Possible Choices of Complex Co-ordinates for a Surface

Moduli Spaces of Riemann Surfaces

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Graduate mentor: Kaleb Ruscitti

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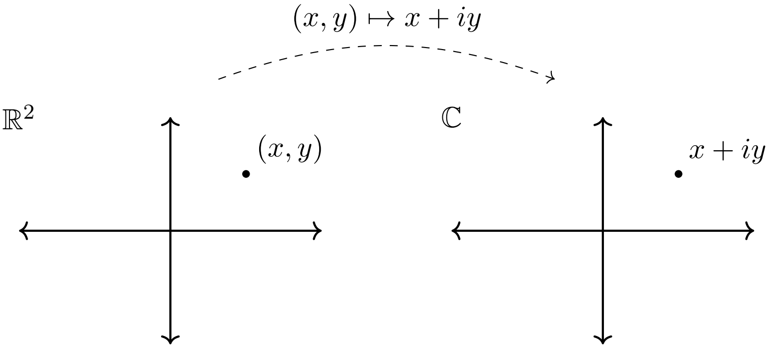
How many complex coordinates on $\mathbb{R}^2$?



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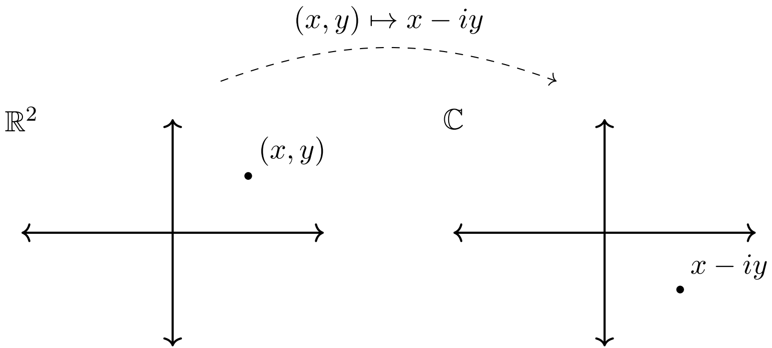

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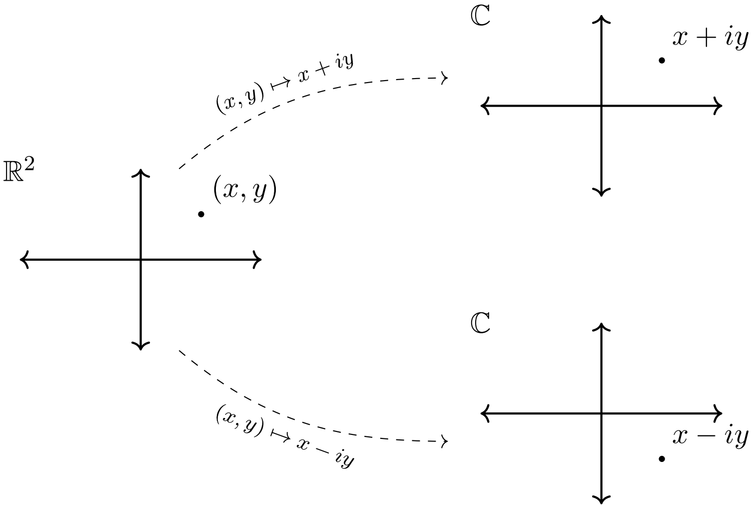

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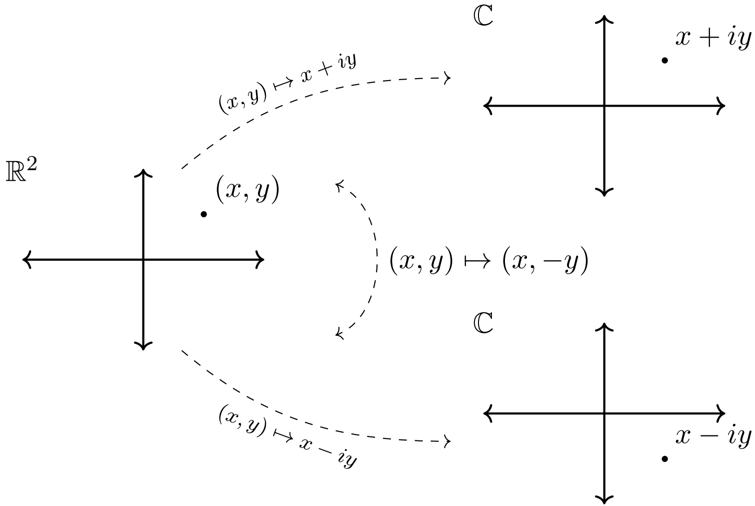

Refined question

How many complex coordinates on $\mathbb{R}^2$?

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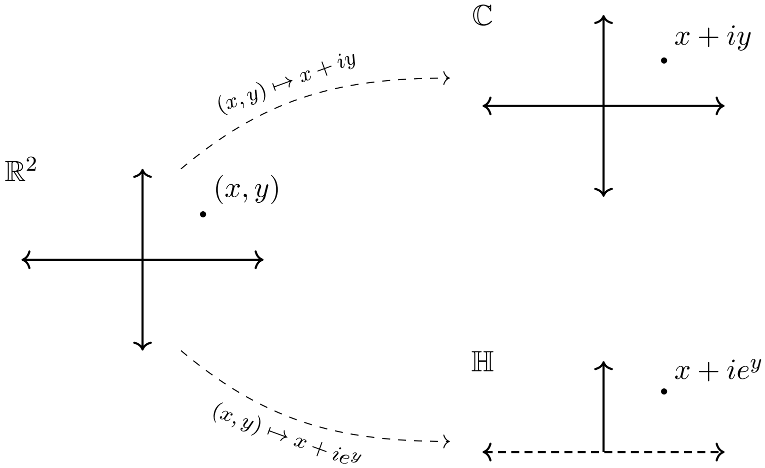
How many complex coordinates on $\mathbb{R}^2$ *up to biholomorphism*?

$\mathbb{C}$ and $\mathbb{H}$



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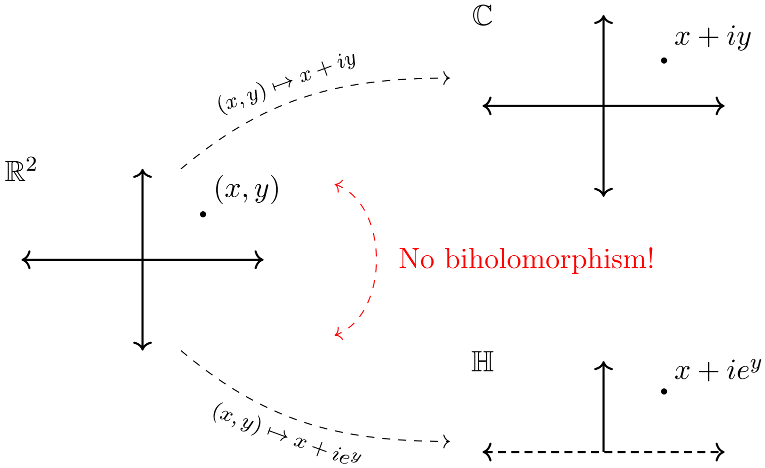


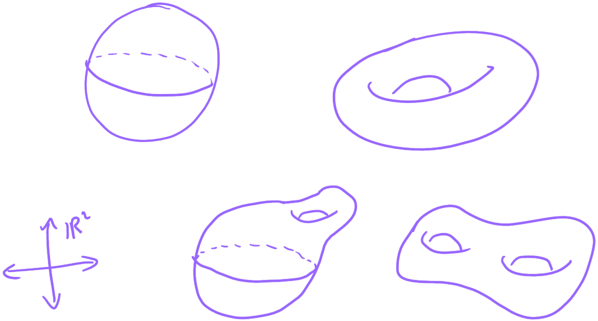
What about for manifolds?

Riemann surfaces

What about for manifolds?

Definition

A Riemann surface is a connected 1-dimensional complex manifold.



Theorem (Classification)

Every compact Riemann surface is classified by its genus.

Compact Riemann surfaces

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Topological torus

Theorem

Every torus is a quotient $\mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2)$ *for some linearly independent* $\omega_1, \omega_2 \in \mathbb{C}$.

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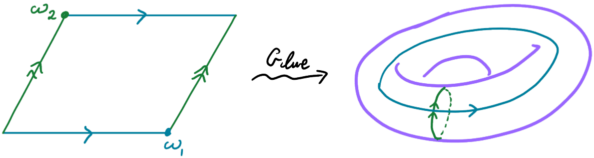

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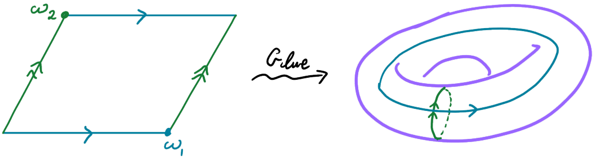

Theorem

There is, up to homeomorphism, only one topological torus.

Complex tori

Questions

- *Up to biholomorphism, in how many ways can we give the torus complex coordinates?*

Complex tori

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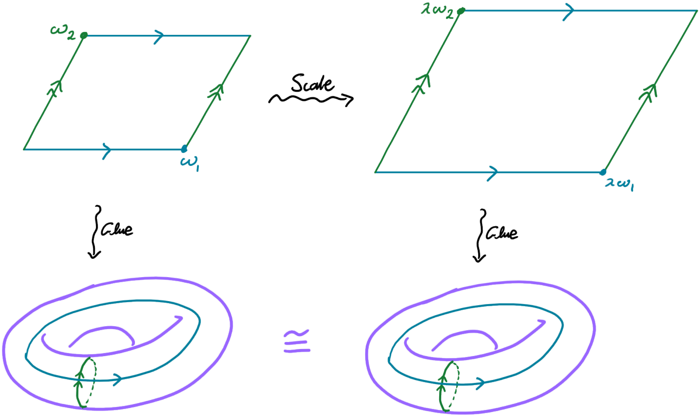
- *Up to biholomorphism, in how many ways can we give the torus complex coordinates?*
- *What is the moduli space of the torus?*

Complex tori



Complex tori

Theorem

Every complex tori is the quotient $X_\tau := \mathbb{C}/\Gamma$ *where* $\Gamma = \mathbb{Z} \oplus \mathbb{Z}\tau$ *for some* $\tau \in \mathbb{H}$.

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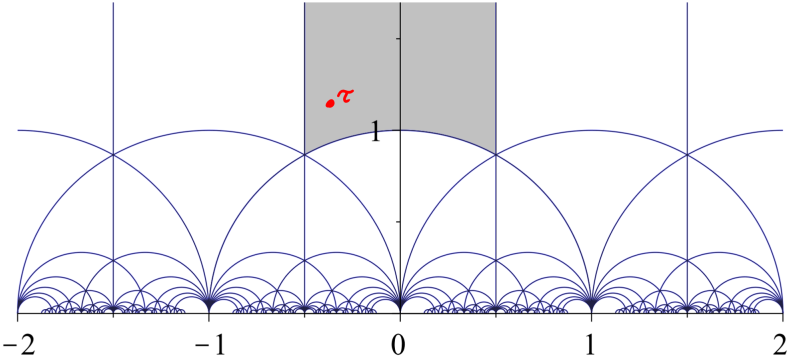
Every complex tori is the quotient $X_\tau := \mathbb{C}/\Gamma$ *where* $\Gamma = \mathbb{Z} \oplus \mathbb{Z}\tau$ *for some* $\tau \in \mathbb{H}$. *Indeed, for all linearly independent* $\omega_1, \omega_2 \in \mathbb{C}$,

$$\mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2) \cong X_\tau$$

for $\tau := \omega_2/\omega_1 \in \mathbb{H}$.

Complex tori



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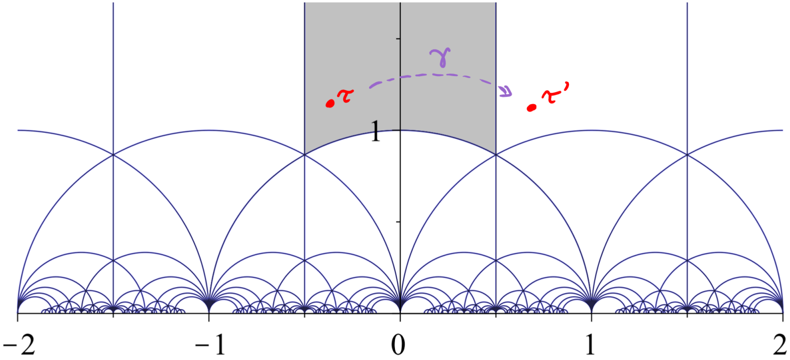

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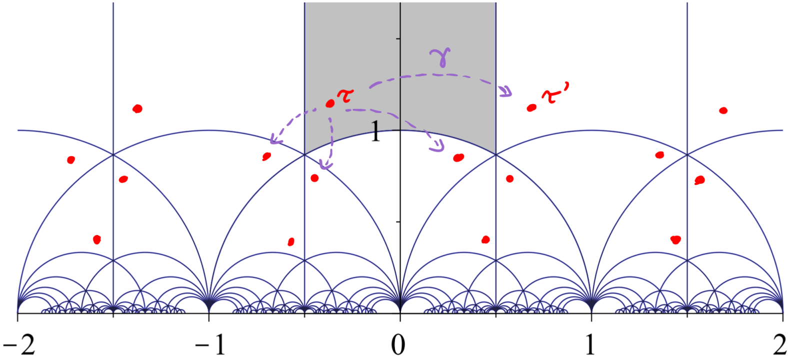

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The modular group $\mathrm{PSL}_2(\mathbb{Z})$ *is the group of functions* $\gamma : \mathbb{H} \to \mathbb{H}$ *mapping*

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}$$

for some $a, b, c, d \in \mathbb{Z}$ *with* $ad - bc = 1$.

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For any $\tau, \tau' \in \mathbb{H}$, *the tori* X_τ *and* X_τ *are biholomorphic iff there exists some* $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$ *such that* $\tau' = \gamma(\tau)$.

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Corollary

The moduli space of complex tori is $\mathbb{H}/\,\mathrm{PSL}_2(\mathbb{Z})$*.*

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- Sheaf cohomology: uhh something. ■

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