter-clockwise from rest) and its angular velocity. Suppose the angular position is



Pendulum State

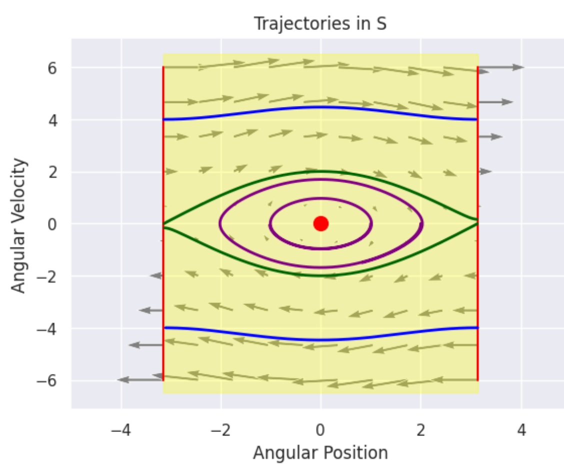
Let the angular position at time t be $\phi(t)$ radians, so angular velocity is $\phi'(t)$. The motion of an ideal pendulum satisfies Newton's second law with gravitation, or

$$\phi''(t) + \sin(\phi(t)) = 0.$$

If ϕ is small, then $\sin(\theta) \approx \theta$, which is identical to the undamped mass-spring system. We transform our ODE into a first order system, where $w(t) = \phi'(t)$,

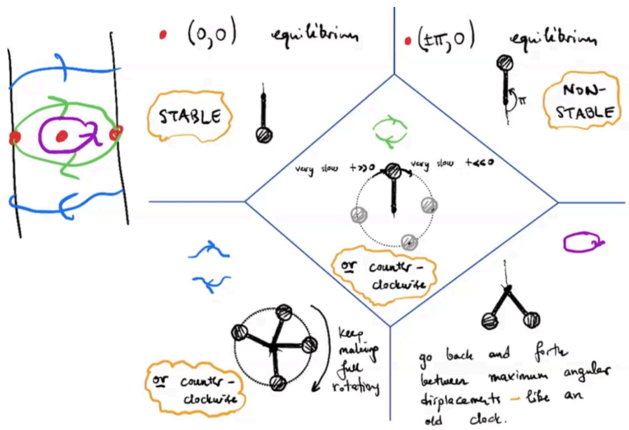
$$\begin{bmatrix} \phi \\ w \end{bmatrix}' = \begin{bmatrix} w \\ -\sin(\phi) \end{bmatrix}.$$

Consider the ϕw-plane which consists of points with first component in $[-\pi, \pi]$. These form an infinite strip S that we can imagine with both sides glued, where $(-\pi, w)$ is matched to (π, w) for any $w \in \mathbb{R}$. Note that S corresponds with all possible states of the pendulum as $-\pi \leq \phi \leq \pi$. When trajectories in S reach a boundary of the strip, they will reappear on the other boundary and keep moving.



A *vector field* on S is just a vector assigned to each point on S, with the mandate that the vector at (π, w) is the same as the vector at $(-\pi, w)$. We can build the vector field by giving (w, ϕ) the vector $\begin{bmatrix} w \\ -\sin(\phi) \end{bmatrix}$, which comes from the ODE system. This means that there are two equilibrium points: $(0, 0)$ and $(-\pi, 0) = (\pi, 0)$. By following the vectors, we can see the various classes of trajectories, as illustrated above: the blue and purple ones are periodic, the green converges to $(\pm\pi, 0)$ as $t \to \pm\infty$, the red is the equilibria.

The corresponding motions are illustrated below from the slides.

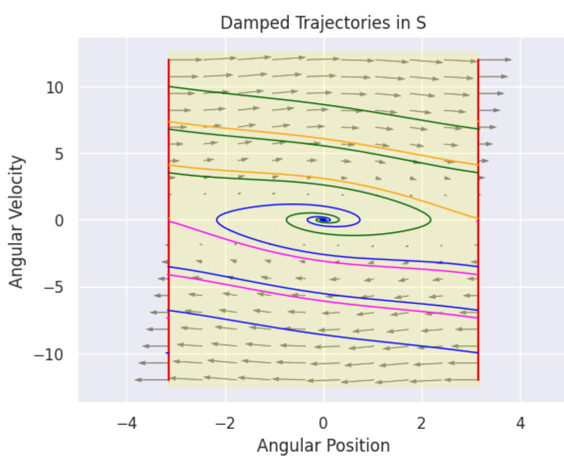


Slightly Damped Pendulum

The motion of this more realistic pendulum follows $\phi'(t) + 0.5\phi'(t) + \sin(\phi(t)) = 0$, which corresponds to the dynamical system

$$\begin{bmatrix}\phi \\ w\end{bmatrix}' = \begin{bmatrix} w \\ -\sin(\phi) - 0.5w\end{bmatrix}$$

We remark the energy $\frac{w^2}{2} - \cos(\phi)$ will dissipate from the damping. We have a vector field in S, with $(0,0)$ as a stable spiral and $(\pm\pi, 0)$ as a saddle. A few trajectories are drawn (the green and blue ones are typical, the orange and pink ones are special in that they converge at the saddles $(\pm\pi, 0)$).



Since the pendulum keeps dissipating energy, eventually, it must reach a maximal height and fall, going back and forth around the equilibrium with its maximal displacement decreasing each time.

Basin of Attraction

For some equilibrium point p, the basin is the set of all points q such that the solution trajectory at q at $t = 0$ converges to p as $t \to \infty$.

The basin of attraction the unstable $(\pm\pi, 0)$ are itself and the orange/pink points. All other points are inside the basin of attraction the stable $(0, 0)$ equilibrium.

18 May 10, 2021 (Butterfly Effect and Poincare-Bendixson)

For an autonomous ODE of form $x' = F(x)$ for $x(0) = x_0$ that generates a unique solution $x(t)$, we consider its possible behaviors as $t \to \infty$. For $n = 1$, $x(t)$ either converges to an equilibrium point or goes to $\pm\infty$.

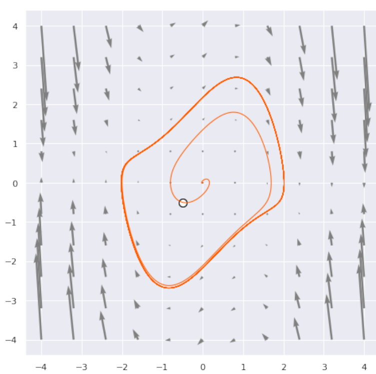
Poincare-Bendixson Theorem

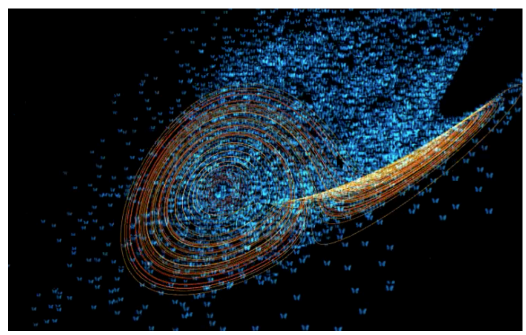
In the plane ($n = 2$), there are 5 possibilities for $(x(t), y(t))$ as $t \to \infty$: (1) it can converge to an equilibrium point, (2) go to ∞, (3) be a periodic trajectory, (4) converge to a periodic trajectory without being one itself, or (5) converge to a cycle of homoclinical and heteroclinical orbits.

For (4), consider Pol's equation.

$$x' = y$$
$$y' = -x + \mu(1 - x^2)y, \quad \mu \neq 0$$

We have a trajectory that spirals out toward a periodic orbit, as visualized below (left).





Orbits

In a homoclinical orbit, as $t \to \pm\infty$, $(x(t), y(t))$ converges to an equilibrium point $p \in \mathbb{R}^2$.

In a heteroclinical orbit, as $t \to \infty$, $(x(t), y(t))$ converges to point $q \in \mathbb{R}^2$, and as $t \to -\infty$, $(x(t), y(t))$ converges to a different point $p \in \mathbb{R}^2$.

Consider vector fields on the surface of a torus/donut. Suppose this vector field is constant with irrational slope. It turns out that a trajectory will never come back to itself while orbiting the torus's surface but come arbitrary close. We see that Poincare-Bendixson may not apply outside the plane.

In the **butterfly effect**, small changes in initial conditions may result in very large changes as $t \to \infty$. This can describe competing species ODE with initial condition dependent domination, the Belousov-Zhabotinsky (BZ) reaction, or the Lorenz attractor.

Lorenz Equation

The Lorenz Equations feature strange attractors, where the trajectories are attracted to a strange shape, as imaged above (right).

$$x' = -10x + 10y, \; y' = -xz + 28x - y, \; z' = xy - 2.5y$$

19 May 12, 2021 (Real World Dynamical Systems)

Mostly midterm review. The official lecture transcript can be found **here**.

Logistic ODE with Allee Effect

Let $x(t)$ be the rabbit population at time t, k be the carrying capacity, and A be the minimal population needed to avoid extinction.

$$x' = x(\frac{x}{A} - 1)(1 - \frac{x}{k})$$

For any $x(0) > k$, then $x(t)$ goes to k as $t \to \infty$. For $0 < x(0) < A$, the population goes exintct as $x(t) \to 0$ as $t \to \infty$.

Damped Mass-Spring System

Let $x(t)$ be the block position, $v(t)$ be velocity, m be mass, k be spring constant, and γ be damping constant.

$$\begin{bmatrix} x \\ v \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\gamma}{m} \end{bmatrix}$$

When $\gamma = 0$, we have undamped motion, so the trajectories are periodic ellipses around $(0, 0)$. Let $\boldsymbol{\Delta}$ be the discriminant of the characteristic polynomial, which is equal to $\frac{\gamma^2}{m^2} - \frac{4k}{m}$. When $\boldsymbol{\Delta}$ is negative, 0, and positive, we have underdamped (inward spirals), critically damped (degenerate node), and overdamped (stable node), respectively.

Predator-Prey ODE Lotka-Volterra

Let $a, b, c, d > 0$, and $r(t)$ and $w(t)$ be rabbit and wolf populations, respectively. Recall that we have level sets of conserved quantity.

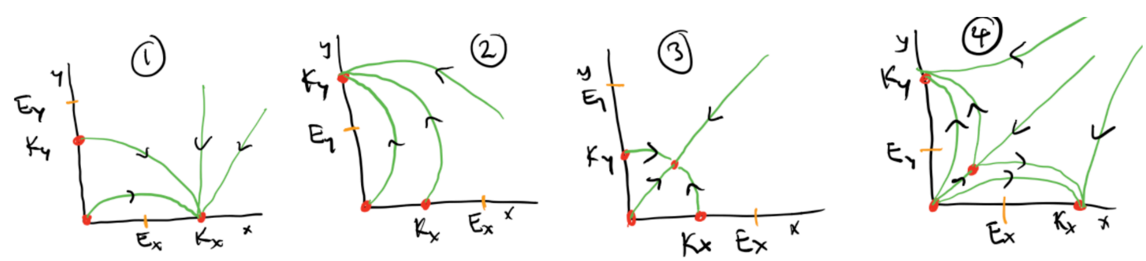
$$\begin{bmatrix} r \\ w \end{bmatrix}' = \begin{bmatrix} ar - brw \\ -cw + drw \end{bmatrix}$$

Competing Species ODE

Let $r_x, r_y, E_x, E_y, K_x, K_y > 0$. We have two species competing for limited resources.

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} r_x x(1 - \frac{x}{K_x} - \frac{y}{E_y}) \\ r_y y(1 - \frac{x}{E_x} - \frac{y}{K_y}) \end{bmatrix}$$

There are four possible outcomes, (1) x-dominated, (2) y-dominated, (3) co-existence, (4) initial-condition dependence, as visualized below.



20 May 17, 2021 (Introduction to Operators)

Definition: Operator

An operator D takes in a function y and returns a function $D(y)$. We can add operators, $(D_1 + D_2)y = D_1 y + D_2 y$, and compose of them, $D_2 \circ D_1(y) = D_2(D_1(y))$. An operator D is called **linear** if it satisfies

- $D(cy) = cD(y)$ for scalar c
- $D(y_1 + y_2) = D(y_1) + D(y_2)$

For example, if D is the derivative operator, then $Dy(t) = y'(t)$. Linear operators include taking the derivative, multiplying with a fixed function, the sum/composition of two linear operators.

LD Operator

For real-valued functions $a_0(t), \cdots, a_n(t)$, the *linear differential (LD) operator* is linear.

$$D = a_n(t)\frac{d^n}{dt^n} + a_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + a_1(t)\frac{d}{dt} + a_0(t).$$

The order of D is defined as n. If the functions $a_0(t)$ through $a_n(t)$ are independent of time, they are *linear constant coefficient (LCD)* operators. The linear scalar ODE $a_n(t)y^{(n)} + \cdots + a_0(t)y = g(t)$ can now be rewritten as $Dy = g$.

Remark: Suppose D is a real-valued LD operator. It can accept complex functions $y(t) = y_1(t) + iy_2(t)$ for real-valued $y_1(t), y_2(t)$, where

$$Dy = Dy_1 + iD_2 y.$$

Linear ODE: General Solutions

Suppose D is an LD operator. If $Dy_h = 0$ and $Dy = g$, then $D(y + y_h) = g$. If $Dy = g$ and $Dy_1 = g$, then $D(y_1 - y) = 0$.

Hence, all solutions of $Dy = g$ inhabit the form $y_p + y_h$, where y_p is a particular solution of $Dy = g$ and y_h is any solution of $Dy = 0$.

Observation

If D is an LD operator, and $Dy_1 = g_1, \cdots, Dy_n = g_n$, then for any complex scalars $c_1, \cdots, c_n$ and $y(t0 = c_1 y_1(t) + \cdots + c_n y_n(t)$,

$$Dy = c_1 g_1 + \cdots c_n g_n$$

This implies that if the input is a linear combination of functions, we can find solutions to each function and take the linear combination of them.

Exponential Inputs

Let D be an LCD operator of order n, or $\sum_{i=0}^{n} a_i \frac{d^i}{dt^i}$. We consider $g(t) = e^{(r+i\boldsymbol{\omega})t}$ for complex $r + i\boldsymbol{\omega}$. To find a solution to $Dy = e^{(r+i\boldsymbol{\omega})t}$, we try the input $y(t) = Ae^{(r+i\boldsymbol{\omega})t}$ for complex A.

The characteristic polynomial of D is equal to

$$p_D(z) = a_n z^n + \cdots + a_1 z + a_0$$

We can compute that

$$D(Ae^{(r+i\boldsymbol{\omega})t}) = Ap_D(r + i\boldsymbol{\omega})e^{(r+i\boldsymbol{\omega})t}$$

Thus, if $p_D(r + i\boldsymbol{\omega}) \neq 0$, then a solution is equal to

$$\boxed{y(t) = \frac{e^{(r+i\boldsymbol{\omega}t)}}{p_D(r+i\boldsymbol{\omega})}}.$$

21 May 19, 2021 (Real and Complex Fourier Series)

Defintion: P-periodic

A real and complex valued function $y(t)$ is P-periodic if $y(t) = y(t+P)$ for all $t \in \mathbb{R}$.

Real Fourier Series (Approximation)

Fix $P > 0$ and define $\omega = \frac{2\pi}{P}$ for convenience. For any $k \in \mathbb{Z}$, note $\sin(k\omega t), \cos(k\omega t)$ are P-periodic functions. All P-periodic real-valued functions have a real **Fourier series** expansion.

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos(k\omega t) + b_k \sin(k\omega t)]$$

To break this down, define the partial sums

$$F_n y(t) = \frac{a_0}{2} + \sum_{k=1}^{n} [a_k \cos(k\omega t) + b_k \sin(k\omega t)]$$

Here, $F_n y(t)$ is a continuous P-periodic real function. If $y(t)$ has no discontinuity at $t_0 \in \mathbb{R}$, then $F_n y(t_0) \to y(t_0)$ as $n \to \infty$, and if $y(t)$ has a jump discontinuity at t_0, then $F_n y(t_0) \to \frac{y(t_0-)+y(t_0+)}{2}$ as $n \to \infty$, where $y(t_0\pm) = \lim_{t \to t_0^{\pm}} y(t)$.

Real Fourier Series Coefficients

The formulas for finding the Fourier coefficients are

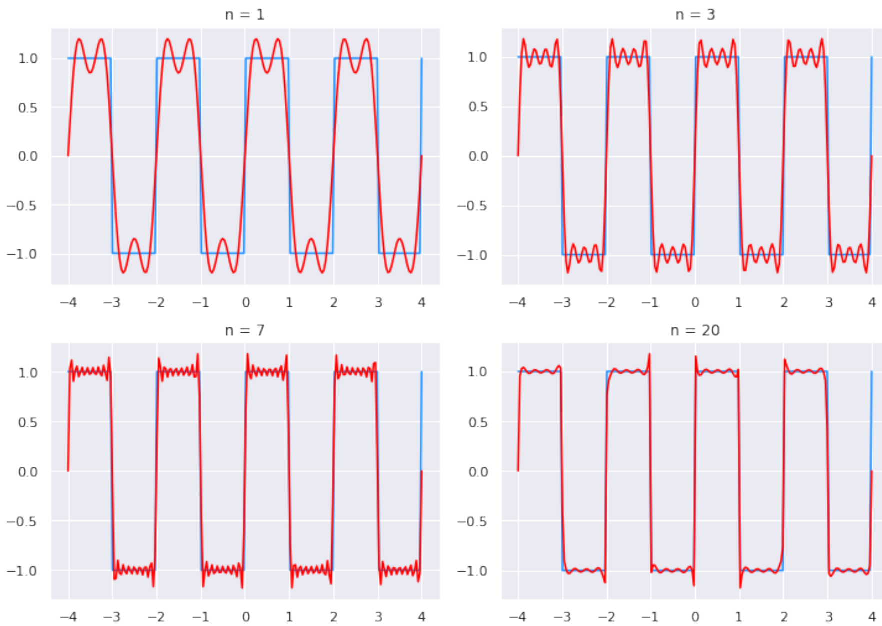
$$a_0 = \frac{2}{P} \int_0^P y(t)\ dt$$
$$a_k = \frac{2}{P} \int_0^P y(t) \cos(k\omega t)\ dt$$
$$b_k = \frac{2}{P} \int_0^P y(t) \sin(k\omega t)\ dt$$

Remark: The integration can be done from A to $A+P$ for any $A \in \mathbb{R}$.

For example, consider the 2-periodic function $u(t)$ where $u(t) = u(t+2)$ and

$$u(t) = \begin{bmatrix} 1, 0 \leq t < 1 \\ -1, 1 \leq t < 2. \end{bmatrix}$$

We have $u(t) = \sum_{k=0}^{\infty} \left[\frac{4}{\pi(2k+1)} \sin((2k+1)\pi t)\right]$. There are just sine functions since $u(t) = -u(-t$ for any $t \in \mathbb{R}$. Note that the partial sums $F_n u(t)$ approximate $u(t)$ with greater and greater fidelity, as visualized below.



Complex Fourier Series

Fix $P > 0$ and define $\omega = \frac{2\pi}{P}$. For any $k \in \mathbb{Z}$, note $e^{ik\omega t} = \cos(k\omega t) + i\sin(k\omega t)$ is P-periodic. Any P-periodic complex valude function has an expansion, for $c_k \in \mathbb{C}$,

$$y(t) = \sum_{k=-\infty}^{\infty} c_k e^{ik\omega t}$$

The integral $\int_0^P e^{ik\omega t}$ is equal to 0 when $k \neq 0$ and P if $k = 0$ by the symmetry of sin and cos. We multiply both sides of $y(t) = \sum_{k=-\infty} \infty c_k e^{ik\omega t}$ with $e^{-m\omega t}$ and integrate both sides to obtain

$$\int_0^P y(t)e^{-im\omega t}\, dt = \sum_{k=-\infty}^{\infty} c_k \int_0^P e^{ik\omega t}e^{-ik\omega t}\, dt = \sum_{k=-\infty}^{\infty} c_k \int_0^P e^{i(k-m)\omega t}\, dt$$

On the right hand side, only the term with $k = m$ supplies a non-zero integral so it is equal to $c_m P$. Thus,

$$\int_0^P y(t)e^{-im\omega t}\, dt = c_m P$$

$$\boxed{c_m = \frac{1}{P}\int_0^P y(t)e^{-im\omega t}\, dt}.$$

This represents a formula to compute coefficient c_m for all $m \in \mathbb{Z}$.

Remark: Every P-periodic real function also has a complex Fourier series expansion $y(t) = \sum_{k=-\infty}^{\infty} c_k e^{ik\omega t}$. Here, c_k is complex but we have $c_{-k} = \overline{c_k}$ for complex conjugation $\overline{a+bi} = a - ib$.

To transform a real Fourier series into a complex one, we leverage the identities $\cos(\theta) = \frac{1}{2}e^{i\theta} + \frac{1}{2}e^{-i\theta}$ and $\sin(\theta) = \frac{1}{2i}e^{i\theta} - \frac{1}{2i}e^{-i\theta}$. Also, the complex Fourier series can be metamorphosed into the real one through Euler's formula.