On Split Coverings of Calabi-Yau Threefolds in Positive Chatacteristics

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ABSTRACT. Based on the existence of Bogomolov-Beauville decomposition on weakly ordinary varieties over perfect fields in positive characteristics [PZ20], we prove the existence of a minimal decomposition in the sense of Beauville [Bea83], under a μ_p-simply connected assumption. We classify also weakly ordinary Calabi-Yau threefolds whose Beauville-Bogomolov decompositions are abelian threefolds through purely algebraic techniques, which extends a classical result of Oguiso and Sakurai [OS01] to any perfect field of characteristic $p > 2$.

CONTENTS

Conventions and Notations. We work always over a perfect field k of characteristic $p > 0$, unless specifically mentioned. A variety means always an integral, separated scheme of finite type over a field. When no possible confusion is made, we write $W = W(k)$ for the ring of Witt vectors over k and $K = W[\frac{1}{p}]$ for the fraction field. For a W-module M of finite type, we write W/Tor for the torsion free part of W. We abbreviate $H^*(X/K) := H^*(X/W) \otimes_W K$ for the crystalline cohomology on a scheme X.

1. INTRODUCTION

A useful tool to study the geometry of compact Kähler manifolds with trivial canonical bundle is the minimal split coverings defined by Beauville in [Bea83]:

Definition 1.1. A finite covering space Y of a compact Kähler manifold X with $c_1(X) = 0$ is called **split** if Y is isomorphic to a product of a simply connected compact Kähler manifold V and a complex torus B. A split covering $V \times B \to X$ is called **minimal** if any other split coverings of X factors through it.

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A split covering of X always exists due to the Beauville-Bogomolov decomposition, which was first established by Bogomolov in [Bog74]. Beauville then proved in [Bea83] that a minimal split covering always exist and is unique up to isomorphism.

Restricting to dimension 3, there are only three possibilities on the dimension of the complex torus B in a split covering:

(1) Type A: $\dim B = 3$, that is, X is covered by a complex torus of dimension 3.
(2) Type K: $\dim B = 1$, that is, X is covered by the product of a K3 surface and an elliptic curve.
(3) Type S: $\dim B = 0$, that is, X has finite fundamental group, and V is the universal covering space of X.

We note that the case $\dim B = 2$ cannot occur, as we have in this case $\dim V = 1$, implying that V is an elliptic curve, which contradicts the assumption that V is simply connected. Since the dimension of B is independent of the choice of a split covering of X, the type of a Kähler threefold is well-defined. Oguiso and Sakurai classified in [OS01] all Calabi-Yau threefolds of Type A, by studying their minimal split coverings:

Theorem 1.2 ([OS01, Theorem 0.1]). *Let X be a Calabi-Yau threefold of Type A, then by definition, $X = B/G$ is an étale quotient of an three-dimensional complex torus B by a finite group G whose action on B is free. We have only two possibilities of G:*

(1) *$G = \langle a\rangle \times \langle b\rangle \cong C_2 \times C_2$, and the action of G on $H^0(B, \Omega_B^1)$ is of the form*

$$a \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, b \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(2) *$G = \langle a, b | a^4 = b^2 = abab = 1\rangle \cong D_8$, and the action of G on $H^0(B, \Omega_B^1)$ is of the form*

$$a \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, b \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Moreover, both cases actually occur.

In positive characteristics, a Beauville-Bogomolov decomposition theorem for weakly ordinary smooth projective varieties with trivial canonical bundle was proven by Patakfalvi and Zdanowicz in [PZ20]. The theorem asserts that there exists always a cover

$$V \times B \to Z \to X$$

such that V is a weakly ordinary projective variety with trivial Albanese, B is an abelian variety, $V \times B \to Z$ is an infinitesimal torsor and $Z \to X$ is étale. Using this decomposition, we may modify the definition of a minimal split covering to positive characteristics:

Definition 1.3. Over an algebraically closed field k of characteristic $p > 0$, a Beauville-Bogomolov decomposition $V \times B \to Z \to X$ with $K_X \sim 0$ is called **split** if V is simply connected and μ_p-simply connected, and B is an abelian variety. A split covering $V \times B \to X$ is called **minimal** if any other split coverings of X factors through it.

Unlike the case over the complex numbers, a Beauville-Bogomolov decomposition is not necessarily split, due to the absence of simply connected assumptions on V. Nevertheless, we are still able show the existence of a minimal split covering, assuming that we have a split covering:

Theorem 1.4 (c.f. Theorem 3.4). *Let X be a globally F-split smooth projective variety over an algebraically closed field of characteristic $p > 0$ with $K_X \sim 0$. Assume that there exists a split covering $V \times B \to Z \to X$, then a minimal split covering of X exists and is unique (up to a non-unique isomorphism).*

Similar to the case of complex manifolds, given a variety X in the above setting, we have three possibilities on $\dim V$, namely $0, 2, 3$, corresponding to $\hat{q}(X) = 3, 1, 0$. We call these varieties of Type A,K and S respectively, see Subsection 2.1 for a more detailed treatment. In this thesis, we study the minimal split coverings of weakly ordinary Calabi-Yau threefolds of Type A. Note that the definition for Type A threefolds, that X is covered by an abelian threefold, is still valid without assuming that the base field is algebraically closed. It eventually turns out that we have the same classification as over complex numbers.

Theorem 1.5 (c.f. Theorem 4.6). *Let X be a weakly ordinary Calabi-Yau threefold of Type A over a perfect field k of characteristic $p > 2$, then by definition, $X = B/G$ is an étale quotient of an abelian threefold B by a finite group G whose action on B is free. We have only two possibilities of G:*

(1) *$G = \langle a \rangle \times \langle b \rangle \cong C_2 \times C_2$, and the action of G on $H^1(B, \mathcal{O}_B)^\vee$ is of the form*

$$a \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, b \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(2) *$G = \langle a, b | a^4 = b^2 = abab = 1 \rangle \cong D_8$, and the action of G on $H^1(B, \mathcal{O}_B)^\vee$ is of the form*

$$a \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, b \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Moreover, both cases actually occur.

The thesis is structured as follows:

In Section 2, we discuss the essential preliminaries for the classification theorem. In Subsection 2.1, we review the results for Beauville-Bogomolov decomposition in [PZ20], and discuss the properties of Type A,K,S threefolds in further details. In Subsection 2.2, we recall the construction of Nori's fundamental group scheme following Nori's original paper [Nor82]. We show that the category of principal bundles of finite groups over a proper connected reduced scheme is Tannakian, hence is equivalent to the category of rational representations of an affine group scheme. In Subsection 2.3, we review the construction of crystalline cohomology via the de Rham-Witt complex established by Illusie in [Ill79]. In particular, this construction gives a Hodge decomposition on the crystalline cohomology, which we will heavily use in the classification. We also list some specific properties of the crystalline cohomology enjoyed by ordinary abelian varieties. In Subsection 2.4, we prove a special version of the Hochschild-Serre spectral sequence, which gives a tool comparing sheaf cohomologies on a scheme and on its quotient by a group.

In Section 3, we discuss the minimal split coverings in positive characteristics. Due to the possible existence of an infinitesimal torsor in the Beauville-Bogomolov decomposition, we need to assume extra that in a split covering $V \times B \to X$, the variety V is simply connected and μ_p-simply connected, that is, it is simply connected in the usual sense and admits also no non-trivial μ_p-torsors over it. Assuming the existence of one split covering, we show that the minimal split covering exists when the base field is algebraically closed and is unique up to a (non-unique) isomorphism. However, due to the absence of simply connected assumptions of V in a Beauville-Bogomolov decomposition, it is unknown whether a split covering always exists. In dimension 3, we show that a split covering always exists for Type A or K via explicit descriptions of K-trivial varieties with trivial Albanese in small dimensions.

In Section 4, we study and give the classification of weakly ordinary Calabi-Yau threefolds of Type A over a perfect field. The basic idea goes as follows: If $X \cong B/G$ is a weakly ordinary Calabi-Yau threefold of Type A, then G acts naturally on the crystalline cohomology $H^1(B/W) \otimes_W K$ and its subspace $H^1(B, W\mathcal{O}_B) \otimes_W K$. The form of eigenvalues of any element $g \in G$ on $H^1(B/W) \otimes_W K$ and $H^1(B, W\mathcal{O}_B) \otimes_W K$ can be described very explicitly. By the bound of the degree of characteristic polynomials, any element of G has at most order 6, and by a theorem of Hall, the group G has order at most order 24. We then consider the irreducible decomposition of the representation $G \curvearrowright H^1(B, W\mathcal{O}_B)^\vee \otimes_W \overline{K}$, through which we can eliminate most possibilities and obtain that G is either $C_2 \times C_2$ or D_8. Finally, two examples are constructed to show that these two cases indeed exist.

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2. Preliminaries

2.1. **Beauville-Bogomolov decomposition in positive characteristics.** The original version of Beauville-Bogomolov decomposition (see e.g. [Bog74]) states that for any compact Kähler manifold X with $c_1(X) = 0$, one can find a finite covering space

$$V \times B \to X,$$

where V is a simply connected Kähler manifold with $c_1(V) = 0$ and B is a complex torus. In [PZ20], an analogue for globally F-split varieties with strongly F-regular singularities is proven.

Definition 2.1. Here, "**globally F-split**" means that the Frobenius $\mathcal{O}_X \to F_*\mathcal{O}_X$ splits as an $\mathcal{O}_X$-module homomorphism, and "**strongly F-regular**" means that for the stalk $\mathcal{O}_{X,x}$ of the singularity and any element $c \in \mathcal{O}_{X,x}$, the $\mathcal{O}_{X,x}$-module homomorphism $\mathcal{O}_{X,x} \to F^e_*\mathcal{O}_{X,x}, 1 \mapsto c^{\frac{1}{p^e}}$ splits.

We review here only the Bogomolov-Beauville decomposition for smooth varieties, for which a definition of augmented irregularity is required.

Definition 2.2. Let X be a projective variety over a field k. The **augmented irregularity** $\hat{q}(X)$ of X is defined as

$$\hat{q}(X) := \sup\{\dim \mathrm{Alb}(X') \mid X' \to X \text{ is étale}\},$$

where $\mathrm{Alb}(X')$ means the Albanese variety of X'.

Theorem 2.3 ([PZ20, Theorem 1.1])**.** *Let X be a globally F-split smooth projective variety over a perfect field of characteristics $p > 0$ with $K_X \sim 0$, then there are morphisms*

$$V \times B \to Z \to X$$

such that

1. *B is an abelian variety with $\dim B = \hat{q}(X)$,*
2. *V is a globally F-split projective Gorenstein variety with strongly F-regular singularities, such that $K_V \sim 0$ and $\hat{q}(V) = 0$.*
3. *$Z \to X$ is étale,*
4. *$V \times B \to Z$ is an infinitesimal torsor under $\prod_{i=1}^{\hat{q}(X)} \mu_{p^{j_i}}$ for some integers $j_i \geq 0$.*

Moreover, one can assume that the action of $\prod_{i=1}^{\hat{q}(X)} \mu_{p^{j_i}}$ on $V \times B$ is a diagonal action.

Remark 2.4. Note that unlike the case over complex numbers, where the cover is always étale, in positive characteristics there is an infinitesimal part. And moreover, V might be singular.

Definition 2.5. Let X be a proper variety over a field k of characteristic p. We say that X is **weakly ordinary** if the Frobenius on the top sheaf cohomology $H^{\dim X}(X, \mathcal{O}_X) \to H^{\dim X}(X, F_*\mathcal{O}_X)$ is bijective.

We show that over a perfect field of positive characteristic, for K-trivial normal projective varieties, "globally F-split" is equivalent to "weakly ordinary".

Lemma 2.6. *Let X be a normal projective variety over a perfect field of characteristic $p > 0$ with $K_X \sim 0$, then X is globally F-split if and only if X is weakly ordinary.*

Proof. Let $j : U \hookrightarrow X$ be the regular locus of X, then we have $\mathcal{O}(K_X) \cong j_*\omega_U$ which is also the dualizing sheaf ω_X° of X. Then

$$\begin{aligned}
\mathcal{O}_X \to F_*\mathcal{O}_X &\text{ splits} \\
&\Updownarrow (\mathcal{O}_X \text{ is } S_2 \text{ and hence reflexive}) \\
\mathcal{O}_U \to F_*\mathcal{O}_U &\text{ splits} \\
&\Updownarrow (\text{Grothendieck duality}) \\
F_*\omega_U \to \omega_U &\text{ splits} \\
&\Updownarrow (\omega_X^\circ \cong \mathcal{O}_X \text{ so } \omega_X^\circ \text{ is reflexive}) \\
F_*\omega_X^\circ \to \omega_X^\circ &\text{ splits} \\
&\Updownarrow (h^0(X, \omega_X^\circ) = 1) \\
H^0(X, F_*\omega_X^\circ) \to H^0(X, \omega_X^\circ) &\text{ is surjective} \\
&\Updownarrow (\text{Grothendieck duality}) \\
H^{\dim X}(X, \mathcal{O}_X) \to H^{\dim X}(X, F_*\mathcal{O}_X) &\text{ is injective} \\
&\Updownarrow (h^{\dim X}(X, \mathcal{O}_X) = 1) \\
H^{\dim X}(X, \mathcal{O}_X) \to H^{\dim X}(X, F_*\mathcal{O}_X) &\text{ is bijective.}
\end{aligned}$$

□

Restricting ourselves to globally F-split K-trivial smooth threefolds, we have only three possible cases on the dimension of V:

(1) Type A: $V = \operatorname{Spec} k, \dim B = 3$. That is, X admits a finite cover from an abelian variety.
(2) Type K: $\dim V = 2, \dim B = 1$. In this case, B is an elliptic curve and V is a globally F-split projective Gorenstein surface with strongly F-regular singularities and $\hat{q}(V) = 0$.
(3) Type S: $\dim V = 3, B = \operatorname{Spec} k$. There is little to say in this case.

There are also extra observations we can make from the categorization above.

(A) In Type A case, we have an infinitesimal quotient of an abelian variety $B \to Z$. Since any infinitesimal group action on an abelian variety is a translation, the quotient Z has to be an abelian variety as well. Moreover, since the quotient is an isogeny, we have an isomorphism $H^d(B, \mathcal{O}_B) \cong H^d(Z, \mathcal{O}_Z)$. This ismorphism is Frobenius-equivariant, which shows that Z is also weakly ordinary. We prove later in Proposition 2.36 that an abelian variety is ordinary if and only if it is weakly ordinary. Therefore, the Type A threefolds are precisely those admitting a finite étale cover from an ordinary abelian threefold.
(K) Let $\varepsilon : \widetilde{V} \to V$ be the minimal resolution of V, then it is easy to see that V being weakly ordinary implies $\widetilde{V}$ being weakly ordinary, hence $\widetilde{V}$ is globally F-split. By the classification of K-trivial regular projective surfaces, a list of which can be found in [PZ20, Lemma 12.1], the surface $\widetilde{V}$ is either a K3 surface or a non-classical Enriques surface in characteristic 2, which is an étale quotient of a K3 surface by the group $\mathbb{Z}/2\mathbb{Z}$. Hara and Watanabe proved in [HW02] that a $\mathbb{Q}$-Gorenstein strongly F-regular local ring has log terminal discrepancies. In our case, the surface V with possibly strongly F-regular singularities is Gorenstein, which means that it has at most canonical singularities. Besides, the morphism $V \times B \to Z$ is a μ_{p^j}-torsor for an integer $j \geq 0$. Since Z is K-trivial and any μ_{p^j}-action on B must preserve the canonical form, we see that the μ_{p^j}-action preserves the canonical form of V as well. In the case where V is a K3 surface with at most canonical singularities, by [Mat23, Theorem 1.2], we may obtain $p^j \leq 8$.
(S) If one considers a smooth compact K-trivial Kähler threefold of Type S over $\mathbb{C}$ (the types are defined similarly), then it is a direct consequence of the Beauville-Bogomolov decomposition that its fundamental group is finite. However, whether a Type S threefold in positive characteristics has finite fundamental group is still an open problem.

2.2. **Nori fundamental group scheme.** Let X be a scheme over a field k, and let $x : \operatorname{Spec} \bar{k} \to X$ be a geometric point. Let $\text{Fét}(X)$ be the category of finite étale coverings of X, and let FSets be the category of finite sets. We can define a functor $T : \text{Fét}(X) \to \text{FSets}$ by sending $\pi : Y \to X$ to $\pi^{-1}(x)$. The category $\text{Fét}(X)$ together with T is a so called "Galois category with fibre functor T". The étale fundamental group $\pi_1(X, x)$ of X with base point x is then defined as the group of invertible natural transformations from T to itself:

$$\pi_1(X, x) := \operatorname{Aut}(T).$$

By some abstract nonsense in category theory, one can show that $\text{Fét}(X)$ is equivalent to the category of finite sets with $\pi_1(X, x)$-actions, and under this identification, the fibre functor T is isomorphic to the forgetful functor $\pi_1(X, x) - \text{FSets} \to \text{FSets}$. We refer to [Sta25, Tag 0BMQ] for a detailed treatment on Galois categories.

The étale fundamental group $\pi_1(X, x)$ is a profinite group and characterizes all Galois covers of X in the following sense: There is a principal $\pi_1(X, x)$-bundle $\mathcal{P}$ on X together with a geometric point $p : \operatorname{Spec} \bar{k} \to \mathcal{P}$ lying over x, such that given a finite group G, a principal G-bundle $\mathcal{Q}$ together with a geometric point $q : \operatorname{Spec} \bar{k} \to \mathcal{Q}$, there exist a unique pair (φ, f) satisfying:

(1) $\varphi : \pi_1^N(X, x) \to G$ is a homomorphism of groups,
(2) $f : \mathcal{P} \to \mathcal{Q}$ is a morphism of X-schemes intertwines $\mathcal{P}$ and $\mathcal{Q}$ with respect to the $\pi_1(X, x)$- and G-action, i.e. the following diagram commutes

$$\begin{tikzcd} \pi_1(X,x) \times \mathcal{P} \arrow[r] \arrow[d, "\varphi\times f"'] & \mathcal{P} \arrow[d, "f"] \\ G \times \mathcal{Q} \arrow[r] & \mathcal{Q}, \end{tikzcd}$$

where the horizontal arrows are group actions of $\pi_1(X, x)$ and G,
(3) $f(q) = p$.

We are interested in a generalized version in positive characteristics: we want a group scheme $\pi_1^N(X, x)$ together with a principal $\pi_1^N(X, x)$-bundle satisfying the same universal property, but with "any finite group G" replaced by "any finite group scheme G". In other words, we would like to have a group scheme that characterizes all finite torsors over X, but not only the étale ones. Nori proved in [Nor82] the following:

Theorem 2.7 ([Nor82]). *Let X be a proper connected reduced scheme over a field k, and let $x : \operatorname{Spec} k \to X$ be a k-rational point. There exists an affine group scheme $\pi_1^N(X, x)$ over k, which is an inverse limit of finite group schemes, together with a principal $\pi_1^N(X, x)$-bundle $\mathcal{P}$ and a k-rational point $p : \operatorname{Spec} k \to \mathcal{P}$ lying over x, such that given a finite group scheme G, a principal G-bundle $\mathcal{Q}$ together with a k-rational point $q : \operatorname{Spec} k \to \mathcal{Q}$, there exists a unique pair (φ, f) satisfying:*

(1) $\varphi : \pi_1^N(X, x) \to G$ is a morphism of group schemes,
(2) $f : \mathcal{P} \to \mathcal{Q}$ is a morphism of X-schemes that intertwines the $\pi_1^N(X, x)$- and G-action,
(3) $f(q) = p$.

Naturally, $\pi_1^N(X, x)$ is called the Nori fundamental group scheme with base point x. It is noteworthy that the base point here is a k-rational point instead of a geometric point in the case of étale fundamental group.

We review now shortly the construction of the Nori fundamental group, following [Nor82]. The sketchy idea is that we replace the role of a Galois category with a Tannakian category, which turns out to be equivalent to the category of finite dimensional representations of an affine group scheme.

Definition 2.8. A **tensor category** $\mathcal{C}$ is a category equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ such that

(1) there exists a functorial isomorphism l_A between $X \otimes (Y \otimes Z)$ and $(X \otimes Y) \otimes Z$,
(2) there exists a functorial isomorphism l_C between $X \otimes Y$ and $Y \otimes X$,
(3) there exists an identity object $1 \in \mathcal{C}$ for the tensor $\otimes$, i.e. $1 \otimes - : \mathcal{C} \to \mathcal{C}$ is an equivalence.

A **functor $F : \mathcal{C} \to \mathcal{D}$ of tensor categories** is a functor of categories such that

(1) F commutes with $\otimes$, i.e. $F \circ \otimes = \otimes \circ (F \times F)$,
(2) F commutes with l_A and l_C in the above sense,
(3) $F(1)$ is isomorphic to $1 \in \mathcal{D}$.

Let k be a field. A **Tannakian category** $\mathcal{C}$ is a small k-lilnear abelian tensor category equipped with a k-linear abelian tensor functor $T : \mathcal{C} \to \mathrm{FVec}_k$, called the **fibre functor**, such that

(1) the k-semialgebra $\mathrm{End}(1)$ is isomorphic to k,
(2) for every $L \in \mathcal{C}$ such that $\dim T(L) = 1$, there exists an object $L^{-1} \in \mathcal{C}$ such that $L \otimes L^{-1} \cong 1$.

A **functor** $F : \mathcal{C} \to \mathcal{D}$ **of Tannakian categories** is a k-linear abelian functor of tensor categories such that $T_{\mathcal{D}} \circ F = T_{\mathcal{C}}$.

Definition 2.9. Let $\mathcal{C}$ be a Tannakian category, and let R be a k-algebra. We can define the pullback $\mathcal{C}_R$ be the category with the same objects as $\mathcal{C}$, but extend the morphisms R-linearly: $\mathrm{Hom}_{\mathcal{C}_R}(X, Y) := \mathrm{Hom}_{\mathcal{C}}(X, Y) \otimes_k R$. The functors $\otimes$ and T can also be extended R-linearly to functors $\otimes_R : \mathcal{C}_R \times \mathcal{C}_R \to \mathcal{C}_R$ and $T_R : \mathcal{C}_R \to \mathrm{FMod}_R$. We then define the **automorphism functor** of T to be

$$\begin{aligned} \mathrm{Aut}_T : k - \mathrm{Alg} &\to \mathrm{Sets} \\ R &\mapsto \mathrm{Aut}(T_R) := \{\text{invertible natural transformations } T_R \to T_R\}. \end{aligned}$$

Theorem 2.10 ([DM82, Theorem 2.11])**.** *Let $\mathcal{C}$ be a Tannakian category over a field k with fibre functor T, then*

(1) Aut_T *is representable by an affine group scheme,*
(2) *$\mathcal{C}$ is equivalent to* $\mathrm{Aut}_T - \mathrm{FRep}_k$, *the category of finite dimensional affine* Aut_T*-representations over k,*
(3) *under the equivalence $\mathcal{C} \cong \mathrm{Aut}_T - \mathrm{FRep}_k$, the fibre functor T is identified with the forgetful functor* $\mathrm{Aut}_T - \mathrm{FRep}_k \to \mathrm{FVec}_k$.

Moreover, let $\mathcal{D}$ be an another Tannakian category over k with fibre functor T', then any functor $\mathcal{C} \to \mathcal{D}$ is induced by a group scheme homomorphism $\mathrm{Aut}_{T'} \to \mathrm{Aut}_T$.

Assume that X is a proper connected reduced scheme over a field k. Our aim now is to find a Tannakian category that encodes all the finite torsors over X. We note the following:

Proposition 2.11 ([Nor82, Proposiiton 2.9])**.** *Let G be an affine group scheme over k. There is a one-to-one-to-one correspondence between*

(1) *tensor functors $F : G - \mathrm{Rep}_k \to \mathrm{QCoh}(X)$, where $G - \mathrm{Rep}_k$ is the category of (possibly infinite dimensional) affine G-representations over k,*
(2) *principal G-bundles $\mathcal{P}$ over X,*
(3) *tensor functors $F : G - \mathrm{FRep}_k \to \mathrm{QCoh}(X)$, where $G - \mathrm{FRep}_k$ is the category of finite dimensional affine G-representations over k.*

Sketch of proof. (1) $\Rightarrow$ (2): Let F be a functor from $G - \mathrm{Rep}_k$ to $\mathrm{QCoh}(X)$, and let A be the coordinate ring of G. Consider the left regular representation $G \curvearrowright A$, which defines a coherent sheaf $F(A)$ on X. One can show (c.f. [Nor82, Lemma 2.2 & Lemma 2.3]) that $F(A)$ has an $\mathcal{O}_X$-algebra structure and $\underline{\mathrm{Spec}}_{\mathcal{O}_X}(F(A))$ is the principal G-bundle that we desire.

(2) $\Rightarrow$ (3): Given a principal G-bundle $\mathcal{P}$ and a finite dimensional G-representation V, we can take the quotient of the diagonal action $G \curvearrowright \mathcal{P} \times V$ and consider the morphism $f : \mathcal{P} \times V/G \to \mathcal{P}/G \cong X$. One defines then $F(V) := f_* \mathcal{O}_{\mathcal{P} \times V/G}$.

(3) $\Rightarrow$ (1): Given a functor $F : G - \mathrm{FRep}_k \to \mathrm{QCoh}(X)$, one can extend it to the category $G - \mathrm{Rep}_k$ by defining $F(W) := \varinjlim_{V \subset W \text{ finite}} F(V)$. $\square$

Note that the proposition indicates also that the essential image of any tensor functor $G - \mathrm{FRep}_k \to \mathrm{QCoh}(X)$ lies in the category $\mathrm{Vec}(X)$ of vector bundles on X. Moreover, Nori observed more constraints on the essential image, as we will discuss now.

Definition 2.12. A vector bundle on X is called **Nori semi-stable** if it is semi-stable of degree 0 when pulled back to the normalization of each integral curve on X. The category of Nori semi-stable vector bundles over X is denoted $\mathrm{SS}(X)$.

Definition 2.13. The set of all vector bundles on X forms a semi-ring with respect to direct sums and tensor products, and the expressions $f(V)$ for $f \in \mathbb{N}[T]$ a polynomial with positive integer coefficients make therefore sense. A vector bundle V is called **finite** if there exist polynomials $f, g \in \mathbb{N}[T]$ such that $f(V) \cong g(V)$.

Proposition 2.14 ([Nor82, Corollary 3.5])**.** *Finite vector bundles are Nori semi-stable.*

Definition 2.15. A vector bundle V is called **essentially finite** if it is in the abelian category EFVec(X) generated by all finite vector bundles in $\mathrm{SS}(X)$.

Proposition 2.16 ([Nor82, Proposition 3.8])**.** *If $F : G-\mathrm{FRep}_k \to \mathrm{Vec}(X)$ is a tensor functor, then the image $F(V)$ is essentially finite for any G-representation V.*

Finally, Nori showed that $\mathrm{EFVec}(X)$ is Tannakian with respect to the fibre functor x^*, sending a vector bundle V to $V|_x$ for a k-raitonal point x. Therefore, the category $\mathrm{EFVec}(X)$ is equivalent to $\mathrm{Aut}_{x^*}-\mathrm{FReps}_k$. We define then $\pi_1^N(X,x) := \mathrm{Aut}_{x^*}$ to be the Nori fundamental group scheme of X with base point x, and define $\mathcal{P}$ to be the principal $\pi_1^N(X,x)$-bundle given by Proposition 2.11. We omit then the verification on their universal properties. The fact that $\pi_1^N(X,x)$ is an inverse limit of finite group schemes corresponds to the fact that $\mathrm{EFVec}(X)$ is the direct limit of its subcategories with finite generators, which are equivalent to the categories of finite dimensional representations of finite group schemes.

As one can expect, the Nori fundamental group scheme enjoys many similar properties as the étale fundamental group does. We need for example the following for the proofs later.

Theorem 2.17 ([MS02, Theorem 2.3])**.** *Let X, Y be two proper connected reduced scheme over an algebraically field k with base points $x \in X, y \in Y$. Then the natural map*

$$\pi_1^N(X \times Y, (x,y)) \to \pi_1^N(X,x) \times \pi_1^N(Y,y)$$

induced by the two projections is an isomorphism.

2.3. **Crystalline cohomology via de Rham-Witt complex.** Let X be a smooth proper scheme of pure dimension d over a perfect field k of characteristic $p > 0$. The analogue of l-adic cohomology over p-adic numbers, namely $H^i(X, \mathbb{Z}_p) := \varprojlim H^i_{\text{ét}}(X, \mathbb{Z}/p^n\mathbb{Z})$, does not behave well as one would expect for a Weil cohomology theory. Indeed, we can consider the Artin-Schreier sequence

$$0 \longrightarrow \mathbb{Z}/p^n\mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{1-F^n} \mathcal{O}_X \longrightarrow 0$$

which is exact over the étale site. Then, we can prove the comparison theorem $H^i_{\text{ét}}(X, \mathcal{F}) \cong H^i_{\text{Zar}}(X, \mathcal{F})$ for any quasi-coherent sheaf $\mathcal{F}$ ([Mil80, Chap. III, Proposition 3.7 & Remark 3.8]). This shows in particular $H^i_{\text{ét}}(X, \mathcal{O}_X) = 0$ for all $i > d$ and hence $H^i_{\text{ét}}(X, \mathbb{Z}/p^n\mathbb{Z}) = 0$ for all $i > d+1$ as well.

As an attempt for a Weil cohomology theory valued over $W(k)$, Serre considered in [Ser58b] for a scheme X the Zariski sheaves $W_n\mathcal{O}_X$, with value $W_n\mathcal{O}_X(\operatorname{Spec} A) = W_n(A)$ the Witt vectors of length n for each affine open in X, and the inverse limit of their sheaf cohomologies $H^i(X, W\mathcal{O}_X) := \varprojlim_n H^i(X, W_n\mathcal{O}_X)$, which is now usually called the **Witt vector cohomology**. It is shown in [Ser58a] that for an abelian variety, one can get a reasonable first cohomology group by considering the direct sum of the first Witt vector cohomology group and the Tate module of the dual abelian variety. Still, this cohomology theory does not behave as a Weil cohomology theory in general, as $H^j(X, W\mathcal{O}_X) = 0$ for $j > d$.

If we consider the crystalline cohomology defined on the crystalline site, this will eventually turn out to be a Weil cohomology theory valued in the Witt numbers $W(k)[\frac{1}{p}]$. In [Ill79], Illusie gave an alternative construction of crystalline cohomology via the so-called de Rham-Witt complex. The de Rham-Witt complex shows not only the relation between crystalline cohomology and Witt vector cohomology, but it also endows a Hodge decomposition on the crystalline cohomology, which we will use in the proof later. We review in this section the construction of de Rham-Witt complex following [CL98].

Definition 2.18. The **de Rham-Witt complex** $W_\bullet\Omega_X^\bullet$ is a projective system $(W_n\Omega_X^\bullet)_{n\in\mathbb{N}}$ of strictly anti-commutative graded algebras together with a differential d such that $d(W_n\Omega_X^i) \subseteq W_n\Omega_X^{i+1}$ and $d^2 = 0$. In other words, the differential gives $W_n\Omega_X^\bullet$ the structure of a complex

$$\cdots \longrightarrow 0 \longrightarrow W_n\Omega_X^0 \xrightarrow{d} W_n\Omega_X^1 \xrightarrow{d} W_n\Omega_X^2 \xrightarrow{d} \cdots .$$

Moreover, $W_\bullet\Omega_X^\bullet$ satisfies also

(1) the projective system $(W_n\Omega_X^0)_n$ is canonically isomorphic to $(W_n\mathcal{O}_X)_n$,
(2) there is an additive operator $V : W_n\Omega_X^i \to W_{n+1}\Omega_X^i$ such that
  - it agrees with the usual Verschiebung on $W_n\Omega_X^0 \cong W_n\mathcal{O}_X$, and
  - there are identities
$$V(x\ dy) = Vx\ d(Vy), \text{ and } (d[x])Vy = V([x]^{p-1}d[x]\ y)$$
    where $[x] \in W_n\mathcal{O}_X$ is the Teich-Müller representative of $x \in \mathcal{O}_X$,
(3) $W_\bullet\Omega_X^\bullet$ is initial in the category of all projective systems satisfying (1) and (2).

We can construct the de Rham-Witt complex $W_n\Omega_X^\bullet$ formally as a quotient of $\Omega^\bullet_{W_n\mathcal{O}_X}$, inductively on n: We define first $W_1\Omega_X^\bullet := \Omega_X^\bullet$, and $V : W_0\Omega_X^\bullet = 0 \to W_1\Omega_X^\bullet$ is just the zero map. Assume that we have constructed $W_n\Omega_X^\bullet$ and the operator $V : W_{n-1}\Omega_X^\bullet \to W_n\Omega_X^\bullet$. Let $\pi_n : \Omega^\bullet_{W_n\mathcal{O}_X} \to W_n\Omega_X^\bullet$ be the quotient map. We define a homomorphism

$$\varepsilon : W_n\mathcal{O}_X^{\otimes i+1} \to \Omega^i_{W_n\mathcal{O}_X}, \quad a \otimes x_1 \otimes \cdots \otimes x_i \mapsto a \cdot dx_1 \cdots dx_i$$

and write K^i for the kernel of the composition $\pi_n \circ \varepsilon : W_n\mathcal{O}_X^{\otimes i+1} \to \Omega^i_{W_n\mathcal{O}_X} \to W_n\Omega_X^i$. Define another homomorphism

$$v : W_n\mathcal{O}_X^{\otimes i+1} \to \Omega^i_{W_{n+1}\mathcal{O}_X}, \quad a \otimes x_1 \otimes \cdots \otimes x_i \mapsto Va \cdot dVx_1 \cdots dVx_i.$$

One can show that $\bigoplus_i v(K^i)$ is a graded ideal in $\Omega^\bullet_{W_{n+1}\mathcal{O}_X}$. Let $I \subset \Omega^1_{W_{n+1}\mathcal{O}_X}$ be the subsheaf generated by elements of the form $(d[x])Vy - V([x]^{p-1}d[x]\ y)$, and let N be the graded ideal in $\Omega^\bullet_{W_{n+1}\mathcal{O}_X}$ generated by $\bigoplus_i v(K^i)$ and I. We define then $W_{n+1}\Omega_X^\bullet := \Omega^\bullet_{W_{n+1}\mathcal{O}_X}/N$. The restriction map $W_{n+1}\Omega_X^\bullet \to W_n\Omega_X^\bullet$ is inherited from the restriction $\mathrm{Res} : \Omega_{W_{n+1}\mathcal{O}_X} \to \Omega_{W_n\mathcal{O}_X}$ by showing that $\pi_n \circ \mathrm{Res}(N) = 0$. Similarly, the map v descends to a map $V : W_n\Omega_X^i \to W_{n+1}\Omega_X^i$ as $\pi_{n+1} \circ v(K^i) = 0$. This finishes the construction of $W_{n+1}\Omega_X^\bullet$. We refer to [Ill79, Théorème I.1.3] for further details.

Proposition 2.19 ([CL98, Proposition 3.1])**.** *There exists a unique additive operator $F : W_n\Omega_X^\bullet \to W_{n-1}\Omega_X^\bullet$ whose restriction on $W_n\mathcal{O}_X$ is the usual Frobenius σ and satisfying*

$$F(ab) = F(a)F(b), \quad FdV = d \text{ on } W_n\Omega_X^0 \cong W_n\mathcal{O}_X, \text{ and } F(d[x]) = [x]^{p-1}d[x].$$

Moreover, there are also the identities

$$FV = VF = p, \quad FdV = d, \text{ and } xVy = V(F(x)y).$$

We can compare the operator F with the endomorphism $\underline{F}$ on $W_\bullet\Omega^\bullet$ induced by the Frobenius σ on $W_\bullet\mathcal{O}_X$. One can show that $\underline{F} = p^iF$ on $W_\bullet\Omega^i$, or in other words, the previous proposition shows that the Frobenius on $W_\bullet\Omega^i$ is divisible by p^i.

Example 2.20 ([CL98, Section 3.3]). We may explicitly describe the de Rham-Witt complex of the ring $A := \mathbb{F}_p[T_1, \ldots, T_N]$. Define two other rings

$$B := \mathbb{Z}_p[T_1, \ldots, T_N], \quad C := \mathbb{Q}_p\left[T_1^{p^{-\infty}}, \ldots, T_N^{p^{-\infty}}\right] = \bigcup_{r\geq 0} \mathbb{Q}_p\left[T_1^{p^{-r}}, \ldots, T_N^{p^{-r}}\right].$$

An m-form $\omega \in \Omega^m_{C/\mathbb{Q}_p}$ can be uniquely written as a sum

$$\omega = \sum_{1\leq i_1\leq\cdots\leq i_m\leq N} a_{i_1,\ldots,i_m}(T_1, \ldots, T_N)\frac{dT_{i_1}}{T_{i_1}} \wedge \cdots \wedge \frac{dT_{i_m}}{T_{i_m}}.$$

We call ω integral if all of the $a_{i_1,\dots,i_m}$ are in the subring $\mathbb{Z}_p\left[T_1^{p^{-\infty}},\dots,T_N^{p^{-\infty}}\right]\subset C$. Let $E^m\subset\Omega^m_{C/\mathbb{Q}_p}$ be the colllection of m-forms ω such that ω and $d\omega$ are integral. For example, $T_1^{\frac{1}{p}}$ is not in E^0 but $pT_1^{\frac{1}{p}}$ is in E^0. Define two operators:

$$F: T_i\mapsto T_i^p,\quad V: T_i\mapsto pT_i^{\frac{1}{p}}$$

and extend them to operators on C such that F and $\frac{1}{p}V$ are homomorphisms of $\mathbb{Q}_p$-algebras. They induce two corresponding operations on $\Omega^\bullet_{C/\mathbb{Q}_p}$, and we denote them also by F and V. One checks that the subscomplex $E^\bullet$ is preserved by F and V. Then we can define $E^m_n:=E^m/(V^nE^m+dV^nE^{m-1})$. The operator V on E^m descends to an operator $E^m_n\to E^m_{n+1}$, which we denote also with V by an abuse of notation. One can verify that $V(xdy)=Vx\ d(Vy), V(d[x])=[x]^{p-1}d[x]$ and $E^0_n\cong W_nA$. By the universal property of de Rham-Witt complex, there is a unique morphism $W_\bullet\Omega^\bullet_A\to E^\bullet_\bullet$. Illusie showed in [Ill79, Théorème I.2.5] that it is an isomorphism.

Denote with $W\Omega^\bullet_X:=\varprojlim W_n\Omega^\bullet_X$ the projective limit of the system $(W_n\Omega^\bullet_X)_{n\in\mathbb{N}}$. Illusie proved the following two results:

Proposition 2.21 ([Ill79, Proposition II.2.1])**.** *Assume that X is a smooth proper variety. The natural maps*

$$R\Gamma(W\Omega^\bullet_X)\to R\varprojlim_n R\Gamma(W_n\Omega^\bullet_X)$$
$$H^j(X,W\Omega^i_X)\to\varprojlim_n H^j(X,W_n\Omega^i_X)$$

are isomorphisms.

Proposition 2.22 ([Ill79, Théorème II.1.4])**.** *Assume the same setting for X. There is a natural isomorphism between the cohomology of X on the crystalline site* $\mathrm{Cris}(X/W_n)$ *and the hypercohomology of the de Rham-Witt complex $W_n\Omega^\bullet_X$:*

$$H^*(X/W_n)\cong\mathbb{H}^*(W_n\Omega^\bullet_X),$$

which is compatible with the restrictions $H^*(X/W_{n+1})\to H^*(X/W_n)$ and $W_{n+1}\Omega^\bullet_X\to W_n\Omega^\bullet_X$

. Combining these two propositions, we obtain

Theorem 2.23. *Let X be a smooth proper variety. There is a natural isomorphism between the crystalline cohomology of X and the hypercohomology of the de Rham-Witt complex of X:*

$$H^*(X/W)\cong\mathbb{H}^*(W\Omega^\bullet_X).$$

Proof. By definition $H^*(X/W)=\varprojlim_n H^*(X/W_n)$, so it suffices to show that the natural map

$$R^i\Gamma(W\Omega^\bullet_X)=\mathbb{H}^i(R\varprojlim_n R\Gamma(W_n\Omega^\bullet_X))\to\varprojlim_n R^i\Gamma(W_n\Omega^\bullet_X)$$

is an isomorphism. It was shown in [Ill79, Théorème II.2.7] that there are isomorphisms

$$R\Gamma(W\Omega^\bullet_X)\otimes^{\mathbb{L}}_W W_n\cong R\Gamma(W_n\Omega^\bullet_X).$$

By [BO78, Proposition B.5], we see that $R\Gamma(W_n\Omega^\bullet_X)$ is a quasi-consistent projective system indexed by n in the sense of [BO78, Definition B.4]. So the result follows from [BO78, Proposition B.7.2]. □

The isomorphism implies in particular that one can compute the crystalline cohomology via the spectral sequence of filtered complex:

$$E_1^{ij}=H^j(X,W\Omega^i_X)\Rightarrow H^{i+j}(X/W),$$

which is equivariant with respect to $\underline{F}=p^iF$ on $H^j(X,W\Omega^i_X)$ and the Frobenius σ on $H^{i+j}(X/W)$.

Inspired by above, we would like to study in details the structure of Frobenius on modules over the Witt ring, hence the following definition.

Definition 2.24. An F**-crystal** is a free W-module M of finite rank along with an injective σ-linear endomorphism $\varphi : M \to M$, i.e. $\varphi(am) = \sigma(a)\varphi(m)$ for all $a \in W, m \in M$.

An F**-isocrystal** is defined identically as an F-crystal but with W replaced by $K := W[\frac{1}{p}]$.

A **morphism of** F**-crystals** (resp. F-isocrystals) $u : M \to N$ is a homomorphism of W-modules (resp. K-vector spaces) such that $\varphi_N \circ u = u \circ \varphi_M$.

An **isogeny of** F**-crystals** is a morphism whose induced homomorphism $M \otimes_W K \to N \otimes_W K$ is an isomorphism.

Example 2.25. (1) The torsion free part $H^j(X, W\Omega^i_X)/\mathrm{Tor}$ of the sheaf cohomology of the de Rham-Witt complex is an F-crystal with respect to F or $\underline{F} = p^iF$. The fact that F is injective can be derived from the equality $FV = VF = p$.

(2) The torsion free part of the crystalline cohomology $H^j(X/W)/\mathrm{Tor}$ is an F-crystal with respect to the Frobenius σ induced by the Frobenii on the crystalline sites. The Poincaré pairing $H^j(X/W) \times H^{2d-j}(X/W) \to H^{2d}(X/W) \xrightarrow{\mathrm{Tr}} W$ satisfies $\langle \sigma(x), \sigma(y)\rangle = p^d\sigma\langle x, y\rangle$, c.f. [Ber74, Proposition VII.3.2.4]. Hence the injectivity of σ follows from the non-degeneracy of the pairing $\langle -, -\rangle$.

(3) Let $W_\sigma[T]$ be the non-commutative ring of one-variable polynomials with coefficients in W subject to the relation $Ta = \sigma(a)T$ for $a \in W$. Let $\alpha = r/s$ be a non-negative rational number where r and s are coprime. The module $M_\alpha = W_\sigma[T]/(T^s - p^r)$ can be made into an F-crystal by letting $\varphi(m) := Tm$. The injectivity of φ follows directly from $T^s = p^r$.

The last example is of importance by the following description on the category of F-isocrystals, due to Manin.

Theorem 2.26 ([Man63, Theorem 2.1])**.** *If k is algebraically closed, then the category of F-isocrystals is semisimple and the simple objects are $M_\alpha \otimes_W K$, i.e. every F-isocrystal is isomorphic to a direct sum $\bigoplus_{\alpha\in\mathbb{Q}^+}(M_\alpha \otimes_W K)^{n_\alpha}$ with finitely many $n_\alpha > 0$.*

As a direct corollary, over an algebraically closed field, every F-crystal is isogenous to a direct sum $\bigoplus_{\alpha\in\mathbb{Q}^+} M_\alpha^{n_\alpha}$.

Definition 2.27. Let $M \sim \bigoplus_{\alpha\in\mathbb{Q}^+} M_\alpha^{n_\alpha}$ be the decomposition of an F-crystal M up to isogeny. The collection of α such that $n_\alpha > 0$ is called the **slopes** of M, and $n_\alpha \cdot \mathrm{rank}_W(M_\alpha)$ is called the **multiplicity of the slope** α. Over an non-algebraically closed field, the slopes and multiplicities of M is defined to be the slopes and multiplicities of $M \otimes_W W(\bar{k})$.

With the language of F-crystals, the spectral sequence of filtered complex is indeed a spectral sequence of F-crystals (up to torsion):

$$E_1^{ij} = \left(H^j(X, W\Omega^i_X), p^iF\right) \Rightarrow \left(H^{i+j}(X/W), \sigma\right).$$

Since $FV = VF = p$, the slope of $\underline{F} = p^iF$ must be in the interval $[i, i+1]$. Illusie proved then

Theorem 2.28 ([Ill79, Théorème II.3.2, Corollaire II.3.5])**.** *Assume that X is a proper smooth variety. The spectral sequence of the filtration on $W\Omega^\bullet$ degenerates at page E_1 up to torsion. In particular, there is an isomorphism of F-isocrystals*

$$(H^j(X/W) \otimes_W K, \sigma)_{[i,i+1)} \cong (H^{j-i}(X, W\Omega^i_X) \otimes_W K, p^iF).$$

Moreover, we can give finer restrictions on the slopes:

Proposition 2.29 ([CL98, Exemple II.1.2])**.** *Assume furthermore that X is projective. The F-crystal $H^j(X/W)/\mathrm{Tor}$ has slopes in $[0, j]$ if $0 \leq j \leq d$, and in $[j-d, d]$ if $d \leq j \leq 2d$.*

Proof. We prove the statement via an induction on d. We first remark that the Poincaré pairing as in Example 2.25 implies that the slopes of $H^*(X/W)/\mathrm{Tor}$ are in $[0, d]$. The statement for the case when X is a curve follows then from the Poincaré pairing. Now assume that the statement holds for varieties of dimension $d-1$. Given X of dimension d, one can pick a general hyperplane section H, and consider the restrictions $H^*(X/W)/\mathrm{Tor} \to H^*(H/W)/\mathrm{Tor}$, which is injective for degree $j \in [0, d-1]$ by the weak Lefschetz theorem (see [Ber73]), showing the statement for $j \in [0, d-1]$. The case $j \in [d+1, 2d]$ follows then by Poincaré duality as in Example 2.25. So we have now only $H^d(X/W)$ left, but the statement for which follows directly from the Poincaré pairing. $\square$

We restrict ourselves then to the case where B is an abelian variety of dimension d. Analogous to the structure of étale cohomology of an abelian variety, we have the following result:

Proposition 2.30 ([Ill79, II.7.1]). *The first crystalline cohomology $H^1(B/W)$ of an abelian variety B of dimension d has no torsion, and there is a natural isomorphism of F-crystals*

$$H^j(B/W) \cong \bigwedge^j H^1(B/W)$$

for all $j \in \mathbb{N}$. In particular, $H^1(B/W)$ has rank $2d$.

Similar to the identification $H^1(B, \mathbb{Q}_l)^\vee \cong \varprojlim B[l^n]$, we have

Proposition 2.31 ([Ill79, Remarque II.3.11.2]). *There is a natural isomorphism*

$$H^1(B/W) \cong \mathbb{D}(B[p^\infty])$$

between the first crystalline cohomology of B and the contravariant Dieudonné module of the p-divisible group $B[p^\infty] := \varinjlim B[p^n]$.

We refer to [Dem06] for a detailed treatment on Dieudonneé modules.

Definition 2.32. Let B be an abelian variety of dimension d. The **p-rank** of B is defined as

$$p - \operatorname{rank}(B) := \dim_{\mathbb{F}_p} B[p](k).$$

An abelian variety B is called **ordinary** if $p - \operatorname{rank}(B) = d$.

We then show that for an abelian variety, the two notions of “ordinary” and “weakly ordinary” agree. To prove this, two small lemmas are needed.

Lemma 2.33. *The natural map $H^j(X, W\mathcal{O}_X)/VH^j(X, W\mathcal{O}_X) \to H^j(X, \mathcal{O}_X)$ is injective for any smooth proper variety X.*

Proof. By Proposition 2.21, we see that the Witt vector cohomology $H^*(X, W\mathcal{O}_X)$ is indeed the sheaf cohomology associated to the sheaf $W\mathcal{O}_X : \operatorname{Spec} A \mapsto W(A)$. It is easy to check that the following sequence is exact:

$$0 \longrightarrow W\mathcal{O}_X \xrightarrow{V} W\mathcal{O}_X \xrightarrow{\text{Res}} \mathcal{O}_X \longrightarrow 0.$$

So the lemma follows from taking cohomology long exact sequence of the above short exact sequence. $\square$

Remark 2.34. A subtlety here is worth mentioning. One need to be careful when distinguishing the operator V on an F-crystal M and the Verschiebung V on the Witt vectors. So VM can be interpreted as the image of the operator V, or the submodule of M generated by the ideal $V \subset W$. The notation V in the lemma above refers to the former one, namely the operator V on the F-crystal $H^j(X, W\mathcal{O}_X)$. If an F-crystal has slope 0, for example $H^*(B, W\mathcal{O}_B)$ for an ordinary abelian variety B, then the two notions of V agree, and we will use this fact in Section 4. In general however, these two notions are not the same. We can take for example $M = W \oplus W \cdot T$, which is a free W-module generated by 1 and T. The operators F and V acts as follows:

$$\begin{aligned} F(a) &= \sigma(a) \cdot T, \\ F(a \cdot T) &= p \cdot \sigma(a), \\ V(a) &= \sigma^{-1}(a) \cdot T, \\ V(a \cdot T) &= p \cdot \sigma^{-1}(a), \end{aligned}$$

where σ is the Frobenius on W. One can check that this F-crystal is of slope $\frac{1}{2}$, and appears as the first crystalline cohomology group of a supersingular elliptic curve. It is then not hard to show that

$M/VM \cong W/pW$ is one-dimensional over the field W/pW, but $M/(VW)\cdot M$ is two-dimensional, as M is of rank 2 over W.

Lemma 2.35 ([Ill79, II.7.1.2]). *We have the following equation on the p-rank of an abelian variety B:*

$$p-\operatorname{rank}(B) = \operatorname{rank} H^1(B/W)_{=0} = \operatorname{rank} H^1(B/W)_{=1},$$

where $H^1(B/W)_{=0}$ (resp. $H^1(B/W)_{=1}$) is the F-subcrystal of $H^1(B/W)$ with slope 0 (resp. 1).

Proposition 2.36. *Let B be an abelian variety of dimension d. The following are equivalent:*

(1) *The p-rank of B is d;*
(2) *The F-crystal $H^1(B, W\mathcal{O}_B)/\mathrm{Tor}$ is purely of slope 0;*
(3) *The F-crystal $H^d(B, W\mathcal{O}_B)/\mathrm{Tor}$ has rank 1 and is of slope 0;*
(4) *The Frobenius action on $H^1(B, W\mathcal{O}_B)/\mathrm{Tor}$ is an isomorphism;*
(5) *The Frobenius action on $H^d(B, W\mathcal{O}_B)/\mathrm{Tor}$ is an isomorphism;*
(6) *The Frobenius action on $H^1(B, \mathcal{O}_B)$ is an isomorphism;*
(7) *The Frobenius action on $H^d(B, \mathcal{O}_B)$ is an isomorphism.*

In particular, B is ordinary if and only if B is weakly ordinary.

Proof. We show the equivalence as indicated in the graph

$$\begin{tikzcd}
(1) \arrow[r, shift left] & (2) \arrow[l, shift left] \arrow[r, shift left] \arrow[d, shift left] & (4) \arrow[l, shift left] \arrow[r, shift left] & (6) \arrow[l, shift left] \\
 & (3) \arrow[u, shift left] \arrow[r, shift left] & (5) \arrow[l, shift left] \arrow[r, shift left] & (7). \arrow[l, shift left]
\end{tikzcd}$$

(1) $\Rightarrow$ (2): The equation $p-\operatorname{rank}(B) = \operatorname{rank} H^1(B/W)_{=0} = \operatorname{rank} H^1(B/W)_{=1} = d$ by Lemma 2.35 together with $\operatorname{rank} H^1(B/W) = 2d$ by Proposition 2.30 implies directly $H^1(B/W) = H^1(B/W)_{=0} \oplus H^1(B/W)_{=1}$ and we can conclude using Theorem 2.28.

(2) $\Rightarrow$ (1): By Proposition 2.29, the slopes of $H^1(B/W)$ are in $[0,1]$. By Lemma 2.35, we have $p-\operatorname{rank}(B) = \operatorname{rank} H^1(B/W)_{=0} = \operatorname{rank} H^1(B/W)_{=1}$. If the p-rank of B is less than d, then $H^1(B/W)$ will have a slope in $(0,1)$, which by Theorem 2.28 contributes to the slopes of $H^1(B, W\mathcal{O}_B)/\mathrm{Tor}$, a contradiction.

(2) $\Rightarrow$ (3): By Proposition 2.29, the F-crystal $H^1(B/W)$ decomposes as $H^1(B/W) = H^1(B/W)_{=0} \oplus H^1(B/W)_{=1}$. Then by Lemma 2.35, we get $\operatorname{rank} H^1(B/W)_{=0} = \operatorname{rank} H^1(B/W)_{=1} = d$. In particular, we have $H^d(B, W\mathcal{O}_B) = \bigwedge^d H^1(B, W\mathcal{O}_B)$ up to torsion, so $H^d(B, W\mathcal{O}_B)$ is of rank 1.

(3) $\Rightarrow$ (2): By Proposition 2.30, we get $\operatorname{rank} H^1(B/W)_{=0} = d$ and hence $\operatorname{rank} H^1(B/W)_{=1} = d$ as well by Lemma 2.35. Since $\operatorname{rank} H^1(B/W) = 2d$, this shows already that $H^1(B/W)$ has no slope in $(0,1)$.

(2) $\Leftrightarrow$ (4): Clear.

(3) $\Leftrightarrow$ (5): Clear.

(4) $\Rightarrow$ (6): Since $\underline{F}V = V\underline{F} = p$ on $H^1(B, W\mathcal{O}_B)/\mathrm{Tor}$ and $\underline{F}$ is an isomorphism, the subset $V(H^1(B, W\mathcal{O}_B)/\mathrm{Tor})$ can be identified as $p(H^1(B, W\mathcal{O}_B)/\mathrm{Tor})$. By Lemma 2.33, the natural map $H^1(B, W\mathcal{O}_B)/pH^1(B, W\mathcal{O}_B) = H^1(B, W\mathcal{O}_B)/VH^1(B, W\mathcal{O}_B) \hookrightarrow H^1(B, \mathcal{O}_B)$ is injective, hence is also surjective by a comparison on ranks. This implies then the projection $H^1(B, W\mathcal{O}_B)/\mathrm{Tor} \to H^1(B, \mathcal{O}_B)$ is also surjective. If $N \subset H^1(B, \mathcal{O}_B)$ is a subspace on which the Frobenius is not an isomorphism, then its preimage in $H^1(B, \mathcal{O}_B)/\mathrm{Tor}$ is a non-zero submodule on which the Frobenius is not an isomorphism. This yields a contradiction.

(6) $\Rightarrow$ (4): The natural map $H^1(B, W\mathcal{O}_B)/VH^1(B, W\mathcal{O}_B) \hookrightarrow H^1(B, \mathcal{O}_B)$ is injective by Lemma 2.33. If we can find a non-zero subcrystal $M \subset H^1(B, W\mathcal{O}_B)$ with non-zero slope, then the Frobenius will be nilpotent on $M/VM \subset H^1(B, \mathcal{O}_B)$. A contradiction.

(5) $\Leftrightarrow$ (7): The proof is identical with (4) $\Leftrightarrow$ (6) ☐

We prove also the following lemma concerning the structure of Witt vector cohomology on an ordinary abelian variety for later use.

Lemma 2.37. *For an ordinary abelian variety B of dimension d, the natural map $\bigwedge^j H^1(B, W\mathcal{O}_B)\otimes_W K \to H^j(B/K)$ is injective and the image is $H^j(B, W\mathcal{O}_B) \otimes_W K$. As a consequence, there is an isomorphism of F-isocrystals $\bigwedge^* H^1(B, W\mathcal{O}_B) \otimes_W K \cong H^*(B, W\mathcal{O}_B) \otimes_W K$.*

Proof. By Proposition 2.30, one has an isomorphism of F-isocrystals $H^*(B/K) \cong \bigwedge^* H^1(B/K)$, from which we may deduce directly the injectivity. By Proposition 2.36, the F-isocrystal $H^1(B, W\mathcal{O}_B)\otimes_W K$ is purely of slope 0, so the image of $\bigwedge^j H^1(B, W\mathcal{O}_B) \otimes_W K$ is precisely the slope 0 part of $H^j(B/K)$. By Theorem 2.28, the slope $[0, 1)$ part of the F-isocrystal $H^j(B/K)$ is $H^j(B, W\mathcal{O}_B) \otimes_W K$. By Proposition 2.36 and Lemma 2.35, the F-isocrystal $H^1(B/K)$ has only slopes 0 and 1, so again by Proposition 2.30, the slopes of $H^j(B/K)$ are all integers, so $H^j(B, W\mathcal{O}_B) \otimes_W K$ is purely of slope 0, whence the surjectivity. □

2.4. **Hochschild-Serre spectral sequence.** The Hochschild-Serre spectral sequence is a tool for comparing various cohomology groups of a scheme with the ones of its quotient by a group. The originial version of the Hochschild-Serre spectral sequence can be found in [HS53], which concerns mainly the case for G-modules. We prove here a version which we need for later use.

Proposition 2.38. *Let X be a scheme over k admitting an action by a group G, and assume that the quotient $\pi : X \to Y = X/G$ exists. Given a sheaf $\mathcal{F}$ of $\mathcal{O}_Y$-modules, and assume that the natural map $H^0(Y, \mathcal{F}) \to H^0(X, \pi^*\mathcal{F})^G$ is an isomorphism (which holds in particular for $\mathcal{F} = \mathcal{O}_Y$), then there is a spectral sequence*

$$E_2^{pq} = H^p(G, H^q(X, \pi^*\mathcal{F})) \Rightarrow H^{p+q}(Y, \mathcal{F})$$

Proof. One can consider the composition of functors

$$\begin{aligned} &\mathcal{O}_Y - \mathrm{Mod} \to G - \mathrm{Mod} \to \mathrm{Ab}, \\ &\qquad \mathcal{F} \mapsto H^0(X, \pi^*\mathcal{F}) \mapsto H^0(X, \pi^*\mathcal{F})^G = H^0(Y, \mathcal{F}). \end{aligned}$$

The claim follows then by applying the Grothendieck spectral sequence, see e.g. [Sta25, Tag 015N]. □

In particular, if G is linearly reductive over k, e.g. if G is finite and $|G|$ is not divisible by p, then all higher group cohomologies of G vanish, hence

Corollary 2.39. *In the above setting, if G is linearly reductive, then we have an isomorphism*

$$H^j(Y, \mathcal{F}) = H^j(X, \mathcal{F})^G.$$

for all $j \geq 0$.

3. Minimal split coverings

Let X be a compact Kähler manifold with $c_1(X) = 0$, then by [Bog74], X admits a Beauville-Bogomolov decomposition $V \times B \to X$, where V is a simply connected Kähler manifold with $c_1(V) = 0$ and B is an abelian variety. We call such a decomposition also a split covering, following [Bea83]. The expression “split” is due to that the Albanese of $V \times B$ is the projection onto B, which admits a section. A natural question is: How many split coverings does X have? And the answer is not hard: X admits infinitely many split coverings. Indeed, let $\varphi : V \times B \to X$ be a split covering, and let $[n] : B \to B$ be the endomorphism of multiplication by n, then $\varphi \circ (\mathrm{Id} \times [n]) : V \times B \to V \times B \to X$ is again étale. By applying different integers n, we obtain infinitely many non-isomorphic split coverings.

An observation from the argument above is, if the relative automorphism group $\mathrm{Aut}(V \times B/X)$ contains a translation of B, then we can take the quotient of that translation and get a new abelian variety B' together with a split covering $V \times B' \to X$, which is smaller than $V \times B \to X$ in the sense that the latter one factors through the former one. To be more precise, consider the two embeddings of groups $\mathrm{Aut}^0(B) \hookrightarrow \mathrm{Aut}(V \times B)$ and $\mathrm{Aut}(V \times B/X) \hookrightarrow \mathrm{Aut}(V \times B)$ and let G be the intersection of their images. Then B/G is again an abelian variety and the natural map $V \times B/G \to X$ is étale, hence a smaller split covering. This motivates the following definition:

Definition 3.1. Let X be a compact Kähler manifold with $c_1(X) = 0$. A split covering $V \times B \to X$ is called minimal if $\mathrm{Aut}(V \times B/X)$ does not contain any translation of B, i.e. the group G defined above is trivial.

Now a more interesting question is: How many minimal split coverings does X admit? The result is due to Beauville, which also explains why the name "minimal":

Theorem 3.2 ([Bea83, Proposition 3])**.** *There exists a unique minimal split covering $V_0 \times B_0 \to X$ (up to a non-unique isomorphism), and any split covering $V' \times B' \to X$ factors through the minimal one:*

$$\begin{tikzcd} V' \times B' \arrow[rr, dashed, "\exists"] \arrow[dr] & & V_0 \times B_0 \arrow[dl] \\ & X. & \end{tikzcd}$$

In positive characteristics, we can define similarly the notion of a split covering:

Definition 3.3. Let X be a globally F-split smooth projective variety with $K_X \sim 0$ over an algebraically closed field of characteristic $p > 0$. A Beauville-Bogomolov decomposition $V \times B \to Z \to X$ (c.f. Theorem 2.3) is called **split** if

(1) V is simply connected, i.e. V has no non-trivial finite étale covers, and μ_p-simply connected, i.e. V admits no non-trivial μ_{p^j}-torsors over it for any $j > 0$,
(2) the infinitesimal part $V \times B \to Z$ is a $\prod_{i=1}^{\widehat{q}(X)} \mu_{p^{j_i}}$-torsor with a diagonal action, such that the action on V is faithful and the action on B is free.

It is called **minimal** if for any other split covering $V' \times B' \to Z' \to X$, there is a factorization:

$$\begin{tikzcd} V' \times B' \arrow[rr, dashed, "\exists"] \arrow[d] & & V \times B \arrow[d] \\ Z' \arrow[rr, dashed, "\exists"] \arrow[dr] & & Z \arrow[dl] \\ & X. & \end{tikzcd}$$

.

In this section, we prove the corresponding existing result of a minimal split covering in positive characteristics:

Theorem 3.4. *Let X be a globally F-split smooth projective variety with $K_X \sim 0$ over an algebraically closed field of characteristic $p > 0$. Assume that there exists a split covering $V \times B \to Z \to X$, then a minimal split covering of X exists and is unique (up to a non-unique isomorphism).*

However, here it is in general not required that in a Beauville-Bogomolov decomposition $V \times B \to Z \to X$, the fundamental group of V is trivial, nor that the Nori fundamental group scheme of V has no μ_p-quotients. So it is possible that there exists a variety admitting no split covering at all. Nevertheless, in dimension 3, we have an explicit description on the possible V that may occur, so we can prove the following:

Corollary 3.5. *Assume that X is 3-dimensional and of Type A or Type K (c.f. Section 2.1), then X admits a unique minimal split covering.*

Proof. By Theorem 3.4, it suffices to show that X admits a split covering. For Type A the assertion is trivial. For Type K, we know that there is a decomposition $V \times B \to Z \to X$ such that V is either a K3 surface or non-classical Enriques surface with at most canonical singularities (c.f. Section 2.1), and B is an elliptic curve.

Assume that V is a K3 surface, and let $\varepsilon : \widetilde{V} \to V$ be its minimal resolution. Now $\widetilde{V}$ is a smooth K3 surface, whose Picard group $\mathrm{Pic}(\widetilde{V})$ is free of torsion (see [Huy16, Proposition 1.2.4]), and $\mathrm{Pic}(V)$

can be embedded into $\mathrm{Pic}(\widetilde{V})$ via pullback ε^*, hence is torsion-free too. The Kummer sequence

$$0 \longrightarrow \mu_{p^j} \longrightarrow \mathbb{G}_m \xrightarrow{(-)^{p^j}} \mathbb{G}_m \longrightarrow 0$$

gives a half short exact sequence

$$0 \longrightarrow H^1_{\mathrm{fppf}}(V, \mu_{p^j}) \longrightarrow \mathrm{Pic}(V) \xrightarrow{p^j \cdot} \mathrm{Pic}(V),$$

showing that V has no non-trivial μ_{p^j}-torsors. Next, assume that $\mu : V' \to V$ is a finite étale cover from an irreducible variety V'. We note that $\mathcal{O}_V$ is the dualizing sheaf on V, hence $\mathcal{O}_{V'}$ is the dualizing sheaf on V'. This implies in particular $h^0(V', \mathcal{O}_{V'}) = h^2(V', \mathcal{O}_{V'}) = 1$. Then we have

$$h^1(V', \mathcal{O}_{V'}) = 2 - \chi(V', \mathcal{O}_V') = 2 - \deg(\mu) \cdot \chi(V, \mathcal{O}_V) = 2 - 2\deg(\mu),$$

implying that $\deg(\mu) = 1$, hence V is simply connected, and $V \times B \to Z \to X$ is a split covering.

Now we assume that V is a non-classical Enriques surface. Note that this case occurs only in characteristic 2.

Claim: The universal cover of V is a K3 surface with at most canonical singularities, which is of degree 2 over V.

Proof of claim: Let $\widetilde{V} \to V$ be the minimal resolution of V, which is a regular non-classical Enriques surface with an étale $\mathbb{Z}/2\mathbb{Z}$-cover by a K3 surface. We denote the étale K3 cover of $\widetilde{V}$ by $\widetilde{W}$. Let E be the exceptional divisor of the resolution $\widetilde{V} \to V$, which is a union of rational (-2)-curves. Therefore, the preimage of E in $\widetilde{W}$ is just two copies of E, as rational curves has no non-trivial étale covers. We may then contract $E \amalg E$ to get a K3 surface W with at most canonical singularities, and there is a map $W \to V$, and we claim it is étale.

$$\begin{tikzcd}
E \amalg E \arrow[r] \arrow[d] & \widetilde{W} \arrow[r] \arrow[d] & W \arrow[d] \\
E \arrow[r] & \widetilde{V} \arrow[r] & V
\end{tikzcd}$$

Indeed, the relative differential $\Omega_{W/V}$ is only supported on the singularities of V, and on the singular points $W \to V$ is also étale, as it is just the contraction of $E \amalg E$ mapping to the contraction of E. Since W is simply connected by what we have shown before, W is the universal cover of V. ■

Let W be the degree two universal K3 cover of V, and consider $W \times B \to X$. We get naturally a factorization $W \times B \to Z' \to X$ such that $W \times B \to Z'$ is purely inseparable and $Z' \to X$ is separable, and we get a diagram as follows:

$$\begin{tikzcd}
W \times B \arrow[rr, "\pi"] \arrow[d, "u'"'] & & V \times B \arrow[d, "u"] \\
Z' \arrow[rr, "\tau"'] \arrow[dr, "v'"'] & & Z \arrow[dl, "v"] \\
& X. &
\end{tikzcd}$$

We claim then

(1) the upper square is Cartesian, and
(2) v' is étale,

from which one can deduce that $W \times B \to Z' \to X$ is a split covering. Let T denote the fibre product $(V \times B) \times_Z Z'$, then we have induced morphisms $\tau' : T \to V \times B$ and $w : W \times B \to T$ as below:

$$\begin{tikzcd} W \times B \arrow[dr, dashed, "w"] \arrow[drr, bend left, "\pi"] \arrow[ddr, "u'"'] & & \\ & T \arrow[r, "\tau'"'] \arrow[d] & V \times B \arrow[d, "u"] \\ & Z' \arrow[r, "\tau"'] & Z \end{tikzcd}$$

First we show that $\deg(\tau) = \deg(\pi) = 2$. Pick $z \in Z(k)$ a general closed point, then its set theoretical preimage $u^{-1}(z) \subset V \times B(k)$ consists also of a single point, as u is purely inseparable. Hence $|\pi^{-1}(u^{-1}(z))| = \deg(\pi) = 2$, as π is étale. But then $2 = \deg(\pi) = |\pi^{-1}(u^{-1}(z))| = |u'^{-1}(\tau^{-1}(z))| = |\tau^{-1}(z)| = \deg(\tau)$, where the latter equality is again because u' is purely inseparable. However, we have also $\deg(\tau') = \deg(\tau) = 2$ as τ' is a pullback. This implies that $\deg(w) = 1$ and hence $T \cong W \times B$, proving the first claim. The second claim follows then directly from the fact that τ is étale by fpqc descent. So $W \times B \to Z' \to X$ is a split covering, as we have shown before that W is simply connected and μ_p-simply connected. In both cases of V we have successfully constructed a split covering, and we can deduce the existence of a minimal split covering using Theorem 3.4. □

Remark 3.6. In the proof of Type K case above, we actually use no assumptions on the dimension of B. So this proof can be directly generalized to all varieties X satisfying the assumptions in Theorem 2.3, such that $\widetilde{q}(X) = \dim X - 2$. This improves in particular the result in [PZ20, Lemma 12.2].

To prove Theorem 3.4, we need the following lemma.

Lemma 3.7. *Let V be a variety with trivial Albanese and let B be an abelian variety. The natural morphism $\mathrm{Aut}_V \times \mathrm{Aut}_B \to \mathrm{Aut}_{V\times B}$ of group schemes is an isomorphism.*

Proof. For a k-scheme S, the relative Albanese of $(V \times B)_S$ over S is simply the projection onto B_S, see [Gro62, Théorème 3.3]. For each $\sigma \in \mathrm{Aut}_{V\times B}(S)$, one can find $\tau \in \mathrm{Aut}_B(S)$ such that the following diagram commutes:

$$\begin{tikzcd} (V \times B)_S \arrow[r, "\sigma"] \arrow[d] & (V \times B)_S \arrow[d] \\ B_S \arrow[r, "\tau"] & B_S. \end{tikzcd}$$

This implies then σ is of the form $(a, b) \mapsto (\xi_b(a), \tau(b))$ for $(a, b) \in V(S) \times B(S)$. Note that the assignment

$$B(S) \to \mathrm{Aut}(V_S),$$
$$b \mapsto \xi_b \xi_0^{-1}$$

is an action of B on V, which has to be trivial (c.f. [Bri18, Corollary 2.19]), and therefore σ must be of the form (ξ, τ) for $\xi \in \mathrm{Aut}_V(S)$ and $\tau \in \mathrm{Aut}_B(S)$. □

Remark 3.8. Lemma 3.7 shows in particular that the infinitesimal torsor $V \times B \to Z$ is always induced by a diagonal action.

Proof of Theorem 3.4. Suppose we are given a split covering $V \times B \to Z \to X$. By Lemma 3.7, we have an isomorphism $\mathrm{Aut}_{V\times B} \cong \mathrm{Aut}_V \times \mathrm{Aut}_B$. Let G be the intersection of $1 \times \mathrm{Aut}^0_B$ and $\mathrm{Aut}_{(V\times B)/X}$ in $\mathrm{Aut}_{(V\times B)}$, i.e. the group scheme of translations of B over X. We can show that G is finite étale: Finiteness follows directly from that $V \times B \to X$ is finite, and étaleness follows from that the action of the infinitesimal part of $\mathrm{Aut}_{(V\times B)/X}$, which is nothing but $\mathrm{Aut}_{(V\times B)/Z}$, is faithful on V. Since G can be regarded as a subgroup of $\mathrm{Aut}_{(V\times B/X)}$, and Z is a quotient of $V \times B$ by the infinitesimal part of $\mathrm{Aut}_{(V\times B)/X}$, which is a normal subgroup scheme, it follows that the action of G on $V \times B$ can be descended to an action on Z, Consider then $V_0 := V, B_0 := B/G, Z_0 := Z/G$, and the natural morphism $V_0 \times B_0 \to Z_0 \to X$. We claim that this is a split covering. Indeed, consider the following diagram

$$\begin{tikzcd}
V \times B \arrow[rr, "\pi"] \arrow[d, "u'"'] & & V_0 \times B_0 \arrow[d, "u"] \\
Z \arrow[rr, "\tau"'] \arrow[dr, "v'"'] & & Z_0 \arrow[dl, "v"] \\
& X, &
\end{tikzcd}$$

and we shall prove

(1) u is an infinitesimal torsor under the same group for the torsor u', and
(2) v is étale.

The first claim follows from the fact that the upper square is Cartesian, which can be proved using the same degree argument in Corollary 3.5. In particular, the morphism τ is étale by fpqc descent on π. We may then use [Sta25, Tag 02K6] to conclude that v is étale.

We then claim that the split covering $V_0 \times B_0 \to Z_0 \to X$ is minimal. Let $V_1 \times B_1 \to Z_1 \to X$ be another split covering. Let Z_2 be a connected scheme finite étale over $Z_0 \times_X Z_1$ such that the tower

$$\begin{tikzcd}
& Z_2 \arrow[dl] \arrow[dr] & \\
Z_0 \arrow[dr] & & Z_1 \arrow[dl] \\
& X &
\end{tikzcd}$$

is Galois, i.e. $Z_2 \to X$ is a torsor for a finite étale group scheme, and Z_0 and Z_1 is induced by quotients of subgroups. For example, we can take Z_2 being the Galois closure of a connected component of $Z_0 \times_X Z_1$ in the Galois category $\mathrm{Fét}(X)$ of schemes finite étale over X. We then build a diagram

$$\begin{tikzcd}
Y_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & V_1 \times B_1 \arrow[d] \\
Y_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Z_2 \arrow[r] \arrow[d] & Z_1 \arrow[d] \\
V_0 \times B_0 \arrow[r] & Z_0 \arrow[r] & X
\end{tikzcd}$$

such that the indicated three squares are Cartesian. The scheme Y_0 is therefore an Aut_{Z_2/Z_0}-torsor over $V_0 \times B_0$. By Theorem 2.17, Y_0 must be of the form $V_0' \times B_0'$,where V_0' (resp. B_0') is an étale torsor over V_0 (resp. B_0). By the simply connected assumption on V_0 in a split covering, V_0' is a trivial torsor over V_0. By replacing V_0' and B_0' with one of its connected components, we may assume that Y_0 is integral and of the form $V_0 \times B_0'$ for B_0' an abelian variety over B_0. Similarly, we may assume that $Y_1 \cong V_1 \times B_1'$ for B_1' an abelian variety over B_1. Now the diagram looks like

$$\begin{tikzcd}
Y_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & V_1 \times B_1' \arrow[r] \arrow[d] & V_1 \times B_1 \arrow[d] \\
V_0 \times B_0' \arrow[r] \arrow[d] & Z_2 \arrow[r] \arrow[d] & Z_1 \arrow[d] \\
V_0 \times B_0 \arrow[r] & Z_0 \arrow[r] & X.
\end{tikzcd}$$

Again by Theorem 2.17 and the assumption that V_0 is free of μ_{p^j}-torsors, Y_2 must be of the form $V_0'' \times B_0''$, where V_0'' is a trivial infinitesimal torsor over V_0 and B_0'' an infinitesimal torsor over B_0'.

Similarly, $Y_2 \cong V_1'' \times B_1''$ with the parallel properties as above. By replacing Y_2 with its reduction, we may assume $Y_2 \cong V_0 \times B_0''$, where B_0'' is an abelian variety over B_0', and $Y_2 \cong V_1 \times B_1''$, where B_1'' is an abelian variety over B_1'. In particular, $V_0 \cong V_1$ and $B_0'' \cong B_1''$ by Lemma 3.7. Now the diagram becomes

$$\begin{tikzcd}
V_0 \times B_0'' \arrow[rr] \arrow[dr] \arrow[dd] & & V_1 \times B_1 \arrow[d] \\
& Z_2 \arrow[r] \arrow[d] & Z_1 \arrow[d] \\
V_0 \times B_0 \arrow[r] & Z_0 \arrow[r] & X.
\end{tikzcd}$$

The morphism $V_0 \times B_0'' \to V_0 \times B_0$ is given by the quotient of the group scheme of translations Aut_{B_0''/B_0}. By the construction of $V_0 \times B_0$, we know that $\mathrm{Aut}_{B_0''/B_0} = \mathrm{Aut}_{(V_0 \times B_0'')/X} \cap \mathrm{Aut}^0_{B_0''}$ is the group scheme of all translations of B_0'' over X. Here, we use again the identification $\mathrm{Aut}_{V_0 \times B_0''} \cong \mathrm{Aut}_{V_0} \times \mathrm{Aut}_{B_0''}$ by Lemma 3.7, and identify $\mathrm{Aut}_{(V_0 \times B_0'')/X}$ and $\mathrm{Aut}_{B_0''}$ as subgroups in $\mathrm{Aut}_{V_0 \times B_0''}$. Similarly, $V_0 \times B_0'' \to V_1 \times B_1$ is given by the quotient of the group scheme Aut_{B_0''/B_1}, which is a subgroup of $\mathrm{Aut}_{(V_0 \times B_0'')/X} \cap \mathrm{Aut}^0_{B_0''}$. The inclusion $\mathrm{Aut}_{B_0''/B_1} \subset \mathrm{Aut}_{(V_0 \times B_0'')/X} \cap \mathrm{Aut}^0_{B_0''}$ induces a factorization $V_1 \times B_1 \to V_0 \times B_0 \to X$, and the factorization $Z_1 \to Z_0 \to X$ is shown simply by taking the separable parts of $V_1 \times B_1 \to X$ and $V_0 \times B_0 \to X$. $\square$

Remark 3.9. We note that the factorization $V_1 \times B_1 \to V_0 \times B_0$ is the quotient by a group of translations of B_1 over X, as $V_0 \cong V_1$. The infinitesimal part of a split covering is assumed to contain no translation of B_1. As a result, we derive that $V_0 \times B_0 \to Z_0$ and $V_1 \times B_1 \to Z_1$ are torsors under a same infinitesimal group scheme $\prod_{i=1}^{\widehat{q}(X)} \mu_{p^{j_i}}$, and the j_i might be invariants of X of interest.

4. Weakly ordinary Calabi-Yau threefolds of Type A

Let X be a globally F-split smooth projective variety over a perfect field k with $K_X \sim 0$ and $\hat{q}(X) = \dim X$. By Theorem 2.3, there is a finite cover $B \to X$ from an abelian variety B. As discussed in the case-by-case study in Section 2.1, we can get the following:

Lemma 4.1. *Let X be of the same setting as above. Then X admits a finite étale cover from an abelian variety.*

Proof. Let $B \to Z \to X$ be a Beauville-Bogomolov decomposition of X, where

(1) B is an abelian variety,
(2) $B \to Z$ is a $\prod_{i=1}^{\dim X} \mu_{p^{j_i}}$-torsor, and
(3) $Z \to X$ is étale.

Let $G \subset \mathrm{Aut}_{B/X}$ be the subgroup scheme consisting of all translations of B over X, then $B/G \to X$ is a smaller abelian variety cover through which $B \to X$ factors. As any infinitesimal group action on an abelian variety is a translation, $\mathrm{Aut}_{B/Z}$ is a subgroup of G, so $B/G \to X$ is étale. $\square$

Similarly to a minimal split covering in Definition 3.3, we call a cover $B \to X$ minimal if $\mathrm{Aut}_{B/X}$ does not contain any translation of B, but here we do not require that k is algebraically closed.

We now confine ourselves to weakly ordinary Calabi-Yau threefolds of Type A.

Definition 4.2. We say a variety X is **Calabi-Yau** if X is a smooth projective variety with trivial canonical bundle, and $H^1(X, \mathcal{O}_X) = 0$. We say that a weakly ordinary Calabi-Yau variety X is of **Type A** if the base field k is perfect, and X admits a finite cover from an abelian variety, not necessarily assuming that k is algebraically closed.

By Lemma 4.1, to classify all weakly ordinary Calabi-Yau threefolds of Type A, it would suffice to classify all the possible finite group actions $G \curvearrowright B$ on abelian threefolds such that

(1) g is fixed point free for all $g \in G$,
(2) G contains no non-zero translations,

(3) $H^0(B, \omega_B)^G = k$,
(4) $H^1(B, \mathcal{O}_B)^G = 0$.

Definition 4.3. Following the notations in [OS01], we call a group G a **C.Y. group of Type A** (resp. a **pre-C.Y. group of Type A**), if it admits an action on an ordinary abelian threefold B satisfying the conditions (1) - (4) (resp. the conditions (1) - (3)) listed above, and the corresponding abelian threefold B is called a **target threefold**.

Since we are discussing only Type A Calabi-Yau threefolds in this section, we will abbreviate a (pre-)C.Y. group of Type A to a (pre-)C.Y. group throughout this section, as long as no potential confusion exists.

Remark 4.4. We remark here that in positive characteristics, the condition (4) does not always imply on the quotient $X = B/G$, we have $H^1(X, \mathcal{O}_X) = 0$, as G might fail to be linearly reductive when $|G|$ is divisible by p. Nevertheless, we have still the injection $H^1(B, \mathcal{O}_B)^G \hookrightarrow H^1(X, \mathcal{O}_X)$ by Proposition 2.38, so condition (4) is essential for B/G being Calabi-Yau. We will see later that the possible groups satisfying condition (1) - (4) have only prime factor 2 in the order, hence are linearly reductive when $p > 2$.

For a group variety X, as the case in differential geometry for Lie groups, the tangent space T_0X at the identity point $0 \in X$ has a Lie algebra structure, and can be identified as $H^0(X, T_X)$, the global vector fields of X. Over the complex numbers, any abelian variety B is a complex torus, and is isomorphic to its Lie algebra modulo a lattice $H^0(B, T_B)/\Gamma$. Suppose we are given a group action $G \curvearrowright B$ on the torus, then for each $g \in G$, the corresponding automorphism $g : B \to B$ can be decomposed into $t \circ g_0$, where t is a translation and g_0 is a homomorphism of abelian varieties. The homomorphism g_0 induces then a Lie algebra endomorphism $T_0B \to T_0B$, and hence is naturally called the Lie part of g. In the case of positive characteristics, it is in general not true that an abelian variety can be expressed in the form of a torus. Nevertheless, the decomposition $g = t \circ g_0$ into a translation and a homomorphism still exists, and g_0 also acts on the Lie algebra T_0B. For more naturality in algebra, we consider the action of g_0 on $H^1(B, \mathcal{O}_B)$ via pullback, which is the same as the action of g on $H^1(B, \mathcal{O}_B)$ via pullback, instead of the action of g_0 on T_0B. The cost is that the induced representation of G on $H^1(B, \mathcal{O}_B)$ is a right representation. So to keep the notations consistent to conventions in representation theory, we take the dual $H^1(B, \mathcal{O}_B)^\vee$ and consider the left representation of G on it.

Definition 4.5. The induced representation $G \curvearrowright H^1(B, \mathcal{O}_B)^\vee$ is called the **Lie representation** of G.

In this section, we aim to prove

Theorem 4.6. *Let k be a perfect field of characteristic $p > 2$. Let X be a weakly ordinary Calabi-Yau threefold of Type A over k, and let $B \to X$ be its minimal cover. Then $X = B/G$ for a C.Y. group G of Type A, and the pair (B, G) is one of the following two cases:*

(1) *$B \cong (E_1 \times E_2 \times E_3)/\Lambda$, where E_i are ordinary elliptic curves and Λ is a finite subgroup of $E_1 \times E_2 \times E_3$, $G = \langle a | a^2 = 1\rangle \oplus \langle b | b^2 = 1\rangle \cong C_2^2$, and its Lie representation is*
$$a \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, b \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$
with respect to a basis of $H^1(B, \mathcal{O}_B)^\vee$ given by the product $E_1 \times E_2 \times E_3$.

(2) *$B \cong (E_1 \times E_2 \times E_2)/\Lambda$, where E_1 and E_2 are ordinary elliptic curves and Λ is a finite subgroup of $E_1 \times E_2 \times E_2$, $G = \langle a, b | a^4 = b^2 = abab = 1\rangle \cong D_8$, and its Lie representation is*
$$a \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, b \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$
with respect to a basis of $H^1(B, \mathcal{O}_B)^\vee$ given by the product $E_1 \times E_2 \times E_2$.

Moreover, both cases indeed exist.

Remark 4.7. If $f : B' \to B$ is an isogeny, then the pullback on the first sheaf cohomology $f^* : H^1(B, \mathcal{O}_B) \to H^1(B', \mathcal{O}_{B'})$ is an isomorphism. In the settings of Theorem 4.6, the domain $B' \sim E_1 \times E_2 \times E_3$ is isogenous to a product of elliptic curves. By Künneth formula, $H^1(B_1, \mathcal{O}_{B_1}) \cong \bigoplus_{i=1}^3 H^1(E_i, \mathcal{O}_{E_i})$. Since $H^1(E_i, \mathcal{O}_{E_i})$ is one-dimensional, this gives a basis of $H^1(B, \mathcal{O}_B)$, unique up to scalars in each basis vector. We also remark that in case (2), the matrix form of the Lie representation will be eventually different if one scalars the two basis vectors given by E_2 by different coefficients. In the proof, we will construct explicitly an isomorphism between the two E_2 components via the D_8 action on B, and the two basis vectors are assumed to be chosen to be compatible along the isomorphism.

By a theorem of canonical lifts to characteristic 0 ([MS87, Appendix Theorem 1]), it is immediate that the result in Theorem 4.6 must agree with the parallel result over $\mathbb{C}$ proved in [OS01]. We will prove the result through a purely algebraic approach. We start by several simple observations.

Lemma 4.8. *If G is a pre-C.Y. group, then so is any subgroup of G. Equivalently, if G contains a non-pre-C.Y. group, then G is not a pre-C.Y. group.*

Proof. It follows directly from Definition 4.3. □

Lemma 4.9. *If B is a target abelian threefold, then B is ordinary.*

Proof. The quotient B/G is globally F-split by assumption, and we can use [PZ20, Lemma 11.1] to deduce that B is globally F-split as well. By Lemma 2.6, B is weakly ordinary. Then by Proposiiton 2.36, B is ordinary. □

4.1. **Crystalline representation of pre-C.Y. groups.** Let G be a group acting on an abelian variety B. Given any $g \in G$ and the corresponding automorphism $g : B \to B$, the dual of the pullback on the first crystalline cohomology $g^{*\vee} : H^1(B/K)^\vee \to H^1(B/K)^\vee$ gives a left representation of G. We will eventually use these representations to restrict the possible orders of a C.Y. group. Nevertheless, most of the results in this subsection can be applied to finite group actions on (possibly non-ordinary) abelian varieties of arbitrary dimensions.

Remark 4.10. We will deal with many automorphisms of abelian varieties in this section. To avoid confusion, we mean by an "**endomorphism**" (resp. an "**automorphism**") of B a morphism (resp. an isomorphism) of schemes $B \to B$, not necessarily fixing the identity point $0_B \in B$. And by a "**homomorphism**" from B_1 to B_2, we mean a morphism of group schemes $B_1 \to B_2$, that is, it sends the identity point 0_{B_1} to the identity point 0_{B_2}. The set of automorphisms of B is denoted $\mathrm{Aut}(B)$, and the set of homomorphisms from B_1 to B_2 is denoted $\mathrm{Hom}(B_1, B_2)$. In particular, $\mathrm{Hom}(B, B)$ is the set of endomorphisms of B which fix the identity point 0_B.

To start with, we show that for a pre-C.Y. group G, the induced representation on crystalline cohomology is faithful.

Lemma 4.11. *Let B_1, B_2 be abelian varieties, then the natural map*

$$\mathrm{Hom}(B_1, B_2) \to \mathrm{Hom}_{F-crystal}\left(H^1(B_1/W)^\vee, H^1(B_2/W)^\vee\right) \cong \mathrm{Hom}_{F-crystal}\left(H^1(B_2/W), H^1(B_1/W)\right)$$

is injective.

Proof. We have the natural identification $H^1(B/W) \cong \mathbb{D}(B[p^\infty])$ by Proposition 2.31. By [CCO14, Proposition 1.2.5.1], the natural map $\mathrm{Hom}(B_1, B_2) \to \mathrm{Hom}_{\mathbb{Z}_p}(B_1[p^\infty], B_2[p^\infty])$ is injective. We have then natural isomorphisms

$$\mathrm{Hom}_{\mathbb{Z}_p}(B_1[p^\infty], B_2[p^\infty]) \cong \mathrm{Hom}_{\mathbb{D}}(\mathbb{D}(B_2[p^\infty]), \mathbb{D}(B_1[p^\infty])) \cong \mathrm{Hom}_{F-\mathrm{crystal}}\left(H^1(B_2/W), H^1(B_1/W)\right)$$

where the first isomorphism is due to that the Dieudonné functor is a contravariant equivalence (c.f. [Dem06, p. 71, Theorem]), and the second isomorphism is from Proposition 2.31. The lemma follows. □

Proposition 4.12. *If a finite group G acts on an abelian variety B and contains no translation, then the crystalline representation $G \curvearrowright H^1(B/K)^\vee$ is faithful.*

Proof. Assume that there is $g \in G$ whose pullback action on $H^1(B/K)$ is trivial. Write $g = t \circ g_0$ where t is a translation and $g_0 \in \mathrm{Hom}(B, B)$ is a homomorphism of abelian varieties. Since the action of g_0 on $H^1(B/K)$ is trivial, we obtain g_0 is the identity homomorphism by Lemma 4.11. So $g = t$ is a translation, contradiction. □

Given any element $g \in G$ of finite order, the induced action g^* on $H^1(B/K)$ is diagonalizable by passing to an algebraic closure, by elementary representation theory. Our next task is to study the eigenvalues of g^*.

Proposition 4.13. *Let $g : B \to B$ be an automorphism of an abelian variety B of finite order, and let f_g be the characteristic polynomial of the induced action g^* on $H^1(B/K)$. Denote $d = \dim B$. Assume that the fixed point scheme $B^{\langle g\rangle}$ is finite, then $\mathrm{length}_k B^{\langle g\rangle} = f_g(1)$. In particular, if furthermore $\mathrm{Fix}(g)$ is reduced, then $|\mathrm{Fix}(g)| = f_g(1)$.*

Proof. The action of g^* on $H^1(B/K)$ admits a diagonalization $g^* = \mathrm{diag}(\lambda_1, \dots, \lambda_{2d})$ over $\overline{K}$, where $q = \dim B$. By the natural isomorphism $H^*(B/K) \cong \bigwedge^* H^1(B/K)$ (c.f. Proposition 2.30), the eigenvalues of the action g^* on $H^j(B/K)$ are just $\prod_{i\in I} \lambda_i$, for all $I \subset \{1, \dots, 2n\}, |I| = j$. Now by Lefschetz fixed point formula,

$$\mathrm{length}_k B^{\langle g\rangle} = \sum_{j=0}^{2d} (-1)^j \mathrm{Tr}(g^*|H^j(B/K)) = \sum_{j=0}^{2d} (-1)^j \left(\sum_{\substack{I\subset\{1,\dots,2n\} \\ |I|=j}} \prod_{i\in I} \lambda_i \right) = \prod_{i=1}^{2d} (1 - \lambda_i) = f_g(1).$$

A proof of the Leftschetz fixed point formula for crystalline cohomology can be found in [Ber74, Théoème VII.3.1.6]. □

Proposition 4.14. *Let $g_0 \in \mathrm{Hom}(B, B)$ be a homomorphism of an abelian variety B to itself, and let g_0^* be the induced endomorphism on $H^1(B/K)$. We have an equality*

$$\deg(g_0) = \det(g_0^*).$$

In particular, if we let m_a denote the multiplication by a on B, then the characteristic polynomial $f_{g_0^*}(T)$ is equal to $\deg(m_T - g_0)$, and has rational coefficients.

Proof. The proof of [Mum74, Theorem 4, p. 180] still applies when we replace $T_l B$ with $\mathbb{D}(B[p^\infty])$ and $\mathbb{Q}_l$ with $\mathbb{Q}_p$. Then use the natural identification $H^1(B/K) \cong \mathbb{D}(B[p^\infty])$, c.f. Proposition 2.31. □

Proposition 4.15. *Let $d = \dim B$ and assume that B is ordinary. Consider the decomposition $H^1(B/K) \cong (H^1(B, W\mathcal{O}_B) \otimes_W K) \oplus (H^0(B, W\Omega^1_B) \otimes_W K)$ given by Theorem 2.28. If the eigenvalues of g^* on $H^1(B, W\mathcal{O}_B) \otimes_W K$ are $\lambda_1, \lambda_2, \dots, \lambda_d$, counted with multiplicities, then the eigenvalues of g^* on $H^0(B, W\Omega^1_B) \otimes_W K$ are of the form $\lambda_1^{-1}, \lambda_2^{-1}, \dots, \lambda_d^{-1}$.*

Proof. We note first that the image of $c_1 : \mathrm{Pic}(B) \to H^2(B/K)$ lies in the slope 1 part of $H^2(B/K)$, as $F^*D = pD$ for any divisor D. The F-isocrystal $H^1(B, W\mathcal{O}_B) \otimes_W K$ is purely of slope 0 and $H^0(B, W\Omega^1_B) \otimes_W K$ is purely of slope 1 by Proppposition 2.36, so by the identification $H^2(B/K) \cong \bigwedge^2 H^1(B/K)$ by Proposition 2.30, the image of c_1 is in $(H^1(B, W\mathcal{O}_B) \otimes_W K) \wedge (H^0(B, W\Omega^1_B) \otimes_W K)$. Let $\mathcal{L}$ be an ample line bundle on $B/\langle g\rangle$, and let $\pi : B \to B/\langle g\rangle$ denote the natural projection, then $\pi^*\mathcal{L}$ is a g-invariant ample line bundle on B. We can write $c_1(\pi^*\mathcal{L}) = \sum a_i(v_i \wedge w_i)$ for v_i an eigenbasis of $H^1(B, W\mathcal{O}_B) \otimes_W \overline{K}$ corresponding to λ_i, and $w_i \in H^0(B, W\Omega^1_B) \otimes_W \overline{K}$. Then $v_i \wedge w_i$ are linearly independent as v_i are linearly independent. Since $c_1(\pi^*\mathcal{L})$ is g-invariant, w_i must be eigenvectors corresponding to the eigenvalues λ_i^{-1}. Moreover, w_i span $H^0(B, W\Omega^1_B) \otimes_W \overline{K}$ as $c_1(\pi^*\mathcal{L})^d \neq 0$ in $H^{2d}(B/K)$. Therefore, w_i form an eigenbasis of $H^0(B, W\Omega^1_B) \otimes_W K$. □

Corollary 4.16. *If B is ordinary, and g is of finite order and fixed point free, then the eigenvalues of g acting on $H^1(B/K)$ are of the form $1, 1, \lambda_2, \lambda_2^{-1}, \dots, \lambda_d, \lambda_d^{-1}$.*

Proof. By Proposition 4.13, 1 is an eigenvalue of g^*, then use Proposition 4.15. □

Proposition 4.17. *If B is ordinary and g is of finite order, then the action g^* is identity on $H^0(B,\omega_B)$ if and only if the determinant of the action of g^* on $H^1(B,W\mathcal{O}_B)$ is 1.*

Proof. If the determinant of g^* is 1 on $H^1(B,W\mathcal{O}_B)\otimes_W K$, then by Lemma 2.37, the action g^* on the one-dimensional space $H^d(B,W\mathcal{O}_B)\otimes_W K$ is the identity, where $d=\dim B$. Lemma 2.33 gives an injection $H^d(B,W\mathcal{O}_B)/VH^d(B,W\mathcal{O}_B)\hookrightarrow H^d(B,\mathcal{O}_B)$, which is an isomorphism here by a comparison on dimensions. Therefore, the action g^* on $H^d(B,\mathcal{O}_B)$ is identity, and so is g^* on $H^0(B,\omega_B)$ by duality.

Conversely we assume that the action g^* is identity on $H^0(B,\omega_B)$, then the action is also identity on $H^d(B,\mathcal{O}_B)$. The action g^* on $H^d(B,W\mathcal{O}_B)$ is then multiplication by ζ, an $\mathrm{ord}(g)$-th root of unity in $W(k)$. The residue of ζ in k is 1, due to the isomorphism $H^d(B,W\mathcal{O}_B)/VH^d(B,W\mathcal{O}_B)\cong H^d(B,\mathcal{O}_B)$. We need the following claim:
Claim: Assume $p>2$. The only p^s-th root of unity in $W(k)$ is 1.
Proof of claim: By induction it suffices to check the case $s=1$. Assume $x=(1,x_2,x_3,\dots)$ is another p-th root of unity. Write $x=1+p^r\cdot y$ such that $y\neq 0$ is not p-divisible. Then $1=x^p=(1+p^r\cdot y)^p=1+p^{r+1}y+p^{2r+1}\cdot f(y)$ for some polynomial f, assuming $p>2$. Hence $y=p^r\cdot f(y)$ and is p-divisible, contradicting to our assumption, so $y=0$. ■

Write $\mathrm{ord}(g)=p^r\cdot n$ for n non-divisible by p. By the claim, we see $\zeta^n=1$. So ζ is an n-th root of unity in $W(k)$ whose residue is 1, then $\zeta=1$ by Hensel's lemma. By Lemma 2.37, the determinant of g^* on $H^1(B,W\mathcal{O}_B)\otimes_W K$ is precisely $\zeta=1$, proving the claim. □

Due to Proposition 4.15, we have also

Corollary 4.18. *If B is ordinary and g is of finite order, then the action g^* is identity on $H^0(B,\omega_B)$ if and only if the determinant of the action of g^* on $H^0(B,W\Omega^1_B)$ is 1.*

Proposition 4.19. *Let B be an ordinary abelian threefold with a $\langle g\rangle$-action. If g is of finite order n and the $\langle g\rangle$-action is pre-C.Y., then the eigenvalues of $g^*\in\mathrm{GL}(H^1(B/K))$ are of the form $1,1,\zeta_n,\zeta_n^{-1},\zeta_n,\zeta_n^{-1}$, where ζ_n is a primitive n-th root of unity in $\overline{K}$.*

Proof. By Corollary 4.16 and that g is of order n, it follows that the eigenvalues are of the form $1,1,\zeta_u,\zeta_u^{-1},\zeta_v,\zeta_v^{-1}$, for integers u and v such that $\mathrm{lcm}(u,v)=n$. Consider the three eigenvalues on $H^1(B,W\mathcal{O}_B)$, one has four possibilities up to replacing the primitive roots of unitiy with another one:

(1) $1,\zeta_u,\zeta_v$: By Proposition 4.17, it follows that $\zeta_v=\zeta_u^{-1}$ and the proposition is readily verified.
(2) $1,1,\zeta_u$: By Proposition 4.17, we must have $u=1$. Then by Proposition 4.15, all the eigenvalues are 1.
(3) $1,\zeta_u,\zeta_u^{-1}$: By Proposition 4.15, we get $\zeta_v=\zeta_u$ or $\zeta_v=\zeta_u^{-1}$.
(4) $\zeta_u,\zeta_u^{-1},\zeta_v$: By Proposition 4.17, we get $\zeta_v=1$ and reduce this case to case (3) above.

□

In particular, the proof shows also

Corollary 4.20. *The eigenvalues of g^* on $H^1(B,W\mathcal{O}_B)\otimes_W K$ (resp. $H^0(B,W\Omega^1_B)\otimes_W K$) are of the form $1,\zeta_n,\zeta_n^{-1}$.*

Lemma 4.21. *Preserving the settings above, the characteristic polynomial $f_{g^*,H^{0,1}}$ of the g^*-action on $H^1(B,W\mathcal{O}_B)\otimes_W K$ has rational coefficients.*

Proof. Let f_{g^*} be the characteristic polynomial of g^* on $H^1(B/K)$, then $f_{g^*}=f^2_{g^*,H^{0,1}}$ by Corollary 4.20. By Proposition 4.14, f_{g^*} has rational coefficients. So it suffices to show that if $f=h^2$, and h is monic and has rational coefficients, then h has rational coefficients as well. Write $h=\sum_{i=0}^l a_iT^i$, and assume that j is the largest index such that a_j is not rational. Since g is monic, $j<l$. Then the coefficient of T^{l+j} in f is $\sum_{i=j}^l a_ia_{l+j-i}=\sum_{i=j+1}^l a_ia_{l+j-i}+2a_j$, which is a sum of a rational and an irrational number. This yields a contradiction. □

4.2. **Classification of (pre-)C.Y. groups.** Using the result on representations of a cyclic pre-C.Y. group on the first crystalline cohomology in the last section, we can now determine the precise possible orders of elements in a pre-C.Y. group, and classify all (pre-)C.Y. groups. Many ideas in the whole classification are from [OS01], but their proofs rely pretty much on complex geometry settings, and we reformulate the proofs using purely algebraic techniques.

Proposition 4.22. *Let $\langle g\rangle$ be a cyclic pre-C.Y. group acting on a target threefold B. Then* $\operatorname{ord}(g) \in \{1, 2, 3, 4, 6\}$.

Proof. Combining Corollary 4.20 and Lemma 4.21, we see that a primitive $\operatorname{ord}(g)$-th root of unity has degree at most 2 over $\mathbb{Q}$. □

Proposition 4.23. *Let G be an abelian pre-C.Y. group, then G is isomorphic to either $C_2 \times C_2$ or C_n for $n = 1, 2, 3, 4, 6$.*

Proof. Let B be a target threefold. Consider the representation of G on $H^1(B, W\mathcal{O}_B)^\vee \otimes_W \overline{K}$, which is faithful by Corollary 4.20. There exists a basis under which the representation is diagonalized. We know that up to isomorphism, G can be written as $C_{n_1} \times \cdots \times C_{n_k}$ such that $n_j | n_{j+1}$. If G is cyclic, then the result follows from Proposition 4.22. If G is not cyclic, pick the generators a of C_{n_1} and b of C_{n_2}. Again by Corollary 4.20 the eigenvalues of a (resp. of b) are $1, \zeta_{n_1}, \zeta_{n_1}^{-1}$ (resp. $1, \zeta_{n_2}, \zeta_{n_2}^{-1}$). We have only two possibilities on the matrix forms of a and b up to an reordering of the basis and replacing ζ_{n_2} by $\zeta_{n_2}^{-1}$:

(1) $a = \operatorname{diag}(1, \zeta_{n_1}, \zeta_{n_1}^{-1}), b = \operatorname{diag}(1, \zeta_{n_2}, \zeta_{n_2}^{-1})$: It follows that b generates a as $n_1|n_2$, a contradiction.
(2) $a = \operatorname{diag}(1, \zeta_{n_1}, \zeta_{n_1}^{-1}), b = \operatorname{diag}(\zeta_{n_2}, 1, \zeta_{n_2}^{-1})$: Consider $ab = \operatorname{diag}(\zeta_{n_2}, \zeta_{n_1}, (\zeta_{n_1}\zeta_{n_2})^{-1})$. By Corollary 4.20, we must have $\zeta_{n_2} = \zeta_{n_1}^{-1}$. Then consider $a^2 b = \operatorname{diag}(\zeta_{n_1}^{-1}, \zeta_{n_1}^{2}, \zeta_{n_1}^{-1})$. Again by Corollary 4.20, we must have $\zeta_{n_1}^2 = 1$ and $n_1 = n_2 = 2$. In this case, $a = \operatorname{diag}(1, -1, -1), b = \operatorname{diag}(-1, 1, -1)$. If there is a third independent generator c of C_{n_3}, one can show similarly that $c = \operatorname{diag}(-1, -1, 1)$, but then $c = ab$, so this is impossible and we have already $G \cong C_2 \times C_2$.

□

Recall that by definition, a C.Y. group is just a pre-C.Y. group G whose action on $H^1(B, \mathcal{O}_B)$ does not fix any non-zero vector. The following proposition is then immediate.

Proposition 4.24. *Assume $p > 2$. The pre-C.Y. group $C_2 \times C_2$ is C.Y. for any target threefold B, and the pre-C.Y. group C_n is never C.Y. for any target threefold*

Proof. The injection $H^1(B, W\mathcal{O}_B)/VH^1(B, W\mathcal{O}_B) \hookrightarrow H^1(B, \mathcal{O}_B)$ given by Lemma 2.33 is natural and hence equivariant for any group action. By a comparison on ranks, the injection is in fact an isomorphism. We can check then directly that $H^1(B, \mathcal{O}_B)^{C_2\times C_2} = 0$ (resp. $H^1(B, \mathcal{O}_B)^{C_n} \neq 0$) using the explicit description of the action on $H^1(B, W\mathcal{O}_B)^\vee$ in the proof of Proposition 4.23. □

In order to pass to non-abelian C.Y. groups, we recall a theorem by Hall on finite groups.

Theorem 4.25 (Hall)**.** *Let q be a prime number, and let G be a q-group of order q^n. If H is a maximal abelian subgroup of G and has order q^h, then $n \leq \frac{h(h+1)}{2}$.*

Proof. See e.g. [Hup67, Satz III.7.3]. □

Corollary 4.26. *Let G be a pre-C.Y. group, then* $\operatorname{ord}(G) = 2^a 3^b$ *for $0 \leq a \leq 3, 0 \leq b \leq 1$. In particular,* $\operatorname{ord}(G) \in \{1, 2, 3, 4, 6, 8, 12, 24\}$.

Proof. By Lemma 4.8, any Sylow q-group is pre-C.Y., hence we can conclude with Proposition 4.23 and Theorem 4.25. □

Let B be a target threefold of G. Our next step is to study the three-dimensional representation of G on $H^1(B, W\mathcal{O}_B)^\vee \otimes_W K$ via explicit classifications of irreducible representations of groups of small orders. The assumption we have for a C.Y. group is that $H^1(B, \mathcal{O}_B)^G = (H^1(B, \mathcal{O}_B)^\vee)^G = 0$, but we will mainly work on $H^1(B, W\mathcal{O}_B)^\vee, (H^1(B, W\mathcal{O}_B)^\vee) \otimes_W K$ and $(H^1(B, W\mathcal{O}_B)^\vee) \otimes_W \overline{K}$, so we need the following lemmas on comparing the invariant subspaces of related representations:

Lemma 4.27. *Let R be a ring, M a free module over R, and let $G \curvearrowright M$ be an R-linear representation of a finite group G. Given a flat R-algebra R', and consider the induced R'-linear representation $G \curvearrowright M \otimes_R \overline{R}'$ We have* $\operatorname{rank}_R M^G = \operatorname{rank}_{R'}(V \otimes_R R')^G$.

Lemma 4.28. *Let M be a free W-module of finite rank, and let $G \curvearrowright M$ be a W-linear representation of a finite group G. Then G descends to an action on M/VM, and we have* $\operatorname{rank}_W M^G \leq \dim_{W/VW}(M/VM)^G$.

Proof. Choose a basis v_i of M^G. The vectors v_i are not p-divisible in M, as if we have a w_j such that $v_j = p \cdot w_j$, then w_j is fixed by G as well, contradicting to the assumption that v_i are a basis. Therefore the quotient classes $\overline{v_i}$ in M/VM are non-zero. We claim that they are linearly independent in $(M/VM)^G$. Assume that we have a relation $\sum_i \overline{a_i} \cdot \overline{v_i} = 0$ in M/VM. Pick preimages a_i of $\overline{a}_i$ in M, then $\sum_i a_i v_i \in VM = pM$. Now $\sum_i a_i v_i$ is G-invariant, and p-divisible, and therefore $\frac{1}{p} \sum_i a_i v_i$ is also G-invariant. Since v_i are a basis of M^G, we can write $\frac{1}{p} \sum_i a_i v_i = \sum b_i v_i$, but this means precisely that $a_i = p b_i$ and hence $\overline{a_i} = 0$. $\square$

If G acts on an abelian variety B, then we have a left representation $G^{\mathrm{op}} \curvearrowright H^1(B, \mathcal{O}_B)$. The lemmas above shows inparticular that if $H^1(B, \mathcal{O}_B)^{G^{\mathrm{op}}} = 0$, then $H^1(B, W\mathcal{O}_B)^{G^{\mathrm{op}}} = 0$, hence $(H^1(B, W\mathcal{O}_B)^\vee)^G = 0$ as well, hence $\dim_{\overline{K}}(H^1(B, W\mathcal{O}_B)^\vee \otimes_W \overline{K})^G = \dim_K(H^1(B, W\mathcal{O}_B)^\vee \otimes_W K)^G = \operatorname{rank}_W(H^1(B, W\mathcal{O}_B)^\vee)^G = 0$. Therefore

Lemma 4.29. *If G is a C.Y. group acting on a target threefold B, then* $\operatorname{rank}_W(H^1(B, W\mathcal{O}_B)^\vee)^G = \dim_K(H^1(B, W\mathcal{O}_B)^\vee \otimes_W K)^G = \dim_{\overline{K}}(H^1(B, W\mathcal{O}_B)^\vee \otimes_W \overline{K})^G = 0$.

Moreover, we will frequently consider the invariant subscheme of an action on an abelian variety B, in the following setting: Let g be a homomorphism $g : B \to B$ of finite order, that is, g has finite order as an automorphism of B. Let $n = \operatorname{ord}(g)$. Consider the maximal abelian subvariety of $\operatorname{Ker}(g - \operatorname{Id})$, denoted with E, or equivalently, E is the reduction of the identity component of $\operatorname{Ker}(g - \operatorname{Id})$.

Lemma 4.30. *In the above setting, we have also* $E = \operatorname{Im}(g^{n-1} + g^{n-2} + \cdots + g + \operatorname{Id})$.

Proof. Indeed, it is clear that $\operatorname{Im}(g^{n-1} + g^{n-2} + \cdots + g + \operatorname{Id}) \subset E$, and for the converse inclusion, it suffices to check on k-rational points. Given $\alpha \in E(k)$, one can find $\alpha' \in E(k)$ such that $n \cdot \alpha' = \alpha$, and then $g^{n-1}(\alpha') + g^{n-2}(\alpha') + \cdots + g(\alpha') + \alpha' = n \cdot \alpha' = \alpha$. $\square$

The dual of pullback $(g^{n-1} + g^{n-2} + \cdots + g + \operatorname{Id})^{*\vee}$ acts on the dual crystalline cohomology group $H^1(B/K)^\vee$. The following lemma turns out to be helpful:

Lemma 4.31. *In the above setting, we have* $\dim E = \frac{1}{2} \operatorname{rank}\left((g^{n-1} + g^{n-2} + \cdots + g + \operatorname{Id})^{*\vee}\right)$*, and the natural embedding $\iota : E \hookrightarrow B$ induces again a map $\iota^{*\vee} : H^1(E/K)^\vee \to H^1(B/K)^\vee$, which is injective and whose image is precisely $(H^1(B/K)^\vee)^{\langle g \rangle}$.*

Proof. The homomorphism $\sum_{i=0}^{n-1} g^i : B \to B$ induces a morphism of p-divisible groups $\left(\sum_{i=0}^{n-1} g^i\right)[p^\infty] : B[p^\infty] \to B[p^\infty]$, hence also a map of Dieudonné modules $\left(\sum_{i=0}^{n-1} g^i\right)^* : \mathbb{D}(B[p^\infty]) \to \mathbb{D}(B[p^\infty])$, and hence a map of dual crystalline cohomology $\left(\sum_{i=0}^{n-1} g^i\right)^{*\vee} : H^1(B/W)^\vee \to H^1(B/W)^\vee$. All the three functors used above are exact, and respects the $\operatorname{Hom}(B, B)$-structure on the objects, hence we have

$$H^1(E/K)^\vee = H^1\left(\operatorname{Im}\left(\sum_{i=0}^{n-1} g^i\right)/K\right)^\vee = \operatorname{Im}\left(\left(\sum_{i=0}^{n-1} g^i\right)^{*\vee}\right) = \operatorname{Im}\left(\sum_{i=0}^{n-1} \left(g^{*\vee}\right)^i\right) \subset H^1(B/K)^\vee.$$

Therefore, we have $\dim E = \frac{1}{2} \operatorname{rank}\left(\sum_{i=0}^{n-1} g^i\right)^{*\vee}$, showing the first claim. For the second claim, we note first that $E[p^\infty] \to B[p^\infty]$ being injective implies $\mathbb{D}(B[p^\infty]) \to \mathbb{D}(E[p^\infty])$ is surjective, as $\mathbb{D}$ is an anti-equivalence between the category of p-divisible groups and the category of Dieudonné modules over W. Therefore by Proposition 2.31, the map $\iota^* : H^1(B/K) \to H^1(E/K)$ is surjective, and hence the dual $\iota^{*\vee} : H^1(E/K)^\vee \to H^1(B/K)^\vee$ is injective. Consider then the following diagram:

$$\begin{tikzcd} E \arrow[r, hook, "\iota"] \arrow[d, "m_n"'] & B \arrow[d, "\sum_{i=0}^{n-1} g^i"] \\ E \arrow[r, hook, "\iota"'] & B \end{tikzcd}$$

where m_n is multiplication by n. The diagram is commutative since g acts as Id on E. The corresponding diagram on the dual crystalline cohomology is

$$\begin{tikzcd} H^1(E/K)^\vee \arrow[r, hook, "\iota^{*\vee}"] \arrow[d, "\mathrm{diag}(n,\dots,n)"'] & H^1(B/K)^\vee \arrow[d, "\left(\sum_{i=0}^{n-1} g^i\right)^{*\vee}"] \\ H^1(E/K)^\vee \arrow[r, hook, "\iota^{*\vee}"'] & H^1(B/K)^\vee, \end{tikzcd}$$

which shows that the image of $\iota^{*\vee}$ lies in the eigenspace of $\left(\sum_{i=0}^{n-1} g^i\right)^{*\vee}$ of the eigenvalue n. Since the eigenvalues of $g^{*\vee}$ are n-th roots of unity, we see that $\left(\sum_{i=0}^{n-1} g^i\right)^{*\vee}$ has only eigenvalues n or 0, where the $\left(\sum_{i=0}^{n-1} g^i\right)^{*\vee}$-eigenspace of eigenvalue n is precisely the $g^{*\vee}$-eigenspace of eigenvalue 1, and the $\left(\sum_{i=0}^{n-1} g^i\right)^{*\vee}$-eigenspace of eigenvalue 0 is the direct sum of the $g^{*\vee}$-eigenspaces of eigenvalues not equal to 1. As $\dim E = \frac{1}{2} \operatorname{rank} \left(\sum_{i=0}^{n-1} g^i\right)^{*\vee}$, the dimension of $H^1(E/K)^\vee$ agrees with the dimension of the $\left(\sum_{i=0}^{n-1} g^i\right)^{*\vee}$-eigenspace of eigenvalue n, which is the same as the $g^{*\vee}$-invariant subspace, and the claim is proved. $\square$

We start by classifying (pre-)C.Y. groups of order ≤ 12. We can consider abelian subgroups of all non-abelian groups of the orders mentioned in Corollary 4.26, a list of which can be found in [CM84, Table 1]. By Lemma 4.8, subgroups of pre-C.Y. groups are again pre-C.Y., so we can exclude many cases using Proposition 4.23, and obtain

Proposition 4.32. *If G is a non-abelian pre-C.Y. group of order ≤ 12, then G is isomorphic to one of the groups $D_6, D_8, Q_8, D_{12}, Q_{12}, A_4$.*

The full character tables of the groups appearing in Proposition 4.32 can be found in [Led87], with a navigation list on page 205. One can easily check that the characters correspond to the following irreducible representations:

Proposition 4.33. *Let ζ_n denote a primitive n-th root of unity. Up to equivalence, the complex irreducible (left) representations of D_{2n}, Q_8, Q_{12} and A_4 over are given as follows:*

(D_0) $D_{2n} = \langle a, b | a^n = b^2 = abab = 1\rangle$ *with* $n \equiv 0 \pmod 2$*:*
- (1) $\rho_{1,0} : a \mapsto 1,\ b \mapsto 1;\ \rho_{1,1} : a \mapsto 1,\ b \mapsto -1;\ \rho_{1,2} : a \mapsto -1,\ b \mapsto 1;\ \rho_{1,3} : a \mapsto -1,\ b \mapsto -1;$
- (2) $\rho_{2,k} : a \mapsto \begin{pmatrix} \zeta_n^k & 0 \\ 0 & \zeta_n^{-k} \end{pmatrix},\ b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$ *for* $1 \leq k \leq \frac{n}{2} - 1$.

(D_1) $D_{2n} = \langle a, b | a^n = b^2 = abab = 1\rangle$ *with* $n \equiv 1 \pmod 2$*:*
- (1) $\rho_{1,0} : a \mapsto 1,\ b \mapsto 1;\ \rho_{1,1} : a \mapsto 1,\ b \mapsto -1;$
- (2) $\rho_{2,k} : a \mapsto \begin{pmatrix} \zeta_n^k & 0 \\ 0 & \zeta_n^{-k} \end{pmatrix},\ b \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$ *for* $1 \leq k \leq \frac{n-1}{2}$.

(Q_8) $Q_8 = \langle a, b | a^4 = 1,\ a^2 = b^2,\ b^{-1}ab = a^{-1}\rangle$*:*
- (1) $\rho_{1,0} : a \mapsto 1,\ b \mapsto 1;\ \rho_{1,1} : a \mapsto 1,\ b \mapsto -1;\ \rho_{1,2} : a \mapsto -1,\ b \mapsto 1;\ \rho_{1,3} : a \mapsto -1,\ b \mapsto -1;$
- (2) $\rho_{2,0} : a \mapsto \begin{pmatrix} \zeta_4 & 0 \\ 0 & \zeta_4 \end{pmatrix},\ b \mapsto \begin{pmatrix} 0 & \zeta_4 \\ \zeta_4 & 0 \end{pmatrix}.$

(Q_{12}) $Q_{12} = \langle a, b | a^6 = 1,\ a^3 = b^2,\ b^{-1}ab = a^{-1}\rangle$*:*

(1) $\rho_{1,0}: a \mapsto 1,\ b \mapsto 1;\ \rho_{1,1}: a \mapsto 1,\ b \mapsto -1;\ \rho_{1,2}: a \mapsto -1,\ b \mapsto \zeta_4;\ \rho_{1,3}: a \mapsto -1,\ b \mapsto -\zeta_4$;

(2) $\rho_{2,0}: a \mapsto \begin{pmatrix} \zeta_6 & 0 \\ 0 & \zeta_6^{-1} \end{pmatrix},\ b \mapsto \begin{pmatrix} 0 & \zeta_4 \\ \zeta_4 & 0 \end{pmatrix}$;

$\rho_{2,1}: a \mapsto \begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^{-1} \end{pmatrix},\ b \mapsto \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

(A_4) *$A_4 = \langle a, b\rangle \subset S_4$, where $a = (123)$ and $b = (12)(34)$:*

(1) $\rho_{1,0}: a \mapsto 1,\ b \mapsto 1;\ \rho_{1,1}: a \mapsto \zeta_3,\ b \mapsto 1;\ \rho_{1,2}: a \mapsto \zeta_3^{-1},\ b \mapsto 1.$

(2) $\rho_3: a \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},\ b \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

We start by excluding the cases D_6, D_{12} and A_4.

Proposition 4.34. *The groups D_6 and D_{12} are not pre-C.Y. groups.*

Proof. By Lemma 4.8, it suffices to check the statement for D_6 since D_{12} has D_6 as a subgroup. Assume that $D_6 = \langle a, b | a^3 = b^2 = abab = 1\rangle$ is pre-C.Y. and let B be a target threefold. We consider the left representation $\rho : D_6 \to \mathrm{GL}(H^1(B, W\mathcal{O}_B)^\vee \otimes_W \overline{K})$, which is faithful by Proposition 4.12. As D_6 is non-abelian, the representation ρ does not split into three one-dimensional irreducible representations. So ρ contains the subrepresentation $\rho_{2,1}$ in Proposition 4.33. By Proposition 4.17, the image of ρ is in $\mathrm{SL}(H^1(B, W\mathcal{O}_B)^\vee \otimes_W \overline{K})$, so the only possible decomposition is $\rho \cong \rho_{1,1} \oplus \rho_{2,1}$. Pick a basis v_1, v_2, v_3 of $H^1(B, W\mathcal{O}_B)^\vee \otimes_W \overline{K}$ under which ρ is of the form

$$a \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3^{-1} \end{pmatrix},\ b \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Denote the identity point of B with 0 and define $\alpha := a(0), \beta := b(0)$. We can decompose $a = t_\alpha \circ a_0, b = t_\beta \circ b_0$, where t_α and t_β are translations on B by α and β, and a_0 and b_0 are group homomorphisms of B to itself. Define $E := \mathrm{Im}(a_0^2 + a_0 + \mathrm{Id})$, which is nothing but the reduction of the identity component of $\mathrm{Ker}(a_0 - \mathrm{Id})$ by Lemma 4.30. By Proposition 4.15, it follows that the action of a_0 on $H^1(B/K)^\vee \otimes_K \overline{K}$ can be written as $\mathrm{diag}(1, 1, \zeta_3, \zeta_3^{-1}, \zeta_3, \zeta_3^{-1})$. So by Lemma 4.31, E is an elliptic curve, and $H^1(E/K)^\vee$ can be identified as the fixed subspace of $H^1(B/K)^\vee$ under $a_0^{*\vee}$. One can then consider the diagram

$$\begin{tikzcd} E \arrow[r, "\iota"] \arrow[d, "m_{-1}"'] & B \arrow[d, "b_0"] \\ E \arrow[r, "\iota"'] & B \end{tikzcd}$$

where m_{-1} is just the involution on E, and the induced diagram on the dual first crystalline cohomology

$$\begin{tikzcd} H^1(E/K)^\vee \arrow[r, "m_{-1}=\mathrm{diag}(-1,-1)"] \arrow[d, "\tau^{*\vee}=\mathrm{diag}(1,1,0,0,0,0)"'] & H^1(E/K)^\vee \arrow[d, "\tau^{*\vee}=\mathrm{diag}(1,1,0,0,0,0)"] \\ H^1(B/K)^\vee \arrow[r, "b_0^{*\vee}=\mathrm{diag}(-1,-1)\oplus\cdots"'] & H^1(B/K)^\vee \end{tikzcd}$$

is commutative, hence b_0 acts as multiplication by -1 on E, by Lemma 4.11. Let $A := B/E$ be the quotient abelian surface and let $\pi := B \to A$ denote the natural projection. The curve E and the surface A are ordinary as B is isogenous to $A \times E$. We claim the following:

<u>Claim:</u> The action of a and b on B descends to actions on the quotient A, i.e. for $a, b \in \mathrm{Aut}(B)$ there exist $\overline{a}, \overline{b} \in \mathrm{Aut}(A)$ such that $\pi \circ a = \overline{a} \circ \pi$ and $\pi \circ b = \overline{b} \circ \pi$. Moreover, $\overline{a}$ and $\overline{b}$ satisfy the relations of a and b in D_6, so they define a D_6-action on A.

<u>Proof of claim:</u> We construct first $\overline{a}$.

$$\begin{tikzcd} B \arrow[r, "a"] \arrow[d, "\pi"'] & B \arrow[d, "\pi"] \\ A = B/E \arrow[r, dashed, "\overline{a}"'] & A \end{tikzcd}$$

By the universal property of categorical quotients, the unique existence of $\overline{a} : S \to S$ such that the diagram above commutes is equivalent to that the composition $\pi \circ a$ is E-invariant. Let S be any k-scheme, and consider S-rational points $x \in B(S), y \in E(S)$. We have

$$\begin{aligned}
\pi \circ a(x+y) &= \pi \circ t_\alpha \circ a_0(x+y) && \text{(Decompose } a = t_\alpha \circ a_0\text{)} \\
&= \pi(a_0(x) + a_0(y) + \alpha) && (a_0 \text{ is a group homomorphism}) \\
&= \pi(a_0(x) + \alpha + y) && (a_0 \text{ is identity on } E) \\
&= \pi(a_0(x) + \alpha) && (\pi \text{ is a group homomorphism, and } y \text{ is in } \operatorname{Ker}\pi = E) \\
&= \pi \circ a(x).
\end{aligned}$$

So $\pi \circ a$ is indeed invariant under translations by any elements in E. The argument for $\overline{b}$ is similar:

$$\begin{aligned}
\pi \circ b(x+y) &= \pi \circ t_\beta \circ b_0(x+y) \\
&= \pi(b_0(x) + b_0(y) + \beta) \\
&= \pi(b_0(x) + \beta - y) && (b_0 \text{ is multiplication by } -1 \text{ on } E) \\
&= \pi(b_0(x) + \beta) \\
&= \pi \circ b(x).
\end{aligned}$$

The fact that $\overline{a}$ and $\overline{b}$ are subject to the same relations as a and b in D_6 follows from the uniqueness of the descent of Id on B, which is just Id on A. ■

The short exact sequence of p-divisible groups

$$1 \longrightarrow E[p^\infty] \longrightarrow B[p^\infty] \longrightarrow A[p^\infty] \longrightarrow 1$$

induces a short exact sequence of Dieudonné modules

$$0 \longrightarrow \mathbb{D}(A[p^\infty]) \longrightarrow \mathbb{D}(B[p^\infty]) \longrightarrow \mathbb{D}(E[p^\infty]) \longrightarrow 0$$

since the Dieudonné functor $\mathbb{D}$ is an equivalence by [Dem06, p. 71, Theorem]. By Proposition 2.31, we see that the pullback $\pi^* : H^1(A/K) \to H^1(B/K)$ is injective with cokernel $H^1(E/K)$. Since $H^1(A, W\mathcal{O}_A), H^1(B, W\mathcal{O}_B), H^1(E, W\mathcal{O}_E)$ are the slope 0 parts of the crystalline cohomology of the three abelian varieties, the pullback $\pi^* : H^1(A, W\mathcal{O}_A) \to H^1(B, W\mathcal{O}_B)$ on Witt vector cohomology is injective with cokernel $H^1(E, W\mathcal{O}_E)$. By taking duals and passing to an algebraic closure, we get $\pi^{*\vee} : H^1(B, W\mathcal{O}_B)^\vee \otimes_W \overline{K} \to H^1(A, W\mathcal{O}_A)^\vee \otimes_W \overline{K}$ is surjective, with kernel $H^1(E, W\mathcal{O}_E)^\vee \otimes_W \overline{K}$. Recall that we wrote v_1, v_2, v_3 for a basis of $H^1(B, W\mathcal{O}_B)^\vee \otimes_W \overline{K}$ and v_1 spans the eigenspace of a_0 with eigenvalue 1, which is noting but $H^1(E, W\mathcal{O}_E)^\vee \otimes_W \overline{K}$. Therefore $H^1(A, W\mathcal{O}_A)^\vee \otimes_W \overline{K}$ is spanned by $\overline{v_2} := \pi^{*\vee}(v_2)$ and $\overline{v_3} := \pi^{*\vee}(v_3)$. Decompose $\overline{a} = t_{\overline{\alpha}} \circ \overline{a_0}$ and $\overline{b} = t_{\overline{\beta}} \circ \overline{b_0}$ into a group homomorphism $\overline{a_0} : A \to A$ and a translation by $\overline{\alpha} \in A$. One can check that $\overline{a_0}$ (resp. $\overline{b_0}$) is indeed the descent of a_0 (resp. b_0) to A and $\overline{\alpha} = \pi(\alpha)$ (resp. $\overline{\beta} = \pi(\beta)$), so the notation does not lead to any misunderstanding. Applying Lemma 4.11 to the diagram

$$\begin{tikzcd} B \arrow[r, "a_0{,}b_0"] \arrow[d, "\pi"'] & B \arrow[d, "\pi"] \\ A \arrow[r, "\overline{a_0}{,}\overline{b_0}"'] & A, \end{tikzcd}$$

we are able to show that under the basis $\overline{v_2}, \overline{v_3}$, the representation of D_6 on $H^1(A, W\mathcal{O}_A)^\vee \otimes_W \overline{K}$ can be written as

$$\overline{a} \mapsto \begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^{-1} \end{pmatrix}, \ \overline{b} \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By Proposition 4.15, the action of $\overline{a}$ on $H^1(A/K)$ has eigenvalues $\zeta_3, \zeta_3^{-1}, \zeta_3, \zeta_3^{-1}$. In particular, by Lemma 4.31, we see the fixed point scheme $A^{\langle \overline{a} \rangle}$ is finite, so we can apply the fixed point formula in Proposition 4.13 and get $\mathrm{length}_k A^{\langle \overline{a} \rangle} = 9$. Translating $A^{\langle \overline{a} \rangle}$ by $-(\overline{a_0})^{-1}(\overline{\alpha})$ gives an isomorphism of schemes $A^{\langle \overline{a} \rangle} \cong \mathrm{Ker}(\overline{a_0} - \mathrm{Id})$. Therefore, the number of k-rational points $\left|A^{\langle \overline{a} \rangle}(k)\right|$ is either $1, 3$ or 9, where the case $\left|A^{\langle \overline{a} \rangle}(k)\right| = 1$ or 3 can only happen in characteristic 3. Since we have the relation $\overline{a}\overline{b} = \overline{b}(\overline{a})^{-1}$, the action $\overline{b}$ on A permutes the points in $A^{\langle \overline{a} \rangle}(k)$. As $\mathrm{ord}(\overline{b}) = 2$, there exists a point $s \in A^{\langle \overline{a} \rangle}(k)$ fixed by $\overline{b}$. Set $F = \pi^{-1}(s)$, then b can be restricted to an automorphism of F, as $\overline{b}(s) = s$. Moreover, by our construction of E and F, the action of b on $H^1(F/K)$ is of the form $\mathrm{diag}(-1, -1)$. By Lemma 4.31 Proposition 4.13, b admits a fixed point on F. However, this contradicts our assumption $B^{\langle b \rangle} = \emptyset$. ◻

Proposition 4.35. *The group A_4 is not a pre-C.Y. group.*

Proof. Let B be a target threefold, and consider the left representation $\rho : A_4 \to \mathrm{GL}(H^1(B, W\mathcal{O}_B)^\vee \otimes_W \overline{K})$. We may argue similarly as the beginning of the proof of Proposition 4.34, and see that the only possible irreducible decomposition of ρ is $\rho \cong \rho_3$, following the notations in Proposition 4.33. Pick a basis $v_1, v_2, v_3 \in H^1(B, W\mathcal{O}_B)^\vee \otimes_W \overline{K}$ such that the matrix forms of $a = (123)$ and $b = (12)(34)$ are the same as in Proposition 4.33:

$$a \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \ b \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We can also easily compute the matrix forms of the following elements:

$$a^2ba \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \ aba^2 \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let 0 be the identity point of B. Define $\alpha := a(0), \beta := b(0)$, then we have decompositions $a = t_\alpha \circ a_0$ and $b = t_\beta \circ b_0$, where t_α and t_β are translations on B by α and β, and a_0 and b_0 are group homomorphisms of B to itself. Define $E_1 := \mathrm{Im}(b_0 + \mathrm{Id}), E_2 := \mathrm{Im}(a_0^2 b_0 a_0 + \mathrm{Id})$ and $E_3 := \mathrm{Im}(a_0 b_0 a_0^2 + \mathrm{Id})$. By Lemma 4.30, we know that E_1 (resp. E_2, resp. E_3) is also the reduction of the identity component of $\mathrm{Ker}(b_0 - \mathrm{Id})$ (resp. $\mathrm{Ker}(a_0^2 b_0 a_0 - \mathrm{Id})$, resp. $\mathrm{Ker}(a_0 b_0 a_0^2 - \mathrm{Id})$). Moreover, a_0 sends E_1 to E_3, sends E_3 to E_2 and sends E_2 to E_1, so E_1, E_2, E_3 are abstractly isomorphic to each other. By Lemma 4.31 and Lemma 4.27, we have $\dim E_1 = \frac{1}{2}(H^1(B/K)^\vee)^{\langle b \rangle} = \frac{1}{2}(H^1(B/W)^\vee \otimes_W \overline{K})^{\langle b \rangle} = 1$. So E_1 is an elliptic curve and is fixed by b_0. Since a_0 permutes E_i, we see that E_2 is an elliptic curve fixed under the action $a_0^2 b_0 a_0$ and E_3 is an elliptic curve fixed under the action of $a_0 b_0 a_0^2$. Let $\iota_i : E_i \hookrightarrow B$ be the natural embeddings, then by Lemma 4.31, under a suitable choice of bases, the dual of pullback $\iota_i^{*\vee} : H^1(E_i/K)^\vee \to H^1(B/K)^\vee$ on dual crystalline cohomology can be written in matrix form

$$\iota_1^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \ \iota_2^* = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \ \iota_3^* = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let us consider the morphism $\pi : E_1 \times E_2 \times E_3 \to B, (a, b, c) \mapsto \iota_1(a) + \iota_1(b) + \iota_1(c)$. Identifying $H^1(E_1 \times E_2 \times E_3/K)^\vee \cong H^1(E_1/K)^\vee \oplus H^1(E_2/K)^\vee \oplus H^1(E_3/K)^\vee$ using Künneth formula and taking the same bases as above, the dual of pullback $\pi^{*\vee} : H^1(E_1 \times E_2 \times E_3/K)^\vee \to H^1(B/K)^\vee$ is just $\mathrm{diag}(1,1,1,1,1,1)$ in matrix form. So by Proposition 2.30, the pullback $\pi^* : H^6(B/K) \to$

$H^6(E_1\times E_2\times E_3/K)$ on the top crystalline cohomology is identity, which shows that π is an isogeny. Let $\Lambda\subset E_1\times E_2\times E_3$ be the kernel of π. Pick $\tilde\alpha := (\alpha_1,\alpha_2,\alpha_3)\in\pi^{-1}(\alpha), \tilde\beta := (\beta_1,\beta_2,\beta_3)\in\pi^{-1}(\beta)$, and define two automorphisms $E_1\times E_2\times E_3\to E_1\times E_2\times E_3$:

$$\tilde a(x_1,x_2,x_3) = (x_2,x_3,x_1)+(\alpha_1,\alpha_2,\alpha_3),\ \tilde b(x_1,x_2,x_3) = (x_1,-x_2,-x_3)+(\beta_1,\beta_2,\beta_3).$$

Claim: We have $\pi\circ\tilde a = a\circ\pi$ and $\pi\circ\tilde b = b\circ\pi$.
Proof of claim: We show the case for a only, and the proof of b is identical. Define $\tilde a_0 : E_1\times E_2\times E_3\to E_1\times E_2\times E_3, (x_1,x_2,x_3)\mapsto(x_2,x_3,x_1)$. Consider the following diagram

$$\begin{tikzcd}
E_1\times E_2\times E_3 \arrow[r, "\tilde a_0"'] \arrow[d, "\pi"'] & E_1\times E_2\times E_3 \arrow[r, "t_{\tilde\alpha}"'] \arrow[d, "\pi"] & E_1\times E_2\times E_3 \arrow[d, "\pi"] \\
B \arrow[r, "a_0"'] & B \arrow[r, "t_\alpha"'] & B.
\end{tikzcd}$$

The commutativity of the right square follows directly from the choice of $\tilde\alpha$. For the commutativity of the left square, we note that the dual of pullback action $(\tilde a_0)^{*\vee}$ on $H^1(E_1\times E_2\times E_3/K)^\vee$ is $\tilde a_0^{*\vee} = \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$. So the commutativity follows from Lemma 4.11. ■

A direct computation shows that $\tilde a^3(x) = \tilde a_0^2(\tilde\alpha)+\tilde a_0(\tilde\alpha)+\tilde\alpha$. Since $a^3 = \mathrm{Id}$ and therefore $\tilde a^3$ is a translation in Λ, we get $\tilde a_0^2(\tilde\alpha)+\tilde a_0(\tilde\alpha)+\tilde\alpha\in\Lambda$. If we write $\alpha_\Sigma = \alpha_1+\alpha_2+\alpha_3$, then a computation shows that $\tilde a_0^2(\tilde\alpha)+\tilde a_0(\tilde\alpha)+\tilde\alpha = (\alpha_\Sigma,\alpha_\Sigma,\alpha_\Sigma)$. Another computation shows $\tilde b^{-1}\circ t_{(\alpha_\Sigma,\alpha_\Sigma,\alpha_\Sigma)}\circ\tilde b(x_1,x_2,x_3) = (x_1,x_2,x_3)+(\alpha_\Sigma,-\alpha_\Sigma,-\alpha_\Sigma)$. Since $(\alpha_\Sigma,\alpha_\Sigma,\alpha_\Sigma)\in\Lambda$ and $b^{-1}\circ b = \mathrm{Id}$, we get also $(\alpha_\Sigma,-\alpha_\Sigma,-\alpha_\Sigma)\in\Lambda$ and hence $(\alpha_\Sigma,\alpha_\Sigma,\alpha_\Sigma)+(\alpha_\Sigma,-\alpha_\Sigma,-\alpha_\Sigma) = (2\alpha_\Sigma,0,0)\in\Lambda$. Then we may compute

$$\begin{aligned}\tilde a^2(0,\alpha_2+\alpha_3,\alpha_1+\alpha_2+2\alpha_3) &= \tilde a(\alpha_1+\alpha_2+\alpha_3,\alpha_1+2\alpha_2+2\alpha_3,\alpha_3)\\ &= (2\alpha_1+2\alpha_2+2\alpha_3,\alpha_2+\alpha_3,\alpha_1+\alpha_2+2\alpha_3)\\ &= (0,\alpha_2+\alpha_3,\alpha_1+\alpha_2+2\alpha_3)+(2\alpha_\Sigma,0,0).\end{aligned}$$

Therefore, $a^2(\pi(0,\alpha_2+\alpha_3,\alpha_1+\alpha_2+2\alpha_3)) = \pi\tilde a^2(0,\alpha_2+\alpha_3,\alpha_1+\alpha_2+2\alpha_3) = \pi(0,\alpha_2+\alpha_3,\alpha_1+\alpha_2+2\alpha_3)$, and this contradicts our assumption that a is fixed point free. □

As a result, we reduce the possible classes of a pre-C.Y. group to D_8, Q_8 and Q_{12}, when the order is ≤ 12. Arguing similarly to the beginning of the proof of Proposition 4.34, we get

Proposition 4.36. *Let G be a non-abelian pre-C.Y. group isomorphic to D_8, Q_8 or Q_{12}, and let B be a target threefold. Then using the notations in Proposition 4.33, the irreducible decomposition of the induced representation $\rho : G\to\mathrm{GL}(H^1(B,W\mathcal{O}_B)^\vee\otimes_W\overline{K})$ is*

(1) $\rho\cong\rho_{1,1}\oplus\rho_{2,1}$, *if* $G\cong D_8$;
(2) $\rho\cong\rho_{1,0}\oplus\rho_{2,1}$, *if* $G\cong Q_8$;
(3) $\rho\cong\rho_{1,0}\oplus\rho_{2,1}$, *if* $G\cong Q_{12}$.

Proof. Since D_8, Q_8, Q_{12} are non-abelian and their representation on $H^1(B,W\mathcal{O}_B)^\vee\otimes_W\overline{K}$ is faithful by Proposition 4.12, we get that the representation ρ does not split into three one-dimensional irreducible representations. Moreover, by Proposition 4.17, the image of ρ lies in $\mathrm{SL}(H^1(B,W\mathcal{O}_B)^\vee\otimes_W\overline{K})$. Then it is easy to check that the decompositions listed in the statement are the only possibilities. □

Corollary 4.37. *If G is a non-abelian C.Y. group of order ≤ 12, then $G\cong D_8$.*

Proof. In the cases $G\cong Q_8$ or $G\cong Q_{12}$, we have $\dim_{\overline{K}}(H^1(B,W\mathcal{O}_B)^\vee\otimes_W\overline{K})^G = 1$, which is contradictory to Lemma 4.29. □

It remains to classify possible C.Y. groups of order 24. It turns out that there is none:

Proposition 4.38. *Let G be a group of order 24, then G is not a C.Y. group.*

The proof is based on the following classification of groups of order 24 categorized by their 2-Sylow subgroups. A proof can be found on [Bur12, p. 115-120], and we follow the notations in [OS01, Proposition 2.16].

Proposition 4.39. *Let G be a group of order 24, and let H be a 2-Sylow subgroup of G. Then H is isomorphic to one of $C_8, C_2 \times C_4, C_2^3, D_8, Q_8$, and G is isomorphic to one of the following 15 groups according to the isomorphism class of H:*

- *(I)* $H = \langle a \rangle \cong C_8$:
  - *(I_1)* $G = \langle c \rangle \times \langle a \rangle \cong C_3 \times C_8$;
  - *(I_2)* $G = \langle c, a \rangle \cong C_3 \rtimes C_8$, *where* $a^{-1}ca = c^{-1}$.
- *(II)* $H = \langle a, b \rangle \cong C_2 \oplus C_4$:
  - *(II_1)* $G = \langle c \rangle \times \langle a, b \rangle \cong C_3 \times (C_2 \oplus C_4)$;
  - *(II_2)* $G = \langle c, a, b \rangle \cong C_3 \rtimes (C_2 \oplus C_4)$, *where* $a^{-1}ca = c$ *and* $b^{-1}cb = c^{-1}$;
  - *(II_3)* $G = \langle c, a, b \rangle \cong C_3 \rtimes (C_2 \oplus C_4)$, *where* $a^{-1}ca = c^{-1}$ *and* $b^{-1}cb = c$.
- *(III)* $H = \langle a_1, a_2, a_3 \rangle \cong C_2^{\oplus 3}$:
  - *(III_1)* $G = \langle c \rangle \times \langle a_1, a_2, a_3 \rangle \cong C_3 \times C_2^{\oplus 3}$;
  - *(III_2)* $G = \langle a_1, a_2, a_3, c \rangle \cong C_2^{\oplus 3} \rtimes C_3$, *where* $c^{-1}a_1c = a_1$, $c^{-1}a_2c = a_3$, $c^{-1}a_3c = a_2a_3$;
  - *(III_3)* $G = \langle c, a_1, a_2, a_3 \rangle \cong C_3 \rtimes C_2^{\oplus 3}$, *where* $a_1^{-1}ca_1 = c$, $a_2^{-1}ca_2 = c$, $a_3^{-1}ca_3 = c^{-1}$.
- *(IV)* $H = \langle a, b \mid a^4 = 1,\ a^2 = b^2,\ b^{-1}ab = a^{-1} \rangle \cong Q_8$:
  - *(IV_1)* $G = \langle c \rangle \times \langle a, b \rangle \cong C_3 \times Q_8$;
  - *(IV_2)* $G = \langle a, b, c \rangle \cong Q_8 \rtimes C_3$, *where* $c^{-1}ac = b$, $c^{-1}bc = ab$;
  - *(IV_3)* $G = \langle c, a, b \rangle \cong C_3 \rtimes Q_8$, *where* $a^{-1}ca = c$, $b^{-1}cb = c^{-1}$.
- *(V)* $H = \langle a, b \mid a^4 = 1,\ b^2 = 1,\ bab = a^{-1} \rangle \cong D_8$:
  - *(V_1)* $G = \langle c \rangle \times \langle a, b \rangle \cong C_3 \times D_8$;
  - *(V_2)* $G = \langle c, a, b \rangle \cong C_3 \rtimes D_8$, *where* $a^{-1}ca = c$, $b^{-1}cb = c^{-1}$;
  - *(V_3)* $G = \langle c, a, b \rangle \cong C_3 \rtimes D_8$, *where* $a^{-1}ca = c^{-1}$, $b^{-1}cb = c$;
  - *(V_4)* $G \cong S_4$.

Proof of Proposition 4.38. Let B be a target threefold. In the cases $(I), (II)$ and (III), the Sylow subgroup H is an abelian group of order 8, and H is pre-C.Y. by Lemma 4.8. This contradicts Proposition 4.23. In the cases $(IV_1), (IV_3), (V_1), (V_2)$, the subgroup in G generated by a, c is isomorphic to C_{12}, which again contradicts Proposition 4.23. In the case (V_4), the group G contains a subgroup isomorphic to A_4, which contradicts Proposition 4.35.

It remains to consider the cases (IV_2) and (V_3). We deal first with the case (IV_2). Consider the representation ρ_H of H on $H^1(B, W\mathcal{O}_B)^\vee \otimes_W \overline{K}$. Since $H \cong Q_8$, the irreducible representation of ρ_H is $\rho_{1,0} \oplus \rho_{2,1}$ under the notations in Proposition 4.33, by Proposition 4.36. Let V_1 be the subspace corresponding to $\rho_{1,0}$ and let x be a basis vector of V_1. Then $a(c(x)) = c(b(x)) = c(x)$ since $ac = cb$, so $c(x)$ is an eigenvector of the action a with eigenvalue 1, hence is in V_1. This shows that V_1 is G-stable. Take V_2 to be a G-stable complement of V_1, then under a suitable choice of basis of V_2, the representation of G on $H^1(B, W\mathcal{O}_B)^\vee \otimes_W \overline{K}$ is of the form

$$a \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_4 & 0 \\ 0 & 0 & -\zeta_4 \end{pmatrix},\ b \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \zeta_4 \\ 0 & \zeta_4 & 0 \end{pmatrix},\ c \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & C \end{pmatrix},$$

where C is a 2×2 matrix. Since c is of order 3, it has three eigenvalues $1, \zeta_3, \zeta_3^2$ by Corollary 4.20. If $\alpha = 1$, then V_1 is G-invariant and this contradicts Lemma 4.29. So $\alpha = \zeta_3$ or ζ_3^2, and by possibly replacing c with c^{-1} we may assume $\alpha = \zeta_3$. So C has now eigenvalues 1 and ζ_3^2. Then the group element a^2c acts as $\begin{pmatrix} \alpha & 0 \\ 0 & -C \end{pmatrix}$ and has no eigenvalue 1, contradicting Corollary 4.20.

We treat then the case (V_3). We consider again the representation ρ_H of H on $H^1(B, W\mathcal{O}_B)^\vee \otimes_W \overline{K}$. Now $H \cong D_8$ and Proposition 4.36 implies that $\rho_H \cong \rho_{1,1} \oplus \rho_{2,1}$ under the notations in Proposition 4.33. Let V_1 be the invariant subspace under c, which is one-dimensional by Corollary 4.20. Since $ca = ac^{-1}$ and $cb = bc$, we see that V_1 is G-stable. Let V_2 be a G-stable complement of V_1, then V_1 is the subspace corresponding to $\rho_{1,1}$ and V_2 is the subspace corresponding to $\rho_{2,1}$. By a suitable choice of bases of V_1 and V_2, the representation of G on $H^1(B, W\mathcal{O}_B)^\vee \otimes_W \overline{K}$ is of the form

$$a \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_4 & 0 \\ 0 & 0 & -\zeta_4 \end{pmatrix}, \; b \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \; c \mapsto \begin{pmatrix} 1 & 0 \\ 0 & C \end{pmatrix},$$

where C is a 2×2 matrix. So the action of bc is of the form $\begin{pmatrix} -1 & 0 \\ 0 & C^T \end{pmatrix}$, and has order 2 by Corollary 4.20. However, the group element bc has order 6, a contradiction. □

Proof of Theorem 4.6. Through the discussion of this section, we see that the only possible C.Y. groups are $C_2 \times C_2$ and D_8. First, let G be $\langle a\rangle \times \langle b\rangle \cong C_2 \times C_2$ and let B be a target threefold of G. By the proof of Proposition 4.23, the representation of G on $H^1(B, W\mathcal{O}_B)^\vee \otimes_W \overline{K}$ is of the form

$$a \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; b \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

under a suitable choice of basis. Then by Proposition 4.15, the representation of G on $H^1(B/K)^\vee$ is $a \mapsto \mathrm{diag}(1,1,-1,-1,-1,-1), b \mapsto \mathrm{diag}(-1,-1,1,1,-1,-1)$. Decompose $a = t_\alpha \circ a_0$ and $b = t_\beta \circ b_0$ such that t_α and t_β are translations by α and β, and a_0 and b_0 are group homomorphisms. Define $E_1 := \mathrm{Im}(a_0 + \mathrm{Id}), E_2 := \mathrm{Im}(b_0 + \mathrm{Id})$ and $E_3 := \mathrm{Im}(a_0 b_0 + \mathrm{Id})$. We see then E_1 (resp. E_2, resp. E_3) is the reduction of the identity component of $\mathrm{Ker}(a_0 - \mathrm{Id})$ (resp. $\mathrm{Ker}(b_0 - \mathrm{Id})$, resp. $\mathrm{Ker}(a_0 b_0 - \mathrm{Id})$) by Lemma 4.30. It follows then $\dim E_1 = \frac{1}{2}\mathrm{rank}(H^1(B/K)^\vee)^{\langle a\rangle} = 1$ by Lemma 4.31, and similarly $\dim E_2 = \dim E_3 = 1$. Let $\iota_i : E_i \hookrightarrow B$ be the natural embeddings, then under a suitable choice of bases of $H^1(E_i/K)^\vee$, the dual of the pullbacks $\iota_i^{*\vee} : H^1(E_i/K)^\vee \to H^1(B/K)^\vee$ on crystalline cohomology can be written in matrix form

$$\iota_1^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \; \iota_2^* = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \; \iota_3^* = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Consider the morphism $\pi : E_1 \times E_2 \times E_3 \to B, (a,b,c) \mapsto \iota_1(a) + \iota_1(b) + \iota_1(c)$. By Künneth formula, we have $H^1(E_1 \times E_2 \times E_3/K)^\vee \cong H^1(E_1/K)^\vee \oplus H^1(E_2/K)^\vee \oplus H^1(E_3/K)^\vee$. So taking the same basis as above, the dual of pullback $\pi^{*\vee} : H^1(E_1 \times E_2 \times E_3/K)^\vee \to H^1(B/K)^\vee$ is just $\mathrm{diag}(1,1,1,1,1,1)$ in matrix form. So by Proposition 2.30, the pullback $\pi^* : H^6(B/K) \to H^6(E_1 \times E_2 \times E_3/K)$ on the top crystalline cohomology is identity. Hence π is an isogeny. As readily mentioned in Proposition 4.24, the fact that the representation of G on $H^1(B, \mathcal{O}_B)^\vee$ is of the same form follows from the fact that the injection $H^1(B, W\mathcal{O}_B)/VH^1(B, W\mathcal{O}_B) \to H^1(B, \mathcal{O}_B)$ in Lemma 2.33 is an isomorphism, and is equivariant under any automorphism. One can then simply check that the constraint $H^1(B, \mathcal{O}_B)^G = 0$ holds in this case. Moreover, as we assume $p > 2$, the group $C_2 \times C_2$ is linearly reductive, hence by Corollary 2.39, the quotient B/G satisfies $H^1(B/G, \mathcal{O}_{B/G}) = 0$.

The proof of case $G = \langle a, b | a^4 = b^2 = abab = 1\rangle \cong D_8$ is identical, by setting $E_1 := \mathrm{Im}(a_0 + \mathrm{Id}), E_2 := \mathrm{Im}(b_0 + \mathrm{Id})$ and $E_3 := \mathrm{Im}(a_0 b_0 a_0^{-1} + \mathrm{Id})$. Moreover, E_2 and E_3 are isomorphic as a_0 sends E_2 to E_3.

It remains to construct examples for both cases:

Example 4.40 (c.f. also [OS01, Example 2.17]). Let E_1, E_2 and E_3 be elliptic curves. Pick non-zero 2-torsion points $\tau_1 \in E_1[2]\backslash\{0\}, \tau_2 \in E_2[2]\backslash\{0\}, \tau_3 \in E_3[2]\backslash\{0\}$ on each curve. Define on $E_1 \times E_2 \times E_3$ automorphisms

$$a(x_1, x_2, x_3) = (x_1 + \tau_1, -x_2, -x_3), \; b(x_1, x_2, x_3) = (-x_1, x_2 + \tau_2, -x_3 + \tau_3).$$

Then a and b are of order 2, and $ab = ba$. Hence $\langle a, b\rangle$ defines an action of $C_2 \times C_2$ on $E_1 \times E_2 \times E_3$, and it is easy to check that this action makes $C_2 \times C_2$ a C.Y. group.

Example 4.41 (c.f. also [OS01, Example 2.18]). Let E_1 and E_2 be elliptic curves. Pick a 4-torsion point $\tau_1 \in E_1[4]\backslash E_1[2]$ on E_1 which is not 2-torsion, and pick two distinct non-zero 2-torsion points $\tau_2, \tau_3 \in E_2[2]\backslash\{0\}$. Define on $E_1 \times E_2 \times E_3$ automorphisms

$$a(x_1, x_2, x_3) = (x_1 + \tau_1, -x_3, x_2),\ b(x_1, x_2, x_3) = (-x_1, x_2 + \tau_2, -x_3 + \tau_3).$$

Let τ denote the point $(0, \tau_2 + \tau_3)$. We see then $a^4 = b^2 = \mathrm{Id}, abab = t_\tau, at_\tau = t_\tau a$ and $bt_\tau = t_\tau b$. So a and b descends to automorphisms on $E_1 \times E_2 \times E_2/\langle t_\tau\rangle$ and defines a D_8 action on it, which is easily verified to be a C.Y. action.

□

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