Topic 7:

Discrete Wavelet Transform

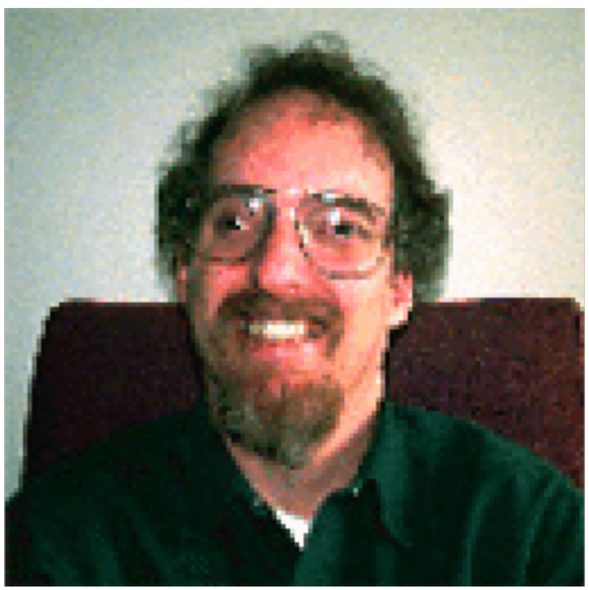
- Wavelet transform vs. Laplacian pyramid
- Basic intuition: a simple wavelet-like 2D transform
- The 1D Haar wavelet transform
- 1D Haar wavelet transform as a matrix product
- Reconstructing a 1D image from its wavelet coefs
- Wavelet-based image compression
- The 2D Haar wavelet transform

A Simple, Minimal 2-D Image Transform

Properties of transformation

- Minimal (no "wasted" pixels)
- Multiple scales represented simultaneously
- Invertible, linear

Input image (2^Nx2^N)

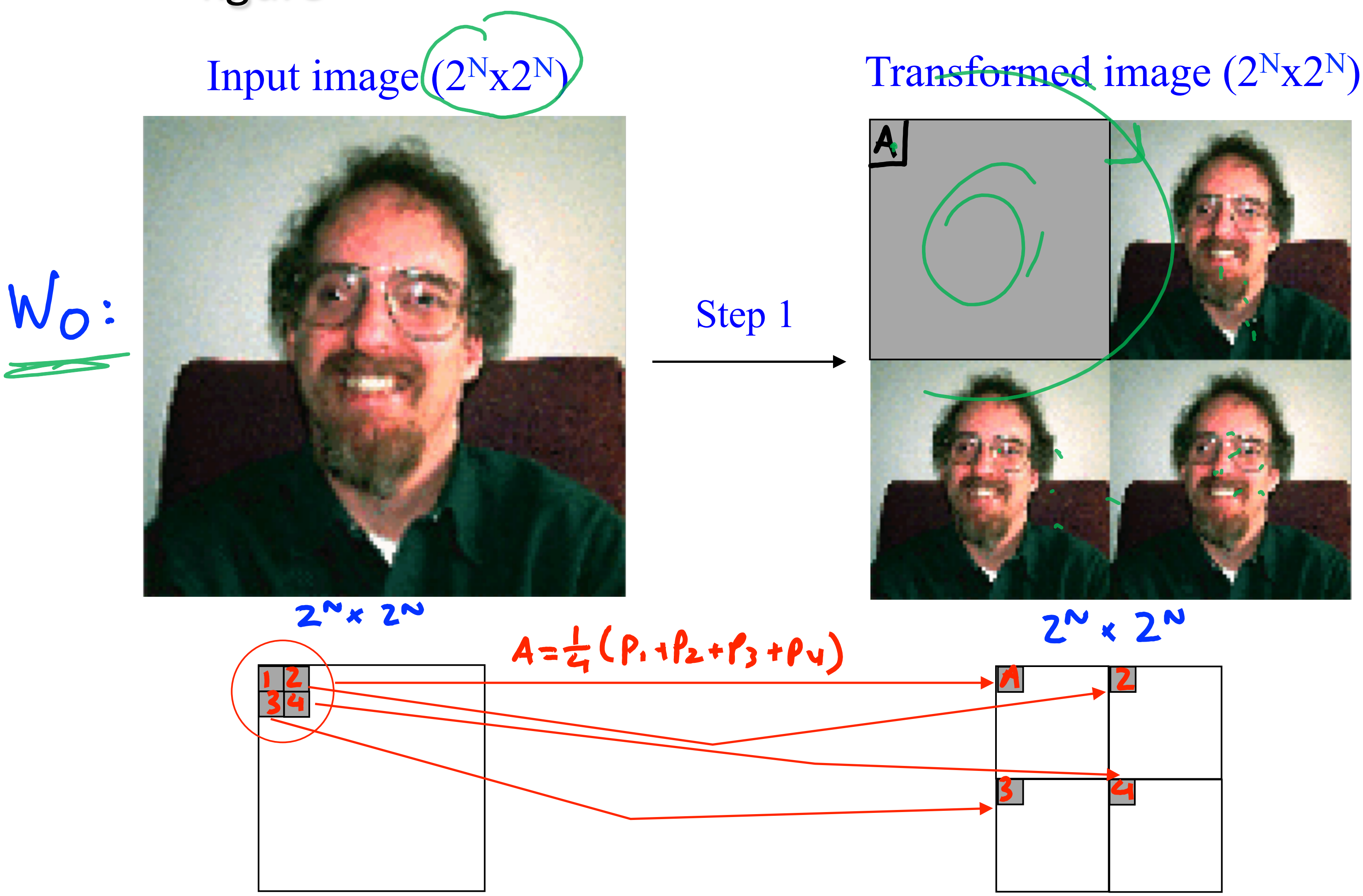


wavelet transform

Transformed image (2^Nx2^N)

A Simple, Minimal 2-D Image Transform

Step 1: Create 4 new images of size $2^{N-1}x2^{N-1}$ as shown in figure

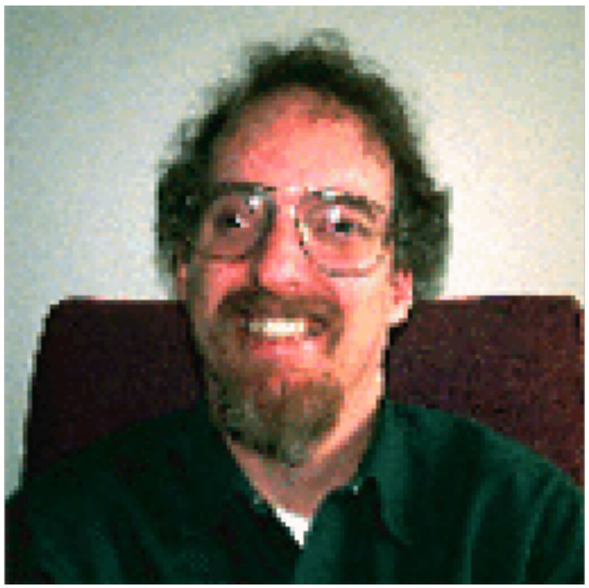


A Simple, Minimal 2-D Image Transform

Step 1: Create 4 new images of size $2^{N-1} x 2^{N-1}$ as shown in figure

Input image ($2^N x 2^N$)

W_0:



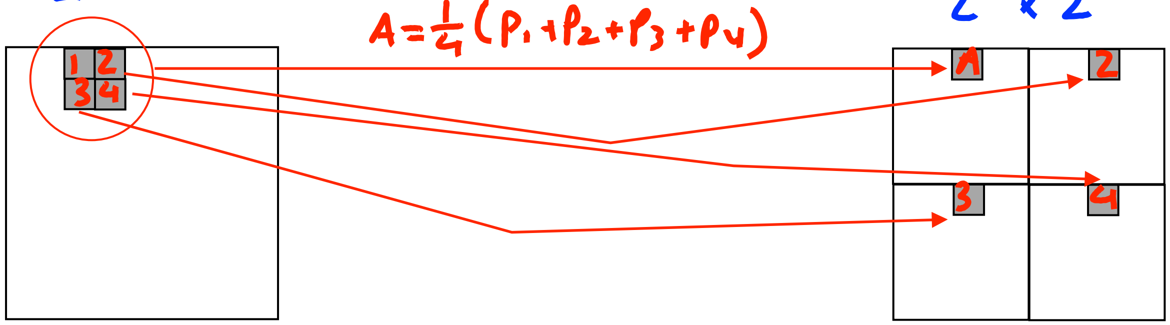
$2^N \times 2^N$

Step 1

Transformed image ($2^N x 2^N$)

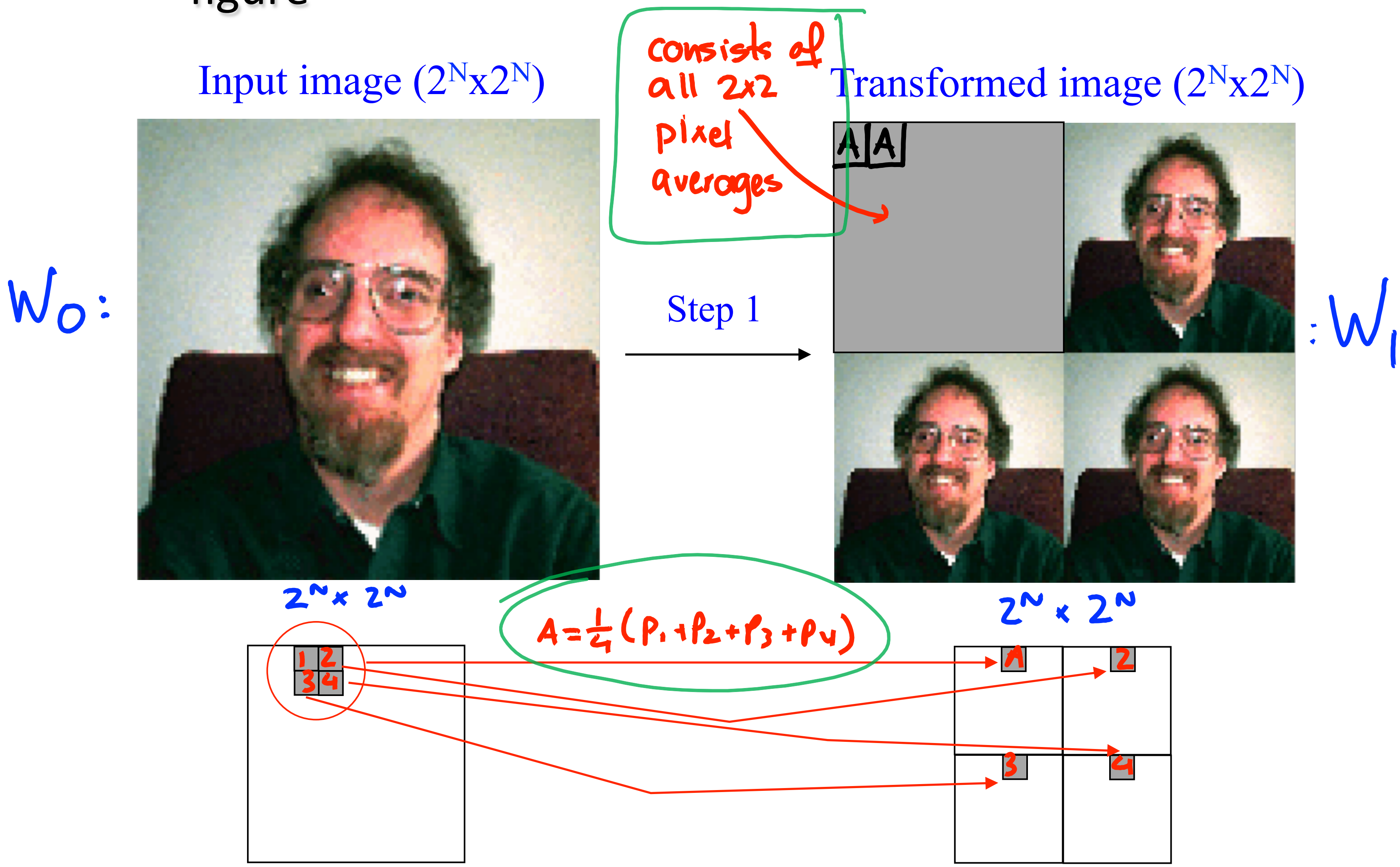
$2^N \times 2^N$

$A = \frac{1}{4}(p_1 + p_2 + p_3 + p_4)$



A Simple, Minimal 2-D Image Transform

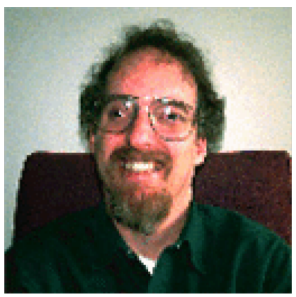
Step 1: Create 4 new images of size $2^{N-1}x2^{N-1}$ as shown in figure



A Simple, Minimal 2-D Image Transform

Step 2: Recursively perform Step 1 for top-left quadrant of result

Result of Step 1 (2^{N-1}x2^{N-1})

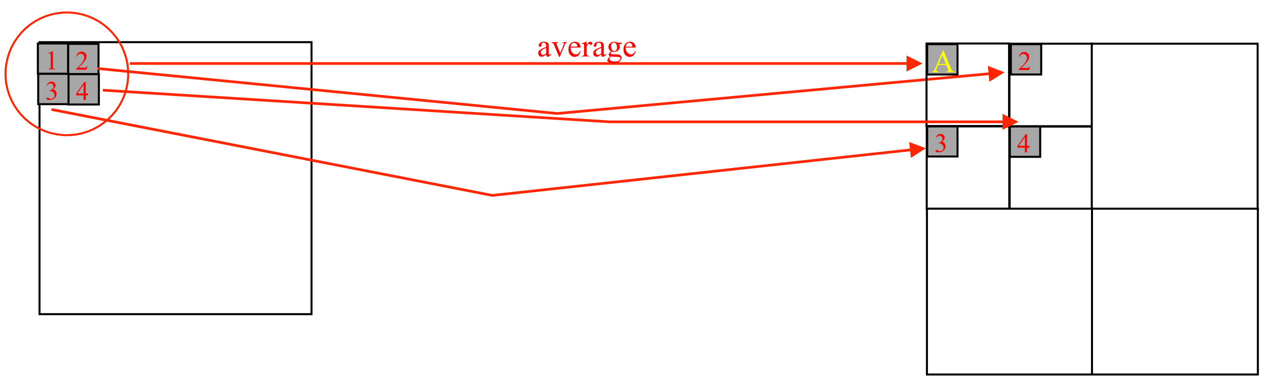


$2^{N-1} \times 2^{N-1}$

Step 2

Transformed image (2^Nx2^N)

: W_2



A Simple, Minimal 2-D Image Transform

Step 3: Recursion stops when average image is 1 pixel

Transformed image (2^Nx2^N)

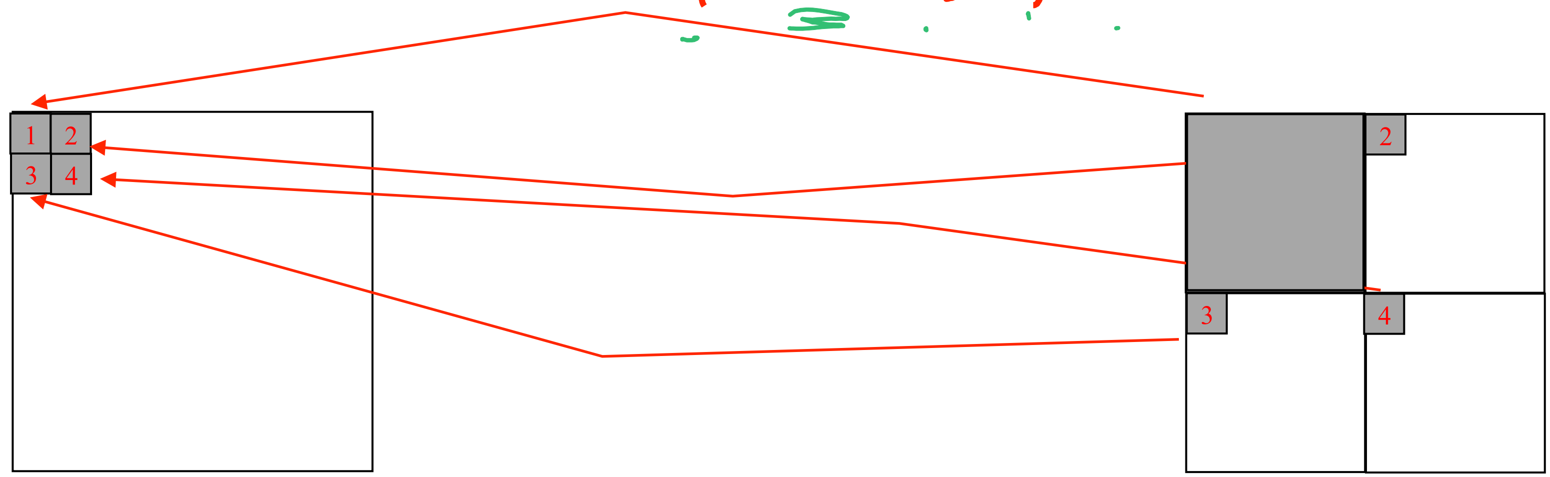
: W_N

Invertibility of the Transformation

Property: W_k can be reconstructed from W_{k+1}

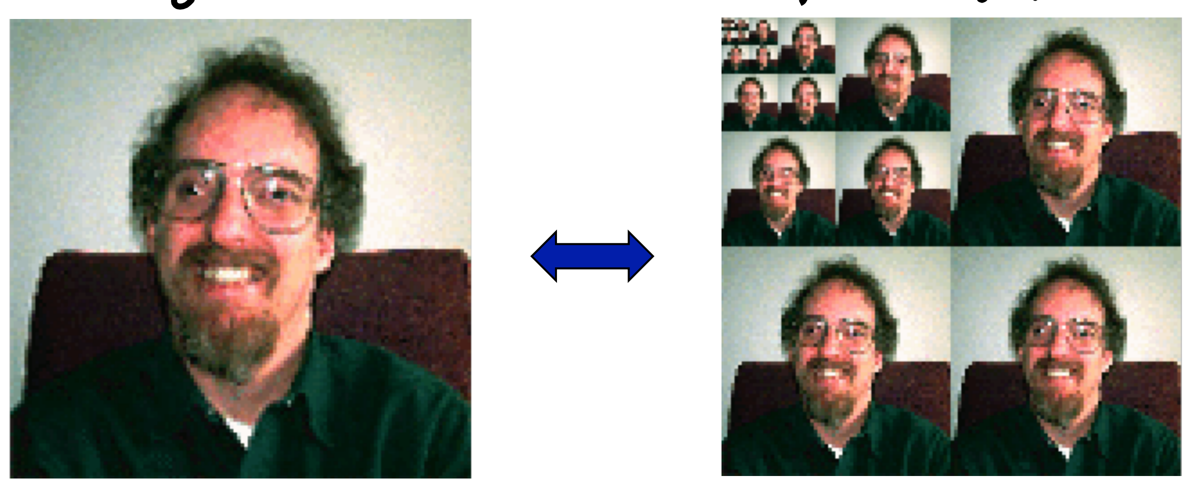
$P_1 = 4P_A - P_2 - P_3 - P_4$

W_k



W_{k+1}

$\Longleftrightarrow$ W_0 reconstructible from W_N



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1D Haar Wavelet Transform: Recursive Definition

pixel # 0 1 2 ... 2^3-1

Input image I^3

9	7	3	5	6	10	2	6

Average I^2

8	4	8	4

Detail coefs D^2

1	-1	-2	-2

(analogous to level g_0 of Gaussian pyramid)

average of first two pixels

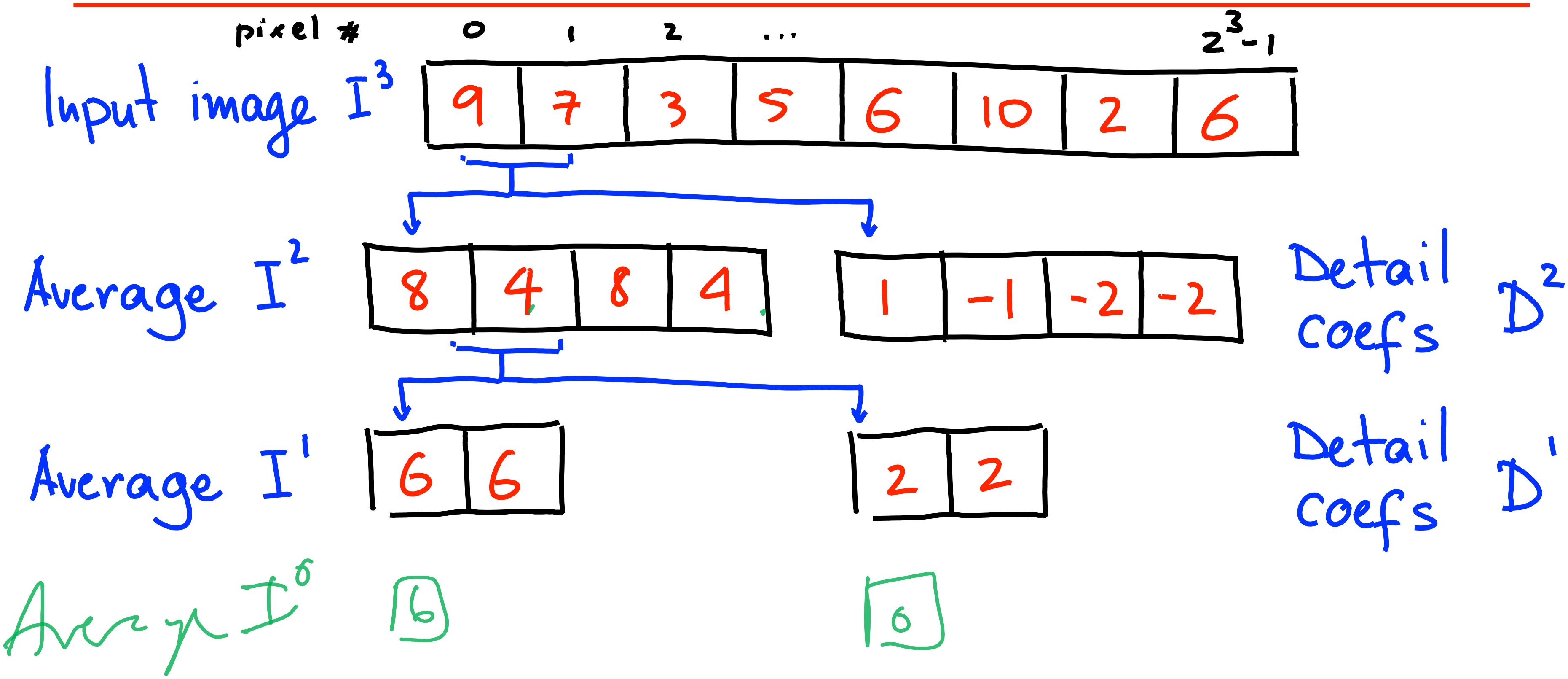
difference of first pixel from the average

(analogous to level L_0 of Laplacian pyramid)

(note: we don't need to store difference of 2nd pixel from average $\Rightarrow$

D^0 has ½ the size of the corresponding Laplacian level L_0!)

1D Haar Wavelet Transform: Recursive Definition

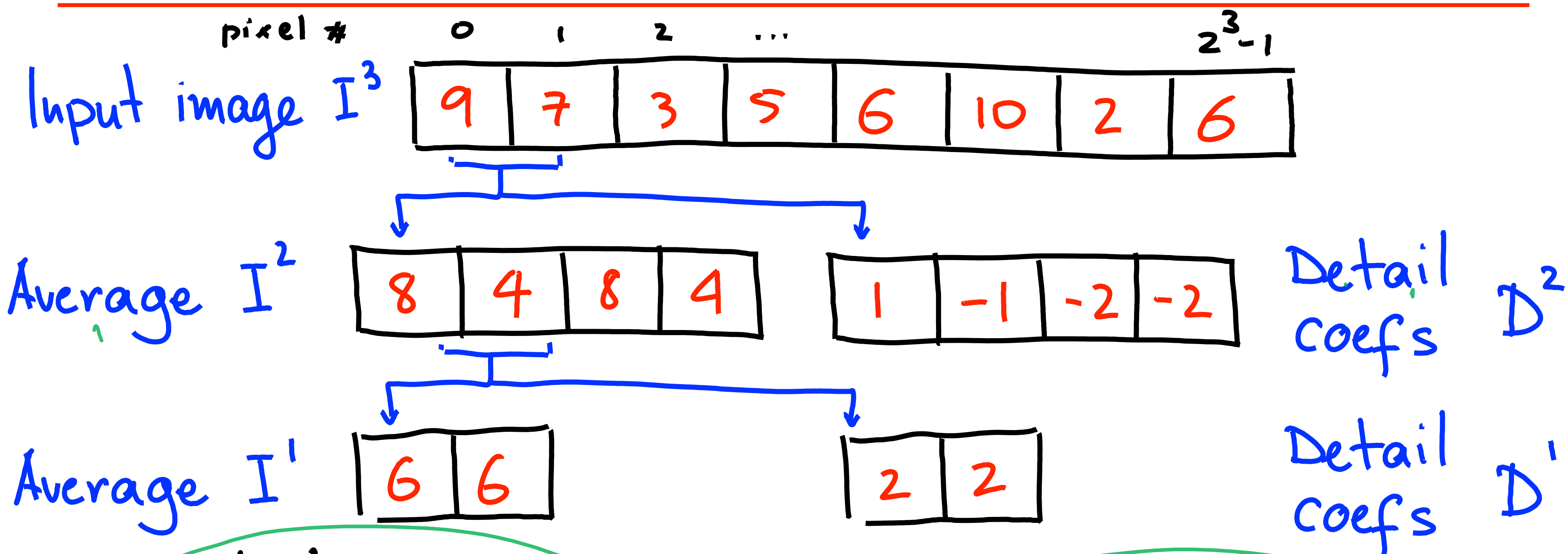


$$I^j_i = \frac{1}{2}\left(I^{j+1}_{2i} + I^{j+1}_{2i+1}\right)$$

j-th level of "pyramid" contains 2^j pixels

$$D^j_i = I^{j+1}_{2i} - \frac{1}{2}\left(I^{j+1}_{2i} + I^{j+1}_{2i+1}\right) = \frac{1}{2}\left(I^{j+1}_{2i} - I^{j+1}_{2i+1}\right)$$

1D Haar Wavelet Transform: Recursive Definition



equivalent convolution mask ϕ: | 1/2 | 1/2 |

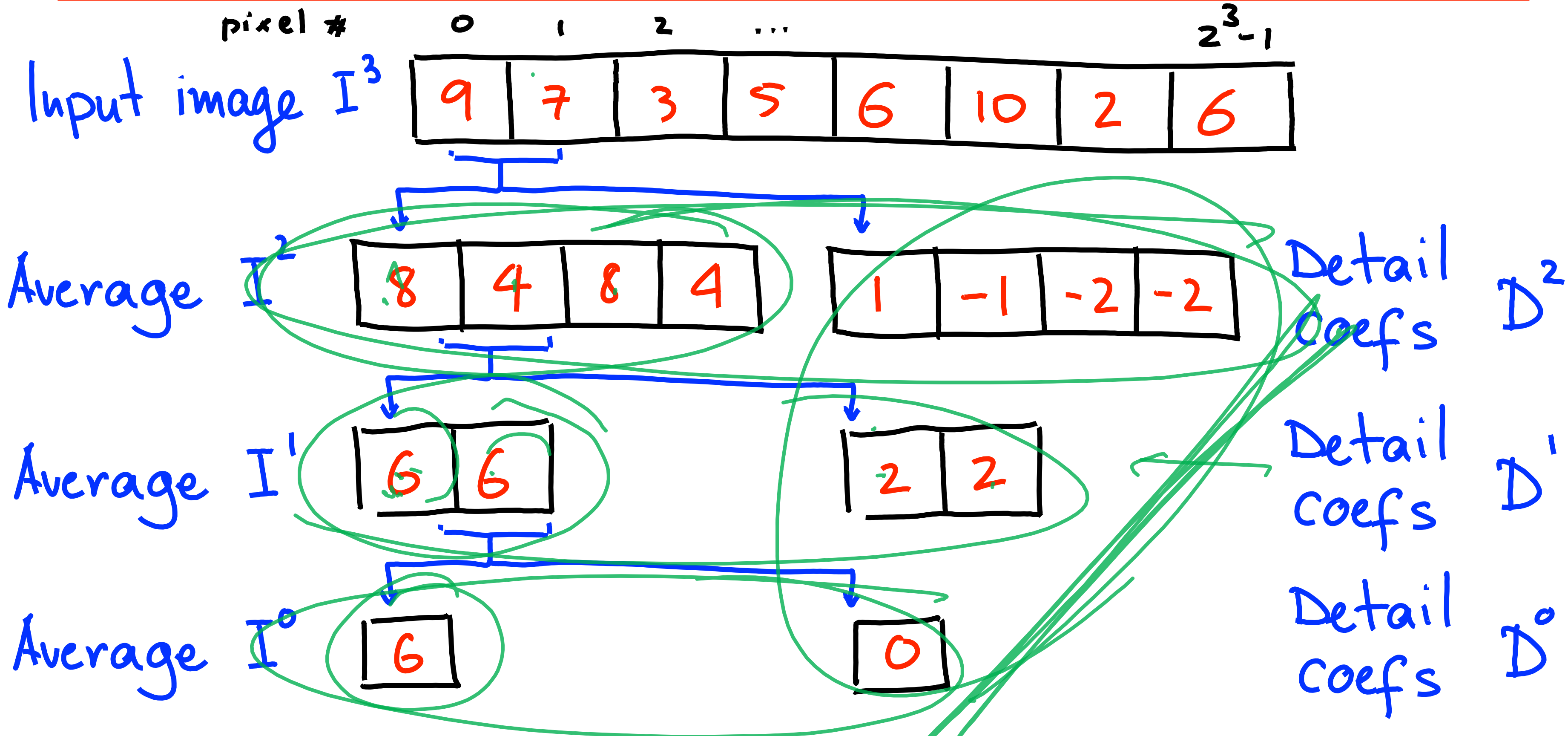
equivalent convolution mask ψ: | -1/2 | 1/2 |

$$I^j_i = \frac{1}{2}\left(I^{j+1}_{2i} + I^{j+1}_{2i+1} \right)$$

$$D^j = \frac{1}{2}\left(I^{j+1}_{2i} - I^{j+1}_{2i+1} \right)$$

j-th level of "pyramid" contains 2^j pixels

1D Haar Wavelet Transform: Recursive Definition

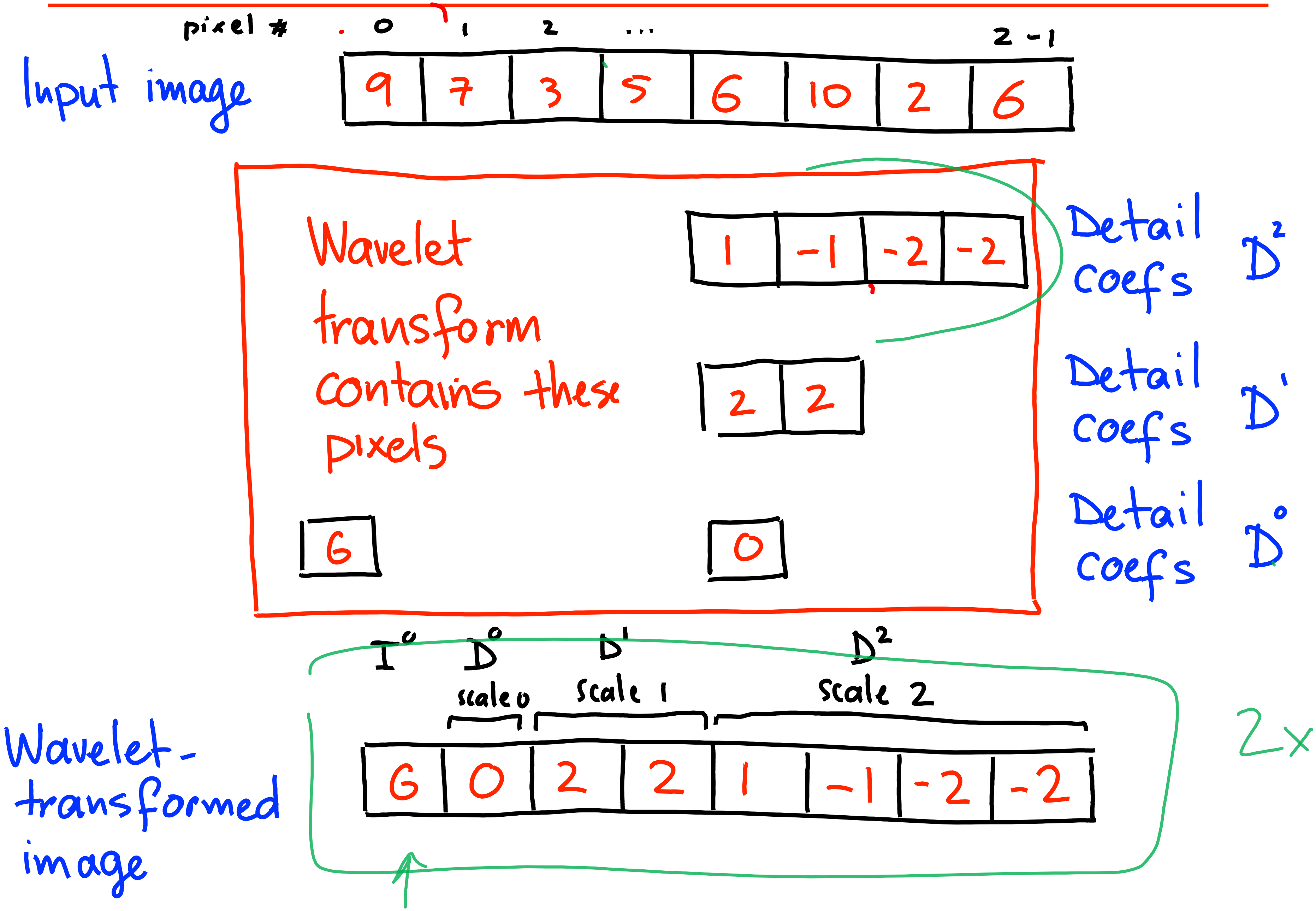


$$I^j_i = \frac{1}{2}\left(I^{j+1}_{2i} + I^{j+1}_{2i+1} \right)$$

$$D^j = \frac{1}{2}\left(I^{j+1}_{2i} - I^{j+1}_{2i+1} \right)$$

j-th level of "pyramid" contains 2^j pixels

1D Haar Wavelet Transform: Recursive Definition



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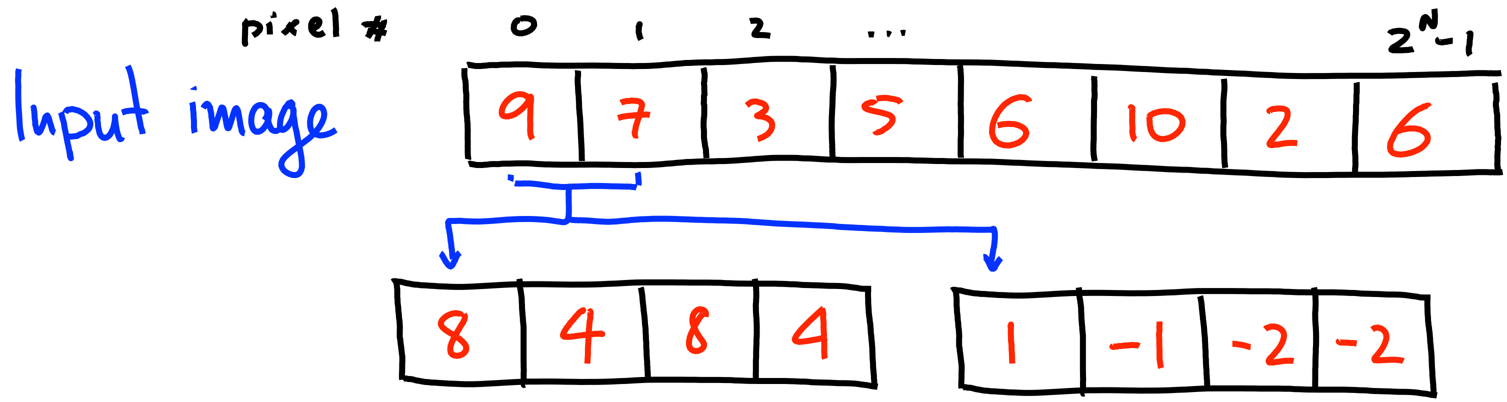
1D Haar Wavelet Transform as a Matrix Product

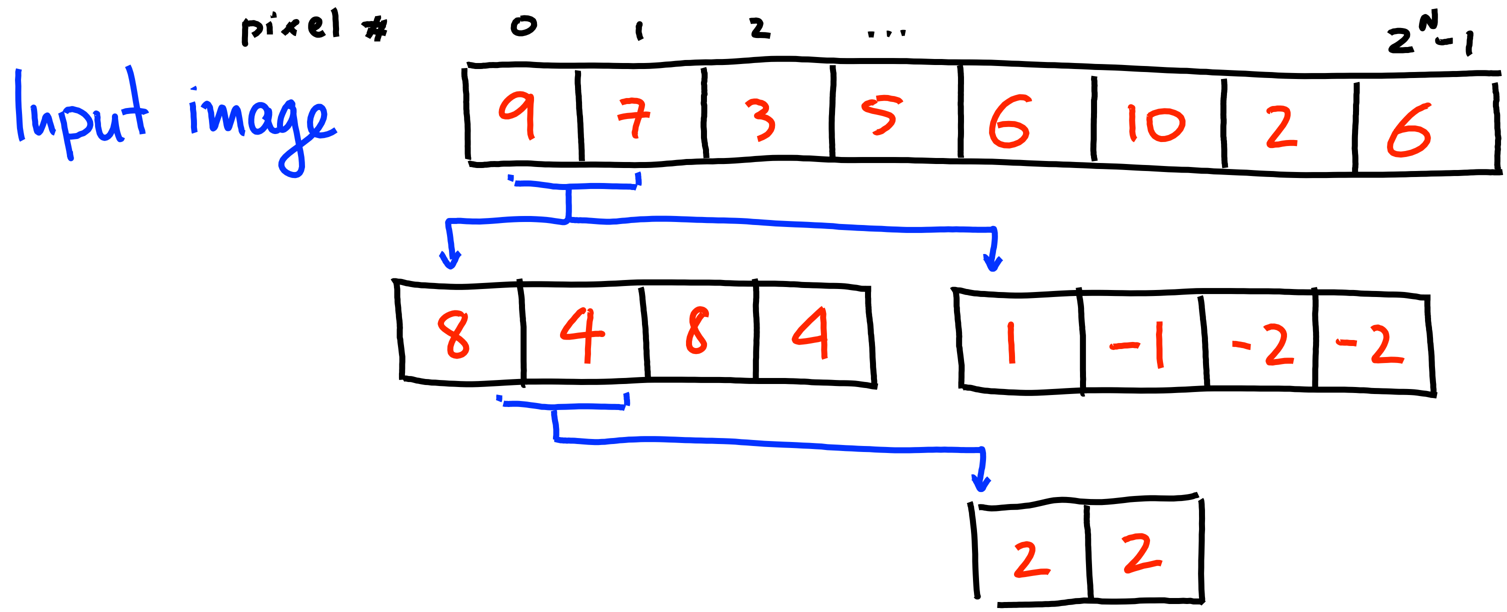


Wavelet transformed image

$$\begin{array}{c} I^0 \\ D^0 \\ D^1 \\ \\ D^2 \\ \\ \\ \\ \end{array} \begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \\ -2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}$$

Original image

1D Haar Wavelet Transform as a Matrix Product

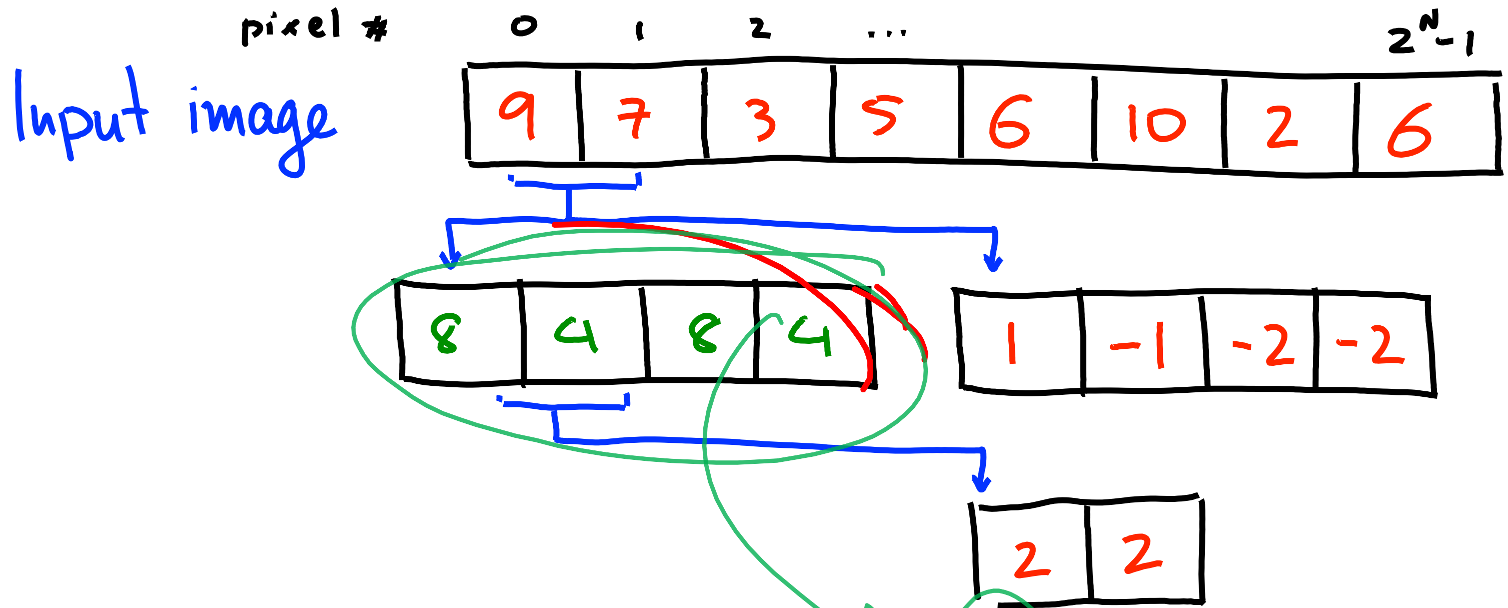


Wavelet transformed image

$$\begin{array}{c} I^0 \\ D^0 \\ D^1 \\ \\ \\ D^2 \\ \\ \\ \end{array}\begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \\ -2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} & & & & & & & \\ & & & ? & & & & \\ & & & & & & & \\ & & & & & & & \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}$$

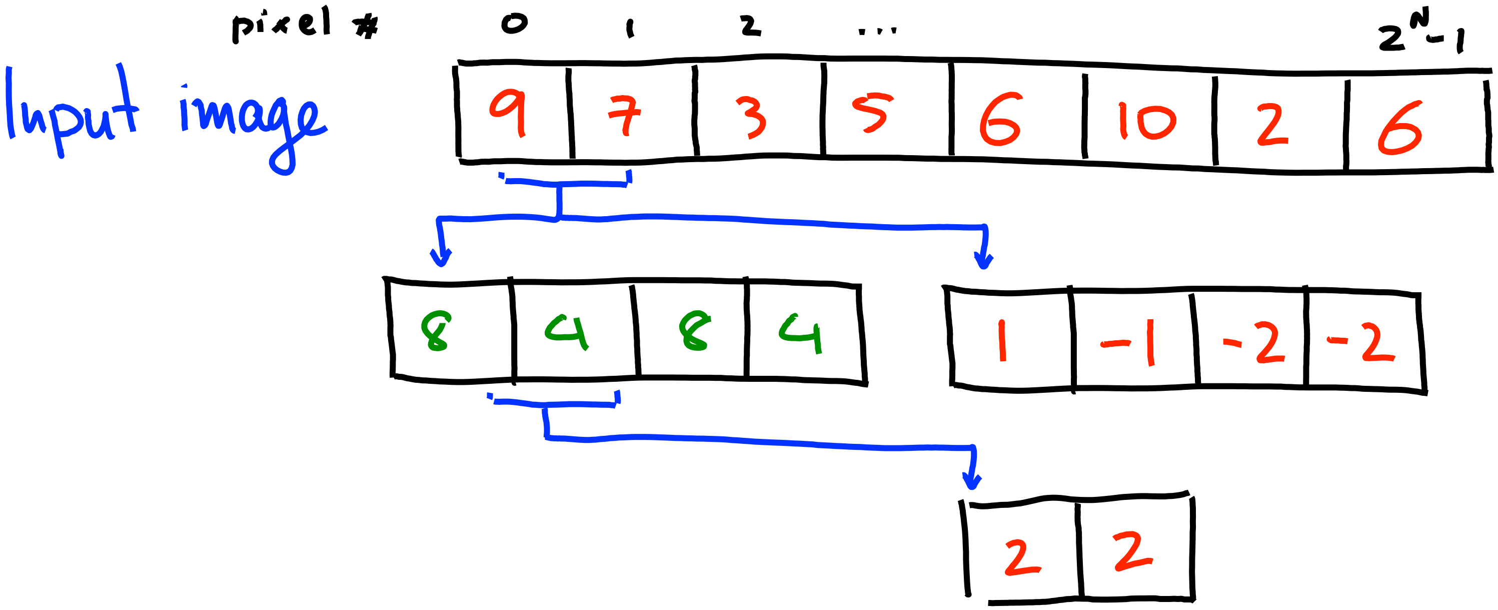
Original image

1D Haar Wavelet Transform as a Matrix Product



$$
\begin{array}{c} I^0 \\ D^0 \\ D^1 \\ \\ D^2 \\ \\ \\ \\ \end{array}
\begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \\ -2 \end{bmatrix}
=
\begin{bmatrix}
 & & & & & & & \\
 & & & & & & & \\
\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} 8 \\ 4 \\ 8 \\ 4 \\ 1 \\ -1 \\ -2 \\ -2 \end{bmatrix}
$$

1D Haar Wavelet Transform as a Matrix Product



$$
\begin{array}{c}
I^0 \\ D^0 \\ \\ D^1 \\ \\ \\ D^2 \\ \\ \\
\end{array}
\begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \\ -2 \end{bmatrix}
=
\begin{bmatrix}
 & & & & & & & \\
 & & & & & & & \\
\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\cdot
\begin{array}{c}
\\ \\ \\ \\ \frac{1}{2}\cdot \\ \\ \\ \\
\end{array}
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{bmatrix}
\begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}
$$

1D Haar Wavelet Transform as a Matrix Product

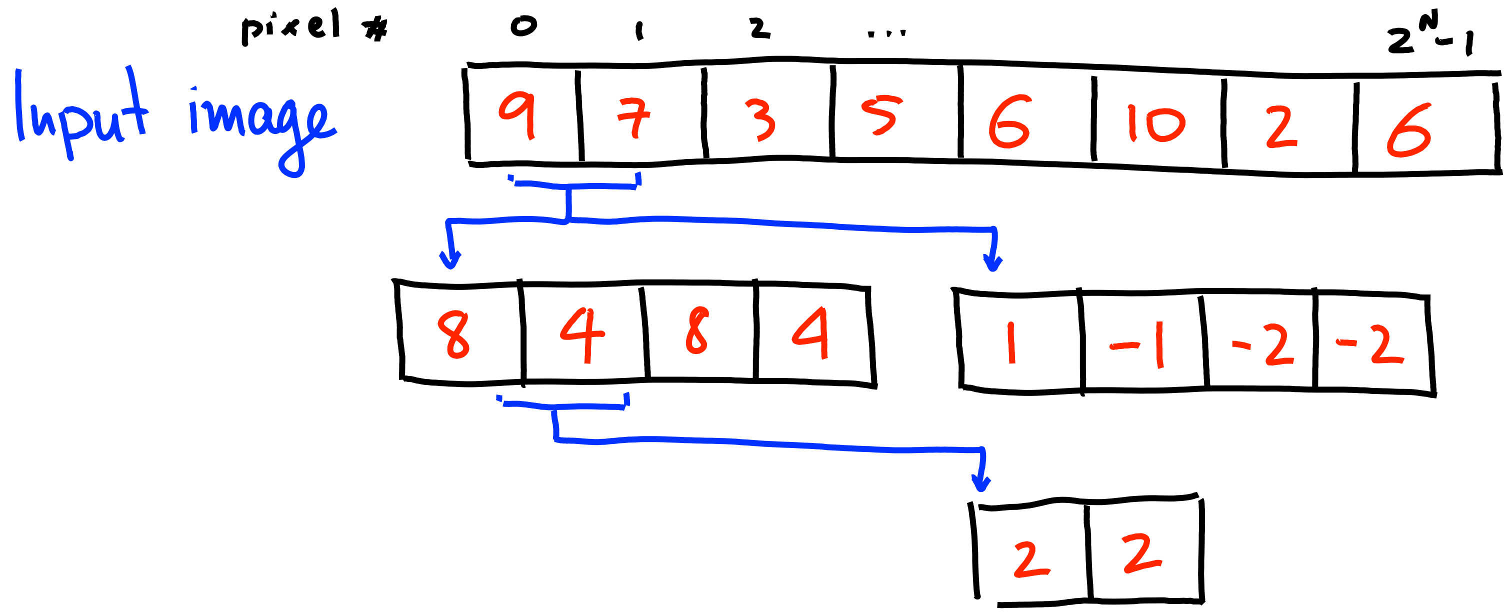
3rd & 4th rows of product

$$\begin{bmatrix} \\ \\ \tfrac14 & \tfrac14 & -\tfrac14 & -\tfrac14 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tfrac14 & \tfrac14 & -\tfrac14 & -\tfrac14 \\ \\ \\ \\ \\ \end{bmatrix}$$

A B

$$\begin{array}{c} I^0 \\ D^0 \\ D^1 \\ \\ D^2 \\ \\ \\ \\ \end{array}\begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} & & & & & & & \\ & & & & & & & \\ \tfrac12 & -\tfrac12 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tfrac12 & -\tfrac12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \tfrac12 \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}$$

1D Haar Wavelet Transform as a Matrix Product



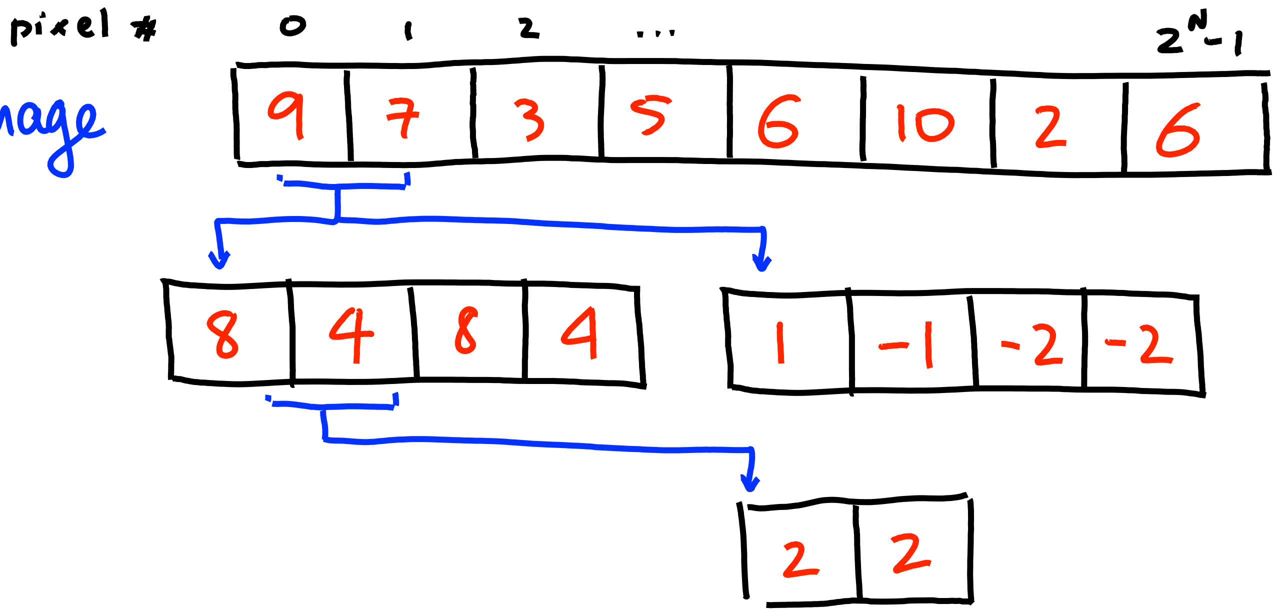
Wavelet transformed image

$$
\begin{array}{c}
I^0 \\ D^0 \\ D^1 \\ \\ D^2 \\ \\ \\ \\
\end{array}
\begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \\ -2 \end{bmatrix}
=
\begin{array}{c}
\frac{1}{8} \\ \frac{1}{8} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2}
\end{array}
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{bmatrix}
\begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}
$$

Original image

1D Haar Wavelet Transform as a Matrix Product

Input image



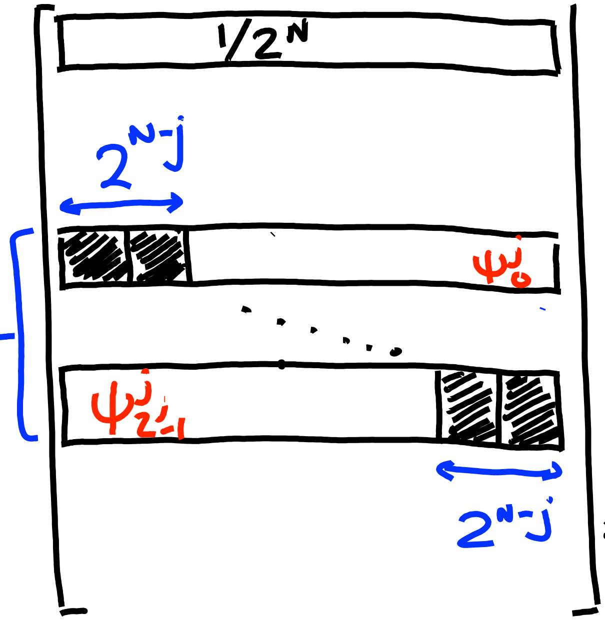
Wavelet transformed image

$$
\begin{array}{c}
I^0 \\ D^0 \\ D^1 \\ \\ D^2 \\ \\ \\ \\
\end{array}
\begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \\ -2 \end{bmatrix}
=
\begin{array}{c}
1/8 \\ 1/8 \\ 1/4 \\ 1/4 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1/2
\end{array}
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{bmatrix}
\begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}
$$

Original image

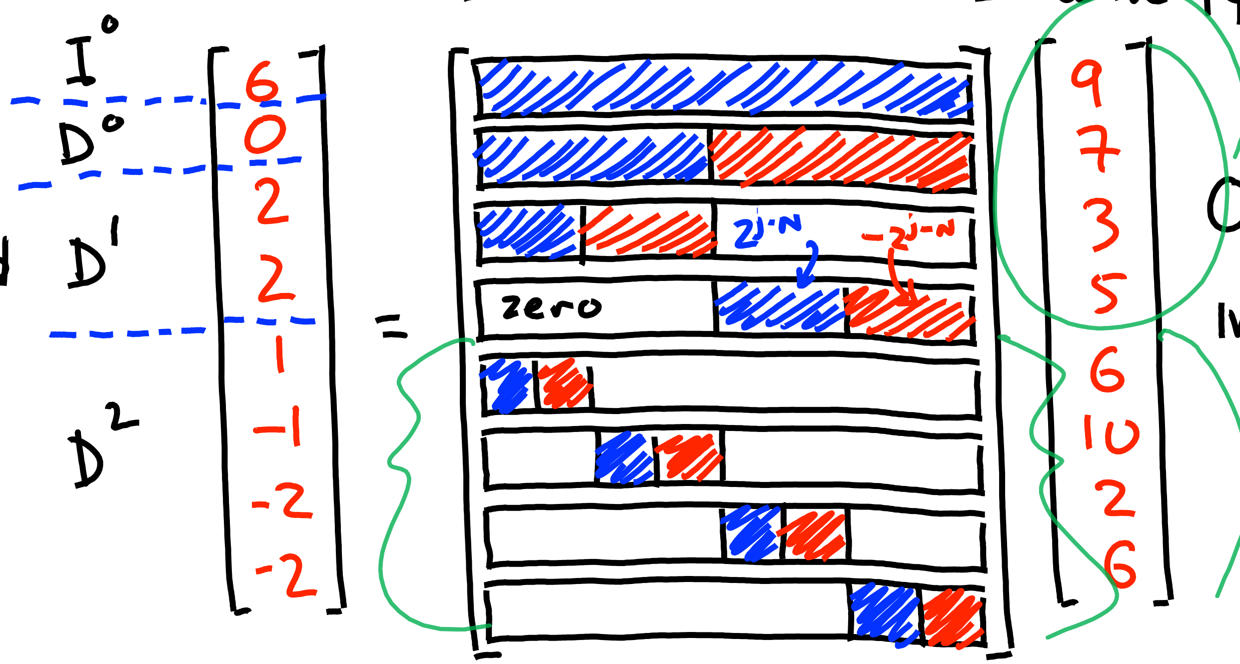
The 1D Haar Wavelet Transform Matrix W

- Matrix contains N-1 scales
- Scale j represented by 2^j rows $\psi_0^j, \dots, \psi_{2^j-1}^j$



- Row ψ_j^i has $\frac{2^N}{2^j} = 2^{N-j}$ non-zero pixels
- They are pixels $x = i\,2^{N-j}, \dots, (i+1)2^{N-j}-1$ with $|\psi_j^i(x)| = \frac{1}{2^{N-j}}$

Wavelet transformed image



Original image

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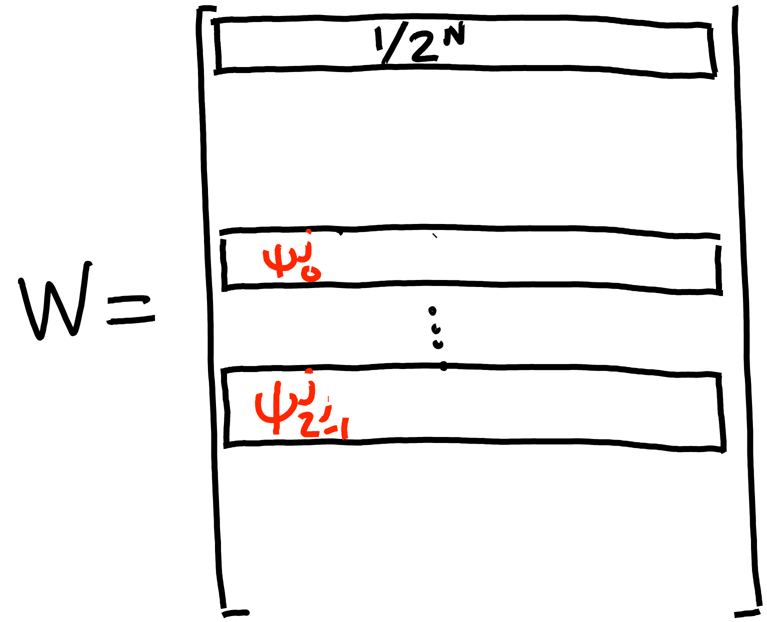
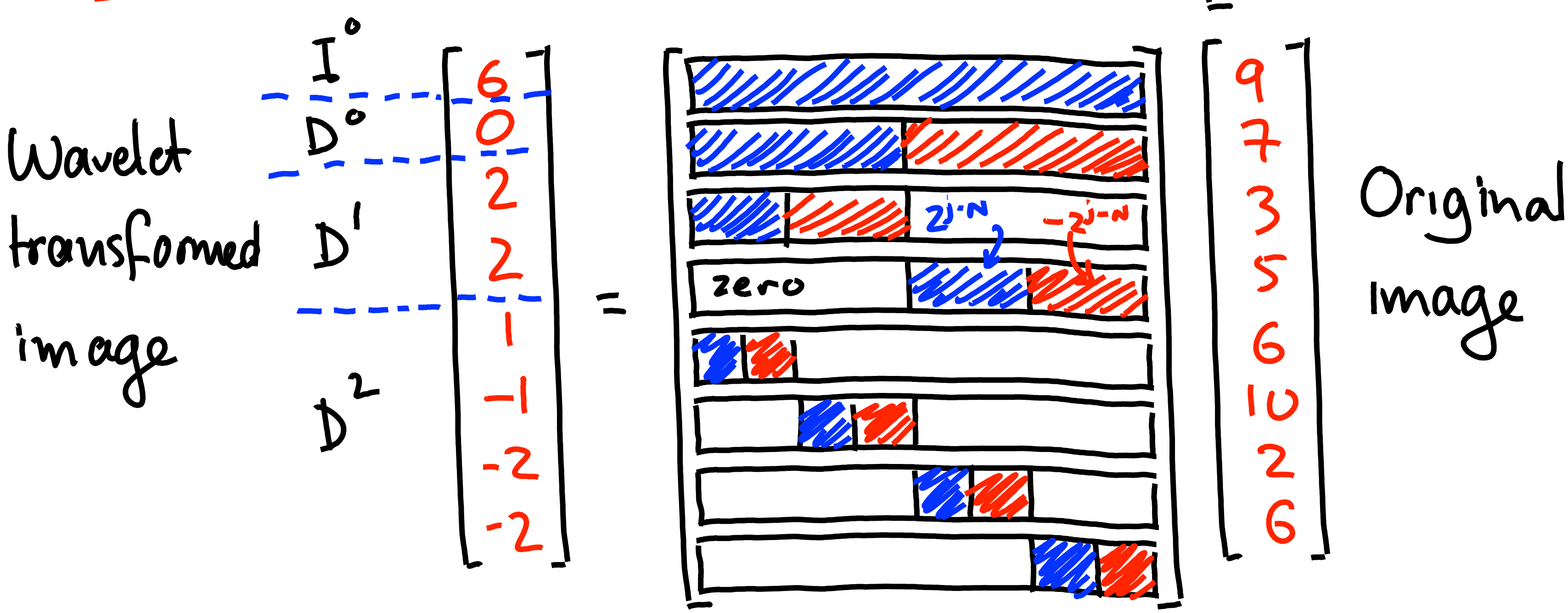
Reconstructing an Image from its Wavelet Coefs

Question: What is the dot product

$$\psi_i^j \cdot \psi_{i'}^{j'}$$

of two distinct rows of W?





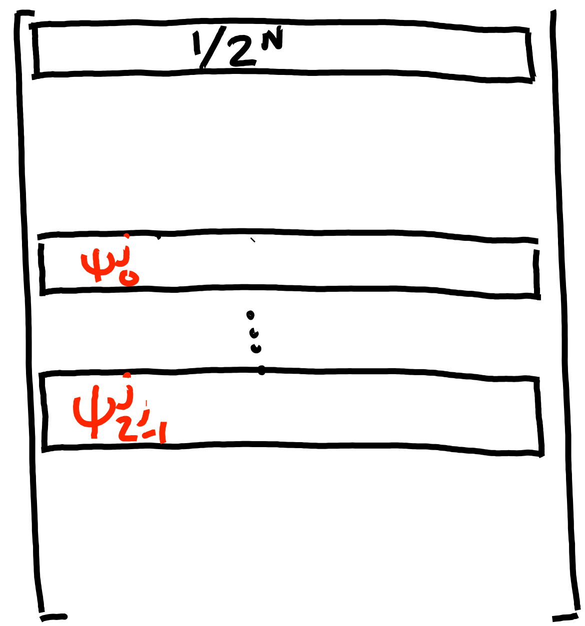
Reconstructing an Image from its Wavelet Coefs

Answer:

$$\psi_i^j \cdot \psi_{i'}^{j'} = 0$$

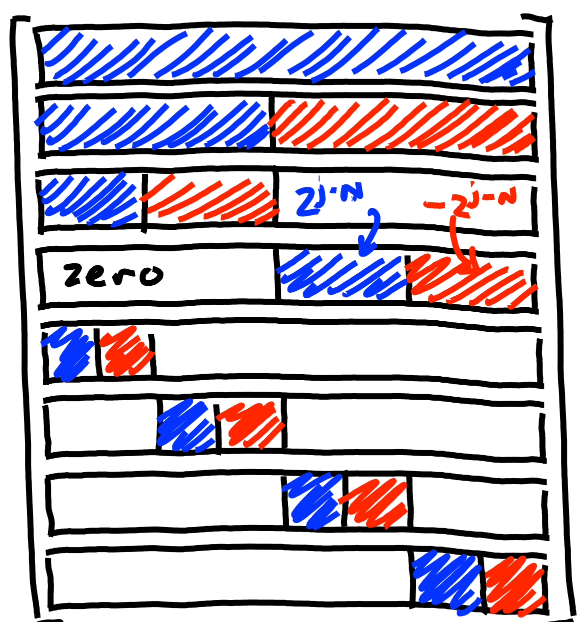
for two distinct rows of W $\Longleftrightarrow$

$$WW^T = \text{diagonal}$$

$W =$ 

Wavelet transformed image

$$\begin{array}{c} I^0 \\ D^0 \\ D^1 \\ \\ D^2 \\ \\ \\ \\ \end{array} \begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \\ -2 \end{bmatrix} =$$



$$\begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}$$

Original image

Reconstructing an Image from its Wavelet Coefs

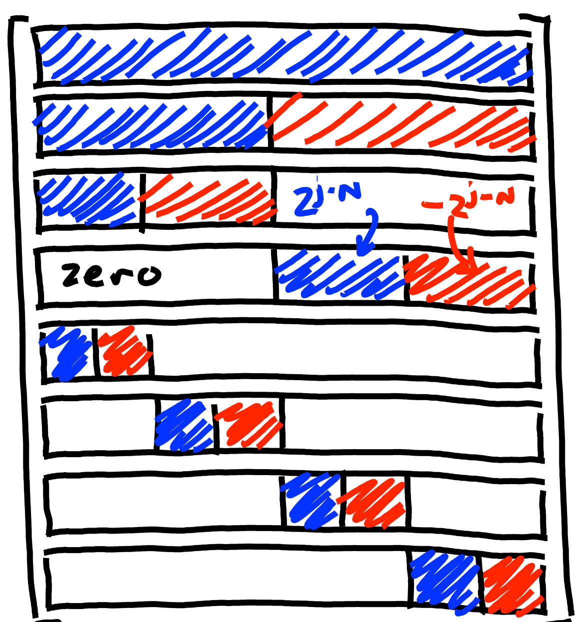
$$WW^T = \text{diagonal}$$

$$(\psi_i^j)\cdot(\psi_i^j)^T \longrightarrow \quad = \begin{bmatrix} \ddots & & \\ & \frac{1}{2^{N-j}} & \\ & & \ddots \end{bmatrix}$$

$$W = \begin{bmatrix} 1/2^N \\ \\ \psi_0^j \\ \vdots \\ \psi_{2^j-1}^j \\ \\ \end{bmatrix}$$

Wavelet transformed image

$$\begin{array}{c} I^0 \\ D^0 \\ D^1 \\ \\ D^2 \\ \\ \\ \\ \end{array} \begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \\ -2 \end{bmatrix} =$$



$$\begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}$$

Original image

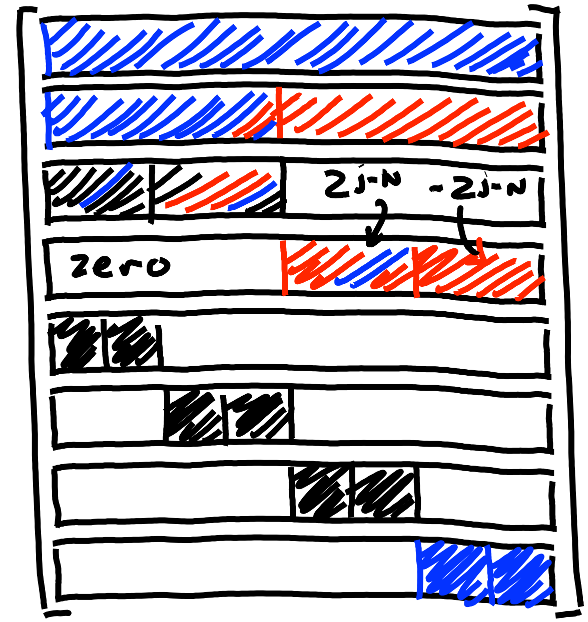
Reconstructing an Image from its Wavelet Coefs

Define $\Lambda = WW^T$

with

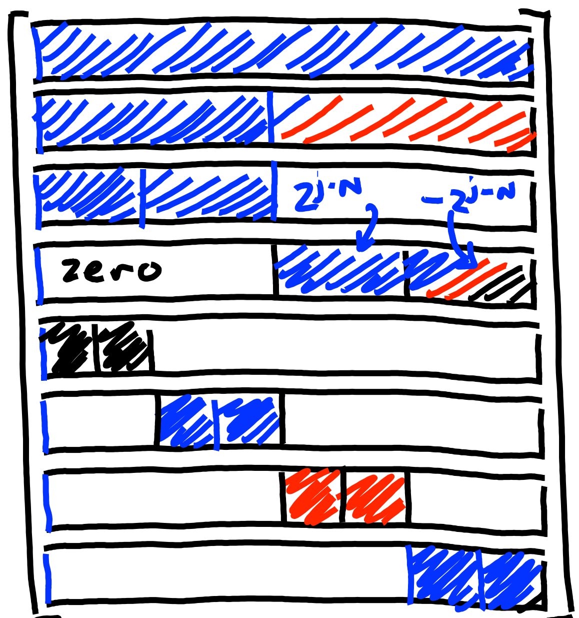
$$\Lambda = \begin{bmatrix} \lambda_0 & & 0 \\ & \ddots & \\ 0 & & \lambda_{2^N-1} \end{bmatrix}$$

of the form $\frac{1}{2^{N-j}}$

$W =$ 

Multiply W^T on both sides:

$$W^T \cdot \begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \\ -2 \end{bmatrix} = W^T \cdot$$



$$\begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}$$

Original image

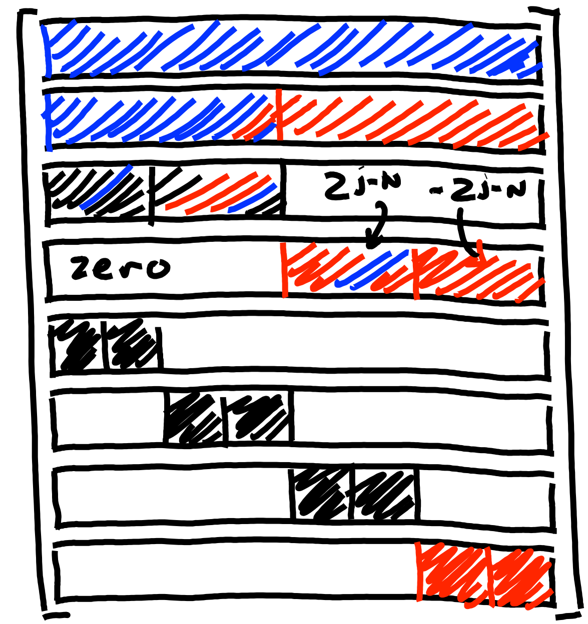
Reconstructing an Image from its Wavelet Coefs

Define $\Lambda = WW^T$

with

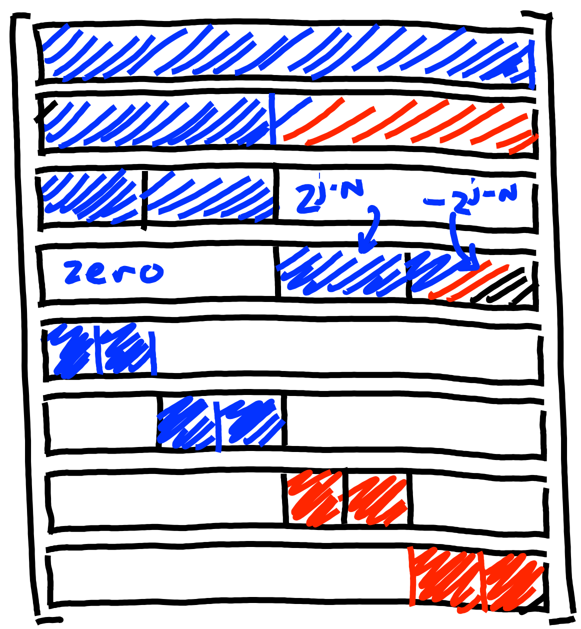
$$\Lambda = \begin{bmatrix} \lambda_0 & & 0 \\ & \ddots & \\ 0 & & \lambda_{2^N-1} \end{bmatrix}$$

of the form $\frac{1}{2^{N-j}}$

$W =$ 

Multiply $\Lambda^{-1} W^T$ on both sides:

$$\Lambda^{-1} W^T \cdot \begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \\ -2 \end{bmatrix} = \Lambda^{-1} W^T \cdot$$



$$\begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}$$

Original image

Reconstructing an Image from its Wavelet Coefs

Observation:

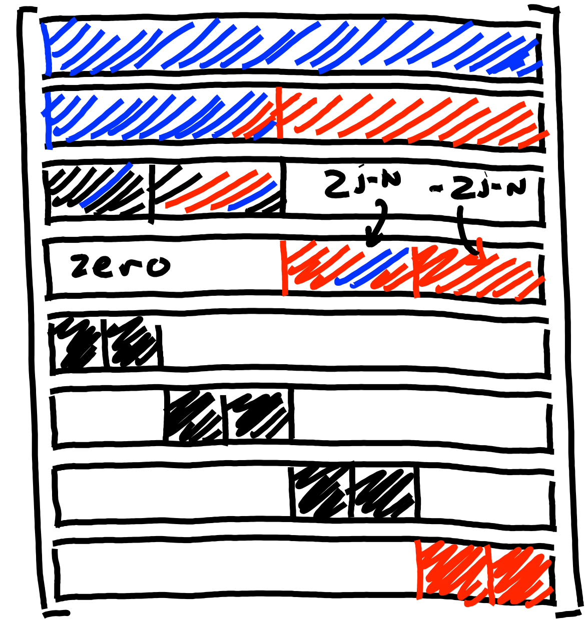
$\Lambda^{-1/2} \cdot W$ is orthogonal because

$$(\Lambda^{-1/2} \cdot W)(\Lambda^{-1/2} \cdot W)^T =$$

$$\Lambda^{-1/2} W \cdot W^T \cdot \Lambda^{-1/2} =$$

$$\Lambda^{-1/2} \cdot \Lambda \cdot \Lambda^{-1/2} = I$$

$W =$



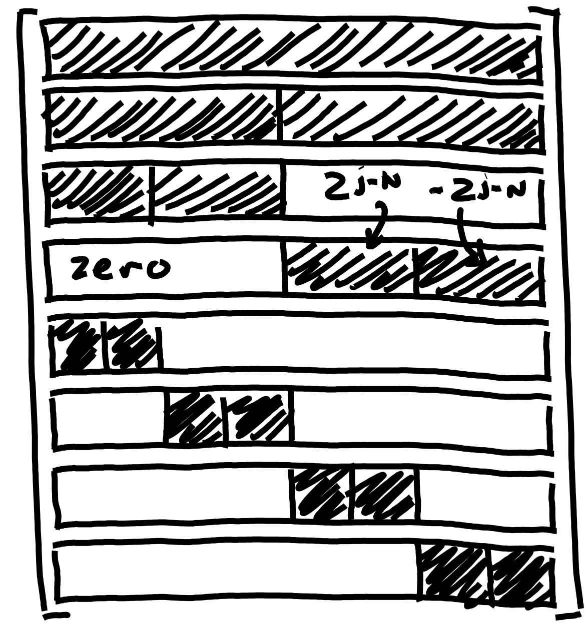
Therefore

$$\Lambda^{-1} W^T \cdot \begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}$$

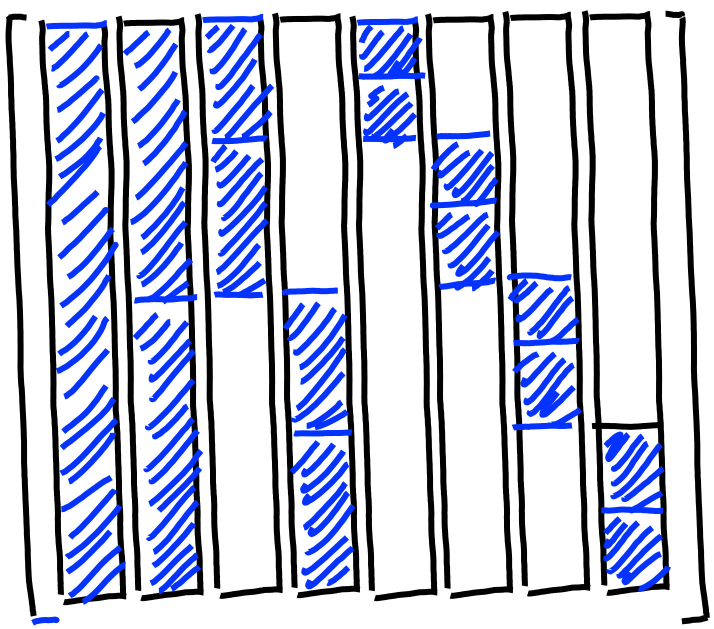
Original image

Reconstructing an Image from its Wavelet Coefs

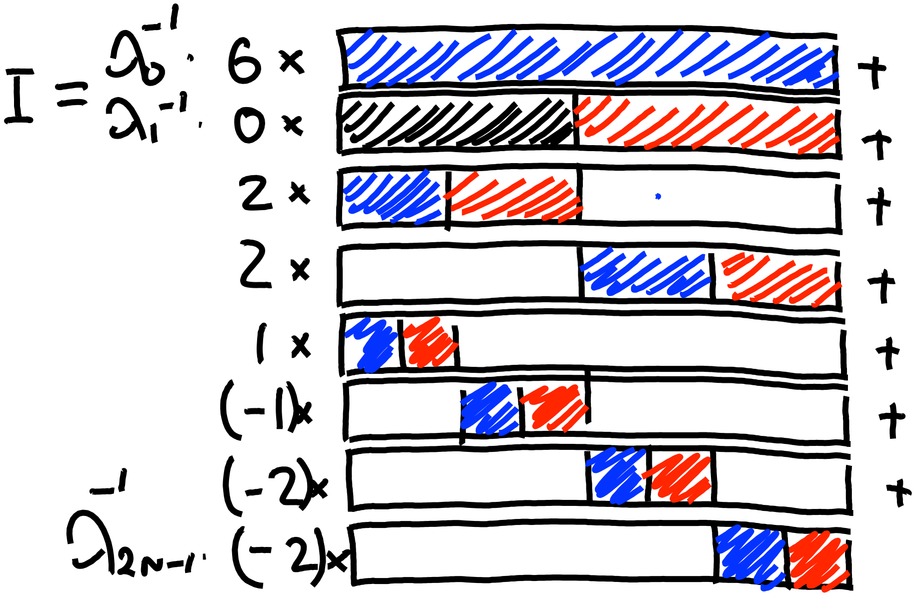
$$\boxed{\Lambda^{-1} = \begin{bmatrix} \lambda_p^{-1} & & 0 \\ & \ddots & \\ 0 & & \lambda_{2^N-1}^{-1} \end{bmatrix}}$$

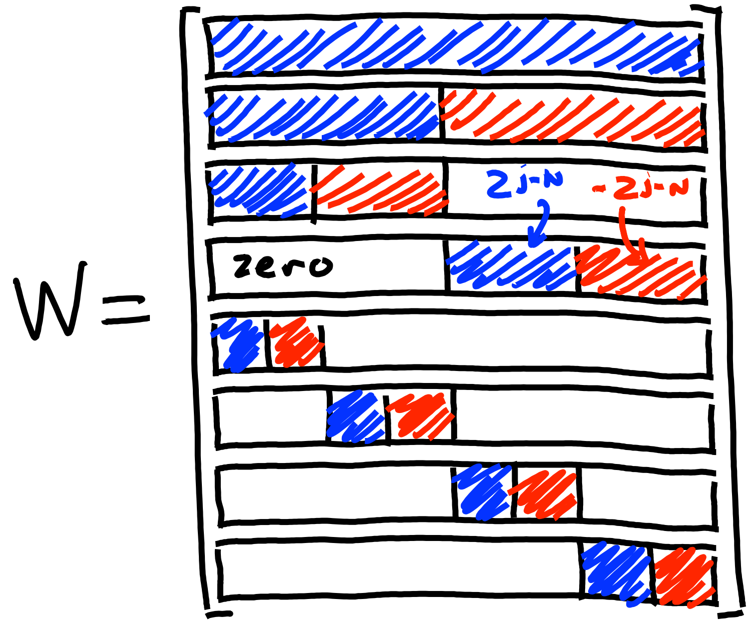
$W =$ 

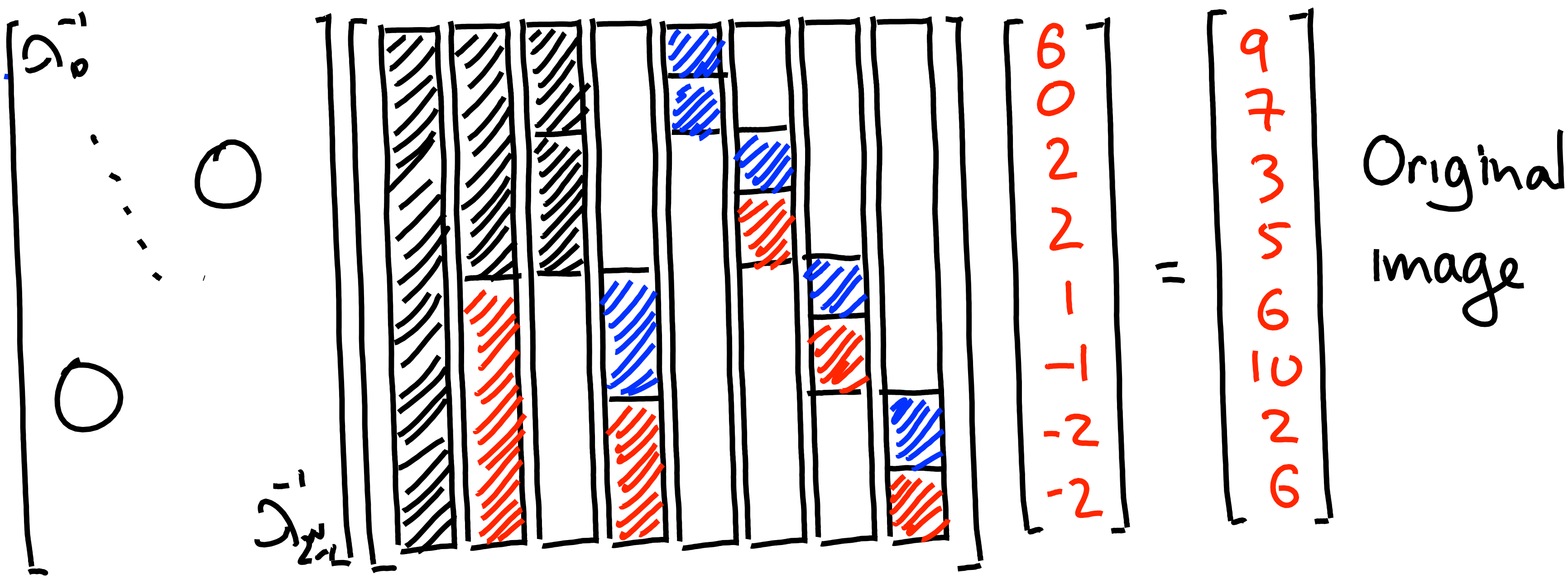
So we have

Λ^{-1}  $\begin{bmatrix} 6 \\ 0 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}$ Original image

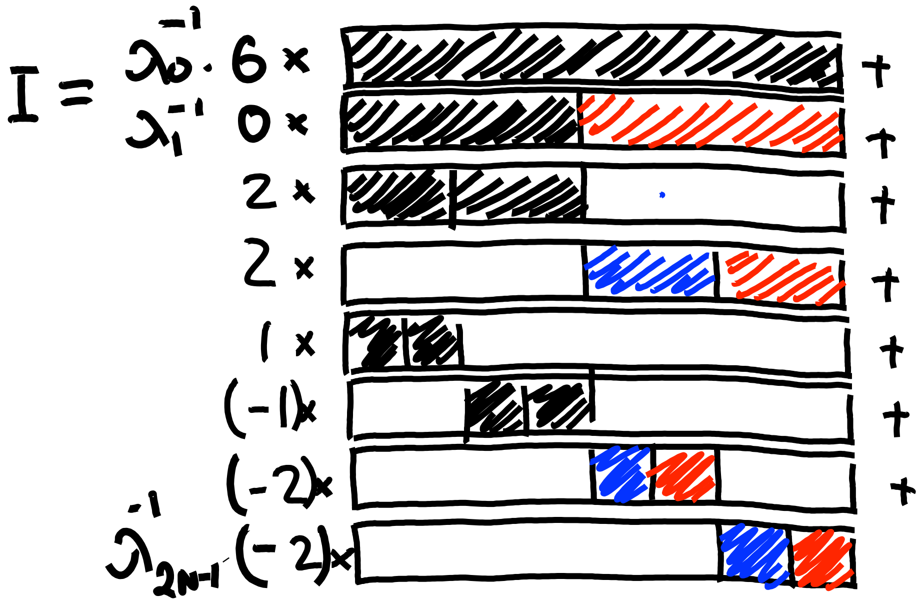
Interpreting the Wavelet Coefficients

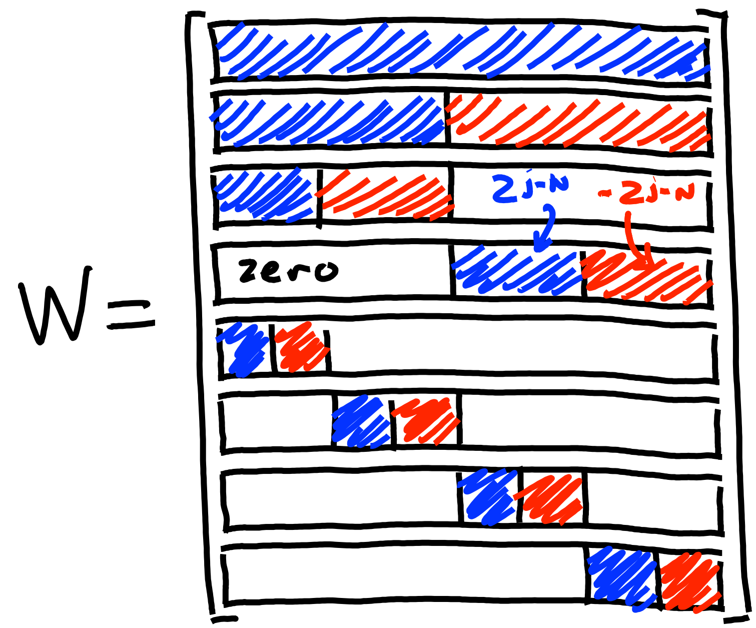






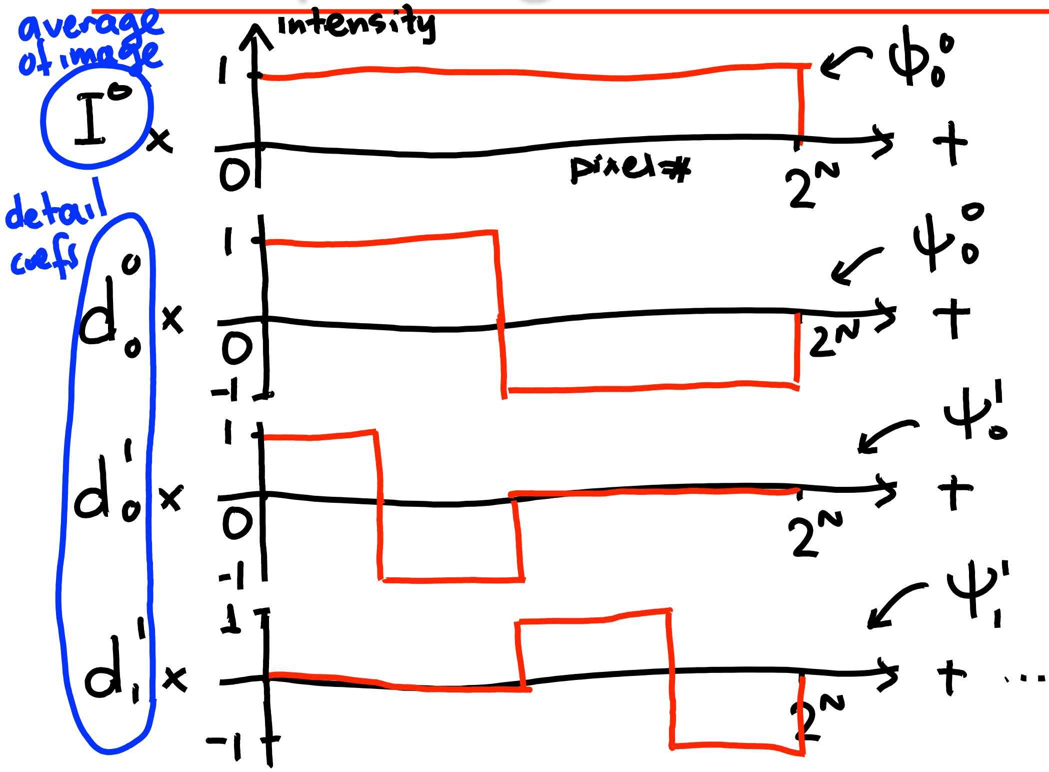
Interpreting the Wavelet Coefficients

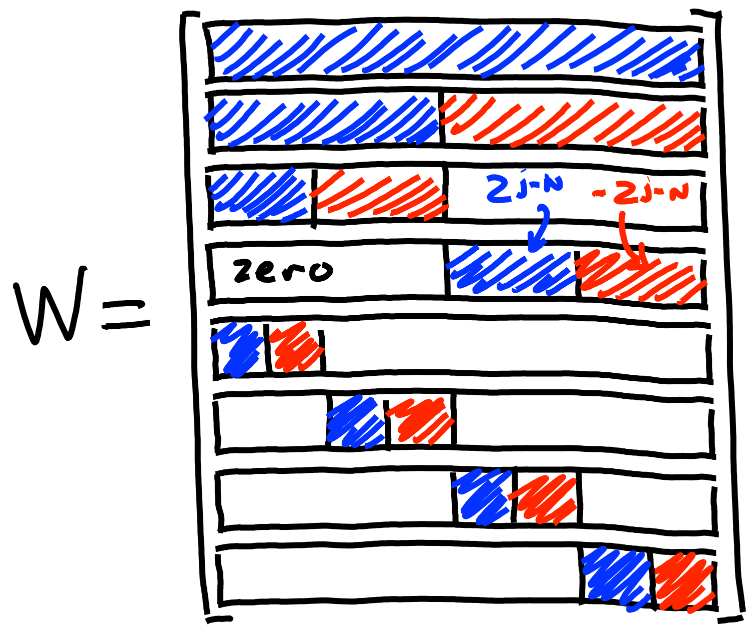




⇒ By multiplying I with W we obtain a decomposition of the image into a sequence of basis images $\psi_0^0, \psi_0^1, \ldots \psi_j^i, \ldots$ that form an orthogonal basis of $\mathbb{R}^{2^N}$

Interpreting the Wavelet Coefficients



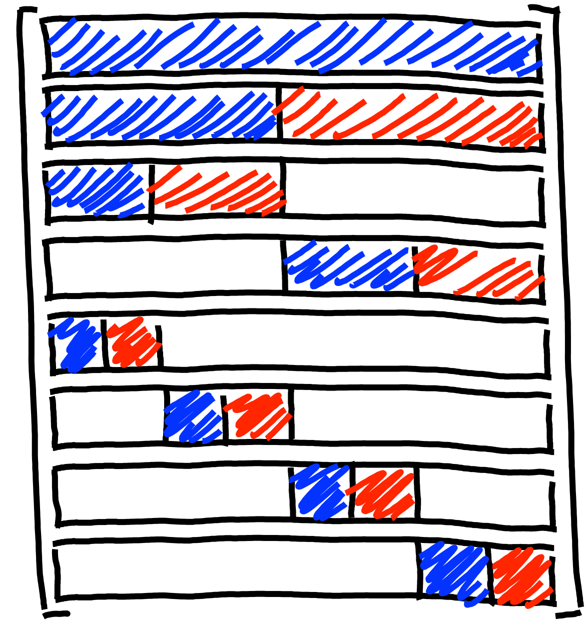


$\Rightarrow$ The wavelet coefficients are the coordinates of the image, considered as a vector in $\mathbb{R}^{2^N}$, in the basis defined by images $\phi_0^0, \psi_0^0, \psi_0^1, \ldots$

The Normalized Haar Wavelet Matrix

We can normalize the wavelet transform matrix by multiplying

$$\tilde{W} = \begin{bmatrix} \sqrt{\lambda_1^{-1}} & & \\ & \ddots & \\ & & \sqrt{\lambda_{2^N-1}^{-1}} \end{bmatrix} \cdot W$$

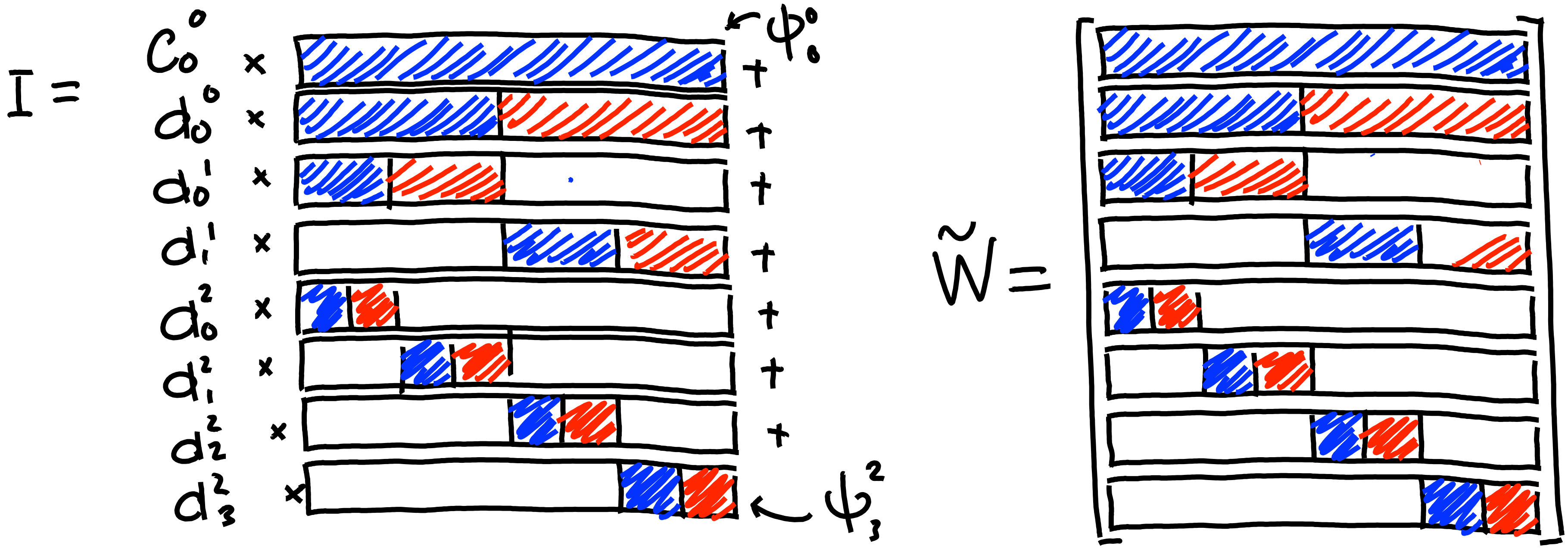
$\tilde{W} =$ 

normalized wavelet coefficients

$$\begin{bmatrix} c_0^0 \\ d_0^1 \\ d_1^1 \\ \vdots \\ \\ \\ \\ \\ \end{bmatrix} = \tilde{W} \cdot \begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}$$

Original image

The Normalized Haar Wavelet Coefficients



$\Rightarrow$ By multiplying I with $\tilde{W}$ we obtain a set of wavelet coefficients $c_0^0, d_0^0, \ldots$ that express I as a linear combination of the basis images $\phi_0^0, \psi_0^0, \psi_0^1, \psi_1^1, \ldots$

Topic 7:

Discrete Wavelet Transform

- Wavelet transform vs. Laplacian pyramid
- Basic intuition: a simple wavelet-like 2D transform
- The 1D Haar wavelet transform
- 1D Haar wavelet transform as a matrix product
- Reconstructing a 1D image from its wavelet coefs
- Wavelet-based image compression
- The 2D Haar wavelet transform

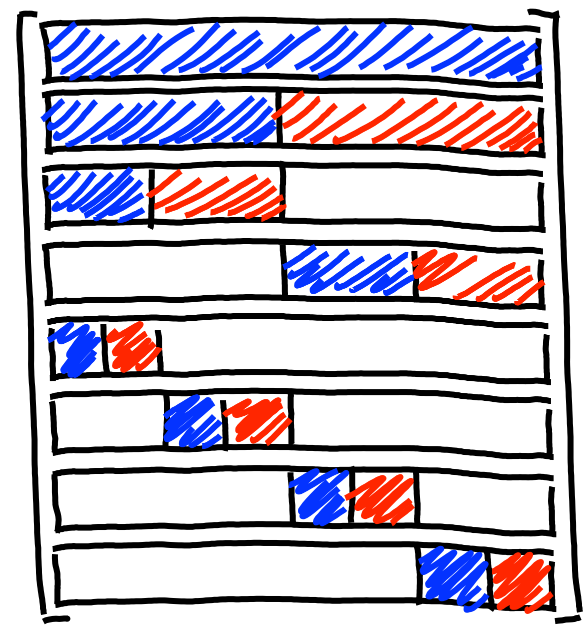
Wavelet Compression Algorithm #1

Input: 1D image I, desired compression k

Output: $k \cdot 2^N$ coefficients

1. Compute $\tilde{W} I$
2. Sort the coefficients $c_0^0, d_0^1, d_1^1, \ldots$ In order of decreasing absolute value
3. Keep the top $k \cdot 2^N$ coefs

$\tilde{W} =$



normalized wavelet coefficients

* Readings show that the algorithm gives the best least-squares approx of the image for the given compression level

$$\begin{bmatrix} c_0^0 \\ d_0^1 \\ d_1^1 \\ \vdots \\ \\ \\ \\ \\ \end{bmatrix} = \tilde{W} \cdot \begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}$$

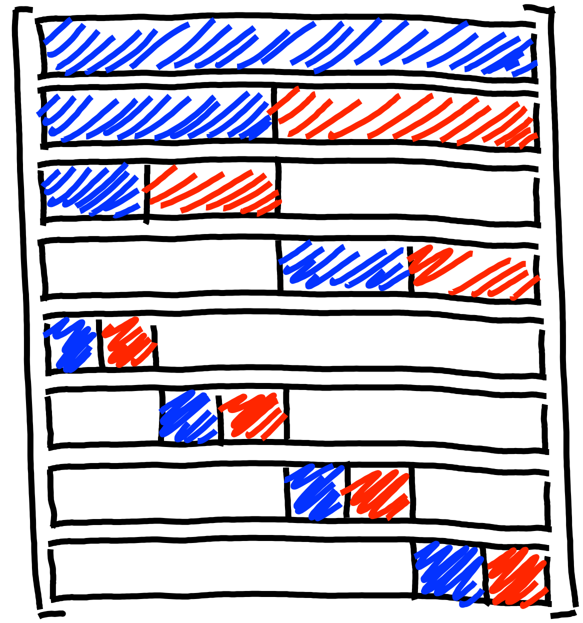
Original image I

Wavelet Compression Algorithm #2

Input : 1D image I, max error ϵ

Output: $\kappa \cdot 2^N$ coefficients

1. Compute $\tilde{W} I$
2. Sort the coefficients $c_0^0, d_0^1, d_1^1, \ldots$ In order of decreasing absolute value
3. keep the top $\kappa \cdot 2^N$ coefs with κ such that $|\tilde{I} - I| < \epsilon$, where $\tilde{I}$ is the image reconstructed from the top $\kappa 2^N$ coefs

$\tilde{W} =$ 

normalized wavelet coefficients

$$\begin{bmatrix} c_0^0 \\ d_0^1 \\ d_1^1 \\ \vdots \end{bmatrix} = \tilde{W} \cdot \begin{bmatrix} 9 \\ 7 \\ 3 \\ 5 \\ 6 \\ 10 \\ 2 \\ 6 \end{bmatrix}$$

Original image I

Topic 7:

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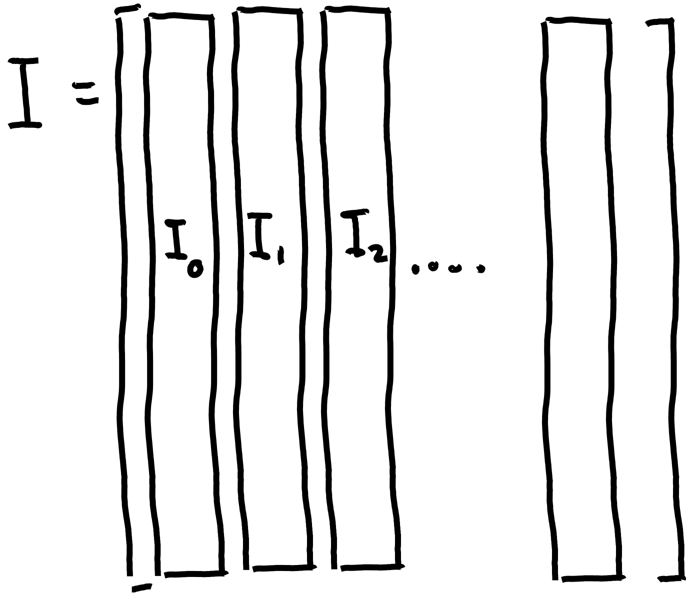
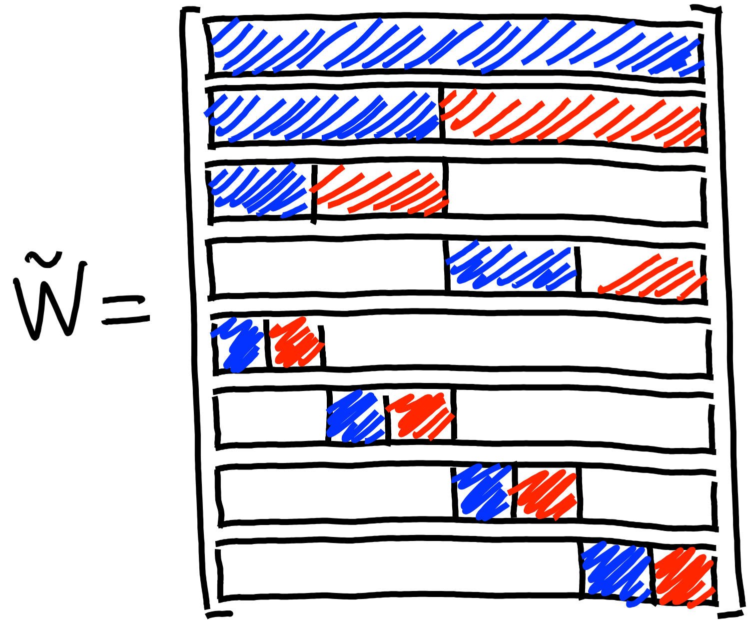
The 2D Haar Wavelet Transform

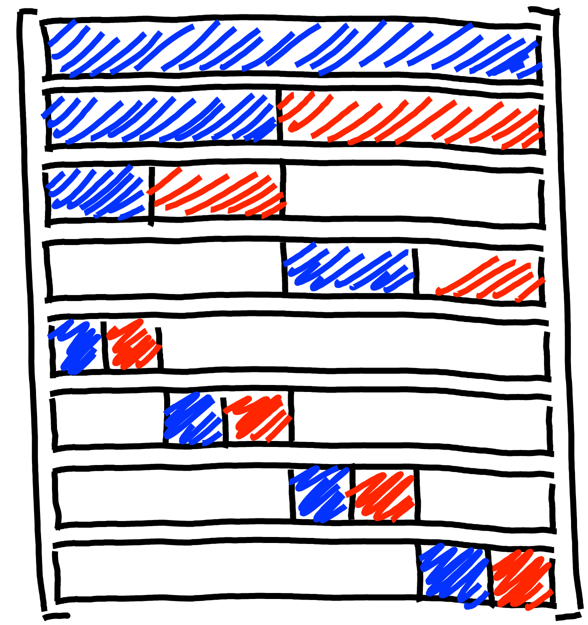




- To compute the wavelet transform of a 2D image.
  1. Compute the 1D transform for each column and place the vectors $\tilde{W} I_i$ in a new image I'
  2. Compute the 1D transform of each row of I'

The 2D Haar Wavelet Transform

Exercise: Show that every 2D wavelet coefficient can be expressed as the result of a dot product of the image I and an image defined by $(\varphi_i^j)^T \cdot (\varphi_{i'}^{j'})$ where φ_i^j are 1D Haar basis images

$\tilde{W} =$ 

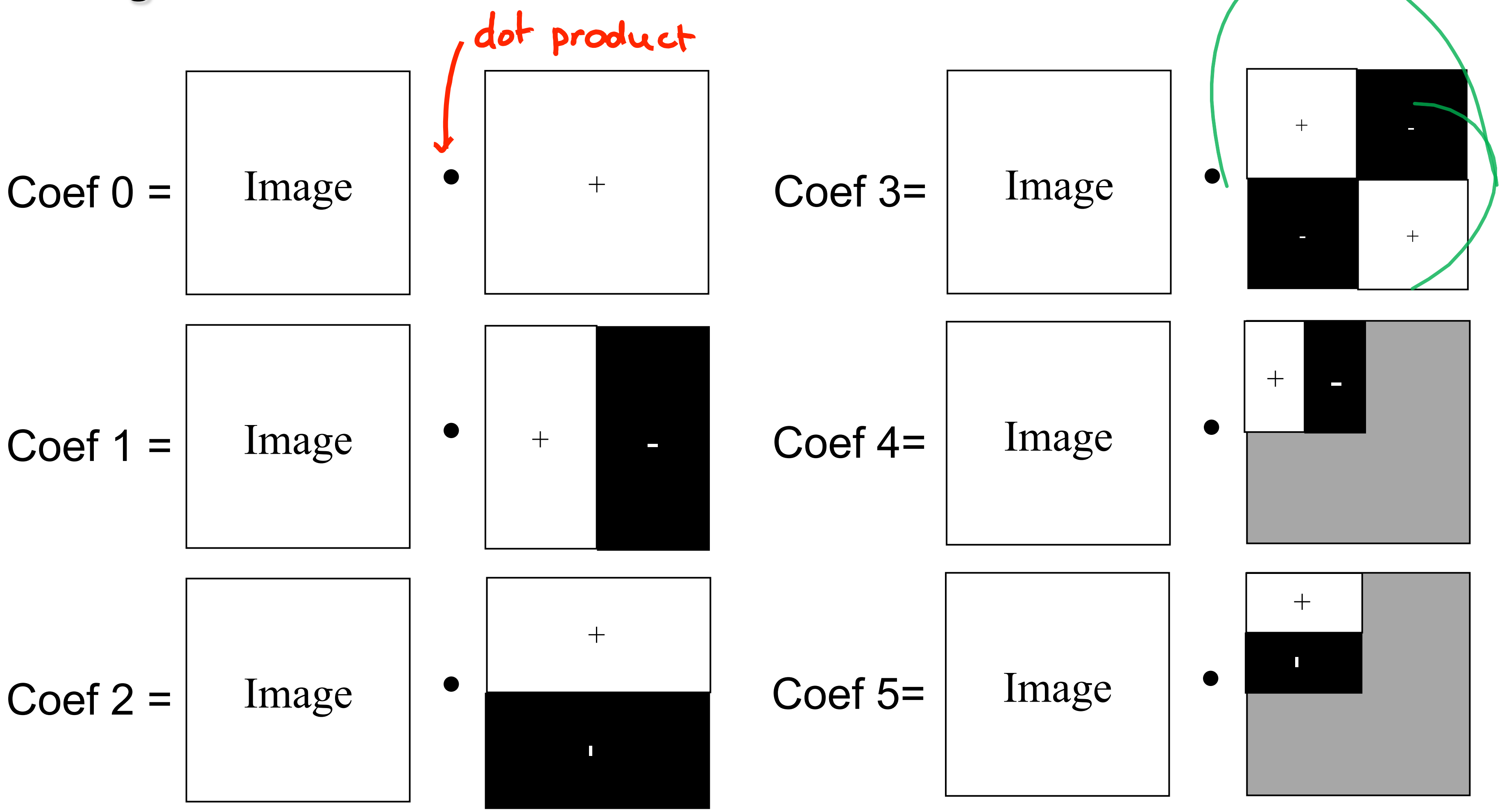
- To compute the wavelet transform of a 2D image.

① Compute the 1D transform for each column and place the vectors $\tilde{W} I_i$ in a new image I'

② Compute the 1D transform of each row of I'

The 2-D Haar Wavelet Basis

Definition of the first few (coarsest scale) wavelet coefficients of an image of dimensions of $2^N \times 2^N$

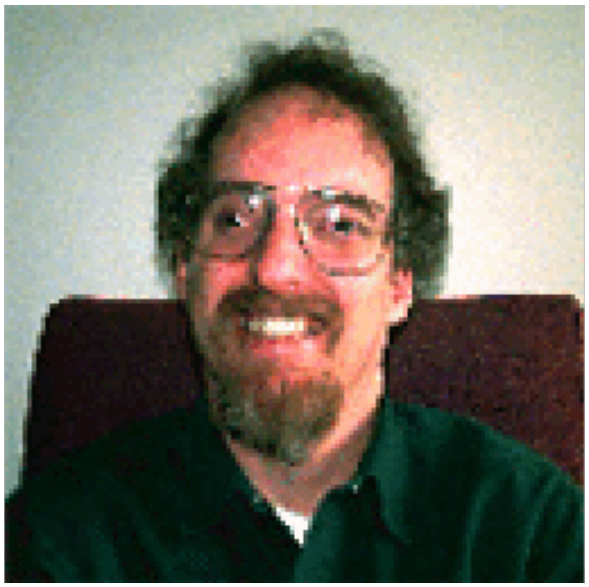


A Simple, Minimal 2-D Image Transform

The Haar 2-D Wavelet Transform

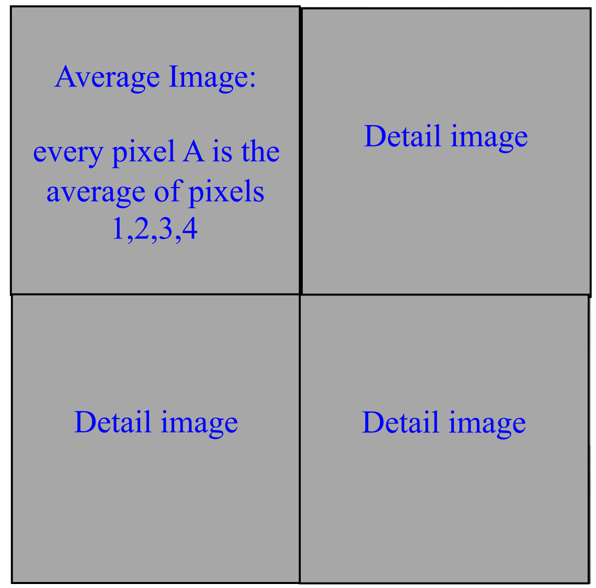
The 2-D Haar Wavelet Transform corresponds to a modification of this minimal recursive transform

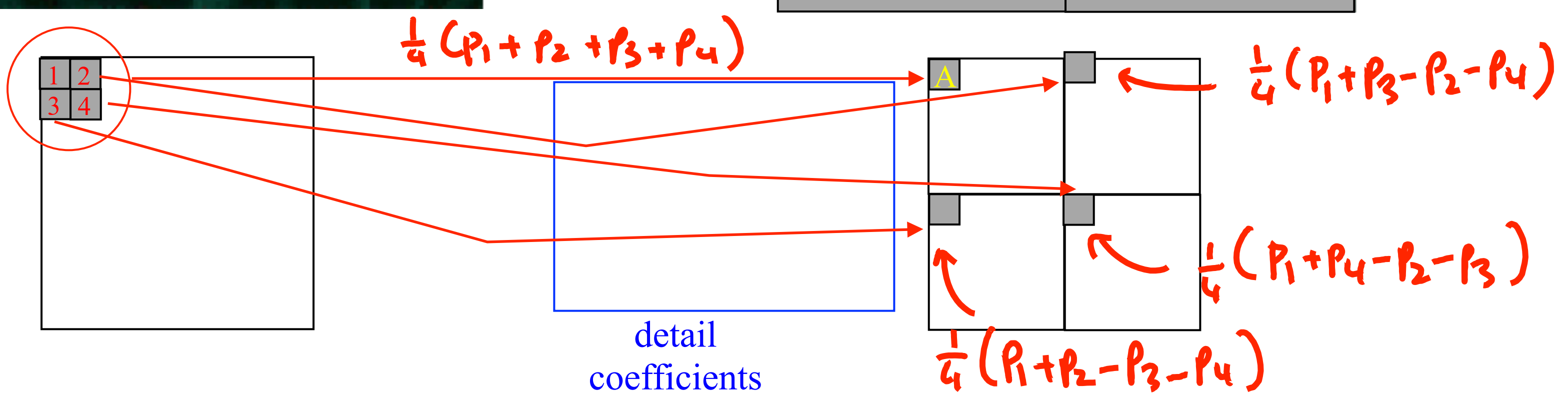
Input image (2^Nx2^N)



Step 1

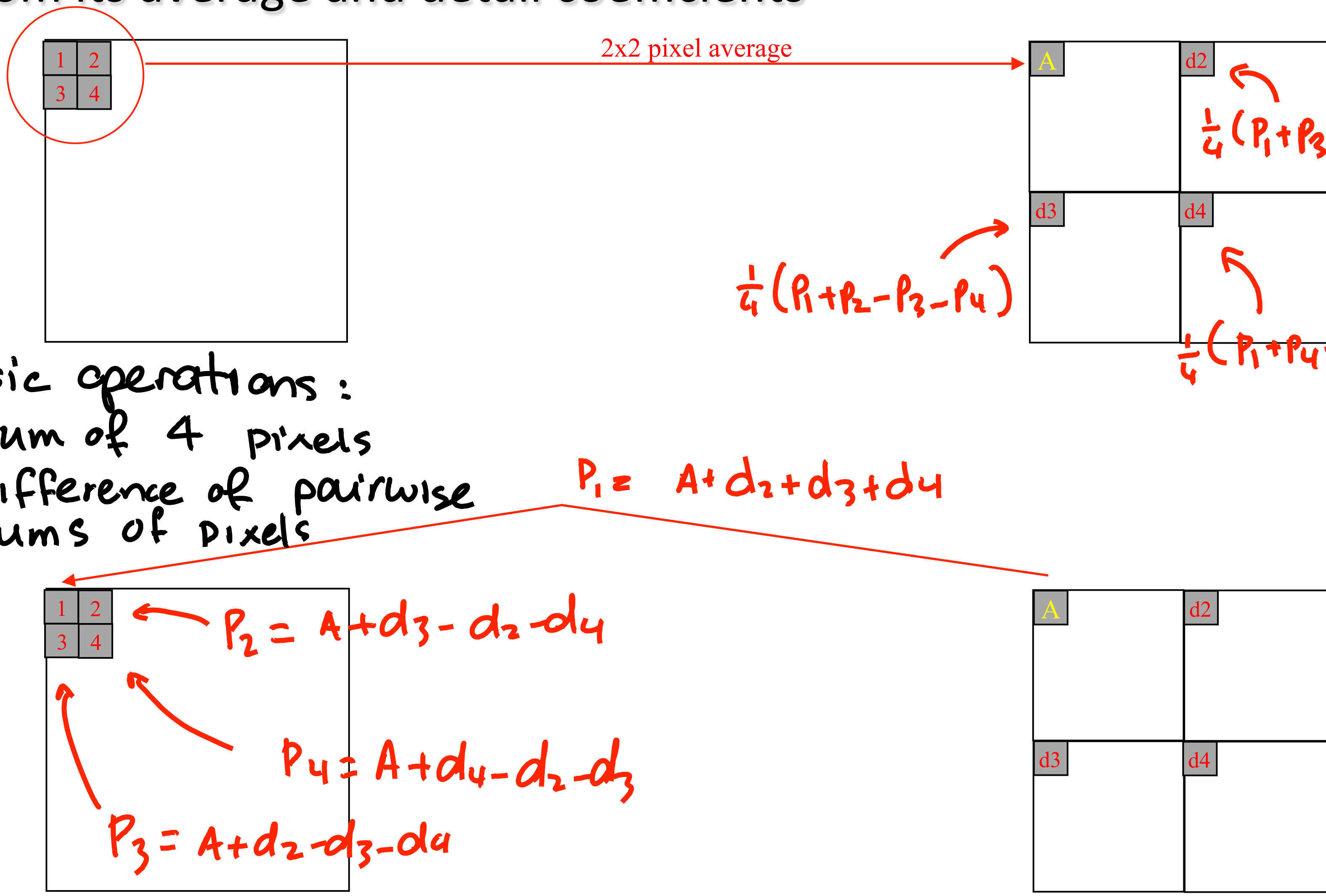
Wavelet image representation (2^Nx2^N)





Invertibility of the 2D Haar Transform

We can recursively reconstruct the intensities of every 2x2 window from its average and detail coefficients



$\frac{1}{4}(P_1+P_3-P_2-P_4)$

$\frac{1}{4}(P_1+P_2-P_3-P_4)$

$\frac{1}{4}(P_1+P_4-P_2-P_3)$

2 basic operations:

- sum of 4 pixels
- difference of pairwise sums of pixels

$P_1 = A+d_2+d_3+d_4$

$P_2 = A+d_3-d_2-d_4$

$P_4 = A+d_4-d_2-d_3$

$P_3 = A+d_2-d_3-d_4$