Possible Choices of Complex Structures for a Surface

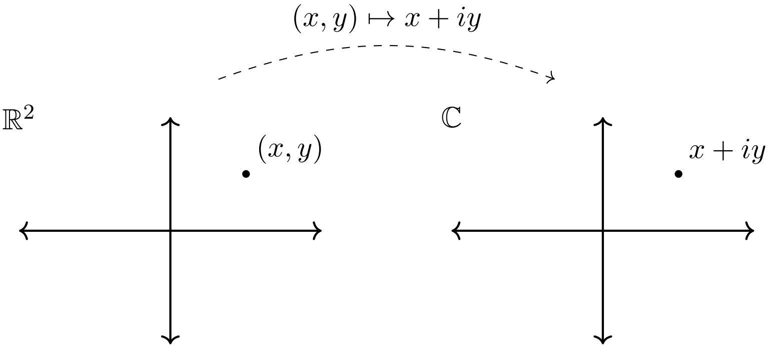
Moduli Spaces of Riemann Surfaces

Zhaoshen Zhai

Graduate mentor: Kaleb Ruscitti

April 28, 2023

Complex structures on $\mathbb{R}^2$



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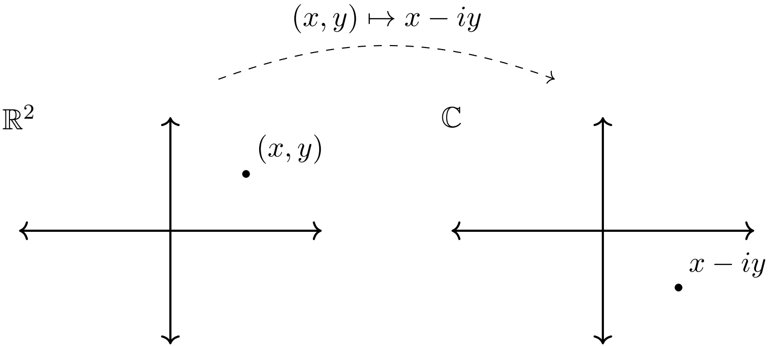

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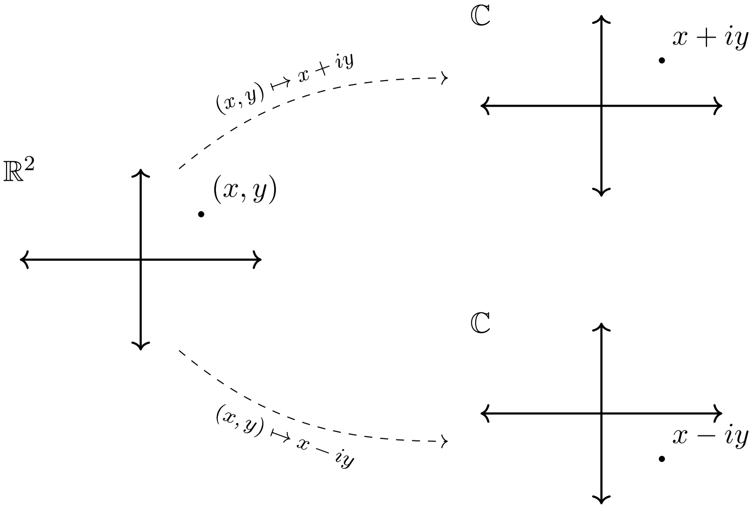

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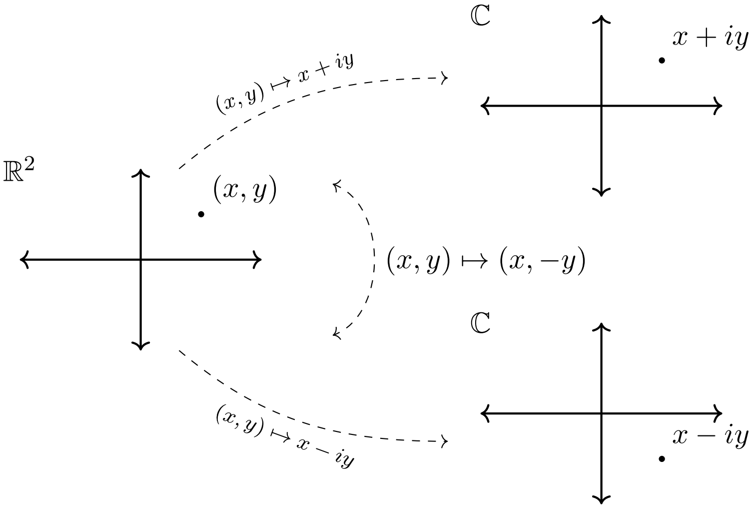

Refined question

How many complex structures on $\mathbb{R}^2$?

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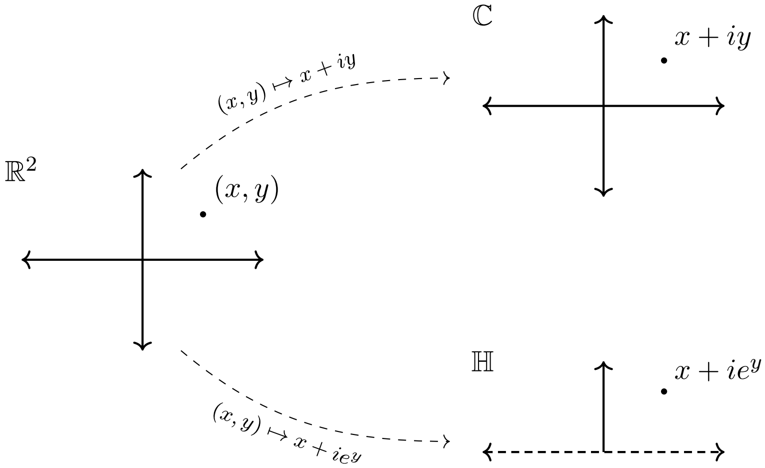
How many complex structures on $\mathbb{R}^2$ *up to biholomorphism*?

$\mathbb{C}$ and $\mathbb{H}$



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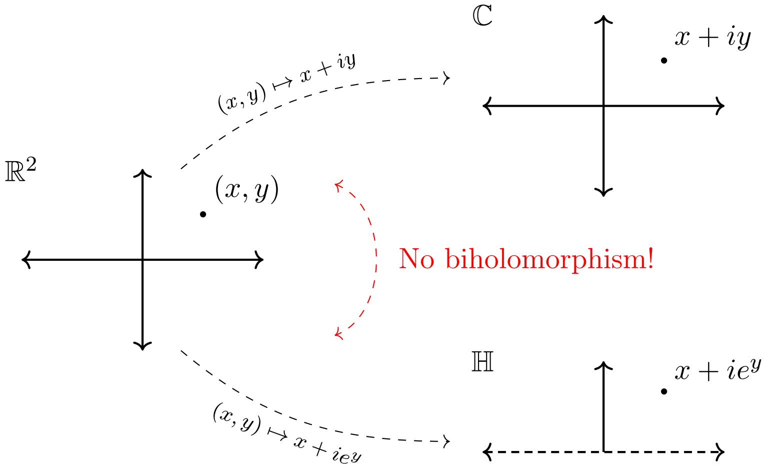
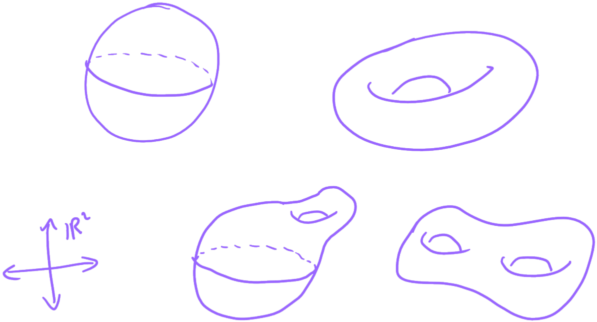


Definition

A <u>*Riemann surface*</u> *is a surface equipped with a complex structure.*



Compact surfaces

Theorem (Classification of Surfaces)

For each natural number g, there exists exactly one compact orientable surface up to homeomorphism.

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Topological torus

Theorem

Every torus is a quotient $\mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2)$ *for some* $\mathbb{R}$*-linearly independent vectors* $\omega_1, \omega_2 \in \mathbb{C}$*.*

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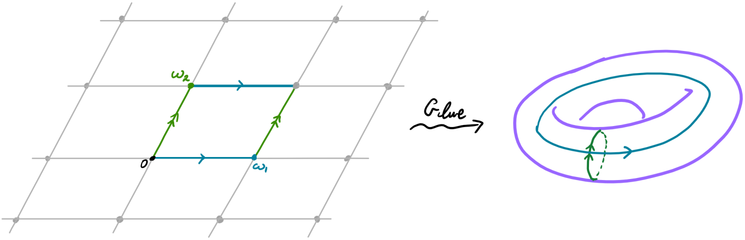
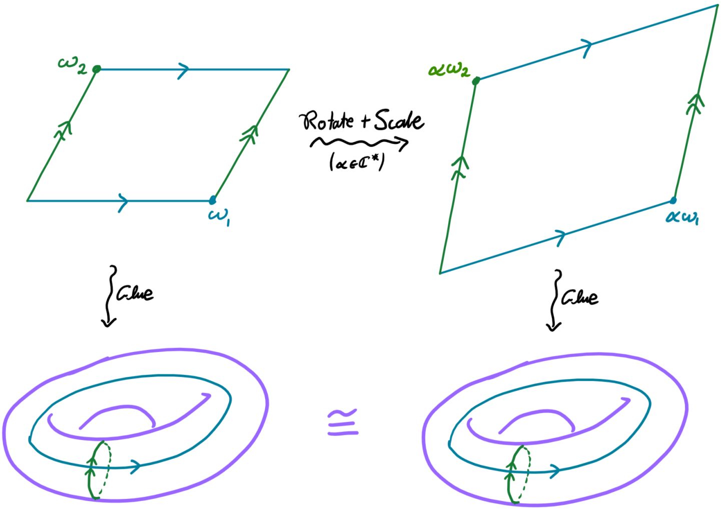


Complex tori



Complex tori

Theorem

Every complex torus is the quotient $X_\tau := \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$ *for some* $\tau \in \mathbb{H}$. *Indeed, for all linearly independent vectors* $\omega_1, \omega_2 \in \mathbb{C}$,

$$\mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2) \cong X_\tau$$

for $\tau := \omega_2/\omega_1 \in \mathbb{H}$.

Fundamental domain

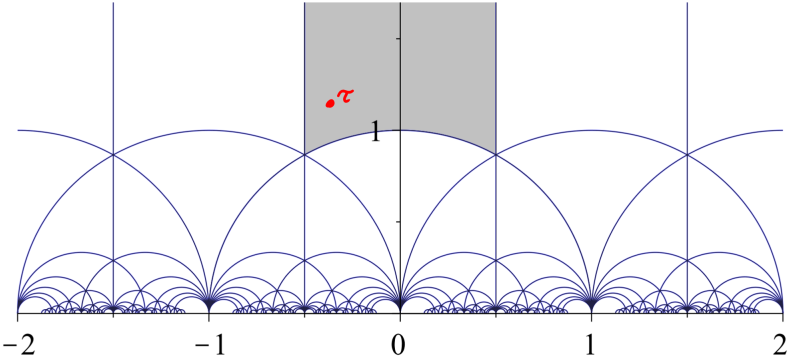


Figure: A fixed $\tau \in \mathbb{H}$.

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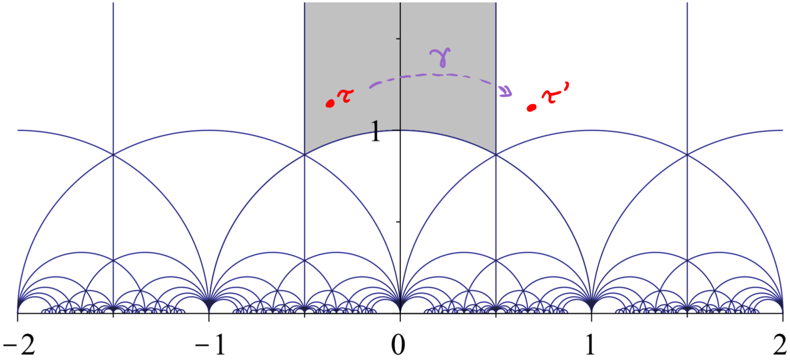

Figure: $X_\tau \cong X_{\tau'}$.

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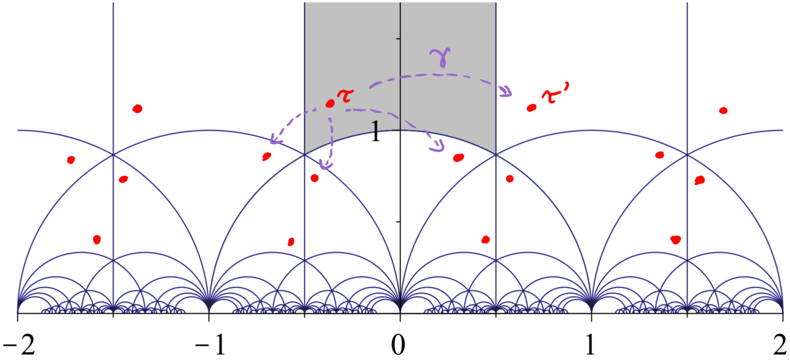

Figure: $X_\tau \cong X_{\color{red}\bullet}$.

Moduli space of the torus

Definition

The modular group $\mathrm{PSL}_2(\mathbb{Z})$ *is the group of functions* $\gamma : \mathbb{H} \to \mathbb{H}$ *mapping*

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}$$

for some $a, b, c, d \in \mathbb{Z}$ *with* $ad - bc = 1$.

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For any $\tau, \tau' \in \mathbb{H}$, *the tori* X_τ *and* $X_{\tau'}$ *are biholomorphic iff there exists some* $\gamma \in \mathrm{PSL}_2(\mathbb{Z})$ *such that* $\tau' = \gamma(\tau)$.

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Corollary

The moduli space of the torus is $\mathbb{H}/\,\mathrm{PSL}_2(\mathbb{Z})$.

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