Preface

These notes were written for the class Math 108: Intro to Combinatorics, taught at Stanford in the 2020-21 spring quarter by Prof. Matthew Kwan.

March 29, 2021 (Combinatorics)

Intro to Discrete Structures

A few definitions.

- A **permutation** of length n is an ordering of the numbers $1, 2, \cdots, n$, such as $1243 \in S_4$. where S_n denotes the set of all such orderings.
- A **Latin square** of order n is an n by n grid filled with the numbers $1, 2, \cdots, n$ such that every number appears in each row and column.

1	2	3
3	1	2
2	3	1

- A **Graeco-Latin square** of order n is an n by n grid in which each cell is a pair of numbers $(x, y) \in \{1, 2, 3, \cdots, n\}^2$ such that (1) the first numbers of each pair form a Latin square (2) the second numbers of each pair form a Latin square, and (3) each of the n^2 possible pairs appears exactly once.

(1,1)	(2,2)	(3,3)
(3,2)	(1,3)	(2,1)
(2,3)	(3,1)	(1,2)

A central question is *whether such discrete structures and objects exist.*

Exercise

For any n, there exists a binary sequence and permutation of length n.

Explication: A simple, trivial example would be just choosing a sequence of n 0s, which is a binary sequence, as well as the ordering $1, 2, 3, \cdots, n \in S_n$, which is a permutation.

Exercise

There exists a Latin square of order n for any $n \in \mathbf{N}$.

Explication: This is less trivial. We can, however, take the first row to be $123 \cdots n$, the second row to be $23 \cdots n1$, and the third row to be $345 \cdots n12$, and so forth. More generally, the element in row i and column j for $1 \le i, j \le n$ is equal to $a_{ij} \equiv i + j - 1 \pmod{n}$. We note that the columns also form an increasing, cyclic permutation of numbers 1 through n, so we have a Latin square. We designate this construction as the *cyclic Latin square.*

Theorem

There exists a Graeco-Latin square for all $n \in \mathbf{N}$ except $n = 2$ or $n = 6$.

Explication: Partial proof reserved for the homework.

We are also interested in *how many distinct objects of each type* can be manufactured. The number of binary sequences is equal to 2^n for length n, as we have two options $\{0, 1\}$ for each of the n binary bits. The number of permutations is equal to $n! = n(n-1)\cdots 1$. Counting Latin squares is more difficult, although $|L_1| = 1$. $|L_2| = 2$, and $|L_3| = 12$. which can be computed by hand.

Latin-Square Theorem (1981)

The number of Latin squares $|L_n|$ of size n belongs in the range:

$$\frac{(n!)^{2n}}{n^{n^2}} \leq |L_n| \leq \prod_{k=1}^{n} (k!)^{\binom{n}{2}}$$

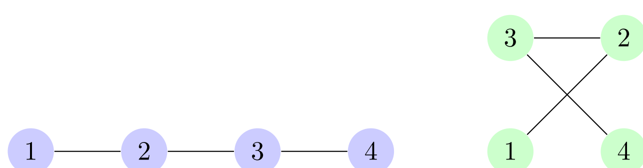
A few definitions. For permutations, a number x is a **fixed point** if the xth number in the permutation is x. For instance, the permutation 12435 has 3 fixed points. The maximum number of fixed points is equal to n ($123\cdots n$), and the minimum number is 0 (just shift each number in $123\cdots n$ by 1). We remark that it is impossible to have $n-1$ fixed points, since that means the nth point is automatically fixed.

Derangements Formula

The number of permutations with zero fixed points is the closest integer to $\frac{n!}{e}$.

Graph Theory

A **graph** G is a pair of finite sets (V, E) such that the elements of V are called vertices and the elements of E are unordered pairs of distinct vertices called edges. An example of a graph is $V = \{1, 2, 3, 4\}$ with edges $\{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$, which is illustrated below in two renditions.



In graphs, we cannot have vertices connected to themselves through loops, but this phenomena is legal under *multigraphs*. If $uv \in E$ for vertices $v, u \in V$, we say that u and v are adjacent, or that u is a **neighbor** of v. An edge e is **incident** to vertex v if $v \in e$ (the edge touches the vertex). Also, edges e and e' are incident if $e \cap e' \neq \emptyset$, which means they touch.

Special Graphs

The **complete graph** K_n contains n vertices with the property that each vertex has an edge to all the other $n-1$ edges. The **empty graph** $\overline{K_n}$ has n vertices but no edges. The **path graph** P_n has vertex set $v_1, \cdots, v_n$ and edge set $v_1v_2, v_2v_3, \cdots, v_{n-1}v_n$. The **cycle graph** C_n has a vertex set $v_1, \cdots, v_n$ and edge set $v_1v_2, v_2v_3, \cdots, v_{n-1}v_n, v_nv_1$.

- For a graph $G = (V, E)$. the **complement** graph $\overline{G}$ of G is the same vertex set and $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.
- Consider graphs $G = (V, E)$ and $G' = (V', E')$. An **isomorphism** $\phi : G \to G'$ is a bijection (relabeling of vertices) from V to V' such that $uv \in E$ if and only if $\phi(u)\phi(v) \in E'$. We call two graphs *isomorphic* if such an isomorphism exists.

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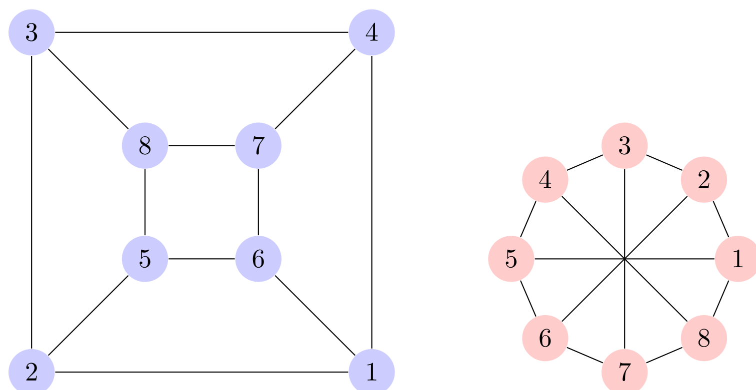
Isomorphism

Recall the definition of isomorphic graphs from the prior lecture.

Isomorphism

Graphs $G = (V, E)$ and $G' = (V', E')$ are ismorphic if there is bijection $\phi : V \to V'$ such that $uv \in E$ if and only if $\phi(u)\phi(v) \in E'$.

Note that isomorphisms cannot alter the degree sequence of the vertices or the existence of cycles and paths with static properties within the graph. For instance, the two graphs cannot be isomorphic since the right graph contains cycles of odd length while these are impossible in the left graph.



We can prove that odd length cycles are impossible in the left graph since in every edge, the vertex numbers of that edge have a different parity. Traversing an edge means changing the parity, so traversing an odd number of edges means the parity of our current vertex is different from the parity of the vertex we started from, so we cannot have a cycle.

Exercise: How many labeled graphs of the vertex set $\{1, 2, \cdots, n\}$ are there.

Solution: There are $\binom{n}{2} = \frac{1}{2}n(n-1)$ total pairs of vertices, which either form an edge or do not form an edge. Thus, we have 2 choices for each pair, meaning there are $2^{\binom{n}{2}}$ labeled graphs.

A few definitions:

- A graph $H = (u, F)$ is a **subgraph** of $G = (V, E)$ if $u \subseteq V$ and $F \subseteq E$.
- For a vertex $v \in V$ in graph $G = (V, E)$, the **degree** of v is $\deg(x) = |y \in Y : xy \in E|$. Essentially, it represents the number of vertices that v is connected to through an edge.
- The minimum degree $S(G) = \min_{x \in V} \deg(x)$ and the maximum degree is $\Delta(G) = \max_{x \in V} \deg(x)$.
- A graph is d-regular if all vertices have degree d.

Handshake Lemma

For every graph $G = (V, E)$, we have
$$2|E| = \sum_{v \in V} \deg(x).$$

Proof: In the sum $\sum_{v \in V} \deg(x)$, each edge $e = uv$ is counted twice, one from vertex u and once from vertex v. Thus, it is equal to $2|E|$.

Walks, Paths, Cycles

- A **walk** in a graph $G = (V, E)$ is a sequence of vertices $v_0, v_1, \cdots, v_k$ such that $v_1 v_{i+1}$ is an edge for all $0 \leq i < k$. This is called a **path** if all v_i are distinct. If $v_0 = v_k$, we have a closed walk. A closed path is a cycle. The number of edges represents the **length** of a walk.
- **Remark:** Having a path of length k is the same as having a sub-graph which is isomorphic to P_{k+1}. A cycle of length k is also a subgraph isomorphic to C_K.

Proposition

Every walk W from u to v contains a path from u to v.

Proof: Consider a shortest walk $w' = w_0 \cdots w_l$ contained in W, where $u = w_0$ and $v = w_l$. Note that this must be a path. If not, there is some $i < j$ with $w_i = w_j$, which means that the walk $w_0, \cdots, w_i, w_{j+1}, \cdots, w_l$ is shorter, which is a contradiction. Thus, w_0 through w_l are unique, and w' is a path.

Proposition

Every graph G with minimum degree $\delta \geq 2$ has a path of length δ.

Proof: Consider a longest path $v_0, \cdots, v_k$ in G. Then, all neighbors of v_k belong to $\{v_0, \cdots, v_{k-1}\}$ because otherwise, we could make the path longer. Thus, $k = |\{v_0, \cdots, v_{k-1}\}| \geq \deg(v_k) \geq \delta$, so $k \geq \delta$, and we have found the path of length at least δ.

A few definitions. (1) A graph $G = (V, E)$ is **connected** if for all pairs $u, v \in V$, there is a path from u to v. This means that there is a walk between any two such vertices. (2) A graph $G = (V, E)$ is **disconnected** if there is a partition $V = X \cup Y$ of the vertices, with $X, Y \neq \emptyset$, such that for all $x \in X$, $y \in Y$, $xy \notin E$.

Remark

Definitions 1 and 2 of connected and disconnected graphs are equivalent.

Proof: Suppose G is connected by definition (1). Let $S \cup T$ be a partition of V, with $S, T \neq \emptyset$. We pick $x \in S, y \in T$, and consider a walk $x_0 \cdots x_k$ for $x = x_0$, $y = x_k$ in G. Let $i = \min\{j : 1 \leq j \leq k : x_j \in T\}$, which is the first point in the walk where we encounter T. This means that $x_{i-1} \notin T$ and $x_i \in T$, so $x_{i-1} x_i \notin E$, which demonstrates that S, T do not disconnect G. So, G is connected by definition 2.

Suppose that G is disconnected by definition 1, which means that there are vertices x, z for which there is no walk between x and z. We define S to be the set of all $y \in V$ such that there exists a walk between x and y, and T to be the complementary set $V - S$. Assume that there is an edge between $y \in S$ and $w \in T$. Note that there is a walk from x to y as $y \in S$, which can be extended to w through edge yw. This creates a walk between x and w, which is a contradiction as $w \in T$. Thus, there is no edge between S and T, so G is disconnected by definition 2.

April 2, 2021 (Combinatorics)

Graph Coloring

Suppose that you have a map of the world and wish to color the countries so that two countries sharing a border receive different colors. Another scenario is that if we have some exams and some students who need to take various exams, how many time slots do we need to avoid exam conflicts?

K-coloring

A K**-coloring** of graph $G = (V, E)$ is a labelling $f : V(G) \to \{1, 2, \cdots, k\}$. It is a proper k-coloring if $xy \in E$ implies $f(x) \neq f(y)$, meaning the vertices of an edge have distinct colors.

- A graph is *k-colorable* if it has a proper k-coloring
- The *chromatic number* $\chi(G)$ is the minimum k such that G is k-colorable.

We compute $\chi(K_n) = n$ since each vertex is connected with its $n - 1$ neighbors, so n colors are needed. Also, $\chi(P_n) = 2$, unless $n = 1$, since we alternate colors along the path. We have $\chi(C_n) = 2$ if n is even and 3 otherwise, to ensure the first and last vertices do not have the same color in the cycle.

Proposition

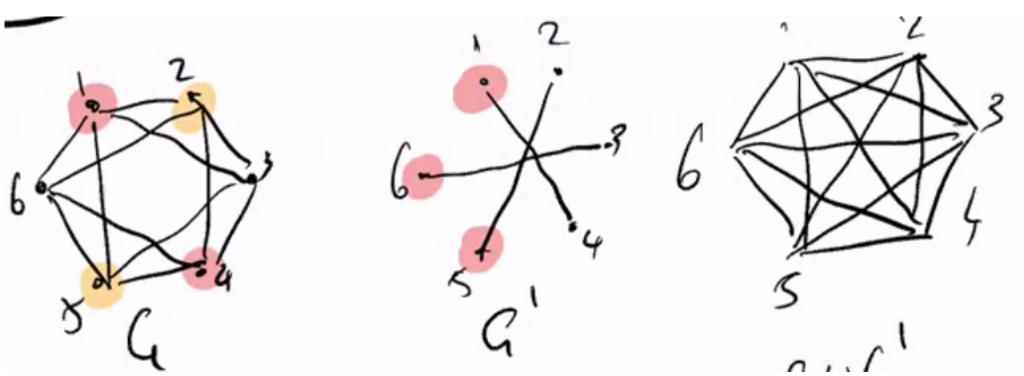
If H is a subgraph of G, then $\chi(H) \leq \chi(G)$.

Proof: Consider any $\chi(G)$ coloring f of G. Restrict this function to vertices of H, which means that we have an $\chi(G)$ coloring of H.

Disprove

Suppose that $G = (V, E)$ and $G = (V, E')$ are graphs on the same set of vertices G. Let $G \cup G' = (V, E \cup E')$. Then, $\chi(G \cup G') \leq \chi(G) + \chi(G')$.

Solution: Here is a counterexample.



Here, $\chi(G) = 3$ (left), $\chi(G') = 2$ (middle), and $\chi(G \cup G') = 6$ (right). This means that $\chi(G \cup G') > \chi(G) + \chi(G')$.

Proposition

We have $\chi(G \cup G') \leq \chi(G)\chi(G')$ for G, G', and $G \cup G'$ defined above.

Proof: Let $k = \chi(G), k' = \chi(G')$. Let f be a k-coloring of G and let f' be a k'-coloring of G'. Define $g : V \to \{1, \cdots, k\}\ x\ \{1, \cdots, k'\}$, with $g(v) = ((f(v), f'(v))$. If $vw \in E$, then $f(v) \neq f(w)$, so $g(v) \neq (w)$. Similarly, if $vw \in E'$, then $f'(v) \neq f'(w)$, so $g(v) \neq g(w)$. This means that g, which assigns vertices to $\chi(G)\chi'(G)$ possible "colors", is a proper coloring.

Theorem (Bipartite Graphs)

If a graph is 2-colorable, we say that it is **bipartite**. A graph is bipartite if and only if it has no cycle of odd length.

Lemma: If a graph has an odd closed walk, then it has an odd cycle.

Proof of Lemma: Consider a minimum-length odd closed walk $x_0 x_1 \cdots x_k$ for $x_0 = x_k$. Suppose that it is not a cycle, which means that there is some $i < j$ with $x_i = x_j$. Consider the closed walks $x_i, \cdots, x_j$ and $x_j, \cdots, x_{k-1}, x_0, x_1, \cdots, x_i$. The sum of the lengths of these walks is the sum of the entire walk, which is odd. This means that one of these closed walks must have odd length and is shorter than $x_0 x_1 \cdots x_k$, which is a contradiction.

Proof of Theorem: If a graph has an odd cycle, it is no bipartite, because odd cycles have chromatic number 3. Suppose that a graph has no odd cycle. We assume the graph is connected since we can color each connected component separately. For any vertex v, we consider the coloring $f(y)$, which is **red** if there is path from x to y of even length and **blue** otherwise.

Suppose the graph is not 2-colorable, which means there is an edge yz with $f(y) = f(z)$. There is a path from x to y and a path from x to z whose lengths have the same parity. Since yz is an edge, there is a closed walk $x \to y \to z \to x$ of odd length, which is a contradiction.

Remark: This proof is, at its core, an algorithm. The computational problem of finding a k-coloring for any $k \geq 3$ if NP-hard.

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Proposition

Show that for a graph $G = (V, E)$, $\chi(G) \leq \Delta(G) + 1$, where $\Delta(G) = \max_{x \in V} \deg(x)$.

Proof: We define the **greedy coloring** with respect to an ordering $v_1, v_2, \cdots, v_n$ of V is obtained by coloring the vertices in order, assigning v_i to the smallest possible color not already used on the previously colored neighbors. We can greedily color with respect to to any ordering. At each step v, at most Δ colors already appear among the neighbors of v, so we never have to use a color greater than $\Delta + 1$. Hence, $\chi(G) \leq \Delta(G) + 1$.

Example

Suppose that we have a group of people and some pairs of them need to have one-on-one meetings. How many time slots are necessary such that the meetings can be scheduled with no time conflicts?

Solution: We have $G_1 = (V_1, E_1)$, where V_1 is the set of people and E_1 represents all edges xy, where $x, y \in V_1$ need to have a meeting together. We also define $G_2 = (V_2, E_2)$, where V_2 is the set of pairs of people $\{x, y\}$ who need to meet together and E_2 represents sets of pairs, $\{\{x, y\}, \{x, z\}\}$ who have a person in common. The answer to the question is $\chi(G_2)$; each color represents a time slot containing pairs of people meeting that do not conflict.

Definition

A k-edge-coloring of $G = (V, E)$ to be a function $f : E \to \{1, 2, \cdots, k\}$, which is proper if incident edges receive different colors. We consider the edge-chromatic-number $\chi'(G)$ to be the minimum k such that G is k-edge-colorable, meaning it has a proper edge coloring.

- $x'(G) \geq \Delta(G)$ since the $\Delta(G)$ edges incident to this maximum-degree vertex must receive distinct colors.
- $x'(G) \leq 2(\Delta - 1) + 1 = 2\Delta - 1$. To see this, consider a greedy algorithm where for any edge v_1v_2 where $\deg(v_1), \deg(v_2) \leq \Delta(G)$, we provide edge v_1v_2 and their collectively incident edges, which is at most $2(\Delta(G) - 1)$, with a distinct color. Hence, we use at most $2(\Delta - 1) + 1$ colors in the algorithm.

We can manufacture stronger bounds using the following theorem.

Vizing Theorem

For graph G, we have $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$. The similar **Konig theorem** states that if G is bipartite, then $\chi'(G) \leq \Delta(G)$. The problem of computing $X'(G)$ is NP-hard.

Hamiltonian Paths and Cycles

Hamiltonian Cycle

Let $G = (V, E)$ be a graph with n vertices. A Hamiltonian cycle in G is a cycle of length n, which visits each veretx exactly once. A Hamiltonian path is a path of length $n - 1$. A graph is *Hamiltonian* if it has a Hamiltonian cycle.

Remark: Hamiltonian cycles are closely related to the famous *Traveling Salesman Problem*, where we search for the shortest route that visits several cities.

Next, we define a **complete bipartite graph** $K_{n,m}$ is the graph G with vertex set $V = \{1, 2, \cdots, m+n\}$ and edges $E = \{ij : i \le m, j > m\}$. Essentially, $K_{5,3}$ consists of a color classes X with 5 vertices Y with 3 vertices, and edges from each vertex in X to every vertex in Y. There are $3 * 5 = 15$ edges in $K_{5,3}$.

Proposition

A complete bipartite graph is not Hamiltonian if $n \neq m$.

Proof: Any cycle in $K_{n,m}$ alternates between vertices on the left $\{1, 2, \cdots, m\}$ and vertices on the right $\{m+1, m+2, \cdots, m+n\}$. Therefore, the number of vertices on the left and right must be equal, which means that $m = |\{1, 2, \cdots, m\}| = |\{m+1, m+2, \cdots, m+n\}| = n$.

Durac's Theorem

Let $G = (V, E)$ be a graph on $n \ge 3$ vertices with $\delta(G) \ge \frac{n}{2}$, where $\delta(G)$ is the minimum degree. Then G is Hamiltonian.

Proof: Let's first prove that G is connected. If n is disconnected and split into at least two components, one component must have at most $\frac{n}{2}$ vertices. This means that each vertex in that component has at most $\frac{n}{2} - 1$ neighbors, which is a contradiction as $\delta(G) \ge \frac{n}{2}$. The rest of the proof is covered in the next lecture.

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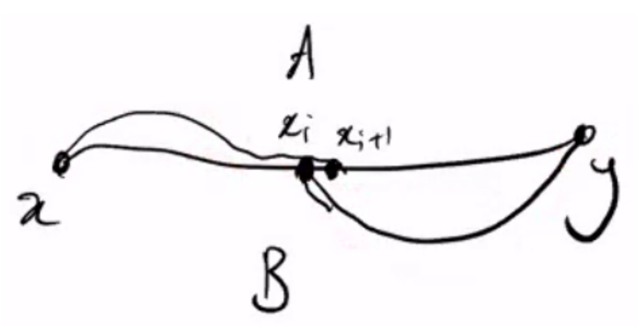
We reiterate the statement of Durac's Theorem.

Durac's Theorem

Let $G = (V, E)$ be a graph on $n \geq 3$ vertices with $\delta(G) \geq \frac{n}{2}$, where $\delta(G)$ is the minimum degree. Then G is Hamiltonian.

Proof: It was previously shown that if $\delta(G) \geq \frac{n}{2}$, then G is connected; otherwise, there would be a connected component with at most $\frac{n}{2}$ vertices, which contradicts $\delta(S) \geq \frac{n}{2}$. Let $x_0x_1x_2 \cdots x_k$ for $x = x_0, y = x_k$ be the largest path in G. Let $S = \{x_0, \cdots, x_k\}$ be the set of vertices in the path. If $x_k = y$ has a neighbor $z \notin S$, then we could extend the path $x \cdots yz$, which means that all neighbors of y are in the path. Similarly, if x had a neighbor $z \notin S$, then we could also extend the path, which means all neighbors of x are in S.



Let $A = \{i : 0 \leq i < k, x_iy \in E\}$ amd $B = \{i : 0 \leq i < k, xx_{i+1} \in E\}$. We have $|A| = \deg(y) \geq \frac{n}{2}$ and $|B| \geq \frac{n}{2}$ as all neighbors of x and y are in S. Additionally, $A \cup B \subseteq \{0, \cdots, k-1\}$, which implies that $|A \cup B| \leq k \leq n-1$. This means that A and B have an element in common (if they were disjointed, $|A \cup B| = |A| + |B| \geq n$). Let $i \in A \cup B$. Consider the cycle $xx_{i+1}x_{i+2}x_{i+3} \cdots yx_ix_{i-1}x_{i-2} \cdots x$, which has length $k + 1$.

Suppose, for the aim of contradiction, that $k + 1 \leq n$. Since G is connected, there must be some vertex $z \notin S$ adjacent to some vertex $x_i \in S$. However, we can attach x_iz to the cycle to get a path of length $k+1$, which is a contradiction. This means that $k + 1 > n$, so we have a maximal path that reaches all n vertices and hence G is *Hamiltonian*.

Example

Consider the set $\{0, 1\}^n$ of all binary sequences of length n. Is it possible to list all elements of $\{0, 1\}^n$ in order $S_1, S_2, \cdots, S_{2^n}$, such that each s_i and s_{i+1} differ in only a single position.

Solution: The answer is yes, and it s called a Grey code. See here for more information. Note that this listing exists if there is a Hamiltonian path for the following hypercube graph.

Definition: Hypercube Graph

The hypercube graph Q_d is the graph with vertex set $\{0, 1\}^d$ and edge set consisting of exactly the pairs of sequences which differ by a single position.

We node that $\delta(Q_d) = d$ (the graph is d-regular), and that the graph has $|V_{Q_d}| = 2^d$ vertices. We will show the following theorem.

Theorem (Gray 1947)

Q_d has a Hamiltonian path for any $d \geq 1$ and a Hamiltonian cycle for $d \geq 2$

Proof: We perform induction on d. The base cases are relatively easy to verify. Suppose Q_d has a Hamiltonian path, and we will show that Q_d has a Hamiltonian cycle. Let $x1, \cdots, x_{2^d}$ be the Hamiltonian path in Q_d, where each $x_i \in \{0,1\}^d$, a binary sequence of length d. Write x_i0 and x_i1 to be the sequences in $\{0,1\}^{d+1}$ defined by appending a 0 or 1, respectively. We remark that

$$x_10, x_20, x_30, \cdots, x_{2^d}0, x_{2^d}1, x_{2^d-1}1, \cdots, x_21, x_11, x_10 \tag{1}$$

is a Hamiltonian cycle is Q_d, as it covers all binary sequences of length $d+1$, and we are done.

Remark: This proof actually offers a construction of the desired sequences. The first few gray sequences are listed below. Note that we create the set of $\{0,1\}^{d+1}$ inductively from the set of $\{0,1\}^d$ by appending either a 0 or 1 to each sequence and ordering them as shown above in equation (1).

$$0, 1$$
$$00, 10, 11, 01$$
$$000, 100, 110, 010, 011, 111, 101, 001$$

Eulerian Trails and Tours

We start with the puzzle that involves drawing the shape without taking your pen off the paper or retracing your line. A strategy is to start with one of the bottom two vertices since they have odd degree and hence must be the starting point of a tracing that covers all edges.

We define the following.

Multigraph Terminologies

- A **multigraph** is a graph where the edge set is allowed to have repetitions.
- A **trail** in a multigraph is a walk along edges such that no edge is visited twice.
- An **Eulerian trail** is a trail that passes through each edge exactly once. An Eulerian trail that is closed is considered an Eulerian trail.

The **degree** of a vertex in a multigraph is the number of edges incident to it.

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Eulerian Trail

An Eulerian trail is a trail passing through each edge of the graph exactly once. An Eulerian tour is a closed Eulerian trail.

We introduce the following theorem relating to such trails.

Theorem

A connected multigraph has a euler tour if and only if each vertex has even degree.

Proof: Suppose that T is an Eulerian tour. If vertex V was visited k times in the tour, then each visit must have used 2 edges incident to v, so $d(v) = 2k$, which is even.

To observe that the condition is sufficient, we first note that in a multigraph with all degrees even, every maximal trail is closed. Let T be the largest trial in graph G, which must be closed. Suppose, for the sake of *contradiction*, that T is open. Since G is connected, there is some edge $e = uv$ outside of T which is incident to a vertex of T. Then, we can obtain a larger trail than T by starting with uv and then tracing along T, which is a contradiction. As we have maximal closed trailsm we will have an Eulerian tour in connected graph G.

Corollary

A connected multi-graph has an Eulerian trail *if and only if* there are 0 or 2 vertices of odd degree.

Proof: If G has 0 vertices of odd degree, then all vertices have even degree and we proved that it has an Euler tour and therefore an Euler trail. If G has 2 vertices of odd degree, say u and v ,then adding uv as a edge gives a multigraph where all vertices have even degree. Thus, this multigraph has an Euler tour. We can delete uv to form a Euler tour that is not closed, but traverses through all of the vertices.

Problem

Let G be a connected graph with an Euler tour. If G is bipartite, then it has an even number of edges.

Solution: Since the graph is bipartite, we simply need to sum the degrees of one of the parts, as the graph G can be cleanly divided into two color classes A and B and each edge goes between a vertex in one class and a vertex in the other class.

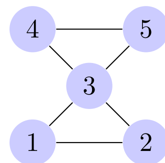
$$|E| = \sum_{a \in A} deg(a)$$

Since we have an Euler tour, then $\deg(a)$ is always even and hence the sum culminating in $|E|$ must be even as well.

Problem

Disprove: For incident edges e and f, there is an Euler tour where e and f appear consecutively.

Solution: See the graph below, noting that we cannot have an Euler tour where vertices 2 and 3 appear consecutively since we split the graph and must either go to 1 or to the upper triangle, and hence not all vertices can be reached.



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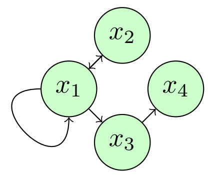
Recall that a **Eulerian trail** is a walk that passes through every edge of a graph exactly once. If the trail is closed, we call it a **Eulerian tour**. Every graph with 0 or 2 vertices of odd degree has an Eulerian trail.

Definition: Directed Graph

A directed graph is a set pair $G = (V, E)$, where V is any finite set and $E \subset \{(x, y) : x, y \in V\}$ is a set of *ordered* pairs of vertices in V. Note $xy \neq yx$.

The graph below shows $V = \{1, 2, 3, 4\}$ and $E = \{12, 21, 13, 34, 11\}$.



Paths, walks, closed walks, cycles, and trails are defined for directed graphs in similar, although directed fashion. We remark that in a graph, the shortest possible cycle length is 3, while we can have cycles of length 1 or 2 in a directed graph.

Directed Degree

In a directed graph $G = (V, E)$ and a vertex V, the **in-degree** $\deg_{in}(v)$ is defined to be $|\{y \in V : yv \in E\}|$. The **out-degree** of this vertex is defined as $\deg_{out}(v) = |\{y \in V : vy \in E\}|$. Finally, we define

$$\deg(v) = \deg_{in}(v) + \deg_{out}(v).$$

We define a directed graph $G = (V, E)$ to be **strongly connected** if for any $x, y \in V$, there is a directed walk from x to y. We say that G is **weakly connected** if its connected when we disregard the arrows on the edges.

Theorem

A weakly connected digraph has an Euler tour if and only if $d_{in}(x) = d_{out}(x)$ for all $x \in V$. In this case, the digraph is actually strongly connected.

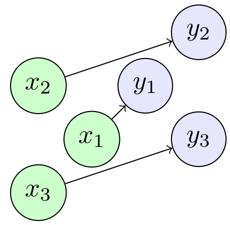
Proof: Similar to the regular graph proof. The gist is that a maximal trail is a closed trail.

Theorem

A weakly connected digraph has an **Euler trail** if and only if $d_{in}(x) = d_{out}(x)$ for all but at most 2 $x \in V$, and $|d_{in}(x) - d_{out}(x)| \leq 1$ for other other vertices.

Proof: Similar to the regular graph proof.

Next, we define a **matching** in a bipartite graph with color classes X, Y to be an injective function $r : X \to Y$ such that $xr(x)$ is an edge for each $x \in X$. Geometrically, a matching is a collection of disjoint edges, which is visualized below for $r(x_1) = y_1, r(x_2) = y_2, r(x_3) = y_3$.



Hall's Theorem

A bipartite graph G has a matching if and only if it satisfies Hall's condition. G satisfies Hall's condition if for every $S \subseteq X$, we have $|\Gamma(S)| \geq |S|$, where $\Gamma(S)$ is the neighborhood, or the set of all vertices adjacent to at least one vertex in S.

A few definitions. Let G be a bipartite graph with color classes X,Y. Let $A \subseteq X$ and $r : A \to Y$ be a (partial) mapping. Let M be the set of edges used in the matching. An **alternating** path is a path $x_0y_0x_1y_1 \cdots x_ky_k$ or $x_0y_0x_1y_1 \cdots x_k$ such that $x_0 \notin A$, $x_iy_i \notin M$, and $y_ix_{i+1} \in M$ for all i. It essentially alternates between matching edges and non-matching edges. If this path ends at some $y_k \in Y$ not covered by $r(A)$, we call it **augmenting**.

Proof of Halls's Theorem: Consider $A \subseteq X$ and let $r : A \to Y$ be a (partial) matching such that $|A|$ is as large as possible. Note that there are no augmenting paths. If $x_0y_0 \cdots x_ky_k$ were an augmenting path, we could define $A' = A \cup \{x_0\}$ and $r' : A' \to Y$ by

$$r'(x) = \begin{bmatrix} y_i & \text{if } x = x_i \text{ for some i} \\ r(x) & \text{otherwise} \end{bmatrix}$$

This would produce a partial matching covering more vertices than A, which is a contradiction. Now, let $S = \{x \in X : \exists \text{ alternating path ending at } x\}$ and $T = \{y \in Y : \exists \text{ alternating path ending at } y\}$. We will show that S violates Hall's condition unless $A = X$.

Note $T \subseteq r(A)$ as the path is not augmenting, and that $\Gamma(s) \subseteq T$, as for every neighbor $y \in \Gamma(S)$ of vertex $x \in S$, there is some alternating path ending at x. Either y already appears in the alternating path ($y \in T$) or we can add y to the end of the path to get an alternating path ending in y ($y \in T$).

We have $S \supseteq X - A$, since we have a trivial path x for $x \notin A$, which happens to be alternating path. Next, we have $|S \cap A| \geq |T|$. For $r(x) \in T$, we claim that $X \in S$. Consider an alternating path ending of $r(x)$. Either x already appeared in the alternating path, so $x \in S$, or we can add x to the end of the path to acquire an alternating path, so $x \in S$ again. Recall that every element in T is the of the form $r(x)$ since $T \subseteq r(A)$. So $S \cap A \supseteq r'(t)$ and hence $|S \cap A| \geq |T|$.

We deduce

$$\begin{aligned} |S| &= |S \cap A| + |X - A)| \\ &\geq |T| + (|X| - |A|) \\ &> |T|(\text{ if } A \neq X) \geq |\Gamma(S)|. \end{aligned}$$

Thus, Hall's condition is violated unless $A = X$, in which case we found a matching $X \to Y$.

April 14, 2021 (Combinatorics)

Definition: Matching

We define a **matching** in a bipartite graph G on the vertex/color classes X, Y to be an injective function $r : X \to Y$ such that each $xr(x)$ is an edge.

One application of bipartite matchings is **Hall's Theorem**, which states that G has a matching if and only if $|\Gamma(S)| \geq |S|$ for all $S \subseteq X$, where $\Gamma(S)$ represents the neighborhood of S. We introduce a defect version of Hall's Theorem.

Defect Version Hall's Theorem

Suppose G is bipartite with vertex classes X, Y and $|X| = n$. Let k be an integer, with matching number $v(G) \geq n - k$ if and only if for every $S \subseteq X$, we have $|\Gamma(S)| \geq |S| - k$.

Proof: The forward direction is relatively straighboard. If there is a matching of size $n - k$ (meaning that at most k vertices go unmatched), then for any $S \subseteq X$, at least $|S| - k$ vertices of S are matched to a vertex in Y. This implies $|\Gamma(S)| \geq |S| - k$. Next, we consider the *backward direction.* We construct a new graph G' by adding k new vertices to Y and joining each of them to everything in X. Then, for any $S \subseteq A$, we have

$$|\Gamma_{G'}(S)| = |\Gamma_G(S)| + k \geq (|S| - k) + k = |S|.$$

This means that Hall's condition is satisfied and G' has a matching $r : X \to Y \cup \{v_1, v_2, \cdots, v_k\}$. Let X' be the set of $x \in X$ such that $r(x) \in Y$. We have $|x'| \geq |x| - k$, which implies that r is a matching $X' \to Y$ in graph G. Thus,

$$v(G) \geq n - k$$

and we are done.

Konig's Theorem

We define a **vertex cover** in a graph G to be a set of vertices u such that every edge in G is incident to some vertex of u. Let $\tau(G)$ be the minimum size of a vertex cover.

For any bipartite G, we have

$$\tau(G) = v(G).$$

Proof Outline: To see that $v(G) \geq \tau(G)$, we need to show that for any partial matching and any vertex cover, the size of the matching is at most the size of the vertex cover. This is true because in any partial matching, we need to use a different vertex to cover each of the matching edges. To show $\tau(G) \geq v(G)$, we use the defect version of Hall's Theorem to show that for every $S \subseteq X$, we have

$$|\Gamma(S)| \geq |S| - (n - \tau(G))$$

or equivalently, with some re-arrangement,

$$\tau(G) \leq (n - |S|) + |\Gamma(S)|$$

We note that $\Gamma(S) \cup (X \backslash S)$ is a vertex cover, since every edge is incident to S is covered bty $\Gamma(S)$ and every other edge is covered by $X \backslash S$. This implies that $\tau(G) \leq (n - |S|) + |\Gamma(S)|$, as $n - |S|$ is the size of $X \backslash S$ and $|\Gamma(S)|$ is the size of $\Gamma(S)$.

Corollary of Hall's Theorem

If a bipartite graph is k-regular with $k \geq 1$, then it has a perfect matching.

Proof: Let $X \cup Y$ be a partition of G into color classes. We observe that $|X| = |Y|$ since the number of edges in G is equal to $\sum_{x \in X} d(x) = k|X|$ and $\sum_{y \in Y} d(y) = k|Y|$, since each edge has a vertex in X and a vertex in Y. This implies $k|X| = k|Y|$, so $|X| = |Y|$. It suffices to verify Hall's Theorem. Consider some $S \subseteq X$. The *number of edges between* $|S|$ and $\Gamma(S)$ is precisely $k|S|$ since the graph is k-regular. On the other hand, it is at most

$$\sum_{x \in \Gamma(S)} d(x) = |\Gamma(S)|k$$

This follows from the fact that every edge incident to S must be in $\Gamma(S)$ by definition but not every edge incident to $\Gamma(S)$ is incident to S. Hence, $k|S| \leq k|\Gamma(S)|$, which implies that $|S| \leq |\Gamma(S)|$, so G has a perfect matching by Hall's Theorem.

Corollary

If G is k-regular, then $\chi'(G) = k$, where $\chi'(G)$ represents the edge chromatic number of G.

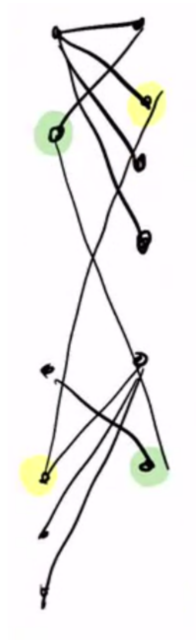
Proof: We present an induction on k, where the cases $k = 0$ and $k = 1$ can be readily handled. Since G is a bipartite and k-regular, it has a perfect matching. Let G' contain the edges not used in the matching, so G' is bipartite and $(k-1)$-regular (since for every vertex v, we deleted a single edge incident to v by removing the matching). So G' has a proper $(k-1)$ coloring by the induction hypothesis, and we can assign the color k to all the edges in our perfect matching to get a proper k-coloring.

Theorem

Every bipartite graph G satisfies $\chi'(G) = \Delta(G)$, where $\Delta(G)$ is the maximal degree of a vertex in G.

Lemma: Every bipartite graph with maximum degree Δ is a subgraph of some Δ-regular bipartite graph G'. Note that this lemma in tandem with the earlier corollary (in cyan box) means that we can take any $\Delta(G)$ coloring of G' and restrict it to G so we have a Δ-coloring of G.

Proof of Lemma: Our first idea is to add $\Delta(G) - \delta(G)$ vertices on both sides and connect these new vertices to the vertices that don't yet have Δ to make them have degree Δ. This idea doesn't quite work since it might create a graph with multiple edges between a pair of vertices since the edges might already exist when we add them. We can alter our construction by making a copy of the entire graph, flip it from left to right (see below), and then we map the vertices with lowest degree to their partners in the other graph, which increases the minimum degree by 1. We do this repeatedly so that the process will eventually create a $\Delta(G)$-regular graph.



Here, the upper half is the original graph and the bottom half is the horizontally inverted copy. We match each vertex in the original with its equivalent in the copy graph, with the pairs shown in green and yellow. We repeatedly create copies of the entire graph and connect low degree vertices until eventually we reach a $\Delta(G)$-regular graph.