Possible Choices of Complex Co-ordinates for a Surface

Moduli Spaces of Riemann Surfaces

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Graduate mentor: Kaleb Ruscitti

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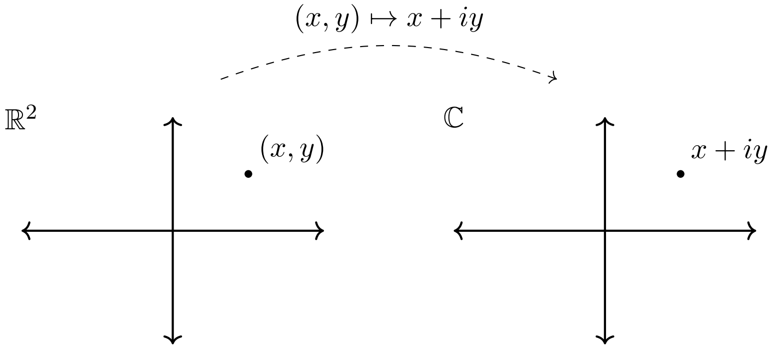
How many complex coordinates on $\mathbb{R}^2$?



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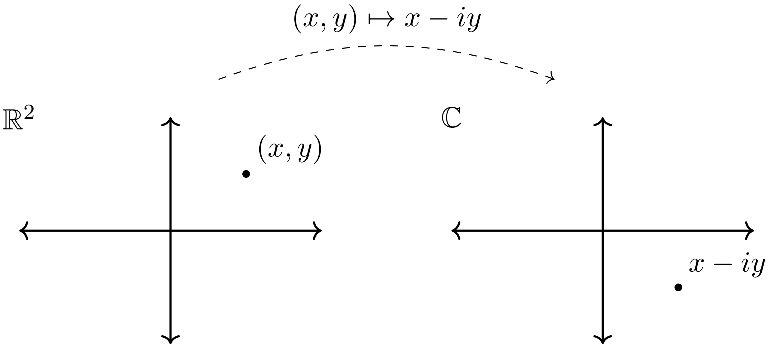

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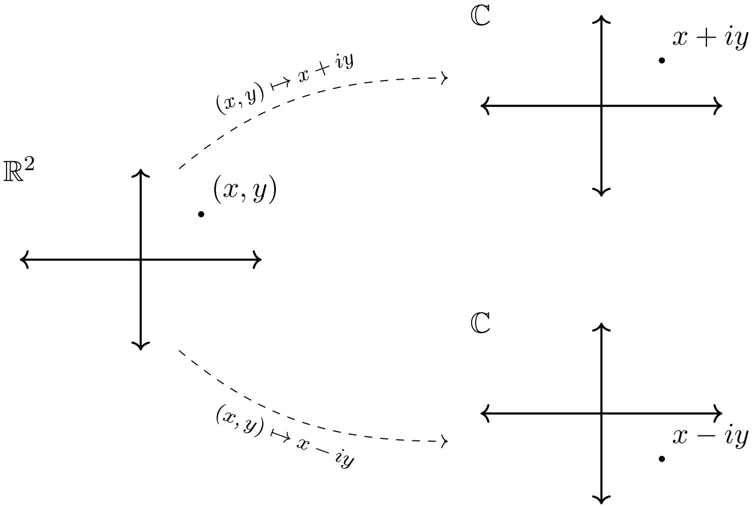

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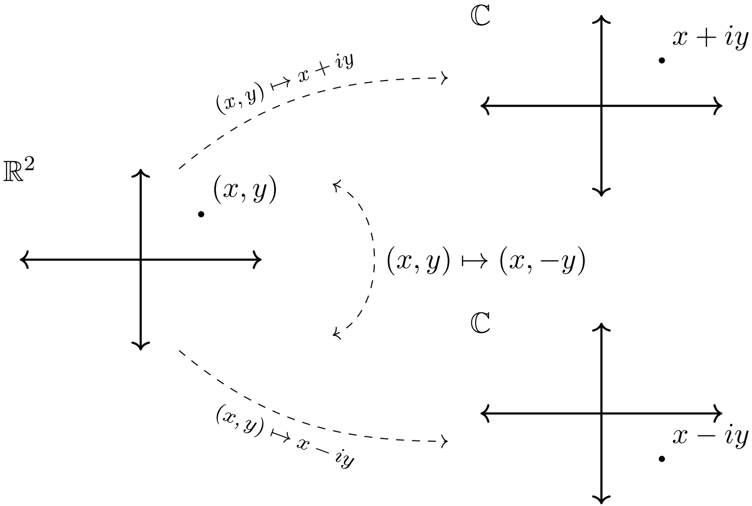

Refined question

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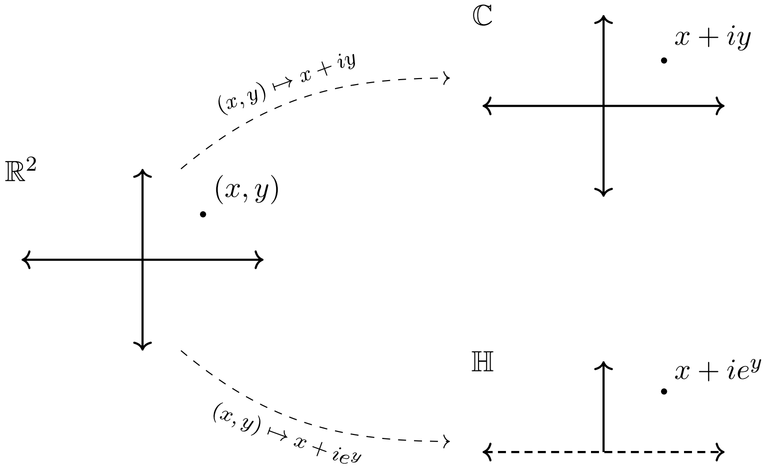
How many complex coordinates on $\mathbb{R}^2$ *up to biholomorphism*?

$\mathbb{C}$ and $\mathbb{H}$



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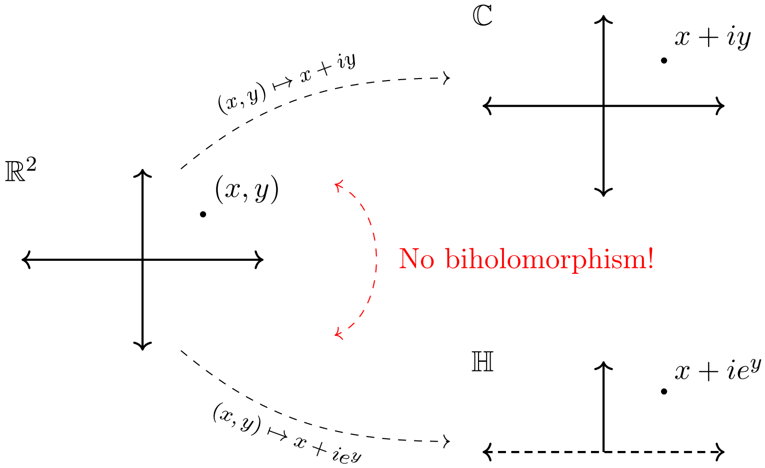
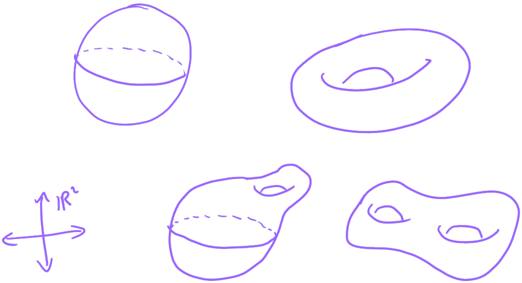


Surfaces

Recall that a surface is a connected 2-dimensional real manifold.



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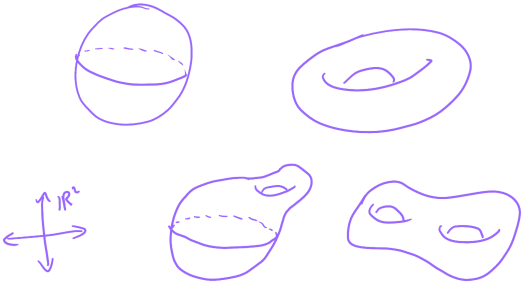

Definition

A <u>*Riemann surface*</u> *is a surface with a choice of complex structure.*

Compact surfaces

Theorem (Classification of Surfaces)

For each natural number g, there exists exactly one compact orientable surface up to homeomorphism. We call g the genus of the surface.

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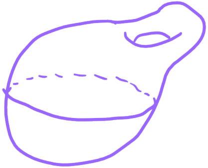

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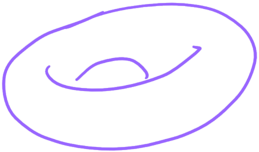

Topological torus

Theorem

Every torus is a quotient $\mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2)$ *for some* $\mathbb{R}$*-linearly independent* $\omega_1, \omega_2 \in \mathbb{C}$.

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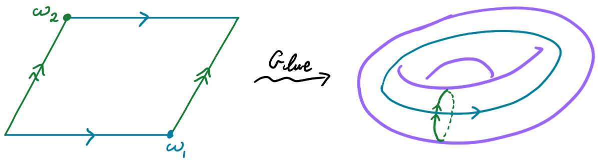

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Questions

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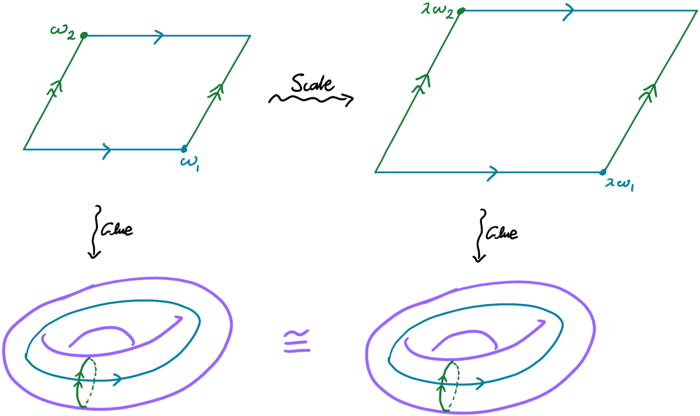
- *Up to biholomorphism, in how many ways can we give the torus complex coordinates?*
- *Can we give the set of all complex coordinates on the torus some geometric structure?*

Complex tori



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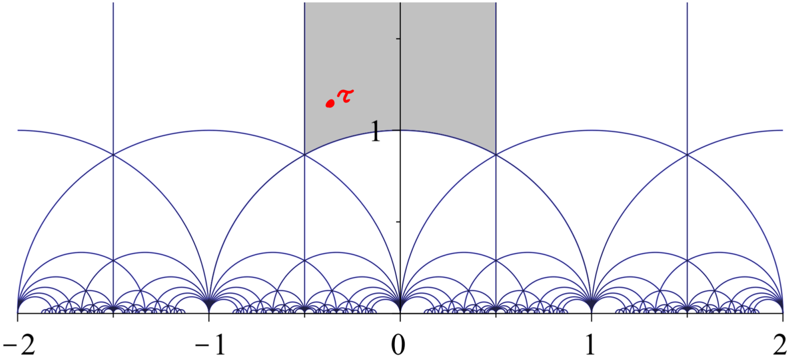
$$\mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2) \cong X_\tau$$

for $\tau := \omega_2/\omega_1 \in \mathbb{H}$*.*

Complex tori



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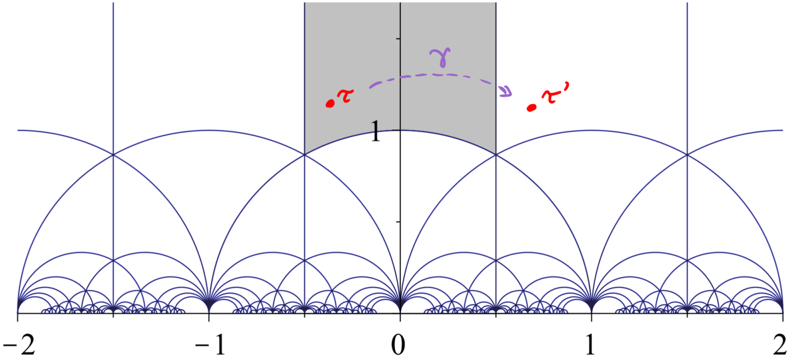

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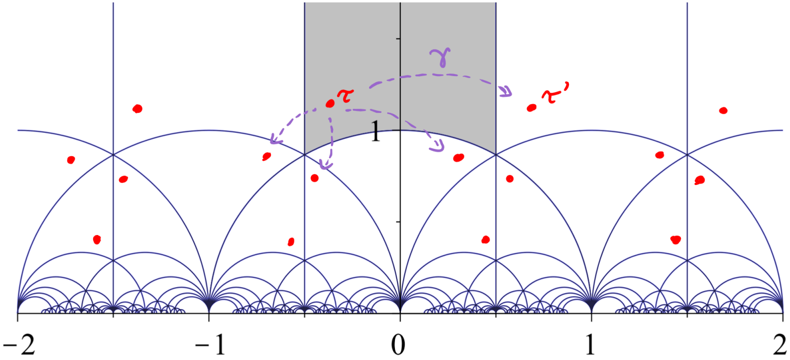

Complex tori

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The modular group $\mathrm{PSL}_2(\mathbb{Z})$ *is the group of functions* $\gamma : \mathbb{H} \to \mathbb{H}$ *mapping*

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}$$

for some $a, b, c, d \in \mathbb{Z}$ *with* $ad - bc = 1$.

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For any $\tau, \tau' \in \mathbb{H}$*, the tori* X_τ *and* X_τ *are biholomorphic iff there exists some* $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$ *such that* $\tau' = \gamma(\tau)$*.*

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Corollary

The moduli space of complex tori is $\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$.

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