

Topic 7:

Discrete Wavelet Transform

- Wavelet transform vs. Laplacian pyramid
- Basic intuition: a simple wavelet-like 2D transform
- The 1D Haar wavelet transform
- 1D Haar wavelet transform as a matrix product
- Reconstructing a 1D image from its wavelet coeffs
- Wavelet-based image compression
- The 2D Haar wavelet transform

A Simple, Minimal 2-D Image Transform

Properties of transformation

- Minimal (no "wasted" pixels)
- • Multiple scales represented simultaneously
- Invertible, linear

Input image ($2^N \times 2^N$)



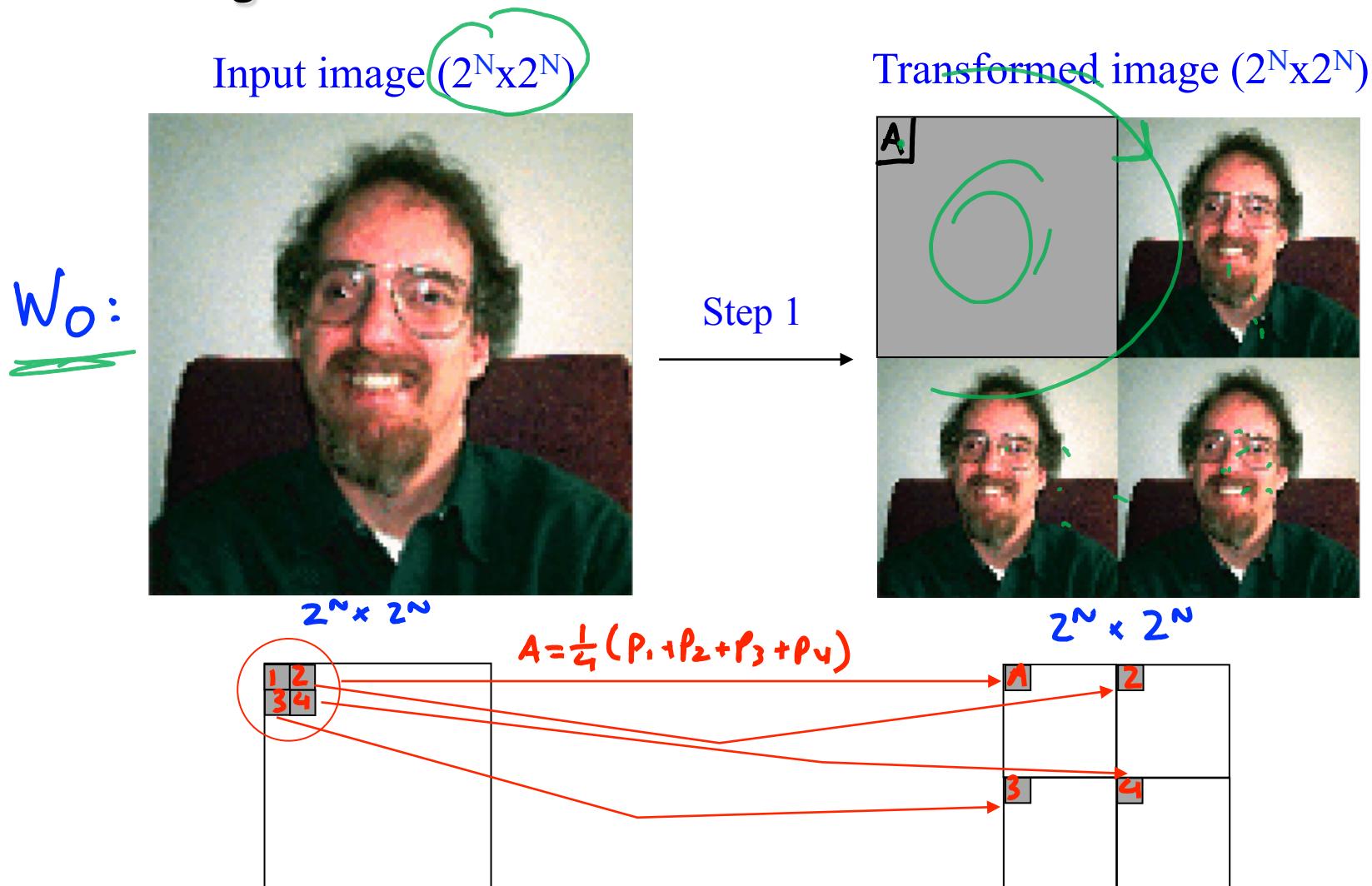
Transformed image ($2^N \times 2^N$)



wavelet
transform

A Simple, Minimal 2-D Image Transform

Step 1: Create 4 new images of size $2^{N-1} \times 2^{N-1}$ as shown in figure



A Simple, Minimal 2-D Image Transform

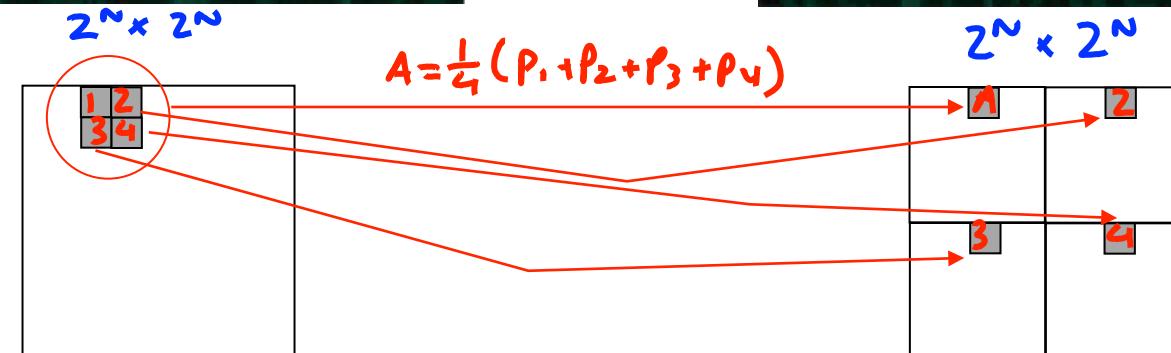
Step 1: Create 4 new images of size $2^{N-1} \times 2^{N-1}$ as shown in figure

W_O:



Step 1

Transformed image ($2^N \times 2^N$)



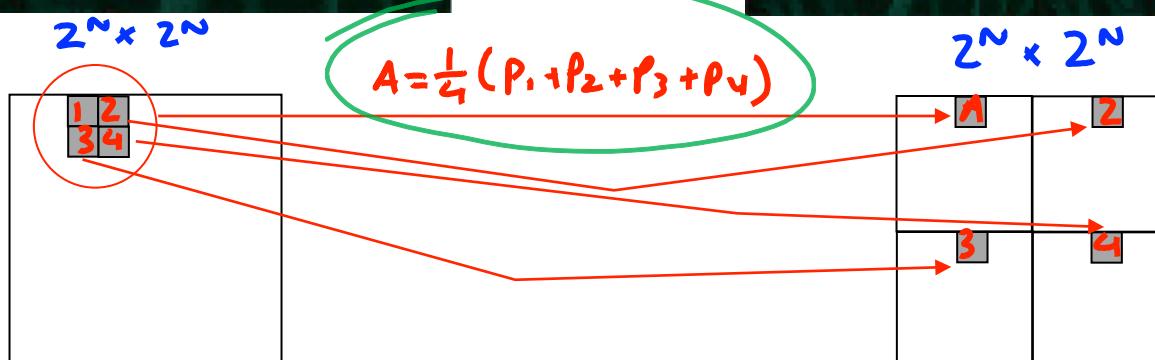
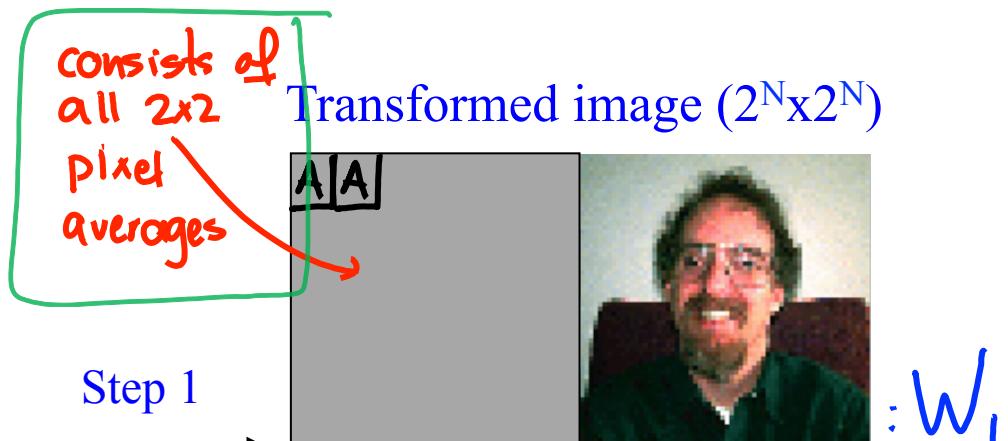
A Simple, Minimal 2-D Image Transform

Step 1: Create 4 new images of size $2^{N-1} \times 2^{N-1}$ as shown in figure

$W_0:$



Input image ($2^N \times 2^N$)



A Simple, Minimal 2-D Image Transform

Step 2: Recursively perform Step 1 for top-left quadrant of result

Result of Step 1 ($2^{N-1} \times 2^{N-1}$)



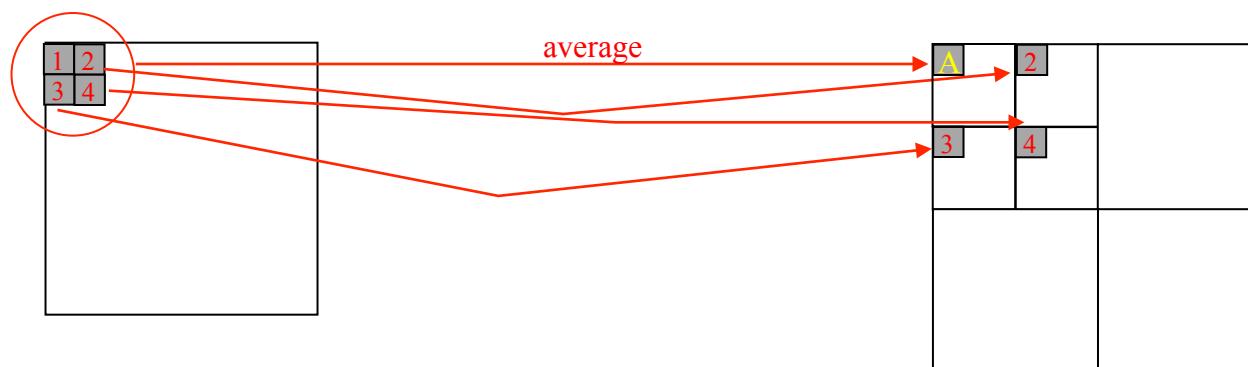
Step 2

Transformed image ($2^N \times 2^N$)



: W_2

$2^{N-1} \times 2^{N-1}$



A Simple, Minimal 2-D Image Transform

Step 3: Recursion stops when average image is 1 pixel

Transformed image ($2^N \times 2^N$)

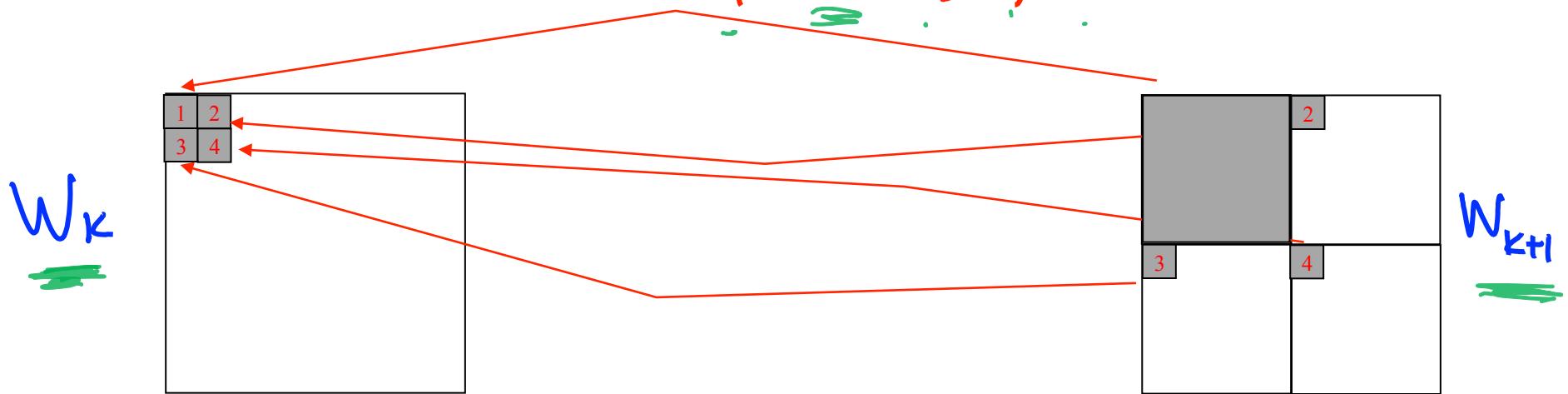


: W_N

Invertibility of the Transformation

Property: W_k can be reconstructed from W_{k+1}

$$P_1 = 4P_A - P_2 - P_3 - P_4$$



$\Leftrightarrow W_0$ reconstructible from W_N

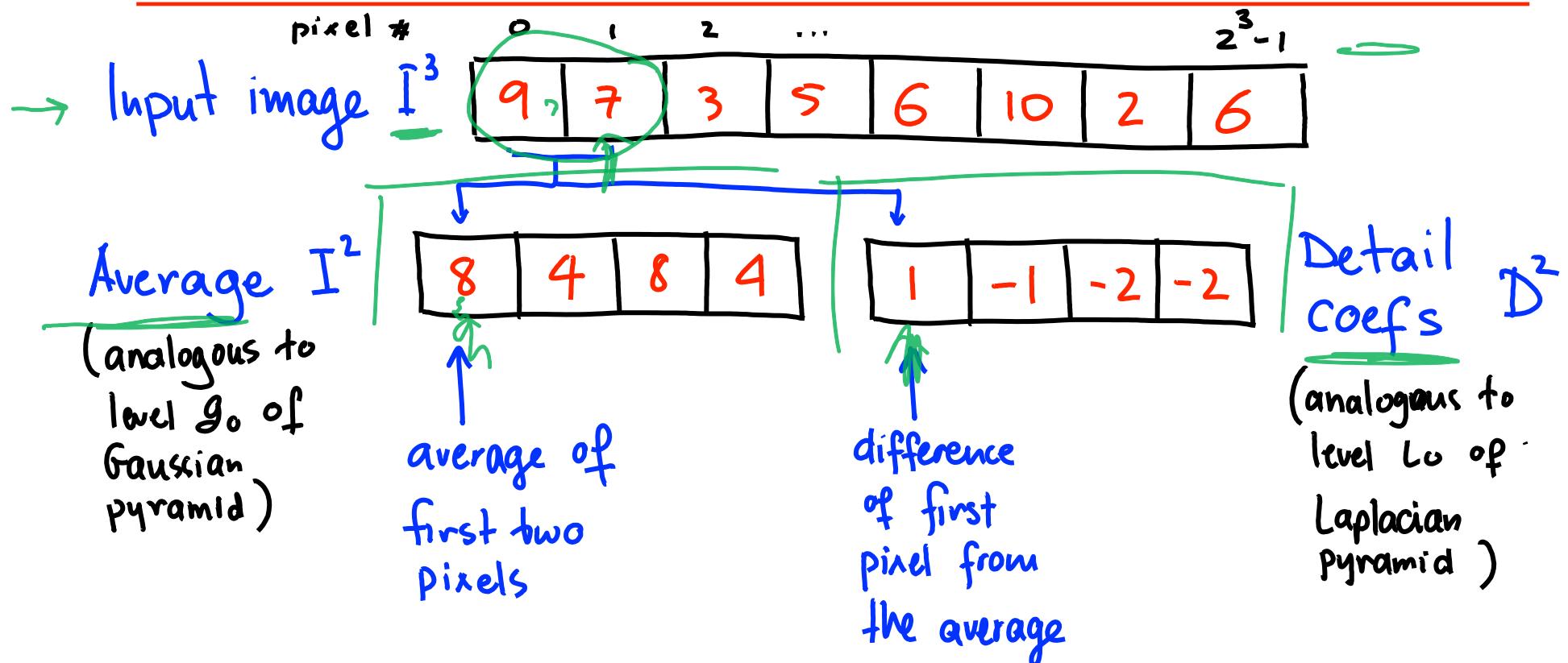


Topic 7:

Discrete Wavelet Transform

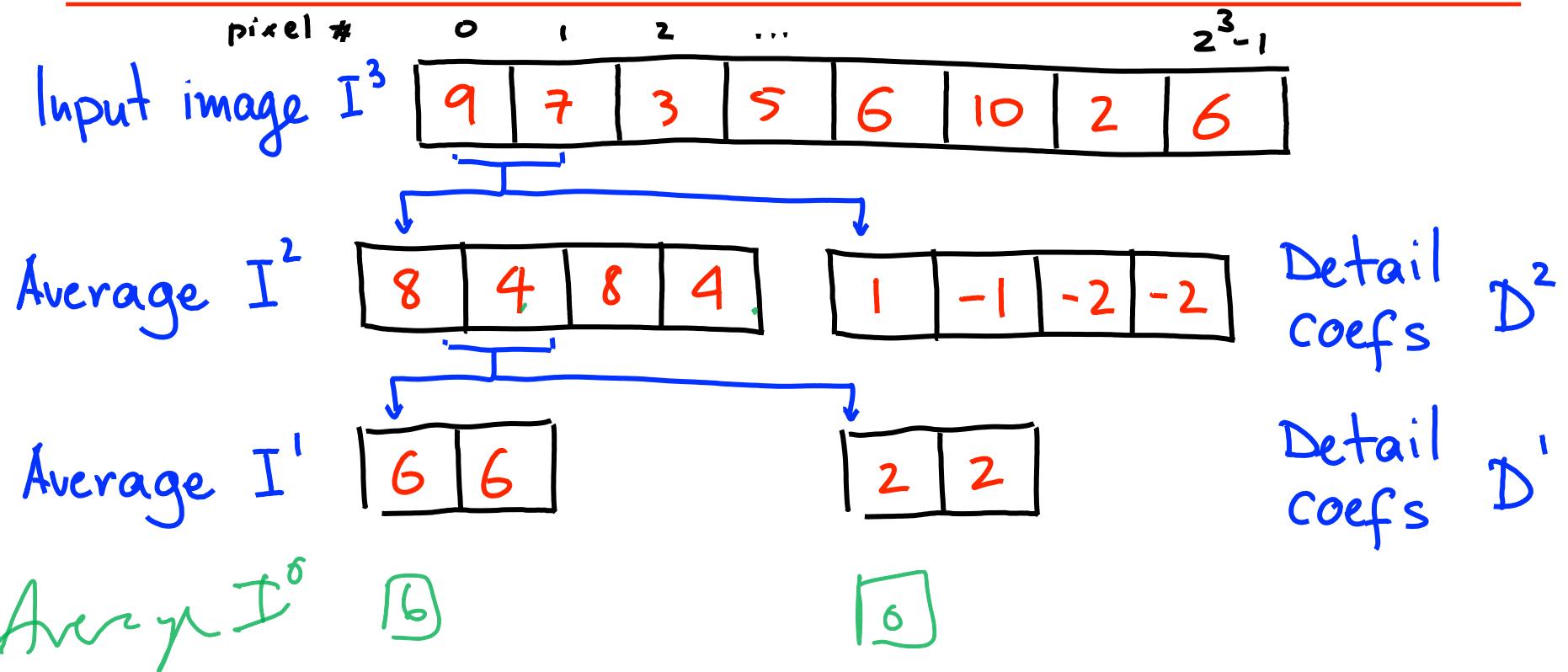
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1D Haar Wavelet Transform: Recursive Definition



(note: we don't need to store difference of 2nd pixel from average \Rightarrow D^0 has $\frac{1}{2}$ the size of the corresponding Laplacian level L_0 !)

1D Haar Wavelet Transform: Recursive Definition



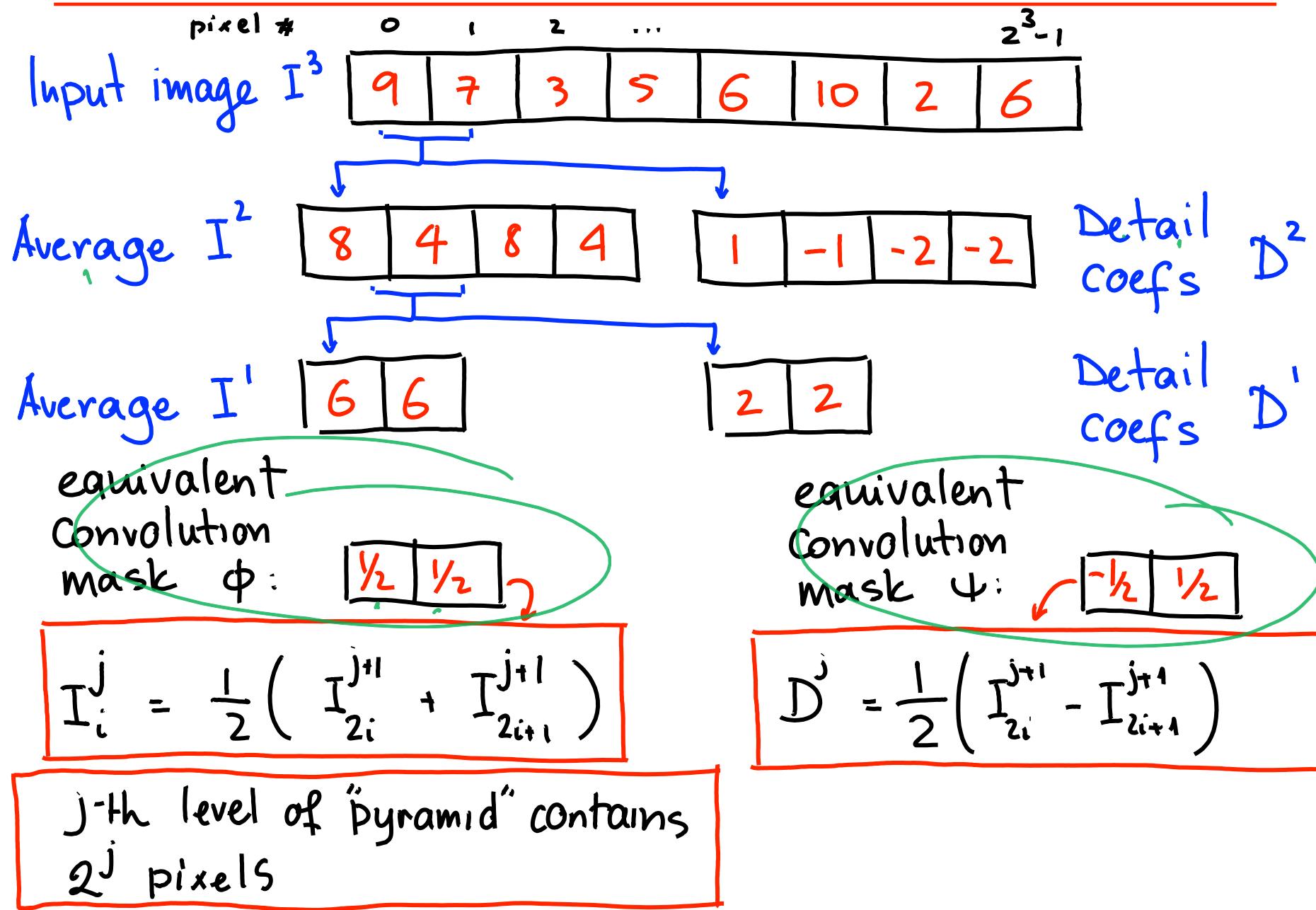
$$I_i^j = \frac{1}{2} (I_{2i}^{j+1} + I_{2i+1}^{j+1})$$

$$D_i^j = I_{2i}^{j+1} - \frac{1}{2} (I_{2i}^{j+1} + I_{2i+1}^{j+1})$$

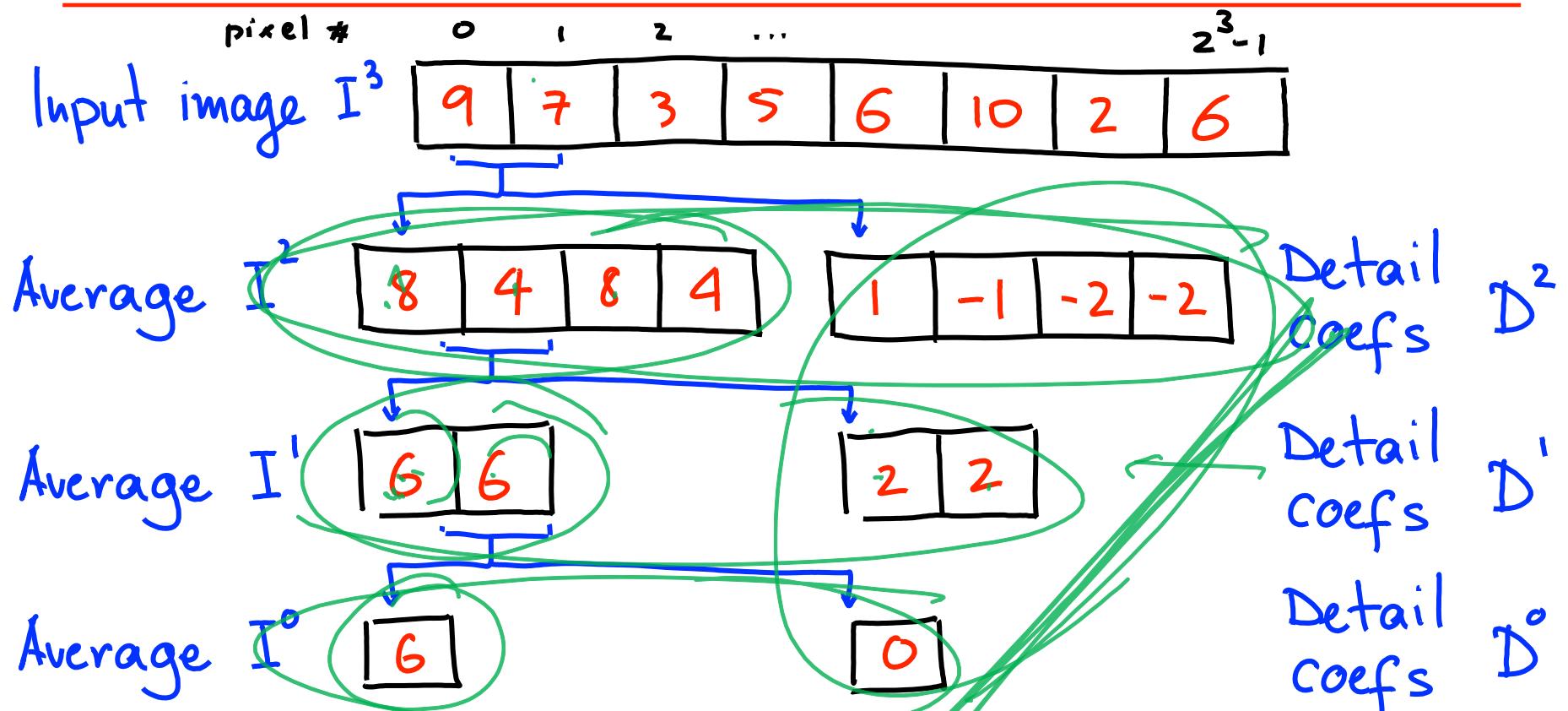
j-th level of "pyramid" contains
 2^j pixels

$$= \frac{1}{2} (I_{2i}^{j+1} - I_{2i+1}^{j+1})$$

1D Haar Wavelet Transform: Recursive Definition



1D Haar Wavelet Transform: Recursive Definition

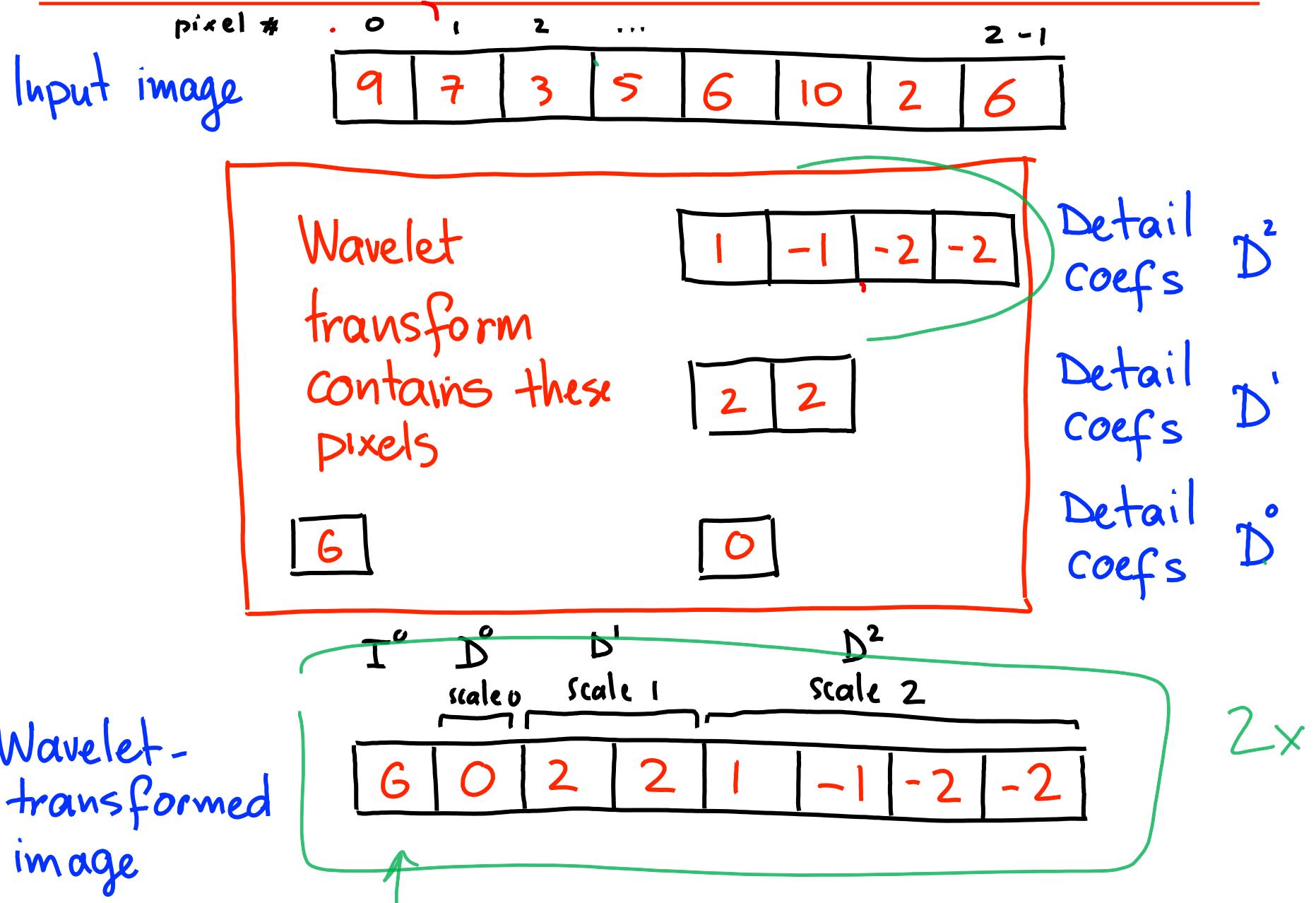


$$I_i^j = \frac{1}{2} \left(I_{2i}^{j+1} + I_{2i+1}^{j+1} \right)$$

$$D^j = \frac{1}{2} \left(I_{2i}^{j+1} - I_{2i+1}^{j+1} \right)$$

j-th level of "pyramid" contains
2^j pixels

1D Haar Wavelet Transform: Recursive Definition

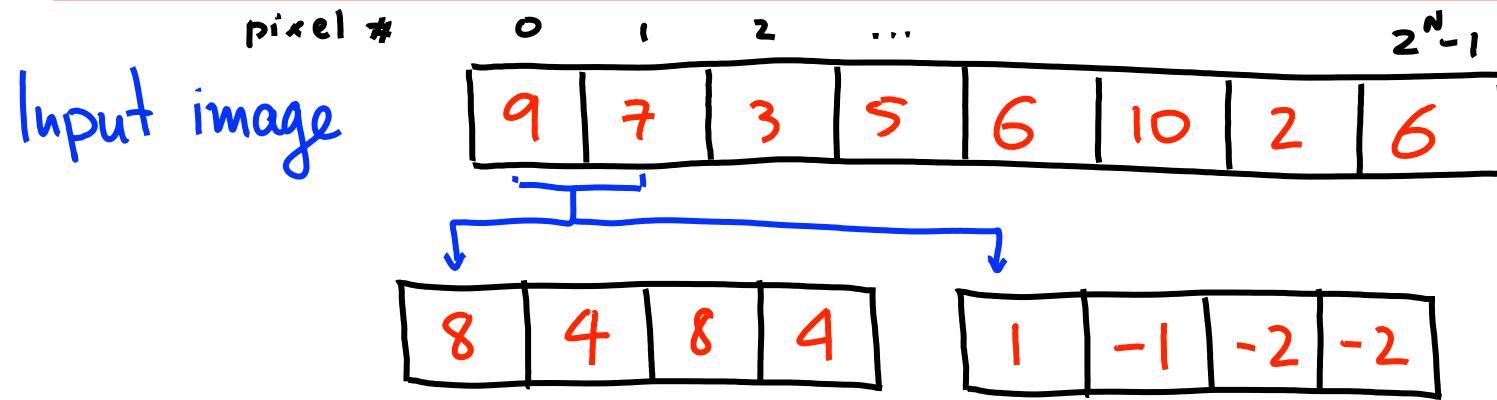


Topic 7:

Discrete Wavelet Transform

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1D Haar Wavelet Transform as a Matrix Product

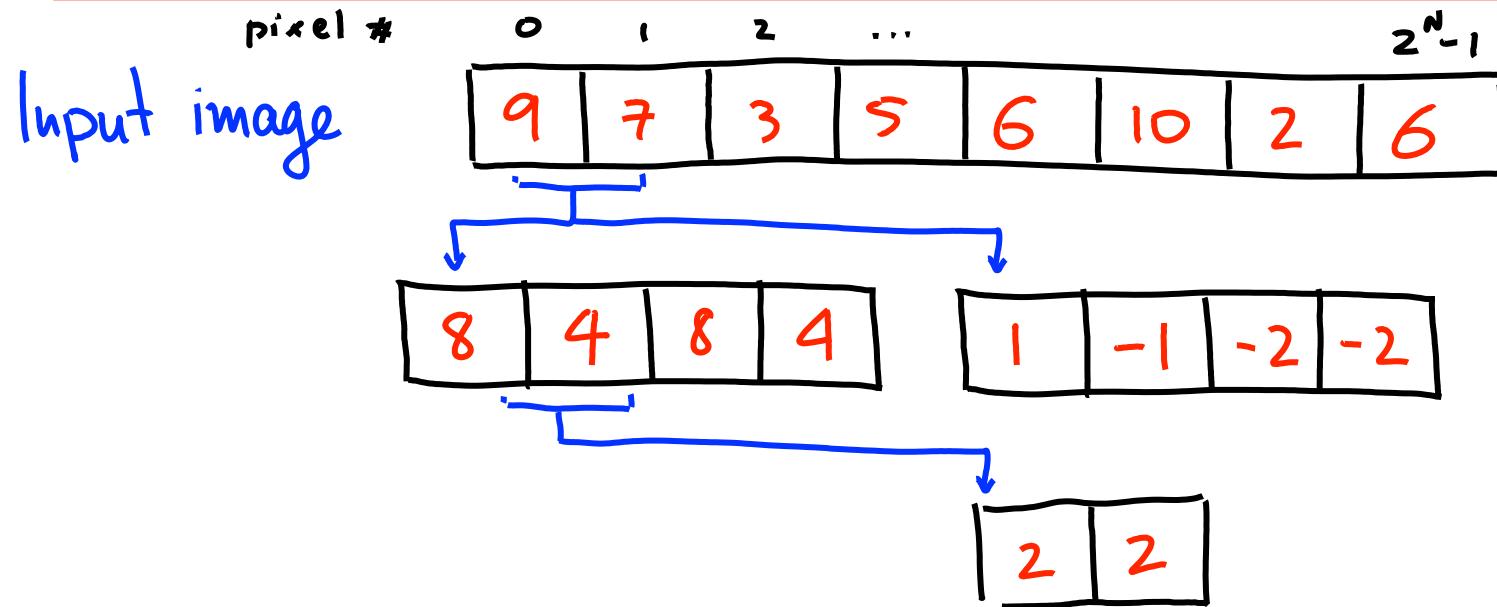


Wavelet transformed image

$$\begin{array}{c}
 \frac{I^0}{D^0} \\
 \frac{D^1}{D^2}
 \end{array}
 \begin{bmatrix}
 6 \\ 0 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \\ -2
 \end{bmatrix}
 = \frac{1}{\sqrt{2}} \begin{bmatrix}
 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
 \end{bmatrix}
 \begin{bmatrix}
 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6
 \end{bmatrix}$$

Original image

1D Haar Wavelet Transform as a Matrix Product



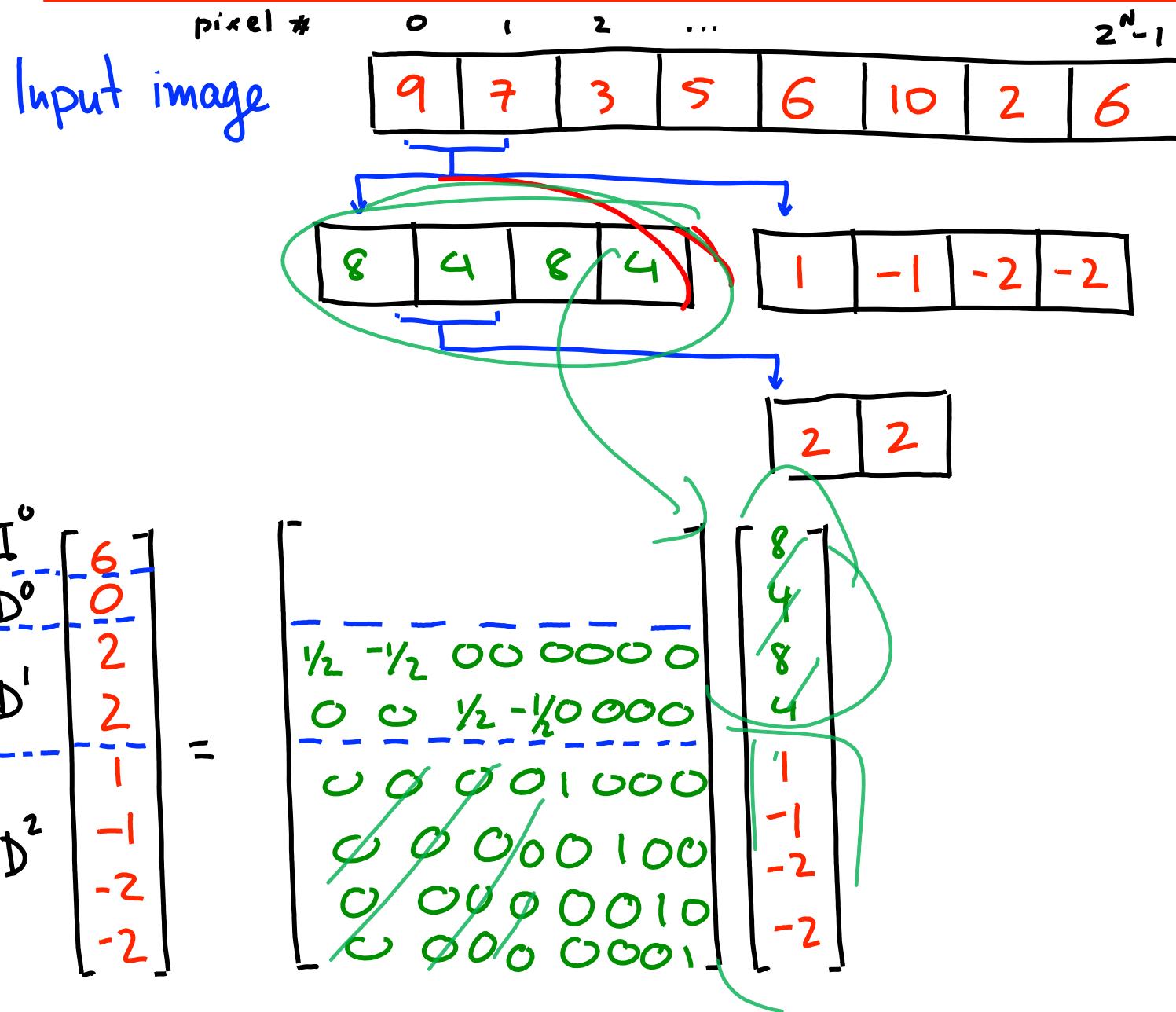
Wavelet transformed image

$$\begin{matrix}
 I^0 \\
 D^0 \\
 D^1 \\
 D^2
 \end{matrix} = \begin{matrix}
 6 \\
 0 \\
 2 \\
 2 \\
 1 \\
 -1 \\
 -2 \\
 -2
 \end{matrix} \cdot \begin{matrix}
 ? \\
 \vdots \\
 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
 \end{matrix} \begin{matrix}
 9 \\
 7 \\
 3 \\
 5 \\
 6 \\
 10 \\
 2 \\
 6
 \end{matrix}$$

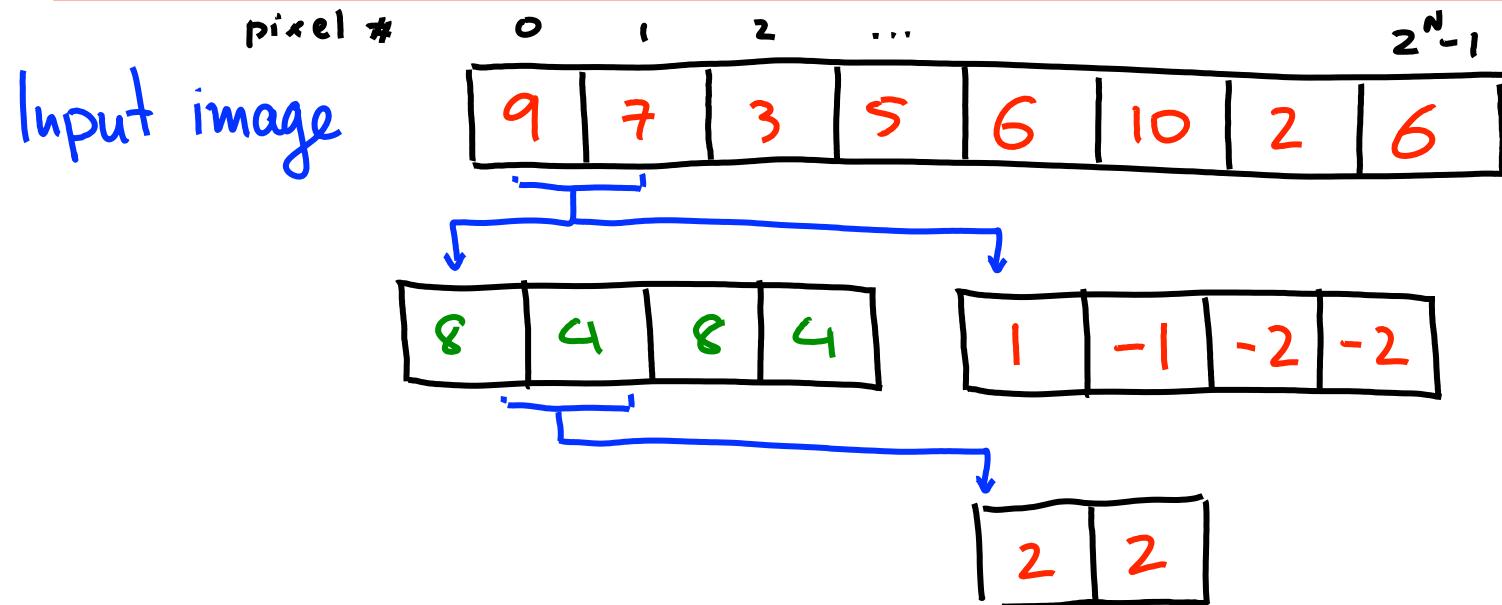
Original image

The equation shows the matrix representation of the 1D Haar Wavelet Transform. The input image is represented as a column vector I^0 . The wavelet transformed image is represented as a column vector containing the coefficients of the approximation and detail components. The transformation matrix is a 8x8 matrix with red entries. The original image is represented as a column vector I^0 .

1D Haar Wavelet Transform as a Matrix Product



1D Haar Wavelet Transform as a Matrix Product



$$\begin{matrix}
 H^0 \\
 D^0 \\
 D^1 \\
 D^2
 \end{matrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{matrix}
 9 \\
 7 \\
 3 \\
 5 \\
 6 \\
 10 \\
 2 \\
 6
 \end{matrix}$$

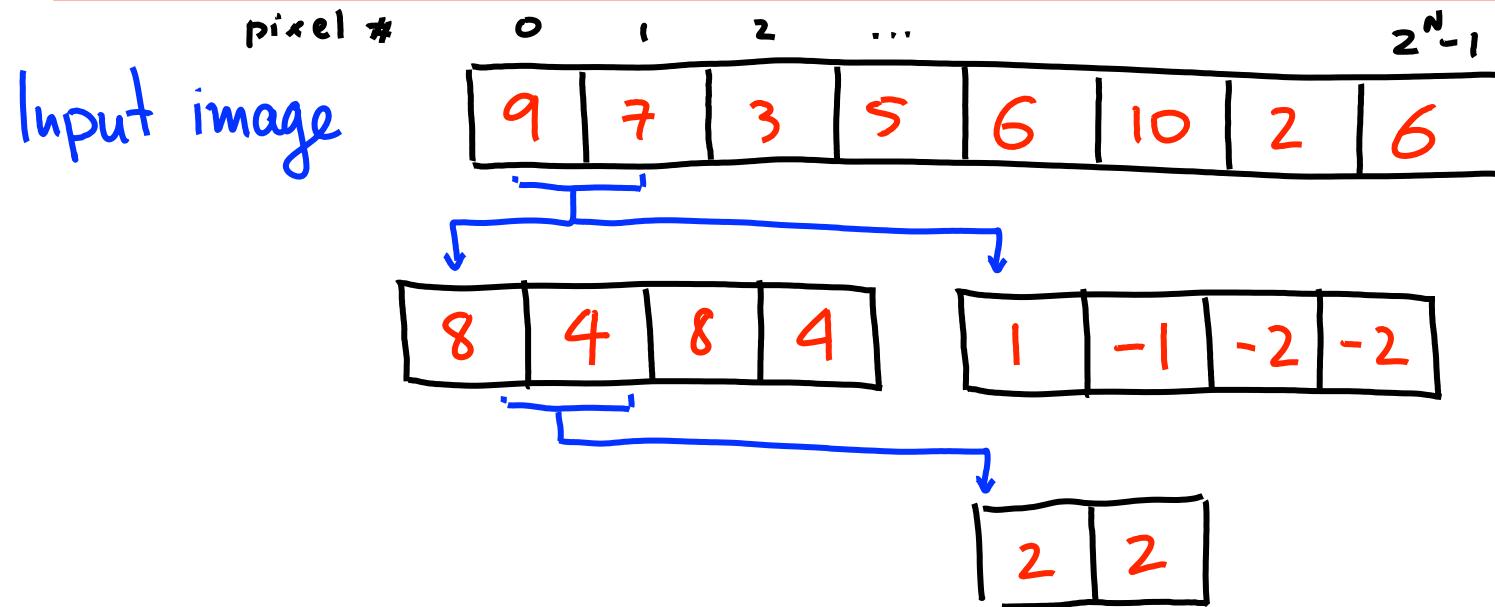
1D Haar Wavelet Transform as a Matrix Product

- 3rd & 4th
rows of
product

$$\begin{matrix}
 I^0 \\
 D^0 \\
 D^1 \\
 D^2 \\
 D^3
 \end{matrix}
 = \begin{matrix}
 A \\
 \frac{1}{2} \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
 \frac{1}{2} \begin{bmatrix} 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \\
 \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \\
 \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}
 \end{matrix} \cdot \begin{matrix}
 B \\
 \frac{1}{2} \\
 \frac{1}{2} \\
 \frac{1}{2} \\
 \frac{1}{2}
 \end{matrix} \cdot \begin{matrix}
 q \\
 7 \\
 3 \\
 5 \\
 6 \\
 10 \\
 2 \\
 6
 \end{matrix}$$

Diagram illustrating the 1D Haar Wavelet Transform as a matrix product. The input vector $\begin{bmatrix} I^0 \\ D^0 \\ D^1 \\ D^2 \\ D^3 \end{bmatrix}$ is multiplied by matrix A , then by matrix B , resulting in the output vector $\begin{bmatrix} q \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}$. Matrix A is a 4x8 matrix with rows: $\frac{1}{4} \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$ and $\frac{1}{4} \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix}$. Matrix B is a 4x4 matrix with diagonal elements $\frac{1}{2}$ and off-diagonal elements 0 or 1. A circled $\frac{1}{2}$ indicates scaling.

1D Haar Wavelet Transform as a Matrix Product



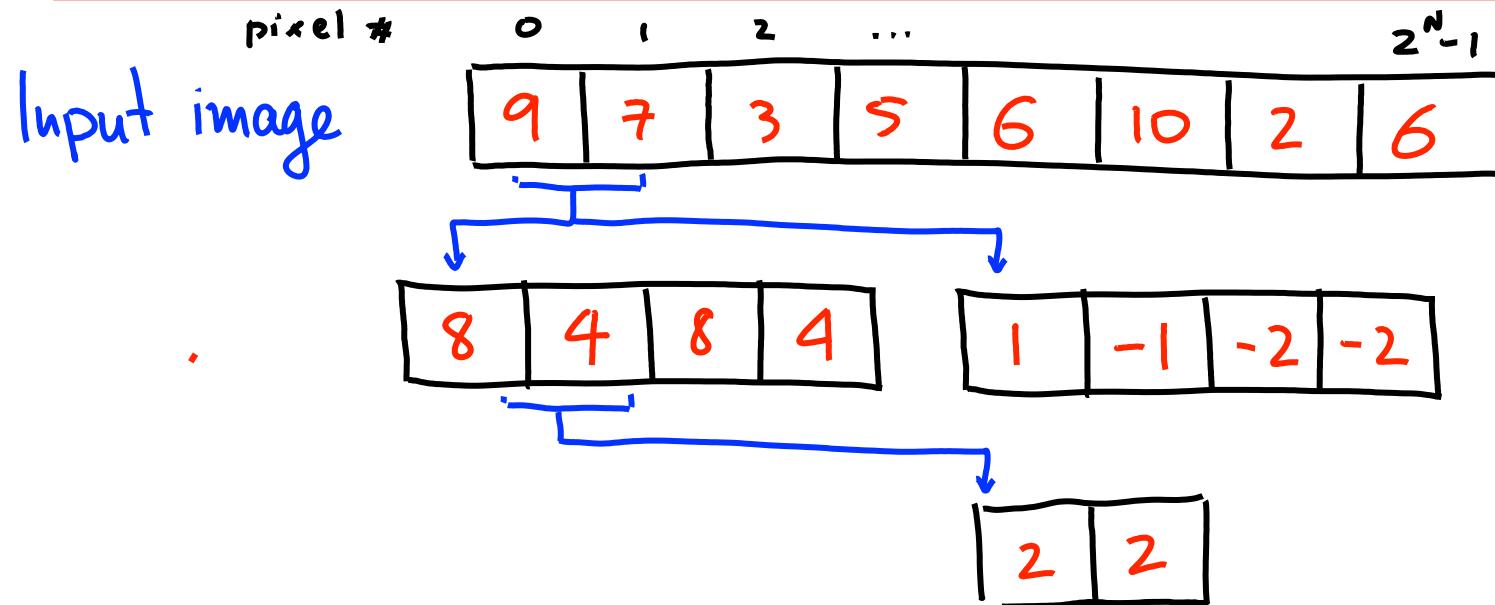
Wavelet transformed image

$$\begin{matrix}
 I^0 \\
 D^0 \\
 D' \\
 D^2
 \end{matrix}
 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \begin{matrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{matrix}$$

Original image

Detailed description: The diagram shows the 1D Haar Wavelet Transform as a matrix product. On the left, the 'Wavelet transformed image' is shown as a vertical stack of four matrices: I^0 , D^0 , D' , and D^2 . The D' matrix is circled in green. To the right, the original image is shown as a vertical stack of 8 red numbers: 9, 7, 3, 5, 6, 10, 2, 6. Between these two stacks is a large matrix equation. The equation consists of three parts: a scalar $\frac{1}{\sqrt{2}}$, a 8x8 matrix with colored dashed lines indicating sub-sampling, and the original image vector. The matrix has colored dashed lines and green arrows indicating the splitting of the original image into approximation and detail components at each level.

1D Haar Wavelet Transform as a Matrix Product



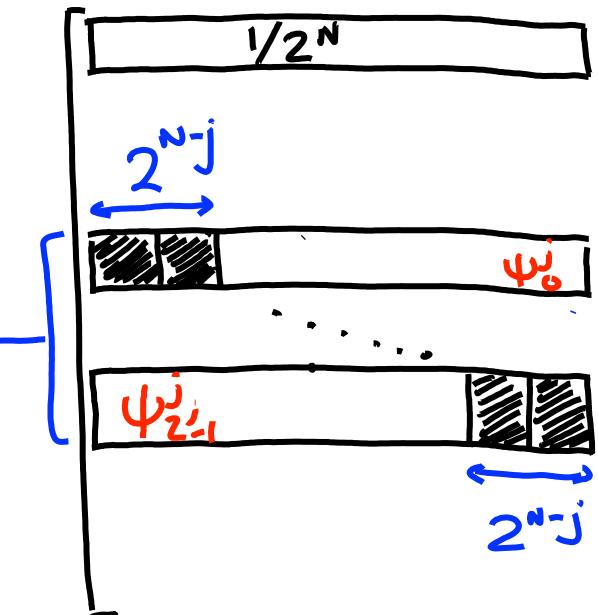
Wavelet transformed image

$$\begin{matrix}
 I^0 \\
 D^0 \\
 D' \\
 D^2
 \end{matrix} = \begin{matrix}
 Y_8 \\
 Y_8 \\
 Y_4 \\
 Y_2
 \end{matrix} = \begin{matrix}
 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\
 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
 \end{matrix} \begin{matrix}
 9 \\
 7 \\
 3 \\
 5 \\
 6 \\
 10 \\
 2 \\
 6
 \end{matrix}$$

Original image

The 1D Haar Wavelet Transform Matrix W

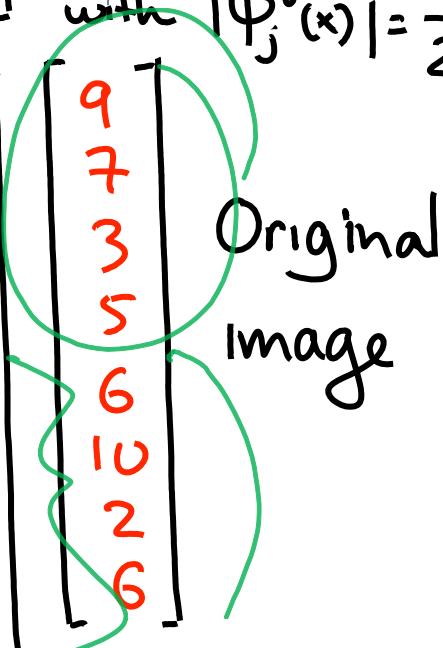
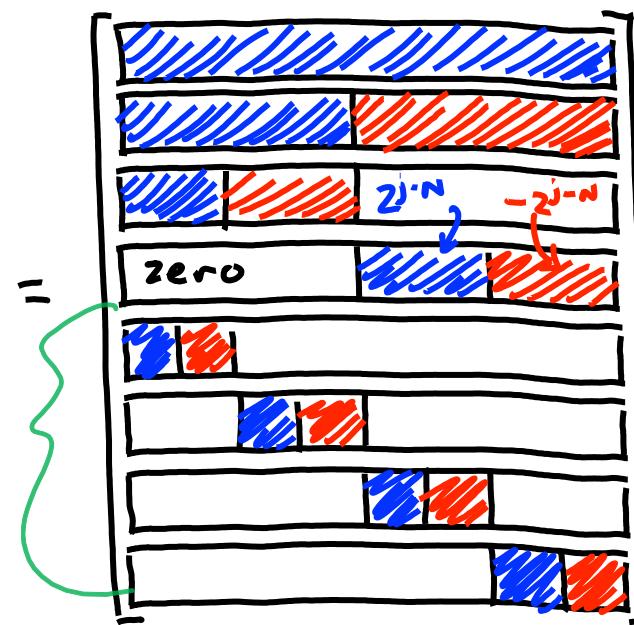
- Matrix contains $N-1$ scales
- Scale j represented by 2^j rows
 $\psi_0^j, \dots, \psi_{2^j-1}^j$



- Row ψ_j^i has $\frac{2^N}{2^j} = 2^{N-j}$ non-zero pixels
- They are pixels $x = i 2^{N-j}, \dots, (i+1)2^{N-j}-1$ with $|\psi_j^i(x)| = \frac{1}{2^{N-j}}$

Wavelet transformed image

$$\begin{bmatrix} I^0 \\ D^0 \\ D^1 \\ D^2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ -1 \\ -1 \\ -2 \\ -2 \end{bmatrix}$$



Original image

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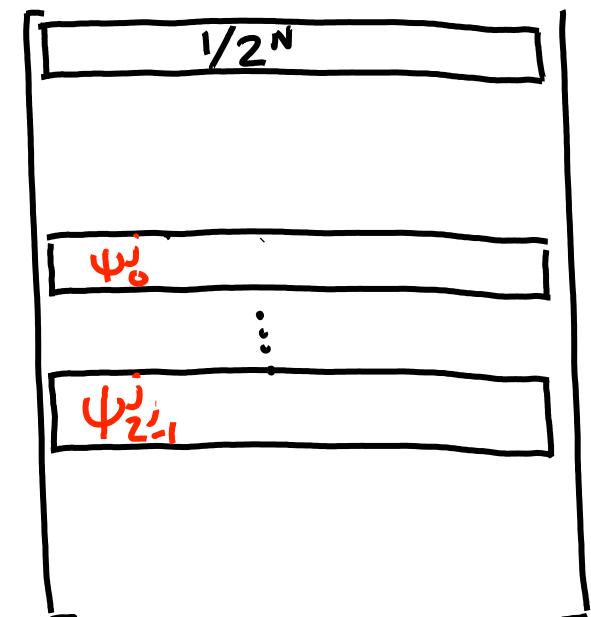
Reconstructing an Image from its Wavelet Coefs

Question: What is the dot product

$$\psi_i^j \cdot \psi_{i'}^{j'}$$

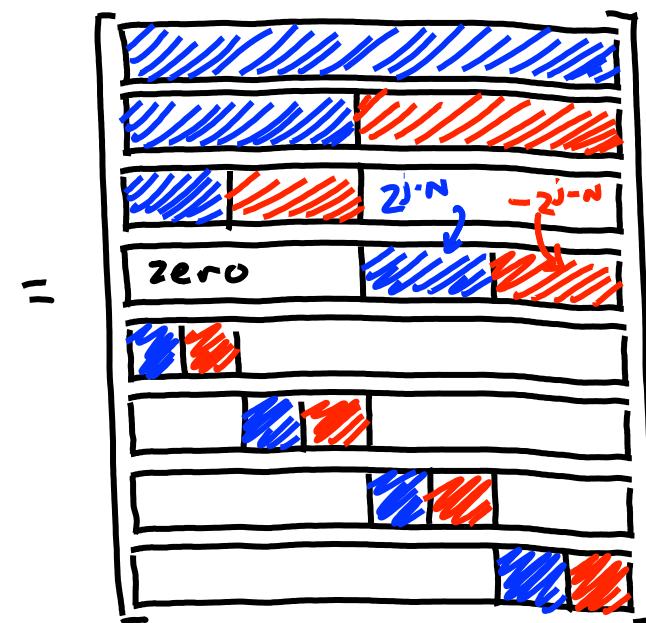
of two distinct rows
of W ?

$$W =$$



Wavelet transformed image

$$\begin{bmatrix} I^0 \\ D^0 \\ D^1 \\ D^2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ -1 \\ -1 \\ -2 \\ -2 \end{bmatrix}$$



Original image

$$\begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}$$

Reconstructing an Image from its Wavelet Coefs

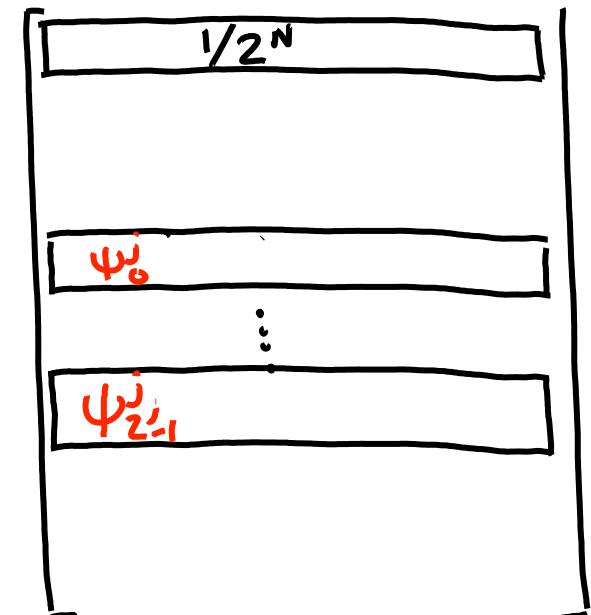
Answer:

$$\psi_i^j \cdot \psi_{i'}^{j'} = 0$$

for two distinct rows
of $W \iff$

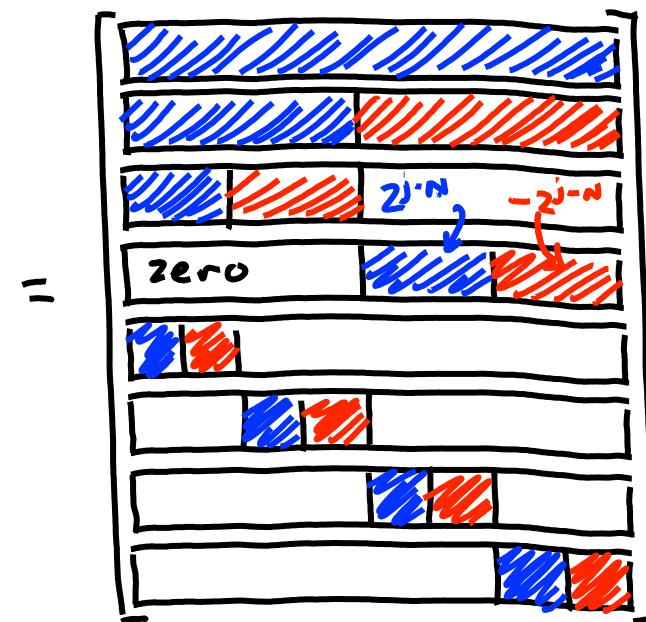
$$WW^T = \text{diagonal}$$

$$W =$$



Wavelet transformed image

$$\begin{bmatrix} I^0 \\ D^0 \\ D^1 \\ D^2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ -1 \\ -1 \\ -2 \\ -2 \end{bmatrix}$$



Original image

Reconstructing an Image from its Wavelet Coefs

$$WW^T = \text{diagonal}$$
$$\left(\psi_i^j \right) \cdot \left(\psi_i^j \right)^T = \frac{1}{2^{N-j}}$$

$W =$

ψ_0^j

\vdots

$\psi_{2^j-1}^j$

Wavelet transformed image

$$\begin{bmatrix} I^0 \\ D^0 \\ D^1 \\ D^2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ -1 \\ -1 \\ -2 \\ -2 \end{bmatrix}$$

=

2^{j-N}

-2^{j-N}

zero

Original image

$$\begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}$$

Reconstructing an Image from its Wavelet Coefs

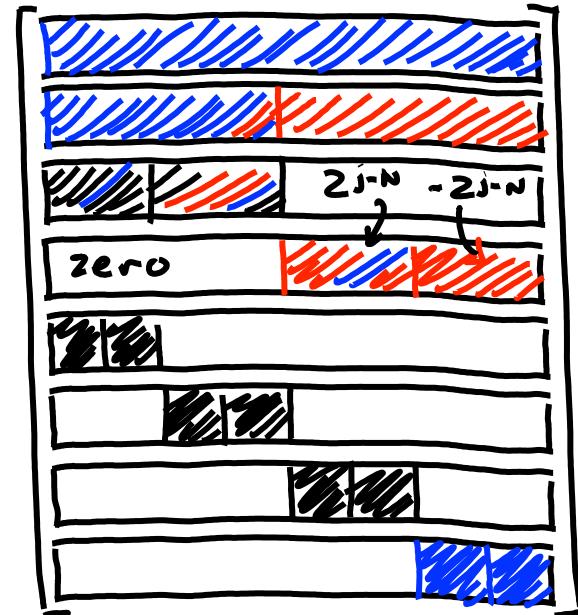
Define $\Lambda = WW^T$

with

$$\Lambda = \begin{bmatrix} D_D & & \\ & \ddots & \\ & & D_{2^{N-1}} \end{bmatrix}$$

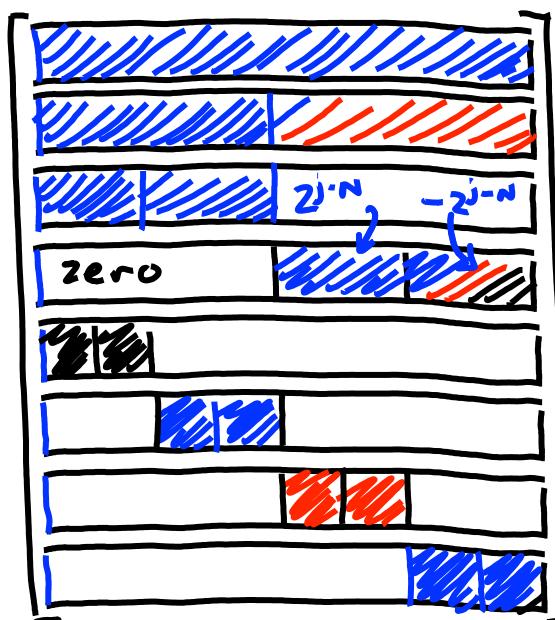
of the form $\frac{1}{2^{N-j}}$

$W =$



Multiply W^T on both sides:

$$W^T \cdot \begin{bmatrix} -6 \\ 0 \\ 2 \\ 2 \\ -1 \\ -1 \\ -2 \\ -2 \end{bmatrix} = W^T \cdot$$



$$\begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}$$

Original
image

Reconstructing an Image from its Wavelet Coefs

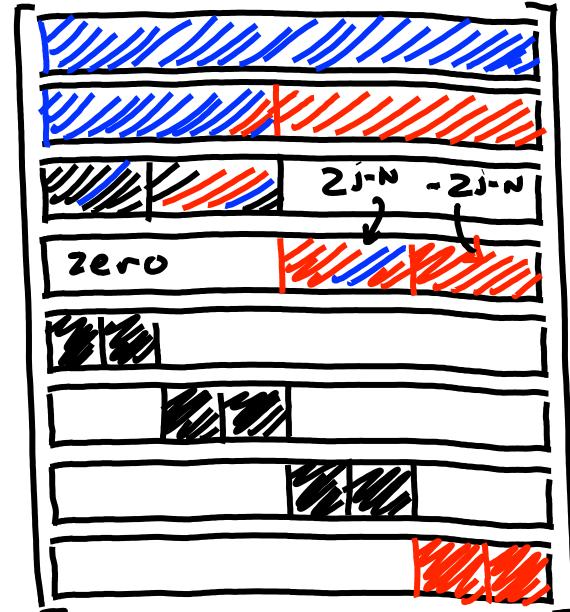
Define $\Lambda = WW^T$

with

$$\Lambda = \begin{bmatrix} D_D & & \\ & \ddots & \\ & & 0 \\ & & & \ddots \\ & & & & \lambda_{2^{N-1}} \end{bmatrix}$$

of the form $\frac{1}{2^{N-j}}$

$W =$

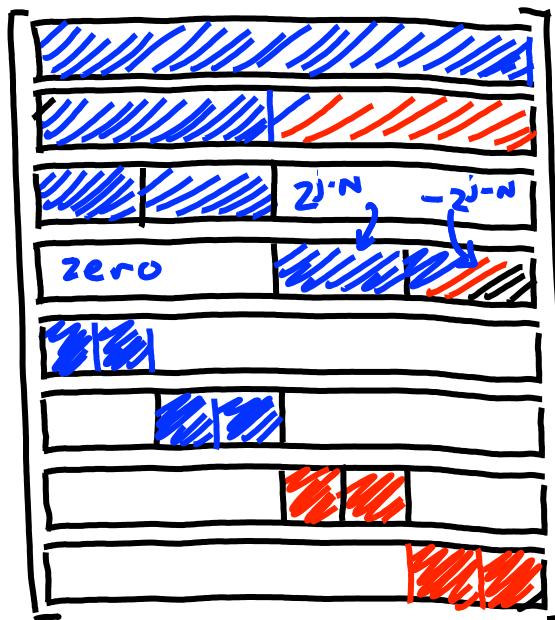


Multiply $\Lambda^{-1} W^T$ on both sides:

$$\Lambda^{-1} W^T = \Lambda^{-1} W^T$$

Vertical vector on the left:

6
0
2
2
-1
-1
-2
-2



9
7
3
5
6
10
2
6

Original
image

Reconstructing an Image from its Wavelet Coefs

Observation:

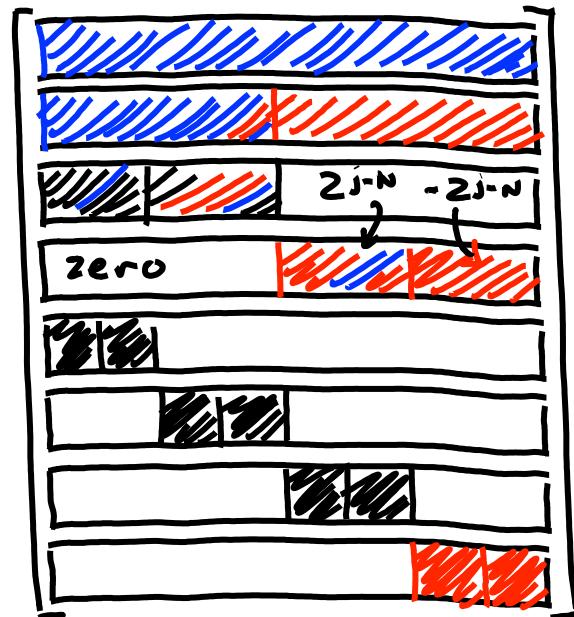
$\tilde{\Lambda}^{1/2} \cdot W$ is orthogonal because

$$(\tilde{\Lambda}^{1/2} \cdot W) (\tilde{\Lambda}^{1/2} \cdot W)^T =$$

$$\tilde{\Lambda}^{-1/2} \cdot W \cdot W^T \cdot \tilde{\Lambda}^{-1/2} =$$

$$\tilde{\Lambda}^{-1/2} \cdot \Lambda \cdot \tilde{\Lambda}^{-1/2} = I$$

$$W =$$



Therefore

$$\tilde{\Lambda}^{-1} W^T \cdot \begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \\ -2 \end{bmatrix} =$$

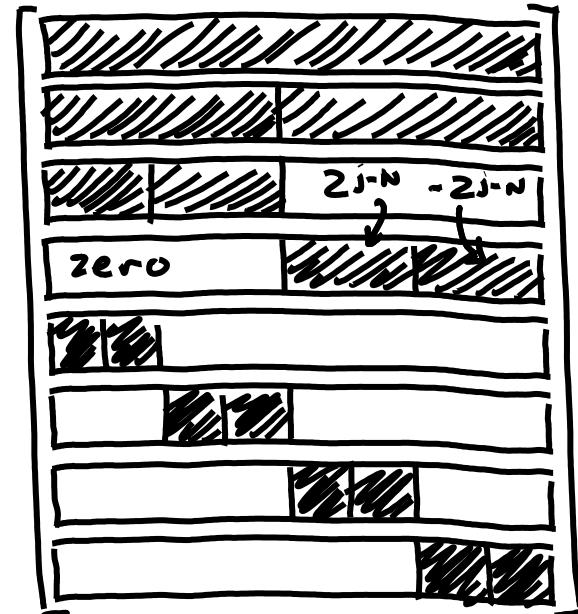
$$\begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}$$

Original
Image

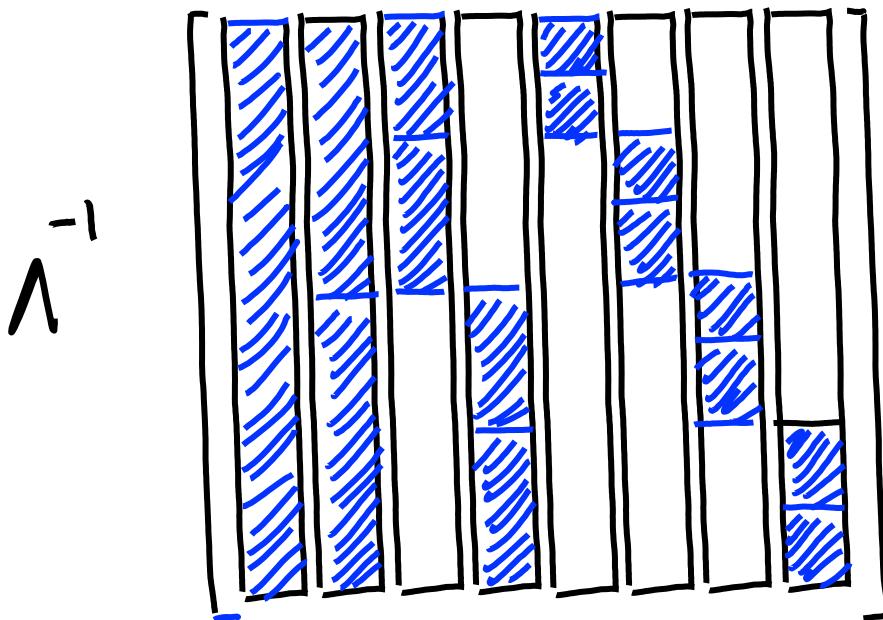
Reconstructing an Image from its Wavelet Coefs

$$\hat{\Lambda}^{-1} = \begin{bmatrix} -1 & & \\ \mathcal{D}_p & \ddots & 0 \\ 0 & \ddots & -1 \\ & & \mathcal{D}_{2^{N-1}} \end{bmatrix}$$

$$W =$$



So we have



$$\begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ -1 \\ -1 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}$$

Original
Image

Interpreting the Wavelet Coefficients

$$I = \frac{D_0^{-1}}{D_{2^n-1}^{-1}} \cdot \begin{matrix} 6 \\ 0 \\ 2 \\ 2 \\ 1 \\ (-1) \\ (-2) \\ D_{2^n-1}^{-1} \end{matrix} \cdot \begin{matrix} + \\ + \\ + \\ + \\ + \\ + \\ + \end{matrix}$$

$$W = \begin{matrix} + \\ + \\ zero \\ + \\ + \\ + \\ + \end{matrix} \cdot \begin{matrix} 2^{j-N} & -2^{j-N} \\ \downarrow & \swarrow \end{matrix} \cdot \begin{matrix} + \\ + \\ + \\ + \\ + \\ + \\ + \end{matrix}$$

$$\begin{matrix} D_0^{-1} \\ \vdots \\ D_{2^n-1}^{-1} \end{matrix} = \begin{matrix} 6 \\ 0 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \\ -2 \end{matrix} = \begin{matrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{matrix}$$

Original
Image

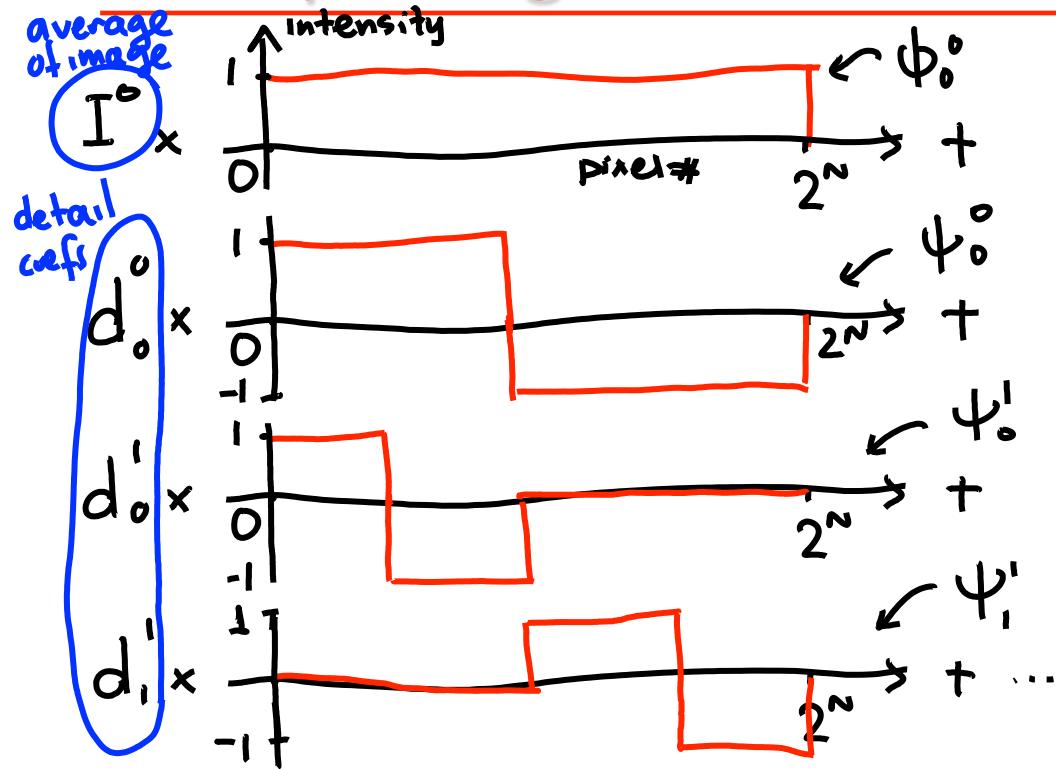
Interpreting the Wavelet Coefficients

$$I = \sum_{j=0}^{-1} 6x + \sum_{j=0}^{-1} 0x + \sum_{j=0}^{-1} 2x + \sum_{j=0}^{-1} 2x + \sum_{j=0}^{-1} 1x + \sum_{j=0}^{-1} (-1)x + \sum_{j=0}^{-1} (-2)x + \sum_{j=0}^{-1} (-2)x$$

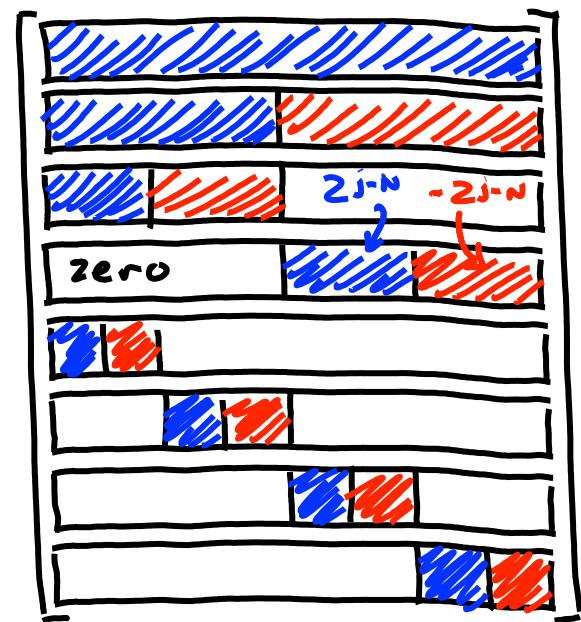
$$W =$$

⇒ By multiplying I with W we obtain a decomposition of the image into a sequence of basis images $\psi_0^0, \psi_0^1, \dots, \psi_j^i, \dots$ that form an orthogonal basis of \mathbb{R}^{2^N}

Interpreting the Wavelet Coefficients



$W =$



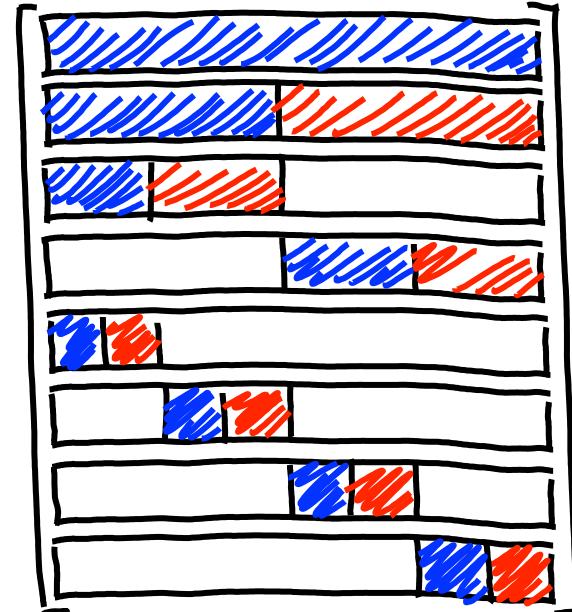
\Rightarrow The wavelet coefficients are the coordinates of the image, considered as a vector in \mathbb{R}^{2^N} , in the basis defined by images $\phi_0^0, \psi_0^0, \psi_0^1, \dots$

The Normalized Haar Wavelet Matrix

We can normalize the wavelet transform matrix by multiplying

$$\tilde{W} = \begin{bmatrix} \sqrt{\alpha_1} & & \\ & \ddots & \\ & & \sqrt{\alpha_{2^n-1}} \end{bmatrix} \cdot W$$

$$\tilde{W} =$$



normalized
wavelet coefficients

$$\begin{bmatrix} c_0 \\ d_0 \\ d_1 \\ \vdots \end{bmatrix}$$

$$=$$

$$\tilde{W} \cdot$$

$$\begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}$$

Original
image

The Normalized Haar Wavelet Coefficients

$$I = \begin{matrix} C_0^0 & \times & \Phi_0^0 \\ d_0^0 & \times & + \\ d_0^1 & \times & + \\ d_1^0 & \times & + \\ d_1^1 & \times & + \\ d_0^2 & \times & + \\ d_1^2 & \times & + \\ d_2^2 & \times & + \\ d_3^2 & \times & \end{matrix}$$

The diagram illustrates the convolution of an input image I with a set of wavelet basis functions $\Phi_0^0, \Phi_0^1, \Phi_1^0, \Phi_1^1, \Phi_0^2, \Phi_1^2, \Phi_2^2, \Phi_3^2$. The input I is a 9x9 grid. The basis functions are 3x3 grids. The convolution results in coefficients $C_0^0, d_0^0, d_0^1, d_1^0, d_1^1, d_0^2, d_1^2, d_2^2, d_3^2$.

$$\tilde{W} = \begin{matrix} \Phi_0^0 \\ \Phi_0^1 \\ \Phi_1^0 \\ \Phi_1^1 \\ \Phi_0^2 \\ \Phi_1^2 \\ \Phi_2^2 \\ \Phi_3^2 \end{matrix}$$

The diagram illustrates the normalized Haar wavelet coefficients \tilde{W} , which are the result of multiplying the input I with the basis functions in W . The coefficients are represented as 3x3 grids.

⇒ By multiplying I with \tilde{W} we obtain a set of wavelet coefficients C_0^0, d_0^0, \dots that express I as a linear combination of the basis images $\Phi_0^0, \Phi_0^1, \Phi_1^0, \Phi_1^1, \Phi_0^2, \Phi_1^2, \Phi_2^2, \Phi_3^2$.

Topic 7:

Discrete Wavelet Transform

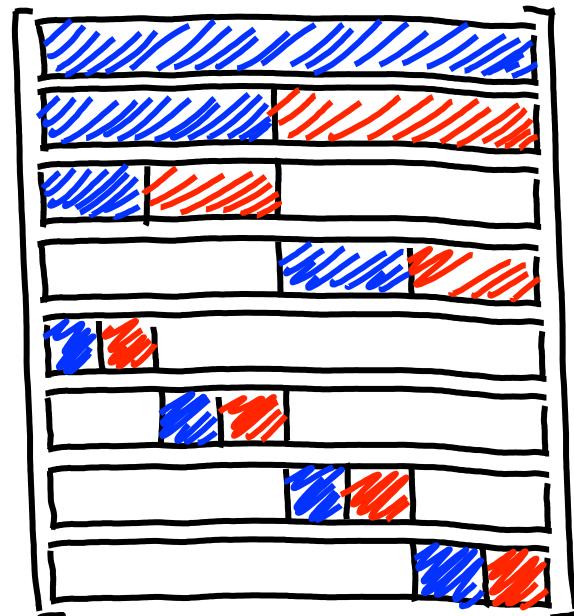
- Wavelet transform vs. Laplacian pyramid
- Basic intuition: a simple wavelet-like 2D transform
- The 1D Haar wavelet transform
- 1D Haar wavelet transform as a matrix product
- Reconstructing a 1D image from its wavelet coeffs
- Wavelet-based image compression
- The 2D Haar wavelet transform

Wavelet Compression Algorithm #1

Input: 1D image I , desired compression K
Output: $K \cdot 2^N$ coefficients

- ① Compute $\tilde{W} I$
- ② Sort the coefficients c_0, d_0, d_1, \dots in order of decreasing absolute value
- ③ keep the top $K \cdot 2^N$ coeffs

$$\tilde{W} =$$



* Readings show that the algorithm gives the best least-squares approx of the image for the given compression level

$$\begin{bmatrix} c_0 \\ d_0 \\ d_1 \\ \vdots \end{bmatrix} = \tilde{W} \cdot \begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}$$

Original image I

Wavelet Compression Algorithm #2

Input: 1D image I , max error ϵ
Output: $K \cdot 2^N$ coefficients

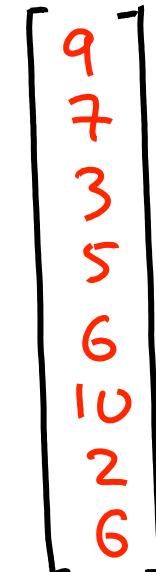
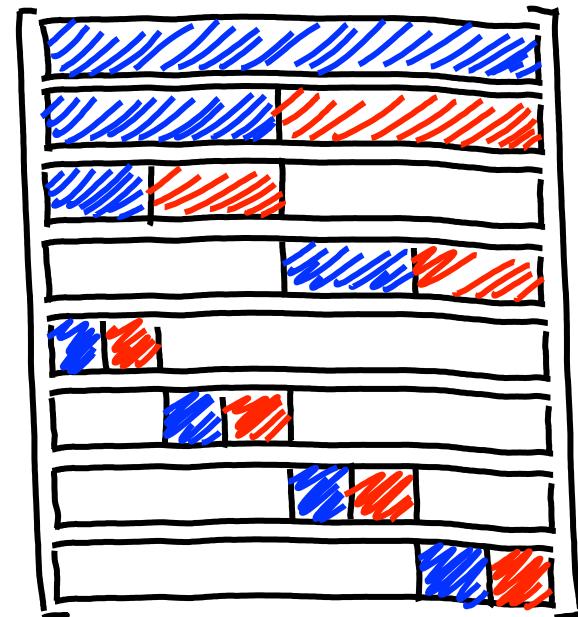
- ① Compute $\tilde{W} I$
- ② Sort the coefficients c_0, d_0, d_1, \dots in order of decreasing absolute value
- ③ keep the top $K \cdot 2^N$ coeffs

with K such that
 $|I - \tilde{I}| < \epsilon$,
where \tilde{I} is the image reconstructed from the top $K \cdot 2^N$ coeffs

normalized wavelet coefficients

$$\begin{bmatrix} c_0 \\ d_0 \\ d_1 \\ \vdots \end{bmatrix} = \tilde{W} \cdot$$

$$\tilde{W} =$$



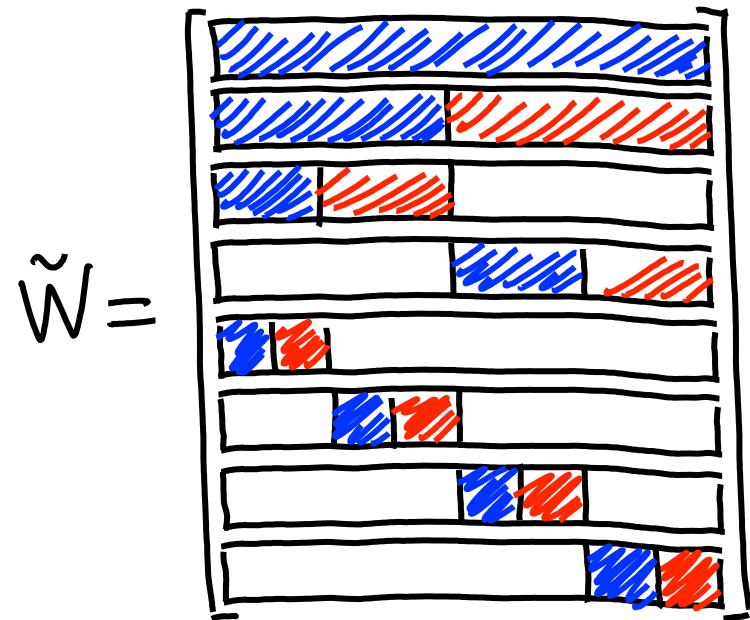
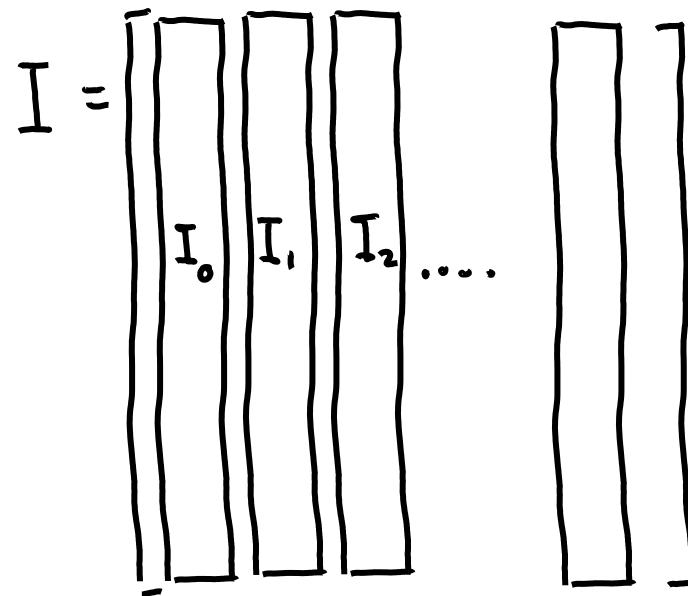
Original
image I

Topic 7:

Discrete Wavelet Transform

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The 2D Haar Wavelet Transform

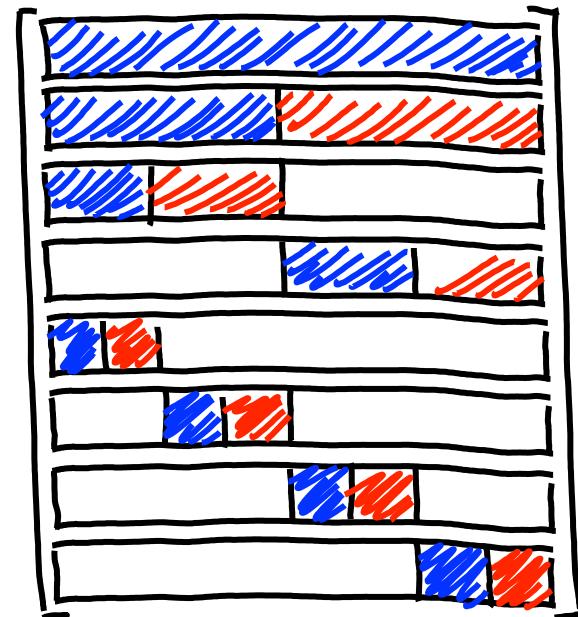


- To compute the wavelet transform of a 2D image:
 - ① Compute the 1D transform for each column and place the vectors $\tilde{W}I_i$ in a new image I'
 - ② Compute the 1D transform of each row of I'

The 2D Haar Wavelet Transform

Exercise: Show that every 2D wavelet coefficient can be expressed as the result of a dot product of the image I and an image defined by $(\psi_i^j)^T \cdot (\psi_i^j)$ where ψ_i are 1D Haar basis images

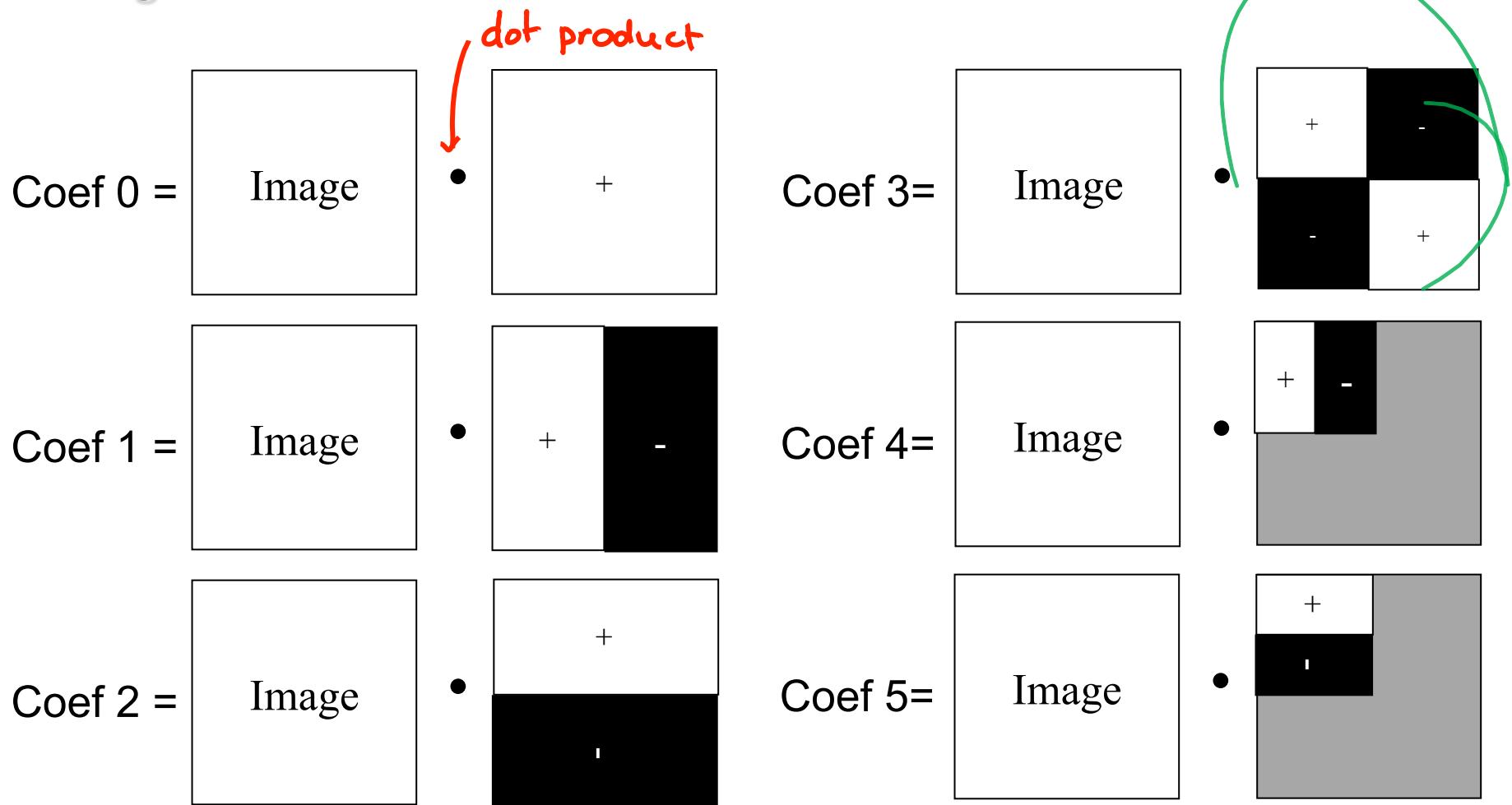
$$\tilde{W} =$$



- To compute the wavelet transform of a 2D image:
 - ① Compute the 1D transform for each column and place the vectors $\tilde{W}I_i$ in a new image I'
 - ② Compute the 1D transform of each row of I'

The 2-D Haar Wavelet Basis

Definition of the first few (coarsest scale) wavelet coefficients of an image of dimensions of $2^N \times 2^N$

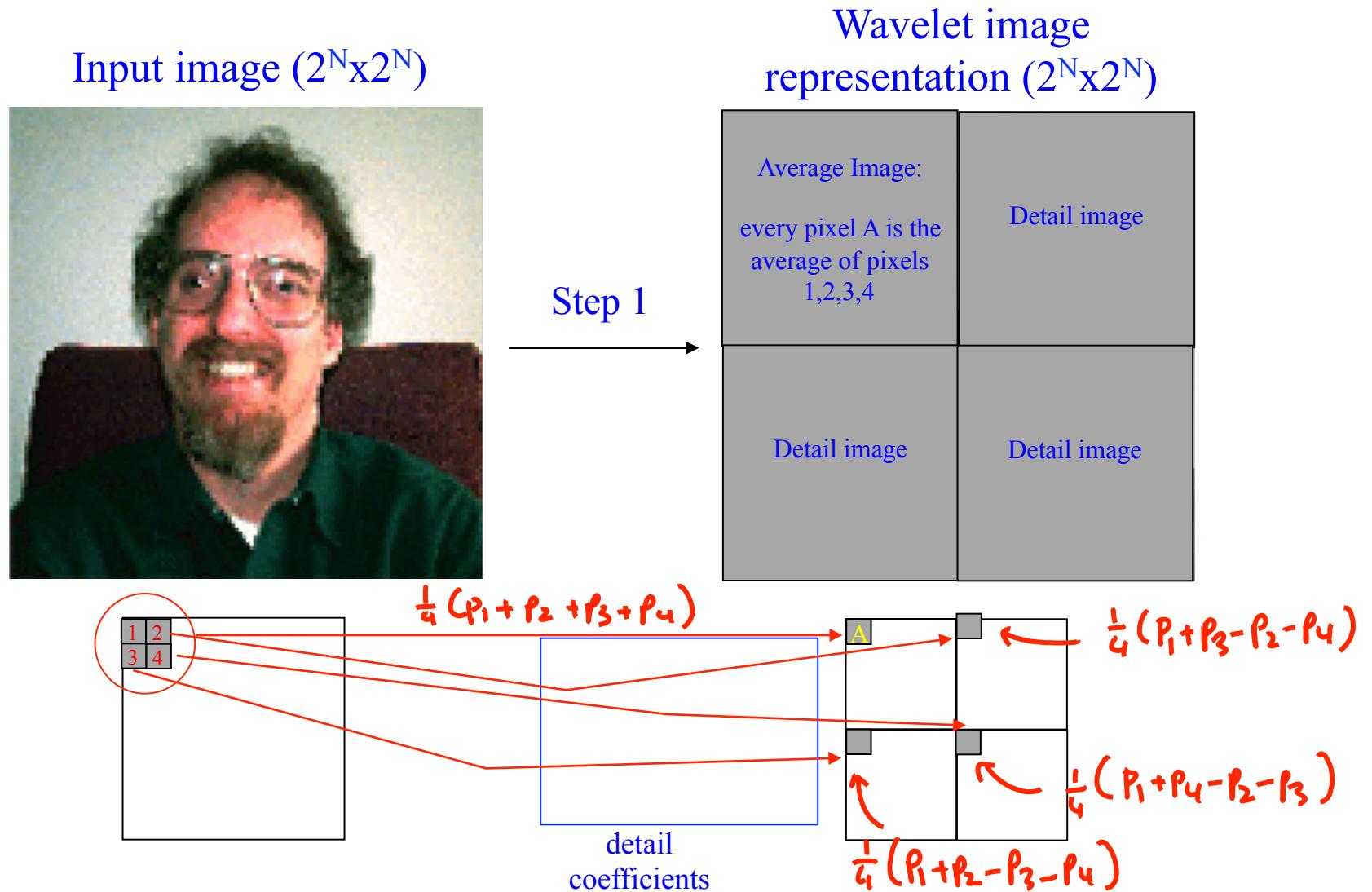


A Simple, Minimal 2-D Image Transform



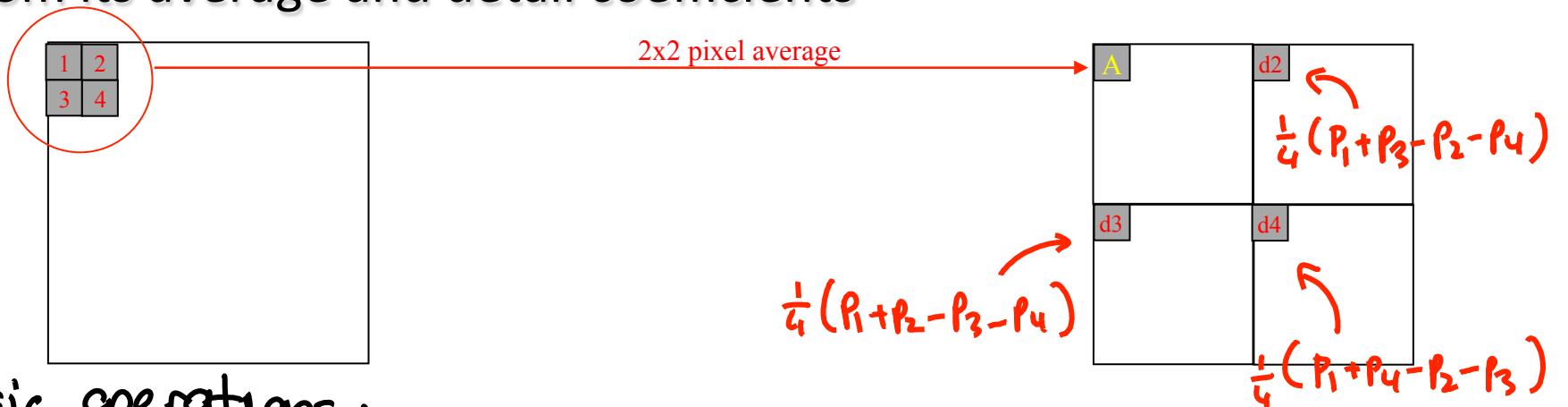
The Haar 2-D Wavelet Transform

The 2-D Haar Wavelet Transform corresponds to a modification of this minimal recursive transform



Invertibility of the 2D Haar Transform

We can recursively reconstruct the intensities of every 2x2 window from its average and detail coefficients



2 basic operations :

- sum of 4 pixels
- difference of pairwise sums of pixels

$$P_1 = A + d_2 + d_3 + d_4$$

