Review of Markov chain

A discrete-time Markov chain is a sequence of random variable $X_1, X_2, X_3 \dots$ with $\boxed{\text{Markov property}}$

$$Pr(X_{n+1} = j \mid X_n = i_n, \dots X_1 = i_1)$$

$$= Pr(X_{n+1} = j \mid X_n = i_n)$$

Def: Transition probability can be represented by a Matrix : Transition Matrix P.

$$P_{ij} = Pr(X_{n+1} = j \mid X_n = i)$$

① $P_{ij} \geq 0$

② $\sum_j P_{ij} = 1$

> P is a right stochastic Matrix.

property : ① $P\vec{1} = \vec{1} \Rightarrow \lambda_1(P) = 1, \ \vec{v}_1 = \vec{1}$

② $\lambda(P)$ are all within unit circle.

$$Pr(X_{n+2} = j \mid X_n = i) = \sum_k P(X_{n+2} = j \mid X_{n+1} = k) \cdot P(X_{n+1} = k \mid X_n = i)$$

P naturaly defines a directed graph.
$e_{ij} \in E \Leftrightarrow P_{ij} > 0$

say $u_i = Pr(X_n = i)$

$v_j = Pr(X_{n+1} = j)$

$\vec{v} = \vec{u} P$ in matrix form

so $w_k = Pr(X_{n+2} = k)$.

$$\boxed{\vec{w} = \vec{u} P^2}$$

Def:

The Markov Chain is irreducible

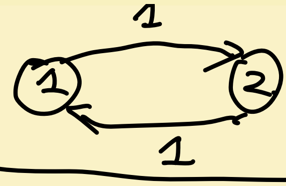
$\Leftrightarrow$ any (i,j), $\exists t$, s.t $P_{ij}^t > 0$

$\Leftrightarrow$ Graph (V, E) is connected.
directed

property: $\lambda_1(P) = 1$ is simple.
if P is irreducible.

Note: this t is not uniform, i.e.
for some (i,j), $\exists t_{ij}$, $P_{ij}^{t_{ij}} > 0$
but for some other (i', j'), $P_{i'j'}^{t_{ij}}$ can be zero!

This is not strong enough for our analysis



$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Def: The Markov chain is primitive.

$\Leftrightarrow$ any $t > 0$, s.t any (i, j),

$A_{ij}^{t} > 0$ // uniformly

$\Leftrightarrow$ Graph $G = (V, E)$ is path-t connected: any pair of Nodes are connected by a path of length no more than t.

$\|$ irreducible + aperiodic.

property: $|\lambda_2(P)| < 1$.

More: P is irreducible and $P_{ii} > 0$

$\Rightarrow$ P is primitive.

A stationary distribution π is a row vector $\pi_i \geq 0$ and $\sum_i \pi_i = 1$.

and $\vec{\pi} P = \vec{\pi}$ ← π is the left eigenvector of P.

If P is irreducible, then π is unique.

If P is primitive, then

$\vec{\pi} = \lim_{n\to\infty} \vec{v} P^n$ for any probability vector $\vec{v}$

$\Longleftrightarrow \lim_{n\to\infty} P^n = \vec{1}\vec{\pi}$

Consider the similarity graph

$G = (V, E, W)$

$$W_{ij} = \exp(-\|x^{(i)} - x^{(j)}\|_2 / 2\varepsilon)$$

☆ This time when $i = j$, $W_{ij} = 1$

$L = D - W$ is the same as LE (why?)

Ex:

$$W_1 = \begin{bmatrix} 0 & 0.5 & 0.1 \\ 0.5 & 0 & 0.4 \\ 0.1 & 0.4 & 0 \end{bmatrix} \qquad W_2 = \begin{bmatrix} 1 & 0.5 & 0.1 \\ 0.5 & 1 & 0.4 \\ 0.1 & 0.4 & 1 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} 0.6 & & \\ & 0.9 & \\ & & 0.5 \end{bmatrix} \qquad D_2 = \begin{bmatrix} 1.6 & & \\ & 1.9 & \\ & & 1.5 \end{bmatrix}$$

$D_1 - W_1$

$$L = \begin{bmatrix} 0.6 & -0.5 & -0.1 \\ -0.5 & 0.9 & -0.4 \\ -0.1 & -0.4 & 0.5 \end{bmatrix}$$

$D_2 - W_2$

But L^{rw} are different.

Look at $L^{rw} = D^{-1}L = D^{-1}(D - W)$

$$= I - \underbrace{D^{-1}W}_{\triangleq P}$$

① P is a Markov transition matrix

② assume G is primitive

the unique $\vec{\pi}$: $\vec{\pi}_i = d_i / vol(V)$

proof: $(d_1, d_2, \cdots, d_N) \begin{pmatrix} 1/d_1 & & & \\ & 1/d_2 & & \\ & & \ddots & \\ & & & 1/d_N \end{pmatrix} W$

$= (1, 1, \cdots, 1) W = (d_1, d_2, \cdots, d_N)$

Normalize it

$$\vec{\pi} = \left(\frac{d_1}{vol(V)}, \frac{d_2}{vol(V)}, \cdots \frac{d_N}{vol(V)} \right)$$

③ P is "reversible" $\Leftarrow$ $\boxed{\text{def: } \pi_i P_{ij} = \pi_j P_{ji}}$

$$P = \begin{pmatrix} \frac{1}{d_1} \vec{w}_1 \\ \frac{1}{d_2} \vec{w}_2 \\ \vdots \\ \frac{1}{d_N} \vec{w}_N \end{pmatrix}$$

$$P_{ij} = \frac{W_{ij}}{d_i}$$

$$\pi_i P_{ij} = \frac{d_i}{vol(V)} \frac{W_{ij}}{d_i} = \frac{W_{ij}}{vol(V)}$$

$$\pi_j P_{ji} = \frac{d_j}{vol(V)} \frac{W_{ji}}{d_j} = \frac{W_{ji}}{vol(V)}$$

④ Left eigenvector $\vec{u}$

Right eigenvector $\vec{v}$

$$\frac{\vec{u}^T}{\vec{\pi}} = \vec{v} \; !!$$

If $\vec{v}$ is the right eigenvector.

$$D^{-1} W \vec{v} = \lambda \vec{v}$$

multiply D $\Downarrow$

$$W \vec{v} = \lambda D \vec{v}.$$

$\Downarrow$ define: $\vec{u} = (D\vec{v})^T = \vec{v}^T D$.

$$W D^{-1} \vec{u}^T = \lambda \vec{u}^T.$$

Take transpose $\Downarrow$

$$\vec{u} D^{-1} W = \lambda \vec{u}.$$

$\Downarrow$

$$\vec{u} P = \lambda \vec{u}$$ so $\vec{u}$ is the left eigenvector.

Since $D_{ii} = \pi_i \, \mathrm{Vol}(V)$

$$\Rightarrow \frac{\vec{u}^T}{\vec{\pi}} = \vec{v}$$