

Math Background.

## Matrices &amp; Eigenvalues/vectors

Rotation matrix (2x2 in plane)

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$



$$p = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r \cdot \cos \alpha \\ r \cdot \sin \alpha \end{bmatrix}$$

$$q = \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} r \cdot \cos(\alpha + \beta) \\ r \cdot \sin(\alpha + \beta) \end{bmatrix}$$

$$q = \begin{bmatrix} r (\cos \alpha \cos \beta - \sin \alpha \sin \beta) \\ r (\sin \alpha \cos \beta + \cos \alpha \sin \beta) \end{bmatrix}$$

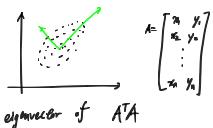
$$= \begin{bmatrix} x \cos \beta - y \sin \beta \\ x \sin \beta + y \cos \beta \end{bmatrix} = \begin{bmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x^2 - 2 \cos \beta \cdot x + 1 = 0$$

$$\frac{2 \cos \beta \pm \sqrt{4 \cos^2 \beta - 4}}{2} = \cos \beta \pm \sqrt{\cos^2 \beta - 1} \\ = \cos \beta \pm i / |\sin \beta|$$

Eigen vectors

$$Av = \lambda v$$



Similarity transform

any non singular S

$$B = S^{-1}AS$$

Taylor's Expansion

let  $f \in C^\infty(a-\varepsilon, a+\varepsilon)$ 

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k \\ = f(a) + f'(a) \cdot (x-a) + \frac{f''(a)}{2!} \cdot (x-a)^2 + \dots$$

Multivariate

$$f(x,y) = f(a,b) + f_x(a,b) \cdot (x-a) + f_y(a,b) \cdot (y-b) \\ = f(a,b) + \begin{pmatrix} \nabla f(a,b) \end{pmatrix}^T \begin{bmatrix} x-a \\ y-b \end{bmatrix}$$

Consider a system of p linear equations in q unknowns.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1q}x_q = y_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2q}x_q = y_2$$

$$\vdots$$

$$a_{p1}x_1 + a_{p2}x_2 + \dots + a_{pq}x_q = y_p$$

$$Ax = y$$

If  $p \leq q$  equations less than unknowns.

$$\begin{cases} p = q \\ p > q \end{cases}$$

equations more than unknowns

Gaussian elimination (unique solution)

(least squares no exact solutions)

$$E(x) = \sum_{i=1}^p (a_{i1}x_1 + \dots + a_{iq}x_q - y_i)^2$$

$$= \|Ax-y\|_2^2 \quad (\text{Euclidean norm})$$

$$x = (A^T A)^{-1} A^T y$$

Three primary topics in the representation of images

1. image as functions
2. image as coefficients/pixels
3. image as graphs.

#### 1. Defining the classes of images

Def 1. A **perfect image** is the continuous image generated by a physical process. We denote them with  $I$ .  
 $I: \mathbb{R}^2 \rightarrow \mathbb{R}^+$ .

Remark: perfect images exist in abstraction only.

Eg. Lambertian Model is a classical model of surface reflectance.

→ the amount of light reflected off the surface

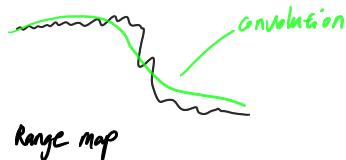
Why do I use Difference of Gaussian instead of Laplacian of Gaussian

$$G \otimes I - G_2 \otimes I \Rightarrow \text{edges}$$

$I_1$

$I_2$

Better than Laplacian since it smooth



Geometric primitives

$$Ax + b \quad | \quad A = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} x \\ 1 \end{bmatrix}$$

Projection plane.  
intersection of two lines

$$\tilde{x} = \tilde{x}_1 \times \tilde{x}_2$$

$$\tilde{l} = \tilde{x}_1 \times \tilde{x}_2$$

degree of freedom

$$\text{rotation} + \text{similarity} \quad \tilde{x}' = \begin{bmatrix} sR & t \\ 0 & 1 \end{bmatrix} \tilde{x}$$

D.o.F = 4

$$\text{Affine} \quad \tilde{x}' = A\tilde{x} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \tilde{x}$$

D.o.F = 6

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} ax_1 + by_1 + cz_1 \\ dx_1 + ey_1 + fz_1 \\ 0x_1 + 0y_1 + 1z_1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} ax_2 + by_2 + cz_2 \\ dx_2 + ey_2 + fz_2 \\ 0x_2 + 0y_2 + 1z_2 \end{bmatrix}$$

$$px + qy + rz = C_1$$

$$px + qy + rz^2 = C_2$$

$$\frac{x_1 - x_2}{r} = \frac{y_1 - y_2}{r} = \frac{z_1 - z_2}{r} \quad \text{luc} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \quad ①$$

$$\frac{x_3 - x_4}{r} = \frac{y_3 - y_4}{r} = \frac{z_3 - z_4}{r} \quad \text{luc} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} \begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix} \quad ②$$

① // ②

New lines: directions

$$\begin{pmatrix} a(x_1 - x_2) + b(y_1 - y_2) + c(z_1 - z_2) \\ d(x_1 - x_2) + e(y_1 - y_2) + f(z_1 - z_2) \\ z_1 - z_2 \end{pmatrix}$$

$$\frac{x}{a(x_1 - x_2) + b(y_1 - y_2) + c(z_1 - z_2)} = \frac{y}{d(x_1 - x_2) + e(y_1 - y_2) + f(z_1 - z_2)}$$

SRO

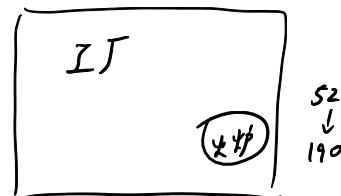
special range operation.

$$\boxed{\text{far}}$$

Avg = 162

$$\boxed{\text{far up}}$$

Avg = 47



RMO

Laplacian of Gaussian.

$$\nabla^2(f(x,y) \otimes G(x,y)) = \nabla^2 G(x,y) \otimes f(x,y)$$

$\underbrace{G \otimes f}_{\text{blurred image}}$

$$f_\sigma(x,y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2+y^2}{2\sigma^2}\right) \quad \sigma > 0 \quad xy \in \mathbb{R}$$

$$\log G(x,y) = -\frac{1}{2\sigma^2} \left[1 - \frac{x^2+y^2}{2\sigma^2}\right] \exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)$$

$$\text{Laplacian} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\text{Gaussian}_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \cdot \frac{1}{9}$$

$$\text{Gaussian}_2 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \cdot \frac{1}{16}$$

Difference of Gaussian

9/27 Discussion

### Image Matching

① Find feature points in both image

→ structure tensor  
(reduction)

$$I(x, y) \rightarrow (x+u, y+v)$$

$$I(x+u, y+v) = I(x, y) + \frac{\partial I(x, y)}{\partial x} \cdot u + \frac{\partial I(x, y)}{\partial y} \cdot v + \dots$$

higher order.  
only use linear terms

Calculate the difference.

$$\begin{aligned} E &= \sum_{x, y \in W} (I(x+u, y+v) - I(x, y))^2 \\ &\approx \sum_{x, y \in W} \left( \frac{\partial I}{\partial x} \cdot u + \frac{\partial I}{\partial y} \cdot v \right)^2 \\ &= [u, v] \underbrace{\begin{bmatrix} \sum_w I_x^2 & \sum_w I_x I_y \\ \sum_w I_x I_y & \sum_w I_y^2 \end{bmatrix}}_{H: \text{tensor structure.}} [u, v]^T \end{aligned}$$

$$\det(H - I\lambda) = \det \begin{pmatrix} h_{11} - \lambda & h_{12} \\ h_{21} & h_{22} - \lambda \end{pmatrix} = 0$$

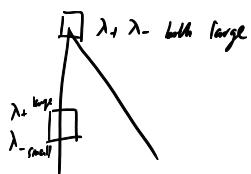
$$(h_{11} - \lambda)(h_{22} - \lambda) - h_{12} \cdot h_{21} = 0$$

$$\lambda = \frac{h_{11} + h_{22} \pm \sqrt{(h_{11} + h_{22})^2 - 4(h_{11}h_{22} - h_{12}h_{21})}}{2}$$

Window size: set manually.

Harris detector property

$\lambda_+$   $\lambda_-$  rotation invariant



$$\frac{\lambda_+ + \lambda_-}{\lambda_+ - \lambda_-} = \frac{\det H}{\text{tr } H}$$

② Match the feature

$$\boxed{P} \quad \boxed{P'}$$

$$P = HP'$$

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

projection

Pick many features  
(parts of P, P')

min  $\|P - HP'\|_F^2$

Recall de fitting

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} M \\ b \end{bmatrix}$$

