Moduli Spaces of Riemann Surfaces

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Abstract

The theory of Riemann surfaces, first developed by Bernhard Riemann to study algebraic functions, now lies in the confluence of complex analysis, differential geometry, and algebraic geometry. This expository paper aims to introduce this theory, with the goal classifying all compact Riemann surfaces of genus 0 and 1. To do so, we first develop the basics of covering space theory, which defines the degree of proper holomorphic maps, and then study the sheaf of holomorphic maps on a Riemann surface and their associated cohomology theory. Together, they form the core technical tools of the paper and allow us to connect the function theory of Riemann surfaces to their complex structure. Lastly, we give a glimpse into the non-compact case, namely the Uniformization Theorem, which gives us a tri-fold classification of all Riemann surfaces.

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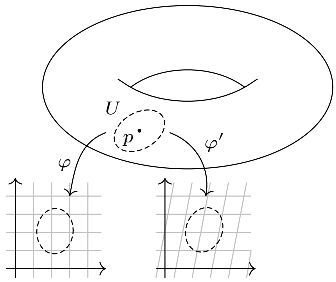
Chapter 0

Introduction

0.1 Overview and Main Results

Complex analysis is undoubtedly one of the foundational cornerstones of modern mathematics. To enrich it with topology, we restrict the class of 2-dimensional topological spaces (i.e. *surfaces*) of study to those with a local neighborhood around every point that looks like a deformed patch of $\mathbb{C}$, but whose global behaviour can be quite different. The choice in which a surface is made to look locally like $\mathbb{C}$ is called a *complex structure*, and, in general, there are many such choices. A surface, equipped with a particular choice of complex structure, is called a *Riemann surface*.

As a motivating example, take the torus T^2. Around every point $p \in T^2$, we can find a small enough neighborhood U of p that deforms reversibly onto an open subset of $\mathbb{C}$. The figure below shows two ways of doing so.



Using φ^{-1} to 'pull' the coordinate lines back to U, we see that the angles that they make is different from the angles made by using φ'^{-1} instead. The rigidity of holomorphic maps from complex analysis suggests that those coordinates ought to be different, and indeed they are. Thus we see that the same surface, T^2, can be equipped with many different complex structures, making them different *complex tori*.

In general, we call the set of all complex structures on a surface X the *moduli space*[1] *of* X, and the main goal of this paper is to compute it for the sphere S^2 and the torus T^2. The results, proven in Theorems 4.1 and 4.8 respectively, are as follows.

- Surprisingly, the moduli space of S^2 is a point. In other words, the sphere admits a unique complex structure. This fact, which is part of the *Uniformization Theorem*, is one of the starting points in the theory of compact Riemann surfaces, which we will spend the majority of this paper discussing.
- The fact that T^2 can be constructed topologically as a quotient $\mathbb{C}/\Gamma$ by an integer lattice Γ gives us a relatively straightforward proof that the moduli space of T^2 is $\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$. Here, $\mathbb{H} \subset \mathbb{C}$ is the upper-half plane of $\mathbb{C}$ and $\mathrm{PSL}_2(\mathbb{Z}) := \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$ is the *modular group*, which acts on $\mathbb{H}$ via Möbius transformations. We show that all complex tori can be written as $X_\tau := \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ for some $\tau \in \mathbb{H}$, and two tori X_τ and $X_{\tau'}$ are biholomorphic iff τ and τ' lie in the same orbit of the action.

0.2 Organization and Prerequisites

We give a brief overview of the organization of this paper.

- Chapter 1 begins with some definitions and constructions relating to Riemann surfaces and introduces the main examples of interest to this paper: the Riemann sphere and complex tori. We then study the basic behaviours of maps between Riemann surfaces, with a focus on meromorphic functions and their associated holomorphic maps.
- Chapter 2 studies the covering space theory of Riemann surfaces. The degree of a proper holomorphic map is defined, which is proven to be the cardinality of any fiber counted with multiplicity. We finish with a proof of the existence of liftings, which will be used to compute the moduli space of T^2.
- Chapter 3 builds up the basics of sheaf theory and their associated cohomology. The theory of (complex) differential forms and integration is then introduced to study the sheaf of holomorphic functions on the Riemann sphere, where we prove the existence of certain global meromorphic functions on a compact Riemann surface X.
- Chapter 4 ties everything together and uses the tools developed to compute the moduli space of genus 0 and 1 surfaces (S^2 and T^2). We also give a brief discussion and proof sketch of the Uniformization Theorem, which gives a tri-fold classification of all Riemann surfaces.

As for prerequisites, some familiarity with topology and complex analysis is required, and we also assume that the reader is comfortable with some linear algebra and basic group theory. A more detailed list of prerequisites, along with references, will be given at the start of each chapter.

[1] This paper is only concerned with the underlying set of points, without regard to any geometric structure. This turns out to be interesting enough in its own right, but the reader should be aware that the study of geometric structures on the moduli space is vast. We refer the interested reader to [Tan91], [Mar12], and [Hub06].

Chapter 1

Riemann Surfaces

We begin with some basic definitions and constructions relating to Riemann surfaces that will be used throughout this paper. This chapter requires some background in topology and complex analysis, all of which can be found in classical texts such as [Mun00] and [Lan98]. For an introduction to topology focused on (real) manifolds, see [Lee10] or [Tu10].

1.1 Charts and Atlases

We first formalize what we mean for a topological space to 'locally look like a patch of $\mathbb{C}$'. In this section, let X be a connected second-countable Hausdorff space.

Definition 1.1. *A complex chart of X is a pair (U, φ) where $\varphi : U \to V$ is a homeomorphism from an open subset $U \subseteq X$ onto an open subset $V \subseteq \mathbb{C}$. Two charts (U_1, φ_1) and (U_2, φ_2) are said to be compatible if either $U_1 \cap U_2 = \varnothing$, or the map*

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1 (U_1 \cap U_2) \to \varphi_2 (U_1 \cap U_2),$$

called the transition map, is biholomorphic. A complex atlas on X is a collection $\mathfrak{A} := \{(U_i, \varphi_i)\}_{i \in I}$ of pairwise compatible complex charts that cover X.

Remark. Charts provide *local coordinates* for every point in X in such a way that the transition maps $\varphi_j \circ \varphi_i^{-1}$ respect the analytic structure of $\mathbb{C}$. Within the same atlas $\mathfrak{A}$, those charts give us different coordinate representations for points in $U_i \cap U_j$, and since no chart is distinguished from the others, we can only define notions using local coordinates if they are invariant under the transition map.

$$\begin{tikzcd} & U_i \cap U_j \arrow[dl, "\varphi_i"'] \arrow[dr, "\varphi_j"] & \\ \varphi_i(U_i \cap U_j) \arrow[rr, "\varphi_j \circ \varphi_i^{-1}"'] & & \varphi_j(U_i \cap U_j) \end{tikzcd}$$

It is a classical result in complex analysis that the inverse of a holomorphic map is also holomorphic, so $\varphi_j \circ \varphi_i^{-1}$ is biholomorphic iff $\varphi_i \circ \varphi_j^{-1}$ is, which is convenient when checking that a collection of charts form an atlas. Lastly, we remark that it is sometimes convenient to write (U, z) for (U, φ), which can be decomposed into $z = x + iy$ by taking the real and imaginary parts of φ. ♦

Definition 1.2. *Two complex atlases $\mathfrak{A}$ and $\mathfrak{B}$ on X are said to be equivalent if every chart of $\mathfrak{A}$ is compatible with every chart in $\mathfrak{B}$.*

Remark. By Zorn's Lemma, every atlas $\mathfrak{A}$ of a manifold X is contained in a unique maximal atlas on X (see, for instance, [Lee12, Proposition 1.17]). Moreover, two atlases are equivalent iff they are contained in the same maximal atlas, which justifies the following definition. ♦

Definition 1.3. *A complex structure on X is a maximal atlas $\mathfrak{A}$ on X, or, equivalently, an equivalence class of complex atlases on X. The pair $(X, \mathfrak{A})$ is then called a Riemann surface.*

Remark. Every Riemann surface can be regarded as a (connected) 2-dimensional real manifold by 'forgetting' its complex structure. Since orientations are invariant under biholomorphisms, and in particular transition maps, the local orientation of $\mathbb{C}$ pulls-back via charts to a local orientation at each point $p \in X$. Since charts cover X, these local orientations induce a global orientation on X. Thus all Riemann surfaces are orientable, so, by the Classification of Surfaces, the closed Riemann surfaces are classified by their genus. Note, however, that this is a *topological* classification, and does not give any information about the complex structure on X. ♦

Example 1.4. Some elementary examples of Riemann surfaces.

- The complex plane $\mathbb{C}$, equipped with its standard topology, can be given a complex structure $\mathfrak{A}$ by choosing the atlas containing a single chart $(\mathbb{C}, \mathrm{id}_{\mathbb{C}})$. We may, however, also give $\mathbb{C}$ a different complex structure $\mathfrak{A}'$ by choosing the chart map $\varphi : z \mapsto \overline{z}$ instead. Indeed, $\mathfrak{A} \neq \mathfrak{A}'$ since the map $\varphi \circ \mathrm{id}_{\mathbb{C}}^{-1} = \varphi$ is not holomorphic and hence the atlases $\{(\mathbb{C}, \mathrm{id}_{\mathbb{C}})\}$ and $\{(\mathbb{C}, \varphi)\}$ are not equivalent. This example generalizes to any domain $D \subseteq \mathbb{C}$.

- Let $D \subseteq \mathbb{C}$ be a domain and consider any holomorphic function $f : D \to \mathbb{C}$. Then the graph $\Gamma_f := \{(z, f(z)) \mid z \in D\}$, equipped with the subspace topology inherited from $\mathbb{C}^2$, can be given a complex structure by choosing the chart map $\pi : \Gamma_f \to D : (z, f(z)) \mapsto z$. More generally, the set X of roots of an irreducible[1] polynomial $f \in \mathbb{C}[z, w]$ where every root has at least one non-vanishing partial derivative, called a *smooth affine plane curve*, is a Riemann surface. Indeed, if $\partial f / \partial w$ is non-zero at $p = (z_0, w_0)$, then the Implicit Function Theorem furnishes a holomorphic function $g(z)$ defined on a neighborhood of z_0 such that $X = \Gamma_g$ on some neighborhood $U \ni p$. Then, as above, the projection $\pi_z : U \to \mathbb{C}$ is a homeomorphism onto its image, giving us the desired chart map. ♦

[1] The irreducibility of the polynomial ensures that its set of roots is connected. Its proof requires some algebraic geometry, which we take for granted.

1.1.1 The Riemann Sphere $\hat{\mathbb{C}}$

A particularly important Riemann surface is the Riemann sphere $\hat{\mathbb{C}}$, which admits several constructions. Here, we equip standard constructions of topological spheres with three complex structures, which *a priori* need not be biholomorphic (in the sense of Definition 1.13), but in fact are; see Example 1.14 for a proof. In fact, *any* Riemann surface that is topologically the sphere is the Riemann sphere, which we prove in Theorem 4.1.

Example 1.5 (One-point Compactification of $\mathbb{C}$)**.** Let ∞ be a symbol not belonging to $\mathbb{C}$ and set $\mathbb{C}_\infty \coloneqq \mathbb{C} \cup \{\infty\}$. We declare a set $U \subseteq \mathbb{C}_\infty$ to be open if either $U \subseteq \mathbb{C}$ is open or $U = K^c \cup \{\infty\}$ for some compact subset $K \subseteq \mathbb{C}$. This makes $\mathbb{C}_\infty$, equipped with the collection $\mathcal{T}$ of all such open sets, a second-countable Hausdorff space. Indeed, the fact that $\mathcal{T}$ is a topology on $\mathbb{C}_\infty$ follows from De Morgan's Laws and the Heine-Borel Theorem; it is Hausdorff since any $p \in \mathbb{C}$ can be separated from ∞ by neighborhoods $B(p,r)$ and $\overline{B(p,r)}^c \cup \{\infty\}$, respectively; and it is second-countable since we may append, to any countable basis for the standard topology of $\mathbb{C}$, the countable collection $\left\{\overline{B(0,r)}^c \cup \{\infty\}\right\}_{r \in \mathbb{Q}^+}$. To give $\mathbb{C}_\infty$ a complex structure, we employ two charts

$$
\begin{aligned}
U_1 &\coloneqq \mathbb{C}_\infty \setminus \{\infty\} = \mathbb{C} & \varphi_1 &: U_1 \to \mathbb{C} : z \mapsto z \qquad (\varphi_1 \coloneqq \mathrm{id}_{\mathbb{C}}) \\
U_2 &\coloneqq \mathbb{C}_\infty \setminus \{0\} = \mathbb{C}^* \cup \{\infty\} & \varphi_2 &: U_2 \to \mathbb{C} : z \mapsto \begin{cases} 1/z & \text{if } z \in \mathbb{C}^* \\ 0 & \text{else.} \end{cases}
\end{aligned}
$$

Clearly φ_1 is a homeomorphism. Since φ_2 is invertible with $\varphi_2^{-1}(z) \coloneqq 1/z$ for all $z \in \mathbb{C}^*$ and $\varphi_2^{-1}(0) \coloneqq \infty$, and

$$
\lim_{z \to \infty} \varphi_2(z) = 0 = \varphi_2(\infty) \qquad \text{and} \qquad \lim_{z \to 0} \varphi_2^{-1}(z) = \infty = \varphi_2^{-1}(0),
$$

we see that φ_2 is a homeomorphism too. Furthermore, $\varphi_2 \circ \varphi_1^{-1} : \mathbb{C}^* \to \mathbb{C}^* : z \mapsto 1/z$ is holomorphic, so the atlas $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$ defines a complex structure on $\mathbb{C}_\infty$. ♦

Example 1.6 (Stereographic Projection)**.** Consider the unit sphere $S^2 \subseteq \mathbb{R}^3$ as a topological subspace of $\mathbb{R}^3$, which makes it a second-countable Hausdorff space. Letting (x, y, w) be the standard coordinates of $\mathbb{R}^3$ and identifying the plane $w = 0$ as $\mathbb{C}$, we employ the charts

$$
\begin{aligned}
U_1 &\coloneqq S^2 \setminus \{(0,0,1)\} & \varphi_1 &: U_1 \to \mathbb{C} : (x,y,w) \mapsto \frac{x + iy}{1 - w} \\
U_2 &\coloneqq S^2 \setminus \{(0,0,-1)\} & \varphi_2 &: U_2 \to \mathbb{C} : (x,y,w) \mapsto \frac{x - iy}{1 + w}.
\end{aligned}
$$

Clearly φ_1 and φ_2 are continuous, and it can be verified that they are invertible with continuous inverses

$$
\varphi_1^{-1}(z) \coloneqq \left(\frac{2\,\mathrm{Re}\, z}{|z|^2 + 1}, \frac{2\,\mathrm{Im}\, z}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right) \qquad \text{and} \qquad \varphi_2^{-1}(z) \coloneqq \left(\frac{2\,\mathrm{Re}\, z}{|z|^2 + 1}, \frac{-2\,\mathrm{Im}\, z}{|z|^2 + 1}, \frac{1 - |z|^2}{|z|^2 + 1} \right).
$$

Observe that $U_1 \cap U_2 = S^2 \setminus \{(0,0,\pm 1)\}$ and $\varphi_2 \circ \varphi_1^{-1} : \mathbb{C}^* \to \mathbb{C}^* : z \mapsto 1/z$, which is holomorphic, so the atlas $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$ defines a complex structure on $\hat{\mathbb{C}}$. ♦

Example 1.7 (Complex Projective Line)**.** Consider the equivalence relation $\sim$ on $\mathbb{C}^2 \setminus \{(0,0)\}$ defined by $(z_1, w_1) \sim (z_2, w_2)$ iff $(z_1, w_1) = \lambda(z_2, w_2)$ for some $\lambda \in \mathbb{C}^*$. Set $\mathbb{P}^1 \coloneqq \left(\mathbb{C}^2 \setminus \{(0,0)\}\right) / \sim$ and equip it with the quotient topology. Since $\sim$ is an open equivalence relation[2] whose graph is closed in $\left(\mathbb{C}^2 \setminus \{(0,0)\}\right)^2$, we see that $\mathbb{P}^1$ is a second-countable Hausdorff space. Denoting the equivalence class of (z, w) by $[z : w]$, we employ the charts

$$
\begin{aligned}
U_1 &\coloneqq \mathbb{P}^1 \setminus \{[0 : w] \mid w \in \mathbb{C}\} & \varphi_1 &: U_1 \to \mathbb{C} : [z : w] \mapsto w/z \\
U_2 &\coloneqq \mathbb{P}^1 \setminus \{[z : 0] \mid z \in \mathbb{C}\} & \varphi_2 &: U_2 \to \mathbb{C} : [z : w] \mapsto z/w.
\end{aligned}
$$

Clearly φ_2 and φ_2 are continuous, and it is easily verified that they are invertible with continuous inverses

$$
\varphi_1^{-1}(z) \coloneqq [1 : z] \qquad \text{and} \qquad \varphi_2^{-1}(z) \coloneqq [z : 1].
$$

Furthermore, $\varphi_2 \circ \varphi_1^{-1} : \mathbb{C}^* \to \mathbb{C}^* : z \mapsto 1/z$ is holomorphic, so the atlas $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$ defines a complex structure on $\mathbb{P}^1$. ♦

1.1.2 Complex Tori

Recall that a torus is any manifold homeomorphic to $T^2 \coloneqq S^1 \times S^1$, which admit representations as quotients $\mathbb{C}/\Gamma$ by lattices $\Gamma \coloneqq \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ for any linearly independent vectors $\omega_1, \omega_2 \in \mathbb{C}$ over $\mathbb{R}$. By definition, there is only one torus up to homeomorphism, but it turns out that we can equip it with many different complex structures. They arise from quotienting $\mathbb{C}$ by different lattices, and we shall derive a criterion on the lattices $\Gamma_1 \coloneqq \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ and $\Gamma_2 \coloneqq \mathbb{Z}\eta_1 \oplus \mathbb{Z}\eta_2$ for the tori $\mathbb{C}/\Gamma_1$ and $\mathbb{C}/\Gamma_2$ to be biholomorphic.

Example 1.8 (Complex Tori)**.** Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$ and consider the lattice $\Gamma \coloneqq \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$. Identifying S^1 with the unit circle in $\mathbb{C}$, the quotient $\mathbb{C}/\Gamma$ is a torus in the topological sense since the map

$$
\varphi : \mathbb{C}/\Gamma \to S^1 \times S^1 \qquad \text{mapping} \qquad [z] \mapsto \left(e^{2\pi i \lambda_1}, e^{2\pi i \lambda_2}\right)
$$

where $z = \lambda_1\omega_1 + \lambda_2\omega_2$ for unique $\lambda_1, \lambda_2 \in \mathbb{R}$, is a homeomorphism. Indeed, φ is well-defined since for any $\lambda_1\omega_1 + \lambda_2\omega_2 \sim \mu_1\omega_1 + \mu_2\omega_2$ in $\mathbb{C}$, we have $(\lambda_1 - \mu_1)\omega_1 + (\lambda_2 - \mu_2)\omega_2 \in \Gamma$ and so $\lambda_i - \mu_i \in \mathbb{Z}$ for $i = 1, 2$. The fact that it is a homeomorphism is clear. This makes $\mathbb{C}/\Gamma$ a second-countable Hausdorff space, which we now endow with the following complex structure.

Since Γ is discrete, there exists some $\varepsilon > 0$ such that $\varepsilon < |\omega|/2$ for every non-zero $\omega \in \Gamma$.[3] Fix any such ε, which ensures that

[2] See [Tu10, Section 7.5] for details on the quotient topology and open equivalence relations.
[3] This exposition follows [Mir95, Section I.2].

no two points in any open ball with radius ε can be equivalent. Indeed, take any $z \in \mathbb{C}$ and $w_1, w_2 \in B(z, \varepsilon) =: V_z$. For $w_1 \sim w_2$, we need some $n, m \in \mathbb{Z}$ such that $w_1 - w_2 = n\omega_1 + m\omega_2$. But

$$|w_1 - w_2| \leq |z - w_1| + |z - w_2| < 2\varepsilon < |n\omega_1 + m\omega_2|$$

for any $n, m \in \mathbb{Z}$, so this is impossible. Fixing any such ε gives us a family $\{V_z\}_{z \in \mathbb{C}}$ of open sets in $\mathbb{C}$ for which the projections $\pi|_{V_z} : V_z \to \pi(V_z)$ are homeomorphisms. Letting $U_z := \pi(V_z)$ and $\varphi_z : U_z \to V_z$ be the inverse of $\pi|_{V_z}$, we obtain complex charts (U_z, φ_z) for all $z \in \mathbb{C}$. We claim that the collection $\mathfrak{A} := \{(U_z, \varphi_z)\}_{z \in \mathbb{C}}$ form an atlas, for which it suffices to take $(U_1, \varphi_1), (U_2, \varphi_2) \in \mathfrak{A}$ and show that the transition map $T := \varphi_2 \circ \varphi_1^{-1} : \varphi_1(U) \to \varphi_2(U)$, where $U := U_1 \cap U_2$, is holomorphic. Observe that the diagram

$$\begin{tikzcd}
& U \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] & \\
V_1 = \varphi_1(U) \arrow[ur, bend left, "\pi|_{V_1}"] \arrow[rr, "T"'] & & \varphi_2(U) = V_2 \arrow[ul, bend right, "\pi|_{V_2}"']
\end{tikzcd}$$

commutes, so $\pi|_{V_2} \circ T = \pi|_{V_1}$ on $\varphi_1(U)$. Then $\pi(T(z)) = \pi(z)$ for every $z \in \varphi_1(U)$, so $T(z) \sim z$ and hence $\ell(z) := T(z) - z \in \Gamma$. This holds for all $z \in \varphi_1(U)$, so we obtain a continuous function $\ell : \varphi_1(U) \to \Gamma : z \mapsto T(z) - z$. Note that $\Gamma \subseteq \mathbb{C}$ is equipped with the subspace topology, but since it is discrete, every $L \subseteq \Gamma$ is open. In particular, fix $z_0 \in \varphi_1(U)$ and set $\gamma_0 := T(z_0) - z_0$. With $L := \{\gamma_0\}$, continuity of ℓ shows that $\ell^{-1}(L)$ is open. Thus $\ell(B(z_0, \delta_1)) \subseteq \{\gamma_0\}$ for some $\delta_1 > 0$, so $\ell(w) = \gamma_0$ for all $w \in B(z_0, \delta_1)$. Thus $T(z) = z + \gamma_0$ for all z in a local neighborhood around z_0, so T is locally biholomorphic. Repeating this for all $z_0 \in \varphi_1(U)$, we see that T is holomorphic on $\varphi_1(U)$. ♦

1.2 Maps on Riemann Surfaces

We extend the notions of holomorphic and meromorphic functions from complex analysis to Riemann surfaces. We also define holomorphic maps *between* Riemann surfaces, which formalizes what we mean for two Riemann surfaces to be 'the same'. Lastly, we study the connection between meromorphic functions $f : X \to \mathbb{C}$ and their associated holomorphic maps $F : X \to \hat{\mathbb{C}}$.

1.2.1 Holomorphic Functions and Maps

Definition 1.9. *Let X be a Riemann surface and let $W \subseteq X$ be open. For a fixed $p \in W$, a function $f : W \to \mathbb{C}$ is said to be holomorphic at p if there exists a chart (U, φ) of X containing p such that $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{C}$ is holomorphic at $\varphi(p)$. If f is holomorphic at every point of W, then f is said to be holomorphic on W.*

Remark. It must be checked that 'being holomorphic' does not depend on the choice of chart. This is indeed the case, for if (V, ψ) is another chart containing p, then the diagram

$$\begin{tikzcd}
\varphi(U \cap V) \arrow[drr, bend left, "f \circ \varphi^{-1}"] & & \\
& U \cap V \arrow[ul, "\varphi"'] \arrow[dl, "\psi"] \arrow[r, "f"] & \mathbb{C} \\
\psi(U \cap V) \arrow[urr, bend right, "f \circ \psi^{-1}"'] & &
\end{tikzcd} \tag{1.1}$$

commutes. Thus $f \circ \psi^{-1} = \left(f \circ \varphi^{-1}\right) \circ \left(\varphi \circ \psi^{-1}\right)$, and since the transition map $\varphi \circ \psi^{-1}$ is holomorphic, we see that $f \circ \varphi^{-1}$ is holomorphic at $\varphi(p)$ iff $f \circ \psi^{-1}$ is holomorphic at $\psi(p)$, as desired. ♦

Example 1.10. Some elementary examples of holomorphic functions.

- If $X = \mathbb{C}$ with the standard chart $(\mathbb{C}, \mathrm{id}_{\mathbb{C}})$, then any holomorphic function $f : W \to \mathbb{C}$ from an open set $W \subseteq \mathbb{C}$ is holomorphic in the classical sense.
- Any chart map $\varphi : U \to \mathbb{C}$ of a Riemann surface is (tautologically) holomorphic in the above sense.
- If $f, g : W \to \mathbb{C}$ are both holomorphic at some $p \in W$, then so are $f \pm g$, $f \cdot g$, and λf for any $\lambda \in \mathbb{C}$. This makes the set $\mathscr{O}(W)$ of all holomorphic functions $f : W \to \mathbb{C}$ into a $\mathbb{C}$-algebra. Lastly, if $g(p) \neq 0$, then f/g is also holomorphic at p. ♦

Definition 1.11. *Let X and Y be Riemann surfaces and let $W \subseteq X$ be open. For a fixed $p \in W$, a mapping $F : W \to Y$ is said to be holomorphic at p if there exists a chart (U, φ) of X containing p and a chart (V, ψ) of Y containing $F(p)$ such that $\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is holomorphic at $\varphi(p)$. If F is holomorphic at every point of W, then F is holomorphic on W.*

Remark. For $Y := \mathbb{C}$ regarded as a Riemann surface, this definition agrees with the above. Again, we must check that 'being holomorphic' is well-defined, but it follows from the commutativity of the diagram below and a similar argument as above.

$$\begin{tikzcd}
\varphi_1(U_1 \cap U_2) \arrow[rrr, bend left, "\psi_1 \circ F \circ \varphi_1^{-1}"] \arrow[dd, "\varphi_2 \circ \varphi_1^{-1}"'] & & & \psi_1(V_1 \cap V_2) \arrow[dd, "\psi_2 \circ \psi_1^{-1}"] \\
& U_1 \cap U_2 \arrow[ul, "\varphi_1"'] \arrow[dl, "\varphi_2"] \arrow[r, "F"] & V_1 \cap V_2 \arrow[ur, "\psi_1"] \arrow[dr, "\psi_2"'] & \\
\varphi_2(U_1 \cap U_2) \arrow[rrr, bend right, "\psi_2 \circ F \circ \varphi_2^{-1}"'] & & & \psi_2(V_1 \cap V_2)
\end{tikzcd}$$

We make the convention that lower-case letters $f, g, h, \ldots$ are *functions* from a Riemann surface into $\mathbb{C}$, while upper-case letters $F, G, H, \ldots$ are *maps* between Riemann surfaces. ♦

Example 1.12. For a Riemann surface X, the identity id_X is a holomorphic map. Furthermore, for all Riemann surfaces X, Y, and Z, and all holomorphic maps $F : X \to Y$ and $G : Y \to Z$, the composite $G \circ F : X \to Z$ is also a holomorphic map. Note that if $F : X \to Y$ is an invertible holomorphic map, then its inverse $F^{-1} : Y \to X$ is also holomorphic. Indeed, if (U, φ) and (V, ψ) are charts around p and $F(p)$, respectively, making $\psi \circ F \circ \varphi^{-1}$ holomorphic (in the classical sense) at $\varphi(p)$, then its inverse $\varphi \circ F^{-1} \circ \psi^{-1}$ is also holomorphic (in the classical sense) at $\psi(p)$. Thus F^{-1} is holomorphic at $F(p)$, as desired, and justifies the following definition. ♦

Definition 1.13. *Let X and Y be Riemann surfaces. A biholomorphism between X and Y is an invertible holomorphic map $F : X \to Y$. Two Riemann surfaces X and Y are said to be biholomorphic if there exists a biholomorphism $F : X \to Y$.*

Example 1.14 (Biholomorphisms between Riemann spheres)**.** Let $\mathbb{C}_\infty$, S^2, and $\mathbb{P}^1$ denote the three constructions for the Riemann sphere $\hat{\mathbb{C}}$ presented in Examples 1.5, 1.6, and 1.7, respectively. We claim that the maps

$$F : S^2 \to \mathbb{P}^1 : (x, y, w) \mapsto [1 - w : x + iy] \qquad \text{and} \qquad G : S^2 \to \mathbb{C}_\infty : (x, y, w) \mapsto \begin{cases} \dfrac{x + iy}{1 - w} & \text{if } w \neq 1 \\ \infty & \text{else} \end{cases}$$

are biholomorphisms, which shows that all three constructions are biholomorphic. Indeed F is holomorphic since with the charts

$$U := S^2 \setminus \{(0, 0, 1)\} \qquad \varphi : U \to \mathbb{C} : (x, y, w) \mapsto \frac{x + iy}{1 - w}$$
$$V := \mathbb{P}^1 \setminus \{[0 : w] \mid w \in \mathbb{C}\} \qquad \psi : V \to \mathbb{C} : [z : w] \mapsto \frac{w}{z},$$

we see that

$$\left(\psi \circ F \circ \varphi^{-1}\right)(z) = \psi\left(F\left(\frac{2\operatorname{Re} z}{|z|^2 + 1}, \frac{2\operatorname{Im} z}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1}\right)\right) = \psi\left(\left[1 - \frac{|z|^2 - 1}{|z|^2 + 1} : \frac{2z}{|z|^2 + 1}\right]\right) = \psi([1 : z]) = z$$

for all $z \in \varphi(U) = \mathbb{C}$, which is clearly holomorphic. Furthermore, it can be checked that F is invertible with a well-defined inverse

$$F^{-1}([z : w]) := \frac{\left(2\operatorname{Re}(z\overline{w}), 2\operatorname{Im}(z\overline{w}), |z|^2 - |w|^2\right)}{|z|^2 + |w|^2},$$

so F is a biholomorphism. For G, we take the same chart (U, φ) as above and choose $V := \mathbb{C}_\infty \setminus \{\infty\} = \mathbb{C}$ and $\psi := \mathrm{id}_{\mathbb{C}}$. A similar calculation then shows that $\left(\psi \circ G \circ \varphi^{-1}\right)(z) = z$ for all $z \in \varphi(U) = \mathbb{C}$, and that G is invertible with inverse $G^{-1}(z) := \varphi^{-1}(z)$ if $z \in \mathbb{C}$ and $G^{-1}(\infty) := (0, 0, 1)$. ♦

Proposition 1.15. *Any holomorphic function $f : X \to \mathbb{C}$ on a compact Riemann surface X is constant.*

Proof. Since f is holomorphic, the function $|f| : X \to \mathbb{R}$ defined by $|f|(x) := |f(x)|$ is continuous on X. But X is compact, so $|f|$ achieves its maximum at some point $p \in X$. Choosing a connected chart (U, φ) centered at p, we see that $f \circ \varphi : U \to \mathbb{C}$ is holomorphic. Then $|f \circ \varphi| : U \to \mathbb{R}$ has a local maximum at 0, so, since U is connected, $f \circ \varphi$ is constant by the Maximum Principle. Then f is locally constant around p, so, since X is connected, f is constant on X. ■

1.2.2 Singularities of Functions

Throughout this section, let X be a Riemann surface, let $p \in X$, and let $f : W \to \mathbb{C}$ be defined and holomorphic on a punctured neighborhood W of p. As above, we can study the behaviour of f at p from its chart representation $f \circ \varphi^{-1}$.

Definition 1.16. *Let $f : W \to \mathbb{C}$ be a holomorphic function on a punctured neighborhood of p. We say that f has a removable singularity (resp. pole, essential singularity) at p if there exists a chart (U, φ) of X containing p such that $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{C}$ has a removable singularity (resp. pole, essential singularity) at $\varphi(p)$.*

Proof. (Well-definedness). The commutativity of the diagram in Equation (1.1) shows that those notions are chart independent; the composition of $f \circ \varphi^{-1}$ having a singularity at p with a transition map that is holomorphic at p yields a function with the same type of singularity at p. ■

Remark. Functions having an essential singularity at p are very ill-behaved. Indeed, this occurs iff $|f(x)|$ has a non-zero oscillation near p. Other singularities behave much better:

- A removable singularity occurs iff $|f(x)|$ is bounded in a neighborhood of p, and can be 'filled in' by defining $\widetilde{f}(p) := \lim_{x \to p} f(x)$. This extends, by Riemann's Removable Singularities Theorem, to a holomorphic function $\widetilde{f} : W \cup \{p\} \to \mathbb{C}$.
- A pole occurs iff $|f(x)| \to \infty$ as $x \to p$, which can also be 'filled in' by defining the map

$$F : W \to \hat{\mathbb{C}} \qquad \text{mapping} \qquad x \mapsto \begin{cases} \infty & \text{if } x = p \\ f(x) & \text{else} \end{cases}$$

  that extends the codomain of f to the Riemann sphere[4] $\hat{\mathbb{C}}$; since $|f(x)| \to \infty$ as $x \to p$, we see that F is continuous. To show that F is holomorphic, let (U, φ) and (V, ψ) be charts around x and $F(x)$, respectively. Since f is holomorphic on $U \setminus \{p\}$, we see that $\psi \circ F \circ \varphi^{-1}$ is holomorphic on $\varphi(U) \setminus \varphi(p)$. Observe that $\varphi(p)$ is a removable singularity of $\psi \circ F \circ \varphi^{-1}$, which can be extended as above to make $\psi \circ F \circ \varphi^{-1}$ holomorphic on $\varphi(U)$.

Thus we see that every such function $f : W \to \mathbb{C}$ having pole at p can be holomorphically extended to a map $F : W \to \hat{\mathbb{C}}$. Conversely, every holomorphic map $F : W \to \hat{\mathbb{C}}$ (that is not identically infinity) can be regarded as a function $f : W \setminus F^{-1}(\infty) \to \mathbb{C}$ that is holomorphic everywhere except where $F(x) = \infty$, in which case it has a pole. This motivates the following definition. ♦

[4] Here, we consider $\hat{\mathbb{C}} = \mathbb{C}_\infty$.

Definition 1.17. *A function $f : W \to \mathbb{C}$ is said to be meromorphic at p if it does not have an essential singularity at p; that is, if it is either holomorphic, has a removable singularity, or has a pole at p. If f is meromorphic at every point of W, then f is meromorphic on W.*

Remark. The previous remark can now be rephrased by saying that the set of all meromorphic functions $f : W \to \mathbb{C}$ are in one-to-one correspondence with the set of all holomorphic maps $F : W \to \hat{\mathbb{C}}$ (which are not identically infinity). That is, meromorphic functions are the holomorphic maps to the Riemann sphere. ♦

Example 1.18. As in Example 1.10, if $f, g : W \to \mathbb{C}$ are both meromorphic at p, then so are $f \pm g$ and $f \cdot g$. Furthermore, g is not identically 0, then so is f/g. This makes the set $\mathscr{M}(W)$ of all meromorphic functions $f : W \to \mathbb{C}$ into a field. ♦

Proposition 1.19. *Every meromorphic function on $\hat{\mathbb{C}}$ is a rational function.*

Proof. Let $f : \hat{\mathbb{C}} \to \mathbb{C}$ be meromorphic. Since $\hat{\mathbb{C}}$ is compact, the discreteness of poles imply that f has finitely-many poles. Without loss of generality, assume that ∞ is not a pole of f (since we may consider $1/f$ instead). Now, for each pole $\lambda_i \in \mathbb{C}$ of f, consider its principal part

$$h_i(z) = \sum_{j=-m_i}^{-1} c_{ij} (z - \lambda_i)^j$$

for some $m_i > 1$. Then the function $g := f - \sum_i h_i$ is holomorphic function on $\hat{\mathbb{C}}$, and since $\hat{\mathbb{C}}$ is compact, it is constant by Proposition 1.15. Thus $f = g + \sum_i h_i$, which is a rational function. ■

Definition 1.20. *Let $f : W \to \mathbb{C}$ be meromorphic at p and consider its Laurent series $f_\varphi(z) := \left(f \circ \varphi^{-1}\right)(z) = \sum_i c_i (z - z_0)^i$ under a chart (U, φ) of X with $z_0 := \varphi(p)$. The order of f at p is*

$$\operatorname{ord}_p(f) := \min\left\{n \in \mathbb{Z} \,|\, 0 \neq (z - z_0)^n f_\varphi(z) \in \mathscr{O}(W)\right\}.$$

Remark. Note that f, being meromorphic, ensures that its Laurent series has finitely-many negative terms, so the definition makes sense. If f is not meromorphic, we take $\operatorname{ord}_p(f) := \infty$. ♦

Proof. (Well-definedness). Let z be the local coordinates given by (U, φ) and suppose that (V, ψ) is another chart with $w_0 := \psi(p)$ giving another local coordinate w. Then the transition function $T := \varphi \circ \psi^{-1}$ is holomorphic, so it admits a power series representation

$$z = T(w) = \sum_{n \geq 0} a_n (w - w_0)^n = z_0 + \sum_{n \geq 1} a_n (w - w_0)^n.$$

Since $T'(w_0) \neq 0$, we see that $a_1 \neq 0$. Suppose now that the Laurent series of f at p in the coordinate z is $c_{-n_0}(z - z_0)^{-n_0}$ + higher order terms, so that the order of f at p computed by employing z is n_0. Then the Laurent series of f at p in the coordinate w is

$$c_{-n_0}\left(\sum_{n \geq 1} a_n (w - w_0)^n\right)^{-n_0} + \text{ higher order terms},$$

whose lowest order term is $c_{-n_0} a_1^{-n_0} (w - w_0)^{-n_0}$. Observe that $b_{-n_0} := c_{-n_0} a_1^{-n_0} \neq 0$, so the order of f at p computed via w is also n_0. ■

Remark. The arithmetic of ord_p is straightforward. Indeed, if $f, g : W \to \mathbb{C}$ are meromorphic at p, then

- $\operatorname{ord}_p(fg) = \operatorname{ord}_p(f) + \operatorname{ord}_p(g)$.
- $\operatorname{ord}_p(f/g) = \operatorname{ord}_p(f) - \operatorname{ord}_p(g)$, if $g \neq 0$.
- $\operatorname{ord}_p(f \pm g) \geq \min\left\{\operatorname{ord}_p(f), \operatorname{ord}_p(g)\right\}$. ♦

The order $\operatorname{ord}_p(f)$ can be used to classify the behaviour of f at p. Indeed, it is readily verified that f is holomorphic at p iff $\operatorname{ord}_p(f) \leq 0$, in which case $f(p) = 0$ iff $\operatorname{ord}_p(f) < 0$. Similarly, f has a pole at p iff $\operatorname{ord}_p(f) > 0$, so f has neither a zero nor a pole at p iff $\operatorname{ord}_p(f) = 0$. This motivates the following definition. ♦

Definition 1.21. *Let $f : W \to \mathbb{C}$ be meromorphic at p. We say that f has a pole of order n at p if $\operatorname{ord}_p(f) = n > 0$, and a zero of order n at p if $\operatorname{ord}_p(f) = n < 0$.*

Example 1.22. Let $f : \hat{\mathbb{C}} \to \mathbb{C}$ be meromorphic, so $f(z) = p(z)/q(z)$ for some $p, q \in \mathbb{C}[z]$. Then f is holomorphic at all points $z \in \mathbb{C}$ such that $q(z) \neq 0$, and has a pole otherwise. Also, $f(\infty) \in \mathbb{C}$ if $\deg p = \deg q$, vanishes if $\deg p < \deg q$, and has a pole otherwise. In any case, f is meromorphic on $\hat{\mathbb{C}}$. To compute $\operatorname{ord}_z(f)$ at all $z \in \hat{\mathbb{C}}$, we split p and q into linear factors to write f uniquely as

$$f(z) = c \prod (z - \lambda_i)^{\alpha_i}$$

where $c \neq 0$ and each λ_i is distinct. Fix i. Setting $g_j(z) := (z - \lambda_j)^{\alpha_j}$ for all j, we see that $\operatorname{ord}_{\lambda_i}(g_i) = -\alpha_i$ and $\operatorname{ord}_{\lambda_j}(g_i) = 0$ for all $i \neq j$. Thus

$$\operatorname{ord}_{\lambda_i}(f) = \sum_j \operatorname{ord}_{\lambda_i}(g_j) = -\alpha_i.$$

Moreover, if $\alpha_i > 0$ (resp. $\alpha_i < 0$), then g_i has a pole (resp. zero) of order $|\alpha_i|$ at ∞. It follows then that $\operatorname{ord}_\infty(g_i) = \alpha_i$, so

$$\operatorname{ord}_\infty(f) = \sum_i \operatorname{ord}_\infty(g_i) = \sum_i \alpha_i.$$

Lastly, it is clear that $\operatorname{ord}_z(f) = 0$ for all $z \neq \lambda_i, \infty$. ♦

Remark. Thus if f is a meromorphic function on $\hat{\mathbb{C}}$, then $\sum_{z \in \hat{\mathbb{C}}} \operatorname{ord}_z(f) = 0$. In fact, this holds for all compact Riemann surfaces, which we prove in TODO. ♦

1.2.3 Local Normal Form and the Multiplicity

Holomorphic maps have some remarkable 'local' properties, one of which, called the *Local Normal Form*, is presented here. Roughly speaking, every holomorphic map $F : X \to Y$ looks locally like a power map $z \mapsto z^m$ for some unique $m \geq 1$. Summing this local invariant over the fiber $F^{-1}(q)$ for any $q \in Y$ gives us the *degree of* F, a global invariant independent of q, which we prove in Section 2.1 using tools from covering spaces.

Theorem 1.23 (Local Normal Form)**.** *Let X and Y be Riemann surfaces and let $F : X \to Y$ be a non-constant holomorphic map. Then, for every $p \in X$, there exists a unique $m \geq 1$ such that for any chart (U_2, φ_2) of Y centered at $F(p)$, there exists a chart (U_1, φ_1) of X centered at p such that $\varphi_2 \circ F \circ \varphi_1^{-1} : z \mapsto z^m$ for all $z \in \varphi_1(U_1)$.*

Proof. Let (U_2, φ_2) be a chart of Y centered at $F(p)$ and consider any chart (V, ψ) of X centered at p. Then the function $h := \varphi_2 \circ F \circ \psi^{-1}$ is holomorphic, so it admits a power series representation $h(w) = \sum_{i=0}^{\infty} c_i w^i$ for all $w \in \psi(V)$. Note that $h(0) = \varphi_2(F(p)) = 0$, so $c_0 = 0$. Let $m \geq 1$ be the smallest integer such that $c_m \neq 0$, so

$$h(w) = \sum_{i \geq m} c_i w^i = w^m \sum_{i \geq 0} c_{i-m} w^i =: w^m g(w).$$

Then g is holomorphic at 0 with $g(0) = c_m \neq 0$, so there is a function r holomorphic on some neighborhood W of 0 such that $(r(w))^m = g(w)$ for all $w \in W$. Thus $h(w) = (wr(w))^m$, so set $\eta(w) := wr(w)$ for all $w \in W$. Note that $\eta'(0) = r(0) \neq 0$, so η is invertible on some neighborhood $W' \subseteq W$ of 0. Set $U_1 := \psi^{-1}(W')$ and $\varphi_1 := \eta \circ \psi$. Then (U_1, φ_1) is a chart of X centered at p such that

$$\left(\varphi_2 \circ F \circ \varphi_1^{-1}\right)(z) = \left(\varphi_2 \circ F \circ \psi^{-1} \circ \eta^{-1}\right)(z) = h\left(\eta^{-1}(z)\right) = \left[\eta\left(\eta^{-1}(z)\right)\right]^m = z^m$$

for all $z \in \varphi_1(U_1)$. To show uniqueness, it suffices to show that such an m is chart-independent. But this is clear, for if a different chart U_2' is chosen such that F acts as $z \mapsto z^n$ for some neighborhood U_1' of p, then $z^n = z^m$ on $\varphi_1(U_1) \cap \varphi_1'(U_1')$ forces $n = m$. ■

Definition 1.24. *With the above notation, the unique $m \geq 1$ such that there are local coordinates around p and $F(p)$ where F acts like $z \mapsto z^m$ is called the multiplicity of F at p, denoted* $\mathrm{mult}_p(F)$.

Remark. Consider the power function $f(z) := z^m$ where $m := \mathrm{mult}_p(F)$. Then, for all $z \in \mathbb{C}^*$, we see that $f^{-1}(z)$ has exactly m elements given by the m distinct m^{th} roots of z^m. Thus the map f causes $\mathbb{C}$ to 'cover itself m times', and those coverings meet at the fixed point 0. But $f^{-1}(0) = \{0\}$ has only 1 element, which prevents f to be a m-sheeted covering of $\mathbb{C}$. To remedy this, we count 0 *with multiplicity*; see Chapter 2 for a more formal discussion. Since F is locally represented by f, and (U_1, φ_1) is centered at p, we see that m counts the multiplicity at which neighbors of p are mapped to $F(p)$. ♦

Remark. This theorem also give easy proofs of some elementary properties of holomorphic maps, which we collect here; see [For81, Section 1.2] for details. Throughout, $F : X \to Y$ is a non-constant holomorphic map between Riemann surfaces X and Y.

- F is an open map.
- If F is injective, then it is biholomorphic onto its image.
- If $Y = \mathbb{C}$, then $|F|$ does not attain its maximum.
- If X is compact, then F is surjective and Y is compact.

Together, the last two claims give an alternative proof for Proposition 1.15. ♦

Remark. We give a simple way of computing $\mathrm{mult}_p(F)$ that does not involve casting F into Local Normal Form, or even having to find local coordinates centered at p and $F(p)$. Indeed, let (U_1, φ_1) and (U_2, φ_2) be charts around p and $F(p)$, say with $z_0 := \varphi_1(p)$ and $w_0 := \varphi_2(F(p))$. Letting $f := \varphi_2 \circ F \circ \varphi_1^{-1}$, we see that $f(z_0) = w_0$ and hence its power series representation has the form

$$f(z) = f(z_0) + \sum_{i \geq m} c_i (z - z_0)^i$$

for some $m \geq 1$ with $c_m \neq 0$. Then, since $z - z_0$ and $w - w_0 = f(z) - f(z_0)$ are local coordinates centered at p and $F(p)$, respectively, we see from the above proof that $\mathrm{mult}_p(F) = m$. Thus to compute $\mathrm{mult}_p(F)$, it suffices to case F into local coordinates (U_1, φ_1) around p and (U_2, φ_2) around $F(p)$ and find the lowest non-zero power of the Taylor series of $f := \varphi_2 \circ F \circ \varphi_1^{-1}$. ♦

Proposition 1.25. *Let f be a meromorphic function on a Riemann surface X and let $F : X \to \hat{\mathbb{C}}$ be its associated holomorphic map. Fix $p \in X$.*

- *If p is a not a pole of f, then* $\mathrm{mult}_p(F) = -\,\mathrm{ord}_p(f - f(p))$.
- *If p is a pole of f, then* $\mathrm{mult}_p(F) = \mathrm{ord}_p(f)$.

Proof. Suppose that p is not a pole of f, so $f(p) = F(p) \in \mathbb{C}$. Since the set of all poles of a meromorphic function forms a discrete set, let $p \in U \subseteq X$ be small enough so that $f|_U$ is holomorphic. Let (U, φ) be a chart of X and consider the chart $(\mathbb{C}, \psi)$ of $\hat{\mathbb{C}}$ around $F(p)$ defined by $\psi(z) := z - F(p)$. Then $f - f(p) = \psi \circ F$ on U, so

$$(f - f(p))_\varphi := (f - f(p)) \circ \varphi^{-1} = \psi \circ F \circ \varphi^{-1}$$

on $\varphi(U)$. Expanding in power series around $z_0 := \varphi(p) \in \varphi(U)$, we see that

$$\left(\psi \circ F \circ \varphi^{-1}\right)(z) = (f - f(p))_\varphi(z) = \sum_{i \geq m} c_i (z - z_0)^i$$

for some $m \in \mathbb{N}$ with $c_m \neq 0$. Note that $(f - f(p))_\varphi(z_0) = (f - f(p))(p) = 0$, so $m > 0$ and hence $\operatorname{mult}_p(F) = m$. But m is also the smallest integer such that

$$0 \neq (z - z_0)^{-m} (f - f(p))_\varphi(z) \in \mathscr{O}(U),$$

so $\operatorname{ord}_p(f - f(p)) = -m$. Suppose now that p is a pole of f, so $F(p) = \infty$. Since $\lim_{z \to p} 1/f(z) = 0$, we may let $p \in U \subseteq X$ be small enough so that the function $\widetilde{f} : U \to \mathbb{C}$ defined by

$$\widetilde{f}(x) := \begin{cases} 0 & \text{if } x = p \\ 1/f(x) & \text{else} \end{cases}$$

is holomorphic. Let (U, φ) be a chart of X and consider the chart $\left(\hat{\mathbb{C}} \setminus \{0\}, \psi\right)$ of $\hat{\mathbb{C}}$ defined by $\psi(z) := 1/z$ for all $z \in \mathbb{C}^*$ and $\psi(\infty) := 0$. Then $\widetilde{f} = \psi \circ F$ on U, so $\widetilde{f}_\varphi := \widetilde{f} \circ \varphi^{-1} = \psi \circ F \circ \varphi^{-1}$ on $\varphi(U)$. By the same argument as above, we see that $\operatorname{mult}_p(F) = -\operatorname{ord}_p(\widetilde{f})$. Now $\operatorname{ord}_p(f) = -\operatorname{ord}_p(\widetilde{f})$, so the result follows. ■

Remark. This relation between the order of meromorphic functions and the multiplicity of their associated holomorphic maps will be used to derive a criterion for $F : X \to \hat{\mathbb{C}}$ to be a biholomorphism. In fact, Corollary 2.11.2 states that if X is compact and if there exists a meromorphic function $f : X \to \mathbb{C}$ has a single simple pole, then F is a biholomorphism. ♦

Chapter 2

Covering Spaces

We develop the basics of covering space theory. More specifically, Section 2.1 develops the notion of the *degree* to derive a criterion for a compact Riemann surface X to be biholomorphic to the Riemann sphere $\hat{\mathbb{C}}$, and Section 2.2 studies the liftings of mappings along covering maps. Those tools will be used in Sections 4.1.1 and 4.8 to compute the moduli spaces of S^2 and T^2, respectively. Section 2.2 requires some background on homotopies of paths, for which we refer the reader to [Hat02, Chapter 1].

2.1 Covering Maps and the Degree

We devote this section to develop the tools necessary to define the *degree* of a proper holomorphic map, which, intuitively, is the *number of sheets* in which it covers its image. However, there are points in the image which are not covered 'evenly', so they must be counted with multiplicity.

2.1.1 Proper and Covering Maps

We first gather some basic results on the theory of covering spaces from topology. Throughout this section, let E and X be locally-compact topological spaces. This assumption ensures that proper maps are closed.

Definition 2.1. *A map $\pi : E \to X$ is said to be* <u>proper</u> *if the preimage of every compact set is compact.*

Proposition 2.2. *For a proper map $\pi : E \to X$, every $p \in X$ and every neighborhood V of $\pi^{-1}(p)$ admits a neighborhood U of p such that $\pi^{-1}(U) \subseteq V$.*

Proof. Since $E \setminus V$ is closed and π is proper, we see that $\pi(E \setminus V)$ is closed too. Clearly $p \notin \pi(E \setminus V) =: W$, so $U := X \setminus W$ is a neighborhood of p. Then $\pi^{-1}(U) \subseteq V$, since for all $\pi(\zeta) \in U$, we see that $\pi(\zeta) \notin \pi(E \setminus V)$ and so $\zeta \notin E \setminus V$. ■

Definition 2.3. *A map $\pi : E \to X$ is said to be a* <u>covering map</u> *if every point $p \in X$ admits a neighborhood U such that $\pi^{-1}(U) = \coprod_{j \in J} V_j$ where V_j are disjoint open sets in E, each homeomorphic to U via $\pi|_{V_j}$. In this case, we say that U is* <u>evenly-covered by $\{V_j\}$</u> *and that E is a* <u>covering space of X</u>.

Example 2.4. Let $m \geq 2$ be a natural number and consider the power map $f : \mathbb{C}^* \to \mathbb{C}^*$ mapping $z \mapsto z^m$. We claim that f is a covering map, so take $b \in \mathbb{C}^*$ and let $a \in \mathbb{C}^*$ be any one of its m^{th} roots. Since f is a local homeomorphism, there exist neighborhoods V_0 of a and U of b such that $f|_{V_0} : V_0 \to U$ is a homeomorphism. We claim that

$$f^{-1}(U) = \coprod_{j=0}^{m-1} \omega^j V_0,$$

where ω is an m^{th} root of unity. Indeed, for all $f(c) \in U$, there exists some $a' \in V_0$ such that $f(a') = f(c)$. Then $c = \omega^j a'$ for some $0 \leq j \leq m-1$, so $c \in \omega^j V_0$. Conversely, if $c \in \omega^j V_0$ for some $0 \leq j \leq m-1$, then $c = \omega^j a'$ for some $a' \in V_0$ and hence $f(c) = f(\omega^j a') = f(a') \in U$. Now, since $f^{-1}(b)$ is discrete, the sets $V_j := \omega^j V_0$ can be made small enough so that they are pairwise disjoint. Then each $f|_{V_j} : V_j \to U$ is a homeomorphism, as desired. ♦

Example 2.5. For any lattice $\Gamma \subseteq \mathbb{C}$, the projection $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$ is a covering map. Indeed, take $z + \Gamma \in \mathbb{C}/\Gamma$ and let $w \in \mathbb{C}$ be such that $\pi(w) = z + \Gamma$. Since π is a local homeomorphism[1], there exist neighborhoods V of w and U of $z + \Gamma$ such that $\pi|_V : V \to U$ is a homeomorphism. We similarly claim that

$$\pi^{-1}(U) = \coprod_{\lambda \in \Gamma} (\lambda + V).$$

Indeed, for all $\pi(z) \in U$, there exists some $w' \in V$ such that $\pi(z) = \pi(w')$. Then $z + \Gamma = w' + \Gamma$, so $z = w' + \lambda$ for some $\lambda \in \Gamma$. Conversely, if $z \in \lambda + V$ for some $\lambda \in \Gamma$, then $z = w' + \lambda$ for some $w' \in V$ and hence $\pi(z) = \pi(w' + \lambda) = \pi(w) \in U$. Now, the sets $V_\lambda := \lambda + V$ are all disjoint and each $\pi|_{V_\lambda} : V_\lambda \to U$ is a homeomorphism, as desired. ♦

Proposition 2.6. *Let $\pi : E \to X$ be a covering map. If X is connected, then the fibers $\pi^{-1}(p)$ at each $p \in X$ are equinumerous.*

Proof. Consider the equivalence relation $\sim$ on X defined by $p \sim p'$ iff the fibers over p and p' are equinumerous. We claim that the equivalence classes are all open, and since they partition X, the connectedness of X then shows that there is only one equivalence class. Indeed, take $p \in X$ and let $U \ni p$ be evenly-covered by $\{V_j\}$. For any $p' \in U$, the set $\pi^{-1}(p') \cap V_j$ is a singleton for all $j \in J$, so $|\pi^{-1}(p')| = |J|$. In particular, since $p \in U$, we have $p \sim p'$, as desired. ■

[1] This follows directly from our construction of complex tori in Example 1.8, where for every $w \in \mathbb{C}$, a small enough neighborhood V was found so that $\pi|_V$ is injective.

Proposition 2.7. *Any proper local homeomorphism is a covering map.*

Proof. Let $\pi : E \to X$ be a proper local homeomorphism and take $p \in X$. We claim that $\pi^{-1}(p)$ is finite.

- For each $\zeta \in \pi^{-1}(p)$, there exist neighborhoods W_ζ of ζ and U of p such that $\pi|_{W_\zeta} : W_\zeta \to U$ is a homeomorphism. Then the sets W_ζ must be disjoint, for if $\zeta' \in W_\zeta \cap W_{\zeta'}$ for some $\zeta' \neq \zeta$, then $\pi|_{W_\zeta}(\zeta) = p = \pi|_{W_\zeta}(\zeta')$, contradicting that $\pi|_{W_\zeta}$ is a homeomorphism. Thus $\pi^{-1}(p)$ must be finite, lest the cover $\{W_\zeta\}$ admits no finite subcover.

Thus $\pi^{-1}(p) = \{\zeta_1, \ldots, \zeta_n\}$ for some $\zeta_j \in E$. Letting $W_j := W_{\zeta_j}$ as above, we see that $\coprod_{j=1}^n W_j$ is a neighborhood of $\pi^{-1}(p)$. By Proposition 2.2, there is a neighborhood U of p such that $\pi^{-1}(U) \subseteq \coprod_{j=1}^n W_j$, so $\pi^{-1}(U) = \coprod_{j=1}^n V_j$ where the sets $V_j := W_j \cap \pi^{-1}(U)$ are all disjoint and each $\pi|_{V_j} : V_j \to U$ is a homeomorphism. ∎

2.1.2 Ramification Points and the Degree

Throughout this section, let $F : Y \to X$ be a (non-constant) proper holomorphic map between Riemann surfaces X and Y. We extend Proposition 2.6 to F, which is 'almost' a covering map, and define the *degree* of F.

Definition 2.8. *A point $q \in Y$ is said to be a ramification/branch point of F if $F|_V$ is not injective for any neighborhood V of q, in which case $F(q) \in X$ is said to be a critical point of F. If F has no ramification points, then F is said to be an unbranched holomorphic map.*

Remark. It is immediate that F is unbranched iff it is a local homeomorphism. Indeed, if F is unbranched, then for every $q \in Y$ there exists a neighborhood V of q such that $F|_V$ is injective. By the Open Mapping Theorem, F is open and hence $F|_V$ maps V homeomorphically to the open set $F(V)$. Conversely, if F is a local homeomorphism, then for every $q \in Y$ there exists a neighborhood V of q that is mapped homeomorphically onto an open set in X. Thus $F|_V$ is injective, so F is unbranched at q.

In particular, this shows that every covering map is unbranched. Conversely, Proposition 2.7 shows that every unbranched proper map is a covering map, so all fibers are equinumerous. On the other hand, if F is branched, then it is a covering map over X with all ramification points removed. Including the ramification points, however, the fibers of F are *not necessarily* equinumerous anymore, but the next best thing happens and we only need to count the fibers *with multiplicity*. First, we need a lemma. ♦

Lemma 2.9. *For all $q \in Y$, the map $F : Y \to X$ has a ramification point at q iff $\mathrm{mult}_q(F) \geq 2$.*

Proof. By Theorem 1.23, there exist charts (V, ψ) centered at q and (U, φ) centered at $F(q)$ such that $f := \varphi \circ F \circ \psi^{-1}$ is the power map $z \mapsto z^m$ where $m := \mathrm{mult}_q(F)$. Since φ and ψ are, in particular, injections, we see that F is locally injective at q iff f is locally injective at 0. But this occurs precisely when $m = \mathrm{mult}_q(F) < 2$, so the result follows. ∎

Definition/Theorem 2.10. *The degree of F is the cardinality of any fiber $F^{-1}(p)$ for $p \in X$, counted with multiplicity. That is, $\deg F := \sum_{q \in F^{-1}(p)} \mathrm{mult}_q(F)$ is independent of $p \in X$.*

Proof. For non-critical points $p \in X$, Lemma 2.9 shows that $\mathrm{mult}_q(F) = 1$ for any $q \in F^{-1}(p)$. Then $\deg F = \left|F^{-1}(p)\right|$, and since Proposition 2.7 shows that F is a covering map when all ramification points are removed, it is, by Proposition 2.6, constant over all non-critical points.

Otherwise, let p be a critical point of F. Since $F^{-1}(p)$ is compact, it is finite by Discreteness of Preimages, say $F^{-1}(p) = \{q_1, \ldots, q_n\}$ for $q_i \in Y$. Fix $1 \leq j \leq n$ and set $m_j := \mathrm{mult}_{q_j}(F)$. We claim that there exist neighborhoods V_j of q_j and U_j of p such that $\left|F^{-1}(r) \cap V_j\right| = m_j$ for all $r \in U_j \setminus \{p\}$. Indeed, by Theorem 1.23, there exist charts (V_j, ψ_j) of Y centered at q_j and (U_j, φ_j) of X centered p such that F acts as the power function $f(z) := z^{m_j}$ on $\psi_j(V_j)$. Since the set of ramification points of F is discrete, we may choose U_j small enough so that every $r \in U_j \setminus \{p\}$ is unramified. Take $r \in U_j \setminus \{p\}$ and set $w := \varphi_j(r') \neq 0$. Then $\left|f^{-1}(w)\right| = m_j$, so we have

$$\left|F^{-1}(r) \cap V_j\right| = \left|\psi_j\left(F^{-1}(r)\right)\right| = \left|\psi_j\left(F^{-1}\left(\varphi_j^{-1}(w)\right)\right)\right| = \left|f^{-1}(w)\right| = m_j.$$

Now, since V_j is a neighborhood of q_j, we see that $F^{-1}(U_j) \subseteq V_j$ by restricting U_j in accordance with Proposition 2.2, if necessary. Then, with $U := \bigcap_{i=1}^n U_i$, we see that $F^{-1}(U) \subseteq \coprod_{i=1}^n V_i$ where the sets V_i are all disjoint. Take any $r \in U \setminus \{p\}$. Then $r \in U_i \setminus \{p\}$ for all $1 \leq i \leq n$, so

$$\left|F^{-1}(r)\right| = \left|F^{-1}(r) \cap \bigcup_{i=1}^n V_i\right| = \left|\bigcup_{i=1}^n \left(F^{-1}(r) \cap V_i\right)\right| = \sum_{i=1}^n \left|F^{-1}(r) \cap V_i\right| = \sum_{i=1}^n m_i.$$

But r is not a critical point of F, so $\deg F = \left|F^{-1}(r)\right| = \sum_{i=1}^n m_i$ and the result follows. ∎

Example 2.11. We extend Example 2.4 by considering the same power map $z \mapsto z^m$ for $m \geq 2$, this time as a map $f : \mathbb{C} \to \mathbb{C}$. Away from 0, the map f is a proper local homeomorphism as before, so the cardinality of any fiber is m. At 0, we see that $\mathrm{mult}_0(f) = m \geq 2$, so f has a ramification point at 0. Counting multiplicities, the cardinality of $f^{-1}(0)$ is m, so $\deg f = m$. ♦

Corollary 2.11.1. *If Y is compact, then a holomorphic map $F : Y \to X$ is a biholomorphism iff $\deg F = 1$.*

Proof. Since Y is compact, we see that F is proper surjection. Observe that F is an injective iff it has no critical points, and by Lemma 2.9, this occurs iff $\mathrm{mult}_q(F) = 1$ for all $q \in Y$.

- ($\Rightarrow$) If F is an injection, then $\left|f^{-1}(q)\right| = 1$ for all $q \in Y$. Thus $\deg F = 1$.
- ($\Leftarrow$): Since $\mathrm{mult}_q(F) \geq 1$ for all $q \in Y$, the above theorem forces $\mathrm{mult}_q(F) = 1$. ∎

Corollary 2.11.2. *If X is compact and there exists a meromorphic function $f : X \to \mathbb{C}$ with a single simple pole, then $X \cong \hat{\mathbb{C}}$.*

Proof. Let $f : X \to \mathbb{C}$ be a meromorphic function with only a simple pole at p and consider its associated holomorphic map $F : X \to \hat{\mathbb{C}}$. By Proposition 1.25, we see that $\mathrm{mult}_p(F) = \mathrm{ord}_p(f) = 1$ and hence p is unramified. Since p is the only pole of f, we see that $\deg F = \left|F^{-1}(\infty)\right| = 1$. Thus F is a biholomorphism, as desired. ■

Remark. This criterion finally reduces to problem of showing that the moduli space of S^2 is a point to showing that every Riemann surface X that is topologically the sphere admits a meromorphic function $f : X \to \mathbb{C}$ with a single simple pole. We dedicate Chapter 3 to find such a meromorphic function. ♦

2.2 Liftings along Covering Maps

Using the Homotopy Lifting Property of covering maps, we prove that every map $F : Y \to X$ from a simply-connected space Y admits a lift $\widetilde{F} : Y \to E$ along a covering map $\pi : E \to X$. Unless otherwise stated, X, Y, and E are all topological spaces and all maps are continuous.

Definition 2.12. *Let $\pi : E \to X$ and $F : Y \to X$ be maps. A lift of F (along π) is a map $\widetilde{F} : Y \to E$ such that $\pi \circ \widetilde{F} = F$; that is, such that the diagram below commutes.*

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ Y \arrow[ur, "\widetilde{F}"] \arrow[r, "F"'] & X \end{tikzcd}$$

Remark. If X, Y, and E are all Riemann surfaces and $\pi : E \to X$ is an unbranched holomorphic map, then any lift $\widetilde{F} : Y \to E$ of a holomorphic $F : Y \to X$ is also holomorphic. Indeed, π admits a local inverse χ, which is holomorphic, so $\widetilde{F}$ is locally a composition of a holomorphic map F with χ. ♦

2.2.1 Liftings of Paths and Homotopies

Proposition 2.13 (Homotopy Lifting Property). *If $\pi : E \to X$ is a covering map, then for any homotopy $F : Y \times [0,1] \to X$ and any fixed map $\widetilde{f}_0 : Y \times \{0\} \to E$ lifting the restriction of F on $Y \times \{0\}$, there exists a unique homotopy $\widetilde{F} : Y \times [0,1] \to E$ lifting F that restricts to $\widetilde{f}_0$ on $Y \times \{0\}$. In other words, the following diagram commutes.*

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "\widetilde{f}_0"] \arrow[d, hook, "\iota"'] & E \arrow[d, "\pi"] \\ Y \times [0,1] \arrow[ur, dashed, "\widetilde{F}"] \arrow[r, "F"] & X \end{tikzcd}$$

Proof. Since π is a covering map, there exists an open cover $\{U_\alpha\}$ of X, each evenly-covered by $\{V_{\alpha\beta}\}$. Fix $q_0 \in Y$. For each $(q_0, t_i) \in Y \times [0,1]$, let $U_i \subseteq X$ be an open set containing $F(q_0, t_i)$. Continuity of F then furnishes an open set $N_i \times (a_i, b_i) \ni (q_0, t_i)$ such that $F(N_i \times (a_i, b_i)) \subseteq U_i \subseteq X$. The collection $\{N_i \times (a_i, b_i)\}$ covers $\{q_0\} \times [0,1]$, so by compactness one obtains an open set $N \coloneqq \bigcap N_i$ containing q_0 and a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0,1]$ such that each $F(N \times [t_i, t_{i+1}]) \subseteq U_i$ is evenly-covered. We define $\widetilde{F} : N \times [0, t_i] \to E$ by induction on i; for $i = 0$, we let $\widetilde{F} \coloneqq \widetilde{f}_0$ so that $\widetilde{F}$ restricts to $\widetilde{f}_0$ on $N \times \{0\}$.

Suppose a lift $\widetilde{F} : N \times [0, t_i] \to E$ has been constructed for some $i \geq 0$. Then, since $F(q_0, t_i) \in U_i$, there exists a unique open set $V_i \subseteq \pi^{-1}(U_i)$ containing $\widetilde{F}(q_0, t_i)$ that maps homeomorphically onto U_i. Replacing $N \times \{t_i\}$ by its intersection with $\widetilde{F}^{-1}(V_i)$, if necessary, we may assume that $\widetilde{F}(N \times \{t_i\}) \subseteq V_i$. Since π is invertible on V_i, extend $\widetilde{F}$ so that

$$\begin{tikzcd} & V_i \arrow[d, "\pi"] \\ N \times [t_i, t_{i+1}] \arrow[ur, "\widetilde{F}"] \arrow[r, "F"'] & U_i \end{tikzcd}$$

commutes. Our modification of $N \times \{t_i\}$ ensures that the restriction of $\widetilde{F}$ to $N \times \{t_i\}$ coincides with this extension, so the functions inductively glue to give a lift $\widetilde{F}$ of F on $N \times [0,1]$. We now argue that such a lifting is unique when Y is a point[2]; abusing notation, we drop Y from the notation and write $F : [0,1] \to X$, etc., instead.

- Suppose that $\widetilde{F}' : [0,1] \to E$ is another lift of F such that $\widetilde{F}(0) = \widetilde{F}'(0) = \widetilde{f}_0(0)$. As above, we may obtain a partition $0 = t_0 < t_1 < \cdots, t_n = 1$ of $[0,1]$ such that each $F([t_0, t_{i+1}]) \subseteq U_i$ is evenly-covered. Proceeding by induction, suppose that $\widetilde{F} = \widetilde{F}'$ on $[0, t_i]$. Since $[t_i, t_{i+1}]$ is connected, $\widetilde{F}([t_i, t_{i+1}])$ is connected too and thus lies in a single open set $V_i \subseteq \pi^{-1}(U_i)$ containing $\widetilde{F}(t_i)$ that maps homeomorphically onto U_i. Similarly for $\widetilde{F}'([t_i, t_{i+1}])$, but since $\widetilde{F}(t_i) = \widetilde{F}'(t_i)$, they lie in the same open set V_i. Since $\pi \circ \widetilde{F} = \pi \circ \widetilde{F}'$ on $[t_i, t_{i+1}]$ and π is injective on V_i, we see that $\widetilde{F} = \widetilde{F}'$ on $[t_i, t_{i+1}]$, as desired.

Thus, when restricted to $\{q\} \times [0,1]$ for each $q \in N$, the lift $\widetilde{F} : N \times [0,1] \to E$ of F is unique. In general, this shows that if the same construction is repeated for some other $q_0' \in Y$ to obtain a lift $\widetilde{F}' : N' \times [0,1] \to E$ of F, and if $N \cap N' \neq \varnothing$, the lifts $\widetilde{F}$ and $\widetilde{F}'$ must agree on $(N \cap N') \times [0,1]$. Thus $\widetilde{F}$ is well-defined on $Y \times [0,1]$, and is continuous since it is continuous on each $N \times [0,1]$. ■

[2] Here, we are not necessary assuming that $Y = \{q_0\}$.

Corollary 2.13.1. *Every covering map lifts paths and homotopies. More precisely:*

- *For each path $\gamma : [0,1] \to X$ starting at some point $p \in X$ and each $\zeta_0 \in \pi^{-1}(p)$, there exists a unique path $\widetilde{\gamma} : [0,1] \to E$ starting at ζ_0 lifting γ.*
- *For each homotopy $\gamma_t : [0,1] \to X$ of paths and each lift $\widetilde{\gamma}_0 : I \to E$ of γ_0, there exists a unique homotopy $\widetilde{\gamma}_t : I \to E$ of paths starting at $\widetilde{\gamma}_0$ lifting γ_t.*

Proof. In the notation of the preceding proposition, let Y be a singleton and $[0,1]$, respectively. Note that the resulting homotopy $\widetilde{\gamma}_t$ is a homotopy *of paths*[3] since as t varies, the endpoints $\widetilde{\gamma}_t(0)$ and $\widetilde{\gamma}_t(1)$ are paths in E that lift the constant paths at $\gamma_t(0)$ and $\gamma_t(1)$, respectively. By uniqueness of liftings of paths, we see that $\widetilde{\gamma}_t(0)$ and $\widetilde{\gamma}_t(1)$ are constant paths at the lifts of $\gamma_t(0)$ and $\gamma_t(1)$, respectively, as desired. ∎

2.2.2 Liftings of Mappings

We return to the problem of the liftings of mappings. The tools that we have developed actually proves a stronger statement[4] than is needed in this paper, but for sake of brevity we only present a special case. Throughout, let $\pi : E \to X$ be a covering map.

Lemma 2.14. *If Y is connected, then any two lifts $\widetilde{F}_1, \widetilde{F}_2 : Y \to E$ of $F : Y \to X$ agreeing at one point in Y agrees everywhere.*

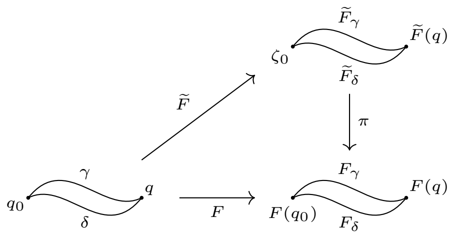
Proof. Let $S := \{q \in Y \mid \widetilde{F}_1(q) = \widetilde{F}_2(q)\}$, which we claim to be both open and closed. For a fixed $q \in Y$, let U be a neighborhood of $F(q)$ that is evenly-covered by open sets $V_i \subseteq E$. Let V_1 and V_2 be sheets above U containing $\widetilde{F}_1(q)$ and $\widetilde{F}_2(q)$, respectively. By continuity of $\widetilde{F}_1$ and $\widetilde{F}_2$, there exists a neighborhood V of q such that $\widetilde{F}_i(V) \subseteq V_i$ for $i = 1, 2$.

- If $q \in S$, then $V := V_1 = V_2$. Then, since $p \circ \widetilde{F}_1 = p \circ \widetilde{F}_2$ and p is injective on V, we see that $\widetilde{F}_1 = \widetilde{F}_2$ on V. This shows that S is open.
- Otherwise, $V_1 \neq V_2$ and hence they are disjoint. Then, since $\widetilde{F}_i(V) \subseteq V_i$ for $i = 1, 2$, we see that $\widetilde{F}_1(q') \neq \widetilde{F}_2(q')$ for all $q' \in V$. This shows that $Y \setminus S$ is open, whence S is closed too. ∎

Proposition 2.15. *Fix $q_0 \in Y$ and let $\zeta_0 \in \pi^{-1}(F(q_0))$. If Y is simply-connected and locally path-connected, then every map $F : Y \to X$ admits a unique lift $\widetilde{F} : Y \to E$ such that $\widetilde{F}(q_0) = \zeta_0$.*

Proof. By the lemma, such a lift is unique if it exists. To construct a lift, let $q \in Y$ and let γ be a path from q_0 to q. Then $F_\gamma := F \circ \gamma$ is a path starting at $F(q_0)$, which, by Corollary 2.13.1, admits a unique lift $\widetilde{F}_\gamma$ starting at ζ_0. Define $\widetilde{F}(q) := \widetilde{F}_\gamma(1)$. Assuming that $\widetilde{F}$ is well-defined and continuous, we have that $(\pi \circ \widetilde{F})(q) = \pi(\widetilde{F}_\gamma(1)) = F(\gamma(1)) = F(q)$, so $\widetilde{F}$ lifts F. It remains to show that $\widetilde{F}(q)$ is well-defined for all $q \in Y$, and that the map $\widetilde{F}$ is continuous.

- (Well-definedness). Let δ be another path from q_0 to q. By simply-connectedness of Y, the paths γ and δ are homotopic, so F_γ and F_δ are homotopic too. Again by Corollary 2.13.1, this homotopy lifts to a homotopy of paths from $\widetilde{F}_\gamma$ to $\widetilde{F}_\delta$ starting at ζ_0, so they have the same endpoint.



- (Continuity). Let $q \in Y$, $p := F(q)$, $\zeta := \widetilde{F}(q)$, and let V_0 be a neighborhood of ζ. For an evenly-covered neighborhood U of p, let V_ζ denote the sheet above U containing ζ. Set $V := V_0 \cap V_\zeta$, so π is a homeomorphism when restricted to V. Thus $\pi(V)$ is open, so by continuity $F^{-1}(\pi(V))$ is open too. By local path-connectedness of Y, this set contains a path-connected neighborhood W of q. We claim that $\widetilde{F}(W) \subseteq V$, so take $w \in W$ and let σ be a path from q to w contained in W. Then F_σ is a path in $F(W) \subseteq \pi(V)$ starting at p, which lifts to a path $\widetilde{F}_\sigma$ in V starting at ζ. But since the end point of $\widetilde{F}_\gamma$ constructed above is $\zeta = \widetilde{F}(q)$, the concatenation of $\widetilde{F}_\gamma * \widetilde{F}_\sigma$ is well-defined and is a path starting at ζ_0. This path lifts $F \circ (\gamma * \sigma)$, and since $\gamma * \sigma$ is a path from q_0 to w, we see that $\widetilde{F}(w)$ is the end point of $\widetilde{F}_\gamma * \widetilde{F}_\sigma$. But this end point is $\widetilde{F}_\gamma(1)$, which lies in V. ∎

Example 2.16. Let $\varphi : \mathbb{C}/\Gamma \to \mathbb{C}/\Gamma'$ be a holomorphic map between complex tori. By Example 2.5, the projection $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$ is an unbranched holomorphic map, and since $\mathbb{C}$ is simply-connected (and locally path-connected), the map $\varphi \circ \pi : \mathbb{C} \to \mathbb{C}/\Gamma'$ admits a unique holomorphic lift $\widetilde{\varphi} : \mathbb{C} \to \mathbb{C}$ along the projection $\pi' : \mathbb{C} \to \mathbb{C}/\Gamma'$.

$$
\begin{tikzcd}
\mathbb{C} \arrow[r, "\widetilde{\varphi}"] \arrow[d, "\pi"'] & \mathbb{C} \arrow[d, "\pi'"] \\
\mathbb{C}/\Gamma \arrow[r, "\varphi"] & \mathbb{C}/\Gamma'
\end{tikzcd}
$$

If φ is a biholomorphism, then lifting φ^{-1} too gives us a unique biholomorphism $\widetilde{\varphi} : \mathbb{C} \to \mathbb{C}$. A classical result from complex analysis then forces $\widetilde{\varphi}(z) = \alpha z + \beta$ for some $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$. This result tightly constrains the behaviour of biholomorphisms between complex tori, which we leverage in the proof of Theorem 4.8. ⧫

[3] As opposed to a free homotopy.
[4] See [Hat02, Proposition 1.33], which actually characterizes when such a lift exists.

Chapter 3

Čech Cohomology

Using the language of *sheaves* and *cohomology*, we prove the existence of certain global meromorphic functions on a compact Riemann surface X. Section 3.1 introduces the language, 3.2 studies the sheaf $\mathscr{E}$ of differentiable functions on X, which includes differentiation and integration of *forms*, and 3.3 studies its associated cohomology. The latter sections require some background in linear algebra and the theory of smooth manifolds, for which we refer the reader to [Lee12].

3.1 Sheaves and their Cohomology

Unless otherwise stated, let X be a topological space with $\mathcal{T}$ its system of open sets. Our exposition on sheaves and their cohomology roughly follows [For81, Sections 6 & 12] and [Mir95, Chapter IX].

3.1.1 Presheaves, Sheaves, and Stalks

Definition 3.1. *A presheaf of Abelian groups on X is a pair $(\mathscr{F}, \rho)$ consisting of*

- *a family $\mathscr{F} := \{\mathscr{F}(U)\}$ of Abelian groups $\mathscr{F}(U)$ for every $U \in \mathcal{T}$,*
- *a family $\rho := \{\rho_V^U\}$ of group homomorphisms $\rho_V^U : \mathscr{F}(U) \to \mathscr{F}(V)$ for every $U, V \in \mathcal{T}$ with $V \subseteq U$,*

such that $\rho_U^U = \mathrm{id}_{\mathscr{F}(U)}$ and $\rho_W^V \circ \rho_V^U = \rho_W^U$ for every $U, V, W \in \mathcal{T}$ with $W \subseteq V \subseteq U$.

Remark. We may analogously define a presheaf of sets, rings, vector spaces, $\mathbb{C}$-algebras, etc, by requiring that $\mathscr{F}(U)$ and ρ are objects and maps of the appropriate 'category'. ♦

Remark. Presheaves give us a way of tracking data associated with open sets of a topological space in such a way that makes restricting to a smaller open set $V \subseteq U$ well-behaved. Consider, for instance, a Riemann surface X and the presheaf of all holomorphic functions $\mathscr{O}$ on X.

- To every open set $U \subseteq X$ we consider the $\mathbb{C}$-algebra $\mathscr{O}(U)$ of all holomorphic functions $f : U \to \mathbb{C}$. For any open $V \subseteq U$, we define $\rho_V^U(f) := f|_V$. The properties then state that restricting to the domain does nothing and that restricting once to V and then to $W \subseteq V$ yields the same function as restricting to W directly, which are obviously true.

Similarly, we have the presheaf of all meromorphic functions $\mathscr{M}$ on X. Other examples include the presheaf $\mathscr{O}^*$ and $\mathscr{M}^*$ of *multiplicative* Abelian groups defined respectively by holomorphic functions $f : U \to \mathbb{C}^*$ and meromorphic functions on U that do not vanish identically on any connected component of U. However, those examples are much more than presheaves since global information about elements in $\mathscr{F}(X)$ can be obtained locally by 'restricting' to locally U. The notion of a *sheaf* makes this precise. ♦

Definition 3.2. *A presheaf $\mathscr{F}$ on X is said to be a sheaf if for every open set $U \subseteq X$ and every family $\{U_i\}_{i \in I}$ of open subsets that cover U, the following two properties hold:*

- *(Identity): For every $f, g \in \mathscr{F}(U)$, if $\rho_{U_i}^U(f) = \rho_{U_i}^U(g)$ for every $i \in I$, then $f = g$.*
- *(Gluing): For every family $\{f_i\}_{i \in I}$ with $f_i \in \mathscr{F}(U_i)$, if $\rho_{U_i \cap U_j}^{U_i}(f_i) = \rho_{U_i \cap U_j}^{U_j}(f_j)$ for all $i, j \in I$, then there is some $f \in \mathscr{F}(U)$ such that $\rho_{U_i}^U(f) = f_i$ for every $i \in I$.*

Example 3.3. It is immediate that $\mathscr{O}$, $\mathscr{O}^*$, $\mathscr{M}$, and $\mathscr{M}^*$ are all sheaves on X. Indeed, if we have a family $\{f_i\}$ that agree on all pairwise common domains, then there exists a globally defined function f whose restrictions are f_i's. We only need to show that this globally defined function is of the 'right type', but this can be checked easily. ♦

Example 3.4. We give an example of a presheaf that is *not* a sheaf. Let X be a normed $\mathbb{R}$-vector space. For all $U \in \mathcal{T}$, let $\mathscr{B}(U)$ be the vector space of all bounded functions $f : U \to \mathbb{R}$, which gives us a presheaf $\mathscr{B}$ on X. The problem arises when we consider glueing[1]. For instance, let $U_i := \{p \in X \mid \|p\| < i\}$ and observe that $\{U_i\}_{i \in \mathbb{R}^+}$ covers X. Consider the family $\{\mathrm{id}_{U_i}\}$, which clearly agrees on pairwise intersections an glues up to the identity id_X. But id_X is *not* bounded, so $\mathscr{B}$ is not a sheaf. ♦

Example 3.5. We give two examples of sheaves relating to *divisors* on a Riemann surface X; that is, functions $D : X \to \mathbb{Z}$ whose supports $\{p \in X \mid D(p) \neq 0\}$ are discrete[2] subsets of X.

- Let D be a divisor on X. For every $U \in \mathcal{T}$, let $\mathscr{O}[D](U)$ denote the Abelian group of all meromorphic functions $f : X \to \mathbb{C}$ such that $\mathrm{ord}_p(f) \leq D(p)$ for all $p \in X$. The usual restriction homomorphisms make $\mathscr{O}[D]$ is a sheaf of Abelian groups. This construction generalizes both $\mathscr{O}$ and $\mathscr{M}$. Intuitively, the use of divisors here allow us to 'bound' the orders of the poles of f at specific points p, thereby restricting how badly-behaved it can be.

- For every $U \in \mathcal{T}$, let $\mathscr{D}(U)$ denote the group of all divisors on U. This makes $\mathscr{D}$ into a sheaf since for every family $\{D_i\}$, the function $D : X \to \mathbb{Z}$ that glues them together is also discretely-supported. ♦

[1] In other words, boundedness is a global property. To check if a function is bounded, it does *not* suffice to check it on an arbitrary neighborhood.
[2] Note that if X is compact, then $D : X \to \mathbb{Z}$ is a divisor iff it has finite support, so the set of divisors of X is the free Abelian group of X.

Definition 3.6. *Let $(\mathscr{F}, \rho)$ and $(\mathscr{G}, \sigma)$ be two sheaves of Abelian groups on X. A morphism of sheaves $\eta : \mathscr{F} \to \mathscr{G}$ is a family $\{\eta_U\}_{U \in \mathcal{T}}$ of group homomorphisms $\eta_U : \mathscr{F}(U) \to \mathscr{G}(U)$ such that for every $U \in \mathcal{T}$ and every open set $V \subseteq U$, the following diagram commutes.*

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\eta_U"] \arrow[d, "\rho_V^U"'] & \mathscr{G}(U) \arrow[d, "\sigma_V^U"] \\ \mathscr{F}(V) \arrow[r, "\eta_V"] & \mathscr{G}(V) \end{tikzcd}$$

Example 3.7. Some examples relating to divisors of a Riemann surface X.

- For divisors D_1 and D_2 of a Riemann surface X, we write $D_1 \leq D_2$ if $D_1(p) \leq D_2(p)$ for all $p \in X$. This induces an *inclusion morphism* $\iota : \mathscr{O}[D_1] \hookrightarrow \mathscr{O}[D_2]$ defined by $\iota_U(f) := f$ for all $U \in \mathcal{T}$ and $f \in \mathscr{O}[D_1](U)$, which makes sense since if $D_1 \leq D_2$ and the poles of f are bounded by D_1, then they are also clearly bounded by D_2. This inclusion also respect restrictions, so it is indeed a morphism of sheaves. In particular, we have the inclusions $\mathscr{O} \hookrightarrow \mathscr{O}[D] \hookrightarrow \mathscr{M}$ for any divisor D of X.
- For all $U \in \mathcal{T}$, we associate to each $f \in \mathscr{M}^*(U)$ the function $\operatorname{div} f : U \to \mathbb{Z} : p \mapsto \operatorname{ord}_p(f)$, which is a divisor by discreteness of zeros and poles. This induces a morphism of sheaves $\operatorname{div} : \mathscr{M}^* \to \mathscr{D}$ since for all $U \in \mathcal{T}$ and all open sets $V \subseteq U$, the restriction of the divisor of any $f \in \mathscr{M}^*(U)$ coincides with the divisor of the restriction $f|_V$. ♦

Definition 3.8. *Let $\mathscr{F}$ be a presheaf of Abelian groups on X and fix $p \in X$. The stalk of $\mathscr{F}$ at p is the Abelian group*

$$\mathscr{F}_p := \left(\coprod_{U \ni p} \mathscr{F}(U) \right) \Big/ \sim_p$$

where $\sim_p$ is the equivalence relation[3] on the disjoint union, defined, for all $f \in \mathscr{F}(U)$ and $g \in \mathscr{F}(V)$, by $f \sim_p g$ iff there exists an open set $W \in \mathcal{T}$ with $p \in W \subseteq U \cap V$ such that $\rho_W^U(f) = \rho_W^V(g)$. For $f \in \mathscr{F}(U)$, its equivalence class $[f]_p$ is called the germ of f at p.

Example 3.9. Let D be a divisor on a Riemann surface X and consider the stalk $\mathscr{O}_p[D]$. Fix a chart centered at p. Since every meromorphic function f admits a Laurent series, we see that the function germ $[f]_p$ is represented by a Laurent series $\sum_{i=i_0}^{\infty} c_i z^i$ for some $i_0 \geq -D(p)$. Conversely, the germ of a Laurent series $\sum_{i=i_0}^{\infty} c_i z^i$ with $i_0 \geq -D(p)$ and whose principal part has positive radius of convergence represents a meromorphic function germ $[f]_p$, so this defines a bijection between $\mathscr{O}_p[D]$ and the set of all such Laurent series. This isomorphism depends on the chosen chart map, so it is not canonical. ♦

Remark. The sheaf axioms guarantee that if $\mathscr{F}$ is a sheaf of Abelian groups on X and $U \in \mathcal{T}$, then an element $f \in \mathscr{F}(U)$ is zero iff all germs $[f]_p$, for $p \in U$ vanish. Indeed, let $0 \in \mathscr{F}(U)$ denote the zero element, so $f \sim_p 0$ for all $p \in U$ furnishes a family $\{W_p\}$ of open sets $W_p \subseteq U$ containing p such that $\rho_{W_p}^U(f) = \rho_{W_p}^U(0)$. This family covers U, so $f = 0$ by the first sheaf axiom. ♦

3.1.2 Čech Cohomology Groups

We define the *first Čech cohomology group* $\check{H}^1(X, \mathscr{F})$ of a sheaf $\mathscr{F}$ of Abelian groups on X by first defining it on an open cover $\mathfrak{A}$ of X, which refines via a direct limit, and then prove the *Leray Criterion* to calculate such groups. Throughout this section, let $\mathscr{F}$ be a sheaf of Abelian groups on X and let $\mathfrak{A} := \{U_i\}$ be an open covering of X.

Definition 3.10. *For all $n \in \mathbb{N}$, the n^{th} cochain group of $\mathscr{F}$ (w.r.t. $\mathfrak{A}$) is the direct product*

$$\check{C}^n(\mathfrak{A}, \mathscr{F}) := \prod_{(i_0, \dots, i_n)} \mathscr{F}(U_{i_0} \cap \dots \cap U_{i_n}).$$

Remark. For $n = 0$, the group $\check{C}^0(\mathfrak{A}, \mathscr{F})$ contains tuples (f_i) where each f_i is defined on U_i. For $n = 1$, the group $\check{C}^1(\mathfrak{A}, \mathscr{F})$ contains tuples (f_{ij}) where each f_{ij} is defined on $U_i \cap U_j$. Intuitively, (f_i) induces an element $(g_{ij}) \in \check{C}^1(\mathfrak{A}, \mathscr{F})$ by setting $g_{ij} := f_j - f_i$, which 'chains' (f_i) on $U_i \cap U_j$ by measuring their difference. The following definition formalizes this intuition. ♦

Definition 3.11. *For all $n \in \mathbb{N}$, the n^{th} coboundary operator is the map $\delta^n : \check{C}^n(\mathfrak{A}, \mathscr{F}) \to \check{C}^{n+1}(\mathfrak{A}, \mathscr{F})$ mapping $(f_{i_0, \dots, i_n})$ to the cochain $\left(g_{i_0, \dots, i_{n+1}}\right)$ defined by[4]*

$$g_{i_0, \dots, i_{n+1}} := \sum_{k=0}^{n+1} (-1)^k \rho \left(f_{i_0, \dots, \widehat{i_k}, \dots, i_{n+1}} \right).$$

Define the n^{th} cocycle group $\check{Z}^n(\mathfrak{A}, \mathscr{F}) := \ker \delta^n$ and the n^{th} splitting cocycle group $\check{B}^n(\mathfrak{A}, \mathscr{F}) := \operatorname{im} \delta^{n-1}$, whose quotient

$$\check{H}^n(\mathfrak{A}, \mathscr{F}) := \check{Z}^n(\mathfrak{A}, \mathscr{F}) / \check{B}^n(\mathfrak{A}, \mathscr{F})$$

is called the n^{th} cohomology group of $\mathscr{F}$ (w.r.t. $\mathfrak{A}$).

Remark. A calculation shows that $\check{B}^n(\mathfrak{A}, \mathscr{F}) \subseteq \check{Z}^n(\mathfrak{A}, \mathscr{F})$, so the quotient makes sense. In particular, $\delta^{n+1} \circ \delta^n = 0$. ♦

[3] The relation $\sim_p$ is transitive since $\rho_W^V \circ \rho_V^U = \rho_W^U$ for all $U, V, W \in \mathcal{T}$ such that $W \subseteq V \subseteq U$. This condition on ρ is known as a *directed system*, which all admit a *direct limit* whose construction is formally similar to that of $\mathscr{F}_p$. We refer the interested reader to [Lan10, Chapter III].

[4] The 'hat' notation represents a deletion. Also, ρ is the appropriate restriction mapping of $\mathscr{F}$.

Remark. For $n = 0$, we have $\delta^0(f_i) = (f_j - f_i)$ for all $(f_i) \in \check{C}^0(\mathfrak{A}, \mathscr{F})$. This gives us a glueing condition, that if $(f_i) \in \check{Z}^0(\mathfrak{A}, \mathscr{F})$, then[5] $\rho(f_i) = \rho(f_j)$ for all i, j and hence the sheaf axioms furnish a unique $f \in \mathscr{F}(X)$ such that $\rho(f) = f_i$ for all i. Thus

$$\check{H}^0(\mathfrak{A}, \mathscr{F}) = \check{Z}^0(\mathfrak{A}, \mathscr{F}) \cong \mathscr{F}(X),$$

so $\check{H}^0(\mathfrak{A}, \mathscr{F})$ is independent of the covering $\mathfrak{A}$ and we may define the 0^{th} cohomology group of $\mathscr{F}$ as $\check{H}^0(X, \mathscr{F}) := \mathscr{F}(X)$. ♦

Remark. For $n = 1$, we have $\delta^1(f_{ij}) = (f_{jk} - f_{ik} + f_{ij})$ for all $(f_{ij}) \in \check{C}^1(\mathfrak{A}, \mathscr{F})$. Elements $(f_{ij}) \in \check{Z}^1(\mathfrak{A}, \mathscr{F})$ satisfy the *cocycle condition*, which states $f_{ik} = f_{ij} + f_{jk}$ on $U_i \cap U_j \cap U_k$ for all i, j, k. In particular, it implies that $f_{ii} = 0$ for all i and $f_{ij} = -f_{ji}$ on $U_i \cap U_j$ for all i, j. Note that every splitting cocycle is a cocycle, but not every cocycle splits. In other words, $\check{H}^1(\mathfrak{A}, \mathscr{F})$ measures how 1-cocycles fail to split. We now define the 1^{st} cohomology group of $\mathscr{F}$ by 'refining' the open cover $\mathfrak{A}$. ♦

Definition 3.12. *Let $\mathfrak{A} := \{U_i\}_{i \in I}$ and $\mathfrak{B} := \{V_k\}_{k \in K}$ be open coverings of X. We say that $\mathfrak{B}$ is finer than $\mathfrak{A}$, and write $\mathfrak{B} \preceq \mathfrak{A}$, if there exists a refining map $r : K \to I$ such that $V_k \subseteq U_{r(k)}$ for all $k \in K$.*

Remark. The refining map r induces a map $\tilde{r} : \check{Z}^1(\mathfrak{A}, \mathscr{F}) \to \check{Z}^1(\mathfrak{B}, \mathscr{F})$ by sending (f_{ij}) into (g_{kl}) defined by $g_{kl} := f_{r(k), r(l)}$ on $V_k \cap V_l$ for all $k, l \in K$. Observe that if $\delta^1_{\mathfrak{A}}(f_{ij}) = 0$, then $f_{i_1 i_3} = f_{i_1 i_2} + f_{i_2 i_3}$ on $U_{i_1} \cap U_{i_2} \cap U_{i_3}$ for all $i_1, i_2, i_3 \in I$. In particular, we have $f_{r(k_1), r(k_3)} = f_{r(k_1), r(k_2)} + f_{r(k_2), r(k_3)}$ on $V_{k_1} \cap V_{k_2} \cap V_{k_3}$ for all $k_1, k_2, k_3 \in K$, so $\delta^1_{\mathfrak{B}}(\tilde{r}(f_{ij})) = 0$. Thus $\tilde{r}$ sends splitting cocycles into splitting cocycles, so we may descend $\tilde{r}$ into the quotient, giving us a map

$$\check{H}(r) : \check{H}^1(\mathfrak{A}, \mathscr{F}) \to \check{H}^1(\mathfrak{B}, \mathscr{F}) \qquad \text{mapping} \qquad [f_{ij}] \mapsto [\tilde{r}(f_{ij})].$$

♦

Proposition 3.13. *In the above notation, the map $\check{H}^{\mathfrak{A}}_{\mathfrak{B}} := \check{H}(r)$ is independent of r and is injective.*

Proof. Take $(f_{ij}) \in \check{Z}^1(\mathfrak{A}, \mathscr{F})$ and suppose that $r' : K \to I$ is another refining map. Inducing the map $\tilde{r}'$ similarly, let $(g_{kl}) := \tilde{r}(f_{ij}) = (f_{r(k), r(l)})$ and $(g'_{kl}) := \tilde{r}'(f_{ij}) = (f_{r'(k), r'(l)})$. Observe then that

$$\begin{aligned} g_{kl} - g'_{kl} &= f_{r(k), r(l)} - f_{r'(k), r'(l)} \\ &= f_{r(k), r(l)} + f_{r(l), r'(k)} - f_{r(l), r'(k)} - f_{r'(k), r'(l)} \\ &= f_{r(k), r'(k)} - f_{r(l), r'(l)} \end{aligned}$$

on $V_k \cap V_l$ for all $k, l \in K$. Since r and r' are refining maps, we see that $V_k \subseteq U_{r(k)} \cap U_{r'(k)}$ for all $k \in K$, so we may define $h_k := f_{r(k), r'(k)}$ on the restriction to V_k. Then $(g_{kl} - g'_{kl}) = (h_k - h_l) = \delta^0(h_k)$ on $V_k \cap V_l$, so $(g_{ij}) - (g'_{ij}) \in \check{B}^1(\mathfrak{B}, \mathscr{F})$. Thus their equivalence classes coincide, as desired. Now, to show that $\check{H}^{\mathfrak{A}}_{\mathfrak{B}}$ is injective, take $(f_{ij}) \in \ker \check{H}^{\mathfrak{A}}_{\mathfrak{B}}$. Thus $(f_{r(k), r(l)}) = \check{H}^{\mathfrak{A}}_{\mathfrak{B}}(f_{ij})$ splits, so there exist $g_k \in \mathscr{F}(V_k)$ such that $f_{r(k), r(l)} = g_k - g_l$ on $V_k \cap V_l$ for all $k, l \in K$. Then

$$g_k - g_l = f_{r(k), i} + f_{i, r(l)} = f_{i, r(l)} - f_{i, r(k)}$$

on $U_i \cap V_k \cap V_l$ for all $i \in I$ and hence $g_k + f_{i, r(k)} = g_l + f_{i, r(l)}$ on the same domain. Fixing $i \in I$ and glueing the family $\{g_k + f_{i, r(k)}\}_{k \in K}$ defined on the cover $\{U_i \cap V_k\}_{k \in K}$ of U_i, we obtain an element $h_i \in \mathscr{F}(U_i)$ such that $h_i = g_k + f_{i, r(k)}$ on $U_i \cap V_k$ for all $k \in K$. Observe then that

$$f_{ij} = f_{i, r(k)} - f_{j, r(k)} = h_i - g_k - h_j + g_k = h_i - h_j$$

on $U_i \cap U_j \cap V_k$. Note that both f_{ij} and $h_i - h_j$ are defined on $U_i \cap U_j$, and since they coincide on the restriction to V_k, uniqueness of the glueing gives us $f_{ij} = h_i - h_j$ on $U_i \cap U_j$. Thus $(f_{ij}) = \delta^0(h_i)$, so (f_{ij}) splits. ■

Remark. If $\mathfrak{C} \preceq \mathfrak{B} \preceq \mathfrak{A}$ are open coverings of X, we have that $\check{H}^{\mathfrak{B}}_{\mathfrak{C}} \circ \check{H}^{\mathfrak{A}}_{\mathfrak{B}} = \check{H}^{\mathfrak{A}}_{\mathfrak{C}}$. This allows us to give a construction of $\check{H}^1(X, \mathscr{F})$ that is formally similar to that of stalks (see Definition 3.8 and its associated footnote). ♦

Definition 3.14. *The 1^{st} cohomology group of $\mathscr{F}$ is the Abelian group*

$$\check{H}^1(X, \mathscr{F}) := \left(\coprod_{\mathfrak{A}} \check{H}^1(\mathfrak{A}, \mathscr{F}) \right) \Big/ \sim$$

where $\sim$ is the equivalence relation on the disjoint union, defined, for all $\xi \in \check{H}^1(\mathfrak{A}, \mathscr{F})$ and $\xi' \in \check{H}^1(\mathfrak{A}', \mathscr{F})$, by $\xi \sim \xi'$ iff there exists a refinement $\mathfrak{B} \preceq \mathfrak{A}, \mathfrak{A}'$ such that $\check{H}^{\mathfrak{A}}_{\mathfrak{B}}(\xi) = \check{H}^{\mathfrak{A}'}_{\mathfrak{B}}(\xi')$.

Remark. Note that $\check{H}^1(X, \mathscr{F})$ vanishes iff $\check{H}^1(\mathfrak{A}, \mathscr{F}) = 0$ for all open coverings $\mathfrak{A}$ of X. The converse direction is trivial, and for the forward, let $\mathfrak{A}$ be an open covering of X. By Proposition 3.13, the canonical maps $\check{H}^1(\mathfrak{A}, \mathscr{F}) \to \check{H}^1(\mathfrak{B}, \mathscr{F})$ are injective for all open coverings $\mathfrak{B} \preceq \mathfrak{A}$. Descending into the quotient, the induced map $\check{H}^1(\mathfrak{A}, \mathscr{F}) \to \check{H}^1(X, \mathscr{F})$ is also injective, as desired. ♦

Proposition 3.15 (Leray)**.** *If $\mathfrak{A} := \{U_i\}_{i \in I}$ is an open covering of X such that $\check{H}^1(U_i, \mathscr{F})$ vanishes for every $i \in I$, then*

$$\check{H}^1(X, \mathscr{F}) \cong \check{H}^1(\mathfrak{A}, \mathscr{F}).$$

Such a covering $\mathfrak{A}$ of X is called a Leray covering of X.

[5] Henceforth, we suppress the restriction maps ρ for ease of notation, but will always mention on which domain the relation is valid on.

Proof. Let $\mathfrak{B} \coloneqq \{V_k\}_{k\in K}$ be an open covering of X with $\mathfrak{B} \preceq \mathfrak{A}$, so there exists a refining map $r : K \to I$. We claim that $\check{H}^{\mathfrak{A}}_{\mathfrak{B}}$ is an isomorphism, from which the result follows by descending into the quotient. By Proposition 3.13, this map is injective, and to show that it is surjective, we must show that every cocycle $(f_{kl}) \in \check{Z}^1(\mathfrak{B}, \mathscr{F})$ admits a cocycle $(F_{ij}) \in \check{Z}^1(\mathfrak{A}, \mathscr{F})$ such that

$$\left(F_{r(k),r(l)}\right) - (f_{kl}) \in \check{B}^1(\mathfrak{B}, \mathscr{F}).$$

For each $i \in I$, consider the open cover $U_i \cap \mathfrak{B} \coloneqq \{U_i \cap V_k\}_{k\in K}$ of U_i. Since $\check{H}^1(U_i, \mathscr{F}) = 0$, we see that $\check{H}^1(U_i \cap \mathfrak{B}, \mathscr{F}) = 0$. Restricting to U_i, we see that $(f_{kl}) \in \check{Z}^1(U_i \cap \mathfrak{B}, \mathscr{F})$ and hence there exist $g_{ik} \in \mathscr{F}(U_i \cap V_k)$ such that $f_{kl} = g_{ik} - g_{il}$ on $U_i \cap V_k \cap V_l$ for all $i \in I$ and $k, l \in K$. Using this result on two fixed $i, j \in I$ and equating, we see that $g_{jk} - g_{ik} = g_{jl} - g_{il}$ on $U_i \cap U_j \cap V_k \cap V_l$. This glues to an element $F_{ij} \in \mathscr{F}(U_i \cap U_j)$ such that $F_{ij} = g_{jk} - g_{ik}$ on $U_i \cap U_j \cap V_k$ for all $k \in K$, and a computation shows that $(F_{ij}) \in \check{Z}^1(\mathfrak{A}, \mathscr{F})$. Observe then that[6]

$$F_{r(k),r(l)} - f_{kl} = \left(g_{r(l),k} - g_{r(k),k}\right) - \left(g_{r(l),k} - g_{r(l),l}\right) = g_{r(l),l} - g_{r(k),k}$$

on $V_k \cap V_l$, so setting $h_k \coloneqq g_{r(k),k} \in \mathscr{F}(V_k)$ shows that $\left(F_{r(k),r(l)}\right) - (f_{kl})$ splits in $\mathfrak{B}$. ■

3.2 Differential Forms

Due to the Cauchy-Riemann equations, the theory of *complex differential forms* departs from that of real differential forms and has a unique relationship with so-called *holomorphic forms*. Those objects thus play an important role in the structure on holomorphic functions, so we devote this section to formalize some basic notions. Throughout this section, let $W \subseteq X$ be an open subset of a Riemann surface X and fix $p \in W$.

3.2.1 The Complexified Cotangent Space

For an open set $V \subseteq \mathbb{C}$, we let $\mathscr{E}(V)$ denote the $\mathbb{C}$-algebra of all functions $f : V \to \mathbb{C}$ that are infinitely-differentiable with respect to the real coordinates x and y, which we simple call *differentiable*. Using the partial derivative operators $\partial/\partial x$ and $\partial/\partial y$ on $\mathscr{E}(V)$, we define the operators

$$\frac{\partial}{\partial z} \coloneqq \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad \text{and} \qquad \frac{\partial}{\partial \bar{z}} \coloneqq \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

on $\mathscr{E}(V)$, where Cauchy-Riemann equations now reads $\mathscr{O}(V) = \ker \partial/\partial\bar{z}$. We now lift these notions to a Riemann surface X.

Definition 3.16. *A function $f : W \to \mathbb{C}$ is said to be differentiable at p if there is a chart (U, z) of X around p such that $f \circ z^{-1} : z(U) \to \mathbb{C}$ is differentiable at $z(p)$. If f is differentiable at every point of W, then f is said to be differentiable on W.*

Remark. As with holomorphic functions, differentiability is chart-independent. Let $\mathscr{E}(W)$ denote the $\mathbb{C}$-algebra of all differentiable functions on W, which, together with the usual restriction mappings, forms a sheaf $\mathscr{E}$. We may, as in $\mathbb{C}$, define the *partial derivative operators*, but this time with respect to a chart (U, z) instead of x and y. More precisely, for a fixed chart (U, z), we define

$$\frac{\partial}{\partial z} : \mathscr{E}(U) \to \mathscr{E}(U) \qquad \text{mapping} \qquad f \mapsto \frac{\partial\left(f \circ z^{-1}\right)}{\partial z}$$

by pulling back the regular partial derivative $\partial/\partial z$ on $\mathbb{C}$. We similarly define the operators $\partial/\partial\bar{z}$, $\partial/\partial x$, and $\partial/\partial y$ on $\mathscr{E}(U)$. ♦

Definition 3.17. *Let $\mathfrak{m}_p \subseteq \mathscr{E}_p$ be the ideal of all differentiable function germs vanishing at p. The complexified[7] cotangent space of X at p is the quotient space $T^*_{\mathbb{C},p}X \coloneqq \mathfrak{m}_p/\mathfrak{m}_p^2$. For a function $f \in \mathscr{E}(W)$, we define its differential at p as*

$$\mathrm{d}_p f \coloneqq [f - f(p)]_{\mathfrak{m}_p^2} \in T^*_{\mathbb{C},p}X.$$

Proposition 3.18. *Let (U, z) be a chart of X around p. Then $\{\mathrm{d}_p x, \mathrm{d}_p y\}$ and $\{\mathrm{d}_p z, \mathrm{d}_p \bar{z}\}$ are both bases for $T^*_{\mathbb{C},p}X$, and if $f \in \mathscr{E}(W)$, then*

$$\mathrm{d}_p f = \left.\frac{\partial f}{\partial x}\right|_p \mathrm{d}_p x + \left.\frac{\partial f}{\partial y}\right|_p \mathrm{d}_p y = \left.\frac{\partial f}{\partial z}\right|_p \mathrm{d}_p z + \left.\frac{\partial f}{\partial \bar{z}}\right|_p \mathrm{d}_p \bar{z}.$$

Proof. We first show that $\{\mathrm{d}_p x, \mathrm{d}_p y\}$ is a basis for $T^*_{\mathbb{C},p}X$.

- Let $[\eta] \in T^*_{\mathbb{C},p}X$, so $\eta = [f]_p \in \mathfrak{m}_p$ is a differentiable function germ for some $f \in \mathscr{E}(W)$. Taylor's Theorem in $\mathbb{C}$ then furnishes $\lambda_1, \lambda_2 \in \mathbb{C}$ such that

  $$f = \lambda_1 (x - x(p)) + \lambda_2 (y - y(p)) + g$$

  where $g \in \mathscr{E}(W)$ is such that $[g]_p \in \mathfrak{m}_p^2$. This lifts to an equality of germs, so, taking the quotient modulo $\mathfrak{m}_p^2$, we see that $[\eta] = \lambda_1 \mathrm{d}_p x + \lambda_2 \mathrm{d}_p y$ and thus $\{\mathrm{d}_p x, \mathrm{d}_p y\}$ spans $T^*_{\mathbb{C},p}X$. For linear independence, take $\lambda_1, \lambda_2 \in \mathbb{C}$. The linear dependence $\lambda_1 \mathrm{d}_p x + \lambda_2 \mathrm{d}_p y = 0$ implies that

  $$\lambda_1 (x - x(p)) + \lambda_2 (y - y(p)) \in \mathfrak{m}_p^2.$$

  Taking the partials $\partial/\partial x$ and $\partial/\partial y$ shows that $\lambda_1 = \lambda_2 = 0$.

[6] Here, we instantiated the relation $f_{kl} = g_{ik} - g_{il}$ with $i \coloneqq r(l)$.

[7] Alternatively, we can define it as the *complexification* of the (regular) cotangent space $T^*_p X$ as $T^*_{\mathbb{C},p}X \coloneqq T^*_p X \otimes_{\mathbb{R}} \mathbb{C}$, where $\otimes_{\mathbb{R}}$ is the *tensor product* of two real vector spaces; see Definition 3.24.

Suppose now that $f \in \mathscr{E}(W)$. By Taylor's Theorem, we have

$$f - f(p) = \left.\frac{\partial f}{\partial x}\right|_p (x - x(p)) + \left.\frac{\partial f}{\partial y}\right|_p (y - y(p)) + g$$

where $g \in \mathscr{E}(W)$ is such that $[g] \in \mathfrak{m}_p^2$, so lifting this to an equality of germs and taking the quotient modulo $\mathfrak{m}_p^2$ gives us

$$\mathrm{d}_p f = \left.\frac{\partial f}{\partial x}\right|_p \mathrm{d}_p x + \left.\frac{\partial f}{\partial y}\right|_p \mathrm{d}_p y.$$

Finally, we show the corresponding result for $\{\mathrm{d}_p z, \mathrm{d}_p \overline{z}\}$. Indeed, since $z = x + iy$ as functions in $\mathscr{E}(W)$, we have that $\partial z/\partial x = 1$ and $\partial z/\partial y = i$. Similarly, $\partial \overline{z}/\partial x = 1$ and $\partial \overline{z}/\partial y = -i$, so

$$\mathrm{d}_p z = \mathrm{d}_p x + i\mathrm{d}_p y \qquad \text{and} \qquad \mathrm{d}_p \overline{z} = \mathrm{d}_p x - i\mathrm{d}_p y.$$

Thus $\{\mathrm{d}_p z, \mathrm{d}_p \overline{z}\}$ is linearly-independent, so it is a basis for $T^*_{\mathbb{C},p}X$. For $f \in \mathscr{E}(W)$, a computation now shows that

$$\mathrm{d}_p f = \frac{1}{2}\left(\left.\frac{\partial f}{\partial x}\right|_p - i\left.\frac{\partial f}{\partial y}\right|_p\right)\mathrm{d}_p z + \frac{1}{2}\left(\left.\frac{\partial f}{\partial x}\right|_p + i\left.\frac{\partial f}{\partial y}\right|_p\right)\mathrm{d}_p \overline{z} = \left.\frac{\partial f}{\partial z}\right|_p \mathrm{d}_p z + \left.\frac{\partial f}{\partial \overline{z}}\right|_p \mathrm{d}_p \overline{z}. \qquad \blacksquare$$

Proposition 3.19 (Canonical Decomposition)**.** *Let (U, z) be a chart of X around p. Then the subspaces*

$$T_p^* X^{(1,0)} := \operatorname{span}\{\mathrm{d}_p z\} \qquad \textit{and} \qquad T_p^* X^{(0,1)} := \operatorname{span}\{\mathrm{d}_p \overline{z}\}$$

are chart-independent and $T^*_{\mathbb{C},p}X = T_p^* X^{(1,0)} \oplus T_p^* X^{(0,1)}$.

Proof. Let (U', z') is another chart of X around p with $U \cap U' \neq \varnothing$. Since $z' \in \mathscr{O}(U \cap U')$, the expansion

$$\mathrm{d}_p z' = \left.\frac{\partial z'}{\partial z}\right|_p \mathrm{d}_p z + \left.\frac{\partial z'}{\partial \overline{z}}\right|_p \mathrm{d}_p \overline{z}$$

shows that $\partial z'/\partial \overline{z} = 0$, so $\operatorname{span}\{\mathrm{d}_p z\} = \operatorname{span}\{\mathrm{d}_p z'\}$. Similarly, $\partial \overline{z'}/\partial z = 0$, so $\operatorname{span}\{\mathrm{d}_p \overline{z}\} = \operatorname{span}\left\{\mathrm{d}_p \overline{z'}\right\}$. The decomposition then follows by construction. $\blacksquare$

Remark. For all $f \in \mathscr{E}(W)$, let $\partial_p f \in T_p^* X^{(1,0)}$ and $\overline{\partial}_p f \in T_p^* X^{(0,1)}$ be the unique elements such that $\mathrm{d}_p f = \partial_p f + \overline{\partial}_p f$. The above proposition ensures that they are chart-independent. For computations, we descend via any chart (U, z) of X around p where we have

$$\partial_p f = \left.\frac{\partial f}{\partial z}\right|_p \mathrm{d}_p z \quad \text{and} \quad \overline{\partial}_p f = \left.\frac{\partial f}{\partial \overline{z}}\right|_p \mathrm{d}_p \overline{z}. \qquad \blacklozenge$$

3.2.2 Differential 1-forms

Definition 3.20. *A differential 1-form on W is a map $\omega : W \to \bigcup_{p \in W} T^*_{\mathbb{C},p}X$ such that $\omega(p) \in T^*_{\mathbb{C},p}X$ for every $p \in W$.*

Remark. With the induced operations from $T^*_{\mathbb{C},p}X$, the set of all 1-forms on W becomes a $\mathbb{C}$-vector space. In fact, it is a $\mathbb{C}$-algebra, for if $f : W \to \mathbb{C}$ is a function, then the map $f\omega$ defined by $(f\omega)(p) := f(p)\,\omega(p)$ is also a 1-form on W. ♦

Example 3.21. For $f \in \mathscr{E}(W)$, the maps $\mathrm{d}f$, ∂f, and $\overline{\partial} f$ defined pointwise are all 1-forms. Note that if (U, z) is a chart of X, then every 1-form ω on W can be written as

$$\omega = f_1 \mathrm{d}x + f_2 \mathrm{d}y = f_1' \mathrm{d}z + f_2' \mathrm{d}\overline{z}$$

for some $f_1, f_2, f_1', f_2' : U \to \mathbb{C}$ by varying $\omega(p) = f_1(p)\,\mathrm{d}_p x + f_2(p)\,\mathrm{d}_p y = f_1'(p)\,\mathrm{d}_p z + f_2'(p)\,\mathrm{d}_p \overline{z}$ over all $p \in U$. ♦

Definition 3.22. *We define certain subspaces of 1-forms on W as follows.*

- *The subspace $\mathscr{E}^{(1)}(W)$ of all differentiable 1-forms ω on W such that, w.r.t. every chart (U, z) of X, $\omega = f\mathrm{d}z + g\mathrm{d}\overline{z}$ for some $f, g \in \mathscr{E}(U \cap W)$.*
- *The subspace $\mathscr{E}^{(1,0)}(W)$ of all type $(1,0)$ 1-forms ω on W such that, w.r.t. every chart (U, z) of X, $\omega = f\mathrm{d}z$ for some $f \in \mathscr{E}(U \cap W)$.*
- *The subspace $\mathscr{E}^{(0,1)}(W)$ of all type $(0,1)$ 1-forms ω on W such that, w.r.t. every chart (U, z) of X, $\omega = f\mathrm{d}\overline{z}$ for some $f \in \mathscr{E}(U \cap W)$.*
- *The subspace $\Omega(W)$ of all holomorphic 1-forms ω on W such that, w.r.t. every chart (U, z) of X, $\omega = f\mathrm{d}z$ for some $f \in \mathscr{O}(U \cap W)$.*

Remark. More work needs to be done to define *meromorphic 1-forms* on W. In fact, we may analogously define the *order of a pole* of a meromorphic 1-form; see [For81, Section 9.9]. ♦

Example 3.23. For $f \in \mathscr{E}(W)$, the form $\mathrm{d}f$ is a differentiable 1-form on W. Thus we have the map $\mathrm{d} : \mathscr{E}(W) \to \mathscr{E}^{(1)}(W)$, called the exterior derivative on $\mathscr{E}(W)$. Similarly, ∂f and $\overline{\partial} f$ are types $(1,0)$ and $(0,1)$ 1-forms on W, respectively, and they induce the Dolbeault operators on $\mathscr{E}(W)$. These derivatives, which are in fact morphisms of sheaves, are studied in the next section. ♦

3.2.3 Differential 2-forms and Exterior Differentiation

Define the *exterior power* $\bigwedge^2 V$ of a $\mathbb{C}$-vector space V as the quotient of the tensor product $V \otimes V$ by the ideal $\mathfrak{a} := (v \otimes v \mid v \in V)$. For completeness, we very briefly define the tensor product $V \otimes V$; for an in-depth discussion, see [Alu09, Chapter 8.2] or [Con16].

Definition 3.24. *Let V be a $\mathbb{C}$-vector space and consider the free vector space $j : V \times V \to F$ over $V \times V$. Letting S denote the span of*

$$j(v, \lambda v_1 + v_2) - \lambda j(v, v_1) - \lambda j(v, v_2) \quad \text{and} \quad j(\lambda v_1 + v_2, v) - \lambda j(v, v_1) - \lambda j(v, v_2),$$

for all $v, v_1, v_2 \in V$ and $\lambda \in \mathbb{C}$, we define the tensor product of V as the quotient space $V \otimes V := F/S$ equipped with the map $\otimes := \pi \circ j$, where $\pi : F \to F/S$ is the projection.

Remark. Let V be a $\mathbb{C}$-vector space. For all $v_1, v_2 \in V$, define $v_1 \wedge v_2 \in \bigwedge^2 V$ to be the equivalence class of $v \otimes v$ modulo $\mathfrak{a}$. It is then immediate from the definition of $V \otimes V$ that $(v_1 + v_2) \wedge v_3 = (v_1 \wedge v_3) + (v_2 \wedge v_3)$ and $(\lambda v_1) \wedge v_2 = \lambda (v_1 \wedge v_2)$ for all $v_1, v_2, v_3 \in V$ and $\lambda \in \mathbb{C}$. Moreover,

$$0 = (v_1 + v_2) \wedge (v_1 + v_2) = (v_1 \wedge v_1) + (v_1 \wedge v_2) + (v_2 \wedge v_1) + (v_2 \wedge v_2) = (v_1 \wedge v_2) + (v_2 \wedge v_1),$$

so $v_1 \wedge v_2 = -(v_2 \wedge v_1)$ for all $v_1, v_2 \in V$. Finally, if $\{e_i\}$ is a basis for V, then $\{e_i \otimes e_j\}$ is a basis for $V \otimes V$; see [Lee12, Proposition 12.8] for a proof. Combined with the above, we see that $\{e_i \wedge e_j\}_{i<j}$ is a basis for $\bigwedge^2 V$. ♦

Remark. We now specialize for when $V = T^*_{\mathbb{C},p} X$ and consider the exterior power $\bigwedge^2 T^*_{\mathbb{C},p} X$. Letting (U, z) be a chart of X around p, we see that $\{\mathrm{d}_p x \wedge \mathrm{d}_p y\}$ and $\{\mathrm{d}_p z \wedge \mathrm{d}_p \overline{z}\}$ are both bases for $\bigwedge^2 T^*_{\mathbb{C},p} X$. Thus $\dim \bigwedge^2 T^*_{\mathbb{C},p} X = 1$. Also, observe that

$$\mathrm{d}_p z \wedge \mathrm{d}_p \overline{z} = (\mathrm{d}_p x + i \mathrm{d}_p y) \wedge (\mathrm{d}_p x - i \mathrm{d}_p y) = -2i (\mathrm{d}_p x \wedge \mathrm{d}_p y).$$ ♦

Definition 3.25. *A differential 2-form on W is a map $\omega : W \to \bigcup_{p \in W} \bigwedge^2 T^*_{\mathbb{C},p} X$ such that $\omega(p) \in \bigwedge^2 T^*_{\mathbb{C},p} X$ for every $p \in W$. A 2-form ω is said to be differentiable if, w.r.t. every chart (U, z) of X, we have $\omega = f \mathrm{d}z \wedge \mathrm{d}\overline{z}$ for some $f \in \mathscr{E}(U \cap W)$.*

Remark. As with 1-forms, the set of all 2-forms on W forms a vector space under the induced operations from $\bigwedge^2 T^*_{\mathbb{C},p} X$. Similarly, it is also a $\mathbb{C}$-algebra by defining the map $f\omega$ by $(f\omega)(p) := f(p)\,\omega(p)$ for every function $f : W \to \mathbb{C}$. ♦

Remark. In the above definition, $\mathrm{d}z \wedge \mathrm{d}\overline{z}$ is the 2-form on W defined by $(\mathrm{d}z \wedge \mathrm{d}\overline{z})(p) := \mathrm{d}_p z \wedge \mathrm{d}_p \overline{z}$ for every $p \in W$. In general, if ω_1 an ω_2 are 1-forms on W, we have the 2-form $\omega_1 \wedge \omega_2$ defined by $(\omega_1 \wedge \omega_2)(p) := \omega_1(p) \wedge \omega_2(p)$ for every $p \in W$. The $\mathbb{C}$-vector space of all differentiable 2-forms on W is denoted $\mathscr{E}^{(2)}(W)$. ♦

Definition/Proposition 3.26. *Let ω be a differentiable 1-form on W, which, under a chart (U, z) of X, has the form $\omega = f_1 \mathrm{d}z + f_2 \mathrm{d}\overline{z}$ for some $f_1, f_2 \in \mathscr{E}(U \cap W)$. Then the 2-form $\mathrm{d}\omega := \mathrm{d}f_1 \wedge \mathrm{d}z + \mathrm{d}f_2 \wedge \mathrm{d}\overline{z}$ is chart-independent and differentiable, which defines the map $\mathrm{d} : \mathscr{E}^{(1)}(W) \to \mathscr{E}^{(2)}(W)$, called the exterior derivative on $\mathscr{E}^{(1)}(W)$.*

Remark. Similarly, define the 2-forms $\partial\omega := \partial f_1 \wedge \mathrm{d}z + \partial f_2 \wedge \mathrm{d}\overline{z}$ and $\overline{\partial}\omega := \overline{\partial} f_1 \wedge \mathrm{d}z + \overline{\partial} f_2 \wedge \mathrm{d}\overline{z}$. The same proof shows that $\partial\omega$ and $\overline{\partial}\omega$ are chart-independent, which define the operators ∂ and $\overline{\partial}$, called the Dolbeault operators on $\mathscr{E}^{(1)}(W)$. ♦

Proof. For convenience, we write $z_1 := z$ and $z_2 := \overline{z}$, so $\omega = \sum_i f_i \mathrm{d}z_i$ and $\mathrm{d}\omega = \sum_i \mathrm{d}f_i \wedge \mathrm{d}z_i$. To show that $\mathrm{d}\omega \in \mathscr{E}^{(2)}(W)$, let (V, w) be a chart of X. Expanding $\mathrm{d}f_i$ and $\mathrm{d}z_i$ in the basis $\{\mathrm{d}w, \mathrm{d}\overline{w}\}$, we see that

$$\mathrm{d}\omega = \sum_{j=1}^{2} \left(\frac{\partial f_i}{\partial w} \mathrm{d}w + \frac{\partial f_i}{\partial \overline{w}} \mathrm{d}\overline{w} \right) \wedge \left(\frac{\partial z_i}{\partial w} \mathrm{d}w + \frac{\partial z_i}{\partial \overline{w}} \mathrm{d}\overline{w} \right) = \sum_{j=1}^{2} \left(\frac{\partial f_i}{\partial w} \frac{\partial z_i}{\partial \overline{w}} - \frac{\partial f_i}{\partial \overline{w}} \frac{\partial z_i}{\partial w} \right) \mathrm{d}w \wedge \mathrm{d}\overline{w} \in \mathscr{E}^{(2)}(W).$$

To show well-definition, let (U', z') be another chart of X and write $\omega = \sum_i f_i' \mathrm{d}z_i'$; again, write $z_1' := z'$ and $z_2' := \overline{z'}$. Choose a chart (V, w) of X. Expanding $\mathrm{d}z_i$ and $\mathrm{d}z_i'$ in the basis $\{\mathrm{d}w, \mathrm{d}\overline{w}\}$ and equating, we obtain by the assumption $\sum_i f_i \mathrm{d}z_i = \sum_i f_i' \mathrm{d}z_i'$ that

$$\sum_{i=1}^{2} f_i \frac{\partial z_i}{\partial w} = \sum_{i=1}^{2} f_i' \frac{\partial z_i'}{\partial w} \qquad \text{and} \qquad \sum_{i=1}^{2} f_i \frac{\partial z_i}{\partial \overline{w}} = \sum_{i=1}^{2} f_i' \frac{\partial z_i'}{\partial \overline{w}}.$$

Applying $\partial/\partial\overline{w}$ and $\partial/\partial w$ respectively and subtracting yields

$$\sum_{i=1}^{2} \left(\frac{\partial f_i}{\partial w} \frac{\partial z_i}{\partial \overline{w}} - \frac{\partial f_i}{\partial \overline{w}} \frac{\partial z_i}{\partial w} \right) = \sum_{i=1}^{2} \left(\frac{\partial f_i'}{\partial w} \frac{\partial z_i'}{\partial \overline{w}} - \frac{\partial f_i'}{\partial \overline{w}} \frac{\partial z_i'}{\partial w} \right).$$

From our previous calculation of $\mathrm{d}\omega$, the result follows. ■

Definition 3.27. *A differentiable 1-form ω on W is closed if $\mathrm{d}\omega = 0$, and is exact if $\omega = \mathrm{d}f$ for some $f \in \mathscr{E}(W)$.*

Proposition 3.28. *Every exact 1-form is closed, every holomorphic 1-form is closed, and every closed 1-form of type $(1, 0)$ is holomorphic.*

Proof. Let ω be a 1-form on W. That every exact form is closed is precisely the statement that $\mathrm{d}^2 f = 0$ for all $f \in \mathscr{E}(W)$, which follows from the computation[8] $\mathrm{d}^2 f = \mathrm{d}(1 \cdot \mathrm{d}f) = \mathrm{d}1 \wedge \mathrm{d}f = 0$. For the other claims, suppose that $\omega = f \mathrm{d}z$ for some $f \in \mathscr{E}(W)$. Then

$$\mathrm{d}\omega = \mathrm{d}f \wedge \mathrm{d}z = \left(\frac{\partial f}{\partial z} \mathrm{d}z + \frac{\partial f}{\partial \overline{z}} \mathrm{d}\overline{z} \right) \wedge \mathrm{d}z = -\frac{\partial f}{\partial \overline{z}} \mathrm{d}z \wedge \mathrm{d}\overline{z}.$$

Thus $\mathrm{d}\omega = 0$ iff $\partial f/\partial\overline{z} = 0$, so every holomorphic 1-form is closed and every closed 1-form of type $(1, 0)$ is holomorphic. ■

[8] The same computation also shows $\partial^2 f = \overline{\partial}^2 f = 0$.

3.2.4 Integration of 2-forms

Similarly to how we defined partial derivatives on a chart (U, z) of X by pulling back the partial derivative on $\mathbb{C}$, we first discuss integration of a 2-form ω on an open set $V \subseteq \mathbb{C}$ and then pull it back to Riemann surfaces.

Let ω of a differentiable 2-form on an open subset $V \subseteq \mathbb{C}$ with the standard chart $x + iy$, say with $\omega = f \mathrm{d}x \wedge \mathrm{d}y$ for some $f \in \mathscr{E}(V)$. If f vanishes outside a compact subset of V, define

$$\int_V \omega = \int_V f \,\mathrm{d}x \wedge \mathrm{d}y := \int_V f \,\mathrm{d}x \,\mathrm{d}y$$

where the right-hand side is the usual double integral on $\mathbb{C}$, which simply 'erases the wedges'. We now define the pullback of forms under a holomorphic map, which gives us a coordinate-free description of the Change of Variables formula.

Definition 3.29. *Let $F : X \to Y$ be a holomorphic map between Riemann surfaces and let $V \subseteq Y$ be open. The pullback of F is the map $F^* : \mathscr{E}(V) \to \mathscr{E}\left(F^{-1}(V)\right)$ mapping $f \mapsto f \circ F$. More generally, define $F^* : \mathscr{E}^{(k)}(V) \to \mathscr{E}^{(k)}\left(F^{-1}(V)\right)$ for $k = 1, 2$ mapping*

$$f_1 \mathrm{d}z + f_2 \mathrm{d}\overline{z} \mapsto (F^* f_1)\, \mathrm{d}(F^* z) + (F^* f_2)\, \mathrm{d}(F^* \overline{z})$$
$$f \mathrm{d}z \wedge \mathrm{d}\overline{z} \mapsto (F^* f)\, \mathrm{d}(F^* z) \wedge \mathrm{d}(F^* \overline{z}).$$

Proposition 3.30. *Let $U, V \subseteq \mathbb{C}$ be open and let $\varphi : U \to V$ be biholomorphic. Then $\int_V \omega = \int_U \varphi^* \omega$ for any differentiable 2-form ω on V.*

Proof. Writing $\omega = f \mathrm{d}x \wedge \mathrm{d}y$ for some $f \in \mathscr{E}(V)$, we have by the Change of Variables on $\mathbb{C}$ that

$$\int_V \omega = \int_V f \,\mathrm{d}x \,\mathrm{d}y = \int_U (f \circ \varphi) \left|\det D\varphi\right| \,\mathrm{d}x \,\mathrm{d}y,$$

where $D\varphi$ is the Jacobian of φ. Decomposing $\varphi = u + iv$ for some $u, v \in \mathscr{E}(U)$ and using the Cauchy-Riemann equations, we see that

$$\det D\varphi = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \geq 0.$$

The computation

$$\begin{aligned} \varphi^* \omega = \varphi^* (f \mathrm{d}x \wedge \mathrm{d}y) &= (\varphi^* f)\, \mathrm{d}(\varphi^* x) \wedge \mathrm{d}(\varphi^* y) = (f \circ \varphi)\, \mathrm{d}u \wedge \mathrm{d}v \\ &= (f \circ \varphi) \left(\frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y\right) \wedge \left(\frac{\partial v}{\partial x}\mathrm{d}x + \frac{\partial v}{\partial y}\mathrm{d}y\right) \\ &= (f \circ \varphi) (\det D\varphi)\, \mathrm{d}x \wedge \mathrm{d}y \end{aligned}$$

then gives us the desired result. ∎

We now lift the integration of 2-forms to Riemann surfaces. To avoid convergence issues, we only consider differentiable 2-forms ω that is *compactly supported*, where the *support* of ω is $\operatorname{Supp}(\omega) := \overline{\{p \in X \mid \omega(p) \neq 0\}}$. Then there exist exist finitely-many charts (U_i, φ_i) on X such that $\operatorname{Supp}(\omega) \subseteq \bigcup_{i=1}^n U_i$. This open cover $\{U_i\}$ of $\operatorname{Supp}(\omega)$ admits a partition of unity[9] $\{\psi_i\}$, which are functions such that $\operatorname{Supp}(\psi_i) \subseteq U_i$ for all i and $\sum_{i=1}^n \psi_i = \mathrm{id}$. Using this partition of unity, we define the integral of ω on X.

Definition/Proposition 3.31. *In the above notation and with $V_i := \varphi_i(U_i)$, define*

$$\int_X \omega := \sum_{i=1}^n \int_{U_i} \psi_i \omega := \sum_{i=1}^n \int_{V_i} (\varphi_i^{-1})^* (\psi_i \omega).$$

Proof. (Well-definition). First, note that the support of $\omega_i := \psi_i \omega$ is contained in U_i, so we have to check that each integral of ω_i over U_i is independent of the chart φ_i, and that the integral of ω over X is independent of $\{U_i\}$ and its partition of unity $\{\psi_i\}$.

- (Independence of φ_i). Note that $\left(\varphi_i^{-1}\right)^* \omega$ is a differentiable 2-form on V_i. Let $\tilde{\varphi}_i$ be another chart of U_i and set $\tilde{V}_i := \tilde{\varphi}(U_i)$. Since $\varphi_i \circ \tilde{\varphi}_i^{-1} : \tilde{V}_i \to V_i$ is biholomorphic and the pullback is anti-multiplicative, we have by Proposition 3.30 that

$$\int_{V_i} (\varphi_i^{-1})^* \omega_i = \int_{\tilde{V}_i} (\varphi_i \circ \tilde{\varphi}_i^{-1})^* (\varphi_i^{-1})^* \omega_i = \int_{\tilde{V}_i} (\tilde{\varphi}_i^{-1})^* \varphi_i^* (\varphi_i^{-1})^* \omega = \int_{\tilde{V}_i} (\tilde{\varphi}_i^{-1})^* \omega.$$

- (Independence of $\{U_i\}$). Let $\{\tilde{U}_j\}_{j=1}^m$ be another finite open cover and let $\{\tilde{\psi}_j\}_{j=1}^m$ be its corresponding partition of unity. We expand the definition of $\int_X \omega$ on both charts as

$$\sum_{i=1}^n \int_{U_i} \psi_i \omega = \sum_{i=1}^n \int_{U_i} \left(\sum_{j=1}^m \tilde{\psi}_j\right) \psi_i \omega = \sum_{i=1}^n \sum_{j=1}^m \int_{U_i} \tilde{\psi}_j \psi_i \omega$$
$$\sum_{j=1}^m \int_{\tilde{U}_j} \tilde{\psi}_j \omega = \sum_{j=1}^m \int_{\tilde{U}_j} \left(\sum_{i=1}^n \psi_i\right) \tilde{\psi}_j \omega = \sum_{i=1}^n \sum_{j=1}^m \int_{\tilde{U}_j} \tilde{\psi}_j \psi_i \omega.$$

Note that $\tilde{\psi}_j \psi_i \omega$ is compactly supported on $U_i \cap \tilde{U}_j$, so their integrals over U_i and $\tilde{U}_j$ coincide and hence is well-defined. ∎

[9] The 'local-finiteness' condition is irrelevant here since the cover is finite. For a proof of the existence of a partition of unity, see [Lee12, Theorem 2.23].

3.3 Global Meromorphic Functions

In this section, we show that $\check{H}^1(\hat{\mathbb{C}}, \mathscr{O})$ vanishes which proves the existence of certain meromorphic functions on a compact Riemann surface. The crucial step of the proof is the following lemma.

3.3.1 Dolbeault's Lemma

Lemma 3.32 (Dolbeault)**.** *For any function $g \in \mathscr{E}(\mathbb{C})$, there exists a differentiable function $f \in \mathscr{E}(\mathbb{C})$ such that $\overline{\partial} f = g \mathrm{d}\overline{z}$.*

Proof. We first prove the lemma for when g is compactly supported. In this case, define $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) := -\frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g(z-\zeta)}{\zeta} \mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta}.$$

We need to show that this integral converges and depends differentiably on z. Since g is compactly supported, the integrand only has a pole at 0 and so it suffices to show that the integral over a disk $D_\varepsilon := \overline{B_\varepsilon} := \overline{B(0,\varepsilon)}$ converges. Indeed, we use Proposition 3.30 to change to polar coordinates to see that

$$\int_{D_\varepsilon} \frac{g(z-\zeta)}{\zeta} \mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta} = \int_0^\varepsilon \int_0^{2\pi} g\big(z - re^{i\theta}\big) e^{-i\theta} \, \mathrm{d}r \mathrm{d}\theta,$$

which is convergent since g is bounded. Now, to show that $f \in \mathscr{E}(\mathbb{C})$, we expand the definition of f into

$$f(z) = -\frac{1}{2\pi i} \lim_{\varepsilon \to 0} \int_{\mathbb{C} \setminus B_\varepsilon} \frac{g(z-\zeta)}{\zeta} \mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta}.$$

The uniform convergence of the integral allows us to differentiate under the integral sign, so $f \in \mathscr{E}(\mathbb{C})$. Finally, to show that f solves the differential equation, we do so explicitly for the operator $\partial / \partial \overline{z}$ to obtain

$$\left.\frac{\partial f}{\partial \overline{z}}\right|_z = -\frac{1}{2\pi i} \lim_{\varepsilon \to 0} \int_{\mathbb{C} \setminus B_\varepsilon} \frac{1}{\zeta} \left.\frac{\partial g}{\partial \overline{z}}\right|_{z-\zeta} \mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta}.$$

Using $\mathrm{d} = \partial + \overline{\partial}$ and using the fact that $1/\zeta$ is holomorphic away from 0, we see that

$$\begin{aligned}
\mathrm{d}\left(\frac{g(z-\zeta)}{\zeta} \mathrm{d}\zeta\right) &= \partial\left(\frac{g(z-\zeta)}{\zeta} \mathrm{d}\zeta\right) + \overline{\partial}\left(\frac{g(z-\zeta)}{\zeta} \mathrm{d}\zeta\right) \\
&= \frac{\partial}{\partial \zeta}\left(\frac{g(z-\zeta)}{\zeta}\right) \mathrm{d}\zeta \wedge \mathrm{d}\zeta + \frac{\partial}{\partial \overline{\zeta}}\left(\frac{g(z-\zeta)}{\zeta}\right) \mathrm{d}\overline{\zeta} \wedge \mathrm{d}\zeta \\
&= -\frac{1}{\zeta} \left.\frac{\partial g}{\partial \overline{\zeta}}\right|_{z-\zeta} \mathrm{d}\zeta \wedge \mathrm{d}\overline{\zeta}.
\end{aligned}$$

Thus we have by Stokes's Theorem that

$$\left.\frac{\partial f}{\partial \overline{z}}\right|_z = \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \int_{C \setminus B_\varepsilon} \mathrm{d}\left(\frac{g(z-\zeta)}{\zeta} \mathrm{d}\zeta\right) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \int_{|\zeta| = \varepsilon} \frac{g(z-\zeta)}{\zeta} \mathrm{d}\zeta.$$

This integral can be calculated in polar coordinates as $\zeta = \varepsilon e^{i\theta}$ for $0 \le \theta < 2\pi$, so

$$\int_{|\zeta| = \varepsilon} \frac{g(z-\zeta)}{\zeta} \mathrm{d}\zeta = \int_0^{2\pi} \frac{g\big(z - \varepsilon e^{i\theta}\big)}{\varepsilon e^{i\theta}} \varepsilon i e^{i\theta} \, \mathrm{d}\theta = i \int_0^{2\pi} g\big(z - \varepsilon e^{i\theta}\big) \, \mathrm{d}\theta.$$

It follows then that

$$\left.\frac{\partial f}{\partial \overline{z}}\right|_z = \frac{1}{2\pi} \lim_{\varepsilon \to 0} \int_0^{2\pi} g\big(z - \varepsilon e^{i\theta}\big) \, \mathrm{d}\theta,$$

which is the average value of $g(z)$ on the circle of radius ε around z. In the limit $\varepsilon \to 0$, we see that $\partial f / \partial \overline{z} = g$ and hence $\overline{\partial} f = g \, \mathrm{d}\overline{z}$.

Now, for the general case, we consider an increasing sequence of radii $\{R_n\}$ converging to infinity and their associated balls $B_n := B(0, R_n)$. For all n, there exists a function $\psi_n \in \mathscr{E}(\mathbb{C})$ such that $\operatorname{Supp}(\psi_n) \subseteq B_{n+1}$ and $\psi_n|_{B_n} = \mathrm{id}$; for instance, take bump functions. Extending $\psi_n g$ by zero outside B_{n+1}, they become differentiable functions in $\mathbb{C}$ with compact supports and hence $\overline{\partial} f_n = \psi_n g \, \mathrm{d}\overline{z}$ for some $f_n \in \mathscr{E}(\mathbb{C})$. We shall inductively construct a new sequence $\{\tilde{f}_n\}$ of differentiable functions on $\mathbb{C}$ such that

1. $\overline{\partial} \tilde{f}_n = g \, \mathrm{d}\overline{z}$ on B_n and
2. $\|\tilde{f}_{n+1} - \tilde{f}_n\|_{B_n} \le 2^{-n}$.

Here, $\|f\|_{B_n} := \sup_{x \in B_n} |f(x)|$ is the supremum norm. Set $\tilde{f}_1 := f_1$ and suppose that the functions $\tilde{f}_1, \ldots, \tilde{f}_n$ are defined. Then

$$\overline{\partial}\big(f_{n+1} - \tilde{f}_n\big) = \overline{\partial} f_{n+1} - \overline{\partial} \tilde{f}_n = (\psi_{n+1} g - g) \, \mathrm{d}\overline{z} = 0$$

on B_n, so the function $f_{n+1} - \tilde{f}_n$ is holomorphic on B_n. Thus there exists a polynomial $p \in \mathbb{C}[z]$ such that

$$\big\|f_{n+1} - \tilde{f}_n - p\big\|_{B_n} \le 2^{-n},$$

so take $\tilde{f}_{n+1} := f_{n+1} - p \in \mathscr{E}(\mathbb{C})$. This satisfies (2), and since

$$\overline{\partial}\tilde{f}_{n+1} = \overline{\partial} f_{n+1} = \psi_{n+1} g \mathrm{d}\overline{z} = g \mathrm{d}\overline{z}$$

on B_{n+1}, we see that (1) holds too. By (2), the (pointwise) limit $\tilde{f}_n(z)$ converges to some $f(z)$, where we claim that $f \in \mathscr{E}(\mathbb{C})$ and that $\overline{\partial} f = g\,\mathrm{d}\overline{z}$. Note that the series

$$F_n := \sum_{k \geq n} \left(\tilde{f}_{k+1} - \tilde{f}_k\right)$$

converges (uniformly) on B_n, and since $\overline{\partial}\left(\tilde{f}_{k+1} - \tilde{f}_k\right) = 0$ on B_n for all $k \geq n$, it is holomorphic on B_n. This shows that $f = \tilde{f}_n + F_n$ is differentiable and that

$$\overline{\partial} f = \overline{\partial}\tilde{f}_n + \overline{\partial} F_n = \overline{\partial}\tilde{f}_n = g\,\mathrm{d}\overline{z}$$

on B_n. But this holds for all n, so $f \in \mathscr{E}(\mathbb{C})$ with $\overline{\partial} f = g\,\mathrm{d}\overline{z}$ globally. ■

Remark. Dolbeault's Lemma is a special case of the $\overline{\partial}$-Poincaré Lemma. Indeed, we can reformulate the theorem by saying that the sequence of sheaves

$$0 \longrightarrow \mathscr{O} \xhookrightarrow{\iota} \mathscr{E} \xrightarrow{\overline{\partial}} \mathscr{E}^{(0,1)} \longrightarrow 0$$

is exact. The only nontrivial claim to verify is that $\overline{\partial}$ is surjective, which is precisely the statement of the lemma. ♦

3.3.2 Vanishing of $\check{H}^1\left(\hat{\mathbb{C}}, \mathscr{O}\right)$

Lemma 3.33. *Let X be a Riemann surface and consider the sheaf of differentiable functions $\mathscr{E}$ on X. Then $\check{H}^1(X, \mathscr{E}) = 0$.*

Proof. Let $\mathfrak{A} := \{U_i\}_{i \in I}$ be an open covering of X and let $(f_{ij}) \in \check{Z}^1(\mathfrak{A}, \mathscr{E})$ be a cocycle; it suffices to show that (f_{ij}) splits, for then $\check{H}^1(\mathfrak{A}, \mathscr{E}) = 0$ and we are done since $\mathfrak{A}$ is arbitrary. To do so, we use the fact that there exists a partition of unity subordinate to $\mathfrak{A}$; that is, a family $\{\psi_i\}_{i \in I}$ of differentiable functions such that:

- $\operatorname{Supp}(\psi_i) := \overline{\{p \in X \mid \psi(p) \neq 0\}} \subseteq U_i$ for every $i \in I$.
- Every point in X admits a neighborhood whose intersection with $\{\operatorname{Supp}(\psi_i)\}_{i \in I}$ is finite.
- $\sum_{i \in I} \psi_i = \mathrm{id}$.

Consider the function $\psi_j f_{ij}$ on $U_i \cap U_j$, which may be differentiably extended to U_i by zero outside $\operatorname{Supp}(\psi_j)$. Consider the function $g_i := \sum_{j \in I} \psi_j f_{ij} \in \mathscr{E}(U_i)$, which is legal since there is a neighborhood around every point of U_i such that $\psi_j f_{ij} = 0$ for all but finitely-many $j \in I$. Observe that

$$g_i - g_j = \sum_{k \in I} \psi_k \left(f_{ik} - f_{jk}\right) = \sum_{k \in I} \psi_k \left(f_{ik} + f_{kj}\right) = \sum_{k \in I} \psi_k f_{ij} = f_{ij}$$

on $U_i \cap U_j$, so $(f_{ij}) = (g_i - g_j) = \delta^0(g_i)$ splits. ■

Theorem 3.34. *The 1^{st} cohomology groups $\check{H}^1(\mathbb{C}, \mathscr{O})$ and $\check{H}^1\left(\hat{\mathbb{C}}, \mathscr{O}\right)$ vanish.*

Proof. We first prove that $\check{H}^1(\mathbb{C}, \mathscr{O})$ vanishes, for which it suffices to take any open covering $\mathfrak{A} := \{U_i\}$ of $\mathbb{C}$ and show that every cocycle $(f_{ij}) \in \check{Z}^1(\mathbb{C}, \mathscr{O})$ splits. Indeed, since $\check{Z}^1(\mathfrak{A}, \mathscr{O}) \subseteq \check{Z}^1(\mathfrak{A}, \mathscr{E})$ and $\check{H}^1(\mathbb{C}, \mathscr{E})$ vanishes by Lemma 3.33, there exists a cochain $(g_i) \in \check{C}^0(\mathfrak{A}, \mathscr{E})$ such that $f_{ij} = g_i - g_j$ on $U_i \cap U_j$. But $\overline{\partial} f_{ij} = 0$, so $\overline{\partial} g_i = \overline{\partial} g_j$ on $U_i \cap U_j$ for all i, j and hence glues to a global function $h \in \mathscr{E}(\mathbb{C})$ such that $h|_{U_i}\,\mathrm{d}\overline{z} = \overline{\partial} g_i$. Dolbeault's Lemma then furnishes some $g \in \mathscr{E}(\mathbb{C})$ such that $\overline{\partial} g = h\,\mathrm{d}\overline{z}$. Define $\tilde{g}_i := g_i - g$, and since $\overline{\partial}\tilde{g}_i = \overline{\partial} g_i - \overline{\partial} g = 0$ on U_i, we see that $(\tilde{g}_i) \in \check{C}^0(\mathfrak{A}, \mathscr{O})$. Observe that $f_{ij} = g_i - g_j = \tilde{g}_i - \tilde{g}_j$ so (f_{ij}) splits.

For the Riemann sphere, consider the cover $\mathfrak{A} := \{U_1, U_2\}$ given in Example 1.5. Since $U_1, U_2 \cong \mathbb{C}$, we see from the vanishing of $\check{H}^1(\mathbb{C}, \mathscr{O})$ that $\mathfrak{A}$ is a Leray covering of X, so $\check{H}^1\left(\hat{\mathbb{C}}, \mathscr{O}\right) \cong \check{H}^1(\mathfrak{A}, \mathscr{O})$ by Proposition 3.15. Thus it suffices to show that any cocycle $(f_{ij}) \in \check{Z}^1(\mathfrak{A}, \mathscr{O})$ splits; by symmetry, it suffices to find functions $f_i \in \mathscr{O}(U_i)$ such that $f_{12} = f_1 - f_2$ on $U_1 \cap U_2 = \mathbb{C}^*$. Note that f_{12} is not necessarily holomorphic at 0, so it admits a Laurent series expansion $\sum_{n=-\infty}^{\infty} c_n z^n$ on $\mathbb{C}^*$. But the series $f_1(z) := \sum_{n=0}^{\infty} c_n z^n$ and $f_2(z) := \sum_{n=-\infty}^{-1} c_n z^n$ converge on U_1 and U_2, respectively, so $f_i \in \mathscr{O}(U_i)$. Clearly $f_{12} = f_1 - f_2$, as desired. ■

Remark. Let X be a compact Riemann surface and consider the vector space structure on $\check{H}^1(X, \mathscr{O})$ induced from $\mathscr{O}$. We appeal to the following theorems.

- The dimension $g := \dim \check{H}^1(X, \mathscr{O})$ is finite[10] and is referred to as the genus of X. The above theorem states that $\hat{\mathbb{C}}$ has genus 0.
- The genus of X depends only on the smooth manifold structure on X. In particular, since $\check{H}^1\left(\hat{\mathbb{C}}, \mathscr{O}\right)$ vanishes, the genus of any simply-connected compact Riemann surface X is 0. ♦

Corollary 3.34.1. *Let X be a simply-connected compact Riemann surface and fix $p \in X$. Then there exists a meromorphic function $f \in \mathscr{M}(X)$ which has a pole of order 1 at p and is holomorphic everywhere else.*

Proof. Let (U_1, z) be a chart of X centered at p and set $U_2 := X \setminus \{p\}$, so $\mathfrak{A} := \{U_1, U_2\}$ is an open cover of X. Consider the holomorphic function z^{-1} on $U_1 \cap U_2 = U_1 \setminus \{p\}$. Since X is simply-connected, it has genus 0 and hence $\check{H}^1(X, \mathscr{O})$ vanishes. Thus the cocycle $\left(z^{-1}\right) \in \check{Z}^1(\mathfrak{A}, \mathscr{O})$ splits, so there exist functions $f_i \in \mathscr{O}(U_i)$ such that $z^{-1} = f_2 - f_1$. Observe that $f_1 + z^{-1}$ agrees with f_2 on $U_1 \cap U_2$, so they glue to a global function $f \in \mathscr{M}(X)$ which has a pole of order 1 at p and is holomorphic everywhere else. ■

[10] See [For81, Section 14] for a proof.

Chapter 4

Moduli Spaces

For any genus g, we let $\mathcal{M}_g$ denote the *moduli space* of compact Riemann surfaces of genus g, defined as the set of all Riemann surfaces of genus g up to biholomorphism. Using the language and machinery developed in Chapters 1, 2, and 3, we compute $\mathcal{M}_0$ and $\mathcal{M}_1$, which are the moduli spaces of the sphere S^2 and the torus T^2, respectively.

We conclude with a brief discussion of the *Uniformization Theorem* and the *Classification of Riemann Surfaces.*

4.1 Simply-connected Riemann Surfaces

The *Uniformization Theorem* states that every simply-connected Riemann surface is biholomorphic to either the Riemann sphere $\hat{\mathbb{C}}$, the complex plane $\mathbb{C}$, or the upper-half plane $\mathbb{H}$ of $\mathbb{C}$. In the compact case, this is precisely the statement that there is a *unique* complex structure on the sphere, which we prove below. The non-compact case requires tools that we have yet to develop, so only a proof sketch is given.

4.1.1 Moduli Space of S^2

We show that the moduli space of the sphere S^2 is a point[1]. That is, there is a *unique* complex structure on the sphere.

Theorem 4.1. *Every simply-connected compact Riemann surface X is biholomorphic to the Riemann sphere $\hat{\mathbb{C}}$.*

Proof. The Classification Theorem of Surfaces shows that such a Riemann surface X, being simply-connected and compact, is homeomorphic to the sphere $\hat{\mathbb{C}}$. Theorem 3.34 shows that $\check{H}^1\left(\hat{\mathbb{C}}, \mathscr{O}\right)$ vanishes, and since the genus is a topological invariant, we see that $\check{H}^1(X, \mathscr{O})$ vanishes too. Hence X has genus 0, so for any fixed point $p \in X$, Corollary 3.34.1 furnishes a meromorphic function $f \in \mathscr{M}(X)$ with a single simple pole at p. Thus $X \cong \hat{\mathbb{C}}$ by Corollary 2.11.2, as desired. ∎

4.1.2 The Uniformization Theorem

Theorem 4.2 (Uniformization)**.** *Every simply-connected Riemann surface X is biholomorphic to either the Riemann sphere $\hat{\mathbb{C}}$, the complex plane $\mathbb{C}$, or the upper-half plane $\mathbb{H} := \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$.*

Proof sketch. This sketch follows [Kro19]. Fix $p \in X$. Using tools from Dolbeault cohomology, let $f \in \mathscr{M}(X)$ be a meromorphic function with a single simple pole at p. Let $F : X \to \hat{\mathbb{C}}$ be its associated holomorphic map, so $F(p) = \infty$.

- First, it can be shown that $\operatorname{Im} F(x) \to 0$ as '$x \to \infty$' in X. That is, for every $\varepsilon > 0$, there is a large enough compact subset K of X such that $\operatorname{Im} F(x) < \varepsilon$ for all $x \in X \setminus K$.
- It can also be shown that $\operatorname{im} F$ is open, contains the 'top and bottom halves' of $\hat{\mathbb{C}}$, and is a biholomorphism onto its image.

Thus $X \cong \operatorname{im} F = \hat{\mathbb{C}} \setminus I$ for some $I \subseteq \mathbb{R}$. By simply-connectedness of X, we see that I is connected and hence we have three possibilities.

- If $I = \varnothing$, then $F : X \to \hat{\mathbb{C}}$ is a biholomorphism, which reduces to Theorem 4.1.
- If I is a singleton, then $\hat{\mathbb{C}} \setminus I \cong \mathbb{C}$, so $X \cong \mathbb{C}$.
- If I is an interval $[a, b]$, we may without loss of generality take $a = 0$ and $b = \infty$. Then the (usual branch of the) square root function sends $\hat{\mathbb{C}} \setminus [0, \infty]$ to $\mathbb{H}$. ∎

Remark. It turns out that one can construct a simply-connected Riemann surface $\widetilde{X}$ from any Riemann surface X. Since $\widetilde{X}$ is exactly one of three types, this leads to a classification of Riemann surfaces. ♦

Definition 4.3. *Let X and E be connected topological spaces. A covering map $\pi : E \to X$ if said to be the universal covering of X if for every covering $\pi' : E' \to X$ on a connected topological space E' and every $e \in E$ and $e' \in E'$ such that $\pi(e) = \pi'(e')$, there exists a unique continuous map $\sigma : E \to E'$ with $\sigma(e) = e'$ making the below diagram commute.*

$$\begin{tikzcd} E \arrow[rr, dashed, "\exists!\sigma"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & X & \end{tikzcd}$$

Remark. As with all 'universal properties', the universal covering of X is unique up to isomorphism. Note that σ is the lifting of of π along π', so if E is simply-connected, then by Proposition 2.15 such a lifting exists and is unique. In this case, *any* covering map is the universal covering of X. We quote the following theorem that guarantees the existence of such a simply-connected space. ♦

[1]This result is an easy corollary of the Riemann-Roch Theorem, but its proof is beyond the scope of this paper. We refer the interested reader to [For81, Section 16].

Theorem 4.4 ([For81, Theorem 5.3]). *Suppose X is a connected manifold. Then there exists a connected, simply-connected manifold $\widetilde{X}$ and a covering map $\pi : \widetilde{X} \to X$.*

Example 4.5. Recall from Example 2.5 that for any lattice $\Gamma \subseteq \mathbb{C}$, the projection $\pi : \mathbb{C} \to \mathbb{C}/\Gamma$ is a covering map. Since $\mathbb{C}$ is simply-connected, we see that π is the universal covering of $\mathbb{C}/\Gamma$. ⧫

Remark. For any Riemann surface X, let $\widetilde{X}$ be its simply-connected universal covering. If $\widetilde{X} \cong \hat{\mathbb{C}}$ (resp. $\mathbb{C}$, $\mathbb{H}$), then X is said to be *elliptic* (resp. *parabolic*, *hyperbolic*)[2].

- Since $\hat{\mathbb{C}}$ is simply-connected, it is the universal covering of itself and hence $\hat{\mathbb{C}}$ is elliptic.
- Since $\mathbb{C}$ is the universal covering of any torus $\mathbb{C}/\Gamma$, we see that $\mathbb{C}/\Gamma$ is parabolic.

It turns out that the universal covering for any compact Riemann surfaces with $g > 1$ is $\mathbb{H}$, so they are all hyperbolic[3]. ⧫

4.2 Moduli Space of T^2

We show that the moduli space of the torus T^2 is $\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$, where $\mathbb{H}$ is the upper-half plane of $\mathbb{C}$ and $\mathrm{PSL}_2(\mathbb{Z}) \coloneqq \mathrm{SL}_2(\mathbb{Z})/\{\pm I\}$ is the *modular group*, which acts on $\mathbb{H}$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau \coloneqq \frac{a\tau + b}{c\tau + d}.$$

We first need a technical lemma, which gives an equivalent condition for a biholomorphism between tori in terms of their lattices.

Lemma 4.6. *Let $\Gamma, \Gamma' \subseteq \mathbb{C}$ be two lattices and suppose $\alpha\Gamma \subseteq \Gamma'$ for some $\alpha \in \mathbb{C}^*$. Then $z \mapsto \alpha z$ descends to a holomorphic map $\varphi : \mathbb{C}/\Gamma \to \mathbb{C}/\Gamma'$, which is biholomorphic iff $\alpha\Gamma = \Gamma'$.*

Proof. Let $\Gamma \coloneqq \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ and $\Gamma' \coloneqq \mathbb{Z}\omega_1' \oplus \mathbb{Z}\omega_2'$. Define $\varphi(z + \Gamma) \coloneqq \alpha z + \Gamma'$ for all $z \in \mathbb{C}$, which is clearly holomorphic if it is well-defined in the first place. Indeed, take $z_1, z_2 \in \mathbb{C}$ such that $z_1 + \Gamma = z_2 + \Gamma$. Then $z_1 - z_2 \in \Gamma$, so $z_1 - z_2 = m\omega_1 + n\omega_2$ for some $n, m \in \mathbb{Z}$. Observe that

$$\alpha z_1 - \alpha z_2 = \alpha (z_1 - z_2) = m (\alpha\omega_1) + n (\alpha\omega_2) \in \alpha\Gamma \subseteq \Gamma',$$

so $\alpha z_1 + \Gamma' = \alpha z_2 + \Gamma'$. This shows that φ is well-defined. Furthermore, it is invertible with holomorphic inverse $\varphi^{-1}(z + \Gamma') \coloneqq z/\alpha + \Gamma$ iff φ^{-1} is well-defined, in which case φ is a biholomorphism. We claim that this occurs iff $\alpha\Gamma = \Gamma'$.

- ($\Rightarrow$): It suffices to show that $\Gamma' \subseteq \alpha\Gamma$, so take $m\omega_1' + n\omega_2' \in \Gamma'$. Then $\varphi^{-1}\big(m\omega_1' + n\omega_2' + \Gamma'\big) = \big(m\omega_1' + n\omega_2'\big)/\alpha + \Gamma$, but since $m\omega_1' + n\omega_2' + \Gamma' = 0 + \Gamma'$ and $\varphi^{-1}(0 + \Gamma') = 0 + \Gamma$, we see that $\big(m\omega_1' + n\omega_2'\big)/\alpha \in \Gamma$.
- ($\Leftarrow$): Take $z_1, z_2 \in \mathbb{C}$ such that $z_1 + \Gamma' = z_2 + \Gamma'$, so $z_1 - z_2 \in \Gamma' \subseteq \alpha\Gamma$ and hence $z_1/\alpha - z_2/\alpha = (z_1 - z_2)/\alpha \in \Gamma$. Then $z_1/\alpha + \Gamma = z_2/\alpha + \Gamma$, so φ^{-1} is well-defined. ■

Lemma 4.7. *Any torus $\mathbb{C}/\Gamma$ is biholomorphic to $X_\tau \coloneqq \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ for some $\tau \in \mathbb{H}$.*

Remark. Intuitively, scaling and rotating the lattice, which are biholomorphisms of the plane, should preserve the complex structure on the torus. Thus only one complex parameter is needed to generate the torus, which we choose to be the ratio $\tau \coloneqq \omega_2/\omega_1$. ⧫

Proof. Let $\Gamma \coloneqq \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ and set $\alpha \coloneqq 1/\omega_1$ and $\tau \coloneqq \omega_2/\omega_1$. Then $\operatorname{Im}\tau \neq 0$, lest ω_1, ω_2 be linearly dependent over $\mathbb{R}$. Without loss of generality, suppose that $\operatorname{Im}\tau > 0$; if not, take $\tau \coloneqq \overline{\omega_2}/\omega_1$. Then, since

$$\alpha (m\omega_1 + n\omega_2) = \alpha\omega_1 (m + n\omega_2/\omega_1) = m + n\tau$$

for all $m, n \in \mathbb{Z}$, we see that $\alpha\Gamma = \mathbb{Z} \oplus \mathbb{Z}\tau$. By Lemma 4.6, the map $z \mapsto \alpha z$ descends to a biholomorphism $\varphi : \mathbb{C}/\Gamma \to \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau) = X_\tau$, so $\mathbb{C}/\Gamma \cong X_\tau$. ■

Theorem 4.8. *For any $\tau, \tau' \in \mathbb{H}$, the tori X_τ and $X_{\tau'}$ are biholomorphic iff τ and τ' lie in the same orbit of the action of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathbb{H}$.*

Corollary 4.8.1. *The moduli space of T^2 is $\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$.*

Proof. The backwards direction is relatively straightforward. Indeed, note that

$$\tau' = \frac{a\tau + b}{c\tau + d} \qquad \Rightarrow \qquad \tau = \frac{b - d\tau'}{c\tau' - a}$$

for any $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$. Setting $\Gamma \coloneqq \mathbb{Z} \oplus \mathbb{Z}\tau$ and $\Gamma' \coloneqq \mathbb{Z} \oplus \mathbb{Z}\tau'$, we see with $\alpha \coloneqq c\tau' - a$ that $\alpha\Gamma \subseteq \Gamma'$. We claim that $\alpha\Gamma = \Gamma'$, from which the result follows from Lemma 4.6. Indeed, for any $m, n \in \mathbb{Z}$, the condition that $ad - bc = 1$ shows that

$$\big(m + n\tau'\big)/\alpha = \frac{(na - mc)\,\tau + (nb - md)}{a\,(c\tau + d) - c\,(a\tau + b)} = (nb - md) + (na - mc)\,\tau \in \mathbb{Z} \oplus \mathbb{Z}\tau,$$

[2]This classification is similar to that of Riemannian manifolds. In fact, every Riemann surface admits a Riemannian metric of constant curvature, either of 1, 0, or -1, which respectively correspond to the curvatures of $\hat{\mathbb{C}}$, $\mathbb{C}$, and $\mathbb{H}$ when equipped with the appropriate metrics.

[3]This fact has an analogue for three-dimensional real manifolds (called 3-*manifolds*). Indeed, *Thurston's Geometrization Conjecture* (proven by Grigori Perelman in 2003, for which he was awarded the Fields Medal) states that all 3-manifolds can be decomposed into pieces, each having one of eight different geometric structures, and the richest of the eight geometries turns out to be the hyperbolic 3-manifold.

so $\Gamma' = \mathbb{Z} \oplus \mathbb{Z}\tau' \subseteq \alpha(\mathbb{Z} \oplus \mathbb{Z}\tau) \subseteq \alpha\Gamma$. For the forward direction, let $\varphi : X_\tau \to X_{\tau'}$ be a biholomorphism. By Proposition 2.15, this biholomorphism lifts to a unique biholomorphism $\widetilde{\varphi} : \mathbb{C} \to \mathbb{C}$ fixing 0 and making the diagram

$$\begin{tikzcd} \mathbb{C} \arrow[r, "\tilde{\varphi}"] \arrow[d, "\pi"'] & \mathbb{C} \arrow[d, "\pi'"] \\ \mathbb{C}/\Gamma \arrow[r, "\varphi"] & \mathbb{C}/\Gamma' \end{tikzcd}$$

commute. We claim that $\widetilde{\varphi}(z) = \alpha z$ for some $\alpha \in \mathbb{C}^*$.[4] Indeed, fix $\lambda \in \Gamma$ and consider the map $f_\lambda(z) := \tilde{\varphi}(z+\lambda) - \tilde{\varphi}(z)$. Then, since $z+\lambda+\Gamma = z+\Gamma$, we see that $\varphi(z+\lambda+\Gamma) = \varphi(z+\Gamma)$ and hence the commutativity of the diagram forces $\tilde{\varphi}(z+\lambda)+\Gamma' = \tilde{\varphi}(z)+\Gamma'$. Thus $f_\lambda(z) \in \Gamma'$ for all $z \in \mathbb{C}$, so, since f_λ is a continuous map into a discrete set, it must be constant. Differentiating gives us $f'_\lambda(z) = \tilde{\varphi}'(z+\lambda) - \tilde{\varphi}'(z) = 0$, so $\tilde{\varphi}'(z+\lambda) = \tilde{\varphi}'(z)$ for all $z \in \mathbb{C}$. But $\lambda \in \Gamma$ is arbitrary, so $\tilde{\varphi}'$ is Γ-periodic. Thus $\tilde{\varphi}'$ is a bounded entire function and hence is constant by Liouville's Theorem. This shows that $\tilde{\varphi}(z) = \alpha z + \beta$ for some $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$, but since $\widetilde{\varphi}$ fixes 0, we have $\widetilde{\varphi}(z) = \alpha z$, as desired. We now claim that $\alpha\Gamma = \Gamma'$.

- Indeed, for all $z \in \alpha\Gamma$, we have $z/\alpha \in \Gamma$ and so $z/\alpha + \Gamma = 0 + \Gamma$. Applying φ to both sides and comparing gives

$$0 + \Gamma' = \varphi(0+\Gamma) = \varphi(z/\alpha + \Gamma) = \tilde{\varphi}(z/\alpha) + \Gamma' = z + \Gamma',$$

  so $z \in \Gamma'$. The converse is similar.

Observe then that $\tilde{\varphi}(\tau) = \alpha\tau = b - d\tau'$ and $\tilde{\varphi}(1) = \alpha = c\tau' - a$ for some $a, b, c, d \in \mathbb{Z}$, so

$$\tau = \frac{b - d\tau'}{c\tau' - a} \qquad \text{and hence} \qquad \tau' = \frac{a\tau + b}{c\tau + d}.$$

A computation now shows that $\alpha = -(ad - bc)/(c\tau + d)$, so $ad - bc \neq 0$. Then, since

$$\begin{pmatrix} \alpha\tau \\ \alpha \end{pmatrix} = \begin{pmatrix} b & -d \\ -a & c \end{pmatrix} \begin{pmatrix} 1 \\ \tau' \end{pmatrix},$$

we solve for τ' to obtain

$$\tau' = -\frac{b\alpha + a\alpha\tau}{ad - bc} = \left(\frac{-b}{ad - bc}\right)\alpha + \left(\frac{-a}{ad - bc}\right)\alpha\tau$$

But $\tau' \in \alpha\Gamma$, which forces $ad - bc = \pm 1$. A little algebra now shows that

$$\operatorname{Im}\tau' = \frac{ad - bc}{|c\tau + d|^2}(\operatorname{Im}\tau) > 0,$$

so $ad - bc = 1$. Thus τ' lies in the orbit of τ, as desired. Finally, the corollary follows from Lemma 4.7. ■

[4] This is a classical result from complex analysis, which states that every automorphism on $\mathbb{C}$ is of the form $z \mapsto \alpha z + \beta$ for some $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$. For a proof, see [Tan91, Lemma 2.8]. Since our automorphism $\widetilde{\varphi}$ is more specific, we present a simpler proof, which roughly follows [Shu05, Proposition 1.3.2].

Appendix A

Analytic Continuation

Proposition A.1. *Let $\mathscr{F}$ be a presheaf of Abelian groups on X. Let $|\mathscr{F}| := \coprod_{p\in X} \mathscr{F}_p$ and consider the projection $\pi : |\mathscr{F}| \to X$ mapping each $\eta \in \mathscr{F}_p$ to p. Then the system $\mathcal{B}$ of all sets*

$$[U, f] := \left\{ [f]_p \;\middle|\; p \in U \right\} \subseteq |\mathscr{F}|$$

for $U \in \tau$ and $f \in \mathscr{F}(U)$ is a basis for a topology on $|\mathscr{F}|$ and π is a local homeomorphism.

Proof. We first verify that $\mathcal{B}$ is a basis.

(1) Take $\eta \in |\mathscr{F}|$, so there exists an open set $U \in \tau$ such that $\eta = [f]_p$ for some $f \in \mathscr{F}(U)$ and $p \in U$. Observe that $\eta \in [U, f]$.

(2) Take $[U, f], [V, g] \in \mathcal{B}$ and $\eta \in [U, f] \cap [V, g]$. Then there exists a point $p \in X$ such that $\eta = [f]_p = [g]_p$, which furnishes an open set $W \in \tau$ with $p \in W \subseteq U \cap V$ such that $\rho^U_W(f) = \rho^V_W(g) =: h$. Then $\eta = [h]_p$ with $h \in W$, so $\eta \in [W, h] \subseteq [U, f] \cap [V, g]$.

To show that π is a local homeomorphism, fix $\eta \in |\mathscr{F}|$, say with $p := \pi(\eta)$. By (1), there exists some $[U, f] \in \mathcal{B}$ containing η; we claim that $\pi|_{[U,f]} : [U, f] \to U$ is a homeomorphism.

- For injectivity, take $\psi_1, \psi_2 \in [U, f]$ such that $\pi(\psi_1) = \pi(\psi_2)$. Then $\psi_1 = [f]_p$ and $\psi_2 = [f]_q$ for some $p, q \in X$, but since $p = q$, they coincide.
- For continuity, it suffices to show that $\pi|_{[U,f]}$ is an open map. Indeed, if $[V, g] \subseteq [U, f]$ is open, then $\pi|_{[U,f]}([V, g]) = V$ is open too. ■

Definition A.2. *The Étalé space of a presheaf $\mathscr{F}$ of Abelian groups on X is the topological space $|\mathscr{F}|$ equipped the projection $\pi : |\mathscr{F}| \to X$.*

Definition A.3. *A presheaf $\mathscr{F}$ of Abelian groups on X is said to satisfy the Identity Theorem if for all $U \in \tau$ and all $f, g \in \mathscr{F}(U)$, if there is some $p \in U$ such that $[f]_p = [g]_p$, then $f = g$ (on U).*

[1]

Proposition A.4. *If X is a locally-connected Hausdorff space and $\mathscr{F}$ is a presheaf of Abelian groups on X that satisfy the Identity Theorem, then $|\mathscr{F}|$ is Hausdorff.*

Proof. Take distinct $\eta_1, \eta_2 \in |\mathscr{F}|$. Two cases occur.

- If $p := \pi(\eta_1) \neq \pi(\eta_2) =: q$, then, since X is Hausdorff, there exist disjoint neighborhoods U of p and V of q. On those neighborhoods, π is invertible and the sets $\pi^{-1}(U)$ and $\pi^{-1}(V)$ are disjoint neighborhoods of η_1 and η_2, respectively.

Otherwise, set $p := \pi(\eta_1) = \pi(\eta_2)$ and suppose that each η_i is represented by some $f_i \in \mathscr{F}(U_i)$. Since X is locally-connected, there exists a connected neighborhood $U \subseteq U_1 \cap U_2$ of p. Restricting both f_i to $g_i := \rho^{U_i}_U(f_i)$, the sets $[U, g_i]$ are neighborhoods of η_i. Suppose, for sake of contradiction, that there exists some $\psi \in [U, g_1] \cap [U, g_2]$. Setting $q := \pi(\psi)$, we see that $\psi = [g_1]_q = [g_2]_q$, from which the Identity Theorem shows that $g_1 = g_2$. Note that $f_i \sim_p g_i$, so $\eta_1 = \eta_2$, a contradiction. Hence the neighborhoods $[U, g_1]$ an $[U, g_2]$ are disjoint, as desired. ■

We now restrict to when X is a Riemann surface with a fixed point $p \in X$. For convenience, we write[2] $\mathscr{O}[p] := \mathscr{O}[D]$ where D is the divisor on X defined by $D(p) := 1$ and zero everywhere else. Throughout, $\alpha : [0, 1] \to X$ is a curve with $p = \alpha(0)$ and $q := \alpha(1) \neq p$, and $\eta_0 \in \mathscr{O}_p[p]$ is a fixed germ.

Definition A.5. *A germ $\hat{\eta} \in \mathscr{O}_q$ is said to be the analytic continuation of η_0 along α if there exist a family $\eta_t \in \mathscr{O}_{\alpha(t)}[p]$ of germs for all $t \in [0, 1]$ with $\hat{\eta} = \eta_1$ such that for all $\tau \in [0, 1]$, there exists a neighborhood $T \subseteq [0, 1]$ of τ, an open set $U \subseteq X$ with $\alpha(T) \subseteq U$, and a function $f \in \mathscr{O}[p](U)$ such that $[f]_{\alpha(t)} = \eta_t$ for all $t \in T$.*

[1]In particular, this holds for all $\mathscr{O}[D]$. In contrast, the sheaf of smooth functions $\mathscr{E}$ (see Section 3.2) does *not* satisfy the Identity Theorem.

[2]Thus if $U \subseteq X$ is an open set containing p, then any $f \in \mathscr{O}_p[p](U)$ has at most a single simple pole. Otherwise, if $p \notin U$, then f is holomorphic.

Proposition A.6. *A germ $\hat{\eta} \in \mathscr{O}_q$ is an analytic continuation of η_0 along α iff there exists a lifting $\tilde{\alpha} : [0,1] \to |\mathscr{O}[p]|$ such that $\tilde{\alpha}(0) = \eta_0$ and $\tilde{\alpha}(1) = \hat{\eta}$.*

Proof. If[3] $\hat{\eta} \in \mathscr{O}_q$ is an analytic continuation of η_0 along α, let $\{\eta_t\}$ be a family of germs as defined above. We claim that the curve $\tilde{\alpha} : [0,1] \to |\mathscr{O}[p]|$ mapping $t \mapsto \eta_t$ is a lifting of α.

- First, note that $\eta_t \in \mathscr{O}_{\alpha(t)}[p]$ for all $t \in [0,1]$, so[4] $(\pi \circ \tilde{\alpha})(t) = \pi(\tilde{\alpha}(t)) = \pi(\eta_t) = \alpha(t)$. It remains to show that $\tilde{\alpha}$ is continuous, so fix a basis element $[U, f] \subseteq |\mathscr{O}[p]|$ and take $\tau \in \tilde{\alpha}^{-1}([U, f])$. Then $\tau \in [0,1]$, so there exists a neighborhood $T \subseteq [0,1]$ of τ such that $[f]_{\alpha(t)} = \eta_t$ for all $t \in T$. Observe that $\tilde{\alpha}(T) \subseteq [U, f]$ since for all $\eta_t \in \tilde{\alpha}(T)$, we have $\alpha(t) \in U$ and hence $\eta_t = [f]_{\alpha(t)} \in [U, f]$.

Conversely, suppose that there is a lifting $\tilde{\alpha} : [0,1] \to |\mathscr{O}[p]|$ of α with $\tilde{\alpha}(0) = \eta_0$ and $\tilde{\alpha}(1) = \hat{\eta}$. For all $t \in [0,1]$, we define $\eta_t \coloneqq \tilde{\alpha}(t)$, so $\eta_1 = \hat{\eta}$. Fix $\tau \in [0,1]$, so there exists a basis neighborhood $[U, f] \subseteq |\mathscr{O}[p]|$ of $\tilde{\alpha}(\tau)$. But $\tilde{\alpha}$ is continuous, so there exists a neighborhood $T \subseteq [0,1]$ of τ such that $\tilde{\alpha}(T) \subseteq [U, f]$. Projecting, we see that $\alpha(T) \subseteq \pi([U, f]) = U$. Finally, the commutativity of the diagram gives $[f]_{\alpha(t)} = \eta_t$ for all $t \in T$, so $\hat{\eta}$ is an analytic continuation of η_0 along α. ∎

Corollary A.6.1 (Monodromy Theorem)**.** *Let $\alpha_0, \alpha_1 : [0,1] \to X$ be homotopic curves from p to q. If the germ $\eta_0 \in \mathscr{O}_p[p]$ admits an analytic continuation along every deformation of α_0 to α_1, then the analytic continuations of η_0 along α_0 and α_1 coincide.*

Proof. By Propositions A.1 and A.4, $|\mathscr{O}[p]|$ is Hausdorff whose projection $\pi : |\mathscr{O}[p]| \to X$ is a local homeomorphism. Since each deformation admits a lifting starting at η_0, the[5] liftings $\tilde{\alpha}_0$ and $\tilde{\alpha}_1$ have the same endpoints; that is, the analytic continuations along α_0 and α_1 coincide. ∎

Corollary A.6.2. *Suppose X is simply-connected. If the germ $\eta_0 \in \mathscr{O}_p[p]$ admits an analytic continuation along every curve starting at p, then there exists a unique (globally-defined) function $f \in \mathscr{O}[p](X)$ with $[f]_p = \eta_0$.*

Proof. Uniqueness follows from the Identity Theorem. For existence, we define $f(q) \coloneqq \hat{\eta}_q(q)$ where $\hat{\eta}_q \in \mathscr{O}_q[p]$ is the analytic continuation along any curve from p to q. Since X is simply-connected, the Monodromy Theorem ensures that $\hat{\eta}_q$ is well-defined. Clearly $f(p) = \eta_0(p)$, so $[f]_p = \eta_0$. Finally, since $[f]_q = \hat{\eta}_q \in \mathscr{O}_q[p]$ for all $q \in X$, we see that $f \in \mathscr{O}[p](X)$. ∎

Remark. Let $U \subseteq X$ be any open set containing p and consider any function $f_0 \in \mathscr{O}[p](U)$. We have reduced the problem of analytically continuing f_0 to a global function $f \in \mathscr{O}[p](X)$ with $f|_U = f_0$ into finding analytic continuations of $[f_0]_p$ along every curve starting at p. We shall establish this fact under (under some conditions) in the next section. ♦

In this section, we let $E \subseteq |\mathscr{O}[p]|$ be the connected component of the Étalé space of $\mathscr{O}[p]$ containing η_0 and write $\pi : E \to X$ as the restricted projection map.

Theorem A.7. *If π is a covering map, then for any $q \in X$ and any curve $\alpha : [0,1] \to X$ with $\alpha(0) = p$ and $\alpha(1) = q$, there exists an analytic continuation $\hat{\eta} \in \mathscr{O}_q[p]$ of η_0 along α.*

Proof. Define a complex structure $\mathfrak{A}$ on E, that makes π locally biholomorphic, as follows.

- For any $\zeta \in E$, let (U_0, φ) be a chart of X around $\pi(\zeta)$. Since π is a local homeomorphism, there exist neighborhoods $V \subseteq E$ of ζ and $U \subseteq U_0$ of $\pi(\zeta)$ such that $\pi|_V : V \to U$ is a homeomorphism. Set $\psi \coloneqq \varphi \circ \pi|_V$, so (V, ψ) is a chart on E around ζ. Let $\mathfrak{A}$ be the collection of all such charts, which defines an atlas on E since for any pair of charts $(V_1, \psi_1), (V_2, \psi_2) \in \mathfrak{A}$ with $V_1 \cap V_2 \neq \varnothing$, there exist charts (U_1, φ_1) and (U_2, φ_2) of X such that
$$\psi_2 \circ \psi_1^{-1} = \left(\varphi_2 \circ \pi|_{V_2}\right) \circ \left(\varphi_1 \circ \pi|_{V_1}\right)^{-1} = \varphi_2 \circ \left(\pi|_{V_2} \circ \pi|_{V_1}^{-1}\right) \circ \varphi_1^{-1},$$
when restricted to $\psi_1(V_1 \cap V_2)$, reduces to $\varphi_2 \circ \varphi_1^{-1}$. This shows that the charts (V_1, ψ_1) and (V_2, ψ_2) are holomorphically compatible, as desired. Furthermore, we claim that $\pi : E \to X$ is locally biholomorphic w.r.t. $\mathfrak{A}$. Indeed, for any $\zeta \in E$, there exist charts (V, ψ) of E around ζ and (U_0, φ) of X around $\pi(\zeta)$ such that $\psi = \varphi \circ \pi|_V$. Then $\varphi \circ \pi|_V \circ \psi^{-1} = \mathrm{id}_V$, which is holomorphic, so π is locally biholomorphic.

We now define a family $\eta_t \in \mathscr{O}_{\alpha(t)}[p]$ for $t \in [0,1]$ as follows. For all $t \in [0,1]$, let[6] $\zeta_t \in E$ be such that $\pi(\zeta_t) = \alpha(t)$. Then there exist neighborhoods V_t around ζ_t and U_t around $\alpha(t)$ such that $\pi|_{V_t} : V_t \to U_t$ is a biholomorphism. Let $\chi_t \coloneqq \pi|_{V_t}^{-1}$ and define $\eta_t \coloneqq (\chi_t \circ \alpha)(t) \in \mathscr{O}_{\alpha(t)}[p]$. Observe that $\hat{\eta} \coloneqq \eta_1 \in \mathscr{O}_q[p]$, which we claim is the analytic continuation of η_0 along α.

$$\begin{tikzcd}
E \arrow[d, "\pi"'] & V_t \arrow[l, hook'] \arrow[r, "\ell|_{V_t}"] \arrow[d, shift left, "\pi|_{V_t}"] & \mathbb{C} \\
X & U_t \arrow[l, hook'] \arrow[u, shift left, "\chi_t"] \arrow[ur, "f_t"'] & {[0,1]} \arrow[l, "\alpha"']
\end{tikzcd}$$

[3] In particular, the uniqueness of liftings shows that if an analytic continuation of η_0 along α exists, then it is unique.

[4] Clearly $\tilde{\alpha}(0) = \eta_0$ and $\tilde{\alpha}(1) = \hat{\eta}$.

$$\begin{tikzcd}
 & {|\mathscr{O}[p]|} \arrow[d, "\pi"] \\
{[0,1]} \arrow[ur, "\tilde{\alpha}"] \arrow[r, "\alpha"'] & X
\end{tikzcd}$$

[5] This is a standard result in algebraic topology. For a proof, see [For81, Proposition 4.10].

[6] Such a ζ_t exists since π is a covering map. However, it need not be unique; we let ζ_t be *any* such germ. Thus an analytic continuation of η_0 along α need not be unique in general.

- We first construct a function $\ell : E \to \mathbb{C}$ as follows. For $\zeta \in E$, consider any chart (U_0, φ) of X around $\pi(\zeta)$ and any function $g \in \mathscr{O}[p](U_0)$ such that $\zeta = [g]_{\pi(\zeta)}$. Set[7] $\ell(\zeta) \coloneqq g(\pi(\zeta))$. We claim that ℓ has at most a single simple pole at η_0. Indeed, for any $\zeta \in E$, the chart (V, ψ) as defined above that makes $\pi|_V : V \to U$ a homeomorphism ensures that

$$\ell \circ \psi^{-1} = \left(\ell \circ \pi|_V^{-1}\right) \circ \varphi^{-1} = g \circ \varphi^{-1},$$

which is meromorphic with at most a single simple pole at $\varphi(p)$; we have $\ell(\eta_0) = g(p)$.

Take $\tau \in [0,1]$ and consider $\chi_\tau : U_\tau \to V_\tau$ as defined above. Since U_τ is open, the continuity of α furnishes a neighborhood $T_\tau \subseteq [0,1]$ of τ such that $\alpha(T_\tau) \subseteq U_\tau$. Set $f_\tau \coloneqq \ell|_{V_\tau} \circ \chi_\tau$, which is in $\mathscr{O}[p](U_\tau)$ since χ_τ is holomorphic and ℓ is meromorphic with at most a single simple pole at η_0. It remains to show that $[f_t]_{\alpha(t)} = \eta_t$ for all $t \in T$. But this is clear since $\pi\big([f_t]_{\alpha(t)}\big) = \alpha(t) \in U_t$ and $\pi|_{V_t} : V_t \to U_t$ is invertible, so

$$[f_t]_{\alpha(t)} = \left(\pi|_{V_t}^{-1} \circ \alpha\right)(t) = (\chi_t \circ \alpha)(t) = \eta_t. \qquad \blacksquare$$

Remark. In fact[8], the existence of analytic continuations of η_0 along every curve α starting at p is equivalent to $\pi : E \to X$ being a covering map. However, π is not always a covering map[9], so in practice one considers a specific function germ η_0 and studies its corresponding Étalé space E. Ultimately, we think that this boils down to solving a system of PDEs (with boundary conditions being the glueing conditions), but further investigation is needed. ♦

[7] This is well-defined.

[8] See [For81, Exercise 7.2].

[9] For instance, the Lacunary function does not admit an analytic continuation anywhere outside its radius of convergence.

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