Moduli Spaces of Riemann Surfaces

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Abstract

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Chapter 1

Riemann Surfaces

1.1 Charts and Atlases

We assume that the reader is familiar with the basic notions of real manifolds. The case for complex manifolds is similar, so our exposition will be brief.

Definition 1.1. *Let X be a second-countable Hausdorff space. A d-dimensional complex chart on X is a pair (U, φ) where $\varphi : U \to V$ is a homeomorphism from an open subset $U \subseteq X$ onto an open subset $V \subseteq \mathbb{C}^d$ for some d. Two d-dimensional charts (U_1, φ_1) and (U_2, φ_2) are said to be holomorphically compatible if either $U_1 \cap U_2 = \varnothing$, or the map*

$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1 (U_1 \cap U_2) \to \varphi_2 (U_1 \cap U_2)$$

is biholomorphic. A d-dimensional complex atlas on X is a collection $\mathscr{A} := \{(U_i, \varphi_i)\}_{i \in I}$ of d-dimensional complex charts such that every two charts (U_i, φ_i) and (U_j, φ_j) are holomorphically compatible and $X = \bigcup_{i \in I} U_i$.

Charts provide us a way of making X 'look like' an open set in $\mathbb{C}^d$. Indeed, they provide local coordinates for every point in X in such a way that the 'change of coordinates' map $\varphi_2 \circ \varphi_1^{-1}$ ensures that local notions of functions in $\mathbb{C}^d$ are well-defined on X too.

$$\begin{tikzcd} & U_1 \cap U_2 \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] & \\ \varphi_1(U_1 \cap U_2) \arrow[rr, "\varphi_2 \circ \varphi_1^{-1}"'] & & \varphi_2(U_1 \cap U_2) \end{tikzcd}$$

It is clear that one only needs $\varphi_2 \circ \varphi_1^{-1}$ to be holomorphic for it to be biholomorphic.

Remark. Two atlases $\mathscr{A}$ and $\mathscr{B}$ on a manifold X are said to be analytically equivalent if every chart in $\mathscr{A}$ is compatible with every chart in $\mathscr{B}$. By Zorn's Lemma, every atlas $\mathscr{A}$ of a manifold X is contained in a unique maximal atlas $\mathfrak{U}$ on X. Moreover, two atlases are equivalent iff they are contained in the same maximal atlas, which justifies the following definition. ♦

Definition 1.2. *Let X be a second-countable Hausdorff space. A d-dimensional complex structure on X is a d-dimensional maximal atlas $\mathfrak{U}$ on X, or, equivalently, an equivalence class of d-dimensional complex atlases on X. The pair $(X, \mathfrak{U})$ is then called a d-dimensional complex manifold.*

To give a complex structure $\mathfrak{U}$ to X, it suffices to give X a complex atlas since it extends to a *unique* complex structure.

Definition 1.3. *A Riemann surface is a connected 1-dimensional complex manifold.*

Every Riemann surface can be regarded as a (connected) 2-dimensional real manifold by 'forgetting' its complex structure; indeed all holomorphic maps are real $\mathcal{C}^\infty$ functions.

Example 1.4. Some elementary examples of Riemann surfaces.

- The complex plane $\mathbb{C}$, equipped with its standard topology, can be given a complex structure $\mathfrak{U}$ by choosing the atlas containing a single chart $(\mathbb{C}, \mathrm{id}_{\mathbb{C}})$. We may, however, also give $\mathbb{C}$ a different complex structure $\mathfrak{U}'$ by choosing the chart map $\varphi : z \mapsto \overline{z}$ instead. Indeed, $\mathfrak{U} \neq \mathfrak{U}'$ since the map $\varphi \circ \mathrm{id}_{\mathbb{C}}^{-1} = \varphi$ is not holomorphic and hence the atlases $\{(\mathbb{C}, \mathrm{id}_{\mathbb{C}})\}$ and $\{(\mathbb{C}, \varphi)\}$ are not equivalent. This example generalizes to any domain $D \subseteq \mathbb{C}$.
- Let $D \subseteq \mathbb{C}$ be a domain and consider any holomorphic function $f : D \to \mathbb{C}$. Then the graph $\Gamma_f := \{(z, f(z)) \mid z \in D\}$, equipped with the subspace topology inherited from $\mathbb{C}^2$, can be given a complex structure by choosing the chart map $\pi : \Gamma_f \to D : (z, f(z)) \mapsto z$. ♦

1.1.1 The Riemann Sphere $\hat{\mathbb{C}}$

A particularly important Riemann surface is the Riemann sphere $\hat{\mathbb{C}}$, which admits several constructions. Here, we give three; see Example 1.14 for a proof that they are all biholomorphic (in the sense of Definition 1.13).

Showing that *every* Riemann surface that is topologically a sphere is biholomorphic to $\hat{\mathbb{C}}$ is a non-trivial task, and it will be the first goal of this paper to establish this fact.

Example 1.5 (One-point Compactification of $\mathbb{C}$)**.** Let ∞ be a symbol not belonging to $\mathbb{C}$ and set $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$. We declare a set $U \subseteq \mathbb{C}_\infty$ to be open if either $U \subseteq \mathbb{C}$ is open or $U = K^c \cup \{\infty\}$ where $K \subseteq \mathbb{C}$ is compact. We employ two charts

$$U_1 := \mathbb{C}_\infty \setminus \{\infty\} = \mathbb{C} \qquad \varphi_1 : U_1 \to \mathbb{C} : z \mapsto z \quad (\varphi_1 := \mathrm{id}_{\mathbb{C}})$$

$$U_2 := \mathbb{C}_\infty \setminus \{0\} = \mathbb{C}^* \cup \{\infty\} \qquad \varphi_2 : U_2 \to \mathbb{C} : z \mapsto \begin{cases} 1/z & \text{if } z \in \mathbb{C}^* \\ 0 & \text{else.} \end{cases}$$

This makes $\mathbb{C}_\infty$, equipped with the collection $\mathcal{T}$ of all such open sets, a second-countable Hausdorff space. Indeed, the fact that $\mathcal{T}$ is a topology on $\mathbb{C}_\infty$ follows from De Morgan's Laws and the Heine-Borel Theorem. It is trivially Hausdorff, and it is second-countable since we may append, to any countable basis for the standard topology of $\mathbb{C}$, the countable collection $\{B_r(0)^c \cup \{\infty\}\}_{r \in \mathbb{Q}^+}$.

Clearly φ_1 is a homeomorphism. Since φ_2 is invertible with $\varphi_2^{-1}(z) := 1/z$ for all $z \in \mathbb{C}^*$ and $\varphi_2^{-1}(0) := \infty$, and

$$\lim_{z \to \infty} \varphi_2(z) = 0 = \varphi_2(\infty) \qquad \text{and} \qquad \lim_{z \to 0} \varphi_2^{-1}(z) = \infty = \varphi_2^{-1}(0),$$

we see that φ_2 is a homeomorphism too. Furthermore,

$$\varphi_2 \circ \varphi_1^{-1} : \mathbb{C}^* \to \mathbb{C}^* : z \mapsto \frac{1}{z}$$

is holomorphic, so the atlas $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$ defines a complex structure on $\mathbb{C}_\infty$. ♦

Example 1.6 (Stereographic Projection)**.** Consider the unit sphere $S^2 \subseteq \mathbb{R}^3$ as a topological subspace of $\mathbb{R}^3$, which makes it a second-countable Hausdorff space. Identifying the plane $w = 0$ as $\mathbb{C}$, we employ the charts

$$
\begin{aligned}
U_1 &\coloneqq S^2 \setminus \{(0,0,1)\} & \qquad \varphi_1 : U_1 \to \mathbb{C} : (x,y,w) &\mapsto \frac{x+iy}{1-w} \\
U_2 &\coloneqq S^2 \setminus \{(0,0,-1)\} & \qquad \varphi_2 : U_2 \to \mathbb{C} : (x,y,w) &\mapsto \frac{x-iy}{1+w}.
\end{aligned}
$$

Clearly φ_1 and φ_2 are continuous, and it can be verified that they are invertible with continuous inverses

$$
\varphi_1^{-1}(z) \coloneqq \left(\frac{2\operatorname{Re} z}{|z|^2+1}, \frac{2\operatorname{Im} z}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right) \quad \text{and} \quad \varphi_2^{-1}(z) \coloneqq \left(\frac{2\operatorname{Re} z}{|z|^2+1}, \frac{-2\operatorname{Im} z}{|z|^2+1}, \frac{1-|z|^2}{|z|^2+1} \right).
$$

Observe that $U_1 \cap U_2 = S^2 \setminus \{(0,0,\pm 1)\}$ and $\varphi_2 \circ \varphi_1^{-1} : \mathbb{C}^* \to \mathbb{C}^* : z \mapsto 1/z$, which is holomorphic, so the atlas $\{(U_1,\varphi_1), (U_2,\varphi_2)\}$ defines a complex structure on $\hat{\mathbb{C}}$. ♦

Example 1.7 (Complex Projective Line)**.** Consider the equivalence relation $\sim$ on $\mathbb{C}^2 \setminus \{(0,0)\}$ defined by $(z_1,w_1) \sim (z_2,w_2)$ iff $(z_1,w_1) = \lambda(z_2,w_2)$ for some $\lambda \in \mathbb{C}^*$. Set $\mathbb{P}^1 \coloneqq \left(\mathbb{C}^2 \setminus \{(0,0)\}\right)/\sim$ and equip it with the quotient topology. Since $\sim$ is an open equivalence relation whose graph is closed in $\left(\mathbb{C}^2 \setminus \{(0,0)\}\right)^2$, we see that $\mathbb{P}^1$ is a second-countable Hausdorff space. Denoting the equivalence class of (z,w) by $[z:w]$, we employ the charts

See [Tu10, section 7.5].

$$
\begin{aligned}
U_1 &\coloneqq \mathbb{P}^1 \setminus \{[0:w] \mid w \in \mathbb{C}\} & \qquad \varphi_1 : U_1 \to \mathbb{C} : [z:w] &\mapsto w/z \\
U_2 &\coloneqq \mathbb{P}^1 \setminus \{[z:0] \mid z \in \mathbb{C}\} & \qquad \varphi_2 : U_2 \to \mathbb{C} : [z:w] &\mapsto z/w.
\end{aligned}
$$

Clearly φ_2 and φ_2 are continuous, and it is easily verified that they are invertible with continuous inverses

$$
\varphi_1^{-1}(z) \coloneqq [1:z] \qquad \text{and} \qquad \varphi_2^{-1}(z) \coloneqq [z:1].
$$

Furthermore, $\varphi_2 \circ \varphi_1^{-1} : \mathbb{C}^* \to \mathbb{C}^* : 1 \mapsto 1/z$ is holomorphic, so the atlas $\{(U_1,\varphi_1), (U_2,\varphi_2)\}$ defines a complex structure on $\mathbb{P}^1$. ♦

1.1.2 Complex Tori

Recall that a torus is any manifold homeomorphic to $T^2 \coloneqq S^1 \times S^1$, which admits a representation as a quotient $\mathbb{C}/\Gamma$ by the lattice $\Gamma \coloneqq \mathbb{Z} \oplus \mathbb{Z}$. Thus (by definition) there is only one torus up to homeomorphism, but it turns out that we can equip it with many different complex structures.

They manifest by quotienting $\mathbb{C}$ by different lattices, and we shall derive a criterion on $\Gamma_1 \coloneqq \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ and $\Gamma_2 \coloneqq \mathbb{Z}\eta_1 \oplus \mathbb{Z}\eta_2$ for the tori $\mathbb{C}/\Gamma_1$ and $\mathbb{C}/\Gamma_2$ to be biholomorphic.

Example 1.8 (Complex Tori)**.** Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$ and consider the lattice $\Gamma \coloneqq \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$. Then the quotient $\mathbb{C}/\Gamma$ is a torus in the topological sense since the map

$$
\varphi : \mathbb{C}/\Gamma \to T^2 \qquad \text{mapping} \qquad [z] \mapsto \left(e^{2\pi i \lambda_1}, e^{2\pi i \lambda_2}\right),
$$

where $z = \lambda_1\omega_1 + \lambda_2\omega_2$ for unique $\lambda_1, \lambda_2 \in \mathbb{R}$, is a homeomorphism. Indeed, φ is well-defined since for any $\lambda_1\omega_1 + \lambda_2\omega_2 \sim \mu_1\omega_1 + \mu_2\omega_2$ in $\mathbb{C}$, we have $(\lambda_1 - \mu_1)\omega_1 + (\lambda_2 - \mu_2)\omega_2 \in \Gamma$ and so $\lambda_i - \mu_i \in \mathbb{Z}$ for $i = 1, 2$. The fact that it is a homeomorphism is clear. This makes $\mathbb{C}/\Gamma$ a second-countable Hausdorff space, which we now endow with the following complex structure.

Since Γ is discrete, there exists some $\varepsilon > 0$ such that $\varepsilon < |\omega|/2$ for every non-zero $\omega \in \Gamma$. Fix any such ε, which ensures that no two points in any open ball with radius ε can be equivalent. Indeed, take any $z \in \mathbb{C}$ and $w_1, w_2 \in B(z,\varepsilon) \eqqcolon V_z$. For $\omega_1 \sim \omega_2$, we need some $n, m \in \mathbb{Z}$ such that $w_1 - w_2 = n\omega_1 + m\omega_2$. But

This exposition follows [Mir95, section I.2].

$$
|w_1 - w_2| \leq |z - w_1| + |z - w_2| < 2\varepsilon < |n\omega_1 + m\omega_2|
$$

for any $n, m \in \mathbb{Z}$, so this is impossible. Fixing any such ε, this gives us a family $\{V_z\}_{z \in \mathbb{C}}$ of open sets in $\mathbb{C}$ for which the projections $\pi|_{V_z} : V_z \to \pi(V_z)$ are homeomorphisms. Letting $U_z \coloneqq \pi(V_z)$ and $\varphi_z : U_z \to V_z$ be the inverse of $\pi|_{V_z}$, we obtain complex charts (U_z, φ_z) for all $z \in \mathbb{C}$. We claim that the collection $\mathfrak{U} \coloneqq \{(U_z,\varphi_z)\}_{z \in \mathbb{C}}$ form an atlas, for which it suffices to take $(U_1,\varphi_1), (U_2,\varphi_2) \in \mathfrak{U}$ and show that the transition map $T \coloneqq \varphi_2 \circ \varphi_1^{-1} : \varphi_1(U) \to \varphi_2(U)$, where $U \coloneqq U_1 \cap U_2$, is holomorphic. Observe that the diagram

The choice of ε ensures that no two points in V_z are equivalent, which make all such projections injective.

$$
\begin{tikzcd}
& U \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] & \\
V_1 = \varphi_1(U) \arrow[ur, bend left, "\pi|_{V_1}"] \arrow[rr, "T"'] & & \varphi_2(U) = V_2 \arrow[ul, bend right, "\pi|_{V_2}"']
\end{tikzcd}
$$

commutes, so $\pi|_{V_2} \circ T = \pi|_{V_1}$ on $\varphi_1(U)$. Then $\pi(T(z)) = \pi(z)$ for every $z \in \varphi_1(U)$, so $T(z) \sim z$ and hence $\ell(z) \coloneqq T(z) - z \in \Gamma$. This holds for all $z \in \varphi_1(U)$, so we obtain a continuous function $\ell : \varphi_1(U) \to \Gamma : z \mapsto T(z) - z$. Note that $\Gamma \subseteq \mathbb{C}$ is equipped with the subspace topology, but since it is discrete, every $L \subseteq \Gamma$ is open. In particular, fix $z_0 \in \varphi_1(U)$ and set $\omega_0 \coloneqq T(z_0) - z_0$. With $L \coloneqq \{\omega_0\}$, continuity of ℓ shows that $\ell^{-1}(L)$ is open. Thus $\ell(B(z_0,\delta_1)) \subseteq \{\omega_0\}$ for some $\delta_1 > 0$, so $\ell(w) = \omega_0$ for all $w \in B(z_0,\delta_1)$. But then $\ell(B(\omega_0,\delta_2)) \subseteq \{\omega_0\}$ for some $\delta_2 > 0$ too, so we may repeat this process to show that ℓ is constant on every connected component of $\varphi_1(U)$. Thus $T(z) = z + \omega_0$ for all $z \in \varphi_1(U)$ in a local neighborhood around z_0, so T is locally holomorphic. But this holds for all $z_0 \in \varphi_1(U)$, so T is holomorphic on $\varphi_1(U)$. ♦

Since $U = \pi(V_1) \cap \pi(V_2)$, it may not be connected. Hence $\varphi_1(U)$ may not be connected, so ℓ may take on multiple values. What matters, hoverer, is that they coincide within every connected component of of $\varphi_1(U)$.

1.2 Maps on Riemann Surfaces

1.2.1 Holomorphic Functions and Maps

Definition 1.9. *Let X be a Riemann surface and let $W \subseteq X$ be open. For a fixed $p \in W$, a function $f : W \to \mathbb{C}$ is said to be holomorphic at p if there exists a chart (U, φ) of X containing p such that $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{C}$ is holomorphic at $\varphi(p)$. If f is holomorphic at every point of W, then f is said to be holomorphic on W.*

Defining some property P of f using charts by transporting f to a function $f \circ \varphi^{-1}$ on a subset of $\mathbb{C}$, and borrowing P from $f \circ \varphi^{-1}$, will be a common theme. However, one must check that P is *independent of charts*; that is, if $f \circ \varphi^{-1}$ satisfies P, then so does $f \circ \psi^{-1}$ for any other chart (V, ψ).

$$\begin{tikzcd}
\varphi(U\cap V) \arrow[drr, bend left, "f\circ\varphi^{-1}"] & & \\
& U\cap V \arrow[ul, "\varphi"'] \arrow[dl, "\psi"] \arrow[r, "f"] & \mathbb{C} \\
\psi(U\cap V) \arrow[urr, bend right, "f\circ\psi^{-1}"'] & &
\end{tikzcd}$$

Remark. It must be checked that 'being holomorphic' does not depend on the choice of chart. This is indeed the case, for if (V, ψ) is another chart containing p, then, since

$$f \circ \psi^{-1} = f \circ \left(\varphi^{-1} \circ \varphi\right) \circ \psi^{-1} = \left(f \circ \varphi^{-1}\right) \circ \left(\varphi \circ \psi^{-1}\right) : \psi(U \cap V) \to \mathbb{C} \tag{1.1}$$

on the intersection $U \cap V$, we see that $f \circ \psi^{-1} : \psi(V) \to \mathbb{C}$ is also holomorphic at p. ♦

Example 1.10. Some elementary examples of holomorphic functions.

- Any holomorphic function $f : W \to \mathbb{C}$ from an open set $W \subseteq \mathbb{C}$, considering $\mathbb{C}$ as a Riemann surface with the standard chart $(\mathbb{C}, \mathrm{id}_{\mathbb{C}})$, is holomorphic in the classical sense.
- Any chart map $\varphi : U \to \mathbb{C}$ of a Riemann surface is (tautologically) holomorphic in the above sense.
- If $f, g : W \to \mathbb{C}$ are both holomorphic at some $p \in W$, then so are $f \pm g$ and $f \cdot g$. If $g(p) \neq 0$, then so is f/g. ♦

This makes the set $\mathcal{O}(W)$ of all holomorphic functions $f : W \to \mathbb{C}$ into a $\mathbb{C}$-algebra.

Definition 1.11. *Let X and Y be Riemann surfaces and let $W \subseteq X$ be open. For a fixed $p \in W$, a mapping $F : W \to Y$ is said to be holomorphic at p if there exists a chart (U, φ) of X containing p and a chart (V, ψ) of Y containing $F(p)$ such that $\psi \circ F \circ \varphi^{-1}$: $\varphi(U) \to \psi(V)$ is holomorphic at $\varphi(p)$. If F is holomorphic at every point of W, then F is holomorphic on W.*

For $Y := \mathbb{C}$ regarded as a Riemann surface, this definition agrees with the above. Again, we must check that 'being holomorphic' is well-defined, but it follows from the commutativity of the diagram below.

$$\begin{tikzcd}
\varphi_1(U_1\cap U_2) \arrow[rrr, bend left, "\psi_1\circ F\circ\varphi_1^{-1}"] \arrow[dd, "\varphi_2\circ\varphi_1^{-1}"'] & & & \psi_1(V_1\cap V_2) \arrow[dd, "\psi_2\circ\psi_1^{-1}"] \\
& U_1\cap U_2 \arrow[ul, "\varphi_1"'] \arrow[dl, "\varphi_2"] \arrow[r, "F"] & V_1\cap V_2 \arrow[ur, "\psi_1"] \arrow[dr, "\psi_2"'] & \\
\varphi_2(U_1\cap U_2) \arrow[rrr, bend right, "\psi_2\circ F\circ\varphi_2^{-1}"'] & & & \psi_2(V_1\cap V_2)
\end{tikzcd}$$

Example 1.12. It is easy to show that the identity map id_X on a Riemann surface X is a holomorphic map. Furthermore, for all Riemann surfaces X, Y and Z and holomorphic maps $F : X \to Y$ and $G : Y \to Z$, their composite $G \circ F : X \to Z$ is also a holomorphic map. This shows that the collection of all Riemann surfaces is a *category*. ♦

Definition 1.13. *Let X and Y be Riemann surfaces. A biholomorphism between X and Y is an invertible holomorphic map $F : X \to Y$ whose inverse $F^{-1} : Y \to X$ is also holomorphic. Two Riemann surfaces X and Y are said to be biholomorphic if there exists a biholomorphism $F : X \to Y$.*

Example 1.14 (Biholomorphisms between Riemann spheres)**.** Let $\mathbb{C}_\infty$, S^2, and $\mathbb{P}^1$ denote the three constructions for the Riemann sphere $\hat{\mathbb{C}}$ presented in Examples 1.5, 1.6, and 1.7, respectively. We claim that the maps

$$F : S^2 \to \mathbb{P}^1 : (x, y, w) \mapsto [1 - w : x + iy] \qquad \text{and} \qquad G : S^2 \to \mathbb{C}_\infty : (x, y, w) \mapsto \frac{x + iy}{1 - w}$$

Take $G(0, 0, 1) := \infty$.

are biholomorphisms, which shows that all three constructions are biholomorphic. Indeed F is holomorphic since with the charts

Since the collection of Riemann surfaces form a category, the 'is isomorphic to' relation is an equivalence relation. Thus we are justified to call all three constructions 'the' Riemann sphere, and, henceforth, we shall denote all three by $\hat{\mathbb{C}}$.

$$\begin{aligned}
&U := S^2 \setminus \{(0, 0, 1)\} && \varphi : U \to \mathbb{C} : (x, y, w) \mapsto \frac{x + iy}{1 - w} \\
&V := \mathbb{P}^1 \setminus \{[0 : w] \mid w \in \mathbb{C}\} && \psi : V \to \mathbb{C} : [z : w] \mapsto \frac{w}{z},
\end{aligned}$$

we see that

$$\begin{aligned}
\left(\psi \circ F \circ \varphi^{-1}\right)(z) &= \psi\left(F\left(\frac{2\operatorname{Re} z}{|z|^2 + 1}, \frac{2\operatorname{Im} z}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1}\right)\right) \\
&= \psi\left(\left[1 - \frac{|z|^2 - 1}{|z|^2 + 1} : \frac{2z}{|z|^2 + 1}\right]\right) \\
&= \psi([1 : z]) \\
&= z
\end{aligned}$$

A similar calculation shows that G is biholomorphic. Indeed, we choose the same chart (U, φ), and choose $V := \mathbb{C}_\infty \setminus \{\infty\} = \mathbb{C}$ with $\psi := \mathrm{id}_{\mathbb{C}}$. Then $\left(\psi \circ G \circ \varphi^{-1}\right)(z) = z$ for all $z \in \varphi(U) = \mathbb{C}$, and G is invertible with inverse

$$G^{-1}(z) := \begin{cases} \varphi^{-1}(z) & \text{if } z \in \mathbb{C} \\ (0, 0, 1) & \text{else.} \end{cases}$$

for all $z \in \varphi(U) = \mathbb{C}$, which is clearly holomorphic. Furthermore, it can be checked that F is invertible with inverse

$$F^{-1}([z : w]) := \frac{\left(2\operatorname{Re}(z\overline{w}), 2\operatorname{Im}(z\overline{w}), |z|^2 - |w|^2\right)}{|z|^2 + |w|^2},$$

which is well-defined, and since $\left(\psi \circ F \circ \varphi^{-1}\right)^{-1} = \varphi \circ F^{-1} \circ \psi^{-1}$, we see that F^{-1} is holomorphic too. ♦

Theorem 1.15. *Any holomorphic function $f : X \to \mathbb{C}$ on a compact Riemann surface X is constant.*

Proof. Since f is holomorphic, the function $|f| : X \to \mathbb{R}$ defined by $|f|(x) \coloneqq |f(x)|$ is continuous on X. But X is compact, so $|f|$ achieves its maximum at some point $p \in X$. Choosing a connected chart (U, φ) centered at p, we see that $f \circ \varphi : U \to \mathbb{C}$ is holomorphic. Then $|f \circ \varphi| : U \to \mathbb{R}$ has a local maximum at 0, so, since U is connected, $f \circ \varphi$ is constant by the Maximum Principle. Then f is locally constant around p, so, since X is connected, f is constant on X. ■

Such a connected U can always be found since we may let V be a chart around p and choose $\varepsilon > 0$ small enough so that $U \coloneqq B(p, \varepsilon) \subseteq V$.

1.2.2 Singularities of Functions

Throughout this section, let X be a Riemann surface, let $p \in X$, and let $f : W \to \mathbb{C}$ be defined and holomorphic on a punctured neighborhood W of p. As above, we can transport the behaviour of f at p from its chart representation $f \circ \varphi^{-1}$.

That is, let f be defined and holomorphic on $B(p, \varepsilon) \setminus \{p\}$ for some $\varepsilon > 0$.

Definition 1.16. *Let $f : W \to \mathbb{C}$ be a holomorphic function in a punctured neighborhood of p. We say that f has a removable singularity (resp. pole, essential singularity) at p if there exists a chart (U, φ) of X containing p such that $f \circ \varphi^{-1} : \varphi(U) \to \mathbb{C}$ has a removable singularity (resp. pole, essential singularity) at $\varphi(p)$.*

We recall those notions from complex analysis. Let $f : W \subseteq \mathbb{C} \to \mathbb{C}$ be a holomorphic function (in the regular sense) in a punctured neighborhood of p. Suppose that f is not holomorphic at p.

- If $\lim_{z \to p} f(z)$ exists, then f has a removable singularity at p.
- If $\lim_{z \to p} f(z) = \pm\infty$, then f has a pole at p. This is equivalent to the existence of some $n > 0$ such that the limit $\lim_{z \to p} (z - p)^n f(z)$ exists. See Definition 1.18.
- Otherwise, f has an essential singularity at p.

Proof. (Well-defined): Equation (1.1) shows that those notions are chart independent; the composition of $f \circ \varphi^{-1}$ having a singularity at p with a transition map that is holomorphic at p yields a function with the same type of singularity at p. ■

Remark. Functions having an essential singularity at p are very ill-behaved. Indeed, this occurs iff $|f(x)|$ has a non-zero oscillation near p. Other singularities behave much better:

- A removable singularity occurs iff $|f(x)|$ is bounded in a neighborhood of p, and can be 'filled in' by defining $\tilde{f}(p) \coloneqq \lim_{x \to p} f(x)$. This makes $\tilde{f} : \tilde{W} \to \mathbb{C}$ into a holomorphic function.
- A pole occurs iff $|f(x)| \to \infty$ as $x \to p$, which can also be 'filled in' by defining the map

$$F : W \to \hat{\mathbb{C}} \qquad \text{mapping} \qquad x \mapsto \begin{cases} \infty & \text{if } x = p \\ f(x) & \text{else} \end{cases}$$

that extends the codomain of f to the Riemann sphere $\hat{\mathbb{C}}$; it is clear that F is holomorphic.

$\tilde{W} \coloneqq W \cup \{p\}$.

Here, we consider $\hat{\mathbb{C}} = \mathbb{C}_\infty$.

Thus we see that every such function $f : W \to \mathbb{C}$ having pole at p can be holomorphically extended to a map $F : W \to \hat{\mathbb{C}}$. Conversely, every holomorphic map $F : W \to \hat{\mathbb{C}}$ (that is not identically zero) can be regarded as a function $f : W \setminus F^{-1}(\infty) \to \mathbb{C}$ that is holomorphic everywhere except where $F(x) = \infty$, in which case it either has a pole. This motivates the following definition. ♦

Definition 1.17. *A function $f : W \to \mathbb{C}$ is said to be meromorphic at p if it does not have an essential singularity at p; that is, if it is either holomorphic, has a removable singularity, or has a pole at p. If f is meromorphic at every point of W, then f is meromorphic on W.*

As in Example 1.10, if $f, g : W \to \mathbb{C}$ are both meromorphic at p, then so are $f \pm g$ and $f \cdot g$. If g is not identically 0, then so is f/g. This makes the set $\mathcal{M}(W)$ of all meromorphic functions $f : W \to \mathbb{C}$ into a $\mathbb{C}$-algebra.

Remark. Remark 1.2.2 can now be rephrased by saying that the set of all meromorphic functions $f : W \to \mathbb{C}$ are in one-to-one correspondence with the set of all holomorphic maps $F : W \to \hat{\mathbb{C}}$ (which are not identically zero). That is, meromorphic functions are the holomorphic maps to the Riemann sphere. ♦

Definition 1.18. *Let $f : W \to \mathbb{C}$ be meromorphic at p and consider its Laurent series $f_\varphi(z) \coloneqq \sum_i c_i (z - z_0)^i$ under a chart (U, φ) of X with $z_0 \coloneqq \varphi(p)$. The order of f at p is*

$$\mathrm{ord}_p(f) \coloneqq \min\{n \in \mathbb{Z} \mid 0 \neq (z - z_0)^n f_\varphi(z) \in \mathcal{O}(W)\}.$$

Note that f, being meromorphic, ensures that its Laurent series has finitely-many negative terms. Thus the set $\{n \in \mathbb{Z} \mid c_n \neq 0\}$ achieves its minimum, so the definition makes sense. Of course, it still remains to show that $\mathrm{ord}_p(f)$ is independent of the local coordinate defining the Laurent series.

The arithmetic of ord_p is straightforward. Indeed, if $f, g : W \to \mathbb{C}$ are meromorphic at p, then

- $\mathrm{ord}_p(fg) = \mathrm{ord}_p(f) + \mathrm{ord}_p(g)$.
- $\mathrm{ord}_p(f/g) = \mathrm{ord}_p(f) - \mathrm{ord}_p(g)$, if $g \neq 0$.
- $\mathrm{ord}_p(1/f) = -\mathrm{ord}_p(f)$, if $f \neq 0$.
- $\mathrm{ord}_p(f \pm g) \geq \min\{\mathrm{ord}_p(f), \mathrm{ord}_p(g)\}$.

Proof. (Well-defined). Let (U, φ) give the local coordinates of z and suppose that (V, ψ) is another chart with $w_0 \coloneqq \psi(p)$ giving another local coordinate w. Then the transition function $T : \varphi \circ \psi^{-1}$ is holomorphic, so it admits a power series representation

$$z = T(w) = \sum_{n \geq 0} a_n (w - w_0)^n = z_0 + \sum_{n \geq 1} a_n (w - w_0)^n.$$

Since $T'(w_0) \neq 0$, we see that $a_1 \neq 0$. Suppose now that the Laurent series of f at p in the coordinate z is $c_{n_0}(z - z_0)^{n_0}$ + higher order terms, so that the order of f at p computed by employing z is n_0. Then the Laurent series of f at p in the coordinate w is

$$c_{n_0} \left(\sum_{n \geq 1} a_n (w - w_0)^n \right)^{n_0} + \text{ higher order terms,}$$

whose lowest order term is $c_{n_0} a_1^{n_0} (w - w_0)^{n_0}$. Observe that $a_{n_0} = c_{n_0} a_1^{n_0} \neq 0$, so the order of f at p computed via w is also n_0. ■

Remark. The order $\mathrm{ord}_p(f)$ can be used to classify the behaviour of f at p. Indeed, it is readily verified that f is holomorphic at p iff $\mathrm{ord}_p(f) \geq 0$, in which case $f(p) = 0$ iff $\mathrm{ord}_p(f) > 0$. Similarly, f has a pole at p iff $\mathrm{ord}_p(f) < 0$, so f has neither a zero nor a pole at p iff $\mathrm{ord}_p(f) = 0$. This motivates the following definition. ⧫

Definition 1.19. *Let $f : W \to \mathbb{C}$ be meromorphic at p. We say that f has a zero (resp. pole) of order n at p if $\mathrm{ord}_p(f) = n > 0$ (resp. $n < 0$).*

1.2.3 Meromorphic Functions on $\hat{\mathbb{C}}$

Example 1.20. Let $f : W \subseteq \hat{\mathbb{C}} \to \mathbb{C}$ be a non-zero rational function $f(z) := p(z)/q(z)$. Then f is holomorphic at all points $z \in \mathbb{C}$ such that $q(z) \neq 0$, and has a pole otherwise. Also, $f(\infty) = c_n/d_m$ if $\deg p = \deg q$, vanishes if $\deg p < \deg q$, and has a pole otherwise. In any case, f is meromorphic on $\hat{\mathbb{C}}$. To compute $\mathrm{ord}_z(f)$ at all $z \in \hat{\mathbb{C}}$, we split p and q into linear factors to write f uniquely as

$$f(z) = c \prod (z - \lambda_i)^{\alpha_i}$$

where $c \neq 0$ and each λ_i is distinct. Fix i. Setting $g_j(z) := (z - \lambda_j)^{\alpha_j}$ for all j, we see that $\mathrm{ord}_{\lambda_i}(g_i) = \alpha_i$ and $\mathrm{ord}_{\lambda_j}(g_i) = 0$ for all $i \neq j$. Thus

$$\mathrm{ord}_{\lambda_i}(f) = \sum_j \mathrm{ord}_{\lambda_i}(g_j) = \alpha_i.$$

Moreover, if $\alpha_i > 0$ (resp. $\alpha_i < 0$), then g_i has a pole (resp. zero) of order $|\alpha_i|$ at ∞. It follows then that $\mathrm{ord}_\infty(g_i) = -\alpha_i$, so

$$\mathrm{ord}_\infty(f) = \sum_i \mathrm{ord}_\infty(g_i) = -\sum_i \alpha_i.$$

Lastly, it is clear that $\mathrm{ord}_z(f) = 0$ for all $z \neq \lambda_i, \infty$. ⧫

In fact, *any* meromorphic function on the Riemann sphere is a rational function; see Theorem 1.21.

Theorem 1.21. *Any meromorphic function on the Riemann sphere is a rational function.*

Proof. Let $f : \hat{\mathbb{C}} \to \mathbb{C}$ be meromorphic. Since $\hat{\mathbb{C}}$ is compact, it has finitely-many poles. W.l.o.g., assume that ∞ is not a pole of f (since we may consider $1/f$ instead). Now, for each pole $\lambda_i \in \mathbb{C}$ of f, consider its principle part

$$h_i(z) = \sum_{j=-m_i}^{-1} c_{ij} (z - \lambda_i)^j$$

for some $m_i > 1$. Then the function $g := f - \sum_i h_i$ is holomorphic function on $\hat{\mathbb{C}}$, and since $\hat{\mathbb{C}}$ is compact, it is constant by Theorem 1.15. Thus $f = g + \sum_i h_i$, which is a rational function. ■

Otherwise, the set of poles would have a limit point, contradicting the discreteness of poles.

Remark. Together with the above computation, this shows that if f is a meromorphic function on $\hat{\mathbb{C}}$, then $\sum_{z \in \hat{\mathbb{C}}} \mathrm{ord}_z(f) = 0$. As we shall see, this is a general fact for all compact Riemann surfaces. ⧫

1.3 Global Properties of Holomorphic Maps

1.3.1 Local Normal Form

Theorem 1.22 (Local Normal Form). *Let X and Y be Riemann surfaces and let $F : X \to Y$ be a non-constant holomorphic map. Then, for every $p \in X$, there exists a unique $m \geq 1$ such that for any chart (U_2, φ_2) of Y centered at $F(p)$, there exists a chart (U_1, φ_1) of X centered at p such that $\varphi_2 \circ F \circ \varphi_1^{-1} : z \mapsto z^m$ for all $z \in \varphi_1(U_1)$.*

This theorem also give easy proofs of some elementary properties of holomorphic maps, which we collect here; see [For81, section 1.2] for details. Throughout, $F : X \to Y$ is a non-constant holomorphic map between Riemann surfaces X and Y.

- F is an open map.
- If F is injective, then it is biholomorphic onto its image.
- If $Y = \mathbb{C}$, then $|F|$ does not attain its maximum.
- If X is compact, then F is surjective and Y is compact.

Together, the last two claims give an alternative proof for Theorem 1.15.

Proof. Let (U_2, φ_2) be a chart of Y centered at $F(p)$ and consider any chart (V, ψ) of X centered at p. Then the function $h := \varphi_2 \circ F \circ \psi^{-1}$ is holomorphic, so it admits a power series representation $h(w) = \sum_{i=0}^\infty c_i w^i$ for all $w \in \psi(V)$. Note that $h(0) = \varphi_2(F(p)) = 0$, so $c_0 = 0$. Let $m \geq 1$ be the smallest integer such that $c_m \neq 0$, so

$$h(w) = \sum_{i \geq m} c_i w^i = w^m \sum_{i \geq 0} c_{i-m} w^i =: w^m g(w).$$

Then g is holomorphic at 0 with $g(0) = c_m \neq 0$, so there is a function h holomorphic on some neighborhood W of 0 such that $(h(w))^m = g(w)$ for all $w \in W$. Thus $h(w) = (wh(w))^m$, so set $\eta(w) := wh(w)$ for all $w \in W$. Note that $\eta'(0) = h(0) \neq 0$, so η is invertible on some neighborhood $W' \subseteq W$ of 0. Set $U_1 := \psi^{-1}(W')$ and $\varphi_1 := \eta \circ \psi$. Then (U_1, φ_1) is a chart of X centered at p such that

$$\left(\varphi_2 \circ F \circ \varphi_1^{-1}\right)(z) = \left(\varphi_2 \circ F \circ \psi^{-1} \circ \eta^{-1}\right)(z) = h\left(\eta^{-1}(z)\right) = \left[\eta\left(\eta^{-1}(z)\right)\right]^m = z^m$$

for all $z \in \varphi_1(U_1)$. To show uniqueness, it suffices to show that such an m is chart-independent. But this is clear, for if a different chart U_2' is chosen such that F acts as $z \mapsto z^n$ for some neighborhood U_1' of p, then $z^n = z^m$ on $\varphi_1(U_1) \cap \varphi_1'(U_1')$ forces $n = m$. ■

Definition 1.23. *With the above notation, the unique $m \geq 1$ such that there are local coordinates around p and $F(p)$ where F acts like $z \mapsto z^m$ is called the multiplicity of f at p, denoted* $\operatorname{mult}_p(f)$.

Remark. We give a simple way of computing $\operatorname{mult}_p(F)$ that does not involve casting F into Local Normal Form, or even having to find local coordinates centered at p and $F(p)$. Indeed, let (U_1, φ_1) and (U_2, φ_2) be charts around p and $F(p)$, say with $z_0 \coloneqq \varphi_1(p)$ and $w_0 \coloneqq \varphi_2(F(p))$. Letting $f \coloneqq \varphi_2 \circ F \circ \varphi_1^{-1}$, we see that $f(z_0) = w_0$ and hence its power series representation has the form

$$f(z) = f(z_0) + \sum_{i \geq m} c_i (z - z_0)^i$$

for some $m \geq 1$ with $c_m \neq 0$. Then, since $z - z_0$ and $w - w_0 = f(z) - f(z_0)$ are local coordinates centered at p and $F(p)$, respectively, we see from the above proof that $\operatorname{mult}_p(F) = m$. But also

$$\frac{\mathrm{d}f}{\mathrm{d}z} = \sum_{i \geq m} i c_i (z - z_0)^{i-1},$$

so $d_{m-1} \coloneqq m c_m$ is the minimal non-zero coefficient its Laurent series. Thus $\operatorname{ord}_p(\mathrm{d}f/\mathrm{d}z) = m - 1$, so

$$\operatorname{mult}_p(F) = 1 + \operatorname{ord}_p\left(\frac{\mathrm{d}f}{\mathrm{d}z}\right).$$

♦

Theorem 1.24. *Let f be a meromorphic function on a Riemann surface X and let $F : X \to \hat{\mathbb{C}}$ be its associated holomorphic map. Fix $p \in X$.*

- *If p is a not a pole of f, then* $\operatorname{mult}_p(F) = \operatorname{ord}_p(f - f(p))$.
- *If p is a pole of f, then* $\operatorname{mult}_p(F) = -\operatorname{ord}_p(f)$.

Proof. Suppose that p is not a pole of f.

Chapter 2

Case for $g = 0$ and $g = 1$

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