

Possible Choices of Complex Co-ordinates for a Surface

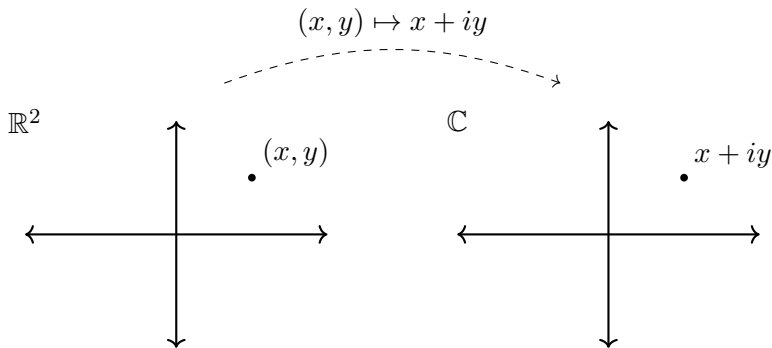
Moduli Spaces of Riemann Surfaces

Zhaoshen Zhai

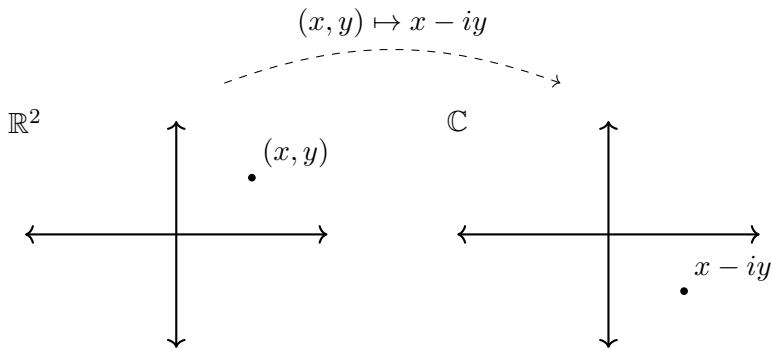
Graduate mentor: Kaleb Ruscitti

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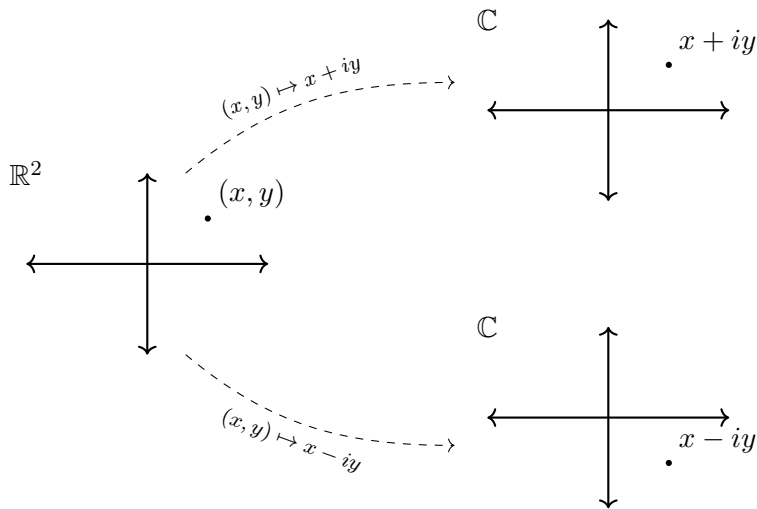
How many complex coordinates on \mathbb{R}^2 ?



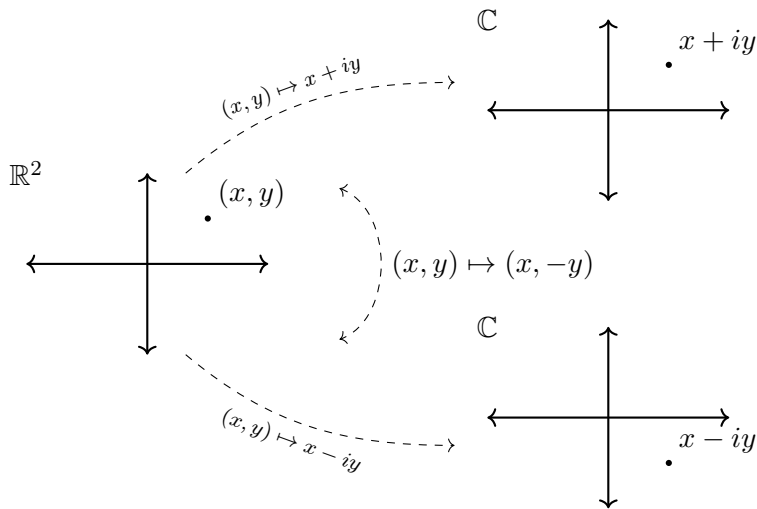
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Refined question

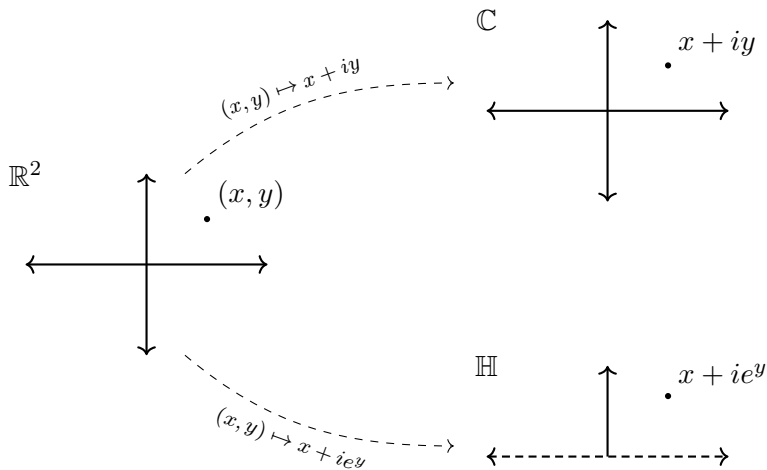
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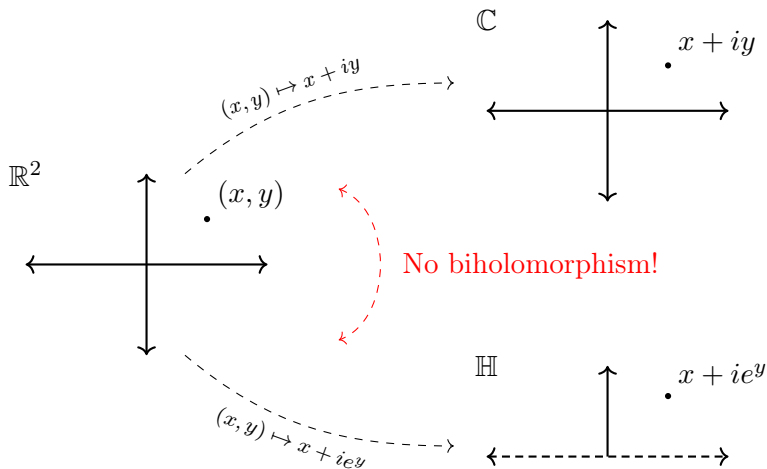
~~How many complex coordinates on \mathbb{R}^2 ?~~

How many complex coordinates on \mathbb{R}^2 *up to biholomorphism*?

\mathbb{C} and \mathbb{H}

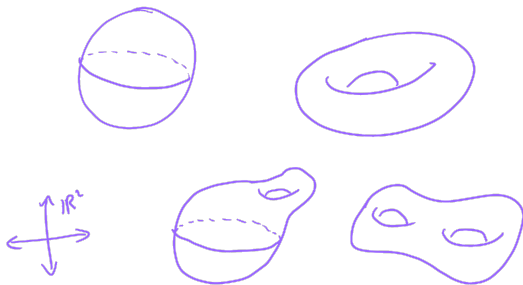


\mathbb{C} and \mathbb{H}



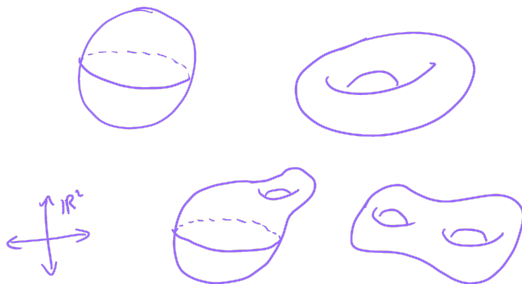
Surfaces

Recall that a surface is a connected 2-dimensional real manifold.



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Definition

A Riemann surface is a surface with a choice of complex structure.

Compact surfaces

Theorem (Classification of Surfaces)

For each natural number g , there exists exactly one compact orientable surface up to homeomorphism. We call g the genus of the surface.

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Topological torus

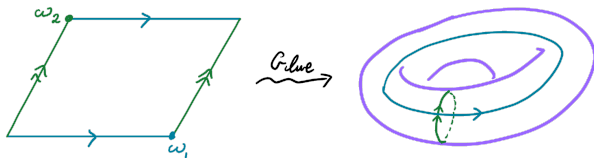
Theorem

Every torus is a quotient $\mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2)$ for some \mathbb{R} -linearly independent $\omega_1, \omega_2 \in \mathbb{C}$.

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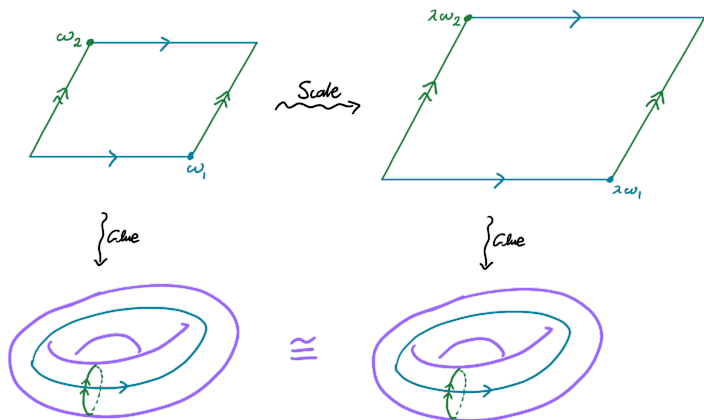
Questions

- *Up to biholomorphism, in how many ways can we give the torus complex coordinates?*

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- *Can we give the set of all complex coordinates on the torus some geometric structure?*

Complex tori



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Theorem

Every complex tori is the quotient $X_\tau := \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$ for some $\tau \in \mathbb{H}$.

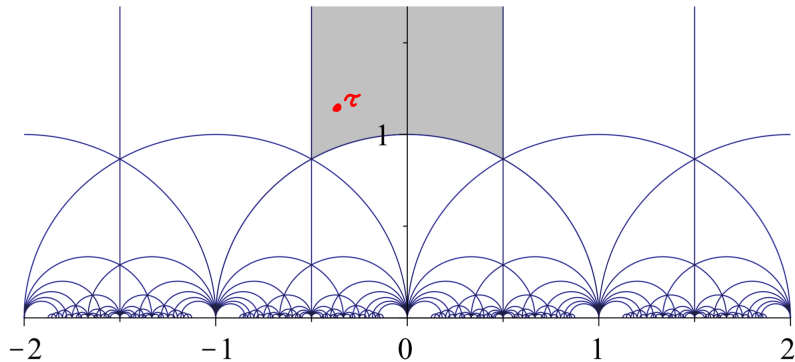
Theorem

Every complex tori is the quotient $X_\tau := \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$ for some $\tau \in \mathbb{H}$. Indeed, for all linearly independent $\omega_1, \omega_2 \in \mathbb{C}$,

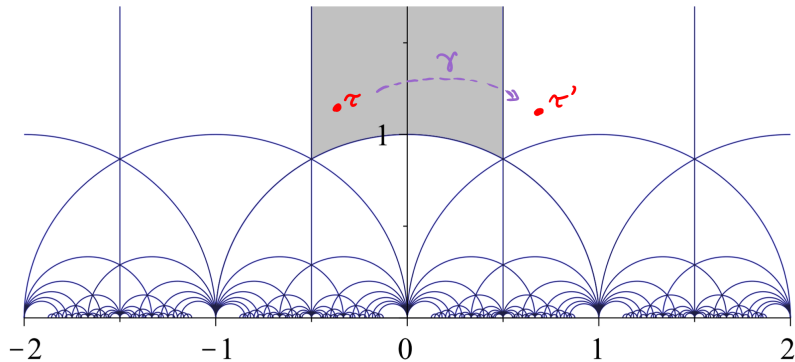
$$\mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2) \cong X_\tau$$

for $\tau := \omega_2/\omega_1 \in \mathbb{H}$.

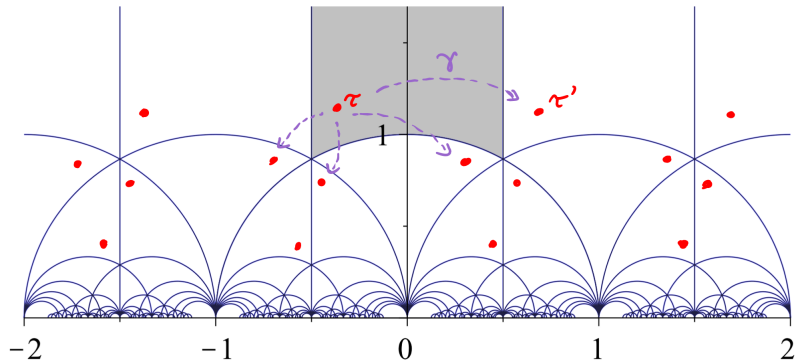
Complex tori



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Complex tori



Definition

The modular group $\mathrm{PSL}_2(\mathbb{Z})$ is the group of functions $\gamma : \mathbb{H} \rightarrow \mathbb{H}$ mapping

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}$$

for some $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$.

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Corollary

The moduli space of complex tori is $\mathbb{H} / \mathrm{PSL}_2(\mathbb{Z})$.

Theorem

There is a unique choice of complex coordinates on the sphere.

Riemann sphere

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- Sheaf cohomology: uhh something. ■

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