

Possible Choices of Complex Co-ordinates for a Surface

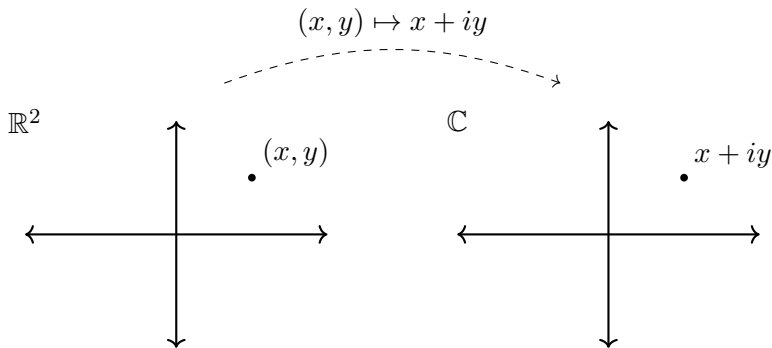
Moduli Spaces of Riemann Surfaces

Zhaoshen Zhai

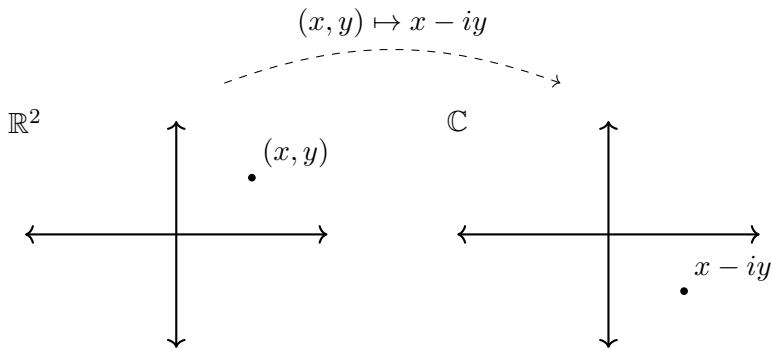
Graduate mentor: Kaleb Ruscitti

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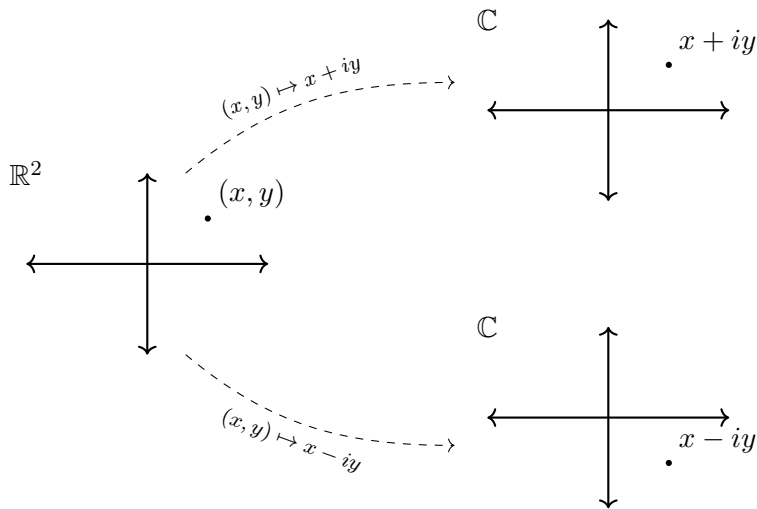
How many complex coordinates on \mathbb{R}^2 ?



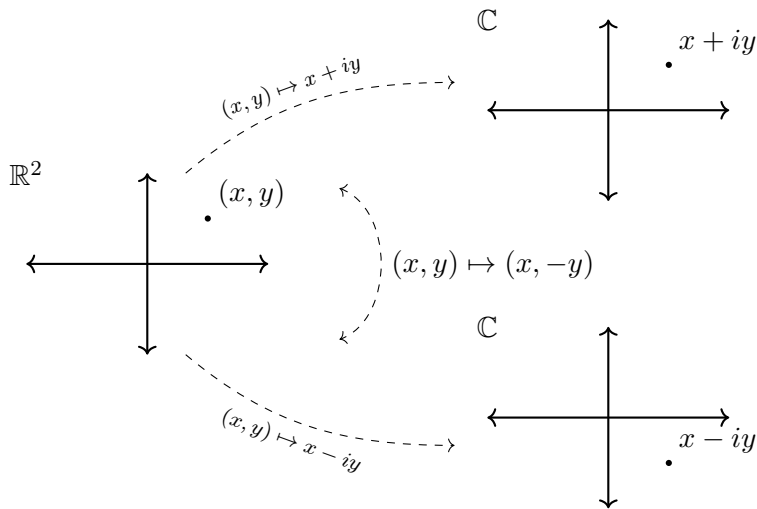
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Refined question

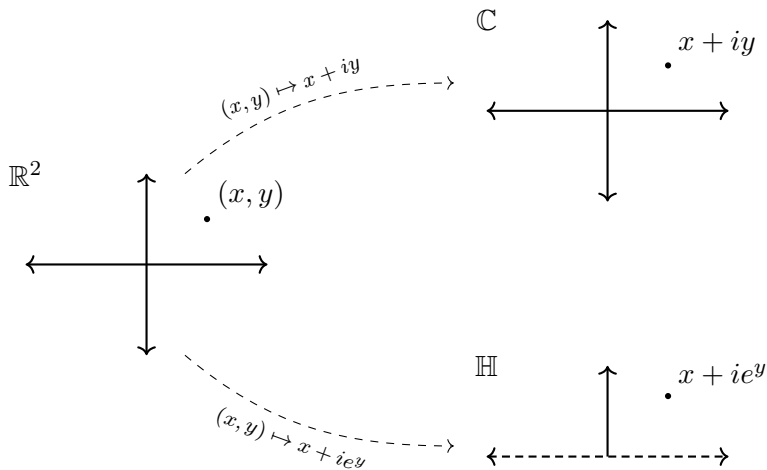
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Refined question

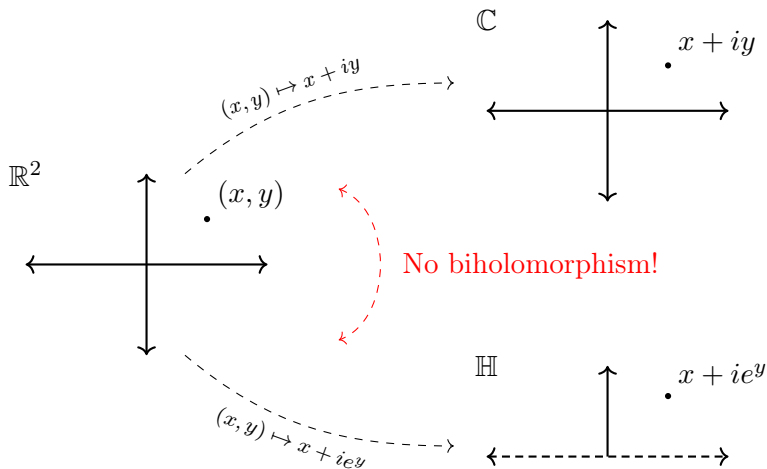
~~How many complex coordinates on \mathbb{R}^2 ?~~

How many complex coordinates on \mathbb{R}^2 *up to biholomorphism*?

\mathbb{C} and \mathbb{H}



\mathbb{C} and \mathbb{H}



Riemann surfaces

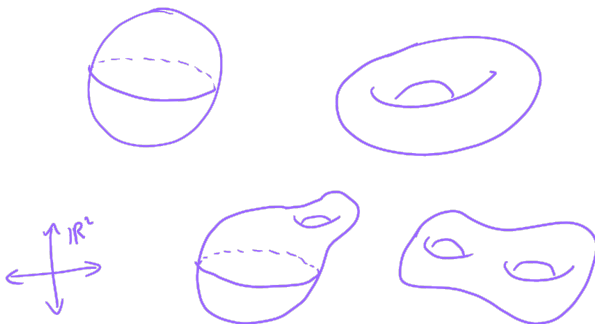
What about for manifolds?

Riemann surfaces

What about for manifolds?

Definition

A Riemann surface is a connected 1-dimensional complex manifold.



Compact Riemann surfaces

Theorem (Classification)

Every compact Riemann surface is classified by its genus.

Compact Riemann surfaces

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Topological torus

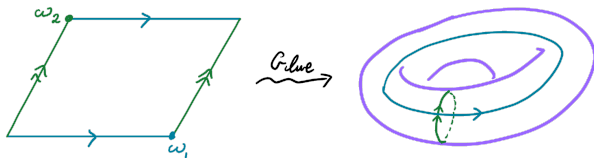
Theorem

Every torus is a quotient $\mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2)$ for some linearly independent $\omega_1, \omega_2 \in \mathbb{C}$.

Topological torus

Theorem

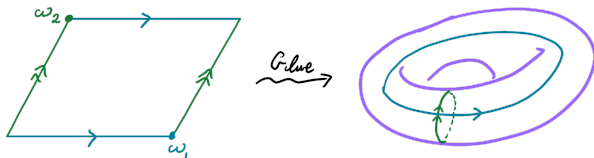
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Topological torus

Theorem

Every torus is a quotient $\mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2)$ for some linearly independent $\omega_1, \omega_2 \in \mathbb{C}$.



Theorem

There is, up to homeomorphism, only one topological torus.

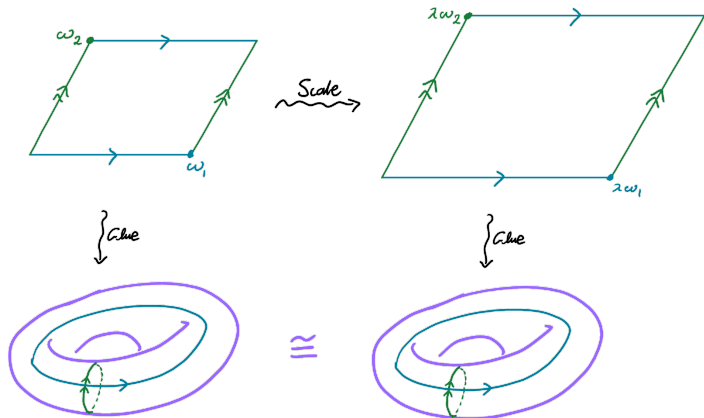
Questions

- *Up to biholomorphism, in how many ways can we give the torus complex coordinates?*

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- *Up to biholomorphism, in how many ways can we give the torus complex coordinates?*
- *What is the moduli space of the torus?*

Complex tori



Complex tori

Theorem

Every complex tori is the quotient $X_\tau := \mathbb{C}/\Gamma$ where $\Gamma = \mathbb{Z} \oplus \mathbb{Z}\tau$ for some $\tau \in \mathbb{H}$.

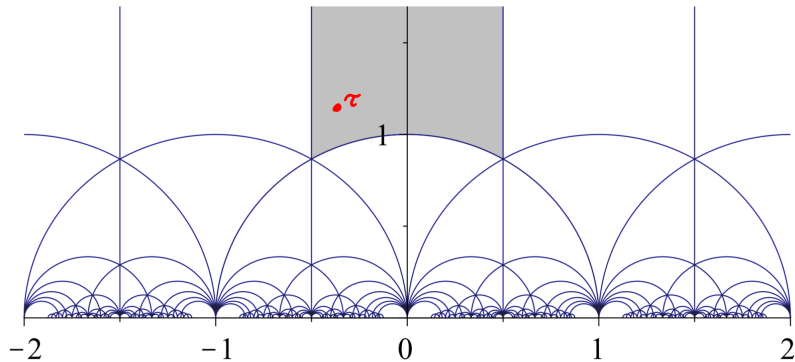
Theorem

Every complex tori is the quotient $X_\tau := \mathbb{C}/\Gamma$ where $\Gamma = \mathbb{Z} \oplus \mathbb{Z}\tau$ for some $\tau \in \mathbb{H}$. Indeed, for all linearly independent $\omega_1, \omega_2 \in \mathbb{C}$,

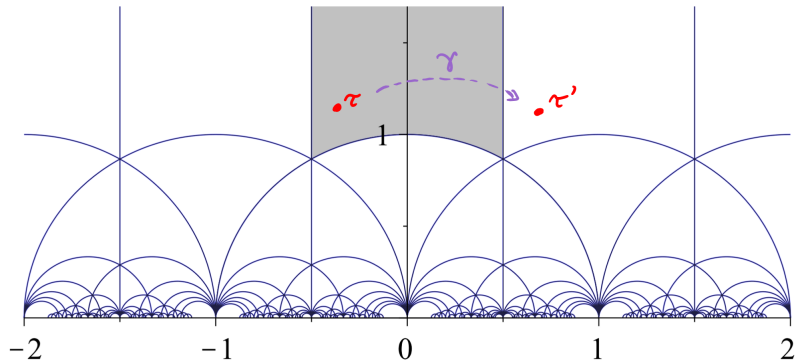
$$\mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2) \cong X_\tau$$

for $\tau := \omega_2/\omega_1 \in \mathbb{H}$.

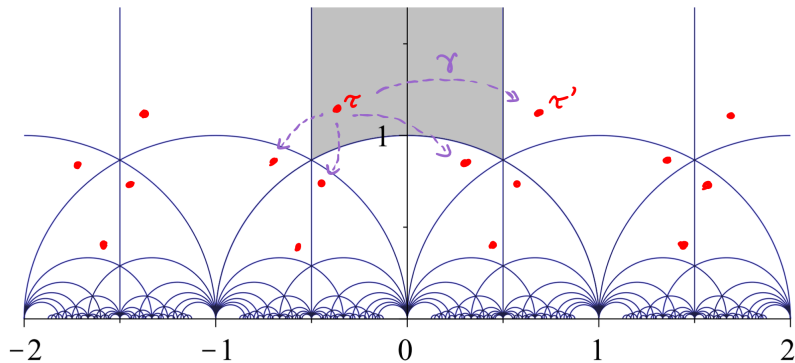
Complex tori



Complex tori



Complex tori



Definition

The modular group $\mathrm{PSL}_2(\mathbb{Z})$ is the group of functions $\gamma : \mathbb{H} \rightarrow \mathbb{H}$ mapping

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}$$

for some $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$.

Complex tori

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Theorem

For any $\tau, \tau' \in \mathbb{H}$, the tori X_τ and $X_{\tau'}$ are biholomorphic iff there exists some $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$ such that $\tau' = \gamma(\tau)$.

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Corollary

The moduli space of complex tori is $\mathbb{H} / \mathrm{PSL}_2(\mathbb{Z})$.

Riemann sphere

Theorem

There is a unique choice of complex coordinates on the sphere.

Riemann sphere

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Proof. Hard! ■

Thank you!

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