

Review of Markov chain

A discrete-time Markov chain is a sequence of random variable $X_1, X_2, X_3 \dots$

with Markov property

$$\Pr(X_{n+1}=j \mid X_n=i_n, \dots, X_1=i_1) \\ = \Pr(X_{n+1}=j \mid X_n=i_n)$$

Def: \hookrightarrow Transition probability can be represented by a Matrix: Transition Matrix P .

$$P_{ij} = \Pr(X_{n+1}=j \mid X_n=i)$$

$$\textcircled{1} P_{ij} \geq 0$$

$$\textcircled{2} \sum_j P_{ij} = 1$$

$\Rightarrow P$ is a right stochastic Matrix.

property: $\textcircled{1} P \vec{1} = \vec{1} \Rightarrow \lambda_1(P) = 1, \vec{v}_1 = \vec{1}$

$$\textcircled{2} \lambda(P) \text{ are all within unit circle.}$$

$$\Pr(X_{n+2}=j \mid X_n=i) = \sum_k \frac{P(X_{n+2}=j \mid X_{n+1}=k) \cdot P(X_{n+1}=k \mid X_n=i)}{P(X_{n+1}=k \mid X_n=i)}$$

say $u_i = \Pr(X_n = i)$

$v_j = \Pr(X_{n+1} = j)$

$\hookrightarrow \vec{v} = \vec{u} P$ in matrix form

so $w_k = \Pr(X_{n+2} = k)$

$$\boxed{\vec{w} = \vec{u} P^2}$$

Def:

The Markov chain is irreducible

$$\Leftrightarrow \text{any } (i, j), \exists t, \text{ s.t. } P_{ij}^t > 0$$

$$\Leftrightarrow \text{Graph } (V, E) \text{ is connected.}$$

directed

property = $\lambda_1(P) = 1$ is simple.
if P is irreducible.

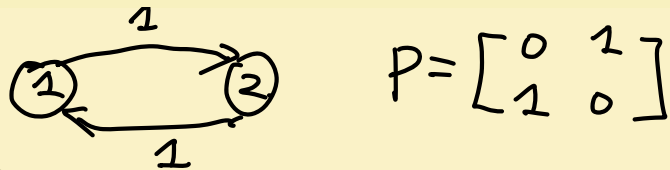
Note: this t is not uniform, i.e.

for some $(i, j), \exists t_{ij}, P_{ij}^{t_{ij}} > 0$

but for some other $(i', j'), P_{i'j'}^{t_{ij}}$ can be zero!

This is not strong enough for our analysis

P naturally defines
a directed graph.
 $e_{ij} \in E \Leftrightarrow P_{ij} > 0$



$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Def: The Markov chain is primitive.

\Leftrightarrow any $t > 0$, s.t any (i, j) ,

$$A_{ij}^t > 0 \text{ // uniformly}$$

\Leftrightarrow Graph $G = (V, E)$ is path- t connected: any pair of Nodes are connected by a path of length no more than t .

// irreducible + aperiodic.

property: $|\lambda_2(P)| < 1$.

More: P is irreducible and $P_{ii} > 0$

$\Rightarrow P$ is primitive.

A stationary distribution π is a row vector $\pi_i \geq 0$ and $\sum_i \pi_i = 1$.

and $\vec{\pi} P = \vec{\pi} \leftarrow \pi$ is the left
eigenvector of P .

If P is irreducible, then π is
unique.

If P is primitive, then

$$\vec{\pi} = \lim_{n \rightarrow \infty} \vec{v}^T P^n \text{ for any probability vector } \vec{v}$$

$$\Leftrightarrow \lim_{n \rightarrow \infty} P^n = \vec{1} \vec{\pi}$$

Consider the similarity graph

$$G = (V, E, W)$$

$$W_{ij} = \exp(-\|x^{(i)} - x^{(j)}\|_2 / 2\epsilon)$$

★ This time when $i=j$, $W_{ij} = 1$

$L = D - W$ is the same as LE (why?)

Ex:

$$W_1 = \begin{bmatrix} 0 & 0.5 & 0.1 \\ 0.5 & 0 & 0.4 \\ 0.1 & 0.4 & 0 \end{bmatrix}$$

$$W_2 = \begin{bmatrix} 1 & 0.5 & 0.1 \\ 0.5 & 1 & 0.4 \\ 0.1 & 0.4 & 1 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} 0.6 & & \\ & 0.9 & \\ & & 0.5 \end{bmatrix} \quad D_2 = \begin{bmatrix} 1.6 & & \\ & 1.9 & \\ & & 1.5 \end{bmatrix}$$

$$\rightarrow L = \begin{bmatrix} 0.6 & -0.5 & -0.1 \\ -0.5 & 0.9 & -0.4 \\ -0.1 & -0.4 & 0.5 \end{bmatrix} \leftarrow D_2 - W_2$$

But L^{rw} are different.

$$\text{Look at } L^{rw} = D^{-1}L = D^{-1}(D - W)$$

$$= I - \underbrace{D^{-1}W}_{\hat{P}}$$

- ① \hat{P} is a Markov transition matrix
- ② assume G is primitive
the unique $\vec{\pi}$: $\pi_i = d_i / \text{vol}(V)$

$$\text{proof: } (d_1, d_2, \dots, d_N) \begin{pmatrix} 1/d_1 & & \\ & 1/d_2 & \\ & & \ddots \\ & & & 1/d_N \end{pmatrix} W$$

$$= (1, 1, \dots, 1) W = (d_1, d_2, \dots, d_N)$$

Normalize it

$$\vec{\pi} = \left(\frac{d_1}{\text{vol}(V)}, \frac{d_2}{\text{vol}(V)}, \dots, \frac{d_N}{\text{vol}(V)} \right)$$

$$\textcircled{3} \text{ } P \text{ is "reversible"} \leftarrow \boxed{\text{def: } \pi_i P_{ij} = \pi_j P_{ji}}$$

$$P = \begin{pmatrix} \frac{1}{d_1} \vec{w}_1 \\ \frac{1}{d_2} \vec{w}_2 \\ \vdots \\ \frac{1}{d_N} \vec{w}_N \end{pmatrix}$$

$$P_{ij} = \frac{w_{ij}}{d_i}$$

$$\pi_i P_{ij} = \frac{d_i}{\text{vol}(V)} \frac{w_{ij}}{d_i}$$

$$= \frac{w_{ij}}{\text{vol}(V)}$$

$$\pi_j P_{ji} = \frac{d_j}{\text{vol}(V)} \frac{w_{ji}}{d_j}$$

$$= \frac{w_{ji}}{\text{vol}(V)}$$

