Student Information

Full Name : Zeynep Özalp
Id Number : 2237691

Answer 1

a.

(i) $G = \{V, \Sigma, R, S\}$
$V = \{a, b, S, S_1, S_2\}$
$\Sigma = \{a, b\}$
$R = \{S- > aSa|bS_1,$
$S_1- > aS_1a|b\}$

(ii) E generates the strings of even length except the empty sting, O generates the strings of odd length. The length of string produced by S is $even + even + 3 = odd$ or $odd + odd + 3 = odd$.
$G = \{V, \Sigma, R, S\}$
$V = \{a, b, S, E, O\}$
$\Sigma = \{a, b\}$
$R = \{S- > aEaEa|bEbEb|aOaOa|bObOb|aaa|bbb,$
$E- > aEa|aEb|bEa|bEb|aa|bb,$
$O- > aE|Ea|bE|Eb|a|b\}$

(iii) S_1 creates the string $a^ib^jc^k$ where $i \neq j$ and S_2 creates the string $a^ib^jc^k$ where $j \neq k$. For S_1, T_1 creates the string a^ib^j where $i \neq j$. T_2 creates the string a^ib^j where $i = j$. A_1 is a^+, B_1 is b^+ and C_1 is c^*. For S_2, T_3 creates the string b^jc^k where $j \neq k$. T_4 creates the string b^jc^k where $k = j$. A_2 is a^*, B_2 is b^+ and C_2 is c^+.
$G = \{V, \Sigma, R, S\}$
$V = \{a, b, S, S_1, S_2, T_1, T_2, T_3, T_4, A_1, A_2, B_1, B_2, C_1, C_2\}$
$\Sigma = \{a, b\}$
$R = \{S- > S_1|S_2,$
$S_1- > T_1C_1,$
$T_1- > A_1T_2|T_2B_1,$
$T_2- > aT_2b|e,$
$A_1- > aA_1,$
$B_1- > bB_1,$
$C_1- > cC_1|e,$
$S_2- > A_2T_3,$
$T_3- > B_2T_4|T_4C_2,$
$T_4- > bT_4c|e,$
$B_2- > bB_2,$
$C_2- > cC_2,$
$A_2- > aA_2|e\}$

(iv) S_1 generates all strings, S_2 generates a $w_i c$, S_3 generates the sequence of $w_i c$'s. S_4 generates $w_j c\{sequence of w_i c's\} c w_j^R$.
$G = \{V, \Sigma, R, S\}$
$V = \{a, b, S, S_1, S_2, S_3, S_4\}$
$\Sigma = \{a, b\}$
$R = \{S- > S_3 S_4,$
$S_1- > aS_1|bS_1|e,$
$S_2- > aS_1c|bS_1c,$
$S_3- > S_2S_3|S_2,$
$S_4- > aS_4a|bS_4b|cS_3c\}$

b.

PART 1

$L(G) \subseteq L$, every string generated by G is in L.
Use mathematical induction.

1. Basis step: Derivation in 2 steps.

$$S \Rightarrow aB \Rightarrow ab \in L$$

$$S \Rightarrow bA \Rightarrow ba \in L$$

2. Inductive Hypothesis: Assume that for every derivation $S \Rightarrow^* w$ with $n \geq 2$ steps, $w \in L$

3. Inductive Step: Let $S \Rightarrow^* w$ be a derivation with $n + 2$ steps, $n \in N$. Since $n + 2 > 2$, the derivation starts with either $abS, baS, aABB$ or $bBAA$.

i)$S \Rightarrow aB \Rightarrow abS \Rightarrow^* abw_1$. By ind. hyp., $w_1 \in L$,w_1 has equal number of a's and b's, so does abw_1 and $abw_1 \in L$.
ii)$S \Rightarrow bA \Rightarrow baS \Rightarrow^* baw_2$. By ind. hyp., $w_2 \in L$,w_2 has equal number of a's and b's, so does baw_2 and $baw_2 \in L$.
iii)$S \Rightarrow aB \Rightarrow abS \Rightarrow abaB \Rightarrow abaABB$. So, the string $aABB$ can be generated by S in two steps and in L.
iv)$S \Rightarrow bA \Rightarrow baS \Rightarrow babA \Rightarrow babBAA$. So, the string $bBAA$ can be generated by S in two steps and in L.
Therefore, every string generated by G is in L.

PART 2

$L \subseteq L(G)$, every string in L can be generated by G.
$L = \{ab, ba, ...\}$ Use mathematical induction.

1. Basis step: For $|w| = 2$, $S \Rightarrow aB \Rightarrow ab$ and $S \Rightarrow bA \Rightarrow ba$. So, $ab \in L(G)$ and $ba \in L(G)$.

2. Inductive Hypothesis: Assume that all strings $|w| \leq k-1$ in L can be generated by G.

3. Inductive Step: Suppose $x = a_1 a_2 a_3 ... a_k \in L$ with $k > 2$, $a_i \in \Sigma$. Now assume that $a_1 a_2 = ab$. Define $N_a(x)$ as the number of a's in x and

$$d_i = N_a(x) - N_b(x), 0 \leq i \leq k$$

$$t = min(i > 0, d_i = 0)$$

Since we choose $a_1 a_2 = ab$, $d_1 = 1, d_2 = 0$. Also, $d_k = 0$ since $x \in L$. Thus, there exists a smallest i such that $d_i = -1$ and $d_{i+1} = 0$. This value is represented as t. Since t was chosen to be the smallest possible and $d_1 > 0$, we must have $d_t < 0$. Since $d_t < 0$ and $d_{t+1} = 0$, we must have $a_t a_{t+1} = ba$. Therefore, we have

$$x = aba_3 ... a_{t-1} baa_{t+2} ... a_k$$

Let $y = a_3 ... a_{t-1}$ and $z = a_{t+2} ... a_k$. Clearly, because $x \in L$, y and z have equal number of a's and b's, $y \in L$ and $z \in L$. By inductive hypothesis, since $|y| < k$ and $|z| < k$, $y, z \in L(G)$. This means $S \Rightarrow^* y$ and $S \Rightarrow^* z$.

$$S \Rightarrow aB \Rightarrow abS \Rightarrow^* abyS \Rightarrow abybA \Rightarrow abybaS \Rightarrow^* abybaz = x$$

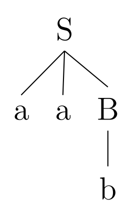
Therefore, $x \in L(G)$ and for all $x \in L \rightarrow x \in L(G)$. This completes the proof that $L(G)$ is the set of all non-empty strings in Σ^* that have equal number of a's and b's.

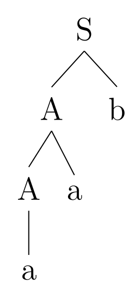
Answer 2

a.

$$s = aab$$

$S \Rightarrow aaB \Rightarrow aab$
$S \Rightarrow Ab \Rightarrow Aab \Rightarrow aab$

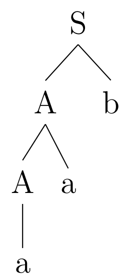




b.

$L(G) = \{ab, aab, aaab, ...\}$
G' is an equivalent unambiguous CFG of G.
$G' = \{V', \Sigma, R', S\}$, $V' = \{a, b, S, A\}$,
$R = \{S \rightarrow Ab, A \rightarrow a|aA\}$

c.



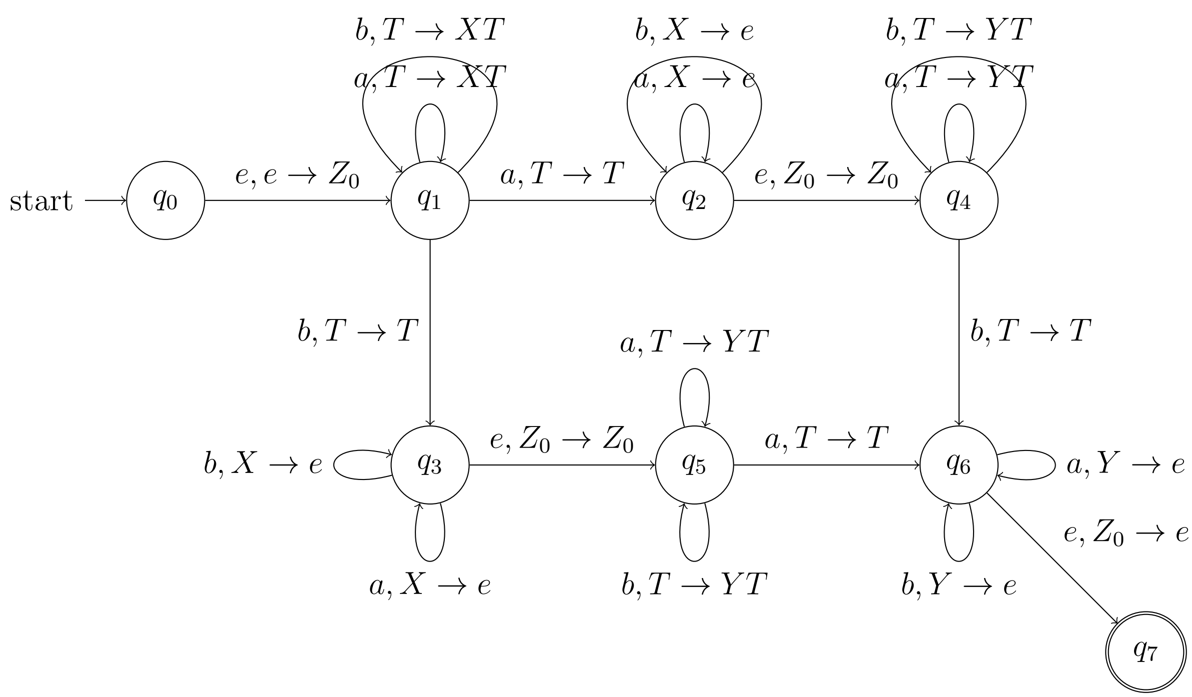
Answer 3

a.

$L = \{w \in \{a,b\}^* | w = (aa)^i(bbb)^i, i \geq 0\}$

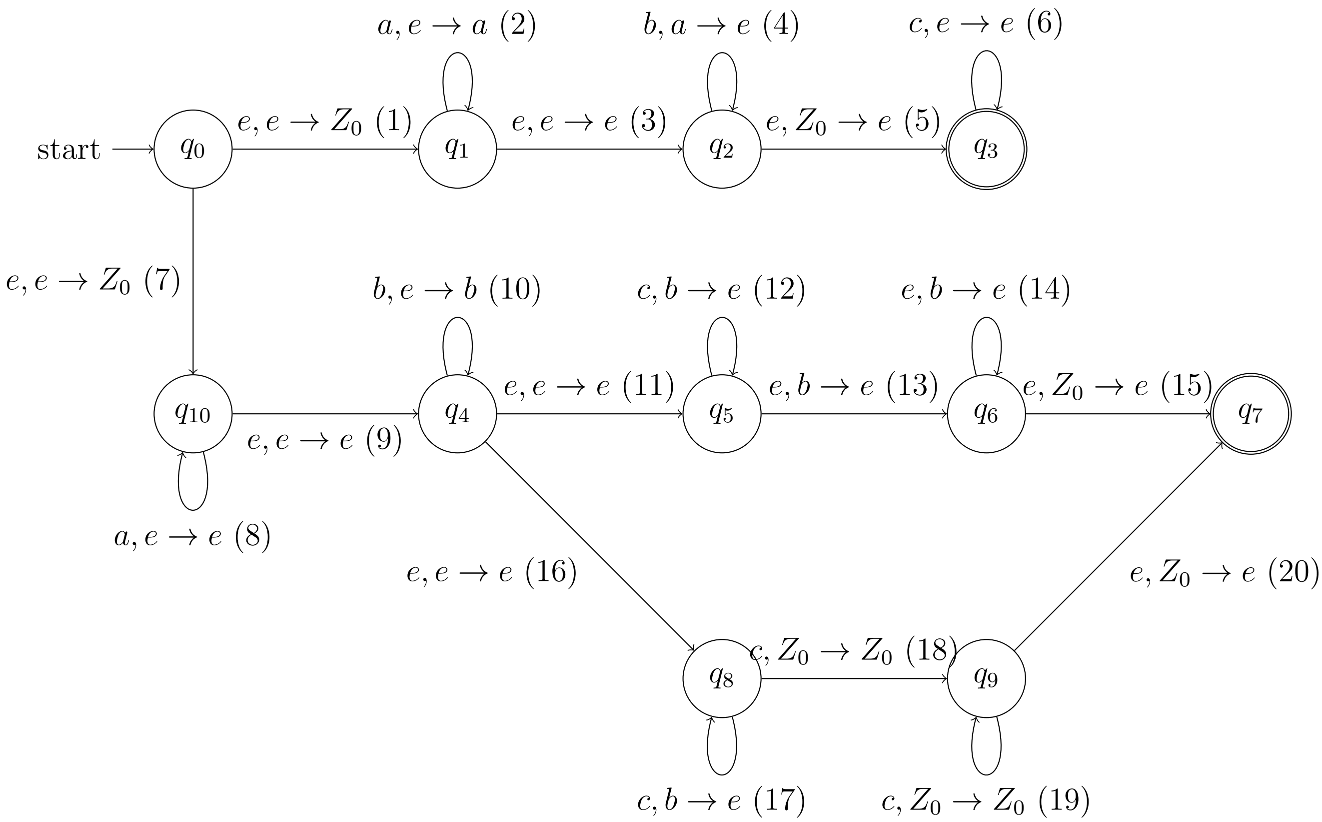
b.

$M = \{K, \Sigma, \Gamma, \Delta, q_0, q_7\}$
$K = \{q_0, q_1, q_2, q_3, q_4, q_5, q_6, q_7\}$, $\Sigma = \{a, b\}, \Gamma = \{Z_0, X, Y\}$
$T = \{Z_0, X\}$



c.

(i) $L(M) = \{a^i b^j c^k \in \{a,b,c\}^* | i,j,k \geq 0, i = j \text{ or } j > k \text{ or } j < k\}$
$M = (K, \{a,b,c\}, \Gamma, \Delta, q_0, \{q_3, q_7\})$
$K = \{q_0, q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9, q_{10}\}$, $\Gamma = \{Z_0, a, b\}$



(ii)

State	Input	Stack	Transition
q_0	aabcc	e	-
q_{10}	aabcc	Z_0	(7)
q_{10}	abcc	Z_0	(8)
q_{10}	bcc	Z_0	(8)
q_4	bcc	Z_0	(9)
q_4	cc	b, Z_0	(10)
q_8	cc	b, Z_0	(16)
q_8	c	Z_0	(17)
q_9	e	Z_0	(18)
q_7	e	e	(20)
q_7	e	e	Accepts.

State	Input	Stack	Transition
q_0	bac	e	-
q_1	bac	Z_0	(1)
q_2	bac	Z_0	(3)
q_3	bac	Z_0	(5)
q_3	bac	Z_0	Rejects.
q_0	bac	e	-
q_{10}	bac	Z_0	(7)
q_4	bac	Z_0	(9)
q_4	ac	b, Z_0	(10)
q_5	ac	b, Z_0	(11)
q_5	ac	b, Z_0	Rejects.
q_0	bac	e	-
q_{10}	bac	Z_0	(7)
q_4	bac	Z_0	(9)
q_4	ac	b, Z_0	(10)
q_8	ac	b, Z_0	(11)
q_8	ac	b, Z_0	Rejects.

Answer 4

a.

Construct a PDA $M = (\{p, q\}, \Sigma, V, \Delta, p, \{q\})$ such that $L(M) = L(G)$. By Lemma 3.4.1,
$\Delta =$
$((p, e, e), (q, E))$,
$((q, e, E), (q, E + T))$,
$((q, e, E), (q, T))$,
$((q, e, T), (q, T \times F))$,
$((q, e, T), (q, F))$,
$((q, e, F), (q, (E)))$,
$((q, e, F), (q, a))$,
$((q, a, a), (q, e))$,
$((q, +, +), (q, e))$,
$((q, \times, \times), (q, e))$,
$((q, (, ()), (q, e))$,
$((q,),)), (q, e))$

b.

First add Δ' the transitions ((s',e,e),(s,Z)) and ((f,e,Z),(f',e)) for each $f \in F$. Now, Δ' consists of these transitions described above and all the transitions in Δ. Next, we replace all the transitions in Δ' that violates the condition $|\beta| + |\gamma| \leq 1$.

Consider any transition $((q, a, \beta), (p, e))$ where $\beta = B_1 B_2 ... B_n$ with $n > 1$. It is replaced by new transitions that pop sequentially the symbols $B_1 ... B_n$. So, we add δ' these transitions:

$$((q, e, B_1), (q_1, e))$$

$$((q_1, e, B_2), (q_2, e))$$

$$\vdots$$

$$((q_{n-2}, e, B_{n-1}), (q_{n-1}, e))$$

$$((q_{n-1}, e, B_n), (p, e))$$

We repeat these replacements for all transitions $((q, a, \beta), (p, e))$ where $|\beta| > 1$ in δ'.

Similarly, we replace transitions $((q, a, e), (p, \gamma))$ where $\gamma = G_1 G_2 ... G_m$ and $m > 1$ by the new transitions:

$$((q, e, e), (q_1, G_1))$$

$$((q_1, e, e), (q_2, G_2))$$

$$\vdots$$

$$((q_{n-2}, e, e), (q_{n-1}, G_{n-1}))$$

$$((q_{n-1}, e, e), (p, G_n))$$

We repeat these replacements for all transitions $((q, a, e), (p, \gamma))$ where $|\gamma| > 1$ in δ'.

Now, consider any transition $((q, a, \beta), (p, \gamma))$ where $\beta = B_1 B_2 ... B_n$ with $n > 1$ and $\gamma = G_1 G_2 ... G_m$ with $m > 1$. We replace these transitions by the new transitions:

$$((q, e, B_1), (q_1, e))$$

$$((q_1, e, B_2), (q_2, e))$$

$$\vdots$$

$$((q_{n-2}, e, B_{n-1}), (q_{n-1}, e))$$

$$((q_{n-1}, e, B_n), (p, e))$$

$$((q_n, e, e), (q_{n+1}, G_1))$$

$$((q_{n+1}, e, e), (q_{n+2}, G_2))$$

$$\vdots$$

$$((q_{n+m-2}, e, e), (q_{n+m-1}, G_{m-1}))$$

$$((q_{n+m-1}, e, e), (p, G_m))$$

It is clear that the resulting pushdown automata is equivalent to the original one.

Answer 5

a.

(i) Let $L_1 = \{a^m b^m \in \{a,b\}^* | m \in N\}$ and $L_2 = \{b^n a^n \in \{a,b\}^* | n \in N\}$. Grammer for L_1 is $G_1 = \{\{a,b,S\}, \{a,b\}, \{S \to aSb|e\}, S\}$ and grammer for L_2 is $G_1 = \{\{a,b,S\}, \{a,b\}, \{S \to bSa|e\}, S\}$. So, L_1 and L_2 are context-free languages. By Theorem 3.5.1, their concatenation L_1L_2 are also context-free.

(ii) $L = \{ba, babaa, babaabaaa, ...\}$
Let L' be the complement of L and use an algorithm to find the complement of L. We can generate strings that is not in L by changing one a or b from the beginning and in order. For $n = 1$, we have the string ba in L. Change a b to an a or change an a to a b in ba, we get aa or bb respectively, which are not in L. Concatenating Σ^* to these two strings we have the languages $aa\Sigma^*$ and $bb\Sigma^*$. These two languages differ from L by the first a or the first b symbols. However, we may have other strings that differs from L by changing the n^th a to a b and n^th b to an a, which are in case, the ones who do not fit the condition of L. Thus, one can generate L' by taking the infinite union of such languages.

$$L' = a\Sigma^* \cup baa\Sigma^* \cup babaaa\Sigma^* \cup ... \cup bb\Sigma^* \cup babb\Sigma^* \cup babab\Sigma^* \cup babaabb\Sigma^* \cup ...$$

Since all regular languages are context-free and context-free languages are closed under union by Theorem 3.5.1, L' is context-free.

b.

(i) Assume L is context-free. Then there exists a pumping length k such that for all $w \in L$ and $|w| > k$ there exists a split $w = uvxyz$, $|vy| > 0$, $|vxy| \leq k$ and $yv^nxy^nz \in L$ for all $n \geq 0$. Choose $w = a^{k^2}b^k$. There are three possible splits:

1) $vxy = a^j$, $j > 0$. $uv^2xy^2z = a^{k^2+2j}b^k$. $k^2 + 2j$ is not less than or equal k^2 since $j > 0$.
2) $vxy = b^j$, $j > 0$. $uv^2xy^2z = a^{k^2}b^{k+2j}$. k^2 is not less than or equal $(k+2j)^2$ since $j > 0$.
3) $vxy = a^ib^j$, $0 < i + j \leq k$. There are 2 cases:
3.1) v and y consist of one symbol, $v = a^l$ and $y = b^m$. If we pump n times, we get $a^{k^2+nl}b^{k+nm}$. $k^2 + nl \leq (k+nm)^2$ should hold for all $n, m, l \in N$. Clearly it does not.
3.2) One of v or y consist of two symbols, say v, $v = a^ib^j$ and $y = b^m$. When we pump, the order of the new string does not match the order of the languages in L that is first only a's then only b's.
For all of these 3 cases, $w \notin L$. Thus, L is not context-free.

(ii) Assume L is context-free. Then there exists a pumping length k such that for all $www \in L$ and $|www| > k$ there exists a split $www = uvxyz$, $|vy| > 0$, $|vxy| < k$ and $yv^nxy^nz \in L$ for all $n \geq 0$. Choose $www = a^kb^ka^kb^ka^kb^k$. There are three possible splits:

1. $vxy = a^i$. For n=0:
1.1: $yv^nxy^nz = a^{k-i}b^ka^kb^ka^kb^k$ is not in L since to get w we need to split this string in 3

pieces with equal lengths. However, when we do this, the first one ends with an a but the last one ends with a b. So, there no such string in L.
1.2: $yv^nxy^nz = a^kb^ka^{k-i}b^ka^kb^k$ is not in L since to get w we need to split this string in 3 pieces with equal lengths. However, when we do this, the first one begins with an a but the middle one begins with a b. So, there no such string in L.
1.3: $yv^nxy^nz = a^kb^ka^kb^ka^{k-i}b^k$ is not in L since to get w we need to split this string in 3 pieces with equal lengths. However, when we do this, the first one begins with an a but the last one begins with a b. So, there no such string in L.
The same discussion and reasons are valid for case 2 and 3.
2. $vxy = a^ib^j$. For n=0:
2.1: $yv^nxy^nz = a^{k-i}b^{k-j}a^kb^ka^kb^k$
2.2: $yv^nxy^nz = a^kb^ka^{k-i}b^{k-j}a^kb^k$
2.3: $yv^nxy^nz = a^kb^ka^kb^ka^{k-i}b^{k-j}$
3. $vxy = b^ja^i$. For n=0:
3.1: $yv^nxy^nz = a^kb^{k-j}a^{k-i}b^ka^kb^k$
3.2: $yv^nxy^nz = a^kb^ka^kb^{k-j}a^{k-i}b^k$
Therefore, L is not context-free.

Answer 6

(i) (T/F)? F

(ii) (T/F)? T

(iii) (T/F)? T

(iv) (T/F)? F