Moduli Spaces of Riemann Surfaces

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Abstract

The theory of Riemann surfaces, first developed by Bernhard Riemann to study algebraic functions, now lies in the confluence of complex analysis, differential geometry, and algebraic geometry. This expository paper aims to introduce this theory, with the goal classifying all compact Riemann surfaces of genus 0 and 1. To do so, we first develop the basics of covering space theory, which defines the degree of proper holomorphic maps, and then study the sheaf of holomorphic maps on a Riemann surface and their associated cohomology theory. Together, they form the core technical tools of the paper and allow us to connect the function theory of Riemann surfaces to their complex structure. Lastly, we give a glimpse into the non-compact case, namely the Uniformization Theorem, which gives us a tri-fold classification of all Riemann surfaces.

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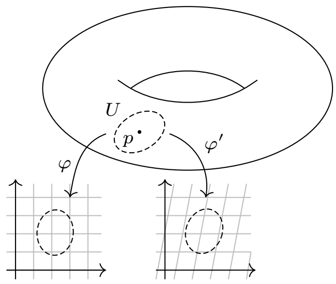
Chapter 1

Introduction

1.1 Overview and Main Results

Complex analysis in the plane is an extremely rich theory with many important features, but is topologically not very interesting. On the other hand, general topological spaces are too badly-behaved for any reasonable analysis to be done, so a middle-ground must be found. To leverage the desired properties of $\mathbb{C}$, we restrict the class of topological spaces of study to those with a local neighborhood around every point that looks like a deformed patch of $\mathbb{C}$, but whose global behaviour can be quite different. The choice in which a topological space is made to look locally like $\mathbb{C}$ is called a *complex structure*, and, in general, there are many such choices. A topological space X, equipped with a particular choice of complex structure, is called a *Riemann surface*.

As a motivating example, take the torus T^2. Around every point $p \in T^2$, we can find a small enough neighborhood U of p that deforms reversibly onto an open subset of $\mathbb{C}$. The figure below shows two ways of doing so.



Using φ to 'pull' the coordinate lines back to U, we see that the angles that they make is different from the angles made by using φ' instead. The rigidity of holomorphic maps from complex analysis suggests that those coordinates ought to be different, and indeed they are. Thus we see that the same topological space, T^2, can be equipped with many different complex structures, making them different *complex tori*.

In general, we call the set of all complex structures on X the *moduli space*[1] *of* X, and the main goal of this paper is to compute it for the sphere S^2 and the torus T^2. The results, proven in Theorem 5.1 and Corollary 5.4.1 respectively, are as follows.

- Surprisingly, the moduli space of S^2 is a point. In other words, the sphere admits a unique complex structure. This fact, which is part of the *Uniformization Theorem*, is one of the starting points in the theory of compact Riemann surfaces, which we will spend the majority of this paper discussing.
- The fact that T^2 can be constructed topologically as a quotient $\mathbb{C}/\Gamma$ by an integer lattice Γ gives us a relatively straightforward proof that the moduli space of T^2 is $\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$. Here, $\mathbb{H} \subset \mathbb{C}$ is the upper-half plane of $\mathbb{C}$ and $\mathrm{PSL}_2(\mathbb{Z}) \coloneqq \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$ is the *modular group*, which acts on $\mathbb{H}$ via Möbius transformations. We show that all complex tori can be written as $X_\tau \coloneqq \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ for some $\tau \in \mathbb{H}$, and two tori X_τ and $X_{\tau'}$ are biholomorphic iff τ and τ' lie in the same orbit of the action.

1.2 Organization and Prerequisites

We give a brief overview of the organization of this paper.

- Chapter 2 begins with some definitions and constructions relating to Riemann surfaces and introduces the main examples of interest to this paper: the Riemann sphere and complex tori. We then study the basic behaviours of maps between Riemann surfaces, with a focus on meromorphic functions and their associated holomorphic maps.
- Chapter 3 studies the covering space theory of Riemann surfaces, with the goal of defining the degree of a proper holomorphic map. This reduces the problem of showing that a simply-connected compact Riemann surface X is biholomorphic to the Riemann sphere to furnishing a global meromorphic function on X.
- Chapter 4 builds up sheaf theory and their associated cohomology to prove the existence of such a meromorphic function. The theory of (complex) differential forms is then introduced to study the sheaf of holomorphic functions on the Riemann sphere, where we prove the existence of such a meromorphic function on X.
- Chapter 5 ties everything together and uses the tools developed to compute the moduli space of genus 0 and 1 surfaces (S^2 and T^2). This chapter closes with a brief discussion of the Uniformization Theorem and its impacts on the theory of Riemann surfaces.

As for prerequisites, some familiarity with topology and complex analysis is required, and an exposure to the theory of real manifolds is nice to have (but not essential). We also assume that the reader is comfortable with some linear algebra and basic group theory. A more complete list of prerequisites, along with references, will be given at the start of each chapter.

[1]This paper is only concerned with the underlying set of points, without regard to any geometric structure. This turns out to be interesting enough in its own right, but the reader should be aware that the study of geometric structures on the moduli space is vast. We refer the interested reader to [Tan91], [Mar12], and [Hub06].

Chapter 2

Riemann Surfaces

We begin with some basic definitions and constructions relating to Riemann surfaces that will be used throughout this paper. This chapter requires some background in topology and complex analysis, all of which can be found in classical texts such as [Mun00] and [Lan98]. For an introduction to topology focused on (real) manifolds, see [Lee10] or [Tu10].

2.1 Charts and Atlases

We first formalize what we mean for a topological space to 'locally look like a patch of $\mathbb{C}$'. In this section, let X be a connected second-countable Hausdorff space.

Definition 2.1. *A complex chart of X is a pair (U, φ) where $\varphi : U \to V$ is a homeomorphism from an open subset $U \subseteq X$ onto an open subset $V \subseteq \mathbb{C}$. Two charts (U_1, φ_1) and (U_2, φ_2) are said to be compatible if either $U_1 \cap U_2 = \varnothing$, or the map*

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1 (U_1 \cap U_2) \to \varphi_2 (U_1 \cap U_2)\,,$$

called the transition map, is biholomorphic. A complex atlas on X is a collection $\mathfrak{A} := \{(U_i, \varphi_i)\}_{i \in I}$ of pairwise compatible complex charts that cover X.

Remark. Charts provide *local coordinates* for every point in X in such a way that the transition maps $\varphi_j \circ \varphi_i^{-1}$ respect the analytic structure of $\mathbb{C}$. Within the same atlas $\mathfrak{A}$, those charts give us different coordinate representations for points in $U_i \cap U_j$, and since no chart is distinguished from the others, we can only define notions using local coordinates if they are invariant under the transition map.

$$\begin{tikzcd} & U_i \cap U_j \arrow[dl, "\varphi_i"'] \arrow[dr, "\varphi_j"] & \\ \varphi_i(U_i \cap U_j) \arrow[rr, "\varphi_j \circ \varphi_i^{-1}"'] & & \varphi_j(U_i \cap U_j) \end{tikzcd}$$

It is a classical result in complex analysis that the inverse of a holomorphic map is also holomorphic, so $\varphi_j \circ \varphi_i^{-1}$ is biholomorphic iff $\varphi_i \circ \varphi_j^{-1}$ is, which is convenient when checking that a collection of charts form an atlas. Lastly, we remark that it is sometimes convenient to write (U, z) for (U, φ), which can be decomposed into $z = x + iy$ by taking the real and imaginary parts of φ. ♦

Definition 2.2. *Two complex atlases $\mathfrak{A}$ and $\mathfrak{B}$ on X are said to be equivalent if every chart of $\mathfrak{A}$ is compatible with every chart in $\mathfrak{B}$.*

Remark. By Zorn's Lemma, every atlas $\mathfrak{A}$ of a manifold X is contained in a unique maximal atlas on X (see, for instance, [Lee12, Proposition 1.17]). Moreover, two atlases are equivalent iff they are contained in the same maximal atlas, which justifies the following definition. ♦

Definition 2.3. *A complex structure on X is a maximal atlas $\mathfrak{A}$ on X, or, equivalently, an equivalence class of complex atlases on X. The pair $(X, \mathfrak{A})$ is then called a Riemann surface.*

Remark. Every Riemann surface can be regarded as a (connected) 2-dimensional real manifold by 'forgetting' its complex structure. Since orientations are invariant under biholomorphisms, and in particular transition maps, the local orientation of $\mathbb{C}$ pulls-back via charts to a local orientation at each point $p \in X$. Since charts cover X, these local orientations induce a global orientation on X. Thus all Riemann surfaces are orientable, so, by the Classification of Surfaces, the closed Riemann surfaces are classified by their genus. Note, however, that this is a *topological* classification, and does not give any information about the complex structure on X. ♦

Example 2.4. Some elementary examples of Riemann surfaces.

- The complex plane $\mathbb{C}$, equipped with its standard topology, can be given a complex structure $\mathfrak{A}$ by choosing the atlas containing a single chart $(\mathbb{C}, \mathrm{id}_{\mathbb{C}})$. We may, however, also give $\mathbb{C}$ a different complex structure $\mathfrak{A}'$ by choosing the chart map $\varphi : z \mapsto \overline{z}$ instead. Indeed, $\mathfrak{A} \neq \mathfrak{A}'$ since the map $\varphi \circ \mathrm{id}_{\mathbb{C}}^{-1} = \varphi$ is not holomorphic and hence the atlases $\{(\mathbb{C}, \mathrm{id}_{\mathbb{C}})\}$ and $\{(\mathbb{C}, \varphi)\}$ are not equivalent. This example generalizes to any domain $D \subseteq \mathbb{C}$.

- Let $D \subseteq \mathbb{C}$ be a domain and consider any holomorphic function $f : D \to \mathbb{C}$. Then the graph $\Gamma_f := \{(z, f(z)) \mid z \in D\}$, equipped with the subspace topology inherited from $\mathbb{C}^2$, can be given a complex structure by choosing the chart map $\pi : \Gamma_f \to D : (z, f(z)) \mapsto z$. More generally, the set X of roots of an irreducible[1] polynomial $f \in \mathbb{C}[z, w]$ where every root has at least one non-vanishing partial derivative, called a *smooth affine plane curve*, is a Riemann surface. Indeed, if $\partial f / \partial w$ is non-zero at $p = (z_0, w_0)$, then the Implicit Function Theorem furnishes a holomorphic function $g(z)$ defined on a neighborhood of z_0 such that $X = \Gamma_g$ on some neighborhood $U \ni p$. Then, as above, the projection $\pi_z : U \to \mathbb{C}$ is a homeomorphism onto its image, giving us the desired chart map. ♦

[1] The irreducibility of the polynomial ensures that its set of roots is connected. Its proof requires some algebraic geometry, which we take for granted.

2.1.1 The Riemann Sphere $\hat{\mathbb{C}}$

A particularly important Riemann surface is the Riemann sphere $\hat{\mathbb{C}}$, which admits several constructions. Here, we equip standard constructions of topological spheres with three complex structures, which *a priori* need not be biholomorphic (in the sense of Definition 2.13), but in fact are; see Example 2.14 for a proof. In fact, *any* Riemann surface that is topologically the sphere is the Riemann sphere, which we prove in Theorem 5.1.

Example 2.5 (One-point Compactification of $\mathbb{C}$)**.** Let ∞ be a symbol not belonging to $\mathbb{C}$ and set $\mathbb{C}_\infty \coloneqq \mathbb{C} \cup \{\infty\}$. We declare a set $U \subseteq \mathbb{C}_\infty$ to be open if either $U \subseteq \mathbb{C}$ is open or $U = K^c \cup \{\infty\}$ for some compact subset $K \subseteq \mathbb{C}$. This makes $\mathbb{C}_\infty$, equipped with the collection $\mathcal{T}$ of all such open sets, a second-countable Hausdorff space. Indeed, the fact that $\mathcal{T}$ is a topology on $\mathbb{C}_\infty$ follows from De Morgan's Laws and the Heine-Borel Theorem; it is Hausdorff since any $p \in \mathbb{C}$ can be separated from ∞ by neighborhoods $B(p,r)$ and $\overline{B(p,r)}^c \cup \{\infty\}$, respectively; and it is second-countable since we may append, to any countable basis for the standard topology of $\mathbb{C}$, the countable collection $\left\{\overline{B(0,r)}^c \cup \{\infty\}\right\}_{r\in\mathbb{Q}^+}$. To give $\mathbb{C}_\infty$ a complex structure, we employ two charts

$$U_1 \coloneqq \mathbb{C}_\infty \setminus \{\infty\} = \mathbb{C} \qquad \varphi_1 : U_1 \to \mathbb{C} : z \mapsto z \qquad (\varphi_1 \coloneqq \mathrm{id}_\mathbb{C})$$

$$U_2 \coloneqq \mathbb{C}_\infty \setminus \{0\} = \mathbb{C}^* \cup \{\infty\} \qquad \varphi_2 : U_2 \to \mathbb{C} : z \mapsto \begin{cases} 1/z & \text{if } z \in \mathbb{C}^* \\ 0 & \text{else.} \end{cases}$$

Clearly φ_1 is a homeomorphism. Since φ_2 is invertible with $\varphi_2^{-1}(z) \coloneqq 1/z$ for all $z \in \mathbb{C}^*$ and $\varphi_2^{-1}(0) \coloneqq \infty$, and

$$\lim_{z\to\infty} \varphi_2(z) = 0 = \varphi_2(\infty) \qquad \text{and} \qquad \lim_{z\to 0} \varphi_2^{-1}(z) = \infty = \varphi_2^{-1}(0)\,,$$

we see that φ_2 is a homeomorphism too. Furthermore, $\varphi_2 \circ \varphi_1^{-1} : \mathbb{C}^* \to \mathbb{C}^* : z \mapsto 1/z$ is holomorphic, so the atlas $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$ defines a complex structure on $\mathbb{C}_\infty$. ♦

Example 2.6 (Stereographic Projection)**.** Consider the unit sphere $S^2 \subseteq \mathbb{R}^3$ as a topological subspace of $\mathbb{R}^3$, which makes it a second-countable Hausdorff space. Letting (x, y, w) be the standard coordinates of $\mathbb{R}^3$ and identifying the plane $w = 0$ as $\mathbb{C}$, we employ the charts

$$U_1 \coloneqq S^2 \setminus \{(0,0,1)\} \qquad \varphi_1 : U_1 \to \mathbb{C} : (x,y,w) \mapsto \frac{x+iy}{1-w}$$

$$U_2 \coloneqq S^2 \setminus \{(0,0,-1)\} \qquad \varphi_2 : U_2 \to \mathbb{C} : (x,y,w) \mapsto \frac{x-iy}{1+w}.$$

Clearly φ_1 and φ_2 are continuous, and it can be verified that they are invertible with continuous inverses

$$\varphi_1^{-1}(z) \coloneqq \left(\frac{2\,\mathrm{Re}\,z}{|z|^2+1}, \frac{2\,\mathrm{Im}\,z}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right) \qquad \text{and} \qquad \varphi_2^{-1}(z) \coloneqq \left(\frac{2\,\mathrm{Re}\,z}{|z|^2+1}, \frac{-2\,\mathrm{Im}\,z}{|z|^2+1}, \frac{1-|z|^2}{|z|^2+1}\right).$$

Observe that $U_1 \cap U_2 = S^2 \setminus \{(0,0,\pm 1)\}$ and $\varphi_2 \circ \varphi_1^{-1} : \mathbb{C}^* \to \mathbb{C}^* : z \mapsto 1/z$, which is holomorphic, so the atlas $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$ defines a complex structure on $\hat{\mathbb{C}}$. ♦

Example 2.7 (Complex Projective Line)**.** Consider the equivalence relation $\sim$ on $\mathbb{C}^2 \setminus \{(0,0)\}$ defined by $(z_1, w_1) \sim (z_2, w_2)$ iff $(z_1, w_1) = \lambda (z_2, w_2)$ for some $\lambda \in \mathbb{C}^*$. Set $\mathbb{P}^1 \coloneqq \left(\mathbb{C}^2 \setminus \{(0,0)\}\right)/\sim$ and equip it with the quotient topology. Since $\sim$ is an open equivalence relation[2] whose graph is closed in $\left(\mathbb{C}^2 \setminus \{(0,0)\}\right)^2$, we see that $\mathbb{P}^1$ is a second-countable Hausdorff space. Denoting the equivalence class of (z, w) by $[z : w]$, we employ the charts

$$U_1 \coloneqq \mathbb{P}^1 \setminus \{[0:w] \mid w \in \mathbb{C}\} \qquad \varphi_1 : U_1 \to \mathbb{C} : [z:w] \mapsto w/z$$

$$U_2 \coloneqq \mathbb{P}^1 \setminus \{[z:0] \mid z \in \mathbb{C}\} \qquad \varphi_2 : U_2 \to \mathbb{C} : [z:w] \mapsto z/w.$$

Clearly φ_2 and φ_2 are continuous, and it is easily verified that they are invertible with continuous inverses

$$\varphi_1^{-1}(z) \coloneqq [1:z] \qquad \text{and} \qquad \varphi_2^{-1}(z) \coloneqq [z:1]\,.$$

Furthermore, $\varphi_2 \circ \varphi_1^{-1} : \mathbb{C}^* \to \mathbb{C}^* : z \mapsto 1/z$ is holomorphic, so the atlas $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$ defines a complex structure on $\mathbb{P}^1$. ♦

2.1.2 Complex Tori

Recall that a torus is any manifold homeomorphic to $T^2 \coloneqq S^1 \times S^1$, which admit representations as quotients $\mathbb{C}/\Gamma$ by lattices $\Gamma \coloneqq \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ for any linearly independent vectors $\omega_1, \omega_2 \in \mathbb{C}$ over $\mathbb{R}$. By definition, there is only one torus up to homeomorphism, but it turns out that we can equip it with many different complex structures. They arise from quotienting $\mathbb{C}$ by different lattices, and we shall derive a criterion on the lattices $\Gamma_1 \coloneqq \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ and $\Gamma_2 \coloneqq \mathbb{Z}\eta_1 \oplus \mathbb{Z}\eta_2$ for the tori $\mathbb{C}/\Gamma_1$ and $\mathbb{C}/\Gamma_2$ to be biholomorphic.

Example 2.8 (Complex Tori)**.** Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$ and consider the lattice $\Gamma \coloneqq \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$. Identifying S^1 with the unit circle in $\mathbb{C}$, the quotient $\mathbb{C}/\Gamma$ is a torus in the topological sense since the map

$$\varphi : \mathbb{C}/\Gamma \to S^1 \times S^1 \qquad \text{mapping} \qquad [z] \mapsto \left(e^{2\pi i\lambda_1}, e^{2\pi i\lambda_2}\right)$$

where $z = \lambda_1\omega_1 + \lambda_2\omega_2$ for unique $\lambda_1, \lambda_2 \in \mathbb{R}$, is a homeomorphism. Indeed, φ is well-defined since for any $\lambda_1\omega_1 + \lambda_2\omega_2 \sim \mu_1\omega_1 + \mu_2\omega_2$ in $\mathbb{C}$, we have $(\lambda_1 - \mu_1)\,\omega_1 + (\lambda_2 - \mu_2)\,\omega_2 \in \Gamma$ and so $\lambda_i - \mu_i \in \mathbb{Z}$ for $i = 1, 2$. The fact that it is a homeomorphism is clear. This makes $\mathbb{C}/\Gamma$ a second-countable Hausdorff space, which we now endow with the following complex structure.

Since Γ is discrete, there exists some $\varepsilon > 0$ such that $\varepsilon < |\omega|/2$ for every non-zero $\omega \in \Gamma$.[3] Fix any such ε, which ensures that

[2] See [Tu10, Section 7.5] for details on the quotient topology and open equivalence relations.

[3] This exposition follows [Mir95, Section I.2].

no two points in any open ball with radius ε can be equivalent. Indeed, take any $z \in \mathbb{C}$ and $w_1, w_2 \in B(z, \varepsilon) =: V_z$. For $w_1 \sim w_2$, we need some $n, m \in \mathbb{Z}$ such that $w_1 - w_2 = n\omega_1 + m\omega_2$. But

$$|w_1 - w_2| \leq |z - w_1| + |z - w_2| < 2\varepsilon < |n\omega_1 + m\omega_2|$$

for any $n, m \in \mathbb{Z}$, so this is impossible. Fixing any such ε gives us a family $\{V_z\}_{z\in\mathbb{C}}$ of open sets in $\mathbb{C}$ for which the projections $\pi|_{V_z} : V_z \to \pi(V_z)$ are homeomorphisms. Letting $U_z := \pi(V_z)$ and $\varphi_z : U_z \to V_z$ be the inverse of $\pi|_{V_z}$, we obtain complex charts (U_z, φ_z) for all $z \in \mathbb{C}$. We claim that the collection $\mathfrak{A} := \{(U_z, \varphi_z)\}_{z\in\mathbb{C}}$ form an atlas, for which it suffices to take $(U_1, \varphi_1), (U_2, \varphi_2) \in \mathfrak{A}$ and show that the transition map $T := \varphi_2 \circ \varphi_1^{-1} : \varphi_1(U) \to \varphi_2(U)$, where $U := U_1 \cap U_2$, is holomorphic. Observe that the diagram

$$\begin{tikzcd}
& U \arrow[dl, "\varphi_1"] \arrow[dr, "\varphi_2"'] & \\
V_1 = \varphi_1(U) \arrow[ur, bend left, "\pi|_{V_1}"] \arrow[rr, "T"'] & & \varphi_2(U) = V_2 \arrow[ul, bend right, "\pi|_{V_2}"']
\end{tikzcd}$$

commutes, so $\pi|_{V_2} \circ T = \pi|_{V_1}$ on $\varphi_1(U)$. Then $\pi(T(z)) = \pi(z)$ for every $z \in \varphi_1(U)$, so $T(z) \sim z$ and hence $\ell(z) := T(z) - z \in \Gamma$. This holds for all $z \in \varphi_1(U)$, so we obtain a continuous function $\ell : \varphi_1(U) \to \Gamma : z \mapsto T(z) - z$. Note that $\Gamma \subseteq \mathbb{C}$ is equipped with the subspace topology, but since it is discrete, every $L \subseteq \Gamma$ is open. In particular, fix $z_0 \in \varphi_1(U)$ and set $\gamma_0 := T(z_0) - z_0$. With $L := \{\gamma_0\}$, continuity of ℓ shows that $\ell^{-1}(L)$ is open. Thus $\ell(B(z_0, \delta_1)) \subseteq \{\gamma_0\}$ for some $\delta_1 > 0$, so $\ell(w) = \gamma_0$ for all $w \in B(z_0, \delta_1)$. Thus $T(z) = z + \gamma_0$ for all z in a local neighborhood around z_0, so T is locally biholomorphic. Repeating this for all $z_0 \in \varphi_1(U)$, we see that T is holomorphic on $\varphi_1(U)$. ♦

2.2 Maps on Riemann Surfaces

We extend the notions of holomorphic and meromorphic functions from complex analysis to Riemann surfaces. We also define holomorphic maps *between* Riemann surfaces, which formalizes what we mean for two Riemann surfaces to be 'the same'. Lastly, we study the connection between meromorphic functions $f : X \to \mathbb{C}$ and their associated holomorphic maps $F : X \to \hat{\mathbb{C}}$.

2.2.1 Holomorphic Functions and Maps

Definition 2.9. *Let X be a Riemann surface and let $W \subseteq X$ be open. For a fixed $p \in W$, a function $f : W \to \mathbb{C}$ is said to be holomorphic at p if there exists a chart (U, φ) of X containing p such that $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{C}$ is holomorphic at $\varphi(p)$. If f is holomorphic at every point of W, then f is said to be holomorphic on W.*

Remark. It must be checked that 'being holomorphic' does not depend on the choice of chart. This is indeed the case, for if (V, ψ) is another chart containing p, then the diagram

$$\begin{tikzcd}
\varphi(U \cap V) \arrow[drr, bend left, "f\circ\varphi^{-1}"] & & \\
& U \cap V \arrow[ul, "\varphi"'] \arrow[dl, "\psi"] \arrow[r, "f"] & \mathbb{C} \\
\psi(U \cap V) \arrow[urr, bend right, "f\circ\psi^{-1}"'] & &
\end{tikzcd} \tag{2.1}$$

commutes. Thus $f \circ \psi^{-1} = (f \circ \varphi^{-1}) \circ (\varphi \circ \psi^{-1})$, and since the transition map $\varphi \circ \psi^{-1}$ is holomorphic, we see that $f \circ \varphi^{-1}$ is holomorphic at $\varphi(p)$ iff $f \circ \psi^{-1}$ is holomorphic at $\psi(p)$, as desired. ♦

Example 2.10. Some elementary examples of holomorphic functions.

- If $X = \mathbb{C}$ with the standard chart $(\mathbb{C}, \mathrm{id}_{\mathbb{C}})$, then any holomorphic function $f : W \to \mathbb{C}$ from an open set $W \subseteq \mathbb{C}$ is holomorphic in the classical sense.
- Any chart map $\varphi : U \to \mathbb{C}$ of a Riemann surface is (tautologically) holomorphic in the above sense.
- If $f, g : W \to \mathbb{C}$ are both holomorphic at some $p \in W$, then so are $f \pm g$, $f \cdot g$, and λf for any $\lambda \in \mathbb{C}$. This makes the set $\mathscr{O}(W)$ of all holomorphic functions $f : W \to \mathbb{C}$ into a $\mathbb{C}$-algebra. Lastly, if $g(p) \neq 0$, then f/g is also holomorphic at p. ♦

Definition 2.11. *Let X and Y be Riemann surfaces and let $W \subseteq X$ be open. For a fixed $p \in W$, a mapping $F : W \to Y$ is said to be holomorphic at p if there exists a chart (U, φ) of X containing p and a chart (V, ψ) of Y containing $F(p)$ such that $\psi \circ F \circ \varphi^{-1}$: $\varphi(U) \to \psi(V)$ is holomorphic at $\varphi(p)$. If F is holomorphic at every point of W, then F is holomorphic on W.*

Remark. For $Y := \mathbb{C}$ regarded as a Riemann surface, this definition agrees with the above. Again, we must check that 'being holomorphic' is well-defined, but it follows from the commutativity of the diagram below and a similar argument as above.

$$\begin{tikzcd}
\varphi_1(U_1 \cap U_2) \arrow[rrr, bend left=10, "\psi_1\circ F\circ\varphi_1^{-1}"] \arrow[dd, "\varphi_2\circ\varphi_1^{-1}"'] & & & \psi_1(V_1 \cap V_2) \arrow[dd, "\psi_2\circ\psi_1^{-1}"] \\
& U_1 \cap U_2 \arrow[ul, "\varphi_1"'] \arrow[dl, "\varphi_2"] \arrow[r, "F"] & V_1 \cap V_2 \arrow[ur, "\psi_1"] \arrow[dr, "\psi_2"'] & \\
\varphi_2(U_1 \cap U_2) \arrow[rrr, bend right=10, "\psi_2\circ F\circ\varphi_2^{-1}"'] & & & \psi_2(V_1 \cap V_2)
\end{tikzcd}$$

We make the convention that lower-case letters $f, g, h, \ldots$ are *functions* from a Riemann surface into $\mathbb{C}$, while upper-case letters $F, G, H, \ldots$ are *maps* between Riemann surfaces. ♦

Example 2.12. For a Riemann surface X, the identity id_X is a holomorphic map. Furthermore, for all Riemann surfaces X, Y, and Z, and all holomorphic maps $F : X \to Y$ and $G : Y \to Z$, the composite $G \circ F : X \to Z$ is also a holomorphic map. Note that if $F : X \to Y$ is an invertible holomorphic map, then its inverse $F^{-1} : Y \to X$ is also holomorphic. Indeed, if (U, φ) and (V, ψ) are charts around p and $F(p)$, respectively, making $\psi \circ F \circ \varphi^{-1}$ holomorphic (in the classical sense) at $\varphi(p)$, then its inverse $\varphi \circ F^{-1} \circ \psi^{-1}$ is also holomorphic (in the classical sense) at $\psi(p)$. Thus F^{-1} is holomorphic at $F(p)$, as desired, and justifies the following definition.⧫

Definition 2.13. *Let X and Y be Riemann surfaces. A biholomorphism between X and Y is an invertible holomorphic map $F : X \to Y$. Two Riemann surfaces X and Y are said to be biholomorphic if there exists a biholomorphism $F : X \to Y$.*

Example 2.14 (Biholomorphisms between Riemann spheres)**.** Let $\mathbb{C}_\infty$, S^2, and $\mathbb{P}^1$ denote the three constructions for the Riemann sphere $\hat{\mathbb{C}}$ presented in Examples 2.5, 2.6, and 2.7, respectively. We claim that the maps

$$F : S^2 \to \mathbb{P}^1 : (x, y, w) \mapsto [1 - w : x + iy] \qquad \text{and} \qquad G : S^2 \to \mathbb{C}_\infty : (x, y, w) \mapsto \begin{cases} \dfrac{x + iy}{1 - w} & \text{if } w \neq 1 \\ \infty & \text{else} \end{cases}$$

are biholomorphisms, which shows that all three constructions are biholomorphic. Indeed F is holomorphic since with the charts

$$\begin{aligned} U &:= S^2 \setminus \{(0,0,1)\} & \qquad \varphi &: U \to \mathbb{C} : (x, y, w) \mapsto \frac{x + iy}{1 - w} \\ V &:= \mathbb{P}^1 \setminus \{[0 : w] \mid w \in \mathbb{C}\} & \qquad \psi &: V \to \mathbb{C} : [z : w] \mapsto \frac{w}{z}, \end{aligned}$$

we see that

$$\left(\psi \circ F \circ \varphi^{-1}\right)(z) = \psi\left(F\left(\frac{2\operatorname{Re} z}{|z|^2 + 1}, \frac{2\operatorname{Im} z}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1}\right)\right) = \psi\left(\left[1 - \frac{|z|^2 - 1}{|z|^2 + 1} : \frac{2z}{|z|^2 + 1}\right]\right) = \psi([1 : z]) = z$$

for all $z \in \varphi(U) = \mathbb{C}$, which is clearly holomorphic. Furthermore, it can be checked that F is invertible with a well-defined inverse

$$F^{-1}([z : w]) := \frac{\left(2\operatorname{Re}(z\overline{w}), 2\operatorname{Im}(z\overline{w}), |z|^2 - |w|^2\right)}{|z|^2 + |w|^2},$$

so F is a biholomorphism. For G, we take the same chart (U, φ) as above and choose $V := \mathbb{C}_\infty \setminus \{\infty\} = \mathbb{C}$ and $\psi := \mathrm{id}_{\mathbb{C}}$. A similar calculation then shows that $\left(\psi \circ G \circ \varphi^{-1}\right)(z) = z$ for all $z \in \varphi(U) = \mathbb{C}$, and that G is invertible with inverse $G^{-1}(z) := \varphi^{-1}(z)$ if $z \in \mathbb{C}$ and $G^{-1}(\infty) := (0, 0, 1)$. ⧫

Proposition 2.15. *Any holomorphic function $f : X \to \mathbb{C}$ on a compact Riemann surface X is constant.*

Proof. Since f is holomorphic, the function $|f| : X \to \mathbb{R}$ defined by $|f|(x) := |f(x)|$ is continuous on X. But X is compact, so $|f|$ achieves its maximum at some point $p \in X$. Choosing a connected chart (U, φ) centered at p, we see that $f \circ \varphi : U \to \mathbb{C}$ is holomorphic. Then $|f \circ \varphi| : U \to \mathbb{R}$ has a local maximum at 0, so, since U is connected, $f \circ \varphi$ is constant by the Maximum Principle. Then f is locally constant around p, so, since X is connected, f is constant on X. ■

2.2.2 Singularities of Functions

Throughout this section, let X be a Riemann surface, let $p \in X$, and let $f : W \to \mathbb{C}$ be defined and holomorphic on a punctured neighborhood W of p. As above, we can study the behaviour of f at p from its chart representation $f \circ \varphi^{-1}$.

Definition 2.16. *Let $f : W \to \mathbb{C}$ be a holomorphic function on a punctured neighborhood of p. We say that f has a removable singularity (resp. pole, essential singularity) at p if there exists a chart (U, φ) of X containing p such that $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{C}$ has a removable singularity (resp. pole, essential singularity) at $\varphi(p)$.*

Proof. (Well-definedness). The commutativity of the diagram in Equation (2.1) shows that those notions are chart independent; the composition of $f \circ \varphi^{-1}$ having a singularity at p with a transition map that is holomorphic at p yields a function with the same type of singularity at p. ■

Remark. Functions having an essential singularity at p are very ill-behaved. Indeed, this occurs iff $|f(x)|$ has a non-zero oscillation near p. Other singularities behave much better:

- A removable singularity occurs iff $|f(x)|$ is bounded in a neighborhood of p, and can be 'filled in' by defining $\widetilde{f}(p) := \lim_{x \to p} f(x)$. This extends, by Riemann's Removable Singularities Theorem, to a holomorphic function $\widetilde{f} : W \cup \{p\} \to \mathbb{C}$.
- A pole occurs iff $|f(x)| \to \infty$ as $x \to p$, which can also be 'filled in' by defining the map

$$F : W \to \hat{\mathbb{C}} \qquad \text{mapping} \qquad x \mapsto \begin{cases} \infty & \text{if } x = p \\ f(x) & \text{else} \end{cases}$$

  that extends the codomain of f to the Riemann sphere[4] $\hat{\mathbb{C}}$; since $|f(x)| \to \infty$ as $x \to p$, we see that F is continuous. To show that F is holomorphic, let (U, φ) and (V, ψ) be charts around x and $F(x)$, respectively. Since f is holomorphic on $U \setminus \{p\}$, we see that $\psi \circ F \circ \varphi^{-1}$ is holomorphic on $\varphi(U) \setminus \varphi(p)$. Observe that $\varphi(p)$ is a removable singularity of $\psi \circ F \circ \varphi^{-1}$, which can be extended as above to make $\psi \circ F \circ \varphi^{-1}$ holomorphic on $\varphi(U)$.

Thus we see that every such function $f : W \to \mathbb{C}$ having pole at p can be holomorphically extended to a map $F : W \to \hat{\mathbb{C}}$. Conversely, every holomorphic map $F : W \to \hat{\mathbb{C}}$ (that is not identically infinity) can be regarded as a function $f : W \setminus F^{-1}(\infty) \to \mathbb{C}$ that is holomorphic everywhere except where $F(x) = \infty$, in which case it has a pole. This motivates the following definition. ⧫

[4]Here, we consider $\hat{\mathbb{C}} = \mathbb{C}_\infty$.

Definition 2.17. *A function $f : W \to \mathbb{C}$ is said to be meromorphic at p if it does not have an essential singularity at p; that is, if it is either holomorphic, has a removable singularity, or has a pole at p. If f is meromorphic at every point of W, then f is meromorphic on W.*

Remark. The previous remark can now be rephrased by saying that the set of all meromorphic functions $f : W \to \mathbb{C}$ are in one-to-one correspondence with the set of all holomorphic maps $F : W \to \hat{\mathbb{C}}$ (which are not identically infinity). That is, meromorphic functions are the holomorphic maps to the Riemann sphere. ♦

Example 2.18. As in Example 2.10, if $f, g : W \to \mathbb{C}$ are both meromorphic at p, then so are $f \pm g$ and $f \cdot g$. Furthermore, g is not identically 0, then so is f/g. This makes the set $\mathscr{M}(W)$ of all meromorphic functions $f : W \to \mathbb{C}$ into a field. ♦

Proposition 2.19. *Every meromorphic function on $\hat{\mathbb{C}}$ is a rational function.*

Proof. Let $f : \hat{\mathbb{C}} \to \mathbb{C}$ be meromorphic. Since $\hat{\mathbb{C}}$ is compact, the discreteness of poles imply that f has finitely-many poles. Without loss of generality, assume that ∞ is not a pole of f (since we may consider $1/f$ instead). Now, for each pole $\lambda_i \in \mathbb{C}$ of f, consider its principal part

$$h_i(z) = \sum_{j=-m_i}^{-1} c_{ij} (z - \lambda_i)^j$$

for some $m_i > 1$. Then the function $g \coloneqq f - \sum_i h_i$ is holomorphic function on $\hat{\mathbb{C}}$, and since $\hat{\mathbb{C}}$ is compact, it is constant by Proposition 2.15. Thus $f = g + \sum_i h_i$, which is a rational function. ■

Definition 2.20. *Let $f : W \to \mathbb{C}$ be meromorphic at p and consider its Laurent series $f_\varphi(z) \coloneqq \left(f \circ \varphi^{-1}\right)(z) = \sum_i c_i (z - z_0)^i$ under a chart (U, φ) of X with $z_0 \coloneqq \varphi(p)$. The order of f at p is*

$$\operatorname{ord}_p(f) \coloneqq \min\left\{n \in \mathbb{Z} \,|\, 0 \neq (z - z_0)^n f_\varphi(z) \in \mathscr{O}(W)\right\}.$$

Remark. Note that f, being meromorphic, ensures that its Laurent series has finitely-many negative terms, so the definition makes sense. If f is not meromorphic, we take $\operatorname{ord}_p(f) \coloneqq \infty$. ♦

Proof. (Well-definedness). Let z be the local coordinates given by (U, φ) and suppose that (V, ψ) is another chart with $w_0 \coloneqq \psi(p)$ giving another local coordinate w. Then the transition function $T \coloneqq \varphi \circ \psi^{-1}$ is holomorphic, so it admits a power series representation

$$z = T(w) = \sum_{n \geq 0} a_n (w - w_0)^n = z_0 + \sum_{n \geq 1} a_n (w - w_0)^n.$$

Since $T'(w_0) \neq 0$, we see that $a_1 \neq 0$. Suppose now that the Laurent series of f at p in the coordinate z is $c_{-n_0}(z - z_0)^{-n_0}$ + higher order terms, so that the order of f at p computed by employing z is n_0. Then the Laurent series of f at p in the coordinate w is

$$c_{-n_0}\left(\sum_{n \geq 1} a_n (w - w_0)^n\right)^{-n_0} + \text{ higher order terms},$$

whose lowest order term is $c_{-n_0} a_1^{-n_0} (w - w_0)^{-n_0}$. Observe that $b_{-n_0} \coloneqq c_{-n_0} a_1^{-n_0} \neq 0$, so the order of f at p computed via w is also n_0. ■

Remark. The arithmetic of ord_p is straightforward. Indeed, if $f, g : W \to \mathbb{C}$ are meromorphic at p, then

- $\operatorname{ord}_p(fg) = \operatorname{ord}_p(f) + \operatorname{ord}_p(g)$.
- $\operatorname{ord}_p(f/g) = \operatorname{ord}_p(f) - \operatorname{ord}_p(g)$, if $g \neq 0$.
- $\operatorname{ord}_p(f \pm g) \geq \min\left\{\operatorname{ord}_p(f), \operatorname{ord}_p(g)\right\}$. ♦

The order $\operatorname{ord}_p(f)$ can be used to classify the behaviour of f at p. Indeed, it is readily verified that f is holomorphic at p iff $\operatorname{ord}_p(f) \leq 0$, in which case $f(p) = 0$ iff $\operatorname{ord}_p(f) < 0$. Similarly, f has a pole at p iff $\operatorname{ord}_p(f) > 0$, so f has neither a zero nor a pole at p iff $\operatorname{ord}_p(f) = 0$. This motivates the following definition. ♦

Definition 2.21. *Let $f : W \to \mathbb{C}$ be meromorphic at p. We say that f has a pole of order n at p if $\operatorname{ord}_p(f) = n > 0$, and a zero of order n at p if $\operatorname{ord}_p(f) = n < 0$.*

Example 2.22. Let $f : \hat{\mathbb{C}} \to \mathbb{C}$ be meromorphic, so $f(z) = p(z)/q(z)$ for some $p, q \in \mathbb{C}[z]$. Then f is holomorphic at all points $z \in \mathbb{C}$ such that $q(z) \neq 0$, and has a pole otherwise. Also, $f(\infty) \in \mathbb{C}$ if $\deg p = \deg q$, vanishes if $\deg p < \deg q$, and has a pole otherwise. In any case, f is meromorphic on $\hat{\mathbb{C}}$. To compute $\operatorname{ord}_z(f)$ at all $z \in \hat{\mathbb{C}}$, we split p and q into linear factors to write f uniquely as

$$f(z) = c \prod (z - \lambda_i)^{\alpha_i}$$

where $c \neq 0$ and each λ_i is distinct. Fix i. Setting $g_j(z) \coloneqq (z - \lambda_j)^{\alpha_j}$ for all j, we see that $\operatorname{ord}_{\lambda_i}(g_i) = -\alpha_i$ and $\operatorname{ord}_{\lambda_j}(g_i) = 0$ for all $i \neq j$. Thus

$$\operatorname{ord}_{\lambda_i}(f) = \sum_j \operatorname{ord}_{\lambda_i}(g_j) = -\alpha_i.$$

Moreover, if $\alpha_i > 0$ (resp. $\alpha_i < 0$), then g_i has a pole (resp. zero) of order $|\alpha_i|$ at ∞. It follows then that $\operatorname{ord}_\infty(g_i) = \alpha_i$, so

$$\operatorname{ord}_\infty(f) = \sum_i \operatorname{ord}_\infty(g_i) = \sum_i \alpha_i.$$

Lastly, it is clear that $\operatorname{ord}_z(f) = 0$ for all $z \neq \lambda_i, \infty$. ♦

Remark. Thus if f is a meromorphic function on $\hat{\mathbb{C}}$, then $\sum_{z \in \hat{\mathbb{C}}} \operatorname{ord}_z(f) = 0$. In fact, this holds for all compact Riemann surfaces, which we prove in TODO. ♦

2.2.3 Local Normal Form and the Multiplicity

Holomorphic maps have some remarkable 'local' properties, one of which is presented here. Roughly speaking, every holomorphic map $F: X \to Y$ looks locally like a power map $z \mapsto z^m$ for some unique $m \geq 1$. Summing this local invariant over the fiber $F^{-1}(q)$ for any $q \in Y$ gives us the *degree of* F, a global invariant independent of q, which we prove in Section 3.1.4 using tools from covering spaces.

Theorem 2.23 (Local Normal Form). *Let X and Y be Riemann surfaces and let $F: X \to Y$ be a non-constant holomorphic map. Then, for every $p \in X$, there exists a unique $m \geq 1$ such that for any chart (U_2, φ_2) of Y centered at $F(p)$, there exists a chart (U_1, φ_1) of X centered at p such that $\varphi_2 \circ F \circ \varphi_1^{-1}: z \mapsto z^m$ for all $z \in \varphi_1(U_1)$.*

Proof. Let (U_2, φ_2) be a chart of Y centered at $F(p)$ and consider any chart (V, ψ) of X centered at p. Then the function $h \coloneqq \varphi_2 \circ F \circ \psi^{-1}$ is holomorphic, so it admits a power series representation $h(w) = \sum_{i=0}^{\infty} c_i w^i$ for all $w \in \psi(V)$. Note that $h(0) = \varphi_2(F(p)) = 0$, so $c_0 = 0$. Let $m \geq 1$ be the smallest integer such that $c_m \neq 0$, so

$$h(w) = \sum_{i \geq m} c_i w^i = w^m \sum_{i \geq 0} c_{i-m} w^i \eqqcolon w^m g(w).$$

Then g is holomorphic at 0 with $g(0) = c_m \neq 0$, so there is a function r holomorphic on some neighborhood W of 0 such that $(r(w))^m = g(w)$ for all $w \in W$. Thus $h(w) = (wr(w))^m$, so set $\eta(w) \coloneqq wr(w)$ for all $w \in W$. Note that $\eta'(0) = r(0) \neq 0$, so η is invertible on some neighborhood $W' \subseteq W$ of 0. Set $U_1 \coloneqq \psi^{-1}(W')$ and $\varphi_1 \coloneqq \eta \circ \psi$. Then (U_1, φ_1) is a chart of X centered at p such that

$$\left(\varphi_2 \circ F \circ \varphi_1^{-1}\right)(z) = \left(\varphi_2 \circ F \circ \psi^{-1} \circ \eta^{-1}\right)(z) = h\left(\eta^{-1}(z)\right) = \left[\eta\left(\eta^{-1}(z)\right)\right]^m = z^m$$

for all $z \in \varphi_1(U_1)$. To show uniqueness, it suffices to show that such an m is chart-independent. But this is clear, for if a different chart U_2' is chosen such that F acts as $z \mapsto z^n$ for some neighborhood U_1' of p, then $z^n = z^m$ on $\varphi_1(U_1) \cap \varphi_1'(U_1')$ forces $n = m$. ■

Definition 2.24. *With the above notation, the unique $m \geq 1$ such that there are local coordinates around p and $F(p)$ where F acts like $z \mapsto z^m$ is called the multiplicity of F at p, denoted* $\mathrm{mult}_p(F)$.

Remark. Consider the power function $f(z) \coloneqq z^m$ where $m \coloneqq \mathrm{mult}_p(F)$. Then, for all $z \in \mathbb{C}^*$, we see that $f^{-1}(z)$ has exactly m elements given by the m distinct m^{th} roots of z^m. Thus the map f causes $\mathbb{C}$ to 'cover itself m times', and those coverings meet at the fixed point 0. But $f^{-1}(0) = \{0\}$ has only 1 element, which prevents f to be a m-sheeted covering of $\mathbb{C}$. To remedy this, we count 0 *with multiplicity*; see Chapter 3 for a more formal discussion. Since F is locally represented by f, and (U_1, φ_1) is centered at p, we see that m counts the multiplicity at which neighbors of p are mapped to $F(p)$. ◆

Remark. This theorem also give easy proofs of some elementary properties of holomorphic maps, which we collect here; see [For81, Section 1.2] for details. Throughout, $F: X \to Y$ is a non-constant holomorphic map between Riemann surfaces X and Y.

- F is an open map.
- If F is injective, then it is biholomorphic onto its image.
- If $Y = \mathbb{C}$, then $|F|$ does not attain its maximum.
- If X is compact, then F is surjective and Y is compact.

Together, the last two claims give an alternative proof for Proposition 2.15. ◆

Remark. We give a simple way of computing $\mathrm{mult}_p(F)$ that does not involve casting F into Local Normal Form, or even having to find local coordinates centered at p and $F(p)$. Indeed, let (U_1, φ_1) and (U_2, φ_2) be charts around p and $F(p)$, say with $z_0 \coloneqq \varphi_1(p)$ and $w_0 \coloneqq \varphi_2(F(p))$. Letting $f \coloneqq \varphi_2 \circ F \circ \varphi_1^{-1}$, we see that $f(z_0) = w_0$ and hence its power series representation has the form

$$f(z) = f(z_0) + \sum_{i \geq m} c_i (z - z_0)^i$$

for some $m \geq 1$ with $c_m \neq 0$. Then, since $z - z_0$ and $w - w_0 = f(z) - f(z_0)$ are local coordinates centered at p and $F(p)$, respectively, we see from the above proof that $\mathrm{mult}_p(F) = m$. Thus to compute $\mathrm{mult}_p(F)$, it suffices to case F into local coordinates (U_1, φ_1) around p and (U_2, φ_2) around $F(p)$ and find the lowest non-zero power of the Taylor series of $f \coloneqq \varphi_2 \circ F \circ \varphi_1^{-1}$. ◆

Proposition 2.25. *Let f be a meromorphic function on a Riemann surface X and let $F: X \to \hat{\mathbb{C}}$ be its associated holomorphic map. Fix $p \in X$.*

- *If p is a not a pole of f, then* $\mathrm{mult}_p(F) = -\mathrm{ord}_p(f - f(p))$.
- *If p is a pole of f, then* $\mathrm{mult}_p(F) = \mathrm{ord}_p(f)$.

Proof. Suppose that p is not a pole of f, so $f(p) = F(p) \in \mathbb{C}$. Since the set of all poles of a meromorphic function forms a discrete set, let $p \in U \subseteq X$ be small enough so that $f|_U$ is holomorphic. Let (U, φ) be a chart of X and consider the chart $(\mathbb{C}, \psi)$ of $\hat{\mathbb{C}}$ around $F(p)$ defined by $\psi(z) \coloneqq z - F(p)$. Then $f - f(p) = \psi \circ F$ on U, so

$$(f - f(p))_\varphi \coloneqq (f - f(p)) \circ \varphi^{-1} = \psi \circ F \circ \varphi^{-1}$$

on $\varphi(U)$. Expanding in power series around $z_0 \coloneqq \varphi(p) \in \varphi(U)$, we see that

$$\left(\psi \circ F \circ \varphi^{-1}\right)(z) = (f - f(p))_\varphi(z) = \sum_{i \geq m} c_i (z - z_0)^i$$

for some $m \in \mathbb{N}$ with $c_m \neq 0$. Note that $(f - f(p))_\varphi(z_0) = (f - f(p))(p) = 0$, so $m > 0$ and hence $\operatorname{mult}_p(F) = m$. But m is also the smallest integer such that

$$0 \neq (z - z_0)^{-m} (f - f(p))_\varphi(z) \in \mathscr{O}(U),$$

so $\operatorname{ord}_p(f - f(p)) = -m$. Suppose now that p is a pole of f, so $F(p) = \infty$. Since $\lim_{z \to p} 1/f(z) = 0$, we may let $p \in U \subseteq X$ be small enough so that the function $\widetilde{f} : U \to \mathbb{C}$ defined by

$$\widetilde{f}(x) \coloneqq \begin{cases} 0 & \text{if } x = p \\ 1/f(x) & \text{else} \end{cases}$$

is holomorphic. Let (U, φ) be a chart of X and consider the chart $\left(\hat{\mathbb{C}} \setminus \{0\}, \psi\right)$ of $\hat{\mathbb{C}}$ defined by $\psi(z) \coloneqq 1/z$ for all $z \in \mathbb{C}^*$ and $\psi(\infty) \coloneqq 0$. Then $\widetilde{f} = \psi \circ F$ on U, so $\widetilde{f}_\varphi \coloneqq \widetilde{f} \circ \varphi^{-1} = \psi \circ F \circ \varphi^{-1}$ on $\varphi(U)$. By the same argument as above, we see that $\operatorname{mult}_p(F) = -\operatorname{ord}_p(\widetilde{f})$. Now $\operatorname{ord}_p(f) = -\operatorname{ord}_p(\widetilde{f})$, so the result follows. ■

Remark. This relation between the order of meromorphic functions and the multiplicity of their associated holomorphic maps will be used to derive a criterion for $F : X \to \hat{\mathbb{C}}$ to be a biholomorphism. In fact, Corollary 3.14.2 states that if X is compact and $f : X \to \mathbb{C}$ has a single simple pole, then F is a biholomorphism. ♦

Chapter 3

Covering Spaces and Analytic Continuation

This chapter assumes that the reader is familiar with the basic notions of liftings and homotopy of curves from algebraic topology, for which we refer the reader to [Hat02, Chapter 1].

3.1 Covering Maps and the Degree

We devote this section to develop the tools necessary to define the *degree* of a proper holomorphic map, which, intuitively, is the *number of sheets* in which it covers its image. However, there are points in the image which are not covered 'uniformly', and they must be taken care of separately.

Using the theory of degrees, we prove a criterion for a compact Riemann surface X to be biholomorphic to the Riemann sphere $\hat{\mathbb{C}}$, which will be used in Section 5.1.1 to calculate the moduli space of $\hat{\mathbb{C}}$.

3.1.1 Ramification and Critical Points

Definition 3.1. *Let X and Y be Riemann surfaces and let $F : X \to Y$ be a non-constant holomorphic map. A point $p \in X$ is said to be a ramification point of F if $F|_U$ is not injective for any neighborhood U of p, in which case $F(p) \in Y$ is said to be a critical value of F. If F has no ramification points, then F is said to be an unbranched holomorphic map.*

[1]

Proposition 3.2. *Let X and Y be Riemann surfaces and fix $p \in X$. A non-constant holomorphic map $F : X \to Y$ has a ramification point at p iff $\mathrm{mult}_p(F) \geq 2$.*

Proof. By Theorem 2.23, there exist charts (U, φ) centered at p and (V, ψ) centered at $F(p)$ such that $f := \psi \circ F \circ \varphi^{-1}$ is the power map $z \mapsto z^m$ where $m := \mathrm{mult}_p(F)$. Since φ and ψ are, in particular, injections, we see that F is locally injective at p iff f is locally injective at 0. But this occurs precisely when $m = \mathrm{mult}_p(F) < 2$, so the result follows. ∎

Example 3.3. For any lattice $\Gamma \subseteq \mathbb{C}$ the projection $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$ is an unbranched holomorphic map. This follows from our construction of complex tori in Example 2.8 where for every $z \in \mathbb{C}$ a small enough neighborhood U was found so that $\pi|_U$ is injective. ♦

Proposition 3.4. *Let X, Y and Z be Riemann surfaces and let $F : X \to Y$ be a holomorphic map. Then any lifting $\tilde{F} : X \to Z$ of F w.r.t. an unbranched holomorphic map $\pi : Z \to Y$ is a holomorphic map.*

[2]

Proof. Take $p \in X$ and set $r := \tilde{F}(p)$ and $q := \pi(r) = F(p)$. Since π is unbranched, there exists a neighborhood W of r such that $\pi|_W : W \to Y$ is holomorphic, so it is biholomorphic onto its image $V := \pi(W)$. Let $\chi := \pi|_W^{-1} : V \to W$. Since $\tilde{F}$ is continuous, its inverse image $U := \tilde{F}^{-1}(W)$ is open. Observe that

$$F|_U = (\pi \circ \tilde{F})|_U = \pi|_W \circ \tilde{F}|_U,$$

so $\tilde{F}|_U = \chi \circ F|_U$. Then $p \in U$ and $\tilde{F}|_U$ is a composition of two holomorphic maps, so $\tilde{F}$ is holomorphic at p. ∎

3.1.2 Proper and Covering Maps

In this section, we gather some basic results on the theory of covering maps from topology. Throughout this section and the next, E and X are locally-compact topological spaces.[3]

Definition 3.5. *A map $\pi : E \to X$ is said to be proper if the preimage of every compact set is compact.*

---

[1]It is immediate that F is unbranched iff it is a local homeomorphism. Indeed, if F is unbranched, then for every $p \in X$ there exists a neighborhood U of p such that $F|_U$ is injective. By the Open Mapping Theorem, F is open and hence $F|_U$ maps U homeomorphically to the open set $F(U)$. Conversely, if F is a local homeomorphism, then for every $p \in X$ there exists a neighborhood U of p that is mapped homeomorphically onto an open set in Y. In particular, $F|_U$ is injective, so F is unbranched at p.

[2]Recall that a continuous map $\tilde{F}$ is a lifting of F w.r.t. P if the diagram

$$\begin{tikzcd} & Z \arrow[d, "\pi"] \\ X \arrow[ur, "\tilde{F}"] \arrow[r, "F"'] & Y \end{tikzcd}$$

commutes.

[3]The assumption that E and X are locally compact ensures that all proper maps are closed; that is, then send closed sets to closed sets.

Proposition 3.6. *Let $\pi : E \to X$ be a proper map. Then for every $p \in X$ and every neighborhood V of $\pi^{-1}(p)$, there exists a neighborhood U of p such that $\pi^{-1}(U) \subseteq V$.*

Proof. Since V is open, the set $E \setminus V$ is closed. Since π is proper, it is closed and hence $\pi(E \setminus V)$ is closed too. Clearly $p \notin \pi(E \setminus V) =: W$, so $U := X \setminus W$ is a neighborhood of p; we claim that $\pi^{-1}(U) \subseteq V$. Indeed, for all $\pi(\zeta) \in U$, we see that $\pi(\zeta) \notin \pi(E \setminus V)$ and so $\zeta \notin E \setminus V$. ■

Definition 3.7. *A map $\pi : E \to X$ is said to be a* covering map *if every point $p \in X$ has a neighborhood U such that $\pi^{-1}(U) = \bigcup_{j \in J} V_j$ where V_j are disjoint open sets in E, each homeomorphic to U via $\pi|_{V_j}$.*

Example 3.8. Let $m \geq 2$ be a natural number and consider the power map $f : \mathbb{C}^* \to \mathbb{C}^*$ mapping $z \mapsto z^m$. We claim that f is a covering map, so take $b \in \mathbb{C}^*$ and let $a \in \mathbb{C}^*$ be any one of its m^{th} roots. Since f is a unbranched, there exist neighborhoods V_0 of a and U of b such that $f|_{V_0} : V_0 \to U$ is a homeomorphism. It is clear then that[4]

$$f^{-1}(U) = \bigcup_{j=0}^{m-1} \omega^j V_0,$$

where ω is an m^{th} root of unity, and since $f^{-1}(b)$ is discrete, the sets $V_j := \omega^j V_0$ can be made small enough so that they are pairwise disjoint. Then each $f|_{V_j} : V_j \to U$ is a homeomorphism, as desired. ♦

Example 3.9. For any lattice $\Gamma \subseteq \mathbb{C}$, the projection $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$ is a covering map. Indeed, take $z + \Gamma \in \mathbb{C}/\Gamma$ and let $w \in \mathbb{C}$ be such that $\pi(w) = z + \Gamma$. Since π is unbranched, there exist neighborhoods V of w and U of $z + \Gamma$ such that $\pi|_V : V \to U$ is a homeomorphism. Then clearly[5]

$$\pi^{-1}(U) = \bigcup_{\lambda \in \Gamma} (\lambda + V)$$

where the sets $V_\lambda := \lambda + V$ are all disjoint and each $\pi|_{V_\lambda} : V_\lambda \to U$ is a homeomorphism. ♦

Proposition 3.10. *Any proper local homeomorphism is a covering map.*

Proof. Let $\pi : E \to X$ be a proper local homeomorphism and take $p \in X$. We claim that $\pi^{-1}(p)$ is finite.

- For each $\zeta \in \pi^{-1}(p)$, there exist neighborhoods W_ζ of ζ and U of p such that $\pi|_{W_\zeta} : W_\zeta \to U$ is a homeomorphism. Then the sets W_ζ must be disjoint, for if $\zeta' \in W_\zeta$ for some $\zeta' \neq \zeta$, then $\pi|_{W_\zeta}(\zeta) = p = \pi|_{W_\zeta}(\zeta')$, contradicting that $\pi|_{W_\zeta}$ is a homeomorphism. Thus $\pi^{-1}(p)$ must be finite, lest the cover $\{W_\zeta\}$ admits no finite subcover.

Thus $\pi^{-1}(p) = \{\zeta_1, \ldots, \zeta_n\}$ for some $\zeta_j \in E$. Letting $W_j := W_{\zeta_j}$ as above, we see that $\bigcup_{j=1}^n W_j$ is a neighborhood of $\pi^{-1}(p)$. By Proposition 3.6, there is a neighborhood U of p such that $\pi^{-1}(U) \subseteq \bigcup_{j=1}^n W_j$, so $\pi^{-1}(U) = \bigcup_{j=1}^n V_j$ where the sets $V_j := W_j \cap \pi^{-1}(U)$ are all disjoint and each $\pi|_{V_j} : V_j \to U$ is a homeomorphism. ■

3.1.3 Liftings of Curves

This section develops some technical tools to define the *number of sheets* of a covering, which in turn is used to define the *degree* of a proper holomorphic map.

Definition 3.11. *A function $\pi : E \to X$ is said to have the* curve lifting property *if for every curve $\alpha : [0,1] \to X$ and every point $\zeta_0 \in E$ with $\pi(\zeta_0) = \alpha(0)$, there exists a lifting $\tilde{\alpha} : [0,1] \to E$ w.r.t. π such that $\tilde{\alpha}(0) = \zeta_0$.*

Proposition 3.12. *Every covering map $\pi : E \to X$ has the curve lifting property.*

Proof. Let[6] $\alpha : [0,1] \to X$ be a curve and let $\zeta_0 \in E$ be a point such that $\pi(\zeta_0) = \alpha(0)$. Consider any open cover $\{U_i\}$ of $\alpha([0,1])$ where each U_i is a connected open set in $\alpha([0,1])$. Thus $\{\alpha^{-1}(U_i)\}$ is an open cover of $[0,1]$, so it admits a finite subcover $\{(t_i, t_{i+1})\}_{i=1}^n := \{\alpha^{-1}(U_i)\}_{i=1}^n$. Reindexing if necessary, we obtain a partition

$$0 =: t_0 < t_1 < \cdots < t_n := 1$$

of $[0,1]$ such that $\alpha([t_{i-1}, t_i]) \subseteq U_i$ for all $1 \leq i \leq n$. Now, since π is a covering map, there exist disjoint open sets V_{ij} in E, each homeomorphic to U_i via $\pi|_{V_{ij}}$, such that $\pi^{-1}(U_i) = \bigcup_{j \in J_i} V_{ij}$. We now construct a lifting $\tilde{\alpha}|_{[0,t_k]} : [0, t_k] \to E$ by induction on $k \in \mathbb{N}$.

- The base case for when $k = 0$ is trivial by defining $\tilde{\alpha}(0) := \zeta_0$.

[4] Indeed, for all $c \in f^{-1}(U)$, $f(c) \in U$ and so there exists some $a' \in V_0$ such that $f(a') = f(c)$. Then $c = \omega^j a'$ for some $0 \leq j \leq m-1$, so $c \in \omega^j V_0$. Conversely, if $c \in \omega^j V_0$ for some $0 \leq j \leq m-1$, then $c = \omega^j a'$ for some $a' \in V_0$ and hence $f(c) = f\left(\omega^j a'\right) = f(a') \in U$.

[5] Similarly, for all $z \in \pi^{-1}(U)$, $\pi(z) \in U$ and so there exists some $w' \in V$ such that $\pi(z) = \pi(w')$. Then $z + \Gamma = w' + \Gamma$, so $z = w' + \lambda$ for some $\lambda \in \Gamma$. Conversely, if $z \in \lambda + V$ for some $\lambda \in \Gamma$, then $z = w' + \lambda$ for some $w' \in V$ and hence $\pi(z) = \pi(\omega' + \lambda) = \pi(w) \in U$.

[6] The idea of this proof is to split $\alpha([0,1])$ into (overlapping) paths $\alpha([t_{k-1}, t_k])$ and construct the lifting $\tilde{\alpha}$ inductively: Given a lifting $\tilde{\alpha}$ defined up to some boundary t_{k-1}, we define it on the next interval $[t_{k-1}, t_k]$ by lifting α (restricted to $[t_{k-1}, t_k]$) via χ. This gives us a 'chain' of paths, which when joined together gives us a global lifting of α.

The base case of this induction simply sets $\tilde{\alpha}(0) := \zeta_0$ in order to start-off this process.

Suppose now that the lifting $\tilde{\alpha}|_{[0,t_{k-1}]} : [0, t_{k-1}] \to E$ has been constructed for some $k \geq 1$. Then[7] $\alpha(t_{k-1}) = \pi(\tilde{\alpha}(t_{k-1})) \in U_k$, so there exists some $j \in J_k$ such that $\tilde{\alpha}(t_{k-1}) \in V_{kj}$. Letting $\chi : U_k \to V_{kj}$ be the inverse of $\pi|_{V_{kj}} : V_{kj} \to U_k$, we set

$$\tilde{\alpha}|_{[t_{k-1},t_k]} := \chi \circ \alpha|_{[t_{k-1},t_k]}\,.$$

Clearly[8], $\tilde{\alpha}(t_{k-1})$ agrees with our existing lifting, which makes the piecewise-defined map $\alpha|_{[0,t_k]}$ a lifting of $\alpha|_{[0,t_k]}$ w.r.t. π. ■

Corollary 3.12.1. *Suppose that X is path-connected and let $\pi : E \to X$ be a covering map. Then, for any $p_1, p_2 \in X$, the sets $\pi^{-1}(p_1)$ and $\pi^{-1}(p_2)$ are equinumerous.*

Proof. Since X is path-connected, there exists a curve $\alpha : [0, 1] \to X$ from p_1 to p_2. We define a map $\varphi : \pi^{-1}(p_1) \to \pi^{-1}(p_2)$ as follows. Every $\zeta \in \pi^{-1}(p_1)$ induces a unique lifting $\tilde{\alpha} : [0, 1] \to E$ such that $\tilde{\alpha}(0) = \zeta$, and since $\pi(\tilde{\alpha}(1)) = \alpha(1) = p_2$, we have $\tilde{\alpha}(1) \in \pi^{-1}(p_2)$. Hence we define $\varphi(\zeta) := \tilde{\alpha}(1)$. The uniqueness of liftings ensures that φ is well-defined and bijective, so $\pi^{-1}(p_1)$ and $\pi^{-1}(p_2)$ are equinumerous. ■

3.1.4 Degrees and Multiplicities

Throughout this section, X and Y are Riemann surfaces and $F : X \to Y$ is a non-constant proper holomorphic map.

Definition 3.13. *The degree of F, denoted $\deg F$, is the cardinality of the fiber $F^{-1}(q)$ of any non-critical point $q \in Y$.*

Proof. (Well-definedness). Since F is a proper map, the fiber $F^{-1}(q)$ is compact and is hence finite by Discreteness of Preimages. Being unramified, we see that F is a local homeomorphism, so it is a covering map by Proposition 3.10. Finally, Corollary 3.12.1 shows that $\deg F$ is well-defined. ■

Remark. Let $n := \deg F$. Then n is referred to as the number of sheets of F and F is said to be an n-sheeted holomorphic covering map. ♦

Theorem 3.14. *Fix an arbitrary $q \in Y$. Then $\deg F$ is the sum of the multiplicities at each $p \in F^{-1}(q)$ of F. That is,*

$$\deg F = \sum_{p \in F^{-1}(q)} \operatorname{mult}_p(F)\,.$$

[9]

Proof. If q is not a critical point, then Proposition 3.2 shows that $\operatorname{mult}_p(F) = 1$ for any $p \in F^{-1}(p)$. Then $\deg F = \left|F^{-1}(q)\right|$, which agrees with our definition.

Otherwise, q is a critical point of F[10]. Since $F^{-1}(q)$ is compact, we see that $F^{-1}(q) = \{p_1, \ldots, p_n\}$ for some $p_i \in X$. Fix $1 \leq j \leq n$ and set $m_j := \operatorname{mult}_{p_j}(F)$. We claim that there exist neighborhoods U_j of p_j and V_j of q such that $\left|F^{-1}(r) \cap U_j\right| = m_j$ for all $r \in V_j \setminus \{q\}$.

- By Theorem 2.23, there exist charts (U_j, φ_j) of X centered at p_j and (V_j, ψ_j) of Y centered q such that F acts as the power function $f(z) := z^{m_j}$ on $\varphi_j(U_j)$. Take[11] $r \in V_j \setminus \{q\}$ and set $w := \psi_j(r) \neq 0$. Then $\left|f^{-1}(w)\right| = m_j$, so we have

$$\left|F^{-1}(r) \cap U_j\right| = \left|\varphi_j\left(F^{-1}(r)\right)\right| = \left|\varphi_j\left(F^{-1}\left(\psi_j^{-1}(w)\right)\right)\right| = \left|f^{-1}(w)\right| = m_j.$$

Since U_j is a neighborhood of p_j, we see that $F^{-1}(V_j) \subseteq U_j$ by restricting V_j in accordance with Proposition 3.6, if necessary. Then, with $V := \bigcap_{i=1}^n V_i$, we see that $F^{-1}(V) \subseteq \bigcup_{i=1}^n U_i$ where the sets U_i are all disjoint. Take any $r \in V \setminus \{q\}$. Then $r \in V_i \setminus \{q\}$ for all $1 \leq i \leq n$, so

$$\left|F^{-1}(r)\right| = \left|F^{-1}(r) \cap \bigcup_{i=1}^n U_i\right| = \left|\bigcup_{i=1}^n \left(F^{-1}(r) \cap U_i\right)\right| = \sum_{i=1}^n \left|F^{-1}(r) \cap U_i\right| = \sum_{i=1}^n m_i.$$

But r is not a critical point of F, so the result follows. ■

Corollary 3.14.1. *If X is compact, then a holomorphic map $F : X \to Y$ is a biholomorphism iff $\deg F = 1$.*

Proof. Since X is compact, we see that F is proper surjection. Observe that F is an injective iff it has no critical points, and by Proposition 3.2, this occurs iff $\operatorname{mult}_p(F) = 1$ for all $p \in X$.

- ($\Rightarrow$) If F is an injection, then $\left|f^{-1}(p)\right| = 1$ for all $p \in X$. Thus $\deg F = 1$.
- ($\Leftarrow$): Since $\operatorname{mult}_p(F) \geq 1$ for all $p \in X$, the above theorem forces $\operatorname{mult}_p(F) = 1$. ■

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[7]

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ {[0,1]} \arrow[ur, "\tilde{\alpha}"] \arrow[r, "\alpha"'] & X \end{tikzcd}$$

[8] $\tilde{\alpha}(t_{k-1}) = \chi(\alpha(t_{k-1})) = \chi(\pi(\tilde{\alpha}(t_{k-1})))$ on the appropriate restrictions.

[9] Instead of simply counting the elements in the fiber, we need count them *with multiplicities*.

[10] Although q is a critical point of F, every point in a small enough neighborhood around it is not a critical point.

[11] Note that V_j can be taken small enough so that r is *not* a critical value of F.

Corollary 3.14.2. *If X is compact and there exists a meromorphic function $f : X \to \mathbb{C}$ with a single simple pole, then $X \cong \hat{\mathbb{C}}$.*

Proof. Let $f : X \to \mathbb{C}$ be a meromorphic function with only a simple pole at p and consider its associated holomorphic map $F : X \to \hat{\mathbb{C}}$. By Proposition 2.25, we see that $\operatorname{mult}_p(F) = \operatorname{ord}_p(f) = 1$ and hence p is unramified. Since p is the only pole of f, we see that $\deg F = \left|F^{-1}(\infty)\right| = 1$. ■

3.2 Sheaves and Function Germs

Unless otherwise stated, in this section, X denotes a topological space with τ its system of open sets. Our exposition on sheaves roughly follows [For81, Section 6] and [Mir95, Chapter IX].

3.2.1 Presheaves and Sheaves

Definition 3.15. *A presheaf of Abelian groups on X is a pair $(\mathscr{F}, \rho)$ consisting of*

- *a family $\mathscr{F} := \{\mathscr{F}(U)\}$ of Abelian groups $\mathscr{F}(U)$ for every $U \in \tau$,*
- *a family $\rho := \left\{\rho_V^U\right\}$ of group homomorphisms $\rho_V^U : \mathscr{F}(U) \to \mathscr{F}(V)$ for every $U, V \in \tau$ with $V \subseteq U$,*

such that the following two properties hold:

- *For every $U \in \tau$, we have $\rho_U^U = \operatorname{id}_{\mathscr{F}(U)}$.*
- *For every $U, V, W \in \tau$ with $W \subseteq V \subseteq U$, we have $\rho_W^V \circ \rho_V^U = \rho_W^U$.*

Remark. [12] Presheaves give us a way of tracking data associated with open sets of a topological space in such a way that makes restricting to a smaller open set $V \subseteq U$ well-behaved. Consider, for instance, a Riemann surface X and the presheaf of all holomorphic functions $\mathscr{O}$ on X.

- To every open set $U \subseteq X$[13] we consider the $\mathbb{C}$-algebra $\mathscr{O}(U)$ of all holomorphic functions $f : U \to \mathbb{C}$. For any open $V \subseteq U$, we define $\rho_V^U(f) := f|_V$. The two properties are then trivial, which respectively states that restricting to the domain does nothing, and that restricting once to V and then to $W \subseteq V$ yields the same function as restricting to W directly.

Similarly, we have the presheaf of all meromorphic functions $\mathscr{M}$ on X. However, those two examples are much more than presheaves since global information about elements in $\mathscr{F}(X)$ can be obtained locally by 'restricting' to U. The notion of a sheaf makes this precise. ⧫

Definition 3.16. *A presheaf $\mathscr{F}$ on X is said to be a sheaf if for every open set $U \subseteq X$ and every family $\{U_i\}_{i\in I}$ of open subsets that cover U, the following two properties hold:*

- *(Identity): For every $f, g \in \mathscr{F}(U)$, if $\rho_{U_i}^U(f) = \rho_{U_i}^U(g)$ for every $i \in I$, then $f = g$.*
- *(Gluing): For every family $\{f_i\}_{i\in I}$ with $f_i \in \mathscr{F}(U_i)$, if $\rho_{U_i\cap U_j}^{U_i}(f_i) = \rho_{U_i\cap U_j}^{U_j}(f_j)$ for all $i, j \in I$, then there is some $f \in \mathscr{F}(U)$ such that $\rho_{U_i}^U(f) = f_i$ for every $i \in I$.*

[14]

Example 3.17. We give an example of a presheaf that is *not* a sheaf. Let X be a normed $\mathbb{R}$-vector space. For all $U \subseteq X$, let $\mathscr{B}(U)$ be the vector space of all bounded functions $f : U \to \mathbb{R}$.

- It is clear from our remarks above that $\mathscr{B}$ is a presheaf. In fact, since $\mathscr{B}(U)$ contains functions, we see that if $f, g \in \mathscr{B}(U)$ agree on all restrictions, then they agree on U.

The problem arises when we consider glueing[15]. For instance, let $U_i := \{p \in X \mid \|p\| < i\}$ and observe that $\{U_i\}_{i\in\mathbb{R}^+}$ covers X. Consider the family $\{f_i\}$ where each $f_i := \operatorname{id}_{U_i}$, which clearly agree on their pairwise intersections. But no function $f : X \to \mathbb{R}$ such that $f|_{U_i} = f_i$ for all $i \in \mathbb{R}^+$ can be bounded, so $\mathscr{B}$ is not a sheaf. ⧫

Example 3.18. We give two examples of sheaves relating to *divisors*[16] on a Riemann surface X; that is, a function $D : X \to \mathbb{Z}$ whose support $\{p \in X \mid D(p) \neq 0\}$ is a discrete subset of X.

- Let D be a divisor on X. For every $U \in \tau$, let $\mathscr{O}[D](U)$ denote the set of all meromorphic functions $f : X \to \mathbb{C}$ such that $\operatorname{ord}_p(f) \leq D(p)$ for all $p \in X$. The usual restriction homomorphisms make $\mathscr{O}[D]$ is a sheaf of Abelian groups, [17] for if $\{f_i\}$ is a family of meromorphic functions having poles bounded by D, then the meromorphic function $f : X \to \mathbb{C}$ that glues them together also has poles bounded by D.

[12] Analogously, we define the presheaf of sets, rings, vector spaces, algebras, etc, on a topological space X.

More generally, fix any category **C**. A **C**-valued presheaf on X is simply a contravariant functor $\mathscr{F} : \tau \to \mathbf{C}$ where τ is the preorder category induced by $(\tau, \subseteq)$. Although many statements are simplified when phrased categorically, no category theory background is needed for this paper. We refer the interested reader to [Lan10].

[13] Similarly, consider the (multiplicative) group $\mathscr{O}^*(U)$ of all holomorphic functions $f : U \to \mathbb{C}^*$, which defines a presheaf $\mathscr{O}^*$ of Abelian groups. We define $\mathscr{M}^*(U)$ similarly, but instead restrict to all meromorphic functions $f : U \to \mathbb{C}$ that do not vanish identically on any connected-component of U.

[14] It is immediate that $\mathscr{O}$, $\mathscr{O}^*$, $\mathscr{M}$, and $\mathscr{M}^*$ are all sheaves on X. Indeed, if we have a family $\{f_i\}$ that agree on all pairwise common domains, then there exists a globally defined function f whose restrictions are f_i's. We only need to show that this globally defined function is of the 'right type', but this can be checked easily.

[15] In other words, boundedness is a global property. To check if a function is bounded, it does *not* suffice to check it on an arbitrary neighborhood.

[16] For compact Riemann surfaces, we see that a function $D : X \to \mathbb{Z}$ is a divisor iff it has finite support, so its set of divisors is the free Abelian group of the points of X.

[17] This construction generalizes both $\mathscr{O}$ and $\mathscr{M}$. Intuitively, the use of divisors here allow us to 'bound' the orders of the poles of f at specific points p, thereby restricting how badly-behaved it can be.

- For every $U \in \tau$, let $\mathscr{D}(U)$ denote the group of all discretely-supported functions from U to $\mathbb{Z}$ (which are exactly the divisors on U). This makes $\mathscr{D}$ into a sheaf since for every family $\{D_i\}$, the function $D : X \to \mathbb{Z}$ that glues them together is also discretely-supported. ♦

Definition 3.19. *Let $(\mathscr{F}, \rho)$ and $(\mathscr{G}, \sigma)$ be two sheaves of Abelian groups on X. A* <u>*morphism of sheaves*</u> *$\eta : \mathscr{F} \to \mathscr{G}$ is a family $\{\eta_U\}_{U\in\tau}$ of group homomorphisms $\eta_U : \mathscr{F}(U) \to \mathscr{G}(U)$ such that for every $U \in \tau$ and every open set $V \subseteq U$, the following diagram commutes.*

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\eta_U"] \arrow[d, "\rho^U_V"'] & \mathscr{G}(U) \arrow[d, "\sigma^U_V"] \\ \mathscr{F}(V) \arrow[r, "\eta_V"] & \mathscr{G}(V) \end{tikzcd}$$

[18]

Example 3.20. Some examples relating to divisors of a Riemann surface X.

- For divisors D_1 and D_2 of a Riemann surface X, we write $D_1 \leq D_2$ if $D_1(p) \leq D_2(p)$ for all $p \in X$. This induces an *inclusion morphism* $\iota : \mathscr{O}[D_1] \hookrightarrow \mathscr{O}[D_2]$ defined[19] by $\iota_U(f) \coloneqq f$ for all $U \in \tau$ and $f \in \mathscr{O}[D_1](U)$, which makes sense since if $D_1 \leq D_2$ and the poles of f are bounded by D_1, then they are also clearly bounded by D_2. This inclusion also respect restrictions, so it is indeed a morphism of sheaves.

- For all $U \in \tau$, we associate to each $f \in \mathscr{M}^*(U)$ the divisor $\operatorname{div} f : U \to \mathbb{Z} : p \mapsto \operatorname{ord}_p(f)$[20]. This induces a morphism of sheaves $\operatorname{div} : \mathscr{M}^* \to \mathscr{D}$ since for all $U \in \tau$ and all open sets $V \subseteq U$, the restriction of the divisor of any $f \in \mathscr{M}^*(U)$ coincides with the divisor of the restriction $f|_V$. ♦

3.2.2 Stalks and the Étalé Space

Throughout this section, $p \in X$ is a fixed point in a topological space X.

Definition 3.21. *Let $\mathscr{F}$ be a presheaf of Abelian groups on X. The* <u>*stalk of $\mathscr{F}$ at p*</u> *is the Abelian group*

$$\mathscr{F}_p \coloneqq \left(\coprod_{U \ni p} \mathscr{F}(U) \right) \Big/ \sim_p$$

where $\sim_p$ is the equivalence relation on the disjoint union, defined, for all $f \in \mathscr{F}(U)$ and $g \in \mathscr{F}(V)$, by $f \sim_p g$ iff there exists an open set $W \in \tau$ with $p \in W \subseteq U \cap V$ such that $\rho^U_W(f) = \rho^V_W(g)$. For any $f \in \mathscr{F}(U)$, its equivalence class $[f]_p$ is called the <u>*germ of f at p*</u>.

[21]

Example 3.22. Let[22][23] D be a divisor on a Riemann surface X and consider the stalk $\mathscr{O}_p[D]$. Fix a chart centered at p. Since any meromorphic function f admits a Laurent series, we see that the function germ $[f]_p$ is represented by a Laurent series $\sum_{i=i_0}^{\infty} c_i z^i$ for some $i_0 \geq -D(p)$ and $c_i \in \mathbb{C}$. Conversely, the germ of any Laurent series $\sum_{i=i_0}^{\infty} c_i z^i$ with $i_0 \geq -D(p)$ and $c_i \in \mathbb{C}$ lifts to a meromorphic function germ $[f]_p$, so this defines a bijection[24] between $\mathscr{O}_p[D]$ and the set of all such Laurent series. ♦

Remark. The sheaf axioms guarantee that if $\mathscr{F}$ is a sheaf of Abelian groups on X and $U \in \tau$, then an element $f \in \mathscr{F}(U)$ is zero iff[25] all germs $[f]_p$, for $p \in U$ vanish. Indeed, let $0 \in \mathscr{F}(U)$ denote the zero element, so $f \sim_p 0$ for all $p \in U$ furnishes a family $\{W_p\}$ of open sets $W_p \subseteq U$ containing p such that $\rho^U_{W_p}(f) = \rho^U_{W_p}(0)$. This family covers U, so $f = 0$ by the first sheaf axiom. ♦

Proposition 3.23. *Let $\mathscr{F}$ be a presheaf of Abelian groups on X. Let $|\mathscr{F}| \coloneqq \coprod_{p\in X} \mathscr{F}_p$ and consider the projection $\pi : |\mathscr{F}| \to X$ mapping each $\eta \in \mathscr{F}_p$ to p. Then the system $\mathcal{B}$ of all sets*

$$[U, f] \coloneqq \left\{ [f]_p \,\middle|\, p \in U \right\} \subseteq |\mathscr{F}|$$

for $U \in \tau$ and $f \in \mathscr{F}(U)$ is a basis for a topology on $|\mathscr{F}|$ and π is a local homeomorphism.

Proof. We first verify that $\mathcal{B}$ is a basis.

(1) Take $\eta \in |\mathscr{F}|$, so there exists an open set $U \in \tau$ such that $\eta = [f]_p$ for some $f \in \mathscr{F}(U)$ and $p \in U$. Observe that $\eta \in [U, f]$.

(2) Take $[U, f], [V, g] \in \mathcal{B}$ and $\eta \in [U, f] \cap [V, g]$. Then there exists a point $p \in X$ such that $\eta = [f]_p = [g]_p$, which furnishes an open set $W \in \tau$ with $p \in W \subseteq U \cap V$ such that $\rho^U_W(f) = \rho^V_W(g) \eqqcolon h$. Then $\eta = [h]_p$ with $h \in W$, so $\eta \in [W, h] \subseteq [U, f] \cap [V, g]$.

[18] Phrased categorically, a morphism of sheaves is simply a natural transformation $\eta : \mathscr{F} \Rightarrow \mathscr{G}$. This makes the collection of all sheaves on X into a category.
[19] In particular, we have the inclusion $\mathscr{O} \hookrightarrow \mathscr{M}$.
[20] Such a function $\operatorname{div} f$ is a divisor by discreteness of zeros and poles.
[21] The relation $\sim_p$ is transitive since $\rho^V_W \circ \rho^U_V = \rho^U_W$ for all $U, V, W \in \tau$ such that $W \subseteq V \subseteq U$.
[22] This construction is analogous to that of the *tangent space* T_pM of a (real) manifold M at some point p.
[23] This equivalence relation allows us to 'evaluate' a function germ $\eta \in \mathscr{O}_p[D]$ as $\eta(p) \coloneqq f(p)$ where $U \ni p$ is any open set and $f \in \mathscr{O}[D](U)$ is any function such that $\eta = [f]_p$.
[24] This isomorphism depends on the chosen chart map, so it is not canonical.
[25] The forward direction is tautological.

To show that π is a local homeomorphism, fix $\eta \in |\mathscr{F}|$, say with $p \coloneqq \pi(\eta)$. By (1), there exists some $[U, f] \in \mathcal{B}$ containing η; we claim that $\pi|_{[U,f]} : [U, f] \to U$ is a homeomorphism.

- For injectivity, take $\psi_1, \psi_2 \in [U, f]$ such that $\pi(\psi_1) = \pi(\psi_2)$. Then $\psi_1 = [f]_p$ and $\psi_2 = [f]_q$ for some $p, q \in X$, but since $p = q$, they coincide.

- For continuity, it suffices to show that $\pi|_{[U,f]}$ is an open map. Indeed, if $[V, g] \subseteq [U, f]$ is open, then $\pi|_{[U,f]}([V, g]) = V$ is open too. ∎

Definition 3.24. *The* <u>*Étalé space*</u> *of a presheaf $\mathscr{F}$ of Abelian groups on X is the topological space $|\mathscr{F}|$ equipped the projection $\pi : |\mathscr{F}| \to X$.*

Definition 3.25. *A presheaf $\mathscr{F}$ of Abelian groups on X is said to satisfy the* <u>*Identity Theorem*</u> *if for all $U \in \tau$ and all $f, g \in \mathscr{F}(U)$, if there is some $p \in U$ such that $[f]_p = [g]_p$, then $f = g$ (on U).*

[26]

Proposition 3.26. *If X is a locally-connected Hausdorff space and $\mathscr{F}$ is a presheaf of Abelian groups on X that satisfy the Identity Theorem, then $|\mathscr{F}|$ is Hausdorff.*

Proof. Take distinct $\eta_1, \eta_2 \in |\mathscr{F}|$. Two cases occur.

- If $p \coloneqq \pi(\eta_1) \neq \pi(\eta_2) \eqqcolon q$, then, since X is Hausdorff, there exist disjoint neighborhoods U of p and V of q. On those neighborhoods, π is invertible and the sets $\pi^{-1}(U)$ and $\pi^{-1}(V)$ are disjoint neighborhoods of η_1 and η_2, respectively.

Otherwise, set $p \coloneqq \pi(\eta_1) = \pi(\eta_2)$ and suppose that each η_i is represented by some $f_i \in \mathscr{F}(U_i)$. Since X is locally-connected, there exists a connected neighborhood $U \subseteq U_1 \cap U_2$ of p. Restricting both f_i to $g_i \coloneqq \rho_U^{U_i}(f_i)$, the sets $[U, g_i]$ are neighborhoods of η_i. Suppose, for sake of contradiction, that there exists some $\psi \in [U, g_1] \cap [U, g_2]$. Setting $q \coloneqq \pi(\psi)$, we see that $\psi = [g_1]_q = [g_2]_q$, from which the Identity Theorem shows that $g_1 = g_2$. Note that $f_i \sim_p g_i$, so $\eta_1 = \eta_2$, a contradiction. Hence the neighborhoods $[U, g_1]$ an $[U, g_2]$ are disjoint, as desired. ∎

3.3 Analytic Continuation

We now restrict to when X is a Riemann surface with a fixed point $p \in X$. For convenience, we write[27] $\mathscr{O}[p] \coloneqq \mathscr{O}[D]$ where D is the divisor on X defined by $D(p) \coloneqq 1$ and zero everywhere else. Throughout, $\alpha : [0, 1] \to X$ is a curve with $p = \alpha(0)$ and $q \coloneqq \alpha(1) \neq p$, and $\eta_0 \in \mathscr{O}_p[p]$ is a fixed germ.

3.3.1 Analytic Continuation of Germs along Curves

Definition 3.27. *A germ $\hat{\eta} \in \mathscr{O}_q$ is said to be the* <u>*analytic continuation of η_0 along α*</u> *if there exist a family $\eta_t \in \mathscr{O}_{\alpha(t)}[p]$ of germs for all $t \in [0, 1]$ with $\hat{\eta} = \eta_1$ such that for all $\tau \in [0, 1]$, there exists a neighborhood $T \subseteq [0, 1]$ of τ, an open set $U \subseteq X$ with $\alpha(T) \subseteq U$, and a function $f \in \mathscr{O}[p](U)$ such that $[f]_{\alpha(t)} = \eta_t$ for all $t \in T$.*

Proposition 3.28. *A germ $\hat{\eta} \in \mathscr{O}_q$ is an analytic continuation of η_0 along α iff there exists a lifting $\tilde{\alpha} : [0, 1] \to |\mathscr{O}[p]|$ such that $\tilde{\alpha}(0) = \eta_0$ and $\tilde{\alpha}(1) = \hat{\eta}$.*

Proof. If[28] $\hat{\eta} \in \mathscr{O}_q$ is an analytic continuation of η_0 along α, let $\{\eta_t\}$ be a family of germs as defined above. We claim that the curve $\tilde{\alpha} : [0, 1] \to |\mathscr{O}[p]|$ mapping $t \mapsto \eta_t$ is a lifting of α.

- First, note that $\eta_t \in \mathscr{O}_{\alpha(t)}[p]$ for all $t \in [0, 1]$, so[29] $(\pi \circ \tilde{\alpha})(t) = \pi(\tilde{\alpha}(t)) = \pi(\eta_t) = \alpha(t)$. It remains to show that $\tilde{\alpha}$ is continuous, so fix a basis element $[U, f] \subseteq |\mathscr{O}[p]|$ and take $\tau \in \tilde{\alpha}^{-1}([U, f])$. Then $\tau \in [0, 1]$, so there exists a neighborhood $T \subseteq [0, 1]$ of τ such that $[f]_{\alpha(t)} = \eta_t$ for all $t \in T$. Observe that $\tilde{\alpha}(T) \subseteq [U, f]$ since for all $\eta_t \in \tilde{\alpha}(T)$, we have $\alpha(t) \in U$ and hence $\eta_t = [f]_{\alpha(t)} \in [U, f]$.

Conversely, suppose that there is a lifting $\tilde{\alpha} : [0, 1] \to |\mathscr{O}[p]|$ of α with $\tilde{\alpha}(0) = \eta_0$ and $\tilde{\alpha}(1) = \hat{\eta}$. For all $t \in [0, 1]$, we define $\eta_t \coloneqq \tilde{\alpha}(t)$, so $\eta_1 = \hat{\eta}$. Fix $\tau \in [0, 1]$, so there exists a basis neighborhood $[U, f] \subseteq |\mathscr{O}[p]|$ of $\tilde{\alpha}(\tau)$. But $\tilde{\alpha}$ is continuous, so there exists a neighborhood $T \subseteq [0, 1]$ of τ such that $\tilde{\alpha}(T) \subseteq [U, f]$. Projecting, we see that $\alpha(T) \subseteq \pi([U, f]) = U$. Finally, the commutativity of the diagram gives $[f]_{\alpha(t)} = \eta_t$ for all $t \in T$, so $\hat{\eta}$ is an analytic continuation of η_0 along α. ∎

[26] In particular, this holds for all $\mathscr{O}[D]$. In contrast, the sheaf of smooth functions $\mathscr{E}$ (see Section 4.1) does *not* satisfy the Identity Theorem.
[27] Thus if $U \subseteq X$ is an open set containing p, then any $f \in \mathscr{O}_p[p](U)$ has at most a single simple pole. Otherwise, if $p \notin U$, then f is holomorphic.
[28] In particular, the uniqueness of liftings shows that if an analytic continuation of η_0 along α exists, then it is unique.
[29] Clearly $\tilde{\alpha}(0) = \eta_0$ and $\tilde{\alpha}(1) = \hat{\eta}$.

$$\begin{tikzcd} & {|\mathscr{O}[p]|} \arrow[d, "\pi"] \\ {[0,1]} \arrow[ur, "\tilde{\alpha}"] \arrow[r, "\alpha"'] & X \end{tikzcd}$$

> **Corollary 3.28.1** (Monodromy Theorem)**.** *Let $\alpha_0, \alpha_1 : [0,1] \to X$ be homotopic curves from p to q. If the germ $\eta_0 \in \mathscr{O}_p[p]$ admits an analytic continuation along every deformation of α_0 to α_1, then the analytic continuations of η_0 along α_0 and α_1 coincide.*

Proof. By Propositions 3.23 and 3.26, $|\mathscr{O}[p]|$ is Hausdorff whose projection $\pi : |\mathscr{O}[p]| \to X$ is a local homeomorphism. Since each deformation admits a lifting starting at η_0, the[30] liftings $\tilde{\alpha}_0$ and $\tilde{\alpha}_1$ have the same endpoints; that is, the analytic continuations along α_0 and α_1 coincide. ■

> **Corollary 3.28.2.** *Suppose X is simply-connected. If the germ $\eta_0 \in \mathscr{O}_p[p]$ admits an analytic continuation along every curve starting at p, then there exists a unique (globally-defined) function $f \in \mathscr{O}[p](X)$ with $[f]_p = \eta_0$.*

Proof. Uniqueness follows from the Identity Theorem. For existence, we define $f(q) := \hat{\eta}_q(q)$ where $\hat{\eta}_q \in \mathscr{O}_q[p]$ is the analytic continuation along any curve from p to q. Since X is simply-connected, the Monodromy Theorem ensures that $\hat{\eta}_q$ is well-defined. Clearly $f(p) = \eta_0(p)$, so $[f]_p = \eta_0$. Finally, since $[f]_q = \hat{\eta}_q \in \mathscr{O}_q[p]$ for all $q \in X$, we see that $f \in \mathscr{O}[p](X)$. ■

Remark. Let $U \subseteq X$ be any open set containing p and consider any function $f_0 \in \mathscr{O}[p](U)$. We have reduced the problem of analytically continuing f_0 to a global function $f \in \mathscr{O}[p](X)$ with $f|_U = f_0$ into finding analytic continuations of $[f_0]_p$ along every curve starting at p. We shall establish this fact under (under some conditions) in the next section. ♦

3.3.2 Existence of Analytic Continuations

In this section, we let $E \subseteq |\mathscr{O}[p]|$ be the connected component of the Étalé space of $\mathscr{O}[p]$ containing η_0 and write $\pi : E \to X$ as the restricted projection map.

> **Theorem 3.29.** *If π is a covering map, then for any $q \in X$ and any curve $\alpha : [0,1] \to X$ with $\alpha(0) = p$ and $\alpha(1) = q$, there exists an analytic continuation $\hat{\eta} \in \mathscr{O}_q[p]$ of η_0 along α.*

Proof. Define a complex structure $\mathfrak{A}$ on E, that makes π locally biholomorphic, as follows.

- For any $\zeta \in E$, let (U_0, φ) be a chart of X around $\pi(\zeta)$. Since π is a local homeomorphism, there exist neighborhoods $V \subseteq E$ of ζ and $U \subseteq U_0$ of $\pi(\zeta)$ such that $\pi|_V : V \to U$ is a homeomorphism. Set $\psi := \varphi \circ \pi|_V$, so (V, ψ) is a chart on E around ζ. Let $\mathfrak{A}$ be the collection of all such charts, which defines an atlas on E since for any pair of charts $(V_1, \psi_1), (V_2, \psi_2) \in \mathfrak{A}$ with $V_1 \cap V_2 \neq \varnothing$, there exist charts (U_1, φ_1) and (U_2, φ_2) of X such that

$$\psi_2 \circ \psi_1^{-1} = \left(\varphi_2 \circ \pi|_{V_2}\right) \circ \left(\varphi_1 \circ \pi|_{V_1}\right)^{-1} = \varphi_2 \circ \left(\pi|_{V_2} \circ \pi|_{V_1}^{-1}\right) \circ \varphi_1^{-1},$$

  when restricted to $\psi_1(V_1 \cap V_2)$, reduces to $\varphi_2 \circ \varphi_1^{-1}$. This shows that the charts (V_1, ψ_1) and (V_2, ψ_2) are holomorphically compatible, as desired. Furthermore, we claim that $\pi : E \to X$ is locally biholomorphic w.r.t. $\mathfrak{A}$. Indeed, for any $\zeta \in E$, there exist charts (V, ψ) of E around ζ and (U_0, φ) of X around $\pi(\zeta)$ such that $\psi = \varphi \circ \pi|_V$. Then $\varphi \circ \pi|_V \circ \psi^{-1} = \mathrm{id}_V$, which is holomorphic, so π is locally biholomorphic.

We now define a family $\eta_t \in \mathscr{O}_{\alpha(t)}[p]$ for $t \in [0,1]$ as follows. For all $t \in [0,1]$, let[31] $\zeta_t \in E$ be such that $\pi(\zeta_t) = \alpha(t)$. Then there exist neighborhoods V_t around ζ_t and U_t around $\alpha(t)$ such that $\pi|_{V_t} : V_t \to U_t$ is a biholomorphism. Let $\chi_t := \pi|_{V_t}^{-1}$ and define $\eta_t := (\chi_t \circ \alpha)(t) \in \mathscr{O}_{\alpha(t)}[p]$. Observe that $\hat{\eta} := \eta_1 \in \mathscr{O}_q[p]$, which we claim is the analytic continuation of η_0 along α.

$$\begin{tikzcd}
E \arrow[d, "\pi"'] & V_t \arrow[l, hook'] \arrow[r, "\ell|_{V_t}"] \arrow[d, shift left, "\pi|_{V_t}"] & \mathbb{C} \\
X & U_t \arrow[l, hook'] \arrow[u, shift left, "\chi_t"] \arrow[ur, "f_t"'] & {[0,1]} \arrow[l, "\alpha"']
\end{tikzcd}$$

- We first construct a function $\ell : E \to \mathbb{C}$ as follows. For $\zeta \in E$, consider any chart (U_0, φ) of X around $\pi(\zeta)$ and any function $g \in \mathscr{O}[p](U_0)$ such that $\zeta = [g]_{\pi(\zeta)}$. Set[32] $\ell(\zeta) := g(\pi(\zeta))$. We claim that ℓ has at most a single simple pole at η_0. Indeed, for any $\zeta \in E$, the chart (V, ψ) as defined above that makes $\pi|_V : V \to U$ a homeomorphism ensures that

$$\ell \circ \psi^{-1} = \left(\ell \circ \pi|_V^{-1}\right) \circ \varphi^{-1} = g \circ \varphi^{-1},$$

  which is meromorphic with at most a single simple pole at $\varphi(p)$; we have $\ell(\eta_0) = g(p)$.

Take $\tau \in [0,1]$ and consider $\chi_\tau : U_\tau \to V_\tau$ as defined above. Since U_τ is open, the continuity of α furnishes a neighborhood $T_\tau \subseteq [0,1]$ of τ such that $\alpha(T_\tau) \subseteq U_\tau$. Set $f_\tau := \ell|_{V_\tau} \circ \chi_\tau$, which is in $\mathscr{O}[p](U_\tau)$ since χ_τ is holomorphic and ℓ is meromorphic with at most a single simple pole at η_0. It remains to show that $[f_t]_{\alpha(t)} = \eta_t$ for all $t \in T$. But this is clear since $\pi([f_t]_{\alpha(t)}) = \alpha(t) \in U_t$ and $\pi|_{V_t} : V_t \to U_t$ is invertible, so

$$[f_t]_{\alpha(t)} = \left(\pi|_{V_t}^{-1} \circ \alpha\right)(t) = (\chi_t \circ \alpha)(t) = \eta_t.$$ ■

[30] This is a standard result in algebraic topology. For a proof, see [For81, Proposition 4.10].

[31] Such a ζ_t exists since π is a covering map. However, it need not be unique; we let ζ_t be *any* such germ. Thus an analytic continuation of η_0 along α need not be unique in general.

[32] This is well-defined.

Remark. In fact[33], the existence of analytic continuations of η_0 along every curve α starting at p is equivalent to $\pi : E \to X$ being a covering map. However, π is not always a covering map[34], so in practice one considers a specific function germ η_0 and studies its corresponding Étalé space E. Ultimately, we think that this boils down to solving a system of PDEs (with boundary conditions being the glueing conditions), but further investigation is needed. ♦

[33] See [For81, Exercise 7.2].

[34] For instance, the Lacunary function does not admit an analytic continuation anywhere outside its radius of convergence.

Chapter 4

Čech Cohomology

4.1 Differential Forms

Throughout this section, let $W \subseteq X$ be an open subset of a Riemann surface X and fix $p \in W$.

4.1.1 The Cotangent Space

For an open set $V \subseteq \mathbb{C}$, we let $\mathscr{E}(V)$ denote the $\mathbb{C}$-algebra of all functions $f : V \to \mathbb{C}$ that are differentiable w.r.t. the real coordinates x and y, which we simple call differentiable[1]. Using the partial derivative operators $\partial/\partial x$ and $\partial/\partial y$, we define

$$\frac{\partial}{\partial z} \coloneqq \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad \text{and} \qquad \frac{\partial}{\partial \bar{z}} \coloneqq \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

In this language, the Cauchy-Riemann equations state that $\mathscr{O}(V) = \ker \partial/\partial \bar{z}$. We now lift these notions to a Riemann surface X.

Definition 4.1. *A function $f : W \to \mathbb{C}$ is said to be differentiable at p if there is a chart (U, z) of X around p such that $f \circ z^{-1} : z(U) \to \mathbb{C}$ is differentiable at $z(p)$. If f is differentiable at every point of W, then f is said to be differentiable on W.*

[2]

Remark. Let $\mathscr{E}(W)$ denote the $\mathbb{C}$-algebra of all differentiable functions $f : W \to \mathbb{C}$ on W. Together with the usual restriction mappings, we obtain a sheaf $\mathscr{E}$ of $\mathbb{C}$-algebras consisting of all differentiable functions on X. ♦

Definition 4.2. *Fix a chart (U, z) of X. The partial derivative operator w.r.t. (U, z) is the operator*

$$\frac{\partial}{\partial z} : \mathscr{E}(U) \to \mathscr{E}(U) \qquad \textit{mapping} \qquad f \mapsto \frac{\partial}{\partial z}\left(f \circ z^{-1}\right).$$

Similarly, we define $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial \bar{z}$.

[3]

Definition 4.3. *Let $\mathfrak{m}_p \subseteq \mathscr{E}_p$ be the ideal of all differentiable functions vanishing at p and let $\mathfrak{m}_p^2$ be its product. The cotangent space of X at p is the quotient space $T_p^* X \coloneqq \mathfrak{m}_p/\mathfrak{m}_p^2$. If $U \ni p$ is open and $f \in \mathscr{E}(U)$, we define the differential of f at p as*

$$\mathrm{d}_p f \coloneqq [f - f(p)]_{\mathfrak{m}_p^2} \in T_p^* X.$$

[4]

Proposition 4.4. *Let (U, z) be a chart of X around p. Then $\{\mathrm{d}_p x, \mathrm{d}_p y\}$ and $\{\mathrm{d}_p z, \mathrm{d}_p \bar{z}\}$ are both bases for $T_p^* X$, and if $f \in \mathscr{E}(W)$, then*

$$\begin{aligned} \mathrm{d}_p f &= \left.\frac{\partial f}{\partial x}\right|_p \mathrm{d}_p x + \left.\frac{\partial f}{\partial y}\right|_p \mathrm{d}_p y \\ &= \left.\frac{\partial f}{\partial z}\right|_p \mathrm{d}_p z + \left.\frac{\partial f}{\partial \bar{z}}\right|_p \mathrm{d}_p \bar{z}. \end{aligned}$$

Proof. We first show that $\{\mathrm{d}_p x, \mathrm{d}_p y\}$ is a basis for $T_p^* X$.

- Let $[\eta] \in T_p^* X$, so $\eta = [f]_p \in \mathfrak{m}_p$ is a differentiable function germ for some $f \in \mathscr{E}(W)$. Taylor's Theorem in $\mathbb{C}$ then furnishes $\lambda_1, \lambda_2 \in \mathbb{C}$ such that[5]

  $$f = \lambda_1 (x - x(p)) + \lambda_2 (y - y(p)) + g$$

  where $g \in \mathscr{E}(W)$ is such that $[g]_p \in \mathfrak{m}_p^2$. This lifts to an equality of germs, so, taking the quotient modulo $\mathfrak{m}_p^2$, we see that $[\eta] = \lambda_1 \mathrm{d}_p x + \lambda_2 \mathrm{d}_p y$.

- For any $\lambda_1, \lambda_2 \in \mathbb{C}$, the linear dependence $\lambda_1 \mathrm{d}_p x + \lambda_2 \mathrm{d}_p y = 0$ implies that

  $$\lambda_1 (x - x(p)) + \lambda_2 (y - y(p)) \in \mathfrak{m}_p^2.$$

  Taking the partials $\partial/\partial x$ and $\partial/\partial y$ shows that $\lambda_1 = \lambda_2 = 0$.

[1] By 'differentiable', we always mean infinitely-differentiable; i.e., smooth.
[2] As with holomorphic functions, differentiability is chart-independent.
[3] That is, we define $\partial f/\partial z$ by pulling back the regular partial derivative $\partial/\partial z$ for the function $f \circ z^{-1}$ on $\mathbb{C}$. Again, this is chart-independent.
[4] That is, let $\mathfrak{m}_p$ contain all germs $[f]_p$ such that $f(p) = 0$ and let $\mathfrak{m}_p^2$ contain all germs $[h]_p$ such that $h = \sum_i f_i g_i$ for some $f_i, g_i \in \mathfrak{m}_p$.
[5] There is no constant term since $[f]_p \in \mathfrak{m}_p$.

Suppose now that $f \in \mathscr{E}(W)$. By Taylor's Theorem, we have

$$f - f(p) = \left.\frac{\partial f}{\partial x}\right|_p (x - x(p)) + \left.\frac{\partial f}{\partial y}\right|_p (y - y(p)) + g$$

where $g \in \mathscr{E}(W)$ is such that $[g] \in \mathfrak{m}_p^2$, so lifting this to an equality of germs and taking the quotient modulo $\mathfrak{m}_p^2$ gives us

$$\mathrm{d}_p f = \left.\frac{\partial f}{\partial x}\right|_p \mathrm{d}_p x + \left.\frac{\partial f}{\partial y}\right|_p \mathrm{d}_p y.$$

Finally, we show the corresponding result for $\{\mathrm{d}_p z, \mathrm{d}_p \overline{z}\}$. Indeed, since $z = x + iy$ as functions in $\mathscr{E}(W)$, we have that $\partial z/\partial x = 1$ and $\partial z/\partial y = i$. Similarly, $\partial \overline{z}/\partial x = 1$ and $\partial \overline{z}/\partial y = -i$, so

$$\mathrm{d}_p z = \mathrm{d}_p x + i\mathrm{d}_p y \qquad \text{and} \qquad \mathrm{d}_p \overline{z} = \mathrm{d}_p x - i\mathrm{d}_p y.$$

Thus $\{\mathrm{d}_p z, \mathrm{d}_p \overline{z}\}$ is linearly-independent, so it is a basis for $T_p^* X$. For $f \in \mathscr{E}(W)$, a computation now shows that

$$\mathrm{d}_p f = \frac{1}{2}\left(\left.\frac{\partial f}{\partial x}\right|_p - i\left.\frac{\partial f}{\partial y}\right|_p\right)\mathrm{d}_p z + \frac{1}{2}\left(\left.\frac{\partial f}{\partial x}\right|_p + i\left.\frac{\partial f}{\partial y}\right|_p\right)\mathrm{d}_p \overline{z} = \left.\frac{\partial f}{\partial z}\right|_p \mathrm{d}_p z + \left.\frac{\partial f}{\partial \overline{z}}\right|_p \mathrm{d}_p \overline{z}. \qquad \blacksquare$$

Proposition 4.5 (Canonical Decomposition)**.** *Let (U, z) be a chart of X around p. Then the subspaces*

$$T_p^* X^{(1,0)} := \operatorname{span}\{\mathrm{d}_p z\} \qquad \textit{and} \qquad T_p^* X^{(0,1)} := \operatorname{span}\{\mathrm{d}_p \overline{z}\}$$

are chart-independent and $T_p^* X = T_p^* X^{(1,0)} \oplus T_p^* X^{(0,1)}$.

Proof. If (U', z') is another chart of X around p. Since $z' \in \mathscr{O}(U \cap U')$, the expansion

$$\mathrm{d}_p z' = \left.\frac{\partial z'}{\partial z}\right|_p \mathrm{d}_p z + \left.\frac{\partial z'}{\partial \overline{z}}\right|_p \mathrm{d}_p \overline{z}$$

shows that $\partial z'/\partial \overline{z} = 0$, so $\operatorname{span}\{\mathrm{d}_p z\} = \operatorname{span}\{\mathrm{d}_p z'\}$. Similarly, $\partial \overline{z'}/\partial z = 0$, so $\operatorname{span}\{\mathrm{d}_p \overline{z}\} = \operatorname{span}\left\{\mathrm{d}_p \overline{z'}\right\}$. The decomposition then follows by construction. $\blacksquare$

Remark. For[6] all $f \in \mathscr{E}(W)$, let $\partial_p f \in T_p^* X^{(1,0)}$ and $\overline{\partial}_p f \in T_p^* X^{(0,1)}$ be the unique elements such that $\mathrm{d}_p f = \partial_p f + \overline{\partial}_p f$. The above proposition ensures that they are chart-independent. ♦

4.1.2 Differential 1-forms

Definition 4.6. *A differential 1-form on W is a map*

$$\omega : W \to \bigcup_{p \in W} T_p^* X$$

such that $\omega(p) \in T_p^* X$ for every $p \in W$.

[7]

Example 4.7. For $f \in \mathscr{E}(W)$, the maps $\mathrm{d}f$, ∂f, and $\overline{\partial} f$ defined by

$$(\mathrm{d}f)(p) := \mathrm{d}_p f, \qquad (\partial f)(p) := \partial_p f, \qquad \text{and} \qquad (\overline{\partial} f)(p) := \overline{\partial}_p f$$

for all $p \in W$ are all 1-forms. Note that if (U, z) is a chart of X, then every 1-form ω on W can be written as

$$\omega = f_1 \mathrm{d}x + f_2 \mathrm{d}y = f_1' \mathrm{d}z + f_2' \mathrm{d}\overline{z}$$

for some[8] $f_1, f_2, f_1', f_2' : U \to \mathbb{C}$. Indeed, for all $p \in U$, we have $\omega(p) = f_1(p)\,\mathrm{d}_p x + f_2(p)\,\mathrm{d}_p y$ for some $f_1(p), f_2(p) \in \mathbb{C}$. Varying over all $p \in U$ gives us functions $f_1, f_2 : U \to \mathbb{C}$. Similarly for f_1' and f_2'. ♦

Definition 4.8. *We define certain subspaces of 1-forms on W as follows.*

- *The subspace $\mathscr{E}^{(1)}(W)$ of all differentiable 1-forms ω on W such that, w.r.t. every chart (U, z) of X, $\omega = f\mathrm{d}z + g\mathrm{d}\overline{z}$ for some $f, g \in \mathscr{E}(U \cap W)$.*
- *The subspace $\mathscr{E}^{(1,0)}(W)$ (resp. $\mathscr{E}^{(0,1)}$) of all type $(1,0)$ (resp. $(0,1)$) 1-forms ω on W such that, w.r.t. every chart (U, z) of X, $\omega = f\mathrm{d}z$ (resp. $\omega = f\mathrm{d}\overline{z}$) for some $f \in \mathscr{E}(U \cap W)$.*
- *The subspace $\Omega(W)$ of all holomorphic 1-forms ω on W such that, w.r.t. every chart (U, z) of X, $\omega = f\mathrm{d}z$ for some $f \in \mathscr{O}(U \cap W)$.*

[6] For computations, we descend via any chart (U, z) of X around p where we have

$$\partial_p f = \left.\frac{\partial f}{\partial z}\right|_p \mathrm{d}_p z \qquad \text{and} \qquad \overline{\partial}_p f = \left.\frac{\partial f}{\partial \overline{z}}\right|_p \mathrm{d}_p \overline{z}.$$

[7] With the induced operations from $T_p^* X$, the set of all 1-forms on W becomes a $\mathbb{C}$-vector space. In fact, it is a $\mathbb{C}$-algebra, for if $f : W \to \mathbb{C}$ is a function, then the map $f\omega$ defined by $(f\omega)(p) := f(p)\,\omega(p)$ is also a 1-form on W.

[8] We note that the functions f_1, f_2, f_1', f_2' are not necessarily continuous.

Remark. More work needs to be done to define *meromorphic* 1-*forms* on W. In fact, we may analogously define the *order of a pole* of a meromorphic 1-form; see [For81, Section 9.9]. ⧫

Example 4.9. For $f \in \mathscr{E}(W)$, the form $\mathrm{d}f$ (resp. $\partial f, \overline{\partial} f$) is a differentiable (resp. type $(1,0)$, type $(0,1)$) 1-form on W. Thus we have the map $\mathrm{d} : \mathscr{E}(W) \to \mathscr{E}^{(1)}(W)$, and similarly the maps ∂ and $\overline{\partial}$, called the exterior derivatives on $\mathscr{E}(W)$. These exterior derivatives, which are in fact morphisms of sheaves, are studied in the next section. ⧫

4.1.3 Differential 2-forms and Exterior Differentiation

Define[9] the *exterior power* $\Lambda^2 V$ of a $\mathbb{C}$-vector space V as the quotient of the tensor product $V \otimes V$ by the ideal $\mathfrak{a} := (v \otimes v \,|\, v \in V)$. For completeness, we very briefly define $V \otimes V$.

Definition 4.10. *Let V be a $\mathbb{C}$-vector space and consider the free vector space (F, j) over $V \times V$. Letting S denote the span of*

$$j(v, \lambda v_1 + v_2) - \lambda j(v, v_1) - \lambda j(v, v_2) \qquad \text{and} \qquad j(\lambda v_1 + v_2, v) - \lambda j(v, v_1) - \lambda j(v, v_2),$$

for all $v, v_1, v_2 \in V$ and $\lambda \in \mathbb{C}$, we define the tensor product of V as the quotient space $V \otimes V := F/S$ equipped with the map $\otimes := \pi \circ j$, where $\pi : F \to F/S$ is the projection.

[10]

Remark. Let V be a $\mathbb{C}$-vector space. For all $v_1, v_2 \in V$, define $v_1 \wedge v_2 \in \Lambda^2 V$ to be the equivalence class of $v \otimes v$ modulo $\mathfrak{a}$. It is then immediate from the definition of $V \otimes V$ that

$$(v_1 + v_2) \wedge v_3 = (v_1 \wedge v_3) + (v_2 \wedge v_3) \qquad \text{and} \qquad (\lambda v_1) \wedge v_2 = \lambda (v_1 \wedge v_2)$$

for all $v_1, v_2, v_3 \in V$ and $\lambda \in \mathbb{C}$. Moreover,

$$\begin{aligned} 0 &= (v_1 + v_2) \wedge (v_1 + v_2) \\ &= (v_1 \wedge v_1) + (v_1 \wedge v_2) + (v_2 \wedge v_1) + (v_2 \wedge v_2) \\ &= (v_1 \wedge v_2) + (v_2 \wedge v_1), \end{aligned}$$

so $v_1 \wedge v_2 = -(v_2 \wedge v_1)$ for all $v_1, v_2 \in V$. Finally, if $\{e_i\}$ is a basis for V, then[11] $\{e_i \otimes e_j\}$ is a basis for $V \otimes V$. Combined with the above, we see that $\{e_i \wedge e_j\}_{i<j}$ is a basis for $\Lambda^2 V$. ⧫

Remark. We now specialize for when $V = T_p^* X$[12] and consider the exterior power $\Lambda^2 T_p^* X$. Letting (U, z) be a chart of X around p, we see that $\{\mathrm{d}_p x \wedge \mathrm{d}_p y\}$ and $\{\mathrm{d}_p z \wedge \mathrm{d}_p \overline{z}\}$ are both bases for $\Lambda^2 T_p^* X$. Thus $\dim \Lambda^2 T_p^* X = 1$. Also, observe that

$$\mathrm{d}_p z \wedge \mathrm{d}_p \overline{z} = (\mathrm{d}_p x + i \mathrm{d}_p y) \wedge (\mathrm{d}_p x - i \mathrm{d}_p y) = -2i (\mathrm{d}_p x \wedge \mathrm{d}_p y).$$ ⧫

Definition 4.11. *A differential 2-form on W is a map*

$$\omega : W \to \bigcup_{p \in W} \Lambda^2 T_p^* X$$

such that $\omega(p) \in \Lambda^2 T_p^* X$ for every $p \in W$. A 2-form ω is said to be differentiable if, w.r.t. every chart (U, z) of X, we have $\omega = f \mathrm{d}z \wedge \mathrm{d}\overline{z}$ for some $f \in \mathscr{E}(U \cap W)$.

[13]

Remark. In the above definition, $\mathrm{d}z \wedge \mathrm{d}\overline{z}$ is the 2-form on W defined by $(\mathrm{d}z \wedge \mathrm{d}\overline{z})(p) := \mathrm{d}_p z \wedge \mathrm{d}_p \overline{z}$ for every $p \in W$. In general, if ω_1 an ω_2 are 1-forms on W, we have the 2-form $\omega_1 \wedge \omega_2$ defined by

$$(\omega_1 \wedge \omega_2)(p) := \omega_1(p) \wedge \omega_2(p)$$

for every $p \in W$. The $\mathbb{C}$-vector space of all differentiable 2-forms on W is denoted $\mathscr{E}^{(2)}(W)$. ⧫

Definition/Proposition 4.12. *Let ω be a differentiable 1-form on W, which, under a chart (U, z) of X, has the form $\omega = f_1 \mathrm{d}z + f_2 \mathrm{d}\overline{z}$ for some $f_1, f_2 \in \mathscr{E}(U \cap W)$. Then the 2-form*

$$\mathrm{d}\omega := \mathrm{d}f_1 \wedge \mathrm{d}z + \mathrm{d}f_2 \wedge \mathrm{d}\overline{z}$$

is chart-independent and differentiable, which defines the map $\mathrm{d} : \mathscr{E}^{(1)}(W) \to \mathscr{E}^{(2)}(W)$, called the exterior derivative on $\mathscr{E}^{(1)}(W)$.

[14]

[9] For an in-depth discussion of the tensor product, see [Alu09, Chapter 8.2] or [Con16].
[10] Here, $j : V \times V \to F$ is a function making (F, j) satisfy the universal property of the free vector space over $V \times V$.
[11] For a proof, see [Lee12, Proposition 12.8].
[12] Recall our notation, where $W \subseteq X$ is an open subset of a Riemann surface X and $p \in W$.
[13] As with 1-forms, the set of all 2-forms on W forms a vector space under the induced operations from $\Lambda^2 T_p^* X$. Similarly, it is also a $\mathbb{C}$-algebra by defining the map $f\omega$ by $(f\omega)(p) := f(p)\,\omega(p)$ for every function $f : W \to \mathbb{C}$.
[14] Similarly, define the 2-forms

$$\begin{aligned} \partial\omega &:= \partial f_1 \wedge \mathrm{d}z + \partial f_2 \wedge \mathrm{d}\overline{z} \\ \overline{\partial}\omega &:= \overline{\partial} f_1 \wedge \mathrm{d}z + \overline{\partial} f_2 \wedge \mathrm{d}\overline{z}. \end{aligned}$$

The same proof shows that $\partial\omega$ and $\overline{\partial}\omega$ are chart-independent, which define the operators ∂ and $\overline{\partial}$.

Proof. For convenience, we write $z_1 := z$ and $z_2 := \overline{z}$, so $\omega = \sum_i f_i \mathrm{d}z_i$ and $\mathrm{d}\omega = \sum_i \mathrm{d}f_i \wedge \mathrm{d}z_i$. To show that $\mathrm{d}\omega \in \mathscr{E}^{(2)}(W)$, let (V, w) be a chart of X. Expanding $\mathrm{d}f_i$ and $\mathrm{d}z_i$ in the basis $\{\mathrm{d}w, \mathrm{d}\overline{w}\}$, we see that

$$\begin{aligned}\mathrm{d}\omega &= \sum_{j=1}^{2} \left(\frac{\partial f_i}{\partial w}\mathrm{d}w + \frac{\partial f_i}{\partial \overline{w}}\mathrm{d}\overline{w}\right) \wedge \left(\frac{\partial z_i}{\partial w}\mathrm{d}w + \frac{\partial z_i}{\partial \overline{w}}\mathrm{d}\overline{w}\right) \\ &= \sum_{j=1}^{2} \left(\frac{\partial f_i}{\partial w}\frac{\partial z_i}{\partial \overline{w}} - \frac{\partial f_i}{\partial \overline{w}}\frac{\partial z_i}{\partial w}\right) \mathrm{d}w \wedge \mathrm{d}\overline{w} \in \mathscr{E}^{(2)}(W).\end{aligned}$$

To show well-definition, let (U', z') be another chart of X and write $\omega = \sum_i f_i' \mathrm{d}z_i'$.[15] Choose a chart (V, w) of X. Expanding $\mathrm{d}z_i$ and $\mathrm{d}z_i'$ in the basis $\{\mathrm{d}w, \mathrm{d}\overline{w}\}$ and equating, we obtain by the assumption $\sum_i f_i \mathrm{d}z_i = \sum_i f_i' \mathrm{d}z_i'$ that

$$\sum_{i=1}^{2} f_i \frac{\partial z_i}{\partial w} = \sum_{i=1}^{2} f_i' \frac{\partial z_i'}{\partial w} \qquad \text{and} \qquad \sum_{i=1}^{2} f_i \frac{\partial z_i}{\partial \overline{w}} = \sum_{i=1}^{2} f_i' \frac{\partial z_i'}{\partial \overline{w}}.$$

Applying $\partial/\partial\overline{w}$ and $\partial/\partial w$ respectively and subtracting yields

$$\sum_{i=1}^{2} \left(\frac{\partial f_i}{\partial w}\frac{\partial z_i}{\partial \overline{w}} - \frac{\partial f_i}{\partial \overline{w}}\frac{\partial z_i}{\partial w}\right) = \sum_{i=1}^{2} \left(\frac{\partial f_i'}{\partial w}\frac{\partial z_i'}{\partial \overline{w}} - \frac{\partial f_i'}{\partial \overline{w}}\frac{\partial z_i'}{\partial w}\right).$$

From our previous calculation of $\mathrm{d}\omega$, the result follows. ■

Definition 4.13. *A differentiable* 1*-form* ω *on* W *is* closed *if* $\mathrm{d}\omega = 0$*, and is* exact *if* $\omega = \mathrm{d}f$ *for some* $f \in \mathscr{E}(W)$*.*

Proposition 4.14.

1. *Every exact form is closed.*
2. *Every holomorphic* 1*-form is closed.*
3. *Every closed* 1*-form of type* $(1, 0)$ *is holomorphic.*

Proof. Let ω be a 1-form on W.

1. This is precisely the statement that $\mathrm{d}^2 f = 0$ for all $f \in \mathscr{E}(W)$, which follows from[16]

$$\mathrm{d}^2 f = \mathrm{d}(1 \cdot \mathrm{d}f) = \mathrm{d}1 \wedge \mathrm{d}f = 0.$$

For 2 and 3, suppose that $\omega = f\mathrm{d}z$ for some $f \in \mathscr{E}(W)$. Then

$$\mathrm{d}\omega = \mathrm{d}f \wedge \mathrm{d}z = \left(\frac{\partial f}{\partial z}\mathrm{d}z + \frac{\partial f}{\partial \overline{z}}\mathrm{d}\overline{z}\right) \wedge \mathrm{d}z = -\frac{\partial f}{\partial \overline{z}}\mathrm{d}z \wedge \mathrm{d}\overline{z}$$

Thus $\mathrm{d}\omega = 0$ iff $\partial f/\partial\overline{z} = 0$, so every holomorphic 1-form is closed and every closed 1-form of type $(1, 0)$ is holomorphic. ■

4.1.4 Integration of 2-forms

Similarly to how we defined partial derivatives on a chart (U, z) of X in Definition 4.2 by pulling back the partial derivative on $\mathbb{C}$, we first discuss integration of a 2-form ω on an open set $V \subseteq \mathbb{C}$ and then pull it back to Riemann surfaces.

Let ω of a differentiable 2-form on an open subset $V \subseteq \mathbb{C}$, say with[17] $\omega = f\mathrm{d}x \wedge \mathrm{d}y$ for some $f \in \mathscr{E}(V)$. If f vanishes outside a compact subset of V, define[18]

$$\int_V \omega = \int_V f\,\mathrm{d}x \wedge \mathrm{d}y := \int_V f\,\mathrm{d}x\,\mathrm{d}y.$$

We now define the pullback of forms under a holomorphic map, which gives us a coordinate-free description of the Change of Variables formula.

Definition 4.15. *Let* $F : X \to Y$ *be a holomorphic map between Riemann surfaces and let* $V \subseteq Y$ *be open. The* pullback of F *is the map* $F^* : \mathscr{E}(V) \to \mathscr{E}\left(F^{-1}(V)\right)$ *mapping* $f \mapsto f \circ F$*. More generally, define* $F^* : \mathscr{E}^{(k)}(V) \to \mathscr{E}^{(k)}\left(F^{-1}(V)\right)$ *for* $k = 1, 2$ *mapping*

$$\begin{aligned} f_1\mathrm{d}z + f_2\mathrm{d}\overline{z} &\mapsto (F^* f_1)\,\mathrm{d}(F^* z) + (F^* f_2)\,\mathrm{d}(F^* \overline{z}) \\ f\mathrm{d}z \wedge \mathrm{d}\overline{z} &\mapsto (F^* f)\,\mathrm{d}(F^* z) \wedge \mathrm{d}(F^* \overline{z}).\end{aligned}$$

[15] Again, write $z_1' := z'$ and $z_2' := \overline{z'}$.
[16] The same computation also shows $\partial^2 f = \overline{\partial}^2 f = 0$.
[17] We take the standard chart on $\mathbb{C}$, so $x + iy = \mathrm{id}_{\mathbb{C}}$.
[18] The right-hand side is the usual double integral on $\mathbb{C}$, which simply 'erases the wedges'.

Proposition 4.16. *Let $U, V \subseteq \mathbb{C}$ be open and let $\varphi : U \to V$ be biholomorphic. Then, for any differentiable 2-form ω on V, $\int_V \omega = \int_U \varphi^* \omega$.*

Proof. Writing $\omega = f \mathrm{d}x \wedge \mathrm{d}y$ for some $f \in \mathscr{E}(V)$, we have by the Change of Variables on $\mathbb{C}$ that

$$\int_V \omega = \int_V f \, \mathrm{d}x \, \mathrm{d}y = \int_U (f \circ \varphi) \left|\det D\varphi\right| \, \mathrm{d}x \, \mathrm{d}y,$$

where $D\varphi$ is the Jacobian of φ. Decomposing[19] $\varphi = u + iv$ and using the Cauchy-Riemann equations, we have that[20]

$$\det D\varphi = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \geq 0$$

and thus the pullback of ω is

$$\begin{aligned}
\varphi^* \omega = \varphi^* (f \mathrm{d}x \wedge \mathrm{d}y) &= (\varphi^* f) \, \mathrm{d}(\varphi^* x) \wedge \mathrm{d}(\varphi^* y) = (f \circ \varphi) \, \mathrm{d}u \wedge \mathrm{d}v \\
&= (f \circ \varphi) \left(\frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y\right) \wedge \left(\frac{\partial v}{\partial x}\mathrm{d}x + \frac{\partial v}{\partial y}\mathrm{d}y\right) \\
&= (f \circ \varphi)(\det D\varphi) \, \mathrm{d}x \wedge \mathrm{d}y.
\end{aligned}$$

Noting that $\varphi^* \omega$ is a differentiable 2-form on U, the result follows by definition of $\int_U \varphi^* \omega$. ∎

Using the above results, let ω be a differentiable 2-form on X with compact support.[21] Then there exist exist finitely-many charts (U_i, φ_i) on X such that $\operatorname{Supp}(\omega) \subseteq \bigcup_{i=1}^n U_i$. This open cover $\{U_i\}$ of $\operatorname{Supp}(\omega)$ admits a partition of unity[22] $\{\psi_i\}$, which are functions such that $\operatorname{Supp}(\psi_i) \subseteq U_i$ for all i and $\sum_{i=1}^n \psi_i = \mathrm{id}$. Using this partition of unity, we define the integral of ω on X.

Definition/Proposition 4.17. *In the above notation and with $V_i \coloneqq \varphi_i(U_i)$, define*

$$\int_X \omega \coloneqq \sum_{i=1}^n \int_{U_i} \psi_i \omega \coloneqq \sum_{i=1}^n \int_{V_i} \left(\varphi_i^{-1}\right)^* (\psi_i \omega).$$

Proof. (Well-definition). First, note that the forms $\omega_i \coloneqq \psi_i \omega$ can all be restricted[23] to U_i, so we have to check that each integral of ω_i over U_i is independent of the chart φ_i, and that the integral of ω over X is independent of $\{U_i\}$ and its partition of unity $\{\psi_i\}$.

- (Independence of φ_i). Note that $\left(\varphi_i^{-1}\right)^* \omega$ is a differentiable 2-form on V_i. Let[24] $\tilde{\varphi}_i$ be another chart of U_i and set $\tilde{V}_i \coloneqq \tilde{\varphi}(U_i)$. The transition map $\varphi_i \circ \tilde{\varphi}_i^{-1} : \tilde{V}_i \to V_i$ is biholomorphic, so we have by Proposition 4.16 that[25]

$$\int_{V_i} \left(\varphi_i^{-1}\right)^* \omega_i = \int_{\tilde{V}_i} \left(\varphi_i \circ \tilde{\varphi}_i^{-1}\right)^* \left(\varphi_i^{-1}\right)^* \omega_i = \int_{\tilde{V}_i} \left(\tilde{\varphi}_i^{-1}\right)^* \varphi_i^* \left(\varphi_i^{-1}\right)^* \omega = \int_{\tilde{V}_i} \left(\tilde{\varphi}_i^{-1}\right)^* \omega.$$

- (Independence of $\{U_i\}$). Let $\{\tilde{U}_j\}_{j=1}^m$ be another finite open cover and let $\{\tilde{\psi}_j\}_{j=1}^m$ be its corresponding partition of unity. We expand the definition of $\int_X \omega$ on both charts as

$$\begin{aligned}
\sum_{i=1}^n \int_{U_i} \psi_i \omega &= \sum_{i=1}^n \int_{U_i} \left(\sum_{j=1}^m \tilde{\psi}_j\right) \psi_i \omega = \sum_{i=1}^n \sum_{j=1}^m \int_{U_i} \tilde{\psi}_j \psi_i \omega \\
\sum_{j=1}^m \int_{\tilde{U}_j} \tilde{\psi}_j \omega &= \sum_{j=1}^m \int_{\tilde{U}_j} \left(\sum_{i=1}^n \psi_i\right) \tilde{\psi}_j \omega = \sum_{i=1}^n \sum_{j=1}^m \int_{\tilde{U}_j} \tilde{\psi}_j \psi_i \omega.
\end{aligned}$$

  Note that $\tilde{\psi}_j \psi_i \omega$ is compactly supported on $U_i \cap \tilde{U}_j$, so their integrals over U_i and $\tilde{U}_j$ coincide and is well-defined. ∎

4.2 Čech Cohomology Groups

Throughout this section, let X be a topological space, $\mathscr{F}$ a sheaf of Abelian groups on X, and $\mathfrak{A} \coloneqq \{U_i\}$ an open covering of X.

[19] Note that $u, v \in \mathscr{E}(U)$.
[20] $\det D\varphi \geq 0$ is related to the fact that Riemann surfaces are all orientable.
[21] $\operatorname{Supp}(\omega) \coloneqq \overline{\{p \in X \mid \omega(p) \neq 0\}}$.
[22] The 'local-finiteness' condition is irrelevant here since the cover is finite.
[23] Indeed, $\operatorname{Supp}(\omega_i) \subseteq U_i$.
[24] We assume that the domain of $\tilde{\varphi}_i$ is also U_i; otherwise take the intersection.
[25] The fact that pullbacks are anti-multiplicative is clear by expanding the definition.

4.2.1 Cochains, Coboundaries, and Cocycles

Definition 4.18. *For all $n \in \mathbb{N}$, the* n^{th} *cochain group of $\mathscr{F}$ w.r.t. $\mathfrak{A}$ is the direct product*

$$\check{C}^n(\mathfrak{A}, \mathscr{F}) := \prod_{(i_0,\dots,i_n)} \mathscr{F}(U_{i_0} \cap \dots \cap U_{i_n}).$$

Remark. For $n = 0$, the group $\check{C}^0(\mathfrak{A}, \mathscr{F})$ contains all tuples (f_i) where each f_i is defined on U_i. For $n = 1$, the group $\check{C}^1(\mathfrak{A}, \mathscr{F})$ contains all tuples (f_{ij}) where each f_{ij} is defined on the pairwise intersection $U_i \cap U_j$. The following discussion formalizes our rough intuition of 'chaining' open sets whose 'boundary'[26] are their pairwise intersections. ♦

Definition 4.19. *For all $n \in \mathbb{N}$, the* n^{th} *coboundary operator w.r.t. $\mathfrak{A}$ is the map*

$$\delta^n : \check{C}^n(\mathfrak{A}, \mathscr{F}) \to \check{C}^{n+1}(\mathfrak{A}, \mathscr{F}) \qquad \textit{mapping} \qquad (f_{i_0,\dots,i_n}) \mapsto (g_{i_0,\dots,i_{n+1}})$$

where

$$g_{i_0,\dots,i_{n+1}} := \sum_{k=0}^{n+1} (-1)^k \rho\left(f_{i_0,\dots,\widehat{i_k},\dots,i_{n+1}}\right).$$

Define the n^{th} *cocycle* $\check{Z}^n(\mathfrak{A}, \mathscr{F}) := \ker \delta^n$ *and the* n^{th} *splitting cocycle* $\check{B}^n(\mathfrak{A}, \mathscr{F}) := \operatorname{im} \delta^{n-1}$, *whose quotient*

$$\check{H}^n(\mathfrak{A}, \mathscr{F}) := \check{Z}^n(\mathfrak{A}, \mathscr{F}) / \check{B}^n(\mathfrak{A}, \mathscr{F})$$

is called the n^{th} *cohomology group of $\mathscr{F}$ w.r.t. $\mathfrak{A}$.*

[27]

Remark. A calculation shows that $\check{B}^n(\mathfrak{A}, \mathscr{F}) \subseteq \check{Z}^n(\mathfrak{A}, \mathscr{F})$, so the quotient makes sense. In particular, $\delta^{n+1} \circ \delta^n = 0$. ♦

Remark. For $n = 0$, we have $\delta^0(f_i) = (f_j - f_i)$ for all $(f_i) \in \check{C}^0(\mathfrak{A}, \mathscr{F})$. This gives us a glueing condition, that if[28] $(f_i) \in \check{Z}^0(\mathfrak{A}, \mathscr{F})$, then the sheaf axioms furnish a unique $f \in \mathscr{F}(X)$ such that $\rho^X_{U_i}(f) = f_i$ for all i. Thus

$$\check{H}^0(\mathfrak{A}, \mathscr{F}) = \check{Z}^0(\mathfrak{A}, \mathscr{F}) \cong \mathscr{F}(X),$$

so $\check{H}^0(\mathfrak{A}, \mathscr{F})$ is independent of the covering $\mathfrak{A}$ and we may define the 0^{th} cohomology group of $\mathscr{F}$ as $\check{H}^0(X, \mathscr{F}) := \mathscr{F}(X)$. ♦

Remark. For $n = 1$[29], we have $\delta^1(f_{ij}) = (f_{jk} - f_{ik} + f_{ij})$ for all $(f_{ij}) \in \check{C}^1(\mathfrak{A}, \mathscr{F})$. Elements $(f_{ij}) \in \check{Z}^1(\mathfrak{A}, \mathscr{F})$ satisfy the *cocycle condition*, which states $f_{ik} = f_{ij} + f_{jk}$ on $U_i \cap U_j \cap U_k$ for all i, j, k. In particular, it implies that $f_{ii} = 0$ for all i and $f_{ij} = -f_{ji}$ on $U_i \cap U_j$ for all i, j. Note that every splitting cocycle is a cocycle, but not every cocycle splits. In other words, $\check{H}^1(\mathfrak{A}, \mathscr{F})$ measures how 1-cocycles fail to split. The next section defines the 1^{st} cohomology group of $\mathscr{F}$, independent of the covering $\mathfrak{A}$. ♦

4.2.2 Refinements and $\check{H}^1(X, \mathscr{F})$

In this section, we specialize to when $n = 1$.

Definition 4.20. *Let $\mathfrak{A} := \{U_i\}_{i \in I}$ and $\mathfrak{B} := \{V_k\}_{k \in K}$ be open coverings of X. We say that* $\mathfrak{B}$ *is finer than* $\mathfrak{A}$, *and write $\mathfrak{B} \preceq \mathfrak{A}$, if there exists a* refining map *$r : K \to I$ such that $V_k \subseteq U_{r(k)}$ for all $k \in K$.*

Remark. The refining map r lifts to a map $\tilde{r} : \check{Z}^1(\mathfrak{A}, \mathscr{F}) \to \check{Z}^1(\mathfrak{B}, \mathscr{F})$ by sending (f_{ij}) into (g_{kl}) defined by $g_{kl} := f_{r(k),r(l)}$ on $V_k \cap V_l$ for all $k, l \in K$. Observe that if $(f_{ij}) \in \check{B}^1(\mathfrak{A}, \mathscr{F})$, then $\delta^1_{\mathfrak{A}}(f_{ij}) = 0$ and hence $f_{i_1 i_3} = f_{i_1 i_2} + f_{i_2 i_3}$ on $U_{i_1} \cap U_{i_2} \cap U_{i_3}$ for all $i_1, i_2, i_3 \in I$. In particular, we have

$$f_{r(k_1),r(k_3)} = f_{r(k_1),r(k_2)} + f_{r(k_2),r(k_3)}$$

on $V_{k_1} \cap V_{k_2} \cap V_{k_3}$ for all $k_1, k_2, k_3 \in K$ and hence $\delta^1_{\mathfrak{B}}(\tilde{r}(f_{ij})) = 0$. Thus $\tilde{r}(f_{ij}) \in \check{B}^1(\mathfrak{B}, \mathscr{F})$ and so $\tilde{r}$ sends splitting cocycles into splitting cocycles. Hence we may descent $\tilde{r}$ into the quotient, giving us a map

$$\check{H}(r) : \check{H}^1(\mathfrak{A}, \mathscr{F}) \to \check{H}^1(\mathfrak{B}, \mathscr{F}) \qquad \text{mapping} \qquad [f_{ij}] \mapsto [\tilde{r}(f_{ij})].$$ ♦

Proposition 4.21. *In the above notation, the map $\check{H}^{\mathfrak{A}}_{\mathfrak{B}} := \check{H}(r)$ is independent of r and is injective.*

Proof. Take $(f_{ij}) \in \check{Z}^1(\mathfrak{A}, \mathscr{F})$ and suppose that $r' : K \to I$ is another refining map. Lifting it to $\tilde{r}'$ similarly, let $(g_{kl}) := \tilde{r}(f_{ij}) = (f_{r(k),r(l)})$ and $(g'_{kl}) := \tilde{r}'(f_{ij}) = (f_{r'(k),r'(l)})$. Observe then that

$$\begin{aligned} g_{kl} - g'_{kl} &= f_{r(k),r(l)} - f_{r'(k),r'(l)} \\ &= f_{r(k),r(l)} + f_{r(l),r'(k)} - f_{r(l),r'(k)} - f_{r'(k),r'(l)} \\ &= f_{r(k),r'(k)} - f_{r(l),r'(l)} \end{aligned}$$

[26] Here, 'boundary' is not the topological boundary.

[27] The 'hat' notation represents a deletion. Also, ρ is the appropriate restriction mapping of $\mathscr{F}$.

Note that $\check{B}^0(\mathfrak{A}, \mathscr{F}) = 0$ since $\check{C}^{-1}(\mathfrak{A}, \mathscr{F}) = 0$.

[28] Indeed, if $(f_i) \in \check{C}^0(\mathfrak{A}, \mathscr{F})$, then $\rho(f_i) = \rho(f_j)$ for all i, j. Henceforth, we suppress the restriction maps ρ for ease of notation, but will always mention on which domain the relation is valid on.

[29] The construction of the 1^{st} cohomology group $\check{H}^1(X, \mathscr{F})$ is more involved. We will not need the general construction for $\check{H}^n(X, \mathscr{F})$, but they can be done similarly as in the $n = 1$ case. With some machinery, we can also define the n^{th} *De Rham* and *Dolbeault* cohomology groups, which relate to $\check{H}^n(X, \mathscr{F})$. See [Mir95, Section IX.4].

on $V_k \cap V_l$ for all $k, l \in K$. Since r and r' are refining maps, we see that $V_k \subseteq U_{r(k)} \cap U_{r'(k)}$ for all $k \in K$, so we may define $h_k := f_{r(k),r'(k)}$ on the restriction to V_k. Then

$$\left(g_{kl} - g'_{kl}\right) = \left(h_k - h_l\right) = \delta^0(h_k)$$

on $V_k \cap V_l$, so $(g_{ij}) - (g'_{ij}) \in \check{B}^1(\mathfrak{B}, \mathscr{F})$. Thus their equivalence classes coincide, as desired. Now, to show that $\check{H}^{\mathfrak{A}}_{\mathfrak{B}}$ is injective, take $(f_{ij}) \in \ker \check{H}^{\mathfrak{A}}_{\mathfrak{B}}$. Thus $\left(f_{r(k),r(l)}\right) = \check{H}^{\mathfrak{A}}_{\mathfrak{B}}(f_{ij})$ splits, so there exist $g_k \in \mathscr{F}(V_k)$ such that $f_{r(k),r(l)} = g_k - g_l$ on $V_k \cap V_l$ for all $k, l \in K$. Then

$$g_k - g_l = f_{r(k),i} + f_{i,r(l)} = f_{i,r(l)} - f_{i,r(k)}$$

on $U_i \cap V_k \cap V_l$ for all $i \in I$ and hence $g_k + f_{i,r(k)} = g_l + f_{i,r(l)}$ on the same domain. Fixing $i \in I$ and glueing the family $\left\{g_k + f_{i,r(k)}\right\}_{k\in K}$ defined on the cover $\{U_i \cap V_k\}_{k\in K}$ of U_i, we obtain an element $h_i \in \mathscr{F}(U_i)$ such that $h_i = g_k + f_{i,r(k)}$ on $U_i \cap V_k$ for all $k \in K$. Observe then that

$$f_{ij} = f_{i,r(k)} - f_{j,r(k)} = h_i - g_k - h_j + g_k = h_i - h_j$$

on $U_i \cap U_j \cap V_k$. Note that both f_{ij} and $h_i - h_j$ are defined on $U_i \cap U_j$, and since they coincide on the restriction to V_k, uniqueness of the glueing gives us $f_{ij} = h_i - h_j$ on $U_i \cap U_j$. Thus $(f_{ij}) = \delta^0(h_i)$, so (f_{ij}) splits. ■

Remark. If $\mathfrak{C} \preceq \mathfrak{B} \preceq \mathfrak{A}$ are open coverings of X, we have that $\check{H}^{\mathfrak{B}}_{\mathfrak{C}} \circ \check{H}^{\mathfrak{A}}_{\mathfrak{B}} = \check{H}^{\mathfrak{A}}_{\mathfrak{C}}$. This[30] allows us to give a construction of $\check{H}^1(X, \mathscr{F})$ similar to that of Definition 3.21. ♦

Definition 4.22. *The* <u>*1^{st} cohomology group of $\mathscr{F}$*</u> *is the Abelian group*

$$\check{H}^1(X, \mathscr{F}) := \left(\coprod_{\mathfrak{A}} \check{H}^1(\mathfrak{A}, \mathscr{F})\right) \Big/ \sim$$

where $\sim$ is the equivalence relation on the disjoint union, defined, for all $\xi \in \check{H}^1(\mathfrak{A}, \mathscr{F})$ and $\xi' \in \check{H}^1(\mathfrak{A}', \mathscr{F})$, by $\xi \sim \xi'$ iff there exists a refinement $\mathfrak{B} \preceq \mathfrak{A}, \mathfrak{A}'$ such that $\check{H}^{\mathfrak{A}}_{\mathfrak{B}}(\xi) = \check{H}^{\mathfrak{A}'}_{\mathfrak{B}}(\xi')$.

[31]

Proposition 4.23. *Let X be a Riemann surface and consider the sheaf of differentiable functions $\mathscr{E}$ on X. Then $\check{H}^1(X, \mathscr{E}) = 0$.*

Proof. Let $\mathfrak{A} := \{U_i\}_{i\in I}$ be an open covering of X and let $(f_{ij}) \in \check{Z}^1(\mathfrak{A}, \mathscr{E})$ be a cocycle; it suffices to show that (f_{ij}) splits, for then $\check{H}^1(\mathfrak{A}, \mathscr{E}) = 0$ and we are done by the remark above. To do so, we use the fact that there exists[32] a partition of unity subordinate to $\mathfrak{A}$; that is, a family $\{\psi_i\}_{i\in I}$ of differentiable functions such that:

- $\operatorname{Supp}(\psi_i) := \overline{\{p \in X \mid \psi(p) \neq 0\}} \subseteq U_i$ for every $i \in I$.
- Every point in X admits a neighborhood whose intersection with $\{\operatorname{Supp}(\psi_i)\}_{i\in I}$ is finite.
- $\sum_{i\in I} \psi_i = \mathrm{id}$.

Consider the function $\psi_j f_{ij}$ on $U_i \cap U_j$, which may be differentiably extended to U_i by zero outside $\operatorname{Supp}(\psi_j)$. Consider the function $g_i := \sum_{j\in I} \psi_j f_{ij} \in \mathscr{E}(U_i)$, which is legal since there is a neighborhood around every point of U_i such that $\psi_j f_{ij} = 0$ for all but finitely-many $j \in I$. Observe that

$$g_i - g_j = \sum_{k\in I} \psi_k \left(f_{ik} - f_{jk}\right) = \sum_{k\in I} \psi_k \left(f_{ik} + f_{kj}\right) = \sum_{k\in I} \psi_k f_{ij} = f_{ij}$$

on $U_i \cap U_j$, so $(f_{ij}) = (g_i - g_j) = \delta^0(g_i)$ splits. ■

Proposition 4.24 (Leray). *If $\mathscr{F}$ is a sheaf of Abelian groups on a topological space X and $\mathfrak{A} := \{U_i\}_{i\in I}$ is an open covering of X such that $\check{H}^1(U_i, \mathscr{F})$ vanishes for every $i \in I$, then*

$$\check{H}^1(X, \mathscr{F}) \cong \check{H}^1(\mathfrak{A}, \mathscr{F}).$$

Such a covering $\mathfrak{A}$ of X is called a <u>*Leray covering of X*</u>.

Proof. Let $\mathfrak{B} := \{V_k\}_{k\in K}$ be an open covering of X with $\mathfrak{B} \preceq \mathfrak{A}$, so there exists a refining map $r : K \to I$. We claim that $\check{H}^{\mathfrak{A}}_{\mathfrak{B}}$ is an isomorphism, from which the result follows by descending into the quotient. By Proposition 4.21, this map is injective, and to show that it is surjective, we must show that every cocycle $(f_{kl}) \in \check{Z}^1(\mathfrak{B}, \mathscr{F})$ admits a cocycle $(F_{ij}) \in \check{Z}^1(\mathfrak{A}, \mathscr{F})$ such that

$$\left(F_{r(k),r(l)}\right) - (f_{kl}) \in \check{B}^1(\mathfrak{B}, \mathscr{F}).$$

For each $i \in I$, consider the open cover $U_i \cap \mathfrak{B} := \{U_i \cap V_k\}_{k\in K}$ of U_i. Since $\check{H}^1(U_i, \mathscr{F}) = 0$, we see that $\check{H}^1(U_i \cap \mathfrak{B}, \mathscr{F}) = 0$. Restricting to U_i, we see that $(f_{kl}) \in \check{Z}^1(U_i \cap \mathfrak{B}, \mathscr{F})$ and hence there exist $g_{ik} \in \mathscr{F}(U_i \cap V_k)$ such that $f_{kl} = g_{ik} - g_{il}$ on $U_i \cap V_k \cap V_l$ for all $i \in I$ and $k, l \in K$. Using this result on two fixed $i, j \in I$ and equating, we see that $g_{jk} - g_{ik} = g_{jl} - g_{il}$ on $U_i \cap U_j \cap V_k \cap V_l$. This glues to an element $F_{ij} \in \mathscr{F}(U_i \cap U_j)$ such that $F_{ij} = g_{jk} - g_{ik}$ on $U_i \cap U_j \cap V_k$ for all $k \in K$, and a computation shows that $(F_{ij}) \in \check{Z}^1(\mathfrak{A}, \mathscr{F})$. Observe then that[33]

$$F_{r(k),r(l)} - f_{kl} = \left(g_{r(l),k} - g_{r(k),k}\right) - \left(g_{r(l),k} - g_{r(l),l}\right) = g_{r(l),l} - g_{r(k),k}$$

on $V_k \cap V_l$, so setting $h_k := g_{r(k),k} \in \mathscr{F}(V_k)$ shows that $\left(F_{r(k),r(l)}\right) - (f_{kl})$ splits in $\mathfrak{B}$. ■

[30] Both constructions are special cases of the so-called *direct limit*; see [Lan10, Chapter III].

[31] Note that $\check{H}^1(X, \mathscr{F})$ vanishes iff $\check{H}^1(\mathfrak{A}, \mathscr{F}) = 0$ for all open coverings $\mathfrak{A}$ of X. Indeed, the converse direction is trivial. For the forward, let $\mathfrak{A}$ be an open covering of X. By Proposition 4.21, the canonical maps $\check{H}^1(\mathfrak{A}, \mathscr{F}) \to \check{H}^1(\mathfrak{B}, \mathscr{F})$ are injective for all open coverings $\mathfrak{B} \preceq \mathfrak{A}$. Descending into the quotient, the induced map $\check{H}^1(\mathfrak{A}, \mathscr{F}) \to \check{H}^1(X, \mathscr{F})$ is also injective, from which the result follows.

[32] For a proof, see [Lee12, Theorem 2.23]. Note that the functions ψ_i are *not necessarily* holomorphic.

[33] Here, we instantiated the relation $f_{kl} = g_{ik} - g_{il}$ with $i := r(l)$.

4.3 Global Meromorphic Functions

In this section, we prove the existence of certain (non-constant) meromorphic functions on a compact Riemann surface. Together with the vanishing of $\check{H}^1\big(\hat{\mathbb{C}}, \mathscr{O}\big)$, we prove that every simply-connected compact Riemann surface admits a meromorphic function with a single simple pole.

4.3.1 Vanishing of $\check{H}^1\big(\hat{\mathbb{C}}, \mathscr{O}\big)$

Theorem 4.25 (Dolbeault)**.** *For any differentiable function $g \in \mathscr{E}(\mathbb{C})$, there exists a differentiable function $f \in \mathscr{E}(\mathbb{C})$ such that $\overline{\partial} f = g \mathrm{d}\overline{z}$.*

Proof. We first prove for when g is compactly supported. In this case, define $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) := -\frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g(z-\zeta)}{\zeta} \, \mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta}.$$

We need to show that this integral converges and depends differentiably on z. Since g is compactly supported, the integrand only has a pole at 0 and so it suffices to show that the integral over a disk $D_\varepsilon := \overline{B_\varepsilon} := \overline{B(0,\varepsilon)}$ converges. Indeed, we change[34] to polar coordinates to see that

$$\int_{D_\varepsilon} \frac{g(z-\zeta)}{\zeta} \, \mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta} = \int_0^\varepsilon \int_0^{2\pi} g\big(z - re^{i\theta}\big) e^{-i\theta} \, \mathrm{d}r \mathrm{d}\theta,$$

which is convergent[35]. Now, to show that $f \in \mathscr{E}(\mathbb{C})$, we expand the definition of f into

$$f(z) = -\frac{1}{2\pi i} \lim_{\varepsilon \to 0} \int_{\mathbb{C} \setminus B_\varepsilon} \frac{g(z-\zeta)}{\zeta} \, \mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta}.$$

The uniform convergence of the integral allows us to differentiate under the integral sign, so $f \in \mathscr{E}(\mathbb{C})$. We do so explicitly for the operator $\partial/\partial\overline{z}$ to obtain

$$\left.\frac{\partial f}{\partial \overline{z}}\right|_z = -\frac{1}{2\pi i} \lim_{\varepsilon \to 0} \int_{\mathbb{C} \setminus B_\varepsilon} \frac{1}{\zeta} \left.\frac{\partial g}{\partial \overline{z}}\right|_{z-\zeta} \mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta}.$$

Using $\mathrm{d} = \partial + \overline{\partial}$ and expanding the definitions, we have that[36]

$$\begin{aligned}
\mathrm{d}\left(\frac{g(z-\zeta)}{\zeta} \, \mathrm{d}\zeta\right) &= \partial\left(\frac{g(z-\zeta)}{\zeta} \, \mathrm{d}\zeta\right) + \overline{\partial}\left(\frac{g(z-\zeta)}{\zeta} \, \mathrm{d}\zeta\right) \\
&= \frac{\partial}{\partial \zeta}\left(\frac{g(z-\zeta)}{\zeta}\right) \mathrm{d}\zeta \wedge \mathrm{d}\zeta + \frac{\partial}{\partial \overline{\zeta}}\left(\frac{g(z-\zeta)}{\zeta}\right) \mathrm{d}\overline{\zeta} \wedge \mathrm{d}\zeta \\
&= -\frac{1}{\zeta} \left.\frac{\partial g}{\partial \overline{\zeta}}\right|_{z-\zeta} \mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta}.
\end{aligned}$$

Thus we have by Stokes's Theorem that

$$\left.\frac{\partial f}{\partial \overline{z}}\right|_z = \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \int_{C \setminus B_\varepsilon} \mathrm{d}\left(\frac{g(z-\zeta)}{\zeta} \, \mathrm{d}\zeta\right) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \int_{|\zeta| = \varepsilon} \frac{g(z-\zeta)}{\zeta} \, \mathrm{d}\zeta.$$

This integral can be calculated in polar coordinates as $\zeta = \varepsilon e^{i\theta}$ for $0 \leq \theta < 2\pi$, so[37]

$$\int_{|\zeta|=\varepsilon} \frac{g(z-\zeta)}{\zeta} \, \mathrm{d}\zeta = \int_0^{2\pi} \frac{g\big(z - \varepsilon e^{i\theta}\big)}{\varepsilon e^{i\theta}} \varepsilon i e^{i\theta} \, \mathrm{d}\theta = i \int_0^{2\pi} g\big(z - \varepsilon e^{i\theta}\big) \, \mathrm{d}\theta.$$

It follows then that

$$\left.\frac{\partial f}{\partial \overline{z}}\right|_z = \frac{1}{2\pi} \lim_{\varepsilon \to 0} \int_0^{2\pi} g\big(z - \varepsilon e^{i\theta}\big) \, \mathrm{d}\theta,$$

which is the average value of $g(z)$ on the circle of radius ε around z. In the limit $\varepsilon \to 0$, we see that $\partial f/\partial\overline{z} = g$ and hence $\overline{\partial} f = g \, \mathrm{d}\overline{z}$.

Now, for the general case, we consider an increasing sequence of radii $\{R_n\}$ such that $R_n \to \infty$ and their associated balls $B_n := B(0, R_n)$. For all n, there exists[38] a function $\psi_n \in \mathscr{E}(\mathbb{C})$ such that $\operatorname{Supp}(\psi_n) \subseteq B_{n+1}$ and $\psi_n|_{B_n} = 1$. Extending $\psi_n g$ by zero outside B_{n+1}, they become differentiable functions in $\mathbb{C}$ with compact supports and hence $\overline{\partial} f_n = \psi_n g \, \mathrm{d}\overline{z}$ for some $f_n \in \mathscr{E}(\mathbb{C})$. We shall inductively construct a new sequence $\big\{\tilde{f}_n\big\}$ of differentiable functions on $\mathbb{C}$ such that

1. $\overline{\partial} \tilde{f}_n = g \, \mathrm{d}\overline{z}$ on B_n and
2. $\big\|\tilde{f}_{n+1} - \tilde{f}_n\big\|_{B_n} \leq 2^{-n}$[39].

[34] Formally, we appeal to Proposition 4.16.
[35] Here, we use the fact that g is bounded.
[36] Use the Product Rule and that $1/\zeta$ is holomorphic away from 0.
[37] $\mathrm{d}\zeta = \frac{\partial \zeta}{\partial \theta} \, \mathrm{d}\theta = \varepsilon i e^{i\theta} \mathrm{d}\theta$.
[38] For instance, take bump functions.
[39] $\|f\|_K := \sup_{x \in K} |f(x)|$ is the supremum norm.

Set $\tilde{f}_1 \coloneqq f_1$ and suppose that the functions $\tilde{f}_1, \dots, \tilde{f}_n$ are defined. Then

$$\overline{\partial}\big(f_{n+1} - \tilde{f}_n\big) = \overline{\partial} f_{n+1} - \overline{\partial}\tilde{f}_n = (\psi_{n+1} g - g)\,\mathrm{d}\overline{z} = 0$$

on B_n, so the function $f_{n+1} - \tilde{f}_n$ is holomorphic on B_n. Thus there exists a polynomial $p \in \mathbb{C}[z]$[40] such that

$$\big\|f_{n+1} - \tilde{f}_n - p\big\|_{B_n} \leq 2^{-n},$$

so take $\tilde{f}_{n+1} \coloneqq f_{n+1} - p \in \mathscr{E}(\mathbb{C})$. This satisfies (2), and since

$$\overline{\partial}\tilde{f}_{n+1} = \overline{\partial} f_{n+1} = \psi_{n+1} g = g$$

on B_{n+1}, we see that (1) holds too. By (2), the (pointwise) limit $\tilde{f}_n(z)$ converges to some $f(z)$, where we claim that $f \in \mathscr{E}(\mathbb{C})$ and that $\overline{\partial} f = g\,\mathrm{d}\overline{z}$. Note that the series

$$F_n \coloneqq \sum_{k \geq n} \big(\tilde{f}_{k+1} - \tilde{f}_k\big)$$

converges (uniformly) on B_n, and since $\overline{\partial}\big(\tilde{f}_{k+1} - \tilde{f}_k\big) = 0$ on B_n for all $k \geq n$, it is holomorphic on B_n. This shows that $f = \tilde{f}_n + F_n$ is differentiable and that

$$\overline{\partial} f = \overline{\partial}\tilde{f}_n + \overline{\partial} F_n = \overline{\partial}\tilde{f}_n = g\,\mathrm{d}\overline{z}$$

on B_n. But this holds for all n, so $f \in \mathscr{E}(\mathbb{C})$ with $\overline{\partial} f = g\,\mathrm{d}\overline{z}$ globally. ■

Remark. This theorem (which we call *Dolbeault's Theorem*) is a special case of the $\overline{\partial}$-Poincaré Lemma. Indeed, we can reformulate the theorem by saying that the sequence of sheaves[41]

$$0 \longrightarrow \mathscr{O} \overset{\iota}{\hookrightarrow} \mathscr{E} \overset{\overline{\partial}}{\twoheadrightarrow} \mathscr{E}^{(0,1)} \longrightarrow 0$$

is exact. The only nontrivial claim to verify is that $\overline{\partial}$ is surjective, which is precisely the statement of the above theorem. ♦

Corollary 4.25.1. *The 1^{st} cohomology groups $\check{H}^1(\mathbb{C}, \mathscr{O})$ and $\check{H}^1(\hat{\mathbb{C}}, \mathscr{O})$ vanish.*

Proof. We first prove that $\check{H}^1(\mathbb{C}, \mathscr{O})$ vanishes, for which it suffices to take any open covering $\mathfrak{A} \coloneqq \{U_i\}$ of $\mathbb{C}$ and show that every cocycle $(f_{ij}) \in \check{Z}^1(\mathbb{C}, \mathscr{O})$ splits. Indeed, since $\check{Z}^1(\mathfrak{A}, \mathscr{O}) \subseteq \check{Z}^1(\mathfrak{A}, \mathscr{E})$ and $\check{H}^1(\mathbb{C}, \mathscr{E})$ vanishes by Proposition 4.23, there exists a cochain $(g_i) \in \check{C}^0(\mathfrak{A}, \mathscr{E})$ such that $f_{ij} = g_i - g_j$ on $U_i \cap U_j$. But $\overline{\partial} f_{ij} = 0$, so $\overline{\partial} g_i = \overline{\partial} g_j$ on $U_i \cap U_j$ for all i, j and hence glues to a global function $h \in \mathscr{E}(\mathbb{C})$ such that $h|_{U_i}\,\mathrm{d}\overline{z} = \overline{\partial} g_i$. Dolbeault's Theorem then furnishes some $g \in \mathscr{E}(\mathbb{C})$ such that $\overline{\partial} g = h\,\mathrm{d}\overline{z}$. Define

$$\tilde{g}_i \coloneqq g_i - g,$$

and since $\overline{\partial}\tilde{g}_i = \overline{\partial} g_i - \overline{\partial} g = 0$ on U_i, we see that $(\tilde{g}_i) \in \check{C}^0(\mathfrak{A}, \mathscr{O})$. Observe that

$$f_{ij} = g_i - g_j = \tilde{g}_i - \tilde{g}_j$$

so (f_{ij}) splits. For the Riemann sphere, consider the cover $\mathfrak{A} \coloneqq \{U_1, U_2\}$ given in Example 2.5. Since $U_1, U_2 \cong \mathbb{C}$, we see from the vanishing of $\check{H}^1(\mathbb{C}, \mathscr{O})$ that $\mathfrak{A}$ is a Leray covering of X, so

$$\check{H}^1\big(\hat{\mathbb{C}}, \mathscr{O}\big) \cong \check{H}^1(\mathfrak{A}, \mathscr{O})$$

by Proposition 4.24. Thus it suffices to show that any cocycle $(f_{ij}) \in \check{Z}^1(\mathfrak{A}, \mathscr{O})$ splits; i.e. it suffices[42] to find functions $f_i \in \mathscr{O}(U_i)$ such that $f_{12} = f_1 - f_2$ on $U_1 \cap U_2 = \mathbb{C}^*$. Note that f_{12} is not necessarily holomorphic at 0, so it admits a Laurent series expansion $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ on $\mathbb{C}^*$. Then the series $f_1(z) \coloneqq \sum_{n=0}^{\infty} c_n z^n$ and $f_2(z) \coloneqq \sum_{n=-\infty}^{-1} c_n z^n$ converges on U_1 and U_2, respectively, so $f_i \in \mathscr{O}(U_i)$. Clearly $f_{12} = f_1 - f_2$. ■

Remark. Let X be a compact Riemann surface and consider the vector space structure on $\check{H}^1(X, \mathscr{O})$ induced from $\mathscr{O}$. We appeal to the following theorems.

- The dimension $g \coloneqq \dim_{\mathbb{C}} \check{H}^1(X, \mathscr{O})$ is finite[43] and is referred to as the genus of X. The above theorem states that $\hat{\mathbb{C}}$ has genus 0.
- The genus of X depends only on the smooth manifold structure on X. In particular, since $\check{H}^1\big(\hat{\mathbb{C}}, \mathscr{O}\big)$ vanishes, the genus of any simply-connected compact Riemann surface X is 0. ♦

[40] Say, some Taylor polynomial.

[41] Here, ι is the inclusion sheaf morphism.

[42] The cases for f_{ii} are trivial and $f_{21} = -f_{12}$.

[43] See [For81, Section 14] for a proof.

4.3.2 Existence of Global Meromorphic Functions

Theorem 4.26. *Let X be a compact Riemann surface of genus g and fix $p \in X$. Then there exists a meromorphic function $f \in \mathscr{M}(X)$ which has a pole at p of order between 1 and $g+1$, and is holomorphic everywhere else.*

Proof. Let (U_1, z) be a chart of X centered at p and set $U_2 := X \setminus \{p\}$, so $\mathfrak{A} := \{U_1, U_2\}$ is an open cover of X. For each $1 \leq i \leq g+1$, consider the holomorphic function z^{-i} on $U_1 \cap U_2 = U_1 \setminus \{p\}$. This gives us $(g+1)$-many cocycles $\left(z^{-i}\right) \in \check{Z}^1(\mathfrak{A}, \mathscr{O})$, but since $\dim \check{H}^1(X, \mathscr{O}) = g$, they are linearly dependent[44]. Thus there are constants $c_1, \ldots, c_{g+1} \in \mathbb{C}$, not all zero, such that

$$\sum_{i=1}^{g+1} c_i z^{-i} = f_2 - f_1$$

on $U_1 \cap U_2$ for some $f_i \in \mathscr{O}(U_i)$. Observe that the function $f := f_1 + \sum_{i=1}^{g+1} c_i z^{-i}$ agrees with f_2 on $U_1 \cap U_2$, so they glue to a global function $f \in \mathscr{M}(X)$ which has a pole at p of order between 1 and p and is holomorphic everywhere else. ■

Remark. By our remarks regarding the genus above, we see that every simply-connected compact Riemann surface X admits a global meromorphic function $f \in \mathscr{O}[p](X)$ for any $p \in X$. ♦

[44] Here, we have linear dependence in the quotient $\check{H}^1(\mathfrak{A}, \mathscr{O})$, whose 0 is a splitting cocycle.

Chapter 5

Moduli Spaces

For any genus g, we let[1] $\mathcal{M}_g$ denote the *moduli space* of compact Riemann surfaces of genus g; that is, the set of all Riemann surfaces of genus g up to biholomorphism. Using the language and machinery developed in Chapters 2, 3, and 4, we compute $\mathcal{M}_0$ and $\mathcal{M}_1$, which are the moduli spaces of the sphere S^2 and the torus T^2, respectively.

We conclude with a brief discussion of the *Uniformization Theorem* and the *Classification of Riemann Surfaces*.

5.1 Case for $g = 0$ and $g = 1$

5.1.1 Moduli Space of S^2

We show that the moduli space of the sphere S^2 is a point. That is, there[2] is a *unique* complex structure on the sphere.

Theorem 5.1. *Every simply-connected compact Riemann surface X is biholomorphic to the Riemann sphere $\hat{\mathbb{C}}$.*

Proof. The Classification Theorem of Surfaces shows that such a Riemann surface X, being simply-connected and compact, is homeomorphic to the sphere $\hat{\mathbb{C}}$. Corollary 4.25.1 shows that $\check{H}^1(\hat{\mathbb{C}}, \mathscr{O})$ vanishes, and since the genus is a topological invariant, we see that $\check{H}^1(X, \mathscr{O})$ vanishes too. Hence X has genus 0, so for any fixed point $p \in X$, Theorem 4.26 furnishes a meromorphic function $f \in \mathscr{M}(X)$ with a single simple pole at p. Thus $X \cong \hat{\mathbb{C}}$ by Corollary 3.14.2, as desired. ■

Remark. This is part of the *Uniformization Theorem*, which states that every simply-connected Riemann surface is biholomorphic to either the Riemann sphere $\hat{\mathbb{C}}$, the complex plane $\mathbb{C}$, or the upper-half plane $\mathbb{H}$ of $\mathbb{C}$. A brief discussion and proof sketch is given in Section 5.2. ♦

5.1.2 Moduli Space of T^2

We show that the moduli space of the torus T^2 is[3] $\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$ where $\mathbb{H}$ is the upper-half plane of $\mathbb{C}$ and $\mathrm{PSL}_2(\mathbb{Z}) := \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$ is the *modular group*, which acts on $\mathbb{H}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau := \frac{a\tau + b}{c\tau + d}.$$

We first need a technical lemma, which gives an equivalent condition for a biholomorphism between tori in terms of their lattices.

Lemma 5.2. *Let $\Gamma, \Gamma' \subseteq \mathbb{C}$ be two lattices and suppose $\alpha\Gamma \subseteq \Gamma'$ for some $\alpha \in \mathbb{C}^*$. Then $z \mapsto \alpha z$ descends to a holomorphic map $\varphi : \mathbb{C}/\Gamma \to \mathbb{C}/\Gamma'$, which is biholomorphic iff $\alpha\Gamma = \Gamma'$.*

Proof. Let $\Gamma := \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ and $\Gamma' := \mathbb{Z}\omega_1' \oplus \mathbb{Z}\omega_2'$. Define $\varphi(z + \Gamma) := \alpha z + \Gamma'$ for all $z \in \mathbb{C}$, which is clearly holomorphic if it is well-defined in the first place. To verify, take $z_1, z_2 \in \mathbb{C}$ such that $z_1 + \Gamma = z_2 + \Gamma$. Then $z_1 - z_2 \in \Gamma$, so $z_1 - z_2 = m\omega_1 + n\omega_2$ for some $n, m \in \mathbb{Z}$. Observe that

$$\alpha z_1 - \alpha z_2 = \alpha(z_1 - z_2) = m(\alpha\omega_1) + n(\alpha\omega_2) \in \alpha\Gamma \subseteq \Gamma',$$

so $\alpha z_1 + \Gamma' = \alpha z_2 + \Gamma'$. This shows that φ is well-defined. Furthermore, it is invertible with holomorphic inverse

$$\varphi^{-1}(z + \Gamma') := z/\alpha + \Gamma$$

iff φ^{-1} is well-defined, in which case φ is a biholomorphism. We claim that this occurs iff $\alpha\Gamma = \Gamma'$.

- ($\Rightarrow$): It suffices to show that $\Gamma' \subseteq \alpha\Gamma$, so take $m\omega_1' + n\omega_2' \in \Gamma'$. Then

  $$\varphi^{-1}(m\omega_1' + n\omega_2' + \Gamma') = (m\omega_1' + n\omega_2')/\alpha + \Gamma,$$

  but since $m\omega_1' + n\omega_2' + \Gamma' = 0 + \Gamma'$ and $\varphi^{-1}(0 + \Gamma') = 0 + \Gamma$, we see that $(m\omega_1' + n\omega_2')/\alpha \in \Gamma$.

- ($\Leftarrow$): Take $z_1, z_2 \in \mathbb{C}$ such that $z_1 + \Gamma' = z_2 + \Gamma'$, so $z_1 - z_2 \in \Gamma' \subseteq \alpha\Gamma$ and hence

  $$z_1/\alpha - z_2/\alpha = (z_1 - z_2)/\alpha \in \Gamma.$$

  Then $z_1/\alpha + \Gamma = z_2/\alpha + \Gamma$, so φ^{-1} is well-defined. ■

[1] This defines $\mathcal{M}_g$ as a set, but it turns out that they can all be equipped with a natural complex structure.

[2] It turns out that this is an easy corollary of the Riemann-Roch Theorem, but its proof is beyond the scope of this paper. Here, we present a more elementary proof.

[3] The complex structure on $\mathbb{H}$ induces a complex structure on the quotient $\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$. This justifies the term 'moduli *space*', as opposed to 'moduli *set*'.

Lemma 5.3. *Any torus $\mathbb{C}/\Gamma$ is biholomorphic to $X_\tau := \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ for some $\tau \in \mathbb{H}$.*

[4]

Proof. Let $\Gamma := \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ and set $\alpha := 1/\omega_1$ and $\tau := \omega_2/\omega_1$. Then $\operatorname{Im}\tau \neq 0$, lest ω_1, ω_2 be linearly dependent over $\mathbb{R}$. Without loss of generality, suppose that $\operatorname{Im}\tau > 0$; if not, take $\tau := \overline{\omega_2}/\omega_1$. Then, since

$$\alpha\left(m\omega_1 + n\omega_2\right) = \alpha\omega_1\left(m + n\omega_2/\omega_1\right) = m + n\tau$$

for all $m, n \in \mathbb{Z}$, we see that $\alpha\Gamma = \mathbb{Z}\oplus\mathbb{Z}\tau$. By Lemma 5.2, the map $z \mapsto \alpha z$ descends to a biholomorphism $\varphi : \mathbb{C}/\Gamma \to \mathbb{C}/(\mathbb{Z}\oplus\mathbb{Z}\tau) = X_\tau$, so $\mathbb{C}/\Gamma \cong X_\tau$. ■

Theorem 5.4. *For any $\tau, \tau' \in \mathbb{H}$, the tori X_τ and $X_{\tau'}$ are biholomorphic iff τ and τ' lie in the same orbit of the action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathbb{H}$.*

Corollary 5.4.1. *The moduli space of T^2 is $\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$.*

Proof. The backwards direction is relatively straightforward. Indeed, note that

$$\tau' = \frac{a\tau + b}{c\tau + d} \qquad \Rightarrow \qquad \tau = \frac{b - d\tau'}{c\tau' - a}$$

for any $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$, so let $\alpha := c\tau' - a$. Then, with $\Gamma := \mathbb{Z}\oplus\mathbb{Z}\tau$ and $\Gamma' := \mathbb{Z}\oplus\mathbb{Z}\tau'$, we proceed by proving that $\alpha\Gamma = \Gamma'$, from which the result follows from Lemma 5.2.

- ($\subseteq$): For any $m, n \in \mathbb{Z}$, our choice of α shows that
$$m\alpha + n\alpha\tau = m\left(c\tau' - a\right) + n\left(b - d\tau'\right) = (nb - ma) + (mc - nd)\,\tau' \in \mathbb{Z}\oplus\mathbb{Z}\tau',$$
so $\alpha\left(\mathbb{Z}\oplus\mathbb{Z}\tau\right) \subseteq \mathbb{Z}\oplus\mathbb{Z}\tau'$.
- ($\supseteq$): For any $m, n \in \mathbb{Z}$, the condition that $ad - bc = 1$ shows that
$$\left(m + n\tau'\right)/\alpha = \frac{(na - mc)\,\tau + (nb - md)}{a\left(c\tau + d\right) - c\left(a\tau + b\right)} = (nb - md) + (na - mc)\,\tau \in \mathbb{Z}\oplus\mathbb{Z}\tau,$$
so $\mathbb{Z}\oplus\mathbb{Z}\tau' \subseteq \alpha\left(\mathbb{Z}\oplus\mathbb{Z}\tau\right)$.

For the forward direction, let $\varphi : X_\tau \to X_{\tau'}$ be a biholomorphism, which lifts to a biholomorphic mapping $\tilde{\varphi} : \mathbb{C} \to \mathbb{C}$ such that

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\tilde{\varphi}"] \arrow[d, "\pi"'] & \mathbb{C} \arrow[d, "\pi'"] \\ \mathbb{C}/\Gamma \arrow[r, "\varphi"] & \mathbb{C}/\Gamma' \end{tikzcd}$$

commutes. Fix[5] $\lambda \in \Gamma$ and consider the map $f_\lambda(z) := \tilde{\varphi}\left(z + \lambda\right) - \tilde{\varphi}\left(z\right)$. Then, since $z + \lambda + \Gamma = z + \Gamma$, we see that $\varphi\left(z + \lambda + \Gamma\right) = \varphi\left(z + \Gamma\right)$ and hence the commutativity of the diagram forces $\tilde{\varphi}\left(z + \lambda\right) + \Gamma' = \tilde{\varphi}\left(z\right) + \Gamma'$. Thus $f_\lambda(z) \in \Gamma'$ for all $z \in \mathbb{C}$, so, since f_λ is a continuous map into a discrete set, it must be constant. Differentiating gives us $f_\lambda'\left(z\right) = \tilde{\varphi}'\left(z + \lambda\right) - \tilde{\varphi}'\left(z\right) = 0$, so $\tilde{\varphi}'\left(z + \lambda\right) = \tilde{\varphi}'\left(z\right)$ for all $z \in \mathbb{C}$. But $\lambda \in \Gamma$ is arbitrary, so $\tilde{\varphi}'$ is Γ-periodic. Thus $\tilde{\varphi}'$ is a bounded entire function and hence is constant by Liouville's Theorem. This shows that $\tilde{\varphi}\left(z\right) = \alpha z + \beta$ for some $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$. Without loss of generality, assume that $\beta = 0$. We now claim that $\alpha\Gamma = \Gamma'$.

- Indeed, for all $z \in \alpha\Gamma$, we have $z/\alpha \in \Gamma$ and so $z/\alpha + \Gamma = 0 + \Gamma$. Applying φ to both sides and comparing gives
$$0 + \Gamma' = \varphi\left(0 + \Gamma\right) = \varphi\left(z/\alpha + \Gamma\right) = \tilde{\varphi}\left(z/\alpha\right) + \Gamma' = z + \Gamma',$$
so $z \in \Gamma'$. The converse is similar.

Observe then that $\tilde{\varphi}\left(\tau\right) = \alpha\tau = b - d\tau'$ and $\tilde{\varphi}\left(1\right) = \alpha = c\tau' - a$ for some $a, b, c, d \in \mathbb{Z}$, so

$$\tau = \frac{b - d\tau'}{c\tau' - a} \qquad \text{and hence} \qquad \tau' = \frac{a\tau + b}{c\tau + d}.$$

A computation now shows that $\alpha = -\left(ad - bc\right)/\left(c\tau + d\right)$, so $ad - bc \neq 0$. Then, since

$$\begin{pmatrix} \alpha\tau \\ \alpha \end{pmatrix} = \begin{pmatrix} b & -d \\ -a & c \end{pmatrix} \begin{pmatrix} 1 \\ \tau' \end{pmatrix},$$

we solve for τ' to obtain

$$\tau' = -\frac{b\alpha + a\alpha\tau}{ad - bc} = \left(\frac{-b}{ad - bc}\right)\alpha + \left(\frac{-a}{ad - bc}\right)\alpha\tau$$

But $\tau' \in \alpha\Gamma$, which forces $ad - bc = \pm 1$. A little algebra now shows that[6]

$$\operatorname{Im}\tau' = \frac{ad - bc}{\left|c\tau + d\right|^2}\left(\operatorname{Im}\tau\right) > 0,$$

so $ad - bc = 1$. ■

[4] Intuitively, scaling and rotating the lattice, which are biholomorphisms of the plane, should preserve the complex structure on the torus. Thus only one complex parameter is needed to generate the torus, which we choose to be the ratio $\tau := \omega_2/\omega_1$.

[5] This proof follows [Shu05, Proposition 1.3.2]. For an alternative proof, see [Tan91, Lemma 2.8].

[6] Let $\tau := e + fi$ and $\tau' := g + hi$ and expand.

5.2 Uniformization and Classification

Results in this section will only be discussed briefly and all proofs presented are sketches.

Theorem 5.5 (Uniformization)**.** *Every simply-connected Riemann surface X is biholomorphic to either the Riemann sphere $\hat{\mathbb{C}}$, the complex plane $\mathbb{C}$, or the upper-half plane $\mathbb{H}$.*

Proof sketch. [7] As in Theorem 5.1, fix $p \in X$ and let $f \in \mathscr{O}[p](X)$ be meromorphic with a single simple pole. Let $F : X \to \hat{\mathbb{C}}$ be its associated holomorphic map, so $F(p) = \infty$. We outline the rest of the proof.[8]

- First, it can be shown that $\operatorname{Im} F(x) \to 0$ as '$x \to \infty$' in X. That is, for every $\varepsilon > 0$, there is a large enough compact subset K of X such that $\operatorname{Im} F(x) < \varepsilon$ for all $x \in X \setminus K$.
- It can also be shown that $\operatorname{im} F$ is open, contains the 'top and bottom halves' of[9] $\hat{\mathbb{C}}$, and is a biholomorphism onto its image.

Thus $X \cong \operatorname{im} F = \hat{\mathbb{C}} \setminus I$ for some $I \subseteq \mathbb{R}$. By simply-connectedness of X, we see that I is connected and hence we have three possibilities.

- If $I = \varnothing$, then $F : X \to \hat{\mathbb{C}}$ is a biholomorphism, which reduces to Theorem 5.1.
- If I is a singleton, then $\hat{\mathbb{C}} \setminus I \cong \mathbb{C}$, so $X \cong \mathbb{C}$.
- If I is an interval $[a, b]$, we may without loss of generality take $a = 0$ and $b = \infty$. Then the (usual branch of the) square root function sends $\hat{\mathbb{C}} \setminus [0, \infty]$ to $\mathbb{H}$. ■

Remark. It turns out that one can construct a simply-connected Riemann surface $\tilde{X}$ from any Riemann surface X. Since $\tilde{X}$ is exactly one of three types, this leads to a classification of Riemann surfaces. ♦

Definition 5.6. *Let X and E be connected topological spaces. A covering map $\pi : E \to X$ if said to be the* <u>*universal covering of X*</u> *if for every covering $\pi' : E' \to X$ on a connected topological space E' and every $e \in E$ and $e' \in E'$ such that $\pi(e) = \pi'(e')$, there exists a unique continuous map $\sigma : E \to E'$ with $\sigma(e) = e'$ making the below diagram commute.*

$$\begin{tikzcd} E \arrow[rr, dashed, "\exists! \sigma"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & X & \end{tikzcd}$$

[10]

Remark. Note that σ is the lifting of of π along π'. Recall that if E is simply-connected, such a lifting exists and is unique, so in this case *any* covering map is the universal covering of X. We quote the following theorem that guarantees the existence of such a simply-connected space. ♦

Theorem 5.7 ([For81, Theorem 5.3])**.** *Suppose X is a connected manifold. Then there exists a connected, simply-connected manifold $\tilde{X}$ and a covering map $\pi : \tilde{X} \to X$.*

Example 5.8. Recall from Example 3.3 that for any lattice $\Gamma \subseteq \mathbb{C}$, the projection $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$ is a covering map. Since $\mathbb{C}$ is simply-connected, we see that π is the universal covering of $\mathbb{C}/\Gamma$. ♦

Remark. For any Riemann surface X, let[11] $\tilde{X}$ be its simply-connected universal covering. If $\tilde{X} \cong \hat{\mathbb{C}}$ (resp. $\mathbb{C}$, $\mathbb{H}$), then X is said to be *elliptic* (resp. *parabolic*, *hyperbolic*).

- Since $\hat{\mathbb{C}}$ is simply-connected, it is the universal covering of itself and hence $\hat{\mathbb{C}}$ is elliptic.
- Since $\mathbb{C}$ is the universal covering of any torus $\mathbb{C}/\Gamma$, we see that $\mathbb{C}/\Gamma$ is parabolic.

It turns out that the universal covering for any compact Riemann surfaces with $g > 1$ is $\mathbb{H}$, so they are all hyperbolic. This is a curious fact (which we do not understand) that, moreover, has an analogue for three-dimensional real manifolds (called 3-*manifolds*). Indeed, *Thurston's Geometrization Conjecture*[12] states that all 3-manifolds can be decomposed into pieces, each having one of eight different geometric structures, and the richest of the eight geometries turns out to be the hyperbolic 3-manifold. ♦

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[7] This sketch follows [Kro19]. The existence of such a meromorphic function f is more involved and uses tools from Dolbeault cohomology.

[8] These claims are nontrivial and are beyond the scope of this paper.

[9] That is, $\{x \in X \mid \operatorname{Im} \varphi(z) \neq 0\} \subseteq \operatorname{im} F$.

[10] As with all 'universal properties', the universal covering of X is unique up to isomorphism.

[11] In fact, every Riemann surface admits a Riemannian metric of constant curvature, either of 1, 0, or -1. We wish to study the connection between the conformal and metric structures on Riemann surfaces in the future.

[12] Proven by Grigori Perelman in 2003, for which he was awarded the Fields Medal.

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