

# MODULI SPACES OF RIEMANN SURFACES

ZHAOSHEN ZHAI

HAOYANG GUO

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*Graduate mentor: Kaleb Ruscitti*

## ABSTRACT

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# Chapter 1

## Riemann Surfaces

### 1.1 Charts and Atlases

We assume that the reader is familiar with the basic notions of real manifolds. The case for complex manifolds is similar, so our exposition will be brief.

**Definition 1.1.** Let  $X$  be a second-countable Hausdorff space. A  $d$ -dimensional complex chart on  $X$  is a pair  $(U, \varphi)$  where  $\varphi : U \rightarrow V$  is a homeomorphism from an open subset  $U \subseteq X$  onto an open subset  $V \subseteq \mathbb{C}^d$  for some  $d$ . Two  $d$ -dimensional charts  $(U_1, \varphi_1)$  and  $(U_2, \varphi_2)$  are said to be holomorphically compatible if either  $U_1 \cap U_2 = \emptyset$ , or the map

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_1 \cap U_2) \rightarrow \varphi_2(U_1 \cap U_2)$$

is biholomorphic. A  $d$ -dimensional complex atlas on  $X$  is a collection  $\mathcal{A} := \{(U_i, \varphi_i)\}_{i \in I}$  of  $d$ -dimensional complex charts such that every two charts  $(U_i, \varphi_i)$  and  $(U_j, \varphi_j)$  are holomorphically compatible and  $X = \bigcup_{i \in I} U_i$ .

**Remark.** Two atlases  $\mathcal{A}$  and  $\mathcal{B}$  on a manifold  $X$  are said to be analytically equivalent if every chart in  $\mathcal{A}$  is compatible with every chart in  $\mathcal{B}$ . By Zorn's Lemma, every atlas  $\mathcal{A}$  of a manifold  $X$  is contained in a unique maximal atlas  $\mathfrak{U}$  on  $X$ . Moreover, two atlases are equivalent iff they are contained in the same maximal atlas, which justifies the following definition. ♦

**Definition 1.2.** Let  $X$  be a second-countable Hausdorff space. A  $d$ -dimensional complex structure on  $X$  is a  $d$ -dimensional maximal atlas  $\mathfrak{U}$  on  $X$ , or, equivalently, an equivalence class of  $d$ -dimensional complex atlases on  $X$ . The pair  $(X, \mathfrak{U})$  is then called a  $d$ -dimensional complex manifold.

**Definition 1.3.** A Riemann surface is a connected 1-dimensional complex manifold.

**Example 1.4.** Some elementary examples of Riemann surfaces.

- The complex plane  $\mathbb{C}$ , equipped with its standard topology, can be given a complex structure  $\mathfrak{U}$  by choosing the atlas containing a single chart  $(\mathbb{C}, \text{id}_{\mathbb{C}})$ . We may, however, also give  $\mathbb{C}$  a different complex structure  $\mathfrak{U}'$  by choosing the chart map  $\varphi : z \mapsto \bar{z}$  instead. Indeed,  $\mathfrak{U} \neq \mathfrak{U}'$  since the map  $\varphi \circ \text{id}_{\mathbb{C}}^{-1} = \varphi$  is not holomorphic and hence the atlases  $\{(\mathbb{C}, \text{id}_{\mathbb{C}})\}$  and  $\{(\mathbb{C}, \varphi)\}$  are not equivalent. This example generalizes to any domain  $D \subseteq \mathbb{C}$ .
- Let  $D \subseteq \mathbb{C}$  be a domain and consider any holomorphic function  $f : D \rightarrow \mathbb{C}$ . Then the graph  $\Gamma_f := \{(z, f(z)) \mid z \in D\}$ , equipped with the subspace topology inherited from  $\mathbb{C}^2$ , can be given a complex structure by choosing the chart map  $\pi : \Gamma_f \rightarrow D : (z, f(z)) \mapsto z$ . ♦

#### 1.1.1 The Riemann Sphere $\hat{\mathbb{C}}$

A particularly important Riemann surface is the Riemann sphere  $\hat{\mathbb{C}}$ , which admits several constructions. Here, we give three; see Example 1.14 for a proof that they are all biholomorphic (in the sense of Definition 1.13).

**Example 1.5** (One-point Compactification of  $\mathbb{C}$ ). Let  $\infty$  be a symbol not belonging to  $\mathbb{C}$  and set  $\mathbb{C}_{\infty} := \mathbb{C} \cup \{\infty\}$ . We declare a set  $U \subseteq \mathbb{C}_{\infty}$  to be open if either  $U \subseteq \mathbb{C}$  is open or  $U = K^c \cup \{\infty\}$  where  $K \subseteq \mathbb{C}$  is compact. We employ two charts

$$\begin{aligned} U_1 &:= \mathbb{C}_{\infty} \setminus \{\infty\} = \mathbb{C} & \varphi_1 : U_1 \rightarrow \mathbb{C} : z &\mapsto z \quad (\varphi_1 := \text{id}_{\mathbb{C}}) \\ U_2 &:= \mathbb{C}_{\infty} \setminus \{0\} = \mathbb{C}^* \cup \{\infty\} & \varphi_2 : U_2 \rightarrow \mathbb{C} : z &\mapsto \begin{cases} 1/z & \text{if } z \in \mathbb{C}^* \\ 0 & \text{else.} \end{cases} \end{aligned}$$

Clearly  $\varphi_1$  is a homeomorphism. Since  $\varphi_2$  is invertible with  $\varphi_2^{-1}(z) := 1/z$  for all  $z \in \mathbb{C}^*$  and  $\varphi_2^{-1}(0) := \infty$ , and

$$\lim_{z \rightarrow \infty} \varphi_2(z) = 0 = \varphi_2(\infty) \quad \text{and} \quad \lim_{z \rightarrow 0} \varphi_2^{-1}(z) = \infty = \varphi_2^{-1}(0),$$

we see that  $\varphi_2$  is a homeomorphism too. Furthermore,

$$\varphi_2 \circ \varphi_1^{-1} : \mathbb{C}^* \rightarrow \mathbb{C}^* : z \mapsto \frac{1}{z}$$

is holomorphic, so the atlas  $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$  defines a complex structure on  $\mathbb{C}_{\infty}$ . ♦

Charts provide us a way of making  $X$  ‘look like’ an open set in  $\mathbb{C}^d$ . Indeed, they provide local coordinates for every point in  $X$  in such a way that the ‘change of coordinates’ map  $\varphi_2 \circ \varphi_1^{-1}$  ensures that local notions of functions in  $\mathbb{C}^d$  are well-defined on  $X$  too.

$$\begin{array}{ccc} & U_1 \cap U_2 & \\ \varphi_1 \swarrow & & \searrow \varphi_2 \\ \varphi_1(U_1 \cap U_2) & \xrightarrow{\varphi_2 \circ \varphi_1^{-1}} & \varphi_2(U_1 \cap U_2) \end{array}$$

It is clear that one only needs  $\varphi_2 \circ \varphi_1^{-1}$  to be holomorphic for it to be biholomorphic.

To give a complex structure  $\mathfrak{U}$  to  $X$ , it suffices to give  $X$  a complex atlas since it extends to a unique complex structure.

Every Riemann surface can be regarded as a (connected) 2-dimensional real manifold by ‘forgetting’ its complex structure; indeed all holomorphic maps are real  $\mathcal{C}^{\infty}$  functions.

Showing that *every* Riemann surface that is topologically a sphere is biholomorphic to  $\hat{\mathbb{C}}$  is a non-trivial task, and it will be the first goal of this paper to establish this fact.

This makes  $\mathbb{C}_{\infty}$ , equipped with the collection  $\mathcal{T}$  of all such open sets, a second-countable Hausdorff space. Indeed, the fact that  $\mathcal{T}$  is a topology on  $\mathbb{C}_{\infty}$  follows from De Morgan's Laws and the Heine-Borel Theorem. It is trivially Hausdorff, and it is second-countable since we may append, to any countable basis for the standard topology of  $\mathbb{C}$ , the countable collection  $\{B_r(0)^c \cup \{\infty\}\}_{r \in \mathbb{Q}^+}$ .

**Example 1.6** (Stereographic Projection). Consider the unit sphere  $S^2 \subseteq \mathbb{R}^3$  as a topological subspace of  $\mathbb{R}^3$ , which makes it a second-countable Hausdorff space. Identifying the plane  $w = 0$  as  $\mathbb{C}$ , we employ the charts

$$\begin{aligned} U_1 &:= S^2 \setminus \{(0, 0, 1)\} & \varphi_1 : U_1 \rightarrow \mathbb{C} : (x, y, w) &\mapsto \frac{x + iy}{1 - w} \\ U_2 &:= S^2 \setminus \{(0, 0, -1)\} & \varphi_2 : U_2 \rightarrow \mathbb{C} : (x, y, w) &\mapsto \frac{x - iy}{1 + w}. \end{aligned}$$

Clearly  $\varphi_1$  and  $\varphi_2$  are continuous, and it can be verified that they are invertible with continuous inverses

$$\varphi_1^{-1}(z) := \left( \frac{2 \operatorname{Re} z}{|z|^2 + 1}, \frac{2 \operatorname{Im} z}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right) \quad \text{and} \quad \varphi_2^{-1}(z) := \left( \frac{2 \operatorname{Re} z}{|z|^2 + 1}, \frac{-2 \operatorname{Im} z}{|z|^2 + 1}, \frac{1 - |z|^2}{|z|^2 + 1} \right).$$

Observe that  $U_1 \cap U_2 = S^2 \setminus \{(0, 0, \pm 1)\}$  and  $\varphi_2 \circ \varphi_1^{-1} : \mathbb{C}^* \rightarrow \mathbb{C}^* : z \mapsto 1/z$ , which is holomorphic, so the atlas  $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$  defines a complex structure on  $\mathbb{C}$ .  $\blacklozenge$

**Example 1.7** (Complex Projective Line). Consider the equivalence relation  $\sim$  on  $\mathbb{C}^2 \setminus \{(0, 0)\}$  defined by  $(z_1, w_1) \sim (z_2, w_2)$  iff  $(z_1, w_1) = \lambda(z_2, w_2)$  for some  $\lambda \in \mathbb{C}^*$ . Set  $\mathbb{P}^1 := (\mathbb{C}^2 \setminus \{(0, 0)\}) / \sim$  and equip it with the quotient topology. Since  $\sim$  is an open equivalence relation whose graph is closed in  $(\mathbb{C}^2 \setminus \{(0, 0)\})^2$ , we see that  $\mathbb{P}^1$  is a second-countable Hausdorff space. Denoting the equivalence class of  $(z, w)$  by  $[z : w]$ , we employ the charts

$$\begin{aligned} U_1 &:= \mathbb{P}^1 \setminus \{[0 : w] \mid w \in \mathbb{C}\} & \varphi_1 : U_1 \rightarrow \mathbb{C} : [z : w] &\mapsto w/z \\ U_2 &:= \mathbb{P}^1 \setminus \{[z : 0] \mid z \in \mathbb{C}\} & \varphi_2 : U_2 \rightarrow \mathbb{C} : [z : w] &\mapsto z/w. \end{aligned}$$

Clearly  $\varphi_1$  and  $\varphi_2$  are continuous, and it is easily verified that they are invertible with continuous inverses

$$\varphi_1^{-1}(z) := [1 : z] \quad \text{and} \quad \varphi_2^{-1}(z) := [z : 1].$$

Furthermore,  $\varphi_2 \circ \varphi_1^{-1} : \mathbb{C}^* \rightarrow \mathbb{C}^* : 1 \mapsto 1/z$  is holomorphic, so the atlas  $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$  defines a complex structure on  $\mathbb{P}^1$ .  $\blacklozenge$

See [Tu10, section 7.5].

## 1.1.2 Complex Tori

Recall that a torus is any manifold homeomorphic to  $T^2 := S^1 \times S^1$ , which admits a representation as a quotient  $\mathbb{C}/\Gamma$  by the lattice  $\Gamma := \mathbb{Z} \oplus \mathbb{Z}$ . Thus (by definition) there is only one torus up to homeomorphism, but it turns out that we can equip it with many different complex structures.

**Example 1.8** (Complex Tori). Let  $\omega_1, \omega_2 \in \mathbb{C}$  be linearly independent over  $\mathbb{R}$  and consider the lattice  $\Gamma := \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ . Then the quotient  $\mathbb{C}/\Gamma$  is a torus in the topological sense since the map

$$\varphi : \mathbb{C}/\Gamma \rightarrow T^2 \quad \text{mapping} \quad [z] \mapsto (e^{2\pi i \lambda_1}, e^{2\pi i \lambda_2}),$$

where  $z = \lambda_1 \omega_1 + \lambda_2 \omega_2$  for unique  $\lambda_1, \lambda_2 \in \mathbb{R}$ , is a homeomorphism. Indeed,  $\varphi$  is well-defined since for any  $\lambda_1 \omega_1 + \lambda_2 \omega_2 \sim \mu_1 \omega_1 + \mu_2 \omega_2$  in  $\mathbb{C}$ , we have  $(\lambda_1 - \mu_1)\omega_1 + (\lambda_2 - \mu_2)\omega_2 \in \Gamma$  and so  $\lambda_i - \mu_i \in \mathbb{Z}$  for  $i = 1, 2$ . The fact that it is a homeomorphism is clear. This makes  $\mathbb{C}/\Gamma$  a second-countable Hausdorff space, which we now endow with the following complex structure.

They manifest by quotienting  $\mathbb{C}$  by different lattices, and we shall derive a criterion on  $\Gamma_1 := \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$  and  $\Gamma_2 := \mathbb{Z}\eta_1 \oplus \mathbb{Z}\eta_2$  for the tori  $\mathbb{C}/\Gamma_1$  and  $\mathbb{C}/\Gamma_2$  to be biholomorphic.

Since  $\Gamma$  is discrete, there exists some  $\varepsilon > 0$  such that  $\varepsilon < |\omega|/2$  for every non-zero  $\omega \in \Gamma$ . Fix any such  $\varepsilon$ , which ensures that no two points in any open ball with radius  $\varepsilon$  can be equivalent. Indeed, take any  $z \in \mathbb{C}$  and  $w_1, w_2 \in B(z, \varepsilon) =: V_z$ . For  $w_1 \sim w_2$ , we need some  $n, m \in \mathbb{Z}$  such that  $w_1 - w_2 = n\omega_1 + m\omega_2$ . But

$$|w_1 - w_2| \leq |z - w_1| + |z - w_2| < 2\varepsilon < |n\omega_1 + m\omega_2|$$

for any  $n, m \in \mathbb{Z}$ , so this is impossible. Fixing any such  $\varepsilon$ , this gives us a family  $\{V_z\}_{z \in \mathbb{C}}$  of open sets in  $\mathbb{C}$  for which the projections  $\pi|_{V_z} : V_z \rightarrow \pi(V_z)$  are homeomorphisms. Letting  $U_z := \pi(V_z)$  and  $\varphi_z : U_z \rightarrow V_z$  be the inverse of  $\pi|_{V_z}$ , we obtain complex charts  $(U_z, \varphi_z)$  for all  $z \in \mathbb{C}$ . We claim that the collection  $\mathfrak{U} := \{(U_z, \varphi_z)\}_{z \in \mathbb{C}}$  form an atlas, for which it suffices to take  $(U_1, \varphi_1), (U_2, \varphi_2) \in \mathfrak{U}$  and show that the transition map  $T := \varphi_2 \circ \varphi_1^{-1} : \varphi_1(U) \rightarrow \varphi_2(U)$ , where  $U := U_1 \cap U_2$ , is holomorphic. Observe that the diagram

$$\begin{array}{ccc} & \pi|_{V_1} & \\ & \swarrow & \searrow \\ V_1 = \varphi_1(U) & \xrightarrow{T} & \varphi_2(U) = V_2 \end{array}$$

commutes, so  $\pi|_{V_2} \circ T = \pi|_{V_1}$  on  $\varphi_1(U)$ . Then  $\pi(T(z)) = \pi(z)$  for every  $z \in \varphi_1(U)$ , so  $T(z) \sim z$  and hence  $\ell(z) := T(z) - z \in \Gamma$ . This holds for all  $z \in \varphi_1(U)$ , so we obtain a continuous function  $\ell : \varphi_1(U) \rightarrow \Gamma : z \mapsto T(z) - z$ . Note that  $\Gamma \subseteq \mathbb{C}$  is equipped with the subspace topology, but since it is discrete, every  $L \subseteq \Gamma$  is open. In particular, fix  $z_0 \in \varphi_1(U)$  and set  $\omega_0 := T(z_0) - z_0$ . With  $L := \{\omega_0\}$ , continuity of  $\ell$  shows that  $\ell^{-1}(L)$  is open. Thus  $\ell(B(z_0, \delta_1)) \subseteq \{\omega_0\}$  for some  $\delta_1 > 0$ , so  $\ell(w) = \omega_0$  for all  $w \in B(z_0, \delta_1)$ . But then  $\ell(B(\omega_0, \delta_2)) \subseteq \{\omega_0\}$  for some  $\delta_2 > 0$  too, so we may repeat this process to show that  $\ell$  is constant on every connected component of  $\varphi_1(U)$ . Thus  $T(z) = z + \omega_0$  for all  $z \in \varphi_1(U)$  in a local neighborhood around  $z_0$ , so  $T$  is locally holomorphic. But this holds for all  $z_0 \in \varphi_1(U)$ , so  $T$  is holomorphic on  $\varphi_1(U)$ .  $\blacklozenge$

This exposition follows [Mir95, section I.2].

The choice of  $\varepsilon$  ensures that no two points in  $V_z$  are equivalent, which make all such projections injective.

Since  $U = \pi(V_1) \cap \pi(V_2)$ , it may not be connected. Hence  $\varphi_1(U)$  may not be connected, so  $\ell$  may take on multiple values. What matters, however, is that they coincide within every connected component of  $\varphi_1(U)$ .

## 1.2 Maps on Riemann Surfaces

### 1.2.1 Holomorphic Functions and Maps

**Definition 1.9.** Let  $X$  be a Riemann surface and let  $W \subseteq X$  be open. For a fixed  $p \in W$ , a function  $f : W \rightarrow \mathbb{C}$  is said to be holomorphic at  $p$  if there exists a chart  $(U, \varphi)$  of  $X$  containing  $p$  such that  $f \circ \varphi^{-1} : \varphi(U) \rightarrow \mathbb{C}$  is holomorphic at  $\varphi(p)$ . If  $f$  is holomorphic at every point of  $W$ , then  $f$  is said to be holomorphic on  $W$ .

**Remark.** It must be checked that ‘being holomorphic’ does not depend on the choice of chart. This is indeed the case, for if  $(V, \psi)$  is another chart containing  $p$ , then, since

$$f \circ \psi^{-1} = f \circ (\varphi^{-1} \circ \varphi) \circ \psi^{-1} = (f \circ \varphi^{-1}) \circ (\varphi \circ \psi^{-1}) : \psi(U \cap V) \rightarrow \mathbb{C} \quad (1.1)$$

on the intersection  $U \cap V$ , we see that  $f \circ \psi^{-1} : \psi(U \cap V) \rightarrow \mathbb{C}$  is also holomorphic at  $p$ .  $\blacklozenge$

**Example 1.10.** Some elementary examples of holomorphic functions.

- Any holomorphic function  $f : W \rightarrow \mathbb{C}$  from an open set  $W \subseteq \mathbb{C}$ , considering  $\mathbb{C}$  as a Riemann surface with the standard chart  $(\mathbb{C}, \text{id}_{\mathbb{C}})$ , is holomorphic in the classical sense.
- Any chart map  $\varphi : U \rightarrow \mathbb{C}$  of a Riemann surface is (tautologically) holomorphic in the above sense.
- If  $f, g : W \rightarrow \mathbb{C}$  are both holomorphic at some  $p \in W$ , then so are  $f \pm g$  and  $f \cdot g$ . If  $g(p) \neq 0$ , then so is  $f/g$ .  $\blacklozenge$

**Definition 1.11.** Let  $X$  and  $Y$  be Riemann surfaces and let  $W \subseteq X$  be open. For a fixed  $p \in W$ , a mapping  $F : W \rightarrow Y$  is said to be holomorphic at  $p$  if there exists a chart  $(U, \varphi)$  of  $X$  containing  $p$  and a chart  $(V, \psi)$  of  $Y$  containing  $F(p)$  such that  $\psi \circ F \circ \varphi^{-1} : \varphi(U) \rightarrow \psi(V)$  is holomorphic at  $\varphi(p)$ . If  $F$  is holomorphic at every point of  $W$ , then  $F$  is holomorphic on  $W$ .

**Example 1.12.** It is easy to show that the identity map  $\text{id}_X$  on a Riemann surface  $X$  is a holomorphic map. Furthermore, for all Riemann surfaces  $X, Y$  and  $Z$  and holomorphic maps  $F : X \rightarrow Y$  and  $G : Y \rightarrow Z$ , their composite  $G \circ F : X \rightarrow Z$  is also a holomorphic map. This shows that the collection of all Riemann surfaces is a *category*.  $\blacklozenge$

**Definition 1.13.** Let  $X$  and  $Y$  be Riemann surfaces. A biholomorphism between  $X$  and  $Y$  is an invertible holomorphic map  $F : X \rightarrow Y$  whose inverse  $F^{-1} : Y \rightarrow X$  is also holomorphic. Two Riemann surfaces  $X$  and  $Y$  are said to be biholomorphic if there exists a biholomorphism  $F : X \rightarrow Y$ .

**Example 1.14** (Biholomorphisms between Riemann spheres). Let  $\mathbb{C}_{\infty}$ ,  $S^2$ , and  $\mathbb{P}^1$  denote the three constructions for the Riemann sphere  $\hat{\mathbb{C}}$  presented in Examples 1.5, 1.6, and 1.7, respectively. We claim that the maps

$$F : S^2 \rightarrow \mathbb{P}^1 : (x, y, w) \mapsto [1 - w : x + iy] \quad \text{and} \quad G : S^2 \rightarrow \mathbb{C}_{\infty} : (x, y, w) \mapsto \frac{x + iy}{1 - w}$$

are biholomorphisms, which shows that all three constructions are biholomorphic. Indeed  $F$  is holomorphic since with the charts

$$\begin{aligned} U &:= S^2 \setminus \{(0, 0, 1)\} & \varphi : U \rightarrow \mathbb{C} : (x, y, w) &\mapsto \frac{x + iy}{1 - w} \\ V &:= \mathbb{P}^1 \setminus \{[0 : w] \mid w \in \mathbb{C}\} & \psi : V \rightarrow \mathbb{C} : [z : w] &\mapsto \frac{w}{z}, \end{aligned}$$

we see that

$$\begin{aligned} (\psi \circ F \circ \varphi^{-1})(z) &= \psi \left( F \left( \frac{2 \operatorname{Re} z}{|z|^2 + 1}, \frac{2 \operatorname{Im} z}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right) \right) \\ &= \psi \left( \left[ 1 - \frac{|z|^2 - 1}{|z|^2 + 1} : \frac{2z}{|z|^2 + 1} \right] \right) \\ &= \psi([1 : z]) \\ &= z \end{aligned}$$

for all  $z \in \varphi(U) = \mathbb{C}$ , which is clearly holomorphic. Furthermore, it can be checked that  $F$  is invertible with inverse

$$F^{-1}([z : w]) := \frac{(2 \operatorname{Re}(z\bar{w}), 2 \operatorname{Im}(z\bar{w}), |z|^2 - |w|^2)}{|z|^2 + |w|^2},$$

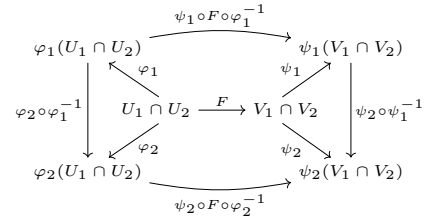
which is well-defined, and since  $(\psi \circ F \circ \varphi^{-1})^{-1} = \varphi \circ F^{-1} \circ \psi^{-1}$ , we see that  $F^{-1}$  is holomorphic too.  $\blacklozenge$

Defining some property  $P$  of  $f$  using charts by transporting  $f$  to a function  $f \circ \varphi^{-1}$  on a subset of  $\mathbb{C}$ , and borrowing  $P$  from  $f \circ \varphi^{-1}$ , will be a common theme. However, one must check that  $P$  is *independent of charts*; that is, if  $f \circ \varphi^{-1}$  satisfies  $P$ , then so does  $f \circ \psi^{-1}$  for any other chart  $(V, \psi)$ .



This makes the set  $\mathcal{O}(W)$  of all holomorphic functions  $f : W \rightarrow \mathbb{C}$  into a  $\mathbb{C}$ -algebra.

For  $Y := \mathbb{C}$  regarded as a Riemann surface, this definition agrees with the above. Again, we must check that ‘being holomorphic’ is well-defined, but it follows from the commutativity of the diagram below.



Take  $G(0, 0, 1) := \infty$ .

Since the collection of Riemann surfaces form a category, the ‘is isomorphic to’ relation is an equivalence relation. Thus we are justified to call all three constructions ‘the’ Riemann sphere, and, henceforth, we shall denote all three by  $\hat{\mathbb{C}}$ .

A similar calculation shows that  $G$  is biholomorphic. Indeed, we choose the same chart  $(U, \varphi)$ , and choose  $V := \mathbb{C}_{\infty} \setminus \{\infty\} = \mathbb{C}$  with  $\psi := \text{id}_{\mathbb{C}}$ . Then  $(\psi \circ G \circ \varphi^{-1})(z) = z$  for all  $z \in \varphi(U) = \mathbb{C}$ , and  $G$  is invertible with inverse

$$G^{-1}(z) := \begin{cases} \varphi^{-1}(z) & \text{if } z \in \mathbb{C} \\ (0, 0, 1) & \text{else.} \end{cases}$$

**Theorem 1.15.** Any holomorphic function  $f : X \rightarrow \mathbb{C}$  on a compact Riemann surface  $X$  is constant.

*Proof.* Since  $f$  is holomorphic, the function  $|f| : X \rightarrow \mathbb{R}$  defined by  $|f|(x) := |f(x)|$  is continuous on  $X$ . But  $X$  is compact, so  $|f|$  achieves its maximum at some point  $p \in X$ . Choosing a connected chart  $(U, \varphi)$  centered at  $p$ , we see that  $f \circ \varphi : U \rightarrow \mathbb{C}$  is holomorphic. Then  $|f \circ \varphi| : U \rightarrow \mathbb{R}$  has a local maximum at 0, so, since  $U$  is connected,  $f \circ \varphi$  is constant by the Maximum Principle. Then  $f$  is locally constant around  $p$ , so, since  $X$  is connected,  $f$  is constant on  $X$ . ■

## 1.2.2 Singularities of Functions

Throughout this section, let  $X$  be a Riemann surface, let  $p \in X$ , and let  $f : W \rightarrow \mathbb{C}$  be defined and holomorphic on a punctured neighborhood  $W$  of  $p$ . As above, we can transport the behaviour of  $f$  at  $p$  from its chart representation  $f \circ \varphi^{-1}$ .

**Definition 1.16.** Let  $f : W \rightarrow \mathbb{C}$  be a holomorphic function in a punctured neighborhood of  $p$ . We say that  $f$  has a removable singularity (resp. pole, essential singularity) at  $p$  if there exists a chart  $(U, \varphi)$  of  $X$  containing  $p$  such that  $f \circ \varphi^{-1} : \varphi(U) \rightarrow \mathbb{C}$  has a removable singularity (resp. pole, essential singularity) at  $\varphi(p)$ .

*Proof.* (Well-defined): Equation (1.1) shows that those notions are chart independent; the composition of  $f \circ \varphi^{-1}$  having a singularity at  $p$  with a transition map that is holomorphic at  $p$  yields a function with the same type of singularity at  $p$ . ■

**Remark.** Functions having an essential singularity at  $p$  are very ill-behaved. Indeed, this occurs iff  $|f(x)|$  has a non-zero oscillation near  $p$ . Other singularities behave much better:

- A removable singularity occurs iff  $|f(x)|$  is bounded in a neighborhood of  $p$ , and can be ‘filled in’ by defining  $\tilde{f}(p) := \lim_{x \rightarrow p} f(x)$ . This makes  $\tilde{f} : \tilde{W} \rightarrow \mathbb{C}$  into a holomorphic function.
- A pole occurs iff  $|f(x)| \rightarrow \infty$  as  $x \rightarrow p$ , which can also be ‘filled in’ by defining the map

$$F : W \rightarrow \hat{\mathbb{C}} \quad \text{mapping} \quad x \mapsto \begin{cases} \infty & \text{if } x = p \\ f(x) & \text{else} \end{cases}$$

that extends the codomain of  $f$  to the Riemann sphere  $\hat{\mathbb{C}}$ ; it is clear that  $F$  is holomorphic.

Thus we see that every such function  $f : W \rightarrow \mathbb{C}$  having pole at  $p$  can be holomorphically extended to a map  $F : W \rightarrow \hat{\mathbb{C}}$ . Conversely, every holomorphic map  $F : W \rightarrow \hat{\mathbb{C}}$  (that is not identically zero) can be regarded as a function  $f : W \setminus F^{-1}(\infty) \rightarrow \mathbb{C}$  that is holomorphic everywhere except where  $F(x) = \infty$ , in which case it either has a pole. This motivates the following definition. ♦

**Definition 1.17.** A function  $f : W \rightarrow \mathbb{C}$  is said to be meromorphic at  $p$  if it does not have an essential singularity at  $p$ ; that is, if it is either holomorphic, has a removable singularity, or has a pole at  $p$ . If  $f$  is meromorphic at every point of  $W$ , then  $f$  is meromorphic on  $W$ .

**Remark.** The previous remark can now be rephrased by saying that the set of all meromorphic functions  $f : W \rightarrow \mathbb{C}$  are in one-to-one correspondence with the set of all holomorphic maps  $F : W \rightarrow \hat{\mathbb{C}}$  (which are not identically zero). That is, meromorphic functions are the holomorphic maps to the Riemann sphere. ♦

**Definition 1.18.** Let  $f : W \rightarrow \mathbb{C}$  be meromorphic at  $p$  and consider its Laurent series  $f_\varphi(z) := (f \circ \varphi^{-1})(z) = \sum_i c_i (z - z_0)^i$  under a chart  $(U, \varphi)$  of  $X$  with  $z_0 := \varphi(p)$ . The order of  $f$  at  $p$  is

$$\text{ord}_p(f) := \min \{n \in \mathbb{Z} \mid 0 \neq (z - z_0)^n f_\varphi(z) \in \mathcal{O}(W)\}.$$

*Proof.* (Well-defined). Let  $z$  be the local coordinates given by  $(U, \varphi)$  and suppose that  $(V, \psi)$  is another chart with  $w_0 := \psi(p)$  giving another local coordinate  $w$ . Then the transition function  $T : \varphi \circ \psi^{-1}$  is holomorphic, so it admits a power series representation

$$z = T(w) = \sum_{n \geq 0} a_n (w - w_0)^n = z_0 + \sum_{n \geq 1} a_n (w - w_0)^n.$$

Since  $T'(w_0) \neq 0$ , we see that  $a_1 \neq 0$ . Suppose now that the Laurent series of  $f$  at  $p$  in the coordinate  $z$  is  $c_{-n_0} (z - z_0)^{-n_0} + \text{higher order terms}$ , so that the order of  $f$  at  $p$  computed by employing  $z$  is  $n_0$ . Then the Laurent series of  $f$  at  $p$  in the coordinate  $w$  is

$$c_{-n_0} \left( \sum_{n \geq 1} a_n (w - w_0)^n \right)^{-n_0} + \text{higher order terms},$$

whose lowest order term is  $c_{-n_0} a_1^{-n_0} (w - w_0)^{-n_0}$ . Observe that  $b_{-n_0} := c_{-n_0} a_1^{-n_0} \neq 0$ , so the order of  $f$  at  $p$  computed via  $w$  is also  $n_0$ . ■

Such a connected  $U$  can always be found since we may let  $V$  be a chart around  $p$  and choose  $\varepsilon > 0$  small enough so that  $U := B(p, \varepsilon) \subseteq V$ .

That is, let  $f$  be defined and holomorphic on  $B(p, \varepsilon) \setminus \{p\}$  for some  $\varepsilon > 0$ .

We recall those notions from complex analysis. Let  $f : W \subseteq \mathbb{C} \rightarrow \mathbb{C}$  be a holomorphic function (in the regular sense) in a punctured neighborhood of  $p$ . Suppose that  $f$  is not holomorphic at  $p$ .

- If  $\lim_{z \rightarrow p} f(z)$  exists, then  $f$  has a removable singularity at  $p$ .
- If  $\lim_{z \rightarrow p} f(z) = \pm\infty$ , then  $f$  has a pole at  $p$ . This is equivalent to the existence of some  $n > 0$  such that the limit  $\lim_{z \rightarrow p} (z - p)^n f(z)$  exists. See Definition 1.18.
- Otherwise,  $f$  has an essential singularity at  $p$ .

$$\tilde{W} := W \cup \{p\}.$$

Here, we consider  $\hat{\mathbb{C}} = \mathbb{C}_\infty$ .

As in Example 1.10, if  $f, g : W \rightarrow \mathbb{C}$  are both meromorphic at  $p$ , then so are  $f \pm g$  and  $f \cdot g$ . If  $g$  is not identically 0, then so is  $f/g$ . This makes the set  $\mathcal{M}(W)$  of all meromorphic functions  $f : W \rightarrow \mathbb{C}$  into a  $\mathbb{C}$ -algebra.

Note that  $f$ , being meromorphic, ensures that its Laurent series has finitely-many negative terms. Thus the set  $\{n \in \mathbb{Z} \mid c_n \neq 0\}$  achieves its minimum, so the definition makes sense. If  $f$  is not meromorphic, we take  $\text{ord}_p(f) := \infty$ .

The arithmetic of  $\text{ord}_p$  is straightforward. Indeed, if  $f, g : W \rightarrow \mathbb{C}$  are meromorphic at  $p$ , then

- $\text{ord}_p(fg) = \text{ord}_p(f) + \text{ord}_p(g)$ .
- $\text{ord}_p(f/g) = \text{ord}_p(f) - \text{ord}_p(g)$ , if  $g \neq 0$ .
- $\text{ord}_p(1/f) = -\text{ord}_p(f)$ , if  $f \neq 0$ .
- $\text{ord}_p(f \pm g) \geq \min \{\text{ord}_p(f), \text{ord}_p(g)\}$ .

**Remark.** The order  $\text{ord}_p(f)$  can be used to classify the behaviour of  $f$  at  $p$ . Indeed, it is readily verified that  $f$  is holomorphic at  $p$  iff  $\text{ord}_p(f) \leq 0$ , in which case  $f(p) = 0$  iff  $\text{ord}_p(f) < 0$ . Similarly,  $f$  has a pole at  $p$  iff  $\text{ord}_p(f) > 0$ , so  $f$  has neither a zero nor a pole at  $p$  iff  $\text{ord}_p(f) = 0$ . This motivates the following definition.  $\blacklozenge$

**Definition 1.19.** Let  $f : W \rightarrow \mathbb{C}$  be meromorphic at  $p$ . We say that  $f$  has a zero (resp. pole) of order  $n$  at  $p$  if  $\text{ord}_p(f) = n < 0$  (resp.  $n > 0$ ).

### 1.2.3 Meromorphic Functions on $\hat{\mathbb{C}}$

**Example 1.20.** Let  $f : W \subseteq \hat{\mathbb{C}} \rightarrow \mathbb{C}$  be a non-zero rational function  $f(z) := p(z)/q(z)$ . Then  $f$  is holomorphic at all points  $z \in \mathbb{C}$  such that  $q(z) \neq 0$ , and has a pole otherwise. Also,  $f(\infty) \in \mathbb{C}$  if  $\deg p = \deg q$ , vanishes if  $\deg p < \deg q$ , and has a pole otherwise. In any case,  $f$  is meromorphic on  $\hat{\mathbb{C}}$ . To compute  $\text{ord}_z(f)$  at all  $z \in \hat{\mathbb{C}}$ , we split  $p$  and  $q$  into linear factors to write  $f$  uniquely as

$$f(z) = c \prod (z - \lambda_i)^{\alpha_i}$$

where  $c \neq 0$  and each  $\lambda_i$  is distinct. Fix  $i$ . Setting  $g_j(z) := (z - \lambda_j)^{\alpha_j}$  for all  $j$ , we see that  $\text{ord}_{\lambda_i}(g_i) = -\alpha_i$  and  $\text{ord}_{\lambda_j}(g_i) = 0$  for all  $i \neq j$ . Thus

$$\text{ord}_{\lambda_i}(f) = \sum_j \text{ord}_{\lambda_i}(g_j) = -\alpha_i.$$

Moreover, if  $\alpha_i > 0$  (resp.  $\alpha_i < 0$ ), then  $g_i$  has a pole (resp. zero) of order  $|\alpha_i|$  at  $\infty$ . It follows then that  $\text{ord}_{\infty}(g_i) = \alpha_i$ , so

$$\text{ord}_{\infty}(f) = \sum_i \text{ord}_{\infty}(g_i) = \sum_i \alpha_i.$$

Lastly, it is clear that  $\text{ord}_z(f) = 0$  for all  $z \neq \lambda_i, \infty$ .  $\blacklozenge$

**Theorem 1.21.** Any meromorphic function on the Riemann sphere is a rational function.

*Proof.* Let  $f : \hat{\mathbb{C}} \rightarrow \mathbb{C}$  be meromorphic. Since  $\hat{\mathbb{C}}$  is compact, it has finitely-many poles. W.l.o.g., assume that  $\infty$  is not a pole of  $f$  (since we may consider  $1/f$  instead). Now, for each pole  $\lambda_i \in \mathbb{C}$  of  $f$ , consider its principle part

$$h_i(z) = \sum_{j=-m_i}^{-1} c_{ij} (z - \lambda_i)^j$$

for some  $m_i > 1$ . Then the function  $g := f - \sum_i h_i$  is holomorphic function on  $\hat{\mathbb{C}}$ , and since  $\hat{\mathbb{C}}$  is compact, it is constant by Theorem 1.15. Thus  $f = g + \sum_i h_i$ , which is a rational function.  $\blacksquare$

**Remark.** Together with the above computation, this shows that if  $f$  is a meromorphic function on  $\hat{\mathbb{C}}$ , then  $\sum_{z \in \hat{\mathbb{C}}} \text{ord}_z(f) = 0$ . As we shall see, this is a general fact for all compact Riemann surfaces.  $\blacklozenge$

## 1.3 Global Properties of Holomorphic Maps

### 1.3.1 Local Normal Form

**Theorem 1.22 (Local Normal Form).** Let  $X$  and  $Y$  be Riemann surfaces and let  $F : X \rightarrow Y$  be a non-constant holomorphic map. Then, for every  $p \in X$ , there exists a unique  $m \geq 1$  such that for any chart  $(U_2, \varphi_2)$  of  $Y$  centered at  $F(p)$ , there exists a chart  $(U_1, \varphi_1)$  of  $X$  centered at  $p$  such that  $\varphi_2 \circ F \circ \varphi_1^{-1} : z \mapsto z^m$  for all  $z \in \varphi_1(U_1)$ .

*Proof.* Let  $(U_2, \varphi_2)$  be a chart of  $Y$  centered at  $F(p)$  and consider any chart  $(V, \psi)$  of  $X$  centered at  $p$ . Then the function  $h := \varphi_2 \circ F \circ \psi^{-1}$  is holomorphic, so it admits a power series representation  $h(w) = \sum_{i=0}^{\infty} c_i w^i$  for all  $w \in \psi(V)$ . Note that  $h(0) = \varphi_2(F(p)) = 0$ , so  $c_0 = 0$ . Let  $m \geq 1$  be the smallest integer such that  $c_m \neq 0$ , so

$$h(w) = \sum_{i \geq m} c_i w^i = w^m \sum_{i \geq 0} c_{i-m} w^i =: w^m g(w).$$

Then  $g$  is holomorphic at 0 with  $g(0) = c_m \neq 0$ , so there is a function  $h$  holomorphic on some neighborhood  $W$  of 0 such that  $(h(w))^m = g(w)$  for all  $w \in W$ . Thus  $h(w) = (wh(w))^m$ , so set  $\eta(w) := wh(w)$  for all  $w \in W$ . Note that  $\eta'(0) = h(0) \neq 0$ , so  $\eta$  is invertible on some neighborhood  $W' \subseteq W$  of 0. Set  $U_1 := \psi^{-1}(W')$  and  $\varphi_1 := \eta \circ \psi$ . Then  $(U_1, \varphi_1)$  is a chart of  $X$  centered at  $p$  such that

$$(\varphi_2 \circ F \circ \varphi_1^{-1})(z) = (\varphi_2 \circ F \circ \psi^{-1} \circ \eta^{-1})(z) = h(\eta^{-1}(z)) = [\eta(\eta^{-1}(z))]^m = z^m$$

for all  $z \in \varphi_1(U_1)$ . To show uniqueness, it suffices to show that such an  $m$  is chart-independent. But this is clear, for if a different chart  $U'_2$  is chosen such that  $F$  acts as  $z \mapsto z^n$  for some neighborhood  $U'_1$  of  $p$ , then  $z^n = z^m$  on  $\varphi_1(U_1) \cap \varphi'_1(U'_1)$  forces  $n = m$ .  $\blacksquare$

In fact, any meromorphic function on the Riemann sphere is a rational function; see Theorem 1.21.

Otherwise, the set of poles would have a limit point, contradicting the discreteness of poles.

This theorem also give easy proofs of some elementary properties of holomorphic maps, which we collect here; see [For81, section 1.2] for details. Throughout,  $F : X \rightarrow Y$  is a non-constant holomorphic map between Riemann surfaces  $X$  and  $Y$ .

- $F$  is an open map.
- If  $F$  is injective, then it is biholomorphic onto its image.
- If  $Y = \mathbb{C}$ , then  $|F|$  does not attain its maximum.
- If  $X$  is compact, then  $F$  is surjective and  $Y$  is compact.

Together, the last two claims give an alternative proof for Theorem 1.15.

**Definition 1.23.** With the above notation, the unique  $m \geq 1$  such that there are local coordinates around  $p$  and  $F(p)$  where  $F$  acts like  $z \mapsto z^m$  is called the multiplicity of  $F$  at  $p$ , denoted  $\text{mult}_p(F)$ .

**Theorem 1.24.** Let  $f$  be a meromorphic function on a Riemann surface  $X$  and let  $F : X \rightarrow \hat{\mathbb{C}}$  be its associated holomorphic map. Fix  $p \in X$ .

- If  $p$  is not a pole of  $f$ , then  $\text{mult}_p(F) = -\text{ord}_p(f - f(p))$ .
- If  $p$  is a pole of  $f$ , then  $\text{mult}_p(F) = \text{ord}_p(f)$ .

*Proof.* Suppose that  $p$  is not a pole of  $f$ , so  $f(p) = F(p) \in \mathbb{C}$ . Since the set of all poles of a meromorphic function forms a discrete set, let  $p \in U \subseteq X$  be small enough so that  $f|_U$  is holomorphic. Let  $(U, \varphi)$  be a chart of  $X$  and consider the chart  $(\mathbb{C}, \psi)$  of  $\hat{\mathbb{C}}$  around  $F(p)$  defined by  $\psi(z) := z - F(p)$ . Then  $f - f(p) = \psi \circ F$  on  $U$ , so

$$(f - f(p))_\varphi := (f - f(p)) \circ \varphi^{-1} = \psi \circ F \circ \varphi^{-1}$$

on  $\varphi(U)$ . Expanding in power series around  $z_0 := \varphi(p) \in \varphi(U)$ , we see that

$$(\psi \circ F \circ \varphi^{-1})(z) = (f - f(p))_\varphi(z) = \sum_{i \geq m} c_i (z - z_0)^i$$

for some  $m \in \mathbb{N}$  with  $c_m \neq 0$ . Note that  $(f - f(p))_\varphi(z_0) = (f - f(p))(p) = 0$ , so  $m > 0$  and hence  $\text{mult}_p(F) = m$ . But  $m$  is also the smallest integer such that

$$0 \neq (z - z_0)^{-m} (f - f(p))_\varphi(z) \in \mathcal{O}(U),$$

so  $\text{ord}_p(f - f(p)) = -m$ .

Suppose now that  $p$  is a pole of  $f$ , so  $F(p) = \infty$ . Since  $\lim_{z \rightarrow p} 1/f(z) = 0$ , we may let  $p \in U \subseteq X$  be small enough so that the function  $\tilde{f} : U \rightarrow \mathbb{C}$  defined by

$$\tilde{f}(x) := \begin{cases} 0 & \text{if } x = p \\ 1/f(x) & \text{else} \end{cases}$$

is holomorphic. Let  $(U, \varphi)$  be a chart of  $X$  and consider the chart  $(\hat{\mathbb{C}} \setminus \{0\}, \psi)$  of  $\hat{\mathbb{C}}$  defined by  $\psi(z) := 1/z$ . Then  $\tilde{f} = \psi \circ F$  on  $U$ , so  $\tilde{f}_\varphi := \tilde{f} \circ \varphi^{-1} = \psi \circ F \circ \varphi^{-1}$  on  $\varphi(U)$ . By the same argument as above, we see that  $\text{mult}_p(F) = -\text{ord}_p(\tilde{f})$ . Now  $\text{ord}_p(f) = -\text{ord}_p(\tilde{f})$ , so the result follows.  $\blacksquare$

We give a simple way of computing  $\text{mult}_p(F)$  that does not involve casting  $F$  into Local Normal Form, or even having to find local coordinates centered at  $p$  and  $F(p)$ . Indeed, let  $(U_1, \varphi_1)$  and  $(U_2, \varphi_2)$  be charts around  $p$  and  $F(p)$ , say with  $z_0 := \varphi_1(p)$  and  $w_0 := \varphi_2(F(p))$ . Letting  $f := \varphi_2 \circ F \circ \varphi_1^{-1}$ , we see that  $f(z_0) = w_0$  and hence its power series representation has the form

$$f(z) = f(z_0) + \sum_{i \geq m} c_i (z - z_0)^i$$

for some  $m \geq 1$  with  $c_m \neq 0$ . Then, since  $z - z_0$  and  $w - w_0 = f(z) - f(z_0)$  are local coordinates centered at  $p$  and  $F(p)$ , respectively, we see from the above proof that  $\text{mult}_p(F) = m$ . Thus to compute  $\text{mult}_p(F)$ , it suffices to case  $F$  into local coordinates  $(U_1, \varphi_1)$  around  $p$  and  $(U_2, \varphi_2)$  around  $F(p)$  and find the lowest non-zero power of the Taylor series of its local representation  $f := \varphi_2 \circ F \circ \varphi_1^{-1}$ .

$$\psi(z) := \begin{cases} 0 & \text{if } z = \infty \\ 1/z & \text{else.} \end{cases}$$

## Chapter 2

### Case for $g = 0$ and $g = 1$

#### 2.1 Moduli Space of $T^2$

In this section, we show that the moduli space of the torus  $T^2$  is  $\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$  where  $\mathbb{H}$  is the upper-half plane of  $\mathbb{C}$  and  $\mathrm{PSL}_2(\mathbb{Z})$  is the *modular group* consisting of all functions  $\gamma : \mathbb{H} \rightarrow \mathbb{H}$  mapping

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}$$

where  $a, b, c, d \in \mathbb{Z}$  with  $ad - bc = 1$ .

**Lemma 2.1.** *Let  $\Gamma, \Gamma' \subseteq \mathbb{C}$  be two lattices and suppose  $\alpha\Gamma \subseteq \Gamma'$  for some  $\alpha \in \mathbb{C}^*$ . Then  $z \mapsto \alpha z$  descends to a holomorphic map  $\varphi : \mathbb{C}/\Gamma \rightarrow \mathbb{C}/\Gamma'$ , which is biholomorphic iff  $\alpha\Gamma \subseteq \Gamma'$ .*

This gives a simple criterion for when two tori are biholomorphic.

*Proof.* Let  $\Gamma := \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$  and  $\Gamma' := \mathbb{Z}\omega'_1 \oplus \mathbb{Z}\omega'_2$ . Define  $\varphi(z + \Gamma) := \alpha z + \Gamma'$  for all  $z \in \mathbb{C}$ , which is clearly holomorphic if it is well-defined in the first place. To verify, take  $z_1, z_2 \in \mathbb{C}$  such that  $z_1 + \Gamma = z_2 + \Gamma$ . Then  $z_1 - z_2 \in \Gamma$ , so  $z_1 - z_2 = m\omega_1 + n\omega_2$  for some  $m, n \in \mathbb{Z}$ . Observe that

$$\alpha z_1 - \alpha z_2 = \alpha(z_1 - z_2) = m(\alpha\omega_1) + n(\alpha\omega_2) \in \alpha\Gamma \subseteq \Gamma',$$

so  $\alpha z_1 + \Gamma' = \alpha z_2 + \Gamma'$ . This shows that  $\varphi$  is well-defined. Furthermore, it is invertible with holomorphic inverse

$$\varphi^{-1}(z + \Gamma') := z/\alpha + \Gamma$$

iff  $\varphi^{-1}$  is well-defined, in which case  $\varphi$  is a biholomorphism. We claim that this occurs iff  $\alpha\Gamma \subseteq \Gamma'$ .

- ( $\Rightarrow$ ): It suffices to show that  $\Gamma' \subseteq \alpha\Gamma$ , so take  $m\omega'_1 + n\omega'_2 \in \Gamma'$ . Then

$$\varphi^{-1}(m\omega'_1 + n\omega'_2 + \Gamma') = (m\omega'_1 + n\omega'_2)/\alpha + \Gamma,$$

but since  $m\omega'_1 + n\omega'_2 + \Gamma' = 0 + \Gamma'$  and  $\varphi^{-1}(0 + \Gamma') = 0 + \Gamma$ , we see that  $(m\omega'_1 + n\omega'_2)/\alpha \in \Gamma$ .

- ( $\Leftarrow$ ): Take  $z_1, z_2 \in \mathbb{C}$  such that  $z_1 + \Gamma' = z_2 + \Gamma'$ , so  $z_1 - z_2 \in \Gamma' \subseteq \alpha\Gamma$  and hence

$$z_1/\alpha - z_2/\alpha = (z_1 - z_2)/\alpha \in \Gamma.$$

Then  $z_1/\alpha + \Gamma = z_2/\alpha + \Gamma$ , so  $\varphi^{-1}$  is well-defined. ■

**Lemma 2.2.** *Any torus  $\mathbb{C}/\Gamma$  is biholomorphic to  $X_\tau := \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$  for some  $\tau \in \mathbb{H}$ .*

This reduces the analysis to just tori of the form  $X_\tau$ , which allows us to leverage Lemma 2.3.

*Proof.* Let  $\Gamma := \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$  and set  $\alpha := 1/\omega_1$  and  $\tau := \omega_2/\omega_1$ . Then  $\mathrm{Im} \tau \neq 0$ , lest  $\omega_1, \omega_2$  be linearly dependent over  $\mathbb{R}$ . Without loss of generality, suppose that  $\mathrm{Im} \tau > 0$ ; if not, take  $\tau := \bar{\omega}_2/\omega_1$ . Then, since

$$\alpha(m\omega_1 + n\omega_2) = \alpha\omega_1(m + n\omega_2/\omega_1) = m + n\tau$$

for all  $m, n \in \mathbb{Z}$ , we see that  $\alpha\Gamma = \mathbb{Z} \oplus \mathbb{Z}\tau$ . By Lemma 2.1, we see that the map  $z \mapsto \alpha z$  descends to a biholomorphism  $\varphi : \mathbb{C}/\Gamma \rightarrow \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau) = X_\tau$ , so  $\mathbb{C}/\Gamma \cong X_\tau$ . ■

**Lemma 2.3.** *For any  $\tau, \tau' \in \mathbb{H}$ , the tori  $X_\tau$  and  $X_{\tau'}$  are biholomorphic iff there exists some  $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$  such that  $\tau' = \gamma(\tau)$ .*

*Proof.* The backwards direction is relatively straightforward. Indeed, note that

$$\tau' = \frac{a\tau + b}{c\tau + d} \quad \Rightarrow \quad \tau = \frac{b - d\tau'}{c\tau' - a}$$

for any  $a, b, c, d \in \mathbb{Z}$  with  $ad - bc = 1$ , so let  $\alpha := c\tau' - a$ . Then, with  $\Gamma := \mathbb{Z} \oplus \mathbb{Z}\tau$  and  $\Gamma' := \mathbb{Z} \oplus \mathbb{Z}\tau'$ , we proceed by proving that  $\alpha\Gamma = \Gamma'$ , from which the result follows from Lemma 2.1.

- ( $\subseteq$ ): For any  $m, n \in \mathbb{Z}$ , our choice of  $\alpha$  shows that

$$m\alpha + n\alpha\tau = m(c\tau' - a) + n(b - d\tau') = (nb - ma) + (mc - nd)\tau' \in \mathbb{Z} \oplus \mathbb{Z}\tau',$$

so  $\alpha(\mathbb{Z} \oplus \mathbb{Z}\tau) \subseteq \mathbb{Z} \oplus \mathbb{Z}\tau'$ .



- ( $\supseteq$ ): For any  $m, n \in \mathbb{Z}$ , the condition that  $ad - bc = 1$  shows that

$$(m + n\tau')/\alpha = \frac{(na - mc)\tau + (nb - md)}{a(c\tau + d) - c(a\tau + b)} = (nb - md) + (na - mc)\tau \in \mathbb{Z} \oplus \mathbb{Z}\tau,$$

so  $\mathbb{Z} \oplus \mathbb{Z}\tau' \subseteq \alpha(\mathbb{Z} \oplus \mathbb{Z}\tau)$ .

For the forward direction, let  $\varphi : X_{\tau'} \rightarrow X_{\tau}$  be a biholomorphism, which lifts to a biholomorphic mapping  $\tilde{\varphi} : \mathbb{C} \rightarrow \mathbb{C}$  between their universal covering spaces such that

This is a standard result in algebraic topology. For a proof, see [Tan91, Theorem 3.4].

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\tilde{\varphi}} & \mathbb{C} \\ \pi \downarrow & & \downarrow \pi' \\ \mathbb{C}/\Gamma & \xrightarrow{\varphi} & \mathbb{C}/\Gamma' \end{array}$$

commutes. Then  $\tilde{\varphi}$  is a meromorphic function on the Riemann sphere with a pole at  $\infty$ , so, by Theorem 1.21,  $\tilde{\varphi}$  is a rational function. Noting that  $\tilde{\varphi}$  is defined on all of  $\mathbb{C}$ , we see that  $\tilde{\varphi}$  must be a polynomial. If  $\deg \tilde{\varphi} > 1$ , then, by the Fundamental Theorem of Algebra, the equation  $\tilde{\varphi}(z) = 0$  has more than one solution, contradicting the fact that  $\tilde{\varphi}$  is a bijection.

**Theorem 2.4.** *The moduli space of  $T^2$  is  $\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$ .*

## 2.2 Moduli Space of $S^2$

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