EECS 442 9/7

Math Background.

Matrices & Eigenvalues/vectors

$$p = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

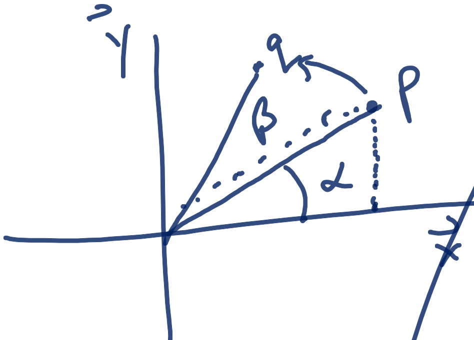
$$q = Ap$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

Rotations in the plane

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



$$p = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r \cos \alpha \\ r \sin \alpha \end{bmatrix}$$

$$q = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} r \cos(\alpha+\beta) \\ r \sin(\alpha+\beta) \end{bmatrix}$$

$$= \begin{bmatrix} r(\cos\alpha \cos\beta - \sin\alpha \sin\beta) \\ r(\sin\alpha \cos\beta - \cos\alpha \sin\beta) \end{bmatrix}$$

$$= \begin{bmatrix} x \cos\beta - y \sin\beta \\ x \sin\beta + y \cos\beta \end{bmatrix}$$

Rotation Matrices

$$= \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

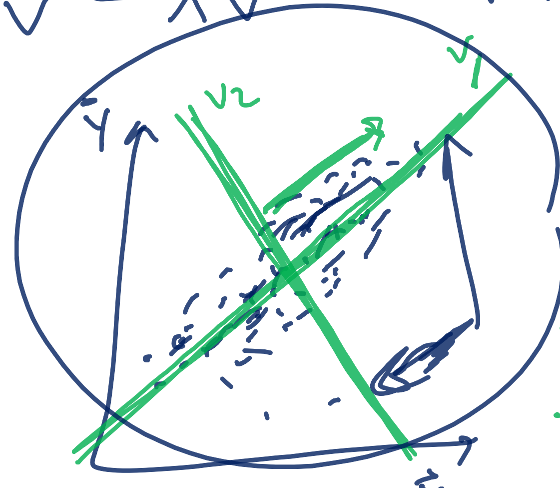
Full Rank

— eigenvalues and eigenvectors

$$Av = \lambda v \qquad v \neq 0, \ \lambda \neq 0$$

$$A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \\ x_n & y_n \end{bmatrix}$$

$A^T A$



$\lambda_1 > \lambda_2$

- eigs
- Laplace Expansion

$$Av = \lambda v \qquad v \neq 0$$

$$\Updownarrow$$

$$Av - \lambda v = 0$$

$$\Updownarrow$$

$$[A - \lambda I] = 0$$

$$\Updownarrow$$

$$\boxed{[\lambda I - A] v = 0} \qquad v \neq 0$$

must be singular

$$\therefore \boxed{\det(\lambda I - A) = 0}$$

Ex $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ Real $\rightarrow$ complex

$$\det(\lambda I - A) = \lambda^2 + 1 = (\lambda + j)(\lambda - j)$$

Similarity Transform

$\hookrightarrow$ any nonsingular matrix S

$B = S^{-1} A S \rightarrow$ similarity transform of A.

Taylor Series Background

Single Variate

Let f be an infinitely differentiable function in an open interval around $x=a$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$= \underbrace{f(a) + f'(a)(x-a)}_{\text{Linear Taylor Expansion}} + \frac{f''(a)}{2}(x-a)^2 + \cdots$$

MVLTE

$$f(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$f(a,b) + \nabla_f \begin{bmatrix} x-a \\ y-b \end{bmatrix}$$

Image Linear Approximation

Linear Least Squares

Consider a system of p linear equations in q unknowns.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1q}x_q &= y_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2q}x_q &= y_2 \\
&\vdots \\
a_{p1}x_1 + a_{p2}x_2 + \cdots \; a_{pq}x_q &= y_p
\end{aligned}
$$

$$Ax = y$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots \\ \vdots & \ddots & \\ & & a_{pq} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_q \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix}$$

$p < q$

$p = q$ unique solution

Gaussian Elimination

$p > q$ many solutions but none are exact

$$E(x) \doteq \sum_{i=1}^{p} \left(a_{i1}x_1 + \cdots + a_{iq}x_q - y_i\right)^2$$

$$\Updownarrow$$

$$\|Ax - y\|_2^2$$

Euclidean Norm

Linear Least Squares Problem

$$x = \underbrace{(A^TA)^{-1}A^T}_{\text{pseudoinverse}}\, y$$

$$A \backslash y$$

$$y_i = m x_i + b \cdot 1$$

$$A \quad \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \\ x_p & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix}$$